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Flows of G_2-structures on 7-manifolds with symmetry

Fluxos de G_2-estruturas sobre 7-variedades com simetria

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Resumo

A tese é composta de duas partes em relação as G_2-estruturas. Na primeira parte, vamos descrever o espaço de G2-estruturas Sp(2)-invariantes de dimensão 10 sobre o espaço homogêneo $\mathbb{S}^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1)$, em que $\mathbb{S}^7$ é a esfera 7-dimensional. Nesta parte, foi formulado um Ansatz geral para as G_2-estruturas que realizam os representantes em cada uma das 7 possíveis classes isométricas de G_2-estruturas homogêneas. Mais ainda, as bem conhecidas G_2-estruturas quase-paralelas na esfera redonda e esmagada acontecem em polos diferentes de uma $\mathbb{S}^3$-família, cujo equador é uma nova $\mathbb{S}^2$-família de G_2-estrutura cofechadas satisfazendo a condição $\mathrm{div} T = 0$.

Na segunda parte, nos estudamos o cofluxo laplaciano de G2-estruturas proposto por Karigiannis, McKay and Tsui in [25] sobre variedades de contato Calabi-Yau usando como valor inicial a G_2-estrutura dada por Habib e Vezoni em [22] encontrando uma singularidade. Nós demostramos que a métrica e o volume colapsam nesta singularidade. Também analisamos soluções, tipo soliton do cofluxo laplaciano de G2-estruturas sobre variedades de contato Calabi-Yau dadas por Sá Earp e Lotay em [32].

Palavras-chave: G_2-estruturas, G_2-fluxo, espaços homogêneos.

Abstract

In this thesis we deal with two topics in G_2-geometry, the first goal is to describe the 10-dimensional space of Sp(2)-invariant G_2-structures on the homogeneous 7-sphere $\mathbb{S}^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1)$ as $\Omega^3_+(\mathbb{S}^7)^{\mathrm{Sp}(2)} \simeq \mathbb{R}^+ \times \mathrm{Gl}^+(3, \mathbb{R})$. In those terms, we formulate a general Ansatz for G_2-structures, which realises representatives in each of the 7 possible isometric classes of homogeneous G_2-structures. Moreover, the well-known nearly parallel *round* and *squashed* metrics are defined by different pairs of poles in an $\mathbb{S}^3$-family, the equator of which is a new $\mathbb{S}^2$-family of coclosed G_2-structures satisfying the harmonicity condition $\mathrm{div}\, T = 0$.

In the second part, we study the Laplacian coflow of G_2-structures proposed by Karigiannis, McKay and Tsui in [25] on Contact Calabi-Yau 7-manifolds using the initial coclosed G_2-structure given by Habib and Vezzoni in [22] and finding a singularity. We show that the metric and the volume collapse at this singularity. Also, we analyze soliton solutions of the Laplacian coflow on Contact Calabi-Yau 7-manifolds using G_2-structure given by Sá Earp and Lotay [32].

Keywords: G_2-structures, G_2-flow, Homogeneous spaces.

Contents

Introduction

Geometric flows have proven to be a powerful tool in geometric analysis to solve a variety of geometric and topological problems. In the case of G_2-geometry, they provide a method to search for metrics with G_2 holonomy, which are then Ricci-flat, on an oriented, spin 7-manifold M by varying a G_2-structure, given by a non-degenerate 3-form φ on M, so that it becomes torsion-free. Flows in G_2-geometry were first introduced by Bryant [9] and have been studied by several authors (see for instance [1, 7, 27, 25, 24, 20, 34]).

We may then define the *full torsion tensor* of the G_2-structure φ as $T := \nabla^{g_\varphi}\varphi$, where ∇^{g_φ} is the Levi-Civita connection of g_φ. Pairs (M^7, φ) which satisfy $T \equiv 0$ (i.e. so that φ is torsion-free) are called *G_2-manifolds* and complete examples are very difficult to construct, especially when M is required to be compact. Fernandez and Gray [16] showed that the torsion-free condition is equivalent to $\varphi \in \Omega^3_+$ being both *closed* and *coclosed*, i.e, $\mathrm{d}\varphi = 0$ and $\mathrm{d}*_\varphi\varphi = 0$ respectively (where $*_\varphi$ is the Hodge star defined by the metric and orientation induced by φ). This alternative viewpoint of the torsion-free condition as a system of nonlinear PDE is fundamental to G_2 geometry and to geometric flows. Thus, given a smooth manifold, are there any best (or nicest, or most distinguished) G_2-structures on M? The question remains natural when restricted to special kinds of manifolds or particular classes of G_2-structures, like the set of all G_2-structures with the same associated metric, left-invariant G_2-structures on a given Lie group, etc.

G_2-Geometry on the sphere has attracted significant interest over the past decade, perhaps most notably in the classification of homogeneous structures by Reidegeld [39], the description of 7-dimensional homogeneous spaces with isotropy representation in G_2 by Munir & Le [42] and the study of the 7-sphere's calibrated geometry by Lotay [30] and Kawai [26]. In particular, G_2-structures on $\mathrm{Sp}(2)/\mathrm{Sp}(1)$ was described in [35]. Since, each $\mathrm{Sp}(2)$-invariant G_2-structure on $\mathbb{S}^7$ is determined by a G_2-structure on $\mathfrak{p} \simeq T_o\mathbb{S}^7$ which is invariant by the $\mathrm{Ad}(\mathrm{Sp}(1))$-action, i.e. $\Omega^3_+(\mathbb{S}^7)^{\mathrm{Sp}(2)} \simeq \Lambda^3_+(\mathfrak{p}^*)^{\mathrm{Ad}(\mathrm{Sp}(1))}$ (see [35]) and $\mathbb{S}^7$ is spinnable, the map (see [23, Eq(11)])

$$B : \varphi \in \Omega^3_+(\mathbb{S}^7) \mapsto g_\varphi \in \mathrm{Sym}^2_+(T^*\mathbb{S}^7),$$

associating a Riemmanian metric to each G_2-structure, is surjective. Each preimage subset $\mathcal{B}_\varphi := B^{-1}(g_\varphi)^{\mathrm{Sp}(2)}$, consisting of the homogeneous G_2-structures defining the same Riemannian metric, is called the *($\mathrm{Sp}(2)$-invariant) isometric class of φ.*

$$\mathcal{B}_\varphi \simeq (a, D \cdot \mathrm{SO}(3)),$$

for each $g_\varphi \in \mathrm{Sym}^2_+(T^*\mathbb{S}^7)^{\mathrm{Sp}(2)}$ and the space of $\mathrm{Sp}(2)$-invariant G_2-structures on $\mathbb{S}^7 \simeq \mathrm{Sp}(2)/\mathrm{Sp}(1)$ is described by the homogeneous manifold $\Omega^3_+(\mathbb{S}^7)^{\mathrm{Sp}(2)} \simeq \mathbb{R}^+ \times \mathrm{GL}^+(3, \mathbb{R})$, via the isomorphism $\Omega^3_+(\mathbb{S}^7)^{\mathrm{Sp}(2)} \simeq \Lambda^3_+(\mathfrak{p}^*)^{\mathrm{Ad(Sp(1))}}$ and the map

$$\begin{aligned}\Theta: \quad \mathbb{R}^+ \times \mathrm{GL}^+(3,\mathbb{R}) &\to \Lambda^3_+(\mathfrak{p}^*)^{\mathrm{Ad(Sp(1))}}\\ (a,D) &\mapsto \Theta(a,D) = \varphi_{a,D}.\end{aligned}$$

We know from [43] that, up to isometry, any Sp(2)-invariant metric is a multiple of the inner product expressed in terms of an oriented left-invariant coframe $e^1,\dots,e^7 \in \Omega^1(\mathbb{S}^7)^{\mathrm{Sp}(2)}$ by

$$g_{\mathbf{r}} = \frac{1}{r_1^6}(e^1)^2 + \frac{1}{r_2^6}(e^2)^2 + \frac{1}{r_3^6}(e^3)^2 + r_1r_2r_3\Big((e^4)^2 + (e^5)^2 + (e^6)^2 + (e^7)^2\Big), \tag{1}$$

with $r_1,\dots,r_3 > 0$ and $\mathbf{r} := (r_1,r_2,r_3)$. The corresponding isometric class of G_2-structures is parametrised by

$$\begin{aligned}\varphi_{(\mathbf{r},h)} = {} & \frac{1}{(r_1r_2r_3)^3}e^{123}\\ & + \Big(\frac{r_2r_3}{r_1^2}(h_0^2+h_1^2-h_2^2-h_3^2)e^1 + 2\frac{r_1r_3}{r_2^2}(h_1h_2-h_0h_3)e^2 + 2\frac{r_1r_2}{r_3^2}(h_1h_3+h_0h_2)e^3\Big)\wedge\omega_1\\ & + \Big(2\frac{r_2r_3}{r_1^2}(h_1h_2+h_0h_3)e^1 + \frac{r_1r_3}{r_2^2}(h_0^2-h_1^2+h_2^2-h_3^2)e^2 + 2\frac{r_1r_2}{r_3^2}(h_2h_3-h_0h_1)e^3\Big)\wedge\omega_2\\ & + \Big(2\frac{r_2r_3}{r_1^2}(h_1h_3-h_0h_2)e^1 + 2\frac{r_1r_3}{r_2^2}(h_2h_3+h_0h_1)e^2 + \frac{r_1r_2}{r_3^2}(h_0^2-h_1^2-h_2^2+h_3^2)e^3\Big)\wedge\omega_3,\end{aligned} \tag{2}$$

with $(h_0,...,h_3) \in \mathbb{H}$ a unit quaternion parametrising a SO(3)-transformation and $\omega_1,\dots,\omega_3 \in \Omega^2(\mathbb{S}^7)^{\mathrm{Sp}(2)}$ as in (2.11).

Let me now briefly describe the contents: Chapter 1 we discuss relevant preliminary results on G_2-structures. In chapter 2 we present a detailed study of the space $\Omega^3_+(\mathbb{S}^7)^{\mathrm{Sp}(2)}$ of Sp(2)-invariant G_2-structures on the homogeneous 7-sphere Sp(2)/Sp(1) obtaining following Theorem:

Theorem 3. *Let $\varphi_{(\mathrm{r},h)}$ be the G_2-structure given by (2) with $r_1 = r_2 = r_3 = r^{-1/3}$. Then each $\mathbb{S}^3$-family of G_2-structures*

$$\mathcal{B}_r := \Theta(\{r\}\times \mathrm{SO}(3)) = B^{-1}(g_r)^{\mathrm{Sp}(2)} \simeq \mathbb{S}^3$$

determines a distinct isometric class. Moreover, in terms of the equator and poles

$$\mathbb{S}^2 \simeq \{(h_0,h_1,0,h_3)\} \quad \textit{and} \quad \mathrm{NS} \simeq \{(0,0,\pm1,0)\},$$

we characterise the following torsion regimes in each isometric class $\mathcal{B}_r$, up to the diffeomorphism Φ:

1. *The coclosed G_2-structures correspond to $\{r\}\times\mathbb{S}^2$ and $\{r\}\times\mathrm{NS}$.*
2. *The nearly parallel G_2-structures correspond to $\{\sqrt[3]{2}\}\times\mathbb{S}^2$ (round) and $\{\sqrt[3]{2/5}\}\times\mathrm{NS}$ (squashed).*
3. *The locally conformal coclosed G_2-structures correspond to $\{1\}\times\mathbb{S}^3$.*

Furthermore, there are no locally conformal closed or purely coclosed structures in $\mathcal{B}_r$.

In Chapter 3, our goal is to study the *Laplacian coflow* of G_2-structures, introduced by Karigiannis, McKay and Tsui [25][1] which is given by

$$\frac{\partial \psi_t}{\partial t} = \Delta_{\psi_t}\psi_t, \tag{4}$$

where $\psi_t := *_t\varphi_t$ is the Hodge dual of the G_2-structure φ_t (here, we write $*_t := *_{\varphi_t}$) and $\Delta_{\psi_t} := (\mathrm{dd}^{*_t} + \mathrm{d}^{*_t}\mathrm{d})$ is the Hodge Laplacian of $g_t := g_{\varphi_t}$ on 4-forms. If M is compact, critical points of this flow (4) are then Hodge duals of non-degenerate 3-forms which are closed and coclosed so that the corresponding G_2-structures are torsion-free.

Motivated by the article of Habib and Vezzoni [22], we consider a *contact Calabi–Yau* 7-manifold, which is a triple $(M^7, \eta, \Phi, \Upsilon)$ where (M^7, η, Φ) is a Sasakian manifold (and η is the contact form) with transverse Kähler form $\omega = \mathrm{d}\eta$, and Υ is a nowhere vanishing transversal form on $\mathcal{D} = \ker \eta$ of type $(3, 0)$ satisfying

$$\Upsilon \wedge \overline{\Upsilon} = -\mathrm{i}\omega^3 \quad \text{and} \quad \mathrm{d}\Upsilon = 0.$$

Hence, on contact Calabi-Yau 7-manifolds there exists a natural G_2-structure defined by

$$\varphi = \eta \wedge \omega + \operatorname{Re}\Upsilon,$$

with corresponding dual 4-form

$$\psi = *_\varphi\varphi = \frac{1}{2}\omega^2 - \eta \wedge \operatorname{Im}\Upsilon.$$

Since ψ is manifestly closed (as ω and Υ are closed and $\omega \wedge \Upsilon = 0$ by type considerations), we see that φ is coclosed and the torsion of φ is encoded in

$$\mathrm{d}\varphi = \omega \wedge \omega.$$

we consider a family of G_2-structures

$$\varphi_t = f_t h_t^2 \eta_0 \wedge \omega_0 + h_t^3 \operatorname{Re}(\Upsilon_0)$$

for $f_t, h_t \in \mathbb{R}$ where $\omega_0 = \mathrm{d}\eta_0$ on a contact Calabi-Yau $(M^7, \eta_0, \Phi_0, \Upsilon_0)$ and we prove the following.

Theorem 5. *Let* $(M^7, \eta_0, \Phi_0, \Upsilon_0)$ *be a contact Calabi-Yau 7-manifold. The family of coclosed* G_2*-structures* φ_t *on* M^7 *given by*

$$\varphi_t = \mathrm{p}(t)^{-1/10}\eta_0 \wedge \omega_0 + \mathrm{p}(t)^{3/10}\operatorname{Re}\Upsilon_0; \tag{6}$$

$$\psi_t = \frac{1}{2}\mathrm{p}(t)^{2/5}\omega_0^2 - \eta_0 \wedge \operatorname{Im}\Upsilon_0, \tag{7}$$

where $\mathrm{p}(t) = 10t + 1$ *and* $t \in (-1/10, \infty)$*, solves the Laplacian coflow* (4) *with initial data determined by* $\varphi_0 = \eta_0 \wedge \omega_0 + \operatorname{Re}\Upsilon_0$.

[1] In [25] the flow (4) is written with an additional minus sign on the right-hand side, but this seems incorrect given the works on the Laplacian flow [7, 34] and the modified Laplacian coflow [18].

Remark 8. We see that if M^7 is compact then the volume of M determined by the G_2-structure φ on M is:

$$\mathcal{H}(\varphi) := \mathrm{Vol}(M, \varphi) = \frac{1}{7}\int_M \phi \wedge \psi.$$

Along the Laplacian coflow solution given in Theorem 3.22 for a compact contact Calabi–Yau 7-manifold we have that

$$\mathcal{H}(\varphi_t) = (10t + 1)^{3/10}\mathcal{H}(\varphi_0).$$

Hence, the Hitchin functional on the cohomology class $[\psi_0]$, which is just $\mathcal{H}(\varphi_t)$, tends to infinity as $t \to \infty$ and tends to 0 as $t \to -1/10$. In particular, the Hitchin functional is unbounded on $[\psi_0]$.

We see that the Laplacian coflow in Theorem 3.22 has singularities. In the analysis of singularities of flows of G_2-structures, as can be seen for example in the work of Lotay and Wei [29] and later work of Gao Chen [11], it is useful to study the quantity

$$\Lambda(x,t) = (|Rm(x,t)|^2_{g_t} + |T(x,t)|^4_{g_t} + |\nabla^{g_t} T(x,t)|^2_{g_t})^{\frac{1}{2}}. \tag{9}$$

Considering this, we compute $\Lambda(x,t)$ for the family (3.23) obtaining the following.

Theorem 10. *Let φ_t be the solution to the Laplacian coflow (4) given by (3.24) on a contact Calabi-Yau manifold $(M_0, \eta_0, \Phi_0, \Upsilon_0)$. Then, for $t \in (-1/10, \infty)$ and $x \in M$ we have $\Lambda(x,t)$ given in (9) is given by*

$$\Lambda(x,t) = (\mathrm{p}(t)^{-2/5}|Rm_0^{\mathcal{D}_0}|^2_0 + k\mathrm{p}(t)^{-2})^{1/2} \tag{11}$$

where $k > 0$ is a constant.In particular, for $t \in [\beta, \gamma] \subset (-1/10, \infty)$ and $\sup_M |Rm_0^{\mathcal{D}_0}|_0 \leq C$ we have

$$\Lambda(t) = \sup_M \Lambda(x,t) \leq K\mathrm{p}(t)^{-1}(1 + c\mathrm{p}(t)^{8/5})^{1/2}$$

for $c, K > 0$ constant.

Also, we present definitions and important results about compactness for Ricci-like flows [12, 20] which will be used to show that the solution of the Laplacian coflow (3.23) with singularity in $t = -1/10$ is collapsing as follows.

Theorem 12. *Let $(M_0^7, \eta_0, \Phi, \Upsilon_0)$ be a contact Calabi–Yau 7-manifold and ψ_t the solution of the Laplacian coflow given by (3.23) with $t \in (-1/10, \infty)$. Then ψ_t is volume collapsing and collapsing with respect to the normalized metric at $-1/10$.*

We also study solitons for the Laplacian coflow of coclosed G_2-structures, which are expected to play an important role in understanding the local behaviour of the flow at singularities. It is well-known, and easy to see from the viewpoint of the Laplacian coflow as the gradient flow of the Hitchin functional, that the only possible compact solitons which are not torsion-free must have $\lambda > 0$ (See section 3.4).

Motivated by the article Lotay and Sá Earp [32], we analyse solitons on a contact Calabi-Yau manifold M. In general, there are no shrinking solitons for the coflow. Furthermore, In particular, we show that for a coclosed G_2-structure φ with associated metric g and a vector field X on M, we have an orthogonal decomposition

$$\mathcal{L}_X\psi = \frac{4}{7}\mathrm{div}(X)\psi + \mathrm{Curl}(X)^\flat \wedge \varphi + *i_\varphi\Big(\frac{3}{49}\mathrm{div}(X)g - \frac{3}{14}(\mathcal{L}_X g)\Big), \tag{13}$$

where $i_\varphi : C^\infty(S^2T^*M) \to \Omega^3(M)$ is the injective map given in (1.6). In particular, any infinitesimal symmetry of the coclosed G_2-structure φ generated by a vector field X satisfies

$$\mathcal{L}_X g = 0 \quad \text{and} \quad \mathrm{Curl}(X) = 0.$$

The above result give an idea about soliton solutions of Laplacian coflow. Concluding that in the particular case of CCY manifolds, we may strengthen this result as follows:

Proposition 14. *Let $(M, \eta, \Phi, \Upsilon)$ be a closed contact Calabi-Yau 7-manifold, fibering by $M^7 \to V$ over the Calabi-Yau 3-fold (V, ω, J, Υ). For each $\varepsilon > 0$, a S^1-invariant G_2-structure given by*

$$\varphi_\varepsilon = \varepsilon\eta \wedge \omega + \mathrm{Re}\,\Upsilon \tag{15}$$

$$\psi_\varepsilon = \frac{1}{2}\omega^2 - \varepsilon\eta \wedge \mathrm{Im}\,\Upsilon. \tag{16}$$

Then, any solitons $(\psi_\varepsilon, X, \lambda)$ for the Laplacian coflow (4) inducing the metric g_ε on M must have $X^\flat \in \Omega^1(M)$ harmonic and

$$\lambda = \varepsilon^2. \tag{17}$$

1 G_2-geometry

In this introductory chapter we briefly review about G_2-structures, introducing important definitions that will be used in the following chapters. These can be found e.g. in [9, 18, 23, 27, 39].

1.1 Linear G_2-structures

In this section, we will give the main results about the group G_2 that will be needed.

The octonion $\mathbb{O}$ is an 8-dimensional real division algebra which is normed, i.e. it admits an inner product $\langle \cdot, \cdot \rangle$ such that $|uv| = |u||v|$ for all $u, v \in \mathbb{O}$. Analogously to the set of quaternion numbers $\mathbb{H}$, the octonion product defines a skew-symmetric bilinear map $\times : \operatorname{Im} \mathbb{O} \times \mathbb{O} \to \operatorname{Im} \mathbb{O}$ by $u \times v := \operatorname{Im} uv$, which turns to be a *cross prodct*, in the sense that

$$u \times v \perp u, v, \quad \text{and} \quad |u \times v|^2 = |u|^2|v|^2 - \langle u, v \rangle^2, \quad \forall u, v \in \operatorname{Im} \mathbb{O}.$$

Then, we can define a 3-form given by

$$\phi(u, v, w) := \langle u \times v, w \rangle.$$

With respect to a suitable oriented and orthonormal basis $\{e_1, \cdots, e_7\}$ of $\operatorname{Im} \mathbb{O}$, the 3-form ϕ is written as

$$\phi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}, \tag{1.1}$$

where $e^{ijk} := e^i \wedge e^j \wedge e^k$ and $\{e^i\}$ is the dual basis of $\{e_i\}$. We identify $\mathbb{R}^7 = \operatorname{Im} \mathbb{O}$. Therefore, the subgroup of $\mathrm{GL}(7, \mathbb{R})$ that fixes ϕ is the compact, connected, simple Lie group G_2 (see [6]).

Definition 1.2 (The group G_2)**.**

$$\mathrm{G}_2 = \{h \in \mathrm{GL}(7, \mathbb{R}) : h^*\phi = \phi\}$$

The group G_2 acts irreducibly on $\mathbb{R}^7$ and preserves the metric and orientation for which the basis $\{e_1, e_2, \cdots, e_7\}$ is an oriented orthonormal basis. The Hodge star operator determined by this metric and orientation will be denoted by $*_\phi$. Note that in particular G_2 also fixes the 4-form

$$*_\phi \phi = e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1245}.$$

The group G_2 acts transitively on the unit sphere $S^6 \subset \mathbb{R}^7$. The stabilizer subgroup of any non-zero vector in $\mathbb{R}^7$ is isomorphic to $\mathrm{SU}(3) \subset \mathrm{SO}(6)$, so that $S^6 = \mathrm{G}_2/\mathrm{SU}(3)$. Since $\mathrm{SU}(3)$ acts transitively on $S^5 \subset \mathbb{R}^6$, it follows that G_2 acts transitively on the set of orthonormal pairs of vectors in $\mathbb{R}^7$.

Sometimes the ε-symbol will be useful to describe the algebra $\mathfrak{g}_2 = \mathrm{Lie}(\mathrm{G}_2)$. It is the totally skew-symmetric symbol which defines ϕ as $\phi = \dfrac{1}{6} \sum_{i,j,k} \varepsilon_{ijk} e^{ijk}$, or equivalently, $e_i \times e_j = \sum_k \varepsilon_{ijk} e_k$.

Then, the Lie algebra $\mathfrak{g}_2$ of G_2 described as a subalgebra of $\mathfrak{so}(7)$ can be given as

$$\mathfrak{g}_2 = \Big\{ A = [a_{ij}] \in \mathfrak{so}(7) : \sum_{j,k} a_{ij}\varepsilon_{ijk} = 0, \forall i \Big\}.$$

On the other hand, cross-product left-multiplication defines a seven-dimensional subspace of $\mathfrak{so}(7)$,

$$\mathfrak{q}_7 = \Big\{ [v_{ij}] : v \in \mathbb{R}^7, v_{ij} := \sum_k \varepsilon_{ijk}\langle v, e_k\rangle \Big\},$$

such that $\mathfrak{so}(7) = \mathfrak{g}_2 \oplus \mathfrak{q}_7$. This is a reductive decomposition for the (symmetric) homogeneous space $\mathbb{RP}^7 = \mathrm{SO}(7)/\mathrm{G}_2$.

1.1.1 Definite form

The dimension of G_2 is 14, therefore, by dimension count, the GL-orbit of ϕ in $\Lambda^3(V^*)$ is open and diffeomorphic to $\mathrm{GL}(V)/\mathrm{G}_2$. We denote the orbit by $\Lambda^3_+(V^*)$ and call the elements of $\Lambda^3_+(V^*)$ as *definite* 3-forms on V. Note that $\Lambda^3_+(V^*)$ has two components, since $\mathrm{GL}(V)$ does and since G_2 is connected. Each component is the negative of the other.

The canonical map $\mathrm{S} : \Lambda^3_+(V^*) \to \Lambda^4(V^*)$ defined by $\mathrm{S}(\varphi) = *_\varphi \varphi$ is a double covering onto an open set $\Lambda^4_+(V^*)$ in $\Lambda^4(V^*)$, which will be referred to as the space of *definite* 4-forms on V. The $\mathrm{GL}(V)$-stabilizer of an element $\psi \in \Lambda^4_+(V^*)$ is then isomorphic to $\pm\mathrm{G}_2 = \mathrm{G}_2 \cup (\mathrm{G}_2 \cdot (-\mathrm{id}_V))$. Thus, a definite 4-form on V defines an inner product on V but not an orientation.

1.1.2 The G_2-decomposition of exterior forms

In this section, we avoid writing $\mathbb{R}^7$ and let V be a vector space of dimension 7. Although G_2 acts irreducibly on V and hence on $\Lambda^1(V^*)$ and $\Lambda^6(V^*)$, it does not act irreducibly on $\Lambda^p(V^*)$ for $2 \le p \le 5$. In order to understand the irreducible decomposition of $\Lambda^p(V^*)$ for p in this range, it suffices to understand the case $p = 2$ and $p = 3$, since the operator $*_\phi$ induces an isomorphism of G_2-modules $\Lambda^p(V^*) = \Lambda^{7-p}(V^*)$.

In [8], it is shown that there are irreducible G_2-modules decompositions

$$\Lambda^2(V^*) = \Lambda^2_{14}(V^*) \oplus \Lambda^2_7(V^*) \tag{1.3}$$

$$\Lambda^3(V^*) = \Lambda^3_1(V^*) \oplus \Lambda^3_7(V^*) \oplus \Lambda^3_{27}(V^*), \tag{1.4}$$

where $\Lambda^p_d(V^*)$ denotes an irreducible G_2-module of dimension d. For $p = 4$ or $p = 5$, adopt the convention that $\Lambda^p_d(V^*) = *(\Lambda^{7-p}_d(V^*))$. These summands can be characterized as follows:

$$\begin{aligned}
\Lambda^2_7(V^*) =& \{ *_\phi(\alpha \wedge *_\phi \phi) : \alpha \in \Lambda^1(V^*) \} \\
=& \{ \alpha \in \Lambda^2(V^*) : \alpha \wedge \phi = 2 *_\phi \alpha \} \\
\Lambda^2_{14}(V^*) =& \{ \alpha \in \Lambda^2(V^*) : \alpha \wedge \phi = - *_\phi \alpha \} = \mathfrak{g}_2^\flat \\
\Lambda^3_1(V^*) =& \{ r\phi : r \in \mathbb{R} \} \\
\Lambda^3_7(V^*) =& \{ *_\phi(\alpha \wedge \phi) : \alpha \in \Lambda^1(V^*) \} \\
\Lambda^3_{27}(V^*) =& \{ \alpha \in \Lambda^3(V^*) : \alpha \wedge \phi = 0 \quad \text{and} \quad \alpha \wedge *_\phi \phi = 0 \} = i_\phi(S^2_0(V^*)),
\end{aligned} \tag{1.5}$$

where,the notation $\mathfrak{g}_2^\flat$ is the *musical isomorphism* $\flat : V \to V^*$ induced by the G_2-invariant inner product $\langle \cdot, \cdot \rangle_\phi$, the Lie algebra of G_2, namely $\mathfrak{g}_2 \subset V \otimes V^*$, is identified with $\mathfrak{g}_2^\flat = (\flat \otimes 1)(\mathfrak{g}_2) \subset \Lambda^2(V^*) \subset V^* \otimes V^*$. This space is an irreducible G_2-module since G_2 is simple. On the other hand, consider the linear mapping $\mathrm{i}_\phi : S^2(V^*) \to \Lambda^3(V^*)$, defined on decomposable elements by

$$\mathrm{i}_\phi(\alpha \circ \beta) = \alpha \wedge *_\phi(\beta \wedge *_\phi \phi) + \beta \wedge (\alpha \wedge *_\phi \phi). \tag{1.6}$$

The mapping i_ϕ is G_2-invariant and one can show that $S^2(V^*) = \mathbb{R}g_\phi \oplus S_0^2(V^*)$ is a decomposition of $S^2(V^*)$ into G_2-irreducible summands which is the scalar multiple of g_ϕ and the traceless factor $S_0^2(V^*) \subset S^2(V^*)$. Evidently, i_ϕ is nonzero on each summand and it is therefore injective. Hence, the image $\mathrm{i}_\phi(S_0^2) \subset \Lambda^3(V^*)$ is 27-dimensional and irreducible. Equation (1.5) defines $\Lambda^3_{27}(V^*)$ as a G_2-invariant, 27-dimensional subspace of $\Lambda^3(V^*)$. Thus, by dimension, it must intersect $\mathrm{i}_\phi(S_0^2(V^*))$ and since it is G_2-irreducible, $\mathrm{i}_\phi(S_0^2(V^*)) = \Lambda^3_{27}(V^*)$. Using the ε-notation, one can express the map i_ϕ as $\mathrm{i}_\phi(h_{ij}e^ie^j) = \varepsilon_{ikl}h_{ij}e^j \wedge e^k \wedge e^l$, making it evident that $\mathrm{i}_\phi(g_\phi) = 3\phi$. It will be useful to have a way to invert the map i_ϕ. Define $\mathrm{j}_\phi : \Lambda^3(V^*) \to S^2(V^*)$ by the formula

$$\mathrm{j}_\phi(\gamma)(v, w) = *_\phi((v \lrcorner \phi) \wedge (w \lrcorner \phi) \wedge \gamma), \tag{1.7}$$

for $\gamma \in \Lambda^3(V^*)$ and $v, w \in V$. It is not difficult to verify that

$$\mathrm{j}_\phi(\mathrm{i}_\phi(h)) = 4h + 2(\mathrm{tr}_{g_\phi}(h))g_\phi,$$

for all $h \in S^2(V^*)$. Note also that $\mathrm{j}_\phi(\phi) = 6g_\phi$, while $\mathrm{j}_\phi(\Lambda^3_7(V^*)) = 0$. Note that i_ϕ and j_ϕ are not isometries when $S_0^2(V^*)$ and $\Lambda^3_{27}(V^*)$ are given their natural metrics. Instead, $\gamma \in \Lambda^3_{27}(V^*)$ satisfies $|\mathrm{j}_\phi(\gamma)|^2 = 8|\gamma|^2$ while $h \in S_0^2(V^*)$ satisfies $|\mathrm{i}_\phi(h)|^2 = 8|h|^2$.

1.2 G_2-structures on manifolds

Let M be a manifold of dimension 7. The union of the subspaces $\Lambda^3_+(T_x^*M)$ is an open subbundle of the bundle $\Lambda^3(T^*M)$ of 3-forms on M.

Definition 1.8. A 3-form φ on M that takes values in $\Lambda^3_+(T^*M)$ will be said to be a *definite* 3-form on M and the set of definite 3-forms on M will be denoted by $\Omega^3_+(M)$.

Each definite 3-form on M defines a G_2-structure on M in the following way: let $\mathcal{F}$ denote the principal right $\mathrm{GL}(V)$-bundle over M consisting of V-coframes $u : TM \to V$. Given any $\varphi \in \Omega^3_+(M)$, define a G_2-bundle

$$F_\varphi = \{u \in \hom(T_xM, V) : x \in M \quad \text{and} \quad u^*(\phi) = \varphi_x\}. \tag{1.9}$$

Every G_2-reduction of $\mathcal{F}$ (i.e., G_2-structure on M in the usual sense) is of the form F_φ for some unique $\varphi \in \Omega^3_+(M)$. For this reason, a 3-form $\varphi \in \Omega^3_+(M)$ by abuse of language, will be usually called a G_2-structure.

Definition 1.10 (*Associated metric, orientation and vector cross product*)**.** For any $\varphi \in \Omega^3_+(M)$, denote by g_φ, $*_\varphi$ $\times_\varphi$ the metric, Hodge star operator, and vector cross product on M that are canonically associated to φ where g_φ and volume form vol_φ satisfy the following

$$6g_\varphi(X, Y)\mathrm{vol}_\varphi := (X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi.$$

The metric g_φ and orientation determine a Hodge star operator $*_\varphi$. Therefore, there exists associated dual 4-form ψ such that $\psi = *\varphi$.

We can define a metric on the space of forms and it will be explained below. Let (M, φ) be a smooth 7-manifold with G_2-structure. In a local coordinate system $(x_1, \cdots, x_7)$, a differential k-form α on M will be written as

$$\alpha = \frac{1}{k!}\alpha_{i_1\cdots i_k} dx^{i_1\cdots i_k},$$

where the sum is taken over all ordered subset $\{i_1, \cdots, i_k\} \subset \{1, \cdots, 7\}$ and $\alpha_{i_1\cdots i_k}$ is skew-symmetric in all indices, i.e., $\alpha_{i_1\cdots i_k} = \alpha(e_{i_1}, \cdots, e_{i_k})$. So, the interior product of a k-form is given by

$$e_j \lrcorner \alpha = \frac{1}{(k-1)!}\alpha_{j i_1\cdots i_{k-1}} dx^{i_1\cdots i_{k-1}}.$$

A Riemannian metric g on M induces on $\Omega^k := \Omega^k(M)$ the metric $g(dx^i, dx^j) := g^{ij}$, where (g^{ij}) denotes the inverse of the matrix (g_{ij}). Then, for decomposable k-forms we have

$$g(dx^{i_1\cdots i_k}, dx^{j_1\cdots j_k}) = \det \begin{pmatrix} g^{i_1 j_1} & \cdots & g^{i_1 j_k} \\ \vdots & \cdots & \vdots \\ g^{i_k j_1} & \cdots & g^{i_k j_k} \end{pmatrix}, \tag{1.11}$$

thus, using this convention, the inner product of two k-forms $\alpha = \frac{1}{k!}\alpha_{i_1\cdots i_k} dx^{i_1\cdots i_k}$ and $\beta = \frac{1}{k!}\beta_{j_1\cdots j_k} dx^{j_1\cdots j_k}$ is given by

$$g(\alpha, \beta) = \alpha_{i_1\cdots i_k}\beta_{j_1\cdots j_k} g^{i_1 j_1} \cdots g^{i_k j_k}.$$

The metric also determines the Levi-Civita connection ∇, and a manifold M is called a G_2-manifold if $\nabla\varphi = 0$. Note that this is a nonlinear partial differential equation for ∇, since ∇ depends on g which depends on φ in a non-linearly way. Such manifolds have Riemannian holonomy $\mathrm{Hol}_g(M)$ contained in the exceptional Lie group $G_2 \subset SO(7)$.

Remark 1.12. On a Riemannian manifold, metric compatible G_2-structures are parametrized by sections of an $\mathbb{RP}^7$-bundle, or alternatively, by sections of an $\mathbb{S}^7$-bundle, with antipodal points identified.

Also, in the context of G_2-structures will be necessary to give some definitions which will be needed later. Given a tensor γ, the rough Laplacian is defined by

$$\Delta\gamma = g^{ab}\nabla_a\nabla_b\gamma = -\nabla^*\nabla\gamma, \tag{1.13}$$

whereas the Hodge Laplacian defined by φ or ψ will be denoted by Δ_φ or Δ_ψ respectively.

For a vector field X, define the *divergence* of X as

$$\mathrm{div}X = \nabla_a X^a. \tag{1.14}$$

This operator can be extended to a 2-tensor β:

$$(\mathrm{div}\beta)_b = \nabla^a\beta_{ab}. \tag{1.15}$$

Furthermore, for a vector X, we can use the G_2-structure φ to define a Curl operator, similar to the standard one on $\mathbb{R}^3$:

$$(\text{Curl}\, X)^a = (\nabla_b X_c)\varphi^{abc}. \tag{1.16}$$

This Curl operator can then also be extended to a 2-tensor β:

$$(\text{Curl}\, \beta)_{ab} = (\nabla_m \beta_{na})\varphi_b{}^{mn}.$$

Note that when β_{ab} is symmetric, $\text{Curl}\, \beta$ is traceless. We can also use the G_2-structure 3-form to define a commutative product $\alpha \circ \beta$ of two 2-tensors α and β

$$(\alpha \circ \beta)_{ab} = \varphi_{amn}\varphi_{bpq}\alpha^{mp}\beta^{nq}. \tag{1.17}$$

Note that $(\alpha \circ \beta)^t = (\alpha^t \circ \beta^t)$. If α and β are both symmetric or both skew-symmetric, then $(\alpha \circ \beta)$ is a symmetric 2-tensor. Also, for a 2-tensor, we have the standard matrix product $(\alpha\beta)_{ab} = \alpha_a{}^k \beta_{kb}$.

Proposition 1.18. *[29, §2] The 3-form φ and the corresponding 4-form ψ satisfy the following identities.*

Contractions of φ with φ

$$\varphi_{abc}\varphi^{abc} = 42 \tag{1.19}$$

$$\varphi_{abj}\varphi^{ab}{}_k = 6g_{jk} \tag{1.20}$$

$$\varphi_{apq}\varphi^a{}_{jk} = g_{pj}g_{qk} - g_{pk}g_{qj} + \psi_{pqjk}. \tag{1.21}$$

Contractions of φ with ψ

$$\begin{aligned} \varphi_{ijk}\psi_a{}^{ijk} &= 0 \\ \varphi_{ijq}\psi^{ij}{}_{kl} &= 4\varphi_{qkl} \end{aligned} \tag{1.22}$$

$$\begin{aligned} \varphi_{ipq}\psi^i{}_{jkl} &= g_{pj}\varphi_{qkl} - g_{jq}\psi_{pkl} + g_{pk}\psi_{jql} \\ &\quad g_{kq}\psi_{jpl} + g_{pl}\psi_{jkq} - g_{lq}\psi_{jkp}. \end{aligned} \tag{1.23}$$

Contractions of ψ with ψ

$$\psi_{abcd}\psi^{ab}{}_{mn} = 4g_{cm}g_{dn} - 4g_{cn}g_{dm} + 2\psi_{abmn} \tag{1.24}$$

$$\psi_{abcd}\psi_m{}^{bcd} = 24g_{am} \tag{1.25}$$

$$\psi_{abcd}\psi^{abcd} = 168$$

Remark 1.26. In [2], Berger classified the possible holonomy groups of simply-connected, irreducible and non-symmetric Riemannian manifolds. Berger's classification is given in the following table

Dimension	Holonomy group	Remarks
n	$\mathrm{SO}(n)$	Generic Riemannian manifold
$2m$	$\mathrm{U}(m)$	Kähler
$2m$	$\mathrm{SU}(n)$	Calabi-Yau
$4q$	$\mathrm{Sp}(q)$	Hyper-Kähler
$4q$	$\mathrm{Sp}(q) \cdot \mathrm{Sp}(1)$	Quaternionic-Kähler
7	G_2	G_2-holonomy
8	Spin(7)	Spin(7)-holonomy

Manifolds with $\mathrm{SU}(m)$, $\mathrm{Sp}(q)$, $\mathrm{Sp}(q) \cdot \mathrm{Sp}(1)$, G_2 or Spin(7) holonomy are called manifolds with special holonomy. With exception of the manifold with holonomy $\mathrm{Sp}(q) \cdot \mathrm{Sp}(1)$, all other special holonomy manifolds are *Ricci-flat*. In particular, G_2-manifolds are always *Ricci-flat*.

Proposition 1.27 ([16])**.** *The G_2-structure corresponding to φ is torsion free if, and only if φ is both closed and coclosed, i.e.*

$$\mathrm{d}\varphi = 0, \quad \mathrm{d} * \varphi = \mathrm{d}\psi = 0.$$

Definition 1.28. There are four independent torsion forms $\tau_0 \in \Omega^0$, $\tau_1 \in \Omega^1$, $\tau_2 \in \Omega^2_{14}$ and $\tau_3 \in \Omega^3_{27}$ corresponding to a G_2-structure $\varphi \in \Omega^3_+$ such that $\mathrm{d}\varphi$ and $\mathrm{d}\psi$ can be expressed as follows

$$\begin{aligned} \mathrm{d}\varphi &= \tau_0\psi + 3\tau_1 \wedge \varphi + *\tau_3, \\ \mathrm{d}\psi &= 4\tau_1 \wedge \psi + \tau_2 \wedge \varphi. \end{aligned} \tag{1.29}$$

We call τ_0 the *scalar torsion*, τ_1 the *vector torsion*, τ_2 the *Lie algebra torsion* and τ_3 the *symmetric traceless torsion*.

The torsion forms can be explicitly computed from φ and ψ by means of the following identities:

$$\begin{aligned} \tau_0 &= \frac{1}{7} *_\varphi (\varphi \wedge \mathrm{d}\varphi), & \tau_1 &= \frac{1}{12} *_\varphi (\varphi \wedge *_\varphi \mathrm{d}\varphi) = \frac{1}{12} *_\varphi (\psi \wedge *_\varphi \mathrm{d}\psi), \\ \tau_2 &= - *_\varphi (\mathrm{d}\psi) + 4 *_\varphi (\tau_1 \wedge \psi), & \tau_3 &= *_\varphi (\mathrm{d}\varphi) - \tau_0\varphi - 3 *_\varphi (\tau_1 \wedge \varphi). \end{aligned} \tag{1.30}$$

Definition 1.31. The *full torsion tensor* is a 2-tensor T satisfying $\nabla_i \varphi_{jkl} = T_i^m \psi_{mjkl}$ and $T_i^j = \frac{1}{24} \nabla_i \varphi_{lmn} \psi^{jlmn}$ where $T_{ij} = T(\partial_i, \partial_j)$ and $T_i^j = T_{ik} g^{jk}$. Since $T \in \Gamma(T^*M \otimes T^*M) \simeq \mathcal{S} \oplus \Omega^2 \simeq \Omega^0 \oplus \mathcal{S}_o \oplus \Omega^2_7 \oplus \Omega^2_{14}$ where $\mathcal{S} = \Gamma(S^2(T^*M))$ and $\mathcal{S}_0$ denote those sections $h \in \mathcal{S}$ that are traceless with respect to the metric g on M, then it is expressed in terms of the torsion forms as

$$T = \frac{\tau_0}{4} g_\varphi - *(\tau_1 \wedge \psi) - \frac{1}{2}\tau_2 - \frac{1}{4}\jmath(\tau_{27}). \tag{1.32}$$

Remark 1.33. Starting from $\nabla_l \varphi_{abc} = T_{lm} g^{mn} \psi_{nabc}$ and using proposition 1.18, we have that

$$\nabla_m \psi_{ijkl} = -T_{mi}\varphi_{jkl} + T_{mj}\varphi_{ikl} - T_{mk}\varphi_{ijl} + T_{ml}\varphi_{ijk} \tag{1.34}$$

Some special classes of G_2-structures are defined or characterised as follows:

- *Parallel* or *torsion-free*: $\mathrm{d}\varphi = 0$ and $\mathrm{d} * \varphi = 0$, or equivalently, φ is parallel with respect to the metric g_φ, i.e., $\nabla^{g_\varphi}\varphi = 0$.
- *closed* or *calibrated*: $\mathrm{d}\varphi = 0$.
- *coclosed* or *cocalibrated*: $\mathrm{d} * \varphi = 0$.
- *Nearly parallel*: $\mathrm{d}\varphi = c * \varphi$ for a constant c.
- *Locally conformal closed* or *locally conformal calibrated*: $\mathrm{d}\varphi = \theta \wedge \varphi$ for some $\theta \in \Omega^1(M)$.
- *Locally conformal coclosed*: $\mathrm{d}\varphi = \tau_0\psi$ and $\mathrm{d}\psi = 0$.
- *Purely coclosed*: $\mathrm{d}\varphi = *\tau_3$ and $\mathrm{d}\psi = 0$.

1.2.1 Ricci curvature

Another interesting topic in this thesis is the singularity analysis which is important to find the quantity $\Lambda(x,t)$ given by (9) where it is composed of Ricci curvature norm. Therefore, we introduce some definitions related to Ricci curvature in this section. Let φ be a G_2-structure which determines a unique metric g on M, then we have the Riemannian curvature tensor Rm of g on M given by

$$R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

and $R(X,Y,Z,W) = g(R(X,Y)W,Z)$ for vector fields X,Y,Z,W on M. In local coordinates denote $R_{ijkl} = R(\partial_i,\partial_j,\partial_k,\partial_l)$. Recall that Rm satisfies the first Bianchi identity:

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0. \tag{1.35}$$

We also have the following Ricci identities when we commute covariant derivatives of a $(0,k)$-tensor α:

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)\alpha_{i_1 i_2\cdots i_k} = \sum_{s=1}^{k} R_{iji_s}{}^{m}\alpha_{i_1\cdots i_{s-1}mi_{s+1}\cdots i_k}.$$

Karigiannis [23] derived the following second Bianchi-type identity for the full torsion tensor.

Lemma 1.36. *[23, §4, Theorem 4.2]*

$$\nabla_i T_{jk} - \nabla_j T_{ik} = \Big(\frac{1}{2}R_{ijmn} - T_{im}T_{jn}\Big)\varphi_k{}^{mn}.$$

We consider the Ricci tensor, given locally as $R_{ik} = R_{ijkl}g^{jl}$.

Proposition 1.37. *[23, §4, Proposition 4.15] The Ricci tensor of the metric g associated to the G_2-structure φ is given locally as*

$$R_{ik} = (\nabla_i T_{jl} - \nabla_j T_{il})\varphi_k{}^{jl} + \mathrm{tr}(T)T_{ik} - T_i{}^{j}T_{jk} + T_{im}T_{jn}\psi_k{}^{jmn}.$$

Lemma 1.38. *[20, §2, Lemma 2.1] The torsion tensor T satisfies the following identities*

$$\begin{aligned}(\nabla T)\lrcorner\psi =& -(T\lrcorner\varphi)\lrcorner T + T^2\lrcorner\varphi + (\operatorname{tr} T)(T\lrcorner\varphi)\\ 0 =& \mathrm{d}(\operatorname{tr} T) - \operatorname{div}(T^t) - (T\lrcorner\varphi)\lrcorner T^t\\ \operatorname{Ric} =& -\operatorname{sym}(\operatorname{Curl} T^t - \nabla(T\lrcorner\varphi) + T^2 - \operatorname{tr}(T)T)\\ \frac{1}{4}\operatorname{Ric}^* =& \operatorname{Curl} T + \frac{1}{2}T\circ T\\ R =& 2\operatorname{tr}(\operatorname{Curl} T) - \psi(T,T) - \operatorname{tr}(T^2) + (\operatorname{tr} T)^2\end{aligned}$$

where $(\operatorname{Ric}^*)_{ab} = R_{mnpq}\varphi^{mn}{}_a\varphi^p{}_b$ *and* $\psi(T,T) = \psi_{abcd}T^{ab}T^{cd}$. *Note that from* (1.32) $\operatorname{tr} T = \frac{7}{4}\tau_0$ *and* $T\lrcorner\varphi = -6\tau_1$.

The following proposition provides useful properties of a coclosed G_2-structure and its full torsion tensor T.

Proposition 1.39. *[18, §2, Proposition 2.3] Suppose we have a coclosed* G_2*-structure on a manifold* M *with* 3*-form* φ. *Let* $\mu = \mathrm{i}_\varphi(h) \in \Omega^3$ *where* h *a symmetric tensor, then the exterior derivative* $\mathrm{d}\mu$ *is given by*

$$\begin{aligned}\mathrm{d}\mu =& \frac{1}{2}(\operatorname{tr} T\operatorname{tr} h - \langle T,h\rangle)\psi - (\nabla\operatorname{tr} h - \operatorname{div} h)^\flat\wedge\varphi\\ &+ *i_\varphi(\operatorname{Curl} h_{(ab)} + \frac{1}{2}T\circ h_{ab} + (Th)_{ab} - \frac{1}{2}(\operatorname{tr} h)T_{ab} - \frac{1}{2}(\operatorname{tr} T)h_{ab}).\end{aligned} \tag{1.40}$$

Furthermore, the torsion tensor T satisfies the following identities

$$\begin{aligned}\operatorname{div} T =& \mathrm{d}(\operatorname{tr} T),\\ \operatorname{Curl} T =& (\operatorname{Curl} T)^t,\\ \operatorname{Ric} =& \operatorname{Curl} T - T^2 + \operatorname{tr}(T),\\ R =& (\operatorname{tr} T)^2 - |T|^2.\end{aligned} \tag{1.41}$$

1.2.2 Nearly parallel G_2-structures

In the above section, we defined nearly parallel G_2-structures, i.e., $\mathrm{d}\varphi = c*\varphi$ for a constant c. Several authors have studied nearly parallel G_2-structures [17, 4, 30, 26] and it will help to understand the behavior of G_2-structures on the homogeneous space $\mathbb{S}^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1)$ in Section 2.4, specifically Theorem 2.54, where the well-known nearly parallel *round* and *squashed* metrics occur naturally on different poles in an $\mathbb{S}^3$-family, the equator of which is a new $\mathbb{S}^2$-family of coclosed G_2-structures.

Proposition 1.42. *[5, §8, Theorem 45]*

Let (M,g) *be a complete* 7*-dimensional Riemannian manifold with a nearly parallel* G_2*-structure. Then the holonomy* $\mathrm{Hol}(\overline{g})$ *of the metric cone* $(C(M),\overline{g})$ *is contained in* $\mathrm{Spin}(7)$ *where* $C(M) = \mathbb{R}^+\times M$ *and* $\overline{g} = \mathrm{d}r^2 + r^2 g_M$ *with* $r\in\mathbb{R}^+$. *In particular,* $C(M)$ *is Ricci-flat and* M *is an Einstein manifold.*

The sphere S^7 with its constant curvature metric is isometric to the isotropy irreducible space $\mathrm{Spin}(7)/\mathrm{G}_2$. The fact that G_2 leaves invariant (up to constant) a unique 3-form and a unique 4-form on $\mathbb{R}^7$ implies immediately that this space has a nearly parallel G_2-structure as explained in detail below. Then, we begin by defining distinguished differential forms on $\mathbb{R}^7$ and $\mathbb{R}^8$.

Definition 1.43. Let $\mathbb{R}^7$ have a coordinates $(x_1, \cdots, x_7)$ and $(x_0.\cdots, x_7)$ a coordinates for $\mathbb{R}^8$. We define a 4-form γ on $\mathbb{R}^8$ by:

$$\begin{aligned}\gamma = dx^{0123} + dx^{0145} + dx^{0167} + dx^{0246} - dx^{0257} - dx^{0347} - dx^{0356} \\ + dx^{4567} + dx^{2367} + dx^{2345} + dx^{1357} - dx^{1346} - dx^{1256} - dx^{1247}.\end{aligned} \tag{1.44}$$

Notice that γ is self-dual. If ϕ is the G_2-structure given by (1.1) and we decompose $\mathbb{R}^8 = \mathbb{R} \oplus \mathbb{R}^7$ then

$$\gamma = dx^0 \wedge \phi + *\phi, \tag{1.45}$$

where $*\phi$ is the Hodge dual of ϕ. By identifying $\mathbb{R}^8$ with $\mathbb{C}^4$ in the usual way, we can decompose γ as follows

$$\gamma = \frac{1}{2}\omega \wedge \omega + \mathrm{Re}(\Upsilon),$$

where ω and Υ are the standard Kahler form and holomorphic volume form on $\mathbb{C}^4$. It is also worth observing that if $\mathbb{H}$ are the quaternions and we identify $\mathbb{R}^8$ with $\mathbb{H}^2$ in an appropriate way, then by [10] we have that

$$\gamma = \frac{1}{2}\omega_I^2 + \frac{1}{2}\omega_J^2 - \frac{1}{2}\omega_K^2,$$

where ω_I, ω_J, ω_K are the Kahler forms associated with the triple of complex structures (I, J, K) given by the standard hyperkahler structures on $\mathbb{H}^2$.

Definition 1.46. Write $\mathbb{R}^8 \setminus \{0\} = \mathbb{R}^+ \times \mathbb{S}^7$ with r being the coordinate $\mathbb{R}^+$ and $\mathbb{S}^7$ the unit 7-sphere. Then, since γ is self-dual, we may define a 3-form on $\mathbb{S}^7$ via the formula

$$\gamma_{(r,p)} = r^3 dr \wedge \varphi|_p + r^4 * \varphi|_p, \tag{1.47}$$

where $*$ here denotes the usual Hodge star on $\mathbb{S}^7$. Moreover, the fact that $d\gamma = 0$ implies that $d\varphi = 4 * \varphi$. By comparing (1.45) and (1.47) it is clear that φ is a positive 3-form on $\mathbb{S}^7$ and the associated metric g_φ is *the round metric* on $\mathbb{S}^7$ with constant curvature 1. Thus, φ is a G_2-structure on $\mathbb{S}^7$.

There are three examples of nearly parallel G_2-structures (see [17, §4, Table 3]). One of them is the so-called *squashed* 7-sphere which we explain below.

If we identify $\mathbb{R}^8 \cong \mathbb{C}^4$ via $\mathbb{R}^8 \ni (x_0, x_1, \cdots, x_7) \mapsto (x_0 + ix_1, x_2 + ix_3, x_4 + ix_5, x_6 + ix_7) := (z_1, z_2, z_3, z_4) \in \mathbb{C}^4$, then γ given by (1.45) is described as

$$\gamma = \frac{1}{2}\omega_0 \wedge \omega_0 + \mathrm{Re}\,\Upsilon_0$$

where $\omega_0 = \frac{i}{2}\sum_{j=1}^{4} dz^j \wedge d\bar{z}^j$ and $\Upsilon_0 = dz^{1234}$ are the standard Kähler form and the holomorphic volume form on $\mathbb{C}^4$, respectively.

In [26], Kawai obtain the second nearly parallel G_2-structure on $\mathbb{S}^7$ by canonical variation. Consider the following Lie groups:

$$\begin{aligned}
\mathrm{Sp}(1) &= \{a_1 + a_2 j \in \mathbb{H} : a_1, a_2 \in \mathbb{C}, |a_1|^2 + |a_2|^2 = 1\} \\
\mathrm{Sp}(2) &= \{C \in \mathrm{GL}(2,\mathbb{H}) : C \quad \text{Preserves the metric on} \quad \mathbb{H}^2\} \\
&= \{C \in \mathrm{U}(4) : C^t J C = J\} \\
&= \{(u, J\overline{u}, J\overline{v}) : u, v \in \mathbb{C}^4, |u| = |v| = 1, \langle v, u\rangle_{\mathbb{C}} = \langle v, J\overline{u}\rangle_{\mathbb{C}} = 0\},
\end{aligned}$$

where $J = \begin{pmatrix} J' & 0 \\ 0 & J' \end{pmatrix}$, $J' = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and $\langle \cdot, \cdot \rangle_{\mathbb{C}} : \mathbb{C}^4 \times \mathbb{C}^4 \to \mathbb{C}$ is the standard Hermitian metric on $\mathbb{C}^4$.

Let $\mathrm{Sp}(1) \times \mathrm{Sp}(2)$ act on $\mathbb{H}^2$ by

$$(q, A) \cdot (q_1, q_2) = q(q_1, q_2)^t \overline{A},$$

where $(q, A) \in \mathrm{Sp}(1) \times \mathrm{Sp}(2)$, $(q_1, q_2) \in \mathbb{H}^2$. Via identification $(z_1, \cdots, z_4) \in \mathbb{C}^4 \mapsto (z_1 + z_2 j, z_3 + z_4 j) \in \mathbb{H}^2$, the Sp(1)-action on $\mathbb{C}^4$ is described as

$$(a_1 + a_2 j) \cdot u = a_1 u + a_2 J\overline{u}, \tag{1.48}$$

where $u \in \mathbb{C}^4$, and $\mathrm{Sp}(2) \subset \mathrm{U}(4)$ acts on $\mathbb{C}^4$ canonically. By definition, the $\mathrm{Sp}(1)-$action commutes with the $\mathrm{Sp}(2)-$action.

The actions of $i, j, k \in \mathrm{Sp}(1)$ induce complex structures I_1, I_2, I_3 on $\mathbb{C}^4$, respectively, and hence induce the 3-Sasakian structure $\{\xi_i, \eta_i, \Phi_i, g\}_{i=1,2,3}$ on $\mathbb{S}^7$, where g is the standard metric $\mathbb{S}^7$, and a vector field $\xi_i \in \mathfrak{X}(\mathbb{S}^7)$, a 1-form $\eta_i \in \Omega^1(\mathbb{S}^7)$ and $(1,1)-$tensor $\Phi_i \in C^\infty(\mathbb{S}^7, \mathrm{End}(T\mathbb{S}^7))$ are defined in Appendix B. Note that the following conditions are satisfied:

$$\begin{aligned}
\Phi_{i+2} &= \Phi_i \circ \Phi_{i+1} - \eta_{i+1} \otimes \xi_i = -\Phi_{i+1} \circ \Phi_i + \eta_i \otimes \xi_{i+1}, \\
\xi_{i+2} &= \Phi_i(\xi_{i+1}) = -\Phi_{i+1}(\xi_i), \\
\eta_{i+2} &= \eta_i \circ \Phi_{i+1} = -\eta_{i+1} \circ \Phi_i
\end{aligned}$$

where $i \in \mathbb{Z}/3$. These tensors are described explicitly as follows.

$$\begin{aligned}
\xi_1 &= -i(z_1, z_2, z_3, z_4)^t \\
\xi_2 &= (\overline{z_2}, -\overline{z_1}, \overline{z_4}, -\overline{z_3})^t \\
\xi_3 &= i(\overline{z_2}, -\overline{z_1}, \overline{z_4}, -\overline{z_3})^t \\
\eta_1 &= \mathrm{Im}\Big(\sum_{j=1}^{4} z_j d\overline{z}^j\Big), \quad \eta_2 + i\eta_3 = -z_1 dz^2 + z_2 dz^1 - z_3 dz^4 + z_4 dz^3 \\
d\eta_1 &= -i \sum_{j=1}^{4} dz_{j\overline{j}} = -2g(\Phi_1(\cdot), \cdot), \quad d(\eta_2 + i\eta_3) = -2(dz^{12} + dz^{34})
\end{aligned}$$

Moreover, we define the canonical variation $\tilde{g}$ of the Riemannian metric g on M by

$$\tilde{g}|_{\mathcal{V}\times\mathcal{V}} = s^2 g|_{\mathcal{V}\times\mathcal{V}}, \quad \tilde{g}|_{\mathcal{H}\times\mathcal{H}} = t^2 g|_{\mathcal{H}\times\mathcal{H}}, \quad \tilde{g}|_{\mathcal{H}\times\mathcal{V}} = 0 \tag{1.49}$$

for $s, t > 0$. Now, applying the canonical variation to a Riemmanian submersion $\pi : \mathbb{S}^7 \to S^4 = \mathbb{H}P^1$, we obtain the second nearly parallel G_2-structure $\tilde{\varphi}$ on $\mathbb{S}^7$. Denote by $\omega_i = \frac{1}{2}d\eta_i((\cdot)^\intercal, (\cdot)^\intercal) \in \Omega^2(\mathbb{S}^7)$ the covariant differentiation of $\frac{1}{2}\eta_i$ where $\intercal : T\mathbb{S}^7 \to \mathcal{H}$ is the canonical projection. In other words, we have

$$\omega_1 = \frac{1}{2}d\eta_1 + \eta_{23}, \quad \omega_2 = \frac{1}{2}d\eta_2 + \eta_{31}, \quad \omega_3 = \frac{1}{2}d\eta_3 + \eta_{12}$$

since $[\xi_i, \xi_{i+1}] = 2\xi_{i+2}$ for $i \in \mathbb{Z}/3$. In general, it is well know that $\frac{1}{2}d\eta_i = -g(\Phi_i(\cdot), \cdot)$. Then we deduce that

$$\omega_i = -g(\Phi_i(\cdot)^\intercal, (\cdot)^\intercal) \quad \text{for} \quad i = 1, 2, 3.$$

Proposition 1.50 ([17])**.** *Define the Riemannian metric $\tilde{g}$, a 3-form $\tilde{\varphi} \in \Omega^3(\mathbb{S}^7)$ and the 4-form $*\tilde{\varphi} \in \Omega^4(\mathbb{S}^7)$ on $\mathbb{S}^7$ by*

$$\begin{aligned}
\tilde{g}|_{\mathcal{V}\times\mathcal{V}} &= \Big(\frac{3}{5}\Big)^2 g|_{\mathcal{V}\times\mathcal{V}}, \quad \tilde{g}|_{\mathcal{H}\times\mathcal{H}} = \Big(\frac{3}{\sqrt{5}}\Big)^2 g|_{\mathcal{H}\times\mathcal{H}}, \quad \tilde{g}|_{\mathcal{H}\times\mathcal{V}} = 0, \\
\tilde{\varphi} &= \frac{27}{25}\Big(\frac{1}{5}\eta_{123} + \sum_{i=1}^{3} \eta_i \wedge \omega_i\Big), \\
*\tilde{\varphi} &= \frac{27}{25}\Big(\frac{1}{2}\sum_{i=1}^{3} \omega_i^3 + \frac{3}{5}(\eta_{23}\wedge\omega_1 + \eta_{31}\wedge\omega_2 + \eta_{12}\wedge\omega_3)\Big).
\end{aligned}$$

Then $\tilde{\varphi}$ is nearly parallel G_2-structure and $*\tilde{\varphi}$ is a Hodge dual of $\tilde{\varphi}$ with respect to $\tilde{g}$. We call $(\mathbb{S}^7, \tilde{\varphi}, \tilde{g})$ the squashed $\mathbb{S}^7$.

Note that in [26], the canonical variation $\overline{g}$ is given by (1.49) in a more general way than in [17], where the canonical variation is denoted by $g^s = \overline{g}$ and $t = 1$. Then, the following Lemma and Proposition with canonical variation g^s will be the key for Theorem 2.54.

Lemma 1.51. *[17, §5, Lemma 5.3] The manifold (M^7, g^s) is Einstein if and only if $s = 1$ or $s = 1/5$.*

Proposition 1.52. *[17, §5, Theorem 5.4] The manifold (M^7, g^s) admits a nearly parallel* G_2*-structure for $s = 1/5$.*

1.3 Hodge Laplacian of G_2-structures

In section 3.4, we study soliton solutions of the Laplacian coflow on contact Calabi-Yau manifolds. We give the decomposition of the Hodge Laplacian of the closed G_2-structure being used in Proposition 3.86 which refers to find properties respect to the following equation $\Delta_\psi\psi - \lambda\psi + \mathcal{L}_X\psi = 0$.

Proposition 1.53. *[20, §2, Proposition 2.2] Suppose φ defines a* G_2*-structure. Then $\Delta_\varphi\varphi = X\lrcorner\psi + i_\varphi(h)$ with*

$$X = -\operatorname{div}T$$

$$\begin{aligned} h = &-\frac{1}{4}\mathrm{Ric}^* + \frac{1}{6}(R + 2|T|^2)g - T^tT - \frac{1}{2}(T\lrcorner\varphi)(T\lrcorner\varphi) \\ &+ \frac{1}{4}T\circ T + \frac{1}{4}T^t\circ T^t - \frac{1}{2}T\circ T^t + \mathrm{sym}\left((T)(T\lrcorner\psi) - (T^t)(T\lrcorner\psi)\right) \end{aligned}$$

In particular,

$$\mathrm{tr}\, h = \frac{2}{3}R + \frac{4}{3}|T|^2$$

Lemma 1.54. *[37, §3, Lemma 25] Let φ be a coclosed G_2-structure. Then, φ has the following properties:*

1. $\mathrm{div}\tau_{27} = \frac{1}{7}\nabla(\mathrm{tr}\, T) - \mathrm{div}T$
2. $\frac{1}{2}(\mathrm{Curl}\,\tau_{27} + (\mathrm{Curl}\,\tau_{27})^{\mathrm{t}}) = -\,\mathrm{Curl}\, T$ *and* $\mathrm{tr}(\mathrm{Curl}\, T) = 0$.
3. $(T\circ\tau_{27}) = \frac{1}{7}((\mathrm{tr}\, T)^2 g - (\mathrm{tr}\, T)T) - T\circ T$ *and* $\mathrm{tr}(T\circ T) = (\mathrm{tr}\, T)^2 - |T|^2$.

Where $\tau_{27} = \frac{1}{7}(\mathrm{tr}\, T)g - T$.

The following Lemma is proved in [18], but with a different orientation for the G_2-structure. Since signs will be crucial for our applications, we work out the proof again, in our convention. Then, let φ be a coclosed G_2-structure, i.e. $\mathrm{d}*\varphi = 0$. Then (1.29) implies τ_1 and τ_2 are both equal to zero. From (1.32) we have that the torsion tensor T satisfies $T_{ij} = T_{ji} = \frac{\tau_0}{4}g_{ij} - j_\varphi(\tau_3)_{ij}$, so T is a symmetric 2-tensor. Since, $d\varphi = \tau_0\psi + *\tau_3$, we have that the Hodge Laplacian of $\psi = *\varphi$ is equal to

$$\begin{aligned} \Delta_\psi\psi =&\mathrm{d}\mathrm{d}^*\psi + \mathrm{d}^*\mathrm{d}\psi = \mathrm{d}*\mathrm{d}\varphi \\ =&\mathrm{d}\tau_0\wedge\varphi + \tau_0^2\psi + \tau_0 *\tau_3 + \mathrm{d}\tau_3. \end{aligned} \tag{1.55}$$

Lemma 1.56. *Let φ be a coclosed G_2-structure on a manifold M with associated metric g. Then,*

$$\begin{aligned} \Delta_\psi\psi = &(\frac{2}{3}R + \frac{4}{3}|T|^2)\psi \oplus (\mathrm{d}\,\mathrm{tr}\, T)\wedge\varphi \\ &\oplus *_\varphi \mathrm{i}_\varphi\Big(-\mathrm{Ric} + \frac{1}{14}(R - 2|T|^2)g + \mathrm{tr}(T)T - 2T^2 - \frac{1}{2}T\circ T\Big). \end{aligned}$$

Proof. For a co-closed G_2-structure, the Laplacian of ψ is

$$\Delta_\psi\psi = \mathrm{d}\tau_0\wedge\psi + \tau_0^2\psi + \tau_0 *\tau_3 + \mathrm{d}\tau_3. \tag{1.57}$$

We can decompose $\Delta_\psi\psi$ as:

$$\Delta_\psi\psi = Y^\flat\wedge\varphi \oplus *\mathrm{i}_\varphi(s) = \frac{3}{7}(\mathrm{tr}\, s)\psi \oplus Y^\flat\wedge\varphi\oplus *\mathrm{i}_\varphi(\overline{s}), \tag{1.58}$$

where Y is a vector field and $\overline{s}$ is the trace-free part of $s\in S^2$. Now, we apply Proposition 1.39 to $\tau_3 = \mathrm{i}_\varphi(\tau_{27})$ and use Lemma 1.54 to obtain

$$\begin{aligned} \mathrm{d}\tau_3 = &\frac{3}{7}\mathrm{d}(\mathrm{tr}\, T)\wedge\varphi \\ &+ *\mathrm{i}_\varphi\Big(-\mathrm{Curl}\, T + \frac{1}{2}T\circ\tau_{27} + \frac{1}{2}(T\tau_{27} + (T\tau_{27})^t) - \frac{1}{2}(\mathrm{tr}\, T)\tau_{27} - \frac{1}{6}\langle T, \tau_{27}\rangle g\Big). \end{aligned} \tag{1.59}$$

Using the identities $\tau_0 = \frac{4}{7}\operatorname{tr}(T)$, $3\psi = *\mathrm{i}_\varphi(g)$, and (1.59) in (1.57), we obtain

$$\begin{aligned}\Delta_\psi\psi =&\mathrm{d}(\operatorname{tr} T)\wedge\varphi + *\mathrm{i}_\varphi\Big(-\operatorname{Curl} T + \frac{1}{2}(T\circ\tau_{27}) + \frac{1}{2}(T\tau_{27} + (T\tau_{27})^t)\\ &+\frac{1}{14}(\operatorname{tr} T)\tau_{27} + \frac{16}{147}(\operatorname{tr} T)^2 g - \frac{1}{6}\langle T,\tau_{27}\rangle g\Big).\end{aligned}$$

Now, replacing τ_{27} using $\tau_{27} = \frac{1}{7}(\operatorname{tr} T)g - T$, as well as using $T\circ g = (\operatorname{tr} T)g - T$, Proposition 1.39 and Lemma 1.54, we have

$$\Delta_\psi\psi = \mathrm{d}(\nabla\operatorname{tr} T)^\flat\wedge\varphi + *\mathrm{i}_\varphi\Big(-\operatorname{Curl} T - \frac{1}{2}T\circ T - T^2 + \frac{1}{6}(\operatorname{tr} T)^2 g + \frac{1}{6}|T|^2 g\Big). \tag{1.60}$$

Therefore

$$\begin{aligned}\operatorname{tr}(s) =&\frac{2}{3}((\operatorname{tr} T)^2 + |T|^2) = \frac{2}{3}R + \frac{4}{3}|T|^2, &(1.61)\\ \overline{s} =&-\operatorname{Ric} + \frac{1}{14}(R - 2|T|^2)g + \operatorname{tr}(T)T - 2T^2 - \frac{1}{2}T\circ T,\\ Y =&\nabla\operatorname{tr} T. &(1.62)\end{aligned}$$

The result then follows. □

1.4 Ricci-Like flows

In this section we state the evolution equations for several important geometric quantities under the G_2-structures following [23, 18, 11, 24], and then write some important remarks. We study the critical points G_2-structures of the energy of φ_r given by (2.58) being Proposition 2.55 a result of what I collaborated. On the other hand, the Laplacian flow and compactness analysis given by [29, 11] was used to study a singularity of the Laplacian coflow, taking the dilaton in our case. For this reason, we give a review about the Laplacian flow. Also, the Laplacian coflow studied in [20] was important in the results of soliton solutions of Laplacian coflow on contact Calabi-Yau manifold. Moreover, I collaborated in every Proposition and Theorem of Section 3.1.

Lemma 1.63. *[23, §3] If $\varphi(t)$ satisfies the equation*

$$\frac{\partial\varphi(t)}{\partial t} = X(t)\lrcorner\psi(t) + i_{\varphi(t)}(h(t)), \tag{1.64}$$

then we also have the following equations

$$\begin{aligned}\frac{\partial g}{\partial t} =&2h,\\ \frac{\partial \mathrm{vol}}{\partial t} =&\operatorname{tr}(h)\mathrm{vol},\\ \frac{\partial\psi}{\partial t} =&i_\psi(h) - X\wedge\varphi,\\ \frac{\partial T}{\partial t} =&\nabla X - \operatorname{Curl} h + Th - (T)(X\lrcorner\varphi). &(1.65)\end{aligned}$$

In [11], Chen defined a class of reasonable flows of G_2-structures that satisfy the following general conditions:

1. The metric should evolve the Ricci flow to leading order and be no more than quadratic in the torsion, that is
$$\frac{\partial g}{\partial t} = 2h = -2\mathrm{Ric} + Cg + L(T) + T \circledast T, \tag{1.66}$$
where $\circledast$ is some multilinear operator involving g, φ, ψ, C is a constant and L is some linear operator involving g, φ, ψ.

2. The vector field X is at most linear in ∇T and at most quadratic in T:
$$X = L(\nabla T) + L(T) + L(Rm) + T \circledast T + C. \tag{1.67}$$

3. The torsion tensor should evolve ΔT to leading order, be at most linear in Rm and ∇T, and at most cubic in T:
$$\begin{aligned}\frac{\partial T}{\partial t} =& \Delta T + L(\nabla T) + L(Rm) + Rm \circledast T + \nabla T \circledast T \\ &+ L(T) + T \circledast T + T \circledast T \circledast T.\end{aligned} \tag{1.68}$$

4. The flow (1.64) has short-time existence and uniqueness.

Thus, the flows that satisfy properties 1-4 are called *Ricci-like flows*. This is appropriate because a variety of techniques which were originated from the study of the Ricci flow have been applied to these flows. In particular, under the Ricci flow, invariants of the metric Rm, Ric, R, all satisfy heat-like equations. Therefore, it is appropriate that for a Ricci-like flow of a G_2–structure, the torsion, which is an invariant of the G_2-structure, also satisfies a heat like equations (1.68). This is important because then $\nabla^k T$ and $|T|^2$ also satisfy heat-like equations and this is necessary to be able to obtain estimates using the maximum principle.

Using techniques developed by Shi for the Ricci flow and their adaptation to G_2-structures given by Lotay and Wei [29], Chen then showed that a reasonable flow must satisfy the following Shi-type estimate.

Proposition 1.69. *[11, §2, Theorem 2.1] Suppose* (1.64) *is a Ricci-like flow of* G_2*-structures such that the coefficients in equations* (1.64)*,* (1.66)*,* (1.67) *and* (1.68) *are bounded by a constant* S*. Let* $B_r(p)$ *be a ball of radius* r *with respect to the initial metric* $g(0)$*. If*
$$|Rm(x,t)|_{g(t)} + |T(x,t)|^2_{g(t)} + |\nabla T(x,t)|_{g(t)} < S \tag{1.70}$$
for any $(x,t) \in B_r(p) \times [0, t_0]$*, then*
$$|\nabla^k Rm(x,t)|_{g(t)} + |\nabla^{k+1} T(x,t)|_{g(t)} < C(k, r, S, T) \tag{1.71}$$
for any $(x,t) \in B_{r/2}(p) \times [t_0/2, t_0]$ *for all* $k = 1, 2, 3, \ldots$

Note that in the Laplacian flow studied by Lotay and Wei in [29], the quantity $\Lambda(x,t) = (|Rm(x,t)|_{g(t)} + |\nabla T(x,t)|_{g(t)})$ is analogous to (1.70). This is because in the case of a closed G_2-structure, $|T|^2 = -R \leq C|Rm|$. Therefore, the norm of the torsion can always be bounded in terms of the norm of Rm. For other torsion classes, and in particular, coclosed G_2-structures, this is no longer true, therefore $|T|^2$ needs to be included in (1.70). Using the estimates from the above theorem, Chen then derived an estimate for the blow-up rate on a compact manifold.

Proposition 1.72. *[11, §5, Theorem 5.1] If φ_t is a solution to a Ricci-like flow of G_2-structures on a compact manifold in a finite maximal time interval $[0, t_0)$, then*

$$\sup_M \left(|Rm(x,t)|^2_{g_t} + |T(x,t)|^4_{g_t} + |\nabla T(x,t)|^2_{g_t}\right)^{\frac{1}{2}} \geq \frac{C}{t_0 - t} \tag{1.73}$$

The estimate 1.73 shows that a solution will exist as long as the quantity of the left-hand side of (1.73) remains bounded.

A classic example of a Ricci-like flow of G_2-structures is the Laplacian flow of G_2-structures that was introduced by Bryant [9]:

$$\frac{\partial \varphi}{\partial t} = \Delta_\varphi \varphi. \tag{1.74}$$

If the initial G_2-structure is closed, then this property is preserved along the flow. It is then natural to think of (1.74) as a flow of closed G_2-structures. In this case, the fact that $T^t = -T$ and Lemma 1.38 give us the following results: $\mathrm{Ric}^* = 4\mathrm{Ric} + T \circledast T$ and $R = 2\,\mathrm{tr}(\mathrm{Curl}\, T) - \psi(T,T) - \mathrm{tr}(T^2) = -|T|^2$; thus by Lemma 1.38, we have $\mathrm{Ric}^* = 4\mathrm{Ric} + T \circledast T$ and $R = 2\,\mathrm{tr}(\mathrm{Curl}\, T) - \psi(T,T) - \mathrm{tr}(T^2) = -|T|^2$; using Proposition 1.53 we have $h = -\mathrm{Ric} + T \circledast T$; and so substituting in Lemma 1.63, we conclude that condition 1 in the Ricci like-flow holds. Moreover, we have that $\mathrm{div} T = 0$ in this case (see [31]), and hence $X = 0$. Using Lemma 1.63 and $h = -\,\mathrm{Curl}\, T + T \circledast T$, we have

$$\frac{\partial T}{\partial t} = \mathrm{Curl}(\mathrm{Curl}\, T) + \nabla T \circledast T + T \circledast T \circledast T. \tag{1.75}$$

On the other hand, in [20] is given the following formula when β is 2-tensor

$$\mathrm{Curl}\left((\mathrm{Curl}\,\beta)^t\right) = -\Delta\beta^t + \nabla(\mathrm{div}\beta) + Rm \circledast \beta + T \circledast \nabla\beta + (\nabla T) \circledast \beta + T \circledast T \circ \beta \tag{1.76}$$

where ^{t} denote transpose and $\circledast$ is some multilinear operator involving g, φ, ψ. Then, using (1.76) into (1.75) together the facts that $\mathrm{Curl}\, T$ is symmetric, T is skewsymmetric, and $\mathrm{div} T = 0$, allows to express (1.75) as

$$\frac{\partial T}{\partial t} = \Delta T + Rm \circledast T + \nabla T \circledast T + T \circledast T \circledast T.$$

Therefore, the Laplacian flow is a Ricci-like flow. Finally, short-term existence and uniqueness of the flow (1.74) has been first proved by Bryant and Xu in [9]. In [29, 31, 28], we can find more properties of this flow, as well as details of the above calculation. The results in Proposition 1.69 and 1.72 are extensions of similar results for the Laplacian flow of closed G_2-structures in [31].

In the study of singularities, a fundamental tool is the Cheeger-Gromov type compactness theorem and using techinques developed by Shi for the Ricci flow, Lotay and Wei proved their adaptation to G_2-structures (see [31]).

Proposition 1.77. *[29, §7, Theorem 7.1] Let M_i be a sequence of smooth 7-manifolds and for each i, let $p_i \in M_i$ and φ_i be a G_2-structure on M_i such that the metric g_i on M_i induced by φ_i is complete on M_i. Suppose that*

$$\sup_i \sup_{x \in M_i} \left(|\nabla_{g_i}^{k+1} T_i(x)|_{g_i}^2 + |\nabla_{g_i}^k Rm_{g_i}(x)|_{g_i}^2 \right)^{1/2} < \infty \tag{1.78}$$

for all $k \leq 0$ and

$$\inf_i \operatorname{inj}(M_i, g_i, p_i) > 0, \tag{1.79}$$

where T_i, Rm_{g_i} are the torsion and curvature tensor of φ_i and g_i respectively, and $\operatorname{inj}(M_i, g_i, p_i)$ denotes the injectivity radius of (M_i, g_i) at p_i.

Then there exists a 7-manifold M, a G_2-structure φ on M and a point $p \in M$ such that, after passing to a subsequence, we have

$$(M_i, \varphi_i, p_i) \to (M, \varphi, p) \quad as \quad i \to \infty.$$

Therefore, Theorem 1.77 reduces the problem of finding singularities models to finding sequences of points and times for which the corresponding dilated solutions have a curvature bound. In [29], Lotay and Wei give shi-types estimates with respect to the quantity $(|Rm(p,t)|_{g_t}^2 + |\nabla T(p,t)|_{g_t}^2)^{\frac{1}{2}}$, which Gao Chen [11] extends as

$$\Lambda(x,t) = (|Rm(p,t)|_{g_t}^2 + |T(p,t)|_{g_t}^4 + |\nabla T(p,t)|_{g_t}^2)^{\frac{1}{2}} \tag{1.80}$$

for a local version of shi-type estimates for Ricci-Like flows of G_2-structures and as in the Ricci flow, we are interested to analyse singularities of the Laplacian coflow from (3.31). The property of being k-noncollapsing below a scale ρ is preserved under Cheeger-Gromov limits, and this is useful in the study of singularity models.

The original k-noncollapsing theorem of Perelman for Ricci flow in [38] requires the Riemannian curvature bound. In [11], Chen use the Perelman's ideas for Ricci-like flow and express in the following proposition.

Proposition 1.81. *[11, §4, Theorem 4.2] Let $\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + E_{ij}$ be the geometric flow on a compact manifold M^n. Then there exists a positive function k of four variables such that if $0 < \rho \leq \rho_0 < \infty$, $0 < \frac{T}{2} \leq t_0 \leq T < \infty$ and*

$$\int_0^{t_0} (t_0 + \rho^2 - t) \sup_M |E|^2 \mathrm{d}t < \infty, \tag{1.82}$$

then $g(t_0)$ is $k(g(0), T, \rho_0, \int_0^{t_0} (t_0 + \rho^2 - t)) \sup_M |E|^2 \mathrm{d}t)$–non-collapsing relative to upper bound of scalar curvature on scale ρ.

We conclude with the following proposition which shows that any blow-up limit at finite time must be a manifold with maximal volume growth rate whose holonomy is contained in G_2.

Proposition 1.83. *Let $\varphi(t)$ be a solution to a reasonable flow of G_2-structures on a compact manifold M^7 in a finite maximal time interval $[0, T)$. If*

$$\int_0^T (T-t) \sup_M |T|^4 dt < \infty,$$

and

$$\sup_M (|R| + |T|^2) = o\left(\frac{1}{T-t}\right),$$

Then there exist a sequence $t_k \to T$, $x_k \in M$ such that

$$Q_k = \Lambda(x_k, t_k) = \sup_{x \in M, t \in [0, t_k]} \left(|Rm|^2 + |T|^4 + |\nabla T|^2\right)^{1/2} \to \infty$$

and $(M, Q_k^{3/2}\varphi(t_k), Q_k g(t_k), x_k)$ converges to a complete manifold M_∞ with a torsion-free G_2-structure $(\varphi_\infty, g_\infty, x_\infty)$ such that

$$\mathrm{vol}_{g_\infty}(B_{g_\infty}(x_\infty, r)) \geq kr^7$$

for some $k > 0$ and all $r > 0$.

2 Homogeneous G_2-structures on $\mathbb{S}^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1)$

In order to understand the isometric flow on homogeneous spaces, we take the particular case $\mathbb{S}^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1)$ and analyse the isometric class of $\mathrm{Sp}(2)$-invariant G_2-structures and then describe the $\mathrm{Sp}(2)$-invariant G_2-structures with $\mathrm{div}T = 0$, i.e., critical points of the Dirichlet energy functional on this class:

$$E : \varphi \in \mathcal{B}_r \mapsto \int_M |T_\varphi|^2 \mathrm{vol}_\varphi \in \mathbb{R}.$$

Note that it is indispensable to know $|T_\varphi|^2$. Therefore, we focus on two aspects, the first one is to describe the isometric class of $\mathrm{Sp}(2)$-invariant G_2-structures which is described by a $\mathrm{SO}(3)$-orbit (Proposition 2.24) showing also explicitly $|T_\varphi|^2$ in this class which is necessary to find the torsion forms τ_0, τ_1, τ_2 and τ_3 and thus conclude Proposition 2.55 and Theorem 2.54 using this results in Section §2.5 and, the second one is to describe the $\mathrm{Sp}(2)$-invariant G_2-structures of $\mathcal{B}_r$ with $\mathrm{div}T = 0$ showed in Proposition 2.66, i.e. critical points of the Dirichlet energy functional. More generally, we will make the blanket assumption that all the variational problems we consider are restricted to a given isometric class concluding Theorem 2.67 about the set $\mathrm{Crit}(E|_{\mathcal{B}_r})$ for $r > 0$ which is proved in §2.5.1.

2.1 G_2-structures on homogeneous spaces

In this section, we will give a survey of Lauret's approach to G-invariant G_2-structures on homogeneous spaces [27]. Also, we use a classification of homogeneous spaces G/H admitting G-invariant G_2-structures which will be useful to describe isometric $\mathrm{Sp}(2)$-invariant G_2-structures on $\mathbb{S}^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1)$.

Consider the action of a Lie group G on a manifold M. A (r,s)-tensor γ on M is G-invariant if $h^*\gamma = \gamma$, for each $h \in G$, where

$$h^*\gamma(X_1, \cdots, X_r, \alpha_1, \cdots, \alpha_s) := \gamma(h_* X_1, \cdots, h_* X_r, (h^{-1})^*\alpha_1, \cdots, (h^{-1})^*\alpha_s).$$

for $X_1, \cdots, X_r \in \Gamma(TM)$ and $\alpha_1, \cdots, \alpha_s \in \Gamma(T^*M)$. In particular, when $M = G/H$ is a reductive homogeneous space, i.e., $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $\mathrm{Ad}(k)\mathfrak{p} \subset \mathfrak{p}$ for all $k \in K$, any G-invariant tensor γ is completely determined by its value γ_{x_0} at the point $x_0 = [1_G] \in G/H$, where γ_{x_0} is an $\mathrm{Ad}(K)$-invariant tensor at $\mathfrak{p} \cong T_{x_0}M$, i.e., $(\mathrm{Ad}(k))^*\gamma_{x_0} = \gamma_{x_0}$ for each $k \in K$. Given $x = [hx_0] \in G/H$, clearly $\gamma_x = (h^{-1})^*\gamma_{x_0}$.

Let $\mathfrak{p}$ be a real vector space of dimension 7 and let φ be a G_2-structure, then it belongs to the open orbit $\mathrm{GL}(\mathfrak{p}) \cdot \phi \subset \Lambda^3\mathfrak{p}^*$. Let us fix a positive 3-form $\varphi \in \mathrm{GL}(\mathfrak{p}) \cdot \phi \subset \Lambda^3\mathfrak{p}^*$. Since the orbit $\mathrm{GL}(\mathfrak{p}) \cdot \varphi$ is open in $\Lambda^3\mathfrak{p}^*$, we have that its tangent space at φ satisfies

$$\theta(\mathfrak{gl}(\mathfrak{p}))\varphi = \Lambda^3\mathfrak{p}^*,$$

where $\theta : \mathfrak{gl}(\mathfrak{p}) \to \mathrm{End}(\Lambda^3\mathfrak{p}^*)$ is the representation obtained as the derivative of the natural left $\mathrm{GL}(\mathfrak{p})$-action on 3-forms $h \cdot \psi = \psi(h^{-1}\cdot, h^{-1}\cdot, h^{-1}\cdot)$, that is,

$$\theta(A)\beta = -\beta(A\cdot,\cdot,\cdot) - \beta(\cdot, A\cdot,\cdot) - \beta(\cdot,\cdot,A\cdot), \quad \forall A \in \mathfrak{gl}(\mathfrak{p}), \quad \beta \in \Lambda^3\mathfrak{p}^*.$$

The Lie algebra of the stabilizer subgroup $G_2(\varphi) := \mathrm{GL}(\mathfrak{p})_\varphi \cong G_2$ is given by

$$\mathfrak{g}_2(\varphi) := \{A \in \mathfrak{gl}(\mathfrak{p}) : \theta(A)\varphi = 0\} \cong \mathfrak{g}_2.$$

We consider the orthogonal complement subspace $\mathfrak{q}(\varphi) \subset \mathfrak{gl}(\mathfrak{p})$ of $\mathfrak{g}_2(\varphi)$ relative to the inner product on $\mathfrak{gl}(\mathfrak{p})$ determined by g_φ (that is, $\mathrm{tr}\, AB^t$). The irreducible $G_2(\varphi)$-components of $\mathfrak{q}(\varphi)$ are $\mathfrak{q}_1(\varphi) = \mathbb{R}I$, the one-dimensional trivial representation, the (seven-dimensional) standard representation $\mathfrak{q}_7(\varphi)$ and $\mathfrak{q}_{27}(\varphi)$, the other fundamental representation which has dimension 27. Summarizing, each positive 3-form φ determines the following $G_2(\varphi)$-invariant decompositions:

$$\begin{aligned} \mathfrak{gl}(\mathfrak{p}) = \mathfrak{g}_2(\varphi) \oplus \mathfrak{q}(\varphi), \quad \mathfrak{q}(\varphi) = \mathfrak{q}_1(\varphi) \oplus \mathfrak{q}_7(\varphi) \oplus \mathfrak{q}_{27}(\varphi), \\ \mathfrak{so}(\mathfrak{p}) = \mathfrak{g}_2 \oplus \mathfrak{q}_7(\varphi), \quad \mathrm{sym}(\mathfrak{p}) = \mathfrak{q}_1(\varphi) \oplus \mathfrak{q}_{27}(\varphi), \quad \mathfrak{q}_{27}(\varphi) = \mathrm{sym}_0(\mathfrak{p}), \end{aligned} \tag{2.1}$$

where $\mathfrak{so}(\mathfrak{p})$ and $\mathrm{sym}(\mathfrak{p})$ are the spaces of skew-symmetric and symmetric linear maps with respect to g_φ, respectively, and $\mathrm{sym}_0(\mathfrak{p}) := \{A \in \mathrm{sym}(\mathfrak{p}) : \mathrm{tr}(A) = 0\}$.

If we identify $\mathrm{sym}(\mathfrak{p})$ with the space $S^2\mathfrak{p}^*$ of symmetric bilinear forms by using $\langle\cdot,\cdot\rangle$, then the linear isomorphism $i_\varphi : S^2\mathfrak{p}^* \to \Lambda^3_1\mathfrak{p}^* \oplus \Lambda^3_{27}\mathfrak{p}^*$ satisfies $i_\varphi(h) = -2\theta(h)\varphi$; in particular, for every $\gamma \in \Lambda^3\mathfrak{p}^*$, there exists a unique operator Q_γ in $\mathfrak{q}(\varphi)$ such that

$$\gamma = \theta(Q_\gamma)\varphi,$$

and we have that $i_\varphi(Q_\gamma) = -2\gamma$.

A 7-manifold endowed with a G_2-structure (M, φ) is said to be *homogeneous* if the Lie group of all symmetries or automorphisms,

$$\mathrm{Aut}(M, \varphi) := \{f \in \mathrm{Diff} : f^*\varphi = \varphi\}$$

acts transitively on M. It is knwon that $\mathrm{Aut}(M, \varphi)$ is a Lie group, it is indeed a closed subgroup of the Lie group $\mathrm{Iso}(M, g_\varphi)$ of all isometries of the Riemannian manifold (M, g_φ). Each Lie subgroup $G \subset \mathrm{Aut}(M, \varphi)$ which is transitive on M gives rise to a presentation of M as a *homogeneous space* G/H, where H is the isotropy subgroup of G at some point $o \in M$, and φ becomes a G-invariant G_2-structure on the homogeneous space $M = G/H$. As in the Riemannian case, G is closed in $\mathrm{Aut}(M, \varphi)$ if and only if K is compact. In the presence of a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ (that is, $\mathrm{Ad}(H)\mathfrak{p} \subset \mathfrak{p}$) for the homogeneous space G/H, where $\mathfrak{g}$ and $\mathfrak{k}$, respectively, denote the Lie algebras of G and H, every G-invariant G_2-structure on G/H is determined by a positive 3-form φ on $\mathfrak{p} \cong T_oG/H$ (the tangent space at the origin o of G/H) which is $\mathrm{Ad}(H)$-invariant. This means that $(\mathrm{Ad}(h)|_\mathfrak{p}) \cdot \varphi = \varphi$ for any $h \in H$, or equivalently if H is connected, $\theta(\mathrm{ad}Z|_\mathfrak{p})\varphi = 0$ for all $Z \in \mathfrak{k}$.

Compact homogeneous manifolds admitting invariant G_2-structures

In this section we use a classification of homogeneous manifolds G/H admitting G-invariant G_2-structures where G is a compact Lie group and H is a closed Lie subgroup (not necessarily connected) of G (see [39, 42]). We describe all connected subgroups of G_2. The semisimple subalgebra of all semisimple Lie algebras including $\mathfrak{g}_2$ have been classified in [15].

Definition 2.2. Let G be a compact connected Lie group and H be a closed connected subgroup of G. We call G/H S^1-reducible if there exists a Lie group $G^{'}$ and a covering map $\pi : G^{'} \times U(1) \to G$ such that $H \subseteq \pi(G^{'})$. Otherwise, G/H is called S^1-irreducible.

Lemma 2.3. *[39, §3, Lemma 3.4] Let G/H be a seven-dimensional homogeneous space which admits a G-invariant G_2-structure. We assume that G acts effectively on G/H. In this situation, there exists a vector space and an isomorphism $\zeta : T_pG/H \to \mathbb{R}^7$ such that $\zeta H \zeta^{-1} \subseteq G_2$, where H is identified with its isotropy representation and G_2 with its seven-dimensional irreducible representation.*

The converse of the above Lemma is also true:

Lemma 2.4. *[39, §3, Lemma 3.5] Let G/H be a seven-dimensional homogeneous space such that G acts effectively and there exists a vector space isomorphism $\zeta : T_pG/H \to \mathbb{R}^7$ with $\zeta H \zeta^{-1} \subseteq G_2$. In this situation, there exists a G-invariant G_2-structure on G/H.*

Lemma 2.5. *[39, §5, Lemma 5.2] Let G/H be a compact homogeneous space which admits a G-invariant G_2-structure. Moreover, let G act almost effectively on G/H. In this situation, the following statements are true:*

1. $\dim \mathfrak{g} = \dim \mathfrak{h} + 7$.
2. *G is a compact and $\mathfrak{g}$ is the direct sum of a semisimple and an abelian Lie aLgebra.*
3. $\operatorname{rank} \mathfrak{h} \in \{0, 1, 2\}$ *and* $\operatorname{rank} \mathfrak{g} \neq \operatorname{rank} \mathfrak{h} (\mod 2)$*. If* $\operatorname{rank} \mathfrak{h} = 1$*, the dimension of the centre* $\mathfrak{z}(\mathfrak{g})$ *of* $\mathfrak{g}$ *is less than or equal* 3*. If* $\operatorname{rank} \mathfrak{h} = 2$, $\dim \mathfrak{z}(\mathfrak{g}) \leq 1$.
4. *Let $G = G^{'} \times U(1)$ and $H = H^{'} \times U(1)$. If the second factor of H is transversely embedded into the product $G^{'} \times U(1)$, G/H is $G^{'}$-equivariantly covered by $G^{'}/H^{'}$.*
5. *Let $\mathfrak{m}$ be the orthonormal complement of $\mathfrak{h}$ and $\mathfrak{g}$ with respect to an Ad_H-invariant metric $\mathfrak{g}$. The restriction of the adjoint action Ad_G to a map $H \to \mathfrak{gl}(\mathfrak{m})$ is equivalent to the isotropy action of H on the tangent space.*

Proposition 2.6. *[39, §1, Theorem 1]*

1. *Let G/H be a seven-dimensional, compact, connected, homogeneous space which admits a G-invariant G_2-structure. We assume that G/H is a product of a circle and another homogeneous space and that G acts almost effectively on G/H. Furthermore, we assume*

G	H	G/H	$\mathfrak{n}_{O(7)}$	$\mathfrak{n}_{G_2}$
$U(1)$	$\{e\}$	T^7	28	35
$SU(2) \times U(1)^4$	$\{e\}$	$S^3 \times T^4$	28	35
$SU(2)^2 \times U(1)$	$\{e\}$	$S^3 \times S^3 \times S^1$	28	35
$SU(2)^2 \times U(1)^2$	$U(1)$	$S^3 \times S^3 \times S^1$	10	13
$SO(4) \times U(1)^2$	$SO(2)$	$V^{4,2} \times T^2$	10	13
$SU(2)^3 \times U(1)$	$SU(2)$	$S^3 \times S^3 \times S^1$	4	5
$SU(3) \times U(1)^2$	$SU(2)$	$S^5 \times T^2$	7	10
$SU(3) \times U(1)$	$U(1)^2$	$SU(3)/U(1)^2 \times S^1$	4	5
$Sp(2) \times U(1)$	$Sp(1) \times U(1)$	$\mathbb{CP}^3 \times S^1$	3	4
$G_2 \times U(1)$	$SU(3)$	$S^6 \times S^1$	2	3

Table 1 – $\mathbb{S}^1$-reducible

G	H	G/H		$\mathfrak{n}_{O(7)}$	$\mathfrak{n}_{G_2}$
$SU(3)$	$U(1)$	$N^{1,1}$		10	13
$SU(3)$	$U(1)$	$N^{1,0}$		6	7
$SU(3)$	$U(1)$	$N^{k,l}$	with $k,l \in \mathbb{Z}, k \leq l \leq 0, kl > 1$	4	5
$SO(5)$	$SO(3)$	$V^{5,2}$		4	5
$Sp(2)$	$Sp(1)$	$\mathbb{S}^7$		7	10
$SO(5)$	$SO(3)$	B^7		1	1
$SU(2)^3$	$U(1)^2$	$Q^{1,1,1}$		4	5
$SU(3) \times U(1)$	$U(1)^2$	$N^{k,l}$	with $l \in \mathbb{Z}$ arbitrary	4	5
$SU(3) \times SU(2)$	$SU(2) \times U(1)$	$M^{1,1,0}$		3	4
$SU(3) \times SU(2)$	$SU(2) \times U(1)$	$N^{1,1}$		2	2
$Sp(2) \times U(1)$	$Sp(1) \times U(1)$	$\mathbb{S}^7$		3	4
$Sp(2) \times Sp(1)$	$Sp(1) \times Sp(1)$	$\mathbb{S}^7$		2	2
$SU(4)$	$SU(3)$	$\mathbb{S}^7$		2	3
$Sp(7)$	G_2	$\mathbb{S}^7$		1	1

Table 2 – $\mathbb{S}^1$-irreducible

that G and H are both connected. In this situation, G, H and G/H are up to a covering one of the space from Table 1 and the dimensions $\mathfrak{n}_{G_2}$ ($\mathfrak{n}_{O(7)}$) of the space of all G-invariant G_2*-structures (metrics) on G/H are shown in Table 1.*

2. *Let G, H and G/H satisfy the same conditions as before with the single exception that G/H is* not *a product of a circle and another homogeneous space. In this situation, G, H and G/H are up to a covering one of the space from Table 2 and the dimensions $\mathfrak{n}_{G_2}$ ($\mathfrak{n}_{O(7)}$) of the space of all G-invariant* G_2*-structures (metrics) on G/H are shown in Table 2.*

3. *Any of the G/H from Table 1 or Table 2 admits a G-invariant cocalibrated* G_2*-structure. If G/H is from Table 2, it even admits a G-invariant nearly parallel* G_2*-structure.*

In Table 1 and 2, $N^{k,l}$ denotes an Aloff-Wallach space, $V^{4,2}$ ($V^{5,2}$) denotes the Stiefel manifold of all orthonormal pairs in $\mathbb{R}^4$ ($\mathbb{R}^5$), and B^7 is the seven-dimensional Berger space. The following Lemma and Theorem answer how many of such structures exist on G/H.

Lemma 2.7. *[39, §7, Lemma 7.1] Let G be a compact Lie group and H be a closed subgroup of G such that G/H admits a G-invariant* G_2*-structure. As usual, let $\mathfrak{g}$ be the Lie algebra of G,*

$\mathfrak{h}$ *of* H *and* $\mathfrak{p}$ *be the orthogonal complement of* $\mathfrak{h}$ *in* $\mathfrak{g}$ *with respect to an* Ad_H*-invariant scalar product on* $\mathfrak{g}$*. We denote the set of all* H*-invariant elements of an* H*-module* V *by* V^H*. The set of all* G*-invariant* G_2*-structures on* G/H *can be bijectively identified with a subset of* $(\Lambda^3\mathfrak{p}^*)^H$*. Moreover, this subset is open.*

Lemma 2.8. *[39, §7, Lemma 7.2] In the situation of above Lemma, the map*

$$\gamma : (\Lambda^3\mathfrak{p}^*)^H \to (S^2\mathfrak{p}^*)^H$$

which maps a G_2*-structure to its associated metric is surjective.*

The statement of the above lemma can be understood as follows: For any G-invariant metric g on G/H there exists a G-invariant G_2-structure such that its associated metric is g. Moreover, the space of all such G_2-structures is of dimension $\mathfrak{n}_{\mathrm{G}_2}$.

2.2 $\mathrm{Sp}(2)$-invariant G_2-structures on $\mathbb{S}^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1)$

G_2-Geometry on the sphere has attracted significant interest over the past decade, perhaps most notably in the classification of homogeneous structures is given by Reidegeld [39] some examples was given in Table 2, the description of 7-dimensional homogeneous spaces with isotropy representation in G_2 by Munir & Le [42] and the study of the 7-sphere's calibrated geometry by Lotay [30] and Kawai [26]. The main results that i collaborated in [35] are given in this section: Proposition 2.24 about isometric $\mathrm{Sp}(2)$-invariant G_2-structures on $\mathbb{S}^7$ and Proposition 2.49 finding torsion forms for an specific case of $\mathrm{Sp}(2)$-invariant G_2-structures.

We begin with a reductive decomposition of $\mathbb{S}^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1)$ as a homogeneous space. Let $A^* = \bar{A}^T$ be the Hermitian transpose, consider the Lie algebra

$$\mathfrak{sp}(2) := \{A \in \mathfrak{gl}(2,\mathbb{H}) \mid A + A^* = 0\},$$

and fix its basis

$$\begin{gathered} v_1 = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}, \\ e_2 = \begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}, \quad e_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_5 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \\ e_6 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}, \quad e_7 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{gathered} \tag{2.9}$$

Define the embedding $\mathfrak{sp}(1) \subset \mathfrak{sp}(2)$ as the subalgebra generated by $\{v_1, v_2, v_3\}$, corresponding to the reductive splitting

$$\mathfrak{sp}(2) = \mathfrak{sp}(1) \oplus \mathfrak{p} \quad \text{with} \quad \mathfrak{p} := \mathfrak{sp}(1)^{\perp} = span(e_1, ..., e_7), \tag{2.10}$$

with respect to the canonical inner product $\langle A_1, A_2\rangle = \mathrm{Re}(\mathrm{tr}(A_1 A_2^*))$. A straightforward computation shows that

$$\mathrm{ad}(\mathfrak{sp}(1))e_l = 0, \quad \text{for} \quad l = 1, 2, 3,$$

$$\mathrm{ad}(\mathfrak{sp}(1))\mathfrak{p}_4 = \mathfrak{p}_4, \quad \text{with} \quad \mathfrak{p}_4 := span(e_4, e_5, e_6, e_7).$$

In terms of the trivial submodules $\mathfrak{p}_l = span(e_l)$, we have the irreducible $\mathrm{ad}(\mathfrak{p}(1))$-decomposition $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3 \oplus \mathfrak{p}_4$. Consider the 2-forms on $\mathfrak{p}_4$

$$\omega_1 = e^{47} + e^{56}, \quad \omega_2 = e^{46} - e^{57}, \quad \omega_3 = e^{45} + e^{67}, \tag{2.11}$$

and notice that $\omega_1^2 = \omega_2^2 = \omega_3^2 = 2e^{4567}$ and $\omega_l \wedge \omega_m = 0$, for any $l \neq m$.

Recall that we denote by $\Omega^3_+(\mathbb{S}^7)^{\mathrm{Sp}(2)}$ the bundle of $\mathrm{Sp}(2)$-invariant G_2-structures on $\mathbb{S}^7$, and by $\mathrm{Sym}^2_+(T^*\mathbb{S}^7)^{\mathrm{Sp}(2)}$ the bundle of $\mathrm{Sp}(2)$-invariant G_2-metrics. As sets, the following equivalences hold (see Lemma 2.7)

$$\Omega^3_+(\mathbb{S}^7)^{\mathrm{Sp}(2)} \simeq \Lambda^3_+(\mathfrak{p}^*)^{\mathrm{Ad}(\mathrm{Sp}(1))} \simeq \Big(\mathrm{GL}(\mathfrak{p}) \cdot \varphi\Big)^{\mathrm{Ad}(\mathrm{Sp}(1))}$$

and

$$\mathrm{Sym}^2_+(T^*\mathbb{S}^7)^{\mathrm{Sp}(2)} \simeq \mathrm{Sym}^2_+(\mathfrak{p}^*)^{\mathrm{Ad}(\mathrm{Sp}(1))},$$

where the left $\mathrm{GL}(\mathfrak{p})$-action is defined by $\Theta \cdot \varphi := (\Theta^{-1})^*\varphi$ for the $\mathrm{Ad}(\mathrm{Sp}(1))$-invariant morphism $\Theta \in \mathrm{GL}(\mathfrak{p})$ and the 3-form $\varphi \in \Lambda^3_+(\mathfrak{p}^*)^{\mathrm{Ad}(\mathrm{Sp}(1))}$. Looking into $\mathrm{GL}(\mathfrak{p})^{\mathrm{Ad}(\mathrm{Sp}(1))}$, the splitting (2.10) establishes the inclusion

$$h \in \mathrm{Sp}(1) \mapsto \left(\begin{array}{c|c} h & \\ \hline & 1 \end{array}\right) \in \mathrm{Sp}(2),$$

in such a way that, for any $(x, y) \in \mathfrak{p}$ with $x \in \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3 \simeq \mathfrak{sp}(1)$ and $y \in \mathfrak{p}_4 \simeq \mathbb{H}$, we have $\mathrm{Ad}(h)(x, y) = (x, hy)$. The equivariance condition $\mathrm{Ad}(h) \circ \Theta = \Theta \circ \mathrm{Ad}(h)$, for each $h \in \mathrm{Sp}(1)$, implies

$$\Theta = \left(\begin{array}{c|c} D & \\ \hline & aI_{4\times 4} \end{array}\right) \quad \text{with} \quad D \in \mathrm{GL}(\mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3) \quad \text{and} \quad a \in \mathbb{R} \setminus \{0\}.$$

We highlight that $\Lambda^3(\mathfrak{p}^*)$ has only two open $\mathrm{GL}(\mathfrak{p})$-orbits, generated by two G_2-structure φ and a $\tilde{\varphi}$. Likewise, $\Lambda^3(\mathfrak{p}^*)^{\mathrm{Ad}(\mathrm{Sp}(1))}$ has two open orbits induced by open subsets of $\mathrm{GL}(\mathfrak{p})^{\mathrm{Ad}(\mathrm{Sp}(1))} \simeq \mathrm{GL}(\mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3) \times \mathbb{R} \setminus \{0\}$, corresponding to the orbit of a $\mathrm{Ad}(\mathrm{Sp}(1))$-invariant G_2-structure φ or $\tilde{\varphi}$.

In accordance with [39], we denote by n_{G_2} the rank of the bundle $\Omega^3_+(\mathbb{S}^7)^{\mathrm{Sp}(2)}$, and by $n_{\mathrm{O}(7)}$ the rank of $\mathrm{Sym}^2_+(T^*\mathbb{S}^7)^{\mathrm{Sp}(2)}$. The following Lemma contains an alternative proof of Reidegeld's assertion that $n_{G_2} = 10$ and $n_{\mathrm{O}(7)} = 7$. It provides moreover an explicit description of all $\mathrm{Sp}(2)$-invariant G_2-structures and their induced $\mathrm{Sp}(2)$-invariant metrics, in which $\mathfrak{p}$ is identified with the tangent space of $\mathbb{S}^7$ at the orbit of the identity $o = 1_{\mathrm{Sp}(2)}\mathrm{Sp}(1)$. Furthermore, it distinguishes the open subset in $\mathrm{GL}(\mathfrak{p})^{\mathrm{Ad}(\mathrm{Sp}(1))}$ which parameterises $\Lambda^3_+(\mathfrak{p}^*)^{\mathrm{Ad}(\mathrm{Sp}(1))}$.

Lemma 2.12. *[35, Lema 1.1] Consider the homogeneous space* $\mathrm{Sp}(2)/\mathrm{Sp}(1)$ *with the reductive decomposition* (2.10)*. The* $\mathrm{Ad}(\mathrm{Sp}(1))$*-invariant* G_2*-structures on* $\mathfrak{p}$ *have the form*

$$\begin{aligned}\varphi = a^3 e^{123} &+ (\alpha_1 e^1 + \alpha_2 e^2 + \alpha_3 e^3) \wedge \omega_1 + (\beta_1 e^1 + \beta_2 e^2 + \beta_3 e^3) \wedge \omega_2 \\ &+ (\gamma_1 e^1 + \gamma_2 e^2 + \gamma_3 e^3) \wedge \omega_3,\end{aligned} \tag{2.13}$$

the coefficients of which satisfy

$$a \cdot \det D^{-1} > 0, \quad \textit{with} \quad D^{-1} := \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix} : \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3 \to \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3.$$

Moreover, the $\mathrm{Ad}(\mathrm{Sp}(1))$*-invariant inner product on* $\mathfrak{p}$*, induced by* (2.13)*, is given by*

$$g = \frac{a^2}{\det(D^{-1})^{2/3}} \left(\begin{array}{c|c} (DD^t)^{-1} & 0 \\ \hline & \frac{\det(D^{-1})}{a^3} I_{4\times 4} \end{array} \right), \tag{2.14}$$

and the induced orientation is parametrised by $a \in \mathbb{R} \setminus \{0\}$*. In particular,* $n_{G_2} = 10$ *and* $n_{O(7)} = 7$*.*

Remark 2.15. We have that the G_2-structure φ induce a symmetric definite bilinear form $B : \mathfrak{p} \times \mathfrak{p} \to \mathbb{R}$, defined by

$$B_{ij} := B(e_i, e_j) = (e_i \lrcorner \varphi) \wedge (e_j \lrcorner \varphi) \wedge \varphi(e_1, e_2, ..., e_7). \tag{2.16}$$

By a long and straightforward computation, we get

$$\begin{aligned} B_{11} &= 6a^3(\alpha_1^2 + \beta_1^2 + \gamma_1^2) & B_{12} = B_{21} &= 6a^3(\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2) \\ B_{22} &= 6a^3(\beta_2^2 + \beta_2^2 + \gamma_2^2) & B_{23} = B_{32} &= 6a^3(\alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3) \\ B_{33} &= 6a^3(\alpha_3^2 + \beta_3^2 + \gamma_3^2) & B_{13} = B_{31} &= 6a^3(\alpha_1\alpha_3 + \beta_1\beta_3 + \gamma_1\gamma_3) \end{aligned}$$

$$\begin{aligned} B_{kk} &= 6\alpha_1(\beta_2\gamma_3 - \beta_3\gamma_2) - 6\alpha_2(\beta_1\gamma_3 - \beta_3\gamma_1) + 6\alpha_3(\beta_1\gamma_2 - \beta_2\gamma_1) \quad \text{for} \quad k = 4,5,6,7, \\ B_{ij} &= 0 \qquad \text{otherwise.} \end{aligned}$$

Assembling the coefficients $\alpha_k, \beta_k, \gamma_k$ $(k = 1,2,3)$, we construct the matrix

$$D^{-1} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}.$$

Expressing the coefficients of (2.16) as

$$\begin{aligned} B_{ij} &= 6a^3[(DD^t)^{-1}]_{ij}, & i,j &= 1,2,3, \\ B_{kk} &= 6\det(D^{-1}), & k &= 4,5,6,7, \end{aligned} \tag{2.17}$$

we see that $\det(B) = 6^7 a^9 (\det(D^{-1}))^6$. For the definiteness of (2.16), we have

$$B_{jj} \in \mathbb{R}_\pm, \quad \forall j \in \{1, ..., 7\} \iff a, \det(D^{-1}) \in \mathbb{R}_\pm. \tag{2.18}$$

For the second part of the Lemma, recall that the inner product on $\mathfrak{p}$ induced by φ is given by (see [24])

$$g_{ij} = \frac{1}{6^{2/9}} \frac{B_{ij}}{\det(B)^{1/9}}.$$

Condition (2.18) then guarantees the positive-definiteness of the inner product, which indeed takes the form (2.14), by (2.17). Finally, the orientation $\sqrt{\det(g)} = 6^{-7/9}\det(B)^{1/9} = a(\det(D^{-1}))^{2/3}$ is determined by the sign of $a \in \mathbb{R} \setminus \{0\}$.

We fix henceforth an orientation by setting $a > 0$. Notice that $\mathrm{Ad}(\mathrm{Sp}(1))$-invariant G_2-structures are parametrised by $G := \mathbb{R}^+ \times \mathrm{GL}^+(3, \mathbb{R})$ (with positive orientation). Indeed, the invariant G_2-structure (2.13) can be written as $\varphi = (D(a)^{-1})^*\varphi_0$ where $\varphi_0 = e^{123} + \sum_{i=1}^{3} e^i \wedge \omega_i$ is the canonical G_2-structure and

$$D(a)^{-1} = \left(\begin{array}{c|c} \frac{a}{(\det(D^{-1}))^{1/3}} D^{-1} & 0 \\ \hline 0 & \frac{(\det(D^{-1}))^{1/6}}{a^{1/2}} I_{4\times 4} \end{array}\right) \quad \text{for} \quad D^{-1} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix},$$

for the pair $(a, D) \in \mathbb{R}^+ \times \mathrm{GL}^+(3, \mathbb{R})$. This yields a surjective map

$$\Theta : (a, D) \in \mathbb{R}^+ \times \mathrm{GL}^+(3, \mathbb{R}) \mapsto \varphi = (D(a)^{-1})^*\varphi_0 \in \Lambda^3_+(\mathfrak{p}^*)^{\mathrm{Ad}(\mathrm{Sp}(1))}. \tag{2.19}$$

2.3 Distinguishing homogeneous G_2-metrics

In [43], W. Ziller describes how the seven-parameter family of Sp(2)-invariant metrics falls into isometry classes, which depend on four parameters. We will now apply his approach to the G_2-metrics of Lemma 2.12, in order to simplify the description of the space of Sp(2)-invariant G_2-structures, as a seven-parameter family.

The Sp(2)-invariant G_2-metrics are described in (2.14) by seven polynomials in ten variables. Given a positive symmetric matrix of the form

$$S = \Big(\frac{1}{a\sqrt[3]{\det D}} D\Big)\Big(\frac{1}{a\sqrt[3]{\det D}} D\Big)^t \in \mathrm{GL}(3, \mathbb{R}), \tag{2.20}$$

straightforward diagonalisation yields $A \in \mathrm{SO}(3)$ such that $S = AQA^t$, with $Q = \mathrm{diag}(r_1^2, r_2^2, r_3^2)$ (if $\det A = -1$ take $\widetilde{A} = -A$). Furthermore, the 3-dimensional group SO(3) can be described using the surjective homomorphism $\Upsilon : \mathrm{Sp}(1) \to \mathrm{SO}(3)$ given by

$$A := \Upsilon(h) = \begin{pmatrix} h_0^2 + h_1^2 - h_2^2 - h_3^2 & 2(h_1h_2 - h_0h_3) & 2(h_1h_3 + h_0h_2) \\ 2(h_1h_2 + h_0h_3) & h_0^2 - h_1^2 + h_2^2 - h_3^2 & 2(h_2h_3 - h_0h_1) \\ 2(h_1h_3 - h_0h_2) & 2(h_2h_3 + h_0h_1) & h_0^2 - h_1^2 - h_2^2 + h_3^2 \end{pmatrix}, \tag{2.21}$$

for $h = h_0 + h_1 i + h_2 j + h_3 k \in \mathrm{Sp}(1)$.

The embedding

$$h \in \mathrm{Sp}(1) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} \in \mathrm{Sp}(2)$$

induces a conjugation $C_h := L_h \circ R_{h^{-1}}$, by elements of Sp(1) and a diffeomorphism of Sp(2)/Sp(1). Its differential acts on $\mathfrak{p}$ by the standard representation on $\mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3 \simeq \mathfrak{sp}(1)$ and the Sp(1)-right action on $\mathfrak{p}_4 \simeq \mathbb{H}$. In particular, its action at the orbit of the identity is

$$(dC_h)_o(x, y) = (hx\bar{h}, y\bar{h}) = (\Upsilon(h)x, y\bar{h}), \quad \text{for} \quad (x, y) \in \mathfrak{p}.$$

Notice that $(dC_h)_o$ identifies the inner product (2.14) with

$$\frac{a^2}{\det(D^{-1})^{2/3}} \left(\begin{array}{c|c} A^t(DD^t)^{-1}A & 0 \\ \hline & \frac{\det(D^{-1})}{a^3} I_{4\times 4} \end{array}\right), \quad \text{where} \quad A = \Upsilon(h).$$

Hence, Sp(1) acts on Sp(2)/Sp(1) by isometries. Setting $r_4^2 = \det D$, this inner product takes the form

$$g = \frac{1}{r_1^2}(f^1)^2 + \frac{1}{r_2^2}(f^2)^2 + \frac{1}{r_3^2}(f^3)^2 + \sqrt[3]{\frac{r_1 r_2 r_3}{r_4^2}}\Big((e^4)^2 + (e^5)^2 + (e^6)^2 + (e^7)^2\Big), \tag{2.22}$$

since $a^6 = \dfrac{1}{\det Q}$ and $f_k = Ae_k$ (for $k = 1,2,3$). Therefore the Sp(2)-invariant metrics of Lemma 2.12 are parametrised by $(r_1, r_2, r_3, r_4, h) \in \mathbb{R}^4 \smallsetminus \{0\} \times \mathrm{Sp}(1)$, provided $r_1 r_2 r_3 > 0$.

The following result reformulates a well-known condition of isometry between homogeneous metrics of the underlying $\mathbb{S}^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1)$, but from the point of view of G_2-structures.

Definition 2.23. Two G_2-structures φ_1, φ_2 will be called *isometric* if they induce the same metric under (2.16).

Proposition 2.24. *Two* Sp(2)*-invariant* G_2*-structures, described by* $(a, D), (\widetilde{a}, \widetilde{D}) \in \mathbb{R}^+ \times \mathrm{GL}^+(3,\mathbb{R})$*, are isometric if, and only if,* $\widetilde{a} = a$ *and* $\widetilde{D} = D \cdot A$ *with* $A \in \mathrm{SO}(3)$*.*

Proof. Let φ and $\widetilde{\varphi}$ be G_2-structures described by (a, D) and $(\widetilde{a}, \widetilde{D})$, respectively. Consider the induced bilinear forms $B, \widetilde{B}$ given by

$$B_{ij} := B(e_i, e_j) = (e_i \lrcorner \varphi) \wedge (e_j \lrcorner \varphi) \wedge \varphi(e_1, e_2, ..., e_7)..$$

According to theRemark 2.15 2.12, if φ and $\widetilde{\varphi}$ are isometric, then

$$\frac{B_{ij}}{a \det(D^{-1})^{2/3}} = \frac{\widetilde{B}_{ij}}{\widetilde{a} \det(\widetilde{D}^{-1})^{2/3}}, \quad i, j = 1, \ldots, 7.$$

Equivalently,

$$\begin{cases} \dfrac{a^2 (DD^t)^{-1}}{\det(D^{-1})^{2/3}} = \dfrac{\widetilde{a}^2 (\widetilde{D}\widetilde{D}^t)^{-1}}{\det(\widetilde{D}^{-1})^{2/3}} \\ \\ \widetilde{a}^3 \det(D^{-1}) = a^3 \det(\widetilde{D}^{-1}) \end{cases}, \tag{2.25}$$

and therefore

$$\begin{cases} \dfrac{DD^t}{a^2 \det(D)^{2/3}} = \dfrac{(\widetilde{D}\widetilde{D}^t)}{\widetilde{a}^2 \det(\widetilde{D})^{2/3}} \\ \\ a^3 \det(D) = \widetilde{a}^3 \det(\widetilde{D}) > 0 \end{cases} \quad \text{and} \quad DD^t = \widetilde{D}\widetilde{D}^t.$$

Define $A := D^{-1}\widetilde{D}$, so that $\widetilde{D} = D \cdot A$. Then $AA^t = I$ and $\det(A) = \dfrac{\det(\widetilde{D})}{\det(D)} > 0$, since $D, \widetilde{D} \in \mathrm{GL}^+(3,\mathbb{R})$, which actually implies $\det(A) = 1$, hence $A \in \mathrm{SO}(3)$ and $a = \widetilde{a}$. Conversely, if $\widetilde{D} = D \cdot A$ and $\widetilde{a} = a$, it is easy to verify that (2.25) holds, thus φ and $\widetilde{\varphi}$ are isometric. □

By definition, the map $\Theta(a, D) = \varphi_{a,D} = \varphi$ given in (2.19) is surjective. Consider $(a, D), (\widetilde{a}, \widetilde{D}) \in \mathbb{R}^+ \times \mathrm{GL}^+(3,\mathbb{R})$ such that $\Theta(a, D) = \Theta(\widetilde{a}, \widetilde{D})$. Then $\widetilde{D}(\widetilde{a})^{-1} D(a) \in G(\varphi_0) \simeq \mathrm{G}_2$, where

$$\varphi_0 = e^{123} + e^1 \wedge \omega_1 + e^2 \wedge \omega_2 + e^3 \wedge \omega_3.$$

In particular, the G_2-structures $\varphi_{a,D}$ and $\varphi_{\widetilde{a},\widetilde{D}}$ are isometric, so Proposition 2.24 implies $\widetilde{a} = a$ and $\widetilde{D} = D \cdot A$, with $A \in \mathrm{SO}(3)$. Then, we have

$$\widetilde{D}(\widetilde{a})^{-1} D(a) = \left(\begin{array}{c|c} A^t & 0 \\ \hline 0 & I_{4\times 4} \end{array}\right) \in \mathrm{G}_2.$$

From the invariance $(\widetilde{D}(\widetilde{a})^{-1}D(a))^*\varphi_0 = \varphi_0$, we deduce that $A = I_{3\times 3}$, so the map Θ is injective. concluding that the space of Sp(2)-invariant G_2-structures on $\mathbb{S}^7 \simeq \mathrm{Sp}(2)/\mathrm{Sp}(1)$ is described by the homogeneous manifold

$$\Omega^3_+(\mathbb{S}^7)^{\mathrm{Sp}(2)} \simeq \mathbb{R}^+ \times \mathrm{GL}^+(3,\mathbb{R}),$$

via the isomorphism $\Omega^3_+(\mathbb{S}^7)^{\mathrm{Sp}(2)} \simeq \Lambda^3_+(\mathfrak{p}^*)^{\mathrm{Ad}(\mathrm{Sp}(1))}$ and the map

$$\begin{aligned} \Theta: \quad \mathbb{R}^+ \times \mathrm{GL}^+(3,\mathbb{R}) &\to \Lambda^3_+(\mathfrak{p}^*)^{\mathrm{Ad}(\mathrm{Sp}(1))} \\ (a,D) &\mapsto \Theta(a,D) = \varphi_{a,D}. \end{aligned}$$

Since $\mathbb{S}^7$ is spinnable, the map [23, eq. (11)]

$$B : \varphi \in \Omega^3_+(\mathbb{S}^7) \mapsto g_\varphi \in \mathrm{Sym}^2_+(T^*\mathbb{S}^7),$$

associating a Riemmanian metric to each G_2-structure, is surjective. Now, using the above affirmation and considering the isomorphism $\mathrm{Sym}^2_+(T^*M)^{\mathrm{Sp}(2)} \simeq \mathrm{Sym}^2_+(\mathfrak{p})^{\mathrm{Ad}(\mathrm{Sp}(1))}$, we obtain the surjective map $B \circ \Theta$, where $B(\varphi) = g_\varphi$ assigns the corresponding Ad(Sp(1))-invariant inner product induced by $\varphi \in \Lambda^3_+(\mathfrak{p}^*)^{\mathrm{Ad}(\mathrm{Sp}(1))}$.Moreover, by Proposition 2.24, $\ker(B\circ\Theta) = \{1\} \times \mathrm{SO}(3)$, and therefore, the corresponding space of Sp(2)-invariant G_2-metrics is an isomorphism which is given by.

$$\mathrm{Sym}^2_+(T^*\mathbb{S}^7)^{\mathrm{Sp}(2)} \simeq \mathbb{R}^+ \times \Big(\mathrm{GL}^+(3,\mathbb{R})/\mathrm{SO}(3)\Big). \tag{2.26}$$

In particular, the moduli space of Sp(2)-invariant Riemannian metrics described in [43] corresponds to the 4-manifold

$$\mathrm{Sym}^2_+(T^*\mathbb{S}^7)^{\mathrm{Sp}(2)}/\mathrm{Iso}(\mathbb{S}^7) \simeq \mathbb{R}^+ \times \mathrm{SO}(3)\backslash\Big(\mathrm{GL}^+(3,\mathbb{R})/\mathrm{SO}(3)\Big).$$

We conclude with some observations regarding the linear algebra of the degrees of freedom contained in a pair $(a,D) \in \mathbb{R}^+ \times \mathrm{GL}^+(3,\mathbb{R})$. By (2.20), the matrix $a^{-1}\sqrt[3]{\det(D^{-1})}D$ is a square root of the positive symmetric matrix S. However, S admits another representation in terms of the eigenvalues r_1^2, r_2^2, r_3^2 and the orthogonal matrix $P = \Upsilon(v)$ $(v \in \mathrm{Sp}(1))$ obtained from eigenvectors:

$$S = CC^t \quad \text{where} \quad C = P\sqrt{Q} \quad \text{and} \quad \sqrt{Q} = \mathrm{diag}(r_1, r_2, r_3). \tag{2.27}$$

If the matrix S can be written in the form (2.27), then was shown in [35, Proposition 1.4] that there exists $A \in \mathrm{SO}(3)$ such that

$$\frac{1}{a\sqrt[3]{\det(D)}} D = CA. \tag{2.28}$$

Moreover, for any $(a, D) \in \mathbb{R}^+ \times \mathrm{GL}^+(3, \mathbb{R})$, there exists $(r_1, ..., r_4, v, h) \in \mathbb{R}^4 \setminus \{0\} \times \mathrm{Sp}(1)^2$ such that

$$a = \frac{1}{\sqrt[3]{r_1 r_2 r_3}} > 0, \quad D = \left[\sqrt[3]{\tfrac{r_4^2}{r_1 r_2 r_3}}\Upsilon(v)\sqrt{Q}\right]\Upsilon(h). \tag{2.29}$$

In summary, from the expression of D in (2.29), the elements inside the square brackets determine the metric. So, up to isometry, we have

$$g_{(r_1,r_2,r_3,r_4)} = \frac{1}{r_1^2}(e^1)^2 + \frac{1}{r_2^2}(e^2)^2 + \frac{1}{r_3^2}(e^3)^2 + \sqrt[3]{\frac{r_1 r_2 r_3}{r_4^2}}\Big((e^4)^2 + (e^5)^2 + (e^6)^2 + (e^7)^2\Big). \tag{2.30}$$

The isometric family of G_2-structures $\varphi := \varphi_{(r_1,r_2,r_3,r_4)}$ is

$$\begin{aligned}
\varphi = \frac{1}{r_1 r_2 r_3}e^{123} &+ \left(\sqrt[3]{\frac{r_2 r_3}{r_4^2 r_1^2}}(h_0^2 + h_1^2 - h_2^2 - h_3^2)e^1\right.\\
&\left.+ 2\sqrt[3]{\frac{r_1 r_3}{r_4^2 r_2^2}}(h_1 h_2 - h_0 h_3)e^2 + 2\sqrt[3]{\frac{r_1 r_2}{r_4^2 r_3^2}}(h_1 h_3 + h_0 h_2)e^3\right) \wedge \omega_1\\
&+ \left(2\sqrt[3]{\frac{r_2 r_3}{r_4^2 r_1^2}}(h_1 h_2 + h_0 h_3)e^1 + \sqrt[3]{\frac{r_1 r_3}{r_4^2 r_2^2}}(h_0^2 - h_1^2 + h_2^2 - h_3^2)e^2\right.\\
&\left.+ 2\sqrt[3]{\frac{r_1 r_2}{r_4^2 r_3^2}}(h_2 h_3 - h_0 h_1)e^3\right) \wedge \omega_2 + \left(2\sqrt[3]{\frac{r_2 r_3}{r_4^2 r_1^2}}(h_1 h_3 - h_0 h_2)e^1\right.\\
&\left.+ 2\sqrt[3]{\frac{r_1 r_3}{r_4^2 r_2^2}}(h_2 h_3 + h_0 h_1)e^2 + \sqrt[3]{\frac{r_1 r_2}{r_4^2 r_3^2}}(h_0^2 - h_1^2 - h_2^2 + h_3^2)e^3\right) \wedge \omega_3,
\end{aligned} \tag{2.31}$$

According to the *reductive decomposition* $\mathfrak{sp}(2) = \mathfrak{sp}(1) \oplus \mathfrak{p}$, where $\mathrm{Ad}(\mathrm{Sp}(1))\mathfrak{p} \subset \mathfrak{p}$, the tangent space of $\mathbb{S}^7$ at the identity class $o = 1_{\mathrm{Sp}(2)}\mathrm{Sp}(1)$ is identified with the $\mathrm{Ad}(\mathrm{Sp}(1))$-invariant complement vector space $\mathfrak{p}$. Moreover, each $\mathrm{Sp}(2)$-invariant G_2-structure on $\mathbb{S}^7$ is determined by a G_2-structure on $\mathfrak{p} \simeq T_o\mathbb{S}^7$ which is invariant by the $\mathrm{Ad}(\mathrm{Sp}(1))$-action, i.e. $\Omega^3_+(\mathbb{S}^7)^{\mathrm{Sp}(2)} \simeq \Lambda^3_+(\mathfrak{p}^*)^{\mathrm{Ad}(\mathrm{Sp}(1))}$. From the identification $\Omega^3_+(\mathbb{S}^7)^{\mathrm{Sp}(2)} \simeq \mathbb{R}^+ \times \mathrm{GL}^+(3, \mathbb{R})$, each such isometric class is described by a $\mathrm{SO}(3)$-orbit (Proposition 2.24):

$$\Omega^3_+(\mathbb{S}^7)^{\mathrm{Sp}(2)} \ni \varphi \longleftrightarrow (a, D) \in \mathbb{R}^+ \times \mathrm{GL}^+(3, \mathbb{R}), \quad \text{with} \quad \mathcal{B}_\varphi \simeq (a, D \cdot \mathrm{SO}(3)),$$

for each $g_\varphi \in \mathrm{Sym}^2_+(T^*\mathbb{S}^7)^{\mathrm{Sp}(2)}$.

2.4 G_2-structures in different isometric classes

We now propose an approach to explicitly parametrise and study different isometric classes of $\mathrm{Sp}(2)$-invariant G_2-structures on $\mathbb{S}^7$. By considering $\mathrm{Sp}(2)$-metrics only up to isometries, we describe the reduced space $\mathrm{Sym}^2_+(\mathfrak{p})^{\mathrm{Ad}(\mathrm{Sp}(1))}$ in terms of just 3 parameters.

Set $\mathbf{r} = (r_1, r_2, r_3)$, with $r_1, r_2, r_3 > 0$, and consider the G_2-structure

$$\varphi_{\mathbf{r}} = \frac{1}{(r_1 r_2 r_3)^3}e^{123} + \frac{r_2 r_3}{r_1^2}e^1 \wedge \omega_1 + \frac{r_1 r_3}{r_2^2}e^2 \wedge \omega_2 + \frac{r_1 r_2}{r_3^2}e^3 \wedge \omega_3, \tag{2.32}$$

with dual 4-form

$$\psi_{\mathbf{r}} := *\varphi_{\mathbf{r}} = (r_1r_2r_3)^2\frac{\omega_1^2}{2} + \frac{r_3}{r_1^2r_2^2}e^{12}\wedge\omega_3 - \frac{r_2}{r_1^2r_3^2}e^{13}\wedge\omega_2 + \frac{r_1}{r_2^2r_3^2}e^{23}\wedge\omega_1,$$

and induced G_2-metric

$$g_{\mathbf{r}} = \frac{1}{r_1^6}(e^1)^2 + \frac{1}{r_2^6}(e^2)^2 + \frac{1}{r_3^6}(e^3)^2 + r_1r_2r_3\Big((e^4)^2 + (e^5)^2 + (e^6)^2 + (e^7)^2\Big). \tag{2.33}$$

Proposition 2.34. *[9, Eq. 3.6] Let* M^7 *be a manifold with* G_2*-structure such that* φ *induces the Riemannian metric* g*. Then all the other* G_2*-structure on* M *inducing the same metric* g *can be parametrized by a pair* $(f, X) \in C^\infty(M)\times\mathfrak{X}(M)$ *where* f *is a function and* X *is a vector field satisfying* $f^2 + |X|^2 = 1$*. The explicit formula for the* G_2*-structure* $\varphi_{(f,X)}$ *corresponding to the pair* (f, X) *is*

$$\varphi_{(f,X)} = (f^2 - |X|^2)\varphi - 2f(X\lrcorner\psi) + 2X^\flat\wedge(X\lrcorner\varphi). \tag{2.35}$$

where $\psi = *\varphi$ *and the norm of* X *is taken with respect to* g*.*

Remark 2.36. In [35, §2, Lemma 2.1], was shown that the G_2-structures of the form (2.52) can be expressed according to Bryant's description of isometric G_2-structures as sections of a $\mathbb{R}P^7$-bundle for Sp(2)-invariant G_2-structures. Then, if the isometric G_2-structures $\varphi_{(f,X)}$ and φ in (2.35) are Sp(2)-invariant, then f must be a constant function and X is a Sp(2)-invariant vector field. In particular, the G_2-structure (2.31) has the form

$$\varphi := \varphi_{(h_0,X)} = (h_0^2 - |X|^2)\varphi_{\mathbf{r}} - 2h_0(X\lrcorner\psi_{\mathbf{r}}) + 2X^\flat\wedge(X\lrcorner\varphi_{\mathbf{r}}) \quad \text{for} \quad h_0^2 + |X|^2 = 1,$$

where $X = r_1^3h_1e_1 + r_2^3h_2e_2 + r_3^3h_3e_3 \in \mathfrak{X}(\mathbb{S}^7)^{\mathrm{Sp}(2)}$ is an $\mathrm{ad}(\mathfrak{sp}(1))$-invariant vector of $\mathfrak{p}$, and $|X|^2 = g_{\mathbf{r}}(X, X) = 1 - h_0^2$.

In particular, back on $M = \mathbb{S}^7$, we can compute the torsion forms of the G_2-structures $\varphi_{\mathbf{r}}$ given by (2.32) using the Chevalley-Eilenberg differential

$$de^i(e_j, e_k) = -\frac{1}{2}e^i([e_j, e_k]_{\mathfrak{p}}),$$

and the Lie bracket in terms of the basis (2.9) given by

$[\cdot,\cdot]$	v_1	v_2	v_3	e_1	e_2	e_3	e_4	e_5	e_6	e_7
v_1	0	$2v_3$	$-2v_2$	0	0	0	$-e_7$	e_6	$-e_5$	e_4
v_2	$-2v_3$	0	$2v_1$	0	0	0	$-e_6$	$-e_7$	e_4	e_5
v_3	$2v_2$	$2v_1$	0	0	0	0	e_5	$-e_4$	$-e_7$	e_6
e_1	0	0	0	0	$2e_3$	$-2e_2$	e_7	e_6	$-e_5$	$-e_4$
e_2	0	0	0	$-2e_3$	0	$2e_1$	$-e_6$	e_7	e_4	$-e_5$
e_3	0	0	0	$2e_2$	$-2e_1$	0	e_5	$-e_4$	e_7	$-e_6$
e_4	e_7	e_6	$-e_5$	$-e_7$	e_6	$-e_5$	0	$v_3 + e_3$	$-v_2 - e_2$	$-v_1 + e_1$
e_5	$-e_6$	e_7	e_4	$-e_6$	$-e_7$	e_4	$-v_3 - e_3$	0	$v_1 + e_1$	$-v_2 + e_2$
e_6	e_5	$-e_4$	e_7	e_5	$-e_4$	$-e_7$	$v_2 + e_2$	$-v_1 - e_1$	0	$-v_3 + e_3$
e_7	$-e_4$	$-e_5$	$-e_6$	e_4	e_5	e_6	$v_1 - e_1$	$v_2 - e_2$	$v_3 - e_3$	0

(2.37)

The differentials of the dual basis are

$$\begin{aligned} \mathrm{d}e^1 &= -2e^{23} - \omega_1, & \mathrm{d}e^4 &= e^{17} - e^{26} + e^{35}, & \mathrm{d}e^7 &= -e^{14} - e^{25} - e^{36}, \\ \mathrm{d}e^2 &= 2e^{13} + \omega_2, & \mathrm{d}e^5 &= e^{16} + e^{27} - e^{34}, & & \\ \mathrm{d}e^3 &= -2e^{12} - \omega_3, & \mathrm{d}e^6 &= -e^{15} + e^{24} + e^{37}, & & \end{aligned} \tag{2.38}$$

and, by the Leibniz rule,

$$\mathrm{d}\omega_1 = -2e^2 \wedge \omega_3 - 2e^3 \wedge \omega_2, \quad \mathrm{d}\omega_2 = -2e^1 \wedge \omega_3 + 2e^3 \wedge \omega_1, \quad \mathrm{d}\omega_3 = 2e^1 \wedge \omega_2 + 2e^2 \wedge \omega_1. \tag{2.39}$$

The Levi-Civita connection ∇ induced by the corresponding left-invariant metric [3] acts on $\mathfrak{p} \simeq T_o\mathbb{S}^7$ in the following way:

$$\begin{aligned} \nabla_p Y_m = g_{\mathbf{r}}(\nabla_p Y, e_m) &= \frac{1}{2} g_{\mathbf{r}}([e_p, Y]_{\mathfrak{p}}, e_m) + g_{\mathbf{r}}(U(e_p, Y), e_m) \\ &= \frac{1}{2} g_{\mathbf{r}}([e_p, Y]_{\mathfrak{p}}, e_m) + \frac{1}{2} g_{\mathbf{r}}([e_m, e_p]_{\mathfrak{p}}, Y) + \frac{1}{2} g_{\mathbf{r}}([e_m, Y]_{\mathfrak{p}}, e_p), \end{aligned} \tag{2.40}$$

where $U : \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$ is defined by

$$2g_{\mathbf{r}}(U(X,Y), Z) = g_{\mathbf{r}}([Z,X]|_{\mathfrak{p}}, Y) + g_{\mathbf{r}}(X, [Z,Y]|_{\mathfrak{p}}), \tag{2.41}$$

and X, Y, Z Killing vector fields in $\mathfrak{p}$, note that U is symmetric tensor. We now can introduce the divergence of a $(p,0)$-tensor ϑ as the $(p-1,0)$-tensor

$$(\mathrm{div}\vartheta)(Y_1, ..., Y_{p-1}) := \mathrm{tr}\left(Z \mapsto (\nabla_Z \vartheta)(\cdot, Y_1, ..., Y_{p-1})\right).$$

The full torsion tensor (1.32) of $\varphi_{(h_0,X)}$, as well as its divergence, can be expressed in terms of (h_0, X) and $(\varphi_{\mathbf{r}}, \psi_{\mathbf{r}}, T_{\mathbf{r}})$ [14]. Let us consider, from now on, *the Ansatz* $r_1 = r_2 = r_3 = r^{-1/3}$, so the metric prescribed in (2.33) is

$$g_r(u,v) = r^2 \langle u, v \rangle, \quad \forall u, v \in \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3 \quad \text{and} \quad g_r(u,v) = \frac{1}{r} \langle u, v \rangle, \quad \forall u, v \in \mathfrak{p}_4. \tag{2.42}$$

The corresponding isometric family is

$$\begin{aligned} \varphi_r = r^3 e^{123} &+ \Big((h_0^2 + h_1^2 - h_2^2 - h_3^2)e^1 + 2(h_1h_2 - h_0h_3)e^2 + 2(h_1h_3 + h_0h_2)e^3\Big) \wedge \omega_1 \\ &+ \Big(2(h_1h_2 + h_0h_3)e^1 + (h_0^2 - h_1^2 + h_2^2 - h_3^2)e^2 + 2(h_2h_3 - h_0h_1)e^3\Big) \wedge \omega_2 \\ &+ \Big(2(h_1h_3 - h_0h_2)e^1 + 2(h_2h_3 + h_0h_1)e^2 + (h_0^2 - h_1^2 - h_2^2 + h_3^2)e^3\Big) \wedge \omega_3. \end{aligned} \tag{2.43}$$

According to the equation (2.29), we have that $\varphi_r = \Theta(r.\Upsilon(\bar{h}))$ where Υ is the double cover homomorphism of SO(3), cf. (2.21). Moreover, the induced dual 4-form is

$$\begin{aligned} \psi_r = \frac{1}{2r^2}\omega_1^2 &+ r\Big((h_0^2 + h_1^2 - h_2^2 - h_3^2)e^{23} - 2(h_1h_2 - h_0h_3)e^{13} + 2(h_1h_3 + h_0h_2)e^{12}\Big) \wedge \omega_1 \\ &+ r\Big(2(h_1h_2 + h_0h_3)e^{23} - (h_0^2 - h_1^2 + h_2^2 - h_3^2)e^{13} + 2(h_2h_3 - h_0h_1)e^{12}\Big) \wedge \omega_2 \\ &+ r\Big(2(h_1h_3 - h_0h_2)e^{23} - 2(h_2h_3 + h_0h_1)e^{13} + (h_0^2 - h_1^2 - h_2^2 + h_3^2)e^{12}\Big) \wedge \omega_3. \end{aligned} \tag{2.44}$$

To compute the terms $d\varphi_r$ and $d\psi_r$, we use the exterior derivatives (2.38) and (2.39):

$$\begin{aligned}
\mathrm{d}(e^{123}) &= -e^{23}\wedge\omega_1 - e^{13}\wedge\omega_2 - e^{12}\wedge\omega_1,\\
\mathrm{d}(e^1\wedge\omega_1) &= -2e^{23}\wedge\omega_1 + 2e^{13}\wedge\omega_2 + 2e^{12}\wedge\omega_3 - \omega_1^2,\\
\mathrm{d}(e^2\wedge\omega_1) &= \quad 2e^{13}\wedge\omega_1 + 2e^{23}\wedge\omega_2,\\
\mathrm{d}(e^3\wedge\omega_1) &= -2e^{12}\wedge\omega_1 - 2e^{23}\wedge\omega_3,\\
\mathrm{d}(e^1\wedge\omega_2) &= -2e^{23}\wedge\omega_2 - 2e^{13}\wedge\omega_1,\\
\mathrm{d}(e^2\wedge\omega_2) &= -2e^{23}\wedge\omega_1 + 2e^{13}\wedge\omega_2 - 2e^{12}\wedge\omega_3 + \omega_2^2,\\
\mathrm{d}(e^3\wedge\omega_2) &= -2e^{12}\wedge\omega_2 - 2e^{13}\wedge\omega_3,\\
\mathrm{d}(e^1\wedge\omega_3) &= -2e^{23}\wedge\omega_3 - 2e^{12}\wedge\omega_1,\\
\mathrm{d}(e^2\wedge\omega_3) &= \quad 2e^{13}\wedge\omega_3 + 2e^{12}\wedge\omega_2,\\
\mathrm{d}(e^3\wedge\omega_3) &= \quad 2e^{23}\wedge\omega_1 + 2e^{13}\wedge\omega_2 - 2e^{12}\wedge\omega_3 - \omega_3^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mathrm{d}(e^{23}\wedge\omega_1) = 0, && \mathrm{d}(e^{13}\wedge\omega_1) = -e^3\wedge\omega_1^2 + 2e^{123}\wedge\omega_3,\\
&\mathrm{d}(e^{12}\wedge\omega_1) = -e^2\wedge\omega_1^2 - 2e^{123}\wedge\omega_2, && \mathrm{d}(e^{23}\wedge\omega_2) = e^3\wedge\omega_1^2 - 2e^{123}\wedge\omega_3,\\
&\mathrm{d}(e^{13}\wedge\omega_2) = 0, && \mathrm{d}(e^{12}\wedge\omega_2) = -e^1\wedge\omega_2^2 + 2e^{123}\wedge\omega_1,\\
&\mathrm{d}(e^{23}\wedge\omega_3) = e^2\wedge\omega_3^2 + 2e^{123}\wedge\omega_2, && \mathrm{d}(e^{13}\wedge\omega_3) = e^1\wedge\omega_3^2 - 2e^{123}\wedge\omega_1,\\
&\mathrm{d}(e^{12}\wedge\omega_3) = 0.
\end{aligned} \tag{2.45}$$

For the G_2-structure φ_r from (2.43), with associated 4-form $\psi_r = *\varphi_r$, substituting the above equations into $d\varphi_r$ and $d\psi_r$ yields:

$$\begin{aligned}
\mathrm{d}\varphi_r \;&=\; -\Big(8h_1h_3e^{12} + 8h_0h_3e^{13} + (r^3 - 8h_3^2 + 2)e^{23}\Big)\wedge\omega_1\\
&\quad +\Big(8h_0h_1e^{12} - (r^3 - 8h_0^2 + 2)e^{13} - 8h_0h_3e^{23}\Big)\wedge\omega_2\\
&\quad -\Big((r^3 - 8h_1^2 + 2)e^{12} - 8h_0h_1e^{13} + 8h_1h_3e^{23}\Big)\wedge\omega_3 - (1 - 4h_2^2)\omega_1^2,\\
\mathrm{d}\psi_r \;&=\; -8h_2r\Big(e^{123}\wedge(h_1\omega_3 + h_0\omega_2 - h_3\omega_1) + (h_3e^1 + h_0e^2 - h_1e^3)\wedge\frac{\omega_1^2}{2}\Big).
\end{aligned} \tag{2.46}$$

Furthermore, we deduce that

$$\begin{aligned}
*\mathrm{d}\varphi_r = &-2r^5(1 - 4h_2^2)e^{123} - \frac{1}{r}\Big((r^3 - 8h_3^2 + 2)e^1 - 8h_0h_3e^2 + 8h_1h_3e^3\Big)\wedge\omega_1\\
&- \frac{1}{r}\Big(8h_0h_3e^1 - (r^3 - 8h_0^2 + 2)e^2 - 8h_0h_1e^3\Big)\wedge\omega_2\\
&- \frac{1}{r}\Big(8h_1h_3e^1 + 8h_0h_1e^2 + (r^3 - 8h_1^2 + 2)e^3\Big)\wedge\omega_3,
\end{aligned} \tag{2.47}$$

$$*\mathrm{d}\psi_r = 8h_2\Big(r^4h_1e^{12} + r^4h_0e^{13} - r^4h_3e^{23} + \frac{1}{r^2}\Big(h_3\omega_1 - h_0\omega_2 - h_1\omega_3\Big)\Big). \tag{2.48}$$

We are now in position to compute the torsion forms of these G_2-structures.

Proposition 2.49. *The torsion forms of the* G_2*-structure* φ_r *defined by* (2.43) *are:*

$$\begin{aligned}
\tau_0 &= -\frac{4}{7r}\Big(r^3(1-4h_2^2)+(5-8h_2^2)\Big),\\
\tau_1 &= -\frac{2(r^3+2)h_2}{3}\Big(h_3e^1+h_0e^2-h_1e^3\Big),\\
\tau_2 &= -\frac{8(r^3-1)h_2}{3}\Big(h_1(2re^{12}+\frac{1}{r^2}\omega_3)+h_0(2re^{13}+\frac{1}{r^2}\omega_2)-h_3(2re^{23}+\frac{1}{r^2}\omega_1)\Big),\\
\tau_3 &= \mathrm{i}_{\varphi_r}((\tau_{27})_{ij}) = (\tau_{27})_{ij}(g_r)^{jl}dx^i\wedge(e_l\lrcorner\varphi_r).
\end{aligned}$$

where the components $(\tau_{27})_{ij} = \frac{1}{4}\mathrm{j}_\varphi(\tau_3)_{ij}$ *define the matrix*

$$\tau_{27} = \frac{2}{7r^2}\left(\begin{array}{ccc|c}
r^3p_3(r) & \mathrm{s}(r)h_0h_3 & -\mathrm{s}(r)h_1h_3 & \\
\mathrm{s}(r)h_0h_3 & r^3p_0(r) & -\mathrm{s}(r)h_0h_1 & \\
-\mathrm{s}(r)h_1h_3 & -\mathrm{s}(r)h_0h_1 & r^3p_1(r) & \\
\hline
 & & & (\mathbf{n}(r)-(5r^3-4)h_2^2)I_{4\times4}
\end{array}\right)$$

with $p_k(r) = 7(r^3-2)h_k^2+(9r^3-10)h_2^2-4(r^3-2)$ *,*$k = 0,1,3.$*,* $\mathrm{s}(r) = 7r^3(r^3-2)$ *and* $\mathbf{n}(r) = \frac{5}{4}(r^3-2)$*.*

Proof. Taking the induced metric g_r induced by (2.43), substituting the equations (2.46) and (2.47) into the formula (1.30), and using the identities $\omega_1^2=\omega_2^2=\omega_3^2=2e^{4567}$ and $\omega_l\wedge\omega_m=0$ $(l\neq m)$, we obtain

$$\begin{aligned}
\tau_0 &= -\frac{4}{7}(r^3(1-4h_2^2)+(5-8h_2^2))*\Big(e^{123}\wedge\frac{\omega_1^2}{2}\Big) = -\frac{4}{7r}\Big(r^3(1-4h_2^2)+(5-8h_2^2)\Big),\\
\tau_1 &= \frac{1}{12}*\Big[\frac{8(r^3+2)}{r}\Big(h_1h_2e^{12}-h_0h_2e^{13}-h_3h_2e^{23}\Big)\wedge\frac{\omega_1^2}{2}\Big]\\
&= -\frac{2(r^3+2)h_2}{3}\Big(h_3e^1+h_0e^2-h_1e^3\Big).
\end{aligned}$$

Now, having in mind the torsion identity for τ_2 in (1.30), we use the above expression for τ_1 to compute:

$$\begin{aligned}
(\tau_1\wedge\psi_r) &= \frac{2(r^3+2)h_2}{3}\Bigg(\Big(r(h_3\omega_1-h_0\omega_2-h_1\omega_3)\wedge e^{123}\\
&\qquad+\frac{1}{r^2}(-h_3e^1-h_0e^2+h_1e^3)\wedge\frac{\omega^2}{2}\Big)\Bigg)\\
&= \frac{2(r^3+2)h_2}{3}\Big(-rh_3e^{23}+rh_0e^{13}+rh_1e^{12}+\frac{1}{r^2}h_3\omega_1-\frac{1}{r^2}h_0\omega_2-\frac{1}{r^2}h_1\omega_3\Big).
\end{aligned}$$

On the other hand, we already have an expression for $*d\psi_r$ in (2.48), so we obtain

$$\tau_2 = -\frac{8(r^3-1)h_2}{3}\Big(h_1(2re^{12}+\frac{1}{r^2}\omega_3)+h_0(2re^{13}+\frac{1}{r^2}\omega_2)-h_3(2re^{23}+\frac{1}{r^2}\omega_1)\Big).$$

Finally, the coefficients of the traceless symmetric 2-tensor τ_{27} are

$$(\tau_{27})_{ab} := \frac{1}{4}\mathrm{j}_\varphi(\tau_3)_{ab} = \frac{1}{4}*(e_a\lrcorner\varphi_r\wedge e_b\lrcorner\varphi_r\wedge\tau_3),$$

where $\tau_3 = *d\varphi_r - \tau_0\varphi_r - 3*(\tau_1 \wedge \varphi_r)$ is the torsion 3-form. In terms of the dual basis to (2.9), the torsion 3-form is

$$\tau_3 = \gamma_{123}e^{123} + \sum_{c,d=1}^{3} \gamma_{cd}e^c \wedge \omega_d,$$

where the coefficients γ_{123} and γ_{cd}, for $c, d = 1, 2, 3$, are obtained from (2.47) and the previous torsion forms τ_0, τ_1. The explicit computation of $(\tau_{27})_{ab}$ involves polynomial operations for $h = (h_0, h_1, h_2, h_3)$, subject to the unitary condition $h\bar{h} = 1$, which quickly get out of hand. We resorted to the MAPLE computer algebra system to obtain the expression of τ_{27}. However, the computation can be carried out by systematically applying the operator j_φ on the basis elements involved in τ_3 (cf. Lemma 2.50). □

Lemma 2.50. *[35, §2, Lemma 2.6] Given $h \in \mathrm{Sp}(1)$, consider the linear map defined by*

$$A(x) = hx\bar{h}, \quad \text{with} \quad x \in \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3,$$

and denote by (A_{ab}) its induced matrix, in the basis $\{e_1, e_2, e_3\}$ from (2.9). Thus, the image of $e^{123} \in \Lambda^3(\mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3)^*$ and $e^m \wedge \omega_n \in \Lambda^3\mathfrak{p}^*$ for $m, n = 1, 2, 3$, under the operator j_φ are

$$\begin{aligned}
\mathrm{j}_\varphi(e^{123})_{ab} &= \frac{1}{r}g_r(e_a, e_b), \quad \text{for} \quad a, b = 1, 2, 3,\\
\mathrm{j}_\varphi(e^m \wedge \omega_n)_{ab} &= 2r(A_{bn}\delta_{ma} + A_{an}\delta_{nb}), \quad \text{for} \quad a, b = 1, 2, 3,\\
\mathrm{j}_\varphi(e^m \wedge \omega_n)_{ab} &= -r^2 A_{pq}A_{uv}\varepsilon_{mpu}\varepsilon_{nqv}g_r(e_a, e_b), \quad \text{for} \quad a, b = 4, 5, 6, 7,\\
\mathrm{j}_\varphi(e^{123})_{ab} = \mathrm{j}_\varphi(e^m \wedge \omega_n)_{ab} &= 0 \quad \text{otherwise},
\end{aligned}$$

where $\varepsilon_{pqn} = \pm 1$ is the sign of the permutation $(p, q, n) \sim (1, 2, 3)$.

We know from [43] that, up to isometry, any Sp(2)-invariant metric is a multiple of the inner product expressed in terms of an oriented left-invariant coframe $e^1, \ldots, e^7 \in \Omega^1(\mathbb{S}^7)^{\mathrm{Sp}(2)}$ by

$$g_{\mathbf{r}} = \frac{1}{r_1^6}(e^1)^2 + \frac{1}{r_2^6}(e^2)^2 + \frac{1}{r_3^6}(e^3)^2 + r_1r_2r_3\Big((e^4)^2 + (e^5)^2 + (e^6)^2 + (e^7)^2\Big), \tag{2.51}$$

with $r_1, \ldots, r_3 > 0$ and $\mathbf{r} := (r_1, r_2, r_3)$. The corresponding isometric class of G_2-structures is parametrised by

$$\begin{aligned}
\varphi_{(\mathbf{r},h)} = &\frac{1}{(r_1r_2r_3)^3}e^{123}\\
&+ \Big(\frac{r_2r_3}{r_1^2}(h_0^2 + h_1^2 - h_2^2 - h_3^2)e^1 + 2\frac{r_1r_3}{r_2^2}(h_1h_2 - h_0h_3)e^2 + 2\frac{r_1r_2}{r_3^2}(h_1h_3 + h_0h_2)e^3\Big) \wedge \omega_1\\
&+ \Big(2\frac{r_2r_3}{r_1^2}(h_1h_2 + h_0h_3)e^1 + \frac{r_1r_3}{r_2^2}(h_0^2 - h_1^2 + h_2^2 - h_3^2)e^2 + 2\frac{r_1r_2}{r_3^2}(h_2h_3 - h_0h_1)e^3\Big) \wedge \omega_2\\
&+ \Big(2\frac{r_2r_3}{r_1^2}(h_1h_3 - h_0h_2)e^1 + 2\frac{r_1r_3}{r_2^2}(h_2h_3 + h_0h_1)e^2 + \frac{r_1r_2}{r_3^2}(h_0^2 - h_1^2 - h_2^2 + h_3^2)e^3\Big) \wedge \omega_3,
\end{aligned} \tag{2.52}$$

with $(h_0, ..., h_3) \in \mathbb{H}$ a unit quaternion parametrising a SO(3)-transformation and $\omega_1, \ldots, \omega_3 \in \Omega^2(\mathbb{S}^7)^{\mathrm{Sp}(2)}$ as in (2.31) below.

Remark 2.53. The curvature formula [3]

$$\langle R(X,Y)Y,X\rangle = -\frac{3}{4}|[X,Y]_{\mathfrak{p}}|^2 - \frac{1}{2}\langle [X,[X,Y]]_{\mathfrak{p}},Y\rangle - \frac{1}{2}\langle [Y,[Y,X]]_{\mathfrak{p}},X\rangle \\ + |U(X,Y)|^2 - \langle U(X,X),U(Y,Y)\rangle$$

implies that the invariant metric induced by the isometric family $(\sqrt[3]{2},h_0,h_1,0,h_3)$ has constant sectional curvature $K(X,Y)=\frac{1}{2}$. On the other hand, Friedrich et al. prove in Lemma 1.51 and Proposition 1.52 that, on a given 3-Sasakian manifold (M^7,g), the Berger metric g^s, obtained from g by conformal deformation along the 3-dimensional foliation, is Einstein if, and only if, $s=1$ or $s=1/\sqrt{5}$. Moreover, it is nearly-parallel for $s=1/\sqrt{5}$. Somewhat similarly, the Einstein metrics $g_{\sqrt[3]{2}}$ and $g_{\sqrt[3]{2/5}}$ induced by the G_2-structures

$$(\sqrt[3]{\frac{2}{5}},0,0,\pm1,0) \quad \text{and} \quad (\sqrt[3]{2},h_0,h_1,0,h_3)$$

are homothetic to the metrics g^1 and $g^{\frac{1}{\sqrt{5}}}$ of [17, Lemma 5.3], respectively:

$$g^1 = \frac{1}{\sqrt[3]{4}}g_{\sqrt[3]{2}} \quad \text{and} \quad g^{\frac{1}{\sqrt{5}}} = \frac{1}{\sqrt[3]{20}}g_{\sqrt[3]{2/5}}.$$

Theorem 2.54. *Let* $r_1=r_2=r_3=r^{-1/3}$ *in* (2.52)*. Then each* $\mathbb{S}^3$*-family of* G_2*-structures*

$$\mathcal{B}_r := \Theta(\{r\}\times \mathrm{SO}(3)) = B^{-1}(g_r)^{\mathrm{Sp}(2)} \simeq \mathbb{S}^3$$

determines a distinct isometric class. Moreover, in terms of the equator and poles

$$\mathbb{S}^2 \simeq \{(h_0,h_1,0,h_3)\} \quad \textit{and} \quad \mathrm{NS}\simeq\{(0,0,\pm1,0)\},$$

we characterise the following torsion regimes in each isometric class $\mathcal{B}_r$*, up to the diffeomorphism* Θ*:*

1. *The coclosed* G_2*-structures correspond to* $\{r\}\times\mathbb{S}^2$ *and* $\{r\}\times\mathrm{NS}$*.*
2. *The nearly parallel* G_2*-structures correspond to* $\{\sqrt[3]{2}\}\times\mathbb{S}^2$ *(round) and* $\{\sqrt[3]{2/5}\}\times\mathrm{NS}$ *(squashed).*
3. *The locally conformal coclosed* G_2*-structures correspond to* $\{1\}\times\mathbb{S}^3$*.*

Furthermore, there are no locally conformal closed or purely coclosed structures in $\mathcal{B}_r$*.*

Proof. Any $D\in\mathrm{SO}(3)$ is described by $h\in\mathbb{S}^3$ using the double cover homomorphism $D=\Upsilon(h)$ from (2.21). Also recall that the torsion forms in the present case are explicitly computed in Proposition 2.49.

Claim (i) follows immediately by solving the equations $\tau_1=0$ and $\tau_2=0$, (i.e. $d\varphi_r=0$), from which we get

$$h_2=0 \quad \text{or} \quad h_0=h_1=h_3=0.$$

For claim (ii), we further impose $\tau_{27} = 0$ (i.e. $d\varphi_r = \tau_0\psi_r$ and $d\psi_r = 0$), which implies $r^3 = 2$, for the case $h_2 = 0$, and $r^3 = 2/5$, for the case $h_0 = h_1 = h_3 = 0$. Finally, claim (iii) stems from the fact that $d\psi_r = 4\tau_1 \wedge \psi_r$ implies $d\tau_1 \wedge \psi_r = 0$.

To conclude the proof, we use (2.38) to check that $d\tau_1 = 0$ implies $\tau_1 = 0$. Now, in the coclosed class, the torsion 0-form can only take the following values:

$$\tau_0 = \begin{cases} -\dfrac{4(r^3+5)}{7r} \neq 0, & \text{if } h_2 = 0 \\ \dfrac{12(r^3+1)}{7r} \neq 0, & \text{if } h_0 = h_1 = h_3 = 0 \end{cases}. \qquad \square$$

Proposition 2.55. *The norm of the full torsion tensor* (1.32) *of φ_r is*

$$|T(r)|^2 = \frac{1}{r^2}\Big(4r^6 - 14r^3 + 19 + 8(r^3+2)(r^3-1)h_2^2\Big). \tag{2.56}$$

Proof. Since each term of the full torsion tensor $T(r) = \frac{\tau_0}{4}g_r - (\tau_1)^\sharp \lrcorner \varphi_r - \frac{1}{2}\tau_2 - \tau_{27}$ belongs to an irreducible component of $\mathfrak{p} \otimes \mathfrak{p}^* = W_1 \oplus W_7 \oplus W_{14} \oplus W_{27}$, respectively, we have

$$|T(r)|^2 = \frac{\tau_0^2}{16}|g_r|^2 + |\tau_{27}|^2 + |(\tau_1)^\sharp \lrcorner \varphi_r|^2 + \frac{1}{4}|\tau_2|^2.$$

Using the expressions for the torsion forms found in Proposition 2.49, we have

$$\frac{\tau_0^2}{16}|g_r|^2 = \frac{1}{7r^2}\Big(r^3(1-4h_2^2) + (5-8h_2^2)\Big)^2 = \frac{1}{7r^2}\Big(r^3 + 5 - 4(r^3+2)h_2^2\Big)^2.$$

For the next term, since τ_{27} is symmetric, we have

$$\begin{aligned} |\tau_{27}|^2 = \frac{4}{49r^2}\Big(& p_3(r)^2 + 49(r^3-2)^2h_0^2h_3^2 + 49(r^3-2)^2h_1^2h_3^2 + 49(r^3-2)^2h_0^2h_3^2 + p_0(r)^2 \\ &+ 49(r^3-2)^2h_0^2h_1^2 + 49(r^3-2)^2h_1^2h_3^2 + 49(r^3-2)^2h_0^2h_1^2 + p_1(r)^2 + \frac{25}{16}(r^3-2)^2 \\ &+ (5r^3-4)^2h_2^4 - \frac{5}{2}(r^3-2)(5r^3-4)h_2^2\Big) \\ = \frac{1}{7r^2}\Big(&(152r^6 - 288r^3 + 160)(1-h_2^2)^2 - (200r^6 - 240r^3 + 64)(1-h_2^2) \\ &+ 75r^6 - 60r^3 + 12\Big). \end{aligned}$$

For the skew-symmetric part of $T(r)$, we use the identity $\varphi_{ajk}\varphi_b^{jk} = 6g_{ab}$:

$$\begin{aligned} |(\tau_1)^\sharp \lrcorner \varphi_r|^2 + \frac{1}{4}|\tau_2|^2 &= 6|(\tau_1)^\sharp|^2 + \frac{16}{9}(r^3-1)^2h_2^2(1-h_2^2)|2re^{12} + \frac{1}{r^2}\omega_3|^2 \\ &= \frac{8}{3r^2}(r^3+2)^2h_2^2(1-h_2^2) + \frac{64}{3r^2}(r^3-1)^2h_2^2(1-h_2^2) \\ &= \frac{8}{r^2}(3r^6 - 4r^3 + 4)h_2^2(1-h_2^2). \end{aligned}$$

Finally, a simple computation yields the norm of the symmetric part of $T(r)$:

$$\frac{\tau_0^2}{16}|g_r|^2 + |\tau_{27}|^2 = \frac{1}{r^2}\Big((24r^6 - 32r^3 + 32)h_2^4 - (16r^6 - 40r^3 + 48)h_2^2 + 4r^6 - 14r^3 + 19\Big). \qquad \square$$

2.5 A gradient flow of isometric G_2-structures

In [24], Dwivedi, Gianniotis and Karigiannis studied the gradient flow of isometric G_2-structures. This isometric flow is the negative gradient of an energy functional. Also, it was shown that at a finite time singularity, the torsion must blow up, so the flow will exist as long as torsion remains bounded. Moreover, in this flow was proved a Cheeger-Gromov type compactness theorem.

Other authors studied this flow in a differently such as Grigorian [21], who regarded this as a flow of octonion sections and Loubeau and Sá Earp [36], who used a more general concept of a harmonic geometric structure, which is the G_2 case reduced to critical points of the energy functional E, that is, G_2-structures with divergence free torsion. This fact was used in [35], where was analysed this flow for a particular example of homogeneous space, which is $\mathbb{S}^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1)$. Moreover, in this section we give important results of the gradient flow of isometric $\mathrm{Sp}(2)$-invariant G_2-structures on sphere $\mathbb{S}^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1)$.

Definition 2.57. Define the energy E on the set $[\varphi]$ by

$$E(\varphi) = \frac{1}{2}\int_M |T_\varphi|^2 \mathrm{vol}_\varphi \tag{2.58}$$

where $[\varphi]$ denote the space of G_2-structures that are isometric to a given G_2-structure φ and where T_φ is the torsion of φ.

Remark 2.59. The energy E on homogeneous space for invariant G_2-structures is given by

$$E(\varphi) = \frac{1}{2}|T_\varphi|^2 \mathrm{vol} \tag{2.60}$$

Definition 2.61. Let M^7 be a compact manifold with a G_2-structure φ_0. Consider the negative gradient flow of the functional $4E$ restricted to the class $[\varphi]$. This evolution is given by

$$\frac{\partial \varphi_t}{\partial t} = (\mathrm{div} T_t)^\sharp \lrcorner \psi_t \quad \text{and} \quad \varphi(0) = \varphi_r, \tag{2.62}$$

We call (2.62) *isometric flow* of G_2-structures.

The existence of critical points of (2.58), and specifically of minimisers, has been studied using the associated gradient flow [14, 20, 36]. In [35], the principal aim was to examine the behavior of the critical points of the energy functional (2.58), restricted to a $\mathrm{Sp}(2)$-invariant isometric class of φ_r. Its critical points are *harmonic* G_2-structures, characterised by a divergence-free torsion [19]

$$\mathrm{div} T(r) = 0.$$

known as the *isometric flow*. In [24], Dwivedi, Gianniotis and Karigiannis shown that the isometric flow preserves isometric classes of G_2-structures which is equivalent to

$$\frac{\partial f}{\partial t} = \frac{1}{2}\langle X, (\mathrm{div} T_t)^\sharp\rangle \quad \text{and} \quad \frac{\partial X}{\partial t} = -\frac{1}{2}f(\mathrm{div} T_t)^\sharp + \frac{1}{2}(\mathrm{div} T_t)^\sharp \times X, \tag{2.63}$$

where $\{\varphi_t = \varphi_{(f,X)}\}$ is a family of isometric G_2-structures defined by (2.35).

2.5.1 Harmonic G_2-structures on the 7-sphere

The aim of this section is to compute the divergence of the full torsion tensor for isometric G_2-structures (2.43) with metric g_r, and then find critical points of the energy functional E from (2.58). We first show that, under our Ansatz, the divergence of the symmetric part of the full torsion tensor will automatically vanish.

Lemma 2.64. *Any* $\mathrm{Sp}(2)$*-invariant symmetric* 2*-tensor is divergence-free, with respect to the metric* g_r *induced by the* G_2*-structure* φ_r *f rom* (2.43)*.*

Proof. The divergence of a 2-tensor S is, by definition, the 1-tensor with coefficients $(\mathrm{div}S)_j = \nabla_i S_{ij}$, where ∇ is the Levi-Civita connection (2.40) of g_r. The basis (2.9) in $\mathfrak{p}$ is identified with a frame in $T\mathbb{S}^7$ generated by the one-parameter subgroups $\exp(te_i) \subset \mathrm{Sp}(2)$. Thus, the symmetric operator U from (2.41) at $o \in \mathbb{S}^7$ is

$$2g_r(U(e_i, e_j), e_k) = g_r([e_i, e_k]_{\mathfrak{p}}, e_j) + g_r([e_j, e_k]_{\mathfrak{p}}, e_i),$$

using the Lie bracket (2.37) and the inner product (2.42), we get

$2U(\cdot,\cdot)$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	0	0	$(1-r^3)e_7$	$(1-r^3)e_6$	$(r^3-1)e_5$	$(r^3-1)e_4$
e_2	0	0	0	$(r^3-1)e_6$	$(1-r^3)e_7$	$(1-r^3)e_4$	$(r^3-1)e_5$
e_3	0	0	0	$(1-r^3)e_5$	$(r^3-1)e_4$	$(1-r^3)e_7$	$(r^3-1)e_6$
e_4	$(1-r^3)e_7$	$(r^3-1)e_6$	$(1-r^3)e_5$	0	0	0	0
e_5	$(1-r^3)e_6$	$(1-r^3)e_7$	$(r^3-1)e_4$	0	0	0	0
e_6	$(r^3-1)e_5$	$(1-r^3)e_4$	$(1-r^3)e_7$	0	0	0	0
e_7	$(r^3-1)e_4$	$(r^3-1)e_5$	$(r^3-1)e_6$	0	0	0	0

(2.65)

We can now compute the divergence of a Sp(2)-invariant 2-tensor S:

$$(\mathrm{div}S)_j = \nabla_i S_{ij} = (\nabla_i S)(e_i, e_j) = e_i(S_{ij}) - S(\nabla_i e_i, e_j) - S(e_i, \nabla_i e_j) = -S(e_i, \nabla_i e_j),$$

as $U(e_i, e_i) = 0$, for any $i = 1, ..., 7$. Now, since S is symmetric,

$$\begin{aligned}
\sum_i S(e_i, [e_1, e_i]_{\mathfrak{p}}) &= -2S_{32} + 2S_{23} + S_{47} + S_{56} - S_{65} - S_{74} = 0\\
\sum_i S(e_i, [e_2, e_i]_{\mathfrak{p}}) &= -2S_{13} + 2S_{31} - S_{46} + S_{57} + S_{64} - S_{75} = 0\\
\sum_i S(e_i, [e_3, e_i]_{\mathfrak{p}}) &= -2S_{21} + 2S_{12} + S_{45} - S_{54} + S_{67} - S_{76} = 0\\
\sum_i S(e_i, [e_4, e_i]_{\mathfrak{p}}) &= -S_{17} + S_{26} - S_{35} + S_{53} - S_{62} + S_{71} = 0\\
\sum_i S(e_i, [e_5, e_i]_{\mathfrak{p}}) &= -S_{16} - S_{27} + S_{34} - S_{43} + S_{61} + S_{72} = 0\\
\sum_i S(e_i, [e_6, e_i]_{\mathfrak{p}}) &= -S_{24} + S_{15} - S_{37} + S_{42} - S_{51} + S_{73} = 0\\
\sum_i S(e_i, [e_7, e_i]_{\mathfrak{p}}) &= -S_{41} - S_{52} - S_{63} + S_{14} + S_{25} + S_{36} = 0.
\end{aligned}$$

Note that $\sum_i S(e_i,[e_j,e_i]_{\mathfrak{p}}) = 0$ implies that $(\mathrm{div}S)_j = -S(e_i, U(e_i,e_j))$. Referring back to table (2.65), we obtain

$$
\begin{aligned}
(\mathrm{div}S)_1 &= -\frac{(1-r^3)}{2}\Big(S_{47}+S_{56}-S_{65}-S_{74}\Big) = 0, & (\mathrm{div}S)_4 &= -\frac{(1-r^3)}{2}\Big(S_{17}-S_{26}+S_{35}\Big),\\
(\mathrm{div}S)_2 &= -\frac{(1-r^3)}{2}\Big(S_{57}-S_{46}+S_{64}-S_{75}\Big) = 0, & (\mathrm{div}S)_5 &= -\frac{(1-r^3)}{2}\Big(S_{16}-S_{27}-S_{34}\Big),\\
(\mathrm{div}S)_3 &= -\frac{(1-r^3)}{2}\Big(S_{45}-S_{54}+S_{67}-S_{76}\Big) = 0, & (\mathrm{div}S)_6 &= -\frac{(1-r^3)}{2}\Big(S_{24}-S_{15}+S_{37}\Big),\\
(\mathrm{div}S)_7 &= -\frac{(1-r^3)}{2}\Big(S_{14}+S_{25}+S_{36}\Big).
\end{aligned}
$$

Since S is an Sp(2)-invariant symmetric 2-tensor, it defines a g_r-selfadjoint linear operator $\beta_r : \mathfrak{p} \to \mathfrak{p}$, which commutes with $\mathrm{Ad}(h)$, for all $h \in \mathrm{Sp}(1)$:

$$S(X,Y) =: g_r(\beta_r(X),Y), \quad \text{for} \quad X,Y \in \mathfrak{p}.$$

Let $\chi : \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3 \to \mathfrak{p}_4$ be the linear map defined by $\chi = \pi_4 \circ \beta_r \circ \mathcal{I}$, where $\pi_4 : \mathfrak{p} \to \mathfrak{p}_4$ denotes the projection onto $\mathfrak{p}_4$ and $\mathcal{I}$ is the inclusion $\mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3 \subset \mathfrak{p}$. It is easy to see that $\mathrm{Ad}(h)\circ\chi = \chi\circ\mathrm{Ad}(h)$, for all $h \in \mathrm{Sp}(1)$, so $\ker\chi \subset \mathfrak{p}_1\oplus\mathfrak{p}_2\oplus\mathfrak{p}_3$ and $\mathrm{im}\,\chi \subset \mathfrak{p}_4$ are $\mathrm{Ad}(h)$-invariant subspaces, for all $h \in \mathrm{Sp}(1)$. Since $\mathfrak{p}_4$ is an irreducible Sp(1)-representation, either $\mathrm{im}\,\chi = 0$ or $\mathrm{im}\,\chi = \mathfrak{p}_4$. Yet, by dimensional reasons, the latter cannot hold:

$$\dim(\mathrm{im}\,\chi) \le \dim(\mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3) = 3 < 4 = \dim(\mathfrak{p}_4).$$

Therefore $\beta_r(\mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3) \subset \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$. One may check analogously that $\beta_r(\mathfrak{p}_4) \subset \mathfrak{p}_4$. Hence

$$S(X,Y) = g_r(\beta_r(X),Y) = 0, \quad \text{for} \quad X \in \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3 \quad \text{and} \quad Y \in \mathfrak{p}_4,$$

so, $S_{ij} = 0$, for $i \in \{1,2,3\}$ and $j \in \{4,5,6,7\}$. In particular, the remaining components $(\mathrm{div}S)_k$, $k = 4,5,6,7$, are also zero. □

Proposition 2.66. *The full torsion tensor $T(r)$ of the* G_2*-structure* (2.43) *has divergence*

$$\mathrm{div}T(r) = \frac{4(r^3+2)(r^3-1)h_2}{r}\Big(h_3e^1 + h_0e^2 - h_1e^3\Big).$$

In particular, in each of the isometric classes $\mathcal{B}_r$ (see Theorem 2.54), the families $\{1\}\times\mathbb{S}^3$, $\{r\}\times\mathbb{S}^2$ and $\{r\}\times\mathrm{NS}$ are exactly the critical points of the energy functional (2.58).

Proof. Since the full torsion tensor (1.32) of φ_r is Ad(Sp(1))-invariant, we know from Lemma 2.64 that its symmetric part (i.e. τ_0 and τ_3) is divergence-free. Hence

$$
\begin{aligned}
\mathrm{div}T &= -\mathrm{d}^* * (\tau_1 \wedge \psi_r) - \frac{1}{2}\mathrm{d}^*\tau_2 = - * \mathrm{d}(\tau_1 \wedge \psi_r) + \frac{1}{2} * \mathrm{d}(\tau_2 \wedge \varphi_r)\\
&= \frac{4(r^3+2)(r^3-1)}{3r}\Big(h_2h_3e^1 + h_0h_2e^2 - h_1h_2e^3\Big)\\
&\quad + \frac{8(r^3+2)(r^3-1)}{3r}\Big(h_2h_3e^1 + h_0h_2e^2 - h_1h_2e^3\Big)\\
&= \frac{4(r^3+2)(r^3-1)h_2}{r}\Big(h_3e^1 + h_0e^2 - h_1e^3\Big),
\end{aligned}
$$

where d^* is the operator which is adjoint to d such that $\mathrm{d}^* : \Omega^p(M) \to \Omega^{p-1}$ is defined by $\mathrm{d}^* = (-1)^{d(p+1)+1} * d*$ and the critical points of the energy functional (2.58) are parametrised by solutions of

$$|\mathrm{div}T(r)|^2 = \frac{16(r^3+2)^2(r^3-1)^3}{r^4} h_2^2(h_0^2+h_1^2+h_3^2) = 0,$$
$$\Leftrightarrow \quad r=1 \quad \text{or} \quad h_2 = 0 \quad \text{or} \quad h_0 = h_1 = h_3 = 0.$$

These cases correspond to the three families in the second part of the statement. □

Remark 2.67. In the context of Theorem 2.54, for a given $r > 0$, the set $\mathrm{Crit}(E|_{\mathcal{B}_r})$ is described as follows:

$\underline{r \neq 1}$: the harmonic G_2-structures in the isometric class $\mathcal{B}_r$ are precisely those parametrised by the equator $\{r\} \times \mathbb{S}^2$ or the poles $\{r\} \times \mathrm{NS}$.

$\underline{r = 1}$: all the G_2-structures $\varphi_{(1,h)}$, with $h \in \mathbb{S}^3$, are harmonic.

2.5.2 The $\mathrm{Ad}(\mathrm{Sp}(1))$-invariant gradient flow of E

In this section we give the main results where the previous theorems and propositions were usedin the isoemtric flow specializing to the case of Sp(2)-invariant G_2-structures, the pointwise norm $|T(r)|^2$ was computed with respect to the Sp(2)-invariant metric of $\mathbb{S}^7$ in Proposition 2.55 . It is therefore everywhere constant and equal to the norm of the torsion of the Ad(Sp(1))-invariant G_2-structure $\varphi_r \in \Lambda^3(\mathfrak{p})^*$, hence

$$E(\varphi_r) = \frac{1}{2}|T(r)|^2 \mathrm{vol}_{\varphi_r}(\mathbb{S}^7).$$

Moreover, the divergence of the full torsion tensor is an Sp(2)-invariant 1-tensor:

$$(\mathrm{div}T \lrcorner \psi)(p) = L_x^*((dL_{x^{-1}})_x(\mathrm{div}T_p \lrcorner \psi)(o) = L_x^*(\mathrm{div}T(r) \lrcorner \psi)(o),$$

where $p = x \cdot \mathrm{Sp}(1) \in \mathbb{S}^7$, in particular $o = 1_{\mathrm{Sp}(2)}\mathrm{Sp}(1)$ is the orbit of the identity, and $\mathrm{div}T(r) := \mathrm{div}T_o \in (\mathfrak{p}^*)^{\mathrm{Ad}(\mathrm{Sp}(1))}$.

Let $\{\varphi_t\}_{t\in(-\varepsilon,\varepsilon)}$ be a solution of the gradient flow (2.62) of the Dirichlet energy functional (2.58). By Proposition 2.24, a solution with initial condition φ_r, as in the Ansatz (2.43), is parametrised by

$$\varphi_t = \Theta(r, \overline{k(t)h}), \quad \text{with} \quad h, k(t) \in \mathrm{Sp}(1) \quad \text{and} \quad k(0) = (1,0,0,0).$$

Letting

$$m(t) := k(t)h = (m_0, m_1, m_2, m_3) \in \mathrm{Sp}(1),$$

the divergence of the full torsion tensor of φ_t is

$$\mathrm{div}T_t := \mathrm{div}T_{m(t)} = \frac{4(r^3+2)(r^3-1)m_2}{r}\Big(m_3e^1 + m_0e^2 - m_1e^3\Big).$$

Applying Remark 2.36 to the equations in (2.63), we find the explicit evolution of each component of $m(t)$:

$$\begin{aligned}
\frac{dm_0}{dt} &= \frac{2(r^3+2)(r^3-1)m_2}{r^2}(m_0m_2),\\
\frac{dm_1}{dt} &= \frac{2(r^3+2)(r^3-1)m_2}{r^2}(m_1m_2),\\
\frac{dm_2}{dt} &= \frac{2(r^3+2)(r^3-1)m_2}{r^2}(-m_0^2-m_1^2-m_3^2),\\
\frac{dm_3}{dt} &= \frac{2(r^3+2)(r^3-1)m_2}{r^2}(m_3m_2).
\end{aligned} \tag{2.68}$$

Remark 2.69. In matrix form, the system (2.68) is a first-order non-linear ODE,

$$\frac{dm(t)}{dt} = \frac{2(r^3+2)(r^3-1)m_2}{r^2}\begin{pmatrix} 0 & 0 & m_0 & 0\\ 0 & 0 & m_1 & 0\\ -m_0 & -m_1 & 0 & -m_3\\ 0 & 0 & m_3 & 0\end{pmatrix} m(t) = \Psi(m(t)). \tag{2.70}$$

The short-time existence and uniqueness of solutions of (2.70), with any initial value $m(0) = h \in \mathrm{Sp}(1)$, follows from the Picard-Lindelöf Theorem, since $\Psi : \mathbb{H} \to \mathbb{H}$ is locally Lipschitz (polynomial, de facto). Namely,

$$m(t) = h + \int_0^t \Psi(m(s))ds, \quad -\varepsilon < t < \varepsilon.$$

This agrees with the short-time existence results in [14] and [36].

Proposition 2.71. *[35, §3, Proposition 3.7] Given any initial value* $m(0) = h = (h_0, h_1, h_2, h_3) \in \mathrm{Sp}(1)$*, the ODE* (2.70) *admits the following unique solution, defined for all* $t \in \mathbb{R}$*:*

$$m_2(t) = \frac{h_2}{\sqrt{(1-h_2^2)e^{\frac{4(r^3+2)(r^3-1)t}{r^2}} + h_2^2}}, \tag{2.72a}$$

$$m_k(t) = \frac{h_k e^{\frac{2(r^3+2)(r^3-1)t}{r^2}}}{\sqrt{(1-h_2^2)e^{\frac{4(r^3+2)(r^3-1)t}{r^2}} + h_2^2}}, \quad \textit{for} \quad k = 0, 1, 3. \tag{2.72b}$$

In the following Remark describes the landscape of limiting harmonic homogeneous G_2-structures for the flow (2.68) at infinity. This can be seen as a concrete instance of the general theory [36, Theorem 3 & Remark 22], which predicts subsequential convergence to a harmonic limit, under uniformly bounded torsion, hence long-time existence for homogeneous structures. However, in the present context, we get to be much more precise. Since we're ultimately studying a gradient flow, solutions are expected to flow, forwards and backwards, between critical regions, in this case the equator and the poles of the 3-sphere. This Remark tells us exactly how such gradient flow lines behave:

Remark 2.73. [35, §3, Theorem 6] Given an initial condition $m(0) \in \mathrm{Sp}(1)$, let $m(t)$ be the all-time solution of (2.70) from Proposition 2.71, and let $\varphi(t) = \Theta(r, \Upsilon(\overline{m(t)})$ be the corresponding solution of the isometric flow (2.62). Then there exist subsequences $t_{n_k^\pm} \to \pm\infty$ such that $m(t_{n_k^\pm}) \to m_{\pm\infty} \in \mathrm{Sp}(1)$, and

$$\varphi_{\pm\infty} = \Phi(r, \Upsilon(\overline{m}_{\pm\infty}))$$

are harmonic $\mathrm{Ad}(\mathrm{Sp}(1))$-invariant coclosed G_2-structures. According to the parameter $r > 0$ and the hemisphere determined by the initial sign $\sigma_2 := \mathrm{sign}(m_2(0)) = \pm 1$, the flow on the 3-sphere behaves asymptotically as follows:

$\underline{r < 1:}$ $m(-\infty) \in \{r\} \times \mathbb{S}^2$ and $m(\infty) = (0, 0, \sigma_2, 0) \in \{r\} \times \mathrm{NS}$.

$\underline{r > 1:}$ $m(\infty) \in \{r\} \times \mathbb{S}^2$ and $m(-\infty) = (0, 0, \sigma_2, 0) \in \{r\} \times \mathrm{NS}$.

3 The Laplacian coflow on contact Calabi-Yau 7 − manifolds

The main problem is that it is not known whether solutions to the Laplacian coflow of G_2-structures actually exist in general, even for an arbitrarily short time. There is a modification of the coflow (3.11) which does have guaranteed short-time existence, but the critical points are no longer closed and coclosed 4-forms, so it does not currently appear to be useful as a tool for studying key problems in G_2 geometry.

Trying to analyse the Laplacian coflow on contact Calabi-Yau manifolds, we find interesting results that will be explained below. We give an introduction about important results for the Laplacian coflow [18, 28] which are going to be very useful in the analysis of the Laplacian coflow of G_2-structures. Then, we show a new example of a solution to the Laplacian coflow of G_2-structures on contact Calabi-Yau manifolds in Theorem 3.22. This solution has a singularity at $t = -1/10$. In particular, in Theorem 12 was shown that it is volume collapsing and collapsing with respect to the normalized metric at $-1/10$. On the other hand, we analyse solitons on a contact Calabi-Yau manifold M given in Proposition 3.96 using the important Theorem 3.75 where we show that for a coclosed G_2-structure φ with associated metric g, dual 4-form ψ and a vector field X on M, we have

$$\mathcal{L}_X\psi = 0 \quad \text{if and only if} \quad \mathcal{L}_X g = 0 \quad \text{and} \quad \mathrm{Curl}(X) = 0.$$

We start with the following definition.

Definition 3.1. A time-dependent G_2-structure φ_t on a 7-manifold M^7, defined for t in some interval $[a, b)$, satisfies the Laplacian coflow if for all $t \in [a, b)$ we have

$$\frac{\partial}{\partial t}\psi_t = \Delta_t \psi_t, \tag{3.2}$$

where $\psi_t = *_t\varphi_t$ is the Hodge dual 4-form of φ_t and $\Delta_t = \mathrm{dd}^{*t} + \mathrm{d}^{*t}\mathrm{d}$ is the Hodge Laplacian with respect to the metric $g_t = g_{\varphi_t}$ and fixed orientation given by φ_0. We will always restrict to solutions so that φ_t is a coclosed G_2-structure for all t, i.e. $\mathrm{d}\psi_t = 0$ for all $t \in [a, b)$.

If M^7 is compact then the volume of M determined by the G_2-structure φ on M is:

$$\mathcal{H}(\varphi) := \mathrm{Vol}(M, \varphi) = \frac{1}{7}\int_M \phi \wedge \psi. \tag{3.3}$$

Proposition 3.4. *[18, Proposition 3.4 and §4] The flow (3.2) for coclosed G_2-structures $*_\varphi\varphi$ is the gradient flow of the volume functional in (3.3) restricted to $[*_\varphi\varphi]$ and the critical points are strict local maxima for the volume functional (modulo diffeomorphisms).*

Therefore the evolution of the metric, the inverse metric and volume form are given in the following proposition:

Proposition 3.5. *[18, §3, Proposition 3.1] Under the flow (3.2), the evolution equations are given by*

$$\begin{aligned}
\frac{\partial g}{\partial t} &= 2\,\mathrm{Curl}\,T + T\circ T + 2T^2 = -2\mathrm{Ric} + T\circ T + 2(\mathrm{tr}\,T)T,\\
\frac{\partial \mathrm{vol}}{\partial t} &= \frac{1}{2}(|T|^2 + (\mathrm{tr}\,T)^2)\mathrm{vol},\\
\frac{\partial}{\partial t}T &= \Delta T - 2\nabla(\mathrm{div}T) + Rm \circledast T + (\nabla T)\circledast T + T\circledast T\circledast T,
\end{aligned}$$

where $\circledast$ is some multilinear operator involving g, φ, ψ and Δ denotes the Lichnerowicz Laplacian..

This follows from [21, Lemma 3.1], Lemma 1.56 and the fact that T is symmetric, so $\mathrm{Curl}(T)$ is traceless and

$$\mathrm{Curl}(\mathrm{Curl}\,T) = -\Delta T + \nabla(\mathrm{div}T) + Rm\circledast T + (\nabla T)\circledast + T\circledast T\circledast T. \tag{3.6}$$

Note that, the $\nabla(\mathrm{div}T)$ term in the above equation is due to the negative sign of $\mathrm{div}T$. As it was shown in [18], the sign of $\mathrm{div}T$ also causes problems at a much more fundamental level: It prevents the flow (3.2) from being parabolic even along closed 4-forms. The following proposition gives the linearization of Δ_ψ. It is easy to see that for closed 4-forms, the symbol will be negative in the Λ^4_7 direction, but non-negative in Λ^4_{27}.

Proposition 3.7. *[18, §4, Proposition 4.4] The linearization of Δ_ψ at ψ is given by*

$$\begin{aligned}
\pi_7(D_\psi\Delta_\psi)(\chi) =& \mathrm{d}(\mathrm{div}X)\wedge\varphi + l.o.t\\
\pi_{1\oplus 27}(D_\psi\Delta_\psi)(\chi) =& \frac{3}{2} * \mathrm{i}_\varphi\Big(\Delta h + \frac{1}{4}\mathrm{Hess}(\mathrm{tr}\,h) - \frac{1}{2}(\Delta\,\mathrm{tr}\,h)g\\
&- \mathrm{sym}(\nabla\mathrm{div}h + \mathrm{Curl}(\nabla X)^t) + l.o.t\Big)
\end{aligned} \tag{3.8}$$

where $\chi = *(X\lrcorner\psi + i_\varphi(h))$. Moreover, if χ is closed, we can write $D_\psi\Delta_\psi$ as

$$D_\psi\Delta_\psi(\chi) = -\Delta_\psi\chi - \mathcal{L}_{V(\chi)}\psi + 2\mathrm{d}((\mathrm{div}X)\varphi) + \mathrm{d}F(\chi) \tag{3.9}$$

where

$$V(\chi) = \frac{3}{4}\nabla\,\mathrm{tr}\,h - 2\,\mathrm{Curl}\,X \tag{3.10}$$

and $F(\chi)$ is a 3-form-valued algebraic function of χ.

We see that the term $\mathrm{d}((\mathrm{div}X)\varphi)$ appears in the linearization (3.9) for exactly the same reason as the term $-\nabla(\mathrm{div}T)$ in Proposition 3.5, namely, the wrong sign of the π_7 component of $\Delta_\psi\psi$. In [18] is fixed in a *modified Laplacian coflow* has been proposed:

$$\frac{\partial\psi}{\partial t} = \Delta_\psi\psi + 2\mathrm{d}((A - \mathrm{tr}\,T)\varphi), \tag{3.11}$$

where A is some constant. Since for coclosed G_2-structures $\nabla\,\mathrm{tr}\,T = \mathrm{div}T$, the leading term in the modification precisely reverses the sign of the Λ^4_7 component of the original flow (3.2). The additional term in (3.11) also allows to prove short-time existence and uniqueness, hence completing the requirement for (3.11) to be a Ricci-like flow. The proof is given in [18], follows a procedure similar to the approach taken by Bryant and Xu [7] for the proof of short-time existence and uniqueness for the Laplacian flow (1.74).

3.1 New solutions of the Laplacian coflow

Motivated by [22], we consider a contact Calabi-Yau manifold $(M^7, \eta, \Phi, \Upsilon)$ (see Appendix A.4), where (M, η, ξ, Φ, g) is a Sasakian manifold with contact form η and transverse Kähler form $\omega = \mathrm{d}\eta$, and Υ is a nowhere vanishing transversal form on $\mathcal{D} = \ker \eta$ of type $(3, 0)$ satisfying

$$\Upsilon \wedge \overline{\Upsilon} = -\mathrm{i}\omega^3 \quad \text{and} \quad \mathrm{d}\Upsilon = 0.$$

In this section, we find a new solution to the Laplacian coflow of G_2-structures on contact Calabi-Yau 7-manifolds having singularity at $t = -1/10$. We shall analyse this solution in Section 3.3.

Proposition 3.12. *[22, §6, Corollary 6.8] Let $(M, \eta, \Phi, \Upsilon)$ be a 7-dimensional contact Calabi-Yau manifold. Then M carries a coclosed G_2-structure defined by*

$$\varphi = \eta \wedge \omega + \operatorname{Re} \Upsilon, \tag{3.13}$$

where $\omega = d\eta$. Furthermore, $d\varphi = \omega \wedge \omega$ and its corresponding dual 4-form is given by

$$\psi = *_\varphi \varphi = \frac{1}{2}\omega^2 - \eta \wedge \operatorname{Im} \Upsilon. \tag{3.14}$$

We want to consider the Laplacian coflow starting at the natural coclosed G_2-structure on a contact Calabi–Yau 7-manifold $(M^7, \eta_0, \Phi_0, \Upsilon_0)$

$$\varphi_0 = \eta_0 \wedge \omega_0 + \operatorname{Re} \Upsilon_0, \tag{3.15}$$

so that

$$\psi_0 = \frac{1}{2}\omega_0^2 - \eta_0 \wedge \operatorname{Im} \Upsilon_0. \tag{3.16}$$

To this end, we consider the family of G_2-structures given by

$$\varphi_t = f_t h_t^2 \eta_0 \wedge \omega_0 + h_t^3 \operatorname{Re} \Upsilon_0, \tag{3.17}$$

for functions f_t, h_t of time only, with

$$f_0 = h_0 = 1. \tag{3.18}$$

Also, we have the induced metric given by

$$g_t = f_t^2 \eta^2 + h_t^2 g_{\mathcal{D}},$$

and its associated volume form

$$\mathrm{vol}_t = f_t h_t^6 \eta \wedge \mathrm{vol}_{\mathcal{D}},$$

where

$$\mathrm{vol}_{\mathcal{D}_0} = \frac{1}{3!}\omega_0^3 = \frac{i}{8}\Upsilon_0 \wedge \overline{\Upsilon}_0 = \frac{1}{4}\operatorname{Re}(\Upsilon_0) \wedge \operatorname{Im}(\overline{\Upsilon}_0),$$

so that evaluating (3.17) at $t = 0$ yields (3.15). We easily see from our calculations above that

$$\psi_t = \frac{1}{2}h_t^4 \omega_0^2 - f_t h_t^3 \eta_0 \wedge \operatorname{Im} \Upsilon_0, \tag{3.19}$$

and observe that $t = 0$ in (3.19) yields (3.16). Therefore, we can calculate the torsion forms of the G_2-structure φ_t:

$$\begin{aligned}(\tau_0)_t &= \frac{1}{7} *_t (\varphi_t \wedge \mathrm{d}\varphi_t)\\ &= \frac{1}{7} *_t ((f_t h_t^2 \eta_0 \wedge \omega_0 + h_t^3 \operatorname{Re}(\Upsilon)) \wedge f_t h_t^2 \omega_0^2)\\ &= \frac{1}{7} *_t f_t^2 h_t^4 \eta_0 \wedge \omega_0^3\\ &= \frac{6 f_t}{7 h_t^2}. \end{aligned} \tag{3.20}$$

Besides this, we have $\tau_1 = \frac{1}{12}(\varphi \wedge \mathrm{d}\varphi) = 0$ and $\tau_2 = *(d\psi) - 4 * (\tau_1 \wedge \psi) = 0$, thus φ_t is coclosed. Furthermore,

$$\begin{aligned}(\tau_3)_t &= *_t \mathrm{d}\varphi_t - (\tau_0)_t \varphi_t\\ &= 2 f_t^2 \eta_0 \wedge \omega_0 - \frac{6 f_t}{7 h_t^2}(f_t h_t^2 \eta_0 \wedge \omega_0 + h_t^3 \operatorname{Re}(\Upsilon))\\ &= \frac{8}{7} f_t^2 \eta_0 \wedge \omega_0 - \frac{6}{7} f_t h_t \operatorname{Re}(\Upsilon_0). \end{aligned} \tag{3.21}$$

We now show a solution of the Laplacian coflow (3.2) on a contact Calabi-Yau manifold given in the following theorem.

Theorem 3.22. *Let $(M^7, \eta_0, \Phi_0, \Upsilon_0)$ be a contact Calabi-Yau 7-manifold. The family of coclosed G_2-structures φ_t on M^7 given by*

$$\varphi_t = \mathrm{p}(t)^{-1/10} \eta_0 \wedge \omega_0 + \mathrm{p}(t)^{3/10} \operatorname{Re} \Upsilon_0; \tag{3.23}$$

$$\psi_t = \frac{1}{2} \mathrm{p}(t)^{2/5} \omega_0^2 - \eta_0 \wedge \operatorname{Im} \Upsilon_0, \tag{3.24}$$

where $\mathrm{p}(t) = 10t + 1$ and $t \in (-1/10, \infty)$, solves the Laplacian coflow (3.2) with initial data determined by $\varphi_0 = \eta_0 \wedge \omega_0 + \operatorname{Re} \Upsilon_0$.

Proof. Let $(M, \eta_0, \Phi_0, \Upsilon_0)$ be a contact Calabi-Yau manifold, $\{\varphi_t\}$ the family of G_2-structures given by (3.17) and $\psi_t = *_t \varphi_t$. We may compute the Laplacian of ψ_t:

$$\begin{aligned}\Delta_t \psi_t &= \mathrm{d} *_t \mathrm{d}\varphi_t\\ &= \mathrm{d} *_t (f_t h_t^2 \omega_0 \wedge \omega_0)\\ &= \mathrm{d} *_t (2 f_t h_t^{-2} \cdot \tfrac{1}{2} h_t^4 \omega_0^2)\\ &= \mathrm{d}(2 f_t h_t^{-2} \cdot f_t h_t^2 \eta_0 \wedge \omega_0)\\ &= \mathrm{d}(2 f_t^2 \eta_0 \wedge \omega_0)\\ &= 2 f_t^2 \omega_0 \wedge \omega_0. \end{aligned} \tag{3.25}$$

Differentiating (3.19) with respect to t and using (3.25), we can compute $\frac{\partial \psi}{\partial t} = \Delta_t \psi_t$ and then equate the coefficients of $\eta_0 \wedge \operatorname{Im} \varepsilon_0$ and ω_0^2 to obtain

$$\frac{\partial}{\partial t} h_t^4 = 4 f_t^2, \quad \frac{\partial}{\partial t} f_t h_t^3 = 0. \tag{3.26}$$

From the second equation in (3.26) and (3.18) we obtain

$$f_t = h_t^{-3}. \tag{3.27}$$

Substituting (3.27) into (3.26), we have

$$\frac{\partial}{\partial t} h_t^4 = 4h_t^{-6}. \tag{3.28}$$

The ODE (3.28) can be easily solved and, together with (3.18) and (3.27), we find that

$$h_t = (10t+1)^{1/10} \quad \text{and} \quad f_t = (10t+1)^{-3/10}. \tag{3.29}$$

In conclusion, we have found a solution to the Laplacian coflow (3.2) with initial condition (3.15) (or (3.16)), which has data:

$$\varphi_t = (10t+1)^{-1/10}\eta_0 \wedge \omega_0 + (10t+1)^{3/10}\operatorname{Re}\Upsilon_0; \tag{3.30}$$

$$\psi_t = \frac{1}{2}(10t+1)^{2/5}\omega_0^2 - \eta_0 \wedge \operatorname{Im}\Upsilon_0; \tag{3.31}$$

$$g_t = (10t+1)^{-3/5}\eta_0^2 + (10t+1)^{1/5}g_{\mathcal{D}_0}; \tag{3.32}$$

$$\mathrm{vol}_t = (10t+1)^{3/10}\eta_0 \wedge \mathrm{vol}_{\mathcal{D}_0}, \tag{3.33}$$

where $\mathcal{D}_0 = \ker \eta_0$, defined for all $t \in (-1/10, \infty)$. □

We notice that this solution to the coflow is *immortal* (i.e. exists for all positive time), but it is not eternal, since it fails to exists for $t \leq -1/10$. If M is compact, this will be the unique solution of the general form (3.17) and (3.18), and we see from (3.33) that the volume is indeed strictly increasing in time, tending to infinity, i.e.,

$$\mathrm{Vol}(M, g_t) \to \infty \quad \text{as } t \to \infty.$$

Definition 3.34. We say that a family of metrics g_t for $t \in [0, T)$ is *uniformly equivalent* to the metric $g_0 = g(x, 0)$ if there exist a constant $C < \infty$ such that

$$\frac{1}{C}g(x,0) \leq g(x,t) \leq Cg(x,0),$$

for all $x \in M$ and $t \in [0, T)$.

Definition 3.35. Let g_t be a family of metrics $t \in [0, T)$. Then, g_t is *uniformly continuous* if for any $\varepsilon > 0$ there exist $\delta > 0$ such that for any $0 < t_0 < t < T$ with $t - t_0 \leq \delta$ we have

$$|g_t - g_{t_0}|_{g_{t_0}} \leq \varepsilon,$$

which implies that as symmetric 2-tensor, we have

$$(1-\varepsilon)g_{t_0} \leq g_t \leq (1+\varepsilon)g_{t_0}. \tag{3.36}$$

Proposition 3.37. *Let φ_t be the solution to the Laplacian coflow given by (3.30) with associated metric (3.32). Then, g_t is uniformly continuous (in t) on any compact interval contained in $(-1/10, \infty)$.*

Proof. Consider the family of metrics $\{g_t\}$ given by (3.32) where

$$\begin{aligned} g_t =&\alpha_t g_0 + \beta_t \eta_0 \otimes \eta_0 = \alpha_t g_{\mathcal{D}_0} + (\alpha_t + \beta_t)\eta_0 \otimes \eta_0 \\ =&\mathrm{p}(t)^{1/5} g_{\mathcal{D}_0} + \mathrm{p}(t)^{-3/5}\eta_0 \otimes \eta_0 \end{aligned} \tag{3.38}$$

with $\mathrm{p}(t) = 10t + 1$. We define

$$F_t = g_t - g_{t_0} = s_1 g_0 + s_2 \eta_0 \otimes \eta_0,$$

where $s_1 = \alpha_t - \alpha_{t_0}$ and $s_2 = \beta_t - \beta_{t_0}$. Therefore

$$|F_t|^2_{g_{t_0}} =6\mathrm{p}(t_0)^{6/5}(\mathrm{p}(t)^{-3/5} - \mathrm{p}(t_0)^{-3/5})^2 + \mathrm{p}(t_0)^{-2/5}(\mathrm{p}(t)^{1/5} - \mathrm{p}(t_0)^{1/5})^2, \tag{3.39}$$

which implies that g_t is uniformly continuous on any compact interval contained in $(-1/10, \infty)$ as required. □

Proposition 3.40. *[33, §3.2] Let φ_t be the solution to the Laplacian coflow given by (3.30) with associated metric g_t as in (3.32). Let Rm_t denote the Riemann curvature tensor of g_t and let $Rm_0^{\mathcal{D}_0}$ denote the curvature of the transverse connection on $\mathcal{D}_0$ induced by the Levi-Civita connection of g_0. Then*

$$|Rm_t|^2_{g_t} = (1 + 10t)^{-2/5}|Rm_0^{\mathcal{D}_0}|^2_{g_0} + c_0(1 + 10t)^{-2}$$

for some constant $c_0 > 0$.

3.2 Full torsion tensor

For convenience, we denote $\mathrm{p}(t) = 10t + 1$. Observe that the full torsion tensor for (3.30) is given by $T_t = \frac{\tau_0}{4} g_t - (\tau_{27})_t$, where $4(\tau_{27})_t = \mathrm{j}_{\varphi_t}(\tau_3)$. Thus,

$$\begin{aligned} (\tau_0)_t &= \frac{6}{7}\mathrm{p}(t)^{-5/10}, \\ (\tau_3)_t &= \frac{8}{7}\mathrm{p}(t)^{-6/10}\eta_0 \wedge \omega_0 - \frac{6}{7}\mathrm{p}(t)^{-2/10}\,\mathrm{Re}(\Upsilon_0). \end{aligned} \tag{3.41}$$

Let ξ_0 be the dual vector field of η_0, $X, Y \in \mathcal{D}_0$ and consider φ_t as in (3.30). Since (ω_0, Υ_0) defines an SU(3)-structure on $\mathcal{D}_0$, we have that $\omega_0 \wedge \mathrm{Re}\,\Upsilon_0 = 0$ and

$$\begin{aligned} (X \lrcorner \mathrm{Re}\,\Upsilon_0) \wedge (Y \lrcorner \mathrm{Re}\,\Upsilon_0) \wedge \omega_0 &= 2g_{\mathcal{D}_0}(X, Y)\mathrm{vol}_{\mathcal{D}_0} \\ (X \lrcorner \omega_0) \wedge (Y \lrcorner \mathrm{Re}\,\Upsilon_0) \wedge \mathrm{Re}\,\Upsilon_0 &= -2g_{\mathcal{D}_0}(X, Y)\mathrm{vol}_{\mathcal{D}_0}. \end{aligned} \tag{3.42}$$

Therefore

$$\begin{aligned} \mathrm{j}_{\varphi_t}((\tau_3)_t)(\xi_0, \xi_0) &= \mathrm{p}(t)^{-2/5} *_t (\omega_0 \wedge \omega_0 \wedge \tau_3) \\ &= \frac{8}{7}\mathrm{p}(t)^{-4/5} *_t (\omega_0^3 \wedge \eta_0) \\ &= \frac{(3!)(8)}{7}\mathrm{p}(t)^{-11/10}, \\ \mathrm{j}_{\varphi_t}((\tau_3)_t)(\xi_0, X) &= 0, \end{aligned} \tag{3.43}$$

$$\begin{aligned} \mathrm{j}_{\varphi_t}((\tau_3)_t)(X,Y) &= -\mathrm{p}(t)^{-1/10} *_t (\eta_0 \wedge X \lrcorner \omega_0 \wedge Y \lrcorner \varphi_t \wedge \tau_3) \\ &\quad +\mathrm{p}(t)^{3/10} *_t (X \lrcorner \mathrm{Re}(\Upsilon_0) \wedge Y \lrcorner \varphi_t \wedge \tau_3). \end{aligned} \tag{3.44}$$

Substituting (3.42) into (3.44), we obtain

$$\mathrm{j}_{\varphi_t}(\tau_3)(X,Y) = -\frac{8}{7}\mathrm{p}(t)^{-3/10} g_{\mathcal{D}}(X,Y). \tag{3.45}$$

Thus, using (3.41), (3.43) and (3.45) we have

$$\begin{aligned} T_t &= \frac{(\tau_0)_t}{4} g_t - (\tau_{27})_t \\ &= \frac{3}{14}\mathrm{p}(t)^{-5/10}\Big(\mathrm{p}(t)^{-3/5}\eta_0 \otimes \eta_0 + \mathrm{p}(t)^{1/5} g_{\mathcal{D}}\Big) \\ &\quad - \frac{12}{7}\mathrm{p}(t)^{-11/10}\eta_0 \otimes \eta_0 + \frac{2}{7}\mathrm{p}(t)^{-3/10} g_{\mathcal{D}} \\ &= -\frac{3}{2}\mathrm{p}(t)^{-11/10}\eta_0 \otimes \eta_0 + \frac{1}{2}\mathrm{p}(t)^{-3/10} g_{\mathcal{D}}. \end{aligned} \tag{3.46}$$

In order to understand the tensor T_t in (3.46), we use as well as some important properties of K-contact manifolds. Since $(M, \eta_0, \Phi_0, \Upsilon_0)$ is contact Calabi-Yau, it is in particular Sasakian and, therefore, a K-contact manifold (see Appendix A). It will be useful to find the full torsion tensor norm.

Proposition 3.47. *Let $(M, \eta_0, \Phi_0, \Upsilon_0)$ be a contact Calabi-Yau manifold and let $\{\varphi_t\}$ be the solution to the Laplacian coflow given by (3.30) with associated metric (3.32) and torsion T_t. Then, we have*

$$|T(p,t)|_t^2 = \frac{15}{4}\mathrm{p}(t)^{-1}. \tag{3.48}$$

Proof. Note that $T_t = \mathrm{r}_1(t) g + \mathrm{r}_2(t)\eta^2$, where

$$\begin{aligned} \mathrm{r}_1(t) &= \frac{1}{2}\mathrm{p}(t)^{-3/10}, \\ \mathrm{r}_1(t) + \mathrm{r}_2(t) &= -\frac{3}{2}\mathrm{p}(t)^{-11/10}. \end{aligned}$$

Using Lemma A.21 and $g_t = \alpha g + \beta\eta^2$ given by (3.38), we compute $|T_t|_t^2$

$$\begin{aligned} |T(p,t)|_t^2 &= T_t(e_i,e_j)T_t(e_m,e_n)g_t^{im}g_t^{jn} \\ &= 6\left(\mathrm{p}(t)^{1/5}\right)^{-2}\left(\frac{1}{2}\mathrm{p}(t)^{-3/10}\right)^2 + \left(\frac{-3}{2}\mathrm{p}(t)^{11/10}\right)^2\left(\mathrm{p}(t)^{-3/5}\right)^{-2} \\ &= \frac{15}{4}\mathrm{p}(t)^{-1}. \end{aligned}$$

Note that $\lim_{t\to\infty} |T_t|_t^2 = 0$. □

Proposition 3.49. *Let $(M, \eta_0, \Phi_0, \Upsilon_0)$ be a contact Calabi-Yau 7-manifold and let $\{\varphi_t\}$ be the solution to the Laplacian coflow given by (3.30) and initial data determined by $\varphi_0 = \eta_0 \wedge \omega_0 + \mathrm{Re}\,\Upsilon_0$, with associated metric (3.32) and torsion T_t. Then $\mathrm{div} T_t = 0$.*

Proof. Let $\{g_t = \alpha_t g_0 + \beta_t \eta^2 = \alpha_t g_{\mathcal{D}} + (\alpha_t + \beta_t)\eta^2\}$ be the family of metrics given by (3.32) where $\alpha_t = \mathrm{p}(t)^{-3/5}$ and $\alpha_t + \beta_t = \mathrm{p}(t)^{1/5}$. Using (3.46), we obtain

$$T_t = -2\mathrm{p}(t)^{-11/10}\eta^2 + \frac{1}{2}\mathrm{p}(t)^{-1/2} g_t. \tag{3.50}$$

Therefore, from Remark A.27, we have, for each k, that

$$\begin{aligned}(\mathrm{div}T_t)_k =& \nabla_i^{g_t}(T_t)_{ij} = \nabla_i^{g_t}(-2\mathrm{p}(t)\eta^2)_{ij} + \frac{1}{2}\nabla_i^{g_t}\mathrm{p}(t)^{-1/2}(g_t)_{ij} \\ =& -2\mathrm{p}(t)^{11/10}\nabla_i^{g_t}(\eta_i\eta_j) = -2\mathrm{p}(t)^{11/10}(\eta_i\nabla_i^{g_t}\eta_j + \eta_j\nabla_i^{g_t}\eta_i) \\ =& -2\mathrm{p}(t)^{11/10}(\eta_i\omega_{ij} + \eta_j\omega_{ij}) = 0.\end{aligned}$$

□

Proposition 3.51. *Let $\{\varphi_t\}$ be the solution to the Laplacian coflow given by (3.30) and initial data determined by $\varphi_0 = \eta_0 \wedge \omega_0 + \mathrm{Re}\,\Upsilon_0$, with associated metric (3.32) and torsion T_t. Then*

$$|\nabla T_t|_t^2 = c_0\mathrm{p}(t)^{-2} \tag{3.52}$$

where $c_0 > 0$ is a constant.

Proof. We know that $\nabla^{g_t} g_t = 0$. On the other hand, by (A.23), we have that $\nabla^{g_t}_{\xi_0}\xi_0 = \nabla^{g_0}_{\xi_0}\xi_0 + A(\xi_0, \xi_0) = 0$. Also, from Proposition 3.49, we know that $\mathrm{div}T_t = 0$, concluding that the only non-zero terms in $\nabla^{g_t}T_t$ arise from $\nabla^{g_t}_X\eta^2$, $X \in \mathcal{D}_0$. Moreover, from Remark A.27, we obtain that $(\nabla^{g_t}_X\eta)^\sharp \in \mathcal{D}_0$, then

$$\begin{aligned}|\nabla^{g_t}_X(\mathrm{p}(t)^{-3/5}\eta^2)|_t^2 =& |\mathrm{p}(t)^{-3/5}\nabla^{g_0}_X(\eta^2)|_t^2 = |\mathrm{p}(t)^{-3/5}\omega(X,\cdot)\eta(\cdot)|_t^2 \\ =& c_0\mathrm{p}(t)^{-3/5}(\mathrm{p}(t))^{-2/5} = c_0\mathrm{p}(t)^{-1}\end{aligned}$$

with $c_0 > 0$ constant and $g_t(X, X) = 1$. Hence, using (3.50), we obtain

$$\begin{aligned}|\nabla^{g_t}_X T_t|_t^2 =& |-2\mathrm{p}(t)^{-11/10}\nabla^t_X\eta^2|_t^2 \\ =& |-2\mathrm{p}(t)^{-1/2}\nabla^{g_t}_X(\mathrm{p}(t)^{-3/5}\eta^2)|_t^2 \\ =& 4c_0\mathrm{p}(t)^{-1}c\mathrm{p}(t)^{-1} = 4c_0\mathrm{p}(t)^{-2}\end{aligned}$$

□

3.3 Collapsing contact Calabi-Yau manifold

In this section, we show that the family of G_2-structures given by (3.30) is volume collapsing and collapsing with respect to the normalized metric at $-1/10$. Inspired by the work of Chen [11], we use the quantity $\Lambda(x, t)$ (see Eq. (1.80)) to show that the volume is collapsed.

Let φ_t be the solution to the Laplacian coflow (4) given by (3.30) on $(M_0, \eta_0, \Phi_0, \Upsilon_0)$ a contact Calabi-Yau manifold. In what follows, we will give proofs for both Theorem 10 and 12.

Proof Theorem 10. Using Proposition 3.40, Proposition 3.47 and Proposition 3.51, we obtain

$$\begin{aligned}\Lambda(x,t) :=& \sup_M(|Rm(y,t)|^2_{g_t} + |T(y,t)|^4_{g_t} + |\nabla T(y,t)|^2_{g_t})^{\frac{1}{2}} \\ =& \sup_M \left(\mathrm{p}(t)^{-2/5}|Rm_0^{\mathcal{D}}|^2_0 + c_0\mathrm{p}(t)^{-2} + \frac{15}{4}\mathrm{p}(t)^{-2} + c_1\mathrm{p}(t)^{-2}\right)^{\frac{1}{2}} \\ =& \sup_M \left(\mathrm{p}(t)^{-2/5}|Rm_0^{\mathcal{D}}|^2_0 + k\mathrm{p}(t)^{-2}\right)^{\frac{1}{2}},\end{aligned} \tag{3.53}$$

whit $k > 0$ constant. Therefore, if $\sup_M |Rm|_0 \leq C$, we have

$$\begin{aligned}\Lambda(t) :=& \sup_M \Lambda(x,t) = \sup_M \left(\mathrm{p}(t)^{-1/5}(|Rm_0^{\mathcal{D}}|^2_0 + k\mathrm{p}(t)^{-8/5}\right)^{\frac{1}{2}}) \\ \leq& k\mathrm{p}(t)^{-1/5}(1 + c\mathrm{p}(t)^{-4/5}),\end{aligned} \tag{3.54}$$

where k, c are constant. □

Definition 3.55. Let $(M, \varphi(t))$, $t \in (T, b)$ be a complete solution to the Laplacian coflow, where $T \in [-\infty, 0)$, $b \in [0, \infty]$ and $0 \in (a, b]$. The solution $\varphi(t)$ with associated metric $g(t)$ is called *locally collapsing* at T if there exists a sequence of points $x_k \in M$, times $t_k \to T$ and radii $r_k \in (0, \infty)$ with r_k^2/t_k uniformly bounded such that the balls $B_{g(t_k)}(x_k, r_k)$ satisfy:

- (*curvature bound comparable to the radius of the ball*)

$$|Rm(g(t_k))| \leq r_k^{-2} \quad \text{in} \quad B_{g(t_k)}(x_k, r_k),$$

- (*volume collapsed of the ball*)

$$\lim_{k\to\infty} \frac{\mathrm{vol}_{g(t_k)} B_{g(t_k)}(x_k, r_k)}{r_k^7} = 0$$

Proof of Theorem 12. We define $\Lambda(t) = \sup_{x\in M} \Lambda(x,t)$. Note that it satisfies $\lim \Lambda(t) = \infty$ as $t \to$-1/10. We have that the solution ψ_t of the Laplacian coflow has a maximal interval $(-1/10, \infty)$. Choose a sequence of points (x_i, t_i) such that $t_i \to -1/10$ and let κ_i be a sequence such that $\kappa_i \to \infty$ we have

$$\Lambda(x_i, t_i) = \sup_{x\in M, t\in[t_i,\kappa_i]} \left(|Rm(x,t)|^2_{g(t)} + |T(x,t)|^4_{g(t)} + |\nabla T(x,t)|^2_{g(t)}\right)^{\frac{1}{2}} \tag{3.56}$$

where T and Rm are the torsion and curvature as usual. We consider a sequence of dilations of the Laplacian coflow

$$\psi_i(t) = *_t\varphi_i = \Lambda(x_i, t_i)^2\psi(t_i + \Lambda(x_i, t_i)^{-1}t). \tag{3.57}$$

Therefore

$$\varphi_i(t) = \Lambda(x_i, t_i)^{3/2}\varphi(t_i + \Lambda(x_i, t_i)^{-1}t). \tag{3.58}$$

and the associated metric $g_i(t)$ of $\varphi_i(t)$ is

$$g_i(t) = \Lambda(x_i, t_i)g(t_i + \Lambda(x_i, t_i)^{-1}t). \tag{3.59}$$

From the conformal property for the 3-form we have

$$\tilde{\psi} = \lambda\psi \Rightarrow \tilde{\varphi} = \lambda^{3/4}\varphi \Rightarrow \tilde{\Delta}\psi = \lambda^{1/2}\Delta\psi.$$

Thus, for each i, $(M, \psi_i(t))$ is a solution of the Laplacian coflow (4) on the time interval

$$t \in ((-1/10 - t_i)\Lambda(x_i, t_i), (\kappa_i - t_i)\Lambda(x_i, t_i)] \tag{3.60}$$

Given that the torsion forms of a conformal G_2-structure are

$$\tilde{\tau_0} = \lambda^{-1/2}\tau_0, \qquad \tilde{\tau_3} = \lambda\tau_3.$$

Then, the full torsion tensor is equal to $\tilde{T} = \lambda^{1/2}T$. Furthermore, we define

$$\Lambda_{\psi_i}(x,t) = \Big(|\nabla_{g_i(t)}T_i(x,t)|^2_{g_i(t)} + |Rm_i(x,t)|^2_{g_i(t)} + |T_i(x,t)|^4_{g_i(t)}\Big)^{1/2}. \tag{3.61}$$

Satisfying $\Lambda_{\psi_i}(x_i, 0) = 1$ so that if $\tilde{t}_i(t) = t_i + \Lambda^{-1}(x_i, t_i)t$

$$\begin{aligned} |Rm_i(x,t)|^2_{g_i(t)} &= \frac{1}{\Lambda^2(x_i,t_i)}|Rm(x,\tilde{t}_i(t))|^2_{g(\tilde{t}_i(t))} \\ |\nabla_{g_i(t)}T_i(x,t)|^2_{g_i(t)} &= \frac{1}{\Lambda^2(x_i,t_i)}|\nabla_{g(\tilde{t}_i(t))}T(x,\tilde{t}_i(t))|^2_{g(\tilde{t}_i(t))} \\ |T_i(x,t)|^2_{g_i(t)} &= \frac{1}{\Lambda(x_i,t_i)}|T(x,\tilde{t}_i(t))|^2_{g(\tilde{t}_i(t))} \end{aligned}$$

and we obtain

$$\Lambda_{\psi_i}(t) = \sup_M |\Lambda_{\psi_i}(x,t)| = \Lambda(x_i,t_i)^{-1} \sup_{x\in M} |\Lambda(x, t_i + \Lambda^{-1}(x_i,t_i)t)| \leq 1 \tag{3.62}$$

for $0 \leq t \leq (\kappa_i - t_i)\Lambda(x_i, t_i)$. Since $\sup_M |\Lambda_{\psi_i}(x, 0)| = 1$. By proposition 3.37, g_t is uniformly continuous. Let $\varepsilon \in (0, 1/2]$ and $\delta > 0$ be given by the definition of uniformly continous, then if $t_0 = -1/10 + \delta$ then (3.36) holds for all $-1/10 < t < t_0$. Suppose i is sufficient large that $t_i \geq t_0$. From (3.36), for any $x, y \in M$ and $t \in (-1/10, t_0]$, we have

$$(1-\varepsilon)^{1/2} d_{g_{t_0}}(x,y) \leq d_{g_t}(x,y) \leq (1+\varepsilon)^{1/2} d_{g_{t_0}}(x,y). \tag{3.63}$$

Therefore, if $B_{g_t}(x, r)$ denotes the geodesic ball of radius r centred at x with respect to the metric g_t, we have

$$B_{g_{t_0}}(x, (1+\varepsilon)^{-1/2}r) \subset B_{g_t}(x, r). \tag{3.64}$$

Along the Laplacian coflow, the volume form (3.33) increases for $t \in (-1/10, t_0]$, so

$$\mathrm{vol}_{g_t}(B_{g_t}(x,r)) \leq \mathrm{vol}_{g_{t_0}}(B_{g_{t_0}}(x, (1+\varepsilon)^{-1/2})r)) \tag{3.65}$$

Then, for $x \in M$ and $r \leq \Lambda(x_i, t_i)^{1/2}$ we have

$$\begin{aligned} \mathrm{vol}_{g_i(0)}\Big(B_{g_i(0)}(x,r)\Big) &= \Lambda(x_i,t_i)^{7/2}\mathrm{vol}_{g(t_i)}\Big(B_{g(t_i)}\Big(x, \Lambda(x_i,t_i)^{-1/2}r\Big)\Big) \\ &\leq \Lambda(x_i,t_i)^{7/2}\mathrm{vol}_{g(t_0)}\Big(B_{g(t_0)}(x,(1+\varepsilon)^{-1/2}r)\Big) \\ &\leq c(1+\varepsilon)^{-7/2}r^7, \end{aligned} \tag{3.66}$$

for some uniform positive constant c. Hence we have

$$\mathrm{vol}_{g_i(0)}\Big(B_{g_i(0)}(x,r)\Big) \le cr^7 \tag{3.67}$$

for all $x \in M$ and $r \in [0, \Lambda(x_i,t_i)^{1/2}]$. Note that by the definition of Λ_{ψ_i}, we have

$$|Rm_{g_i}(x,0)| \le \sup_{M\times[0,(\kappa_i-t_i)\Lambda(x_i,t_i)]} |\Lambda_{\psi_i}(x,t)| \le 1 \tag{3.68}$$

on M. We define $r_i = \Lambda(x_i,t_i)^{-1/2}$ and using $\Lambda(x_i,t_i) \to \infty$ and $t_i \to -1/10$, we have $r_i^2/t_i \le C$ and

- (*curvature bound comparable to the radius of the ball*)

$$|Rm(g(t_i))| \le \Lambda(x_i,t_i) = r_i^{-2},$$

- (*volume collapsed of the ball*) Using (3.67), we have

$$\frac{\mathrm{vol}_{g_i(0)}\Big(B_{g_i(0)}(x_i,r_i)\Big)}{r_i^7} \le C$$
$$\frac{\mathrm{vol}_{\Lambda(x_i,t_i)g(t_i)}\Big(B_{\Lambda(x_i,t_i)g(t_i)}(x_i,r_i)\Big)}{r_i^7} \le C$$

Therefore, if $\tilde{g} = \alpha^2 g$, then $B_g(x,r) = B_{\alpha^2 g}(x,\alpha r)$ and $\mathrm{vol}_{\alpha^2 g}B_{\alpha^2 g}(x,\alpha r) = \alpha^7 \mathrm{vol}_g B_g(x,r)$. So that if $\alpha = \Lambda(x_i,t_i)^{1/2}$, we obtain

$$\frac{\Lambda(x_i,t_i)^{7/2}\mathrm{vol}_{g(t_i)}\Big(B_{g(t_i)}(x_i,r_i^2)\Big)}{r_i^7} \le \frac{\Lambda(x_i,t_i)^{7/2}\mathrm{vol}_{g(t_i)}\Big(B_{g(t_i)}(x_i,r_i)\Big)}{r_i^7} \le C$$

Thus, we have

$$\lim_{i\to\infty} \frac{\mathrm{vol}_{g(t_i)}B_{g(t_i)}(x_i,r_i)}{r_i^7} = 0.$$

□

Remark 3.69. Let $(M_0,\eta_0,\Phi_0,\Upsilon_0)$ be a contact Calabi-Yau manifold and $\psi(t)$, $t \in (-1/10,\infty)$ be the solution (3.31) to the Laplacian coflow (4), (x_i,t_i) a sequence of points such that $t_i \to -1/10$ and $\Lambda(x_i,t_i)$ in (3.56) and $p(t) = 10t+1$. Taking the family of metrics $g_i(t)$ given by

$$g_i(t) = \Lambda(x_i,t_i)\mathrm{p}_i(t)^{-3/5}\eta_0^2 + \Lambda(x_i,t_i)\mathrm{p}_i(t)^{1/5}g_{\mathcal{D}_0}, \tag{3.70}$$

where $\mathrm{p}_i(t) = \mathrm{p}(t_i + \Lambda(x_i,t_i)^{-1}t)$ for $t \in ((-1/10-t_i)\Lambda(x_i,t_i),(\kappa_i-t_i)\Lambda(x_i,t_i)]$, We can rescale the family $g_i(t)$ and see that

$$\tilde{g}_i = p_i(t)^{1/5}\Lambda(x_i,t_i)^{-1}g_i(t).$$

As $t \to (-1/10-t_i)\Lambda(x_i,t_i)$ we have that the circle Reeb orbit decrease whilst the transverse geometry is constant.

3.4 The soliton equations on contact Calabi-Yau manifolds

In this section, Theorem 3.75 is one of the main results which will be used in soliton solutions in a particular case of contact Calabi-yau manifolds.

As for many geometric flows, we are interested in considering *self-similar solutions*, (which only evolve diffeomorphisms and scalings) since these are expected to be related to singularities in the flow. Given a 7-manifold M, a *Laplacian soliton* for the Laplacian coflow (4) for coclosed G_2-structures on M is a triple (ψ, X, λ) satisfying

$$\Delta_\psi \psi = \mathcal{L}_X \psi + \lambda\psi = \mathrm{d}(X \lrcorner \psi) + \lambda\psi, \tag{3.71}$$

where $\mathrm{d}\psi = 0$, $\lambda \in \mathbb{R}$ and X is a vector field on M. Given a Laplacian soliton (ψ, X, λ) we can define, whenever $1 + \lambda t > 0$, a function $\rho(t)$ and a t-dependent family of vector fields $X(t)$ by

$$\rho(t) = 1 + \lambda t \quad \text{and} \quad X(t) = \rho(t)^{-1} X. \tag{3.72}$$

If we let $f(t)$ be the 1-parameter family of diffeomorphisms generated by $X(t)$ such that $f(0)$ is the identity, then we may define

$$\psi_t = \rho(t) f(t)^* \psi \tag{3.73}$$

and see that

$$\begin{aligned}
\frac{\partial}{\partial t}\psi_t &= f(t)^*(\rho'(t)\psi + \rho(t)\mathcal{L}_{X(t)}\psi) \\
&= f(t)^*(\lambda\psi + \rho(t)\mathrm{d}(\rho(t)^{-1} X \lrcorner \psi)) \\
&= f(t)^*(\Delta_\psi \psi) \\
&= \Delta_{\psi_t} \psi_t,
\end{aligned}$$

where we used the fact that

$$\Delta_{\kappa\psi}(\kappa\psi) = \kappa^{1/2} \Delta_\psi \psi$$

for any $\kappa > 0$. Hence, ψ_t solves the Laplacian coflow (4) and simply evolves scaling and pullback by diffeomorphisms. From the behaviour of ψ_t, we see that it is natural to call a Laplacian soliton (ψ, X, λ) expanding if $\lambda > 0$; steady if $\lambda = 0$ and shrinking if $\lambda < 0$.

Proposition 3.74. *[25, §4, Proposition 4.3] If M^7 is compact, then there are no shrinking or steady soliton solutions of (3.71), other than the trivial case of a torsion free G_2-structure in the steady case.*

The following theorem is of key importance in the analysis of soliton solutions and it will be used in Proposition 3.86.

Theorem 3.75. *Let φ be a coclosed G_2-structure on a compact manifold M with associated metric g and let X be a vector field on M. Then,*

$$\mathcal{L}_X \psi = \frac{4}{7}\mathrm{div}(X)\psi \oplus \mathrm{Curl}(X)^\flat \wedge \varphi \oplus *i_\varphi\Big(\frac{3}{49}\mathrm{div}(X) g - \frac{3}{14}(\mathcal{L}_X g)\Big) \in \Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{27}, \tag{3.76}$$

where $i_\varphi : S^2T^*M \to \Omega^3_1(M) \oplus \Omega^3_{27}(M)$ is the injective map given by (1.6). In particular, any symmetry X of the coclosed G_2-structure φ must be a Killing vector field of the asociated metric g on M.

Proof. Since φ is coclosed, i.e. $\mathrm{d} * \varphi = 0$, we have

$$\mathcal{L}_X\psi = \mathrm{d}(X \lrcorner \psi) + X \lrcorner \mathrm{d}\psi = \mathrm{d}(X \lrcorner \psi).$$

Let $\alpha = X \lrcorner \psi$ so that $\alpha_{ijk} = X^l \psi_{lijk}$ and

$$\mathcal{L}_X\psi = \mathrm{d}\alpha = \frac{1}{24}(\nabla_i\alpha_{jkl} - \nabla_j\alpha_{ikl} + \nabla_k\alpha_{jil} - \nabla_l\alpha_{jki})dx^{ijkl},$$

i.e,

$$(\mathcal{L}_X\psi)_{ijkl} = \nabla_i\alpha_{jkl} - \nabla_j\alpha_{ikl} + \nabla_k\alpha_{jil} - \nabla_l\alpha_{jki}. \tag{3.77}$$

We decompose $\mathcal{L}_X\psi$ as

$$\mathcal{L}_X\psi = \pi^4_1(\mathcal{L}_x\psi) + \pi^4_7(\mathcal{L}_x\psi) + \pi^4_{27}(\mathcal{L}_X\psi) = a\psi + W^\flat \wedge \varphi + *i_\varphi(h), \tag{3.78}$$

where $\pi^k_l : \Omega^k(M) \to \Omega^k_l(M)$ denotes the projection onto $\Omega^k_l(M)$, a is a function, $W^\flat = g_\varphi(W, \cdot)$ and h is a trace-free symetric 2-tensor on M. We compute a as follows:

$$\begin{aligned} a &= \frac{1}{7}\langle\mathcal{L}_X\psi, \psi\rangle = \frac{1}{168}(\nabla_i\alpha_{jkl} - \nabla_j\alpha_{ikl} + \nabla_k\alpha_{jil} - \nabla_l\alpha_{jki})\psi^{ijkl} \\ &= \frac{1}{42}\nabla_i\alpha_{jkl}\psi^{ijkl} = \frac{1}{42}\nabla_i(\alpha_{jkl}\psi^{ijkl}) - \frac{1}{42}\alpha_{jkl}\nabla_i\psi^{ijkl} \\ &= \frac{24}{42}\nabla_i(X^m g_{mi}) - \frac{1}{42}X^m\psi_{mjkl}(\nabla_i\psi^{ijkl}) = \frac{4}{7}\nabla_i X_i = \frac{4}{7}\mathrm{div}(X), \end{aligned} \tag{3.79}$$

where we used (1.22) and the fact that T_{ij} is a symmetric tensor. To compute $W^\flat$, note that

$$\langle *((*\mathcal{L}_X\psi) \wedge \varphi), e^m\rangle = 4\langle W^\flat, e^m\rangle,$$

thus

$$\begin{aligned} 4W^m &= *\big((*\mathcal{L}_X\psi) \wedge \varphi \wedge e^m\big) = *\big(\langle\varphi \wedge e^m, \mathcal{L}_X\psi\rangle \mathrm{vol}\big) \\ &= \frac{1}{3!}(\mathcal{L}_X\psi)^{ijkm}\varphi_{ijk}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} W^m &= \frac{1}{4!}(\mathcal{L}_X\psi)^{ijkm}\varphi_{ijk} = \frac{1}{3!}(g^{si}\nabla_s\alpha^{jkm}\varphi_{ijk}) \\ &= \frac{1}{3!}(g^{si}\nabla_s(X^l\psi_l^{\ jkm}\varphi_{ijk}) - X^l\psi_l^{\ jkm}g^{si}\nabla_s\varphi_{ijk}) \\ &= \frac{1}{3!}(4g^{si}\nabla_s(X^l\varphi_{il}^{\ \ m}) - X^l\psi_l^{\ jkm}g^{si}T_{sa}g^{an}\psi_{nijk}) \\ &= \frac{4}{3!}g^{si}\nabla_s(X^l\varphi_{il}^{\ \ m}) = \frac{4}{3!}g^{si}\nabla_s X^l\varphi_{il}^{\ \ m} + \frac{4}{3!}X^l g^{si}\nabla_s\varphi_{ir}^{\ \ m} \\ &= \frac{4}{3!}\varphi_{ir}^{\ \ m}\nabla^i X^l + \frac{4}{3!}X^l g^{si}\nabla_s\varphi_{ir}^{\ \ m} \\ &= \frac{4}{3!}\varphi_{ila}(\nabla^i X^l)g^{am} + \frac{4}{3!}X^l g^{si}T_{sn}g^{nl}\psi_{lirb}g^{bm} \\ &= \frac{4}{3!}\,\mathrm{Curl}(X)_a g^{am}. \end{aligned} \tag{3.80}$$

Finally, to calculate h observe that

$$\begin{aligned} &(\mathcal{L}_X\psi)_{mnpi}\psi_j^{\ mnp} + (\mathcal{L}_X\psi)_{mnpj}\psi_i^{\ mnp} \\ &= a(\psi_{mnpi}\psi_j^{\ mnp} + \psi_{mnpj}\psi_i^{\ mnp}) + (*i_\varphi(h))_{mnpi}\psi_j^{\ mnp} + (*i_\varphi(h))_{mnpj}\psi_i^{\ mnp}, \end{aligned} \tag{3.81}$$

where

$$(*i_\varphi(h))_{mnpi} = (i_\psi(h))_{mnpi} = h_m^q\psi_{qnpi} - h_n^q\psi_{mqpi} + h_p^q\psi_{mnqi} - h_i^q\psi_{mnpq}.$$

Equation (3.81) becomes

$$\begin{aligned}
&a\psi_{mnpi}\psi_j^{mnp} + (h_m^q\psi_{qnpi} - h_n^q\psi_{mqpi} + h_p^q\psi_{mnqi} - h_i^q\psi_{mnpq})\psi_j^{mnp} \\
&a\psi_{mnpj}\psi_i^{mnp} + (h_m^q\psi_{qnpj} - h_n^q\psi_{mqpj} + h_p^q\psi_{mnqj} - h_j^q\psi_{mnpq})\psi_i^{mnp} \\
&= -48ag_{ij} + h_m^q(4g_{qj}g_{im} - 4g_{qm}g_{ij} + 4g_{qi}g_{jm} - 4g_{qm}g_{ij}) \\
&\quad - h_n^q(4g_{qj}g_{in} - 4g_{qn}g_{ij} + 4g_{qi}g_{jn} - 4g_{qn}g_{ij}) + 24h_i^q g_{qj} \\
&\quad + h_p^q(4g_{qj}g_{ip} - 4g_{qp}g_{ji} + 4g_{qi}g_{jp} - 4g_{qp}g_{ij}) + 24h_j^q g_{qi} \\
&= -48ag_{ij} + 56h_{ij}.
\end{aligned} \tag{3.82}$$

We can calculate the left hand side of (3.82) as follows

$$\begin{aligned}
& (\mathcal{L}_X\psi)_{mnpi}\psi_j^{mnp} + (\mathcal{L}_X\psi)_{mnpj}\psi_i^{mnp} \\
= \;& (\nabla_m\alpha_{npi} - \nabla_n\alpha_{mpi} + \nabla_p\alpha_{nmi} - \nabla_i\alpha_{npm})\psi_j^{mnp} \\
& (\nabla_m\alpha_{npj} - \nabla_n\alpha_{mpj} + \nabla_p\alpha_{mmj} - \nabla_j\alpha_{npm})\psi_i^{mnp} \\
= \;& 3(\nabla_m\alpha_{npj}\psi_i^{mnp} + \nabla_m\alpha_{npi}\psi_j^m np) - \nabla_i\alpha_{npm}\psi_j^{mnp} - \nabla_j\alpha_{npm}\psi_i^{mnp} \\
= \;& \underbrace{3\nabla_m(\alpha_{npi}\psi_j^{mnp})}_{I(i)} - \underbrace{3\alpha_{npi}\nabla_m\psi_j^{mnp}}_{II(i)} - \underbrace{\nabla_i(\alpha_{mnp}\psi_j^{mnp})}_{III(i)} + \underbrace{\alpha_{mnp}\nabla_i\psi_j^{mnp}}_{IV(i)} \\
& \underbrace{3\nabla_m(\alpha_{npj}\psi_i^{mnp})}_{I(j)} - \underbrace{3\alpha_{npj}\nabla_m\psi_i^{mnp}}_{II(j)} - \underbrace{\nabla_j(\alpha_{mnp}\psi_i^{mnp})}_{III(j)} + \underbrace{\alpha_{mnp}\nabla_j\psi_i^{mnp}}_{IV(j)}
\end{aligned} \tag{3.83}$$

Using (1.24), (1.25), (1.34) and the following expression

$$X^l T_i^m \varphi_{ljm} = (X\lrcorner\varphi)_{jm}T_i^m = (T\cdot(X\lrcorner\varphi))_{ji} = (T\cdot(X\lrcorner\varphi))_{ij}^T = -((X\lrcorner\varphi)\cdot T)_{ij}, \tag{3.84}$$

we obtain that

$$\begin{aligned}
I(i) &= \nabla_m(\alpha_{npi}\psi_j^{mnp}) = \nabla_m(X^l\psi_{lnpi}\psi_j^{mnp}) \\
&= \nabla_m(X^l(4g_{lj}g_i{}^m - 4g_l{}^m g_{ij} + 2\psi_{lij}{}^m)) \\
&= 4\nabla_i X_j - 4\mathrm{div}X g_{ij} + 2\nabla_m(X^l\psi_{lij}{}^m), \\
I(j) + I(i) &= 4\nabla_j X_i - 4\mathrm{div}X g_{ij} + \nabla_m(X^l\psi_{lji}{}^m) + 4\nabla_i X_j - 4\mathrm{div}X g_{ji} + \nabla_m(X^l\psi_{lij}{}^m) \\
&= 4(\nabla_i X_j + \nabla_j X_i) - 8\mathrm{div}X g_{ij}.
\end{aligned}$$

For a coclosed G_2-structure, we have that T is symmetric. Therefore

$$\begin{aligned}
II(i) + II(j) &= \alpha_{npi}\nabla_m\psi_j^{mnp} + \alpha_{npj}\nabla_m\psi_i^{mnp} = X^l\psi_{lnpi}\nabla_m\psi_j^{mnp} + X^l\psi_{lnpj}\nabla_m\psi_i^{mnp} \\
&= X^l\psi_{lnpi}(-T_{mj}\varphi^{mnp} + T_m^m\varphi_j^{np} - T_m^n\varphi_j^{mp} + T_m^p\varphi_j^{mn}) \\
&\quad + X^l\psi_{lnpj}(-T_{mi}\varphi^{mnp} + T_m^m\varphi_i^{np} - T_m^n\varphi_i^{mp} + T_m^p\varphi_i^{mn}) \\
&= 4(X\lrcorner\varphi\cdot T)_{ij} - 4(T\cdot X\lrcorner\varphi)_{ij} = 4[X\lrcorner\varphi, T]_{ij}, \\
III(i) + III(j) &= \nabla_i(X^l\psi_{lmnp}\psi_j^{mnp}) + \nabla_j(X^l\psi_{lmnp}\psi_i^{mnp}) \\
&= 24(\nabla_i X_j + \nabla_j X_i),
\end{aligned}$$

$$\begin{aligned}
IV(i) &= \alpha_{mnp}\nabla_i\psi_j^{\ mnp} = X^l\psi_{lmnp}\nabla_i\psi_j^{\ mnp}\\
&= X^l\psi_{lmnp}(-T_{ij}\varphi^{mnp} + T_i^m\varphi_j^{\ np} - T_i^n\varphi_j^{\ mp} + T_i^p\varphi_j^{\ mn}) = 12(X\lrcorner\varphi\cdot T)_{ij},\\
IV(i)+IV(j) &= 12(X\lrcorner\varphi\cdot T)_{ij} + 12(X\lrcorner\varphi\cdot T)_{ji} = 12[X\lrcorner\varphi, T]_{ij}.
\end{aligned}$$

So, using (3.79) , (3.82) and the above expressions, we obtain

$$\begin{aligned}
-48ag_{ij} + 56h_{ij} &= -12(\nabla_iX_j + \nabla_jX_i) - 24\mathrm{div}(X)g_{ij}\\
h_{ij} &= \frac{3}{49}\mathrm{div}(X)g_{ij} - \frac{3}{14}(\nabla_iX_j + \nabla_jX_i)\\
&= \frac{3}{49}\mathrm{div}(X)g_{ij} - \frac{3}{14}(\mathcal{L}_Xg)
\end{aligned}\tag{3.85}$$

Hence, substituting (3.79), (3.80) and (3.85) into (3.78) we obtain (3.76). □

Proposition 3.74 was given in [25] with a different and lengthier proof, and with an additional minus sign in the Laplacian operator, so the roles of shrinking and expanding solitons are reversed. On the other hand, in order to understand soliton conditions of coclosed G_2-structures on contact Calabi-Yau manifolds, we need to undertand better the properties of T and the Laplacian coflow.

Proposition 3.86. *Let φ be a coclosed G_2-structure on a compact manifold M with associated metric g_φ. If (ψ, X, λ) is a soliton of the Laplacian coflow as in (3.71), then it satisfies*

$$\begin{aligned}
&\mathrm{Curl}\,X = 0, \quad \mathrm{div}T = 0, \quad \nabla\,\mathrm{tr}\,T = 0,\\
&\lambda = \frac{2}{3}R + \frac{4}{3}|T|^2 - \frac{4}{7}\mathrm{div}(X),\\
&0 = -\,\mathrm{Ric} + \Big(\frac{1}{14}\mathrm{R} - \frac{1}{7}|T|^2 - \frac{3}{49}\mathrm{div}(X)\Big)g + \mathrm{tr}(T)T - 2T^2\\
&\quad -\frac{1}{2}T\circ T + \frac{3}{14}(\mathcal{L}_Xg).
\end{aligned}\tag{3.87}$$

Proof. Using Lemma 1.56 and Theorem 3.75, we obtain

$$\begin{aligned}
0 =&\Delta_\psi\psi - \lambda\psi - \mathcal{L}_X\psi\\
0 =&(\frac{2}{3}R + \frac{4}{3}|T|^2)\psi + (\mathrm{d}\,\mathrm{tr}\,T)\wedge\varphi + *_\varphi \mathrm{i}_\varphi\Big(-\mathrm{Ric} + \frac{1}{14}(R - 2|T|^2)g + \mathrm{tr}(T)T - 2T^2 - \frac{1}{2}T\circ T\Big)\\
&-\Big(\frac{4}{7}\mathrm{div}(X)\psi + \mathrm{Curl}(X)^\flat\wedge\varphi + *\mathrm{i}_\varphi\Big(\frac{3}{49}\mathrm{div}(X)g - \frac{3}{14}(\mathcal{L}_Xg)\Big)\Big) - \lambda\psi\\
=&(\frac{2}{3}R + \frac{4}{3}|T|^2 - \frac{4}{7}\mathrm{div}(X) - \lambda)\psi + (\mathrm{d}\,\mathrm{tr}\,T - \mathrm{Curl}(X)^\flat)\wedge\varphi\\
&+\mathrm{i}_\varphi\Big(-\mathrm{Ric} + \frac{1}{14}(R - 2|T|^2)g + \mathrm{tr}(T)T - 2T^2 - \frac{1}{2}T\circ T - \frac{3}{49}\mathrm{div}(X)g + \frac{3}{14}(\mathcal{L}_Xg)\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
0 =&\frac{2}{3}R + \frac{4}{3}|T|^2 - \frac{4}{7}\mathrm{div}(X) - \lambda\\
0 =&\mathrm{d}\,\mathrm{tr}\,T - \mathrm{Curl}(X)^\flat\\
0 =&-\mathrm{Ric} + \frac{1}{14}(R - 2|T|^2 - \frac{3}{49}\mathrm{div}(X))g + \mathrm{tr}(T)T - 2T^2 - \frac{1}{2}T\circ T + \frac{3}{14}(\mathcal{L}_Xg),
\end{aligned}$$

and thus we get (3.87). □

Moreover, Proposition 3.86 will be useful in the following case which was studied by Lotay and Sá Earp from the perspective of Heterotic string theory. Therefore, let V be a Calabi-Yau 3-orbifold with metric g_V, volume form vol_V, kahler form ω and holomorphic volume form Υ satisfying

$$\mathrm{vol}_V = \frac{1}{3!}\omega^3 = \frac{1}{4}\operatorname{Re}\Upsilon \wedge \operatorname{Im}\Upsilon$$

Let M that the total space of $\pi : K \to V$ is a contact Calabi-Yau 7-manifold. For every $\varepsilon > 0$, we define a S^1-invariant G_2-structure on K by

$$\varphi_\varepsilon = \varepsilon\eta \wedge \omega + \operatorname{Re}\Upsilon \tag{3.88}$$

$$\psi_\varepsilon = \frac{1}{2}\omega^2 - \varepsilon\eta \wedge \operatorname{Im}\Upsilon. \tag{3.89}$$

The metric induced from this G_2-structure and its corresponding volume form are

$$g_\varepsilon = \varepsilon^2\eta^2 + g_V \quad \text{and} \quad \mathrm{vol}_\varepsilon = \varepsilon\eta \wedge \mathrm{vol}_V \tag{3.90}$$

We have from (3.88), (3.89) and Theorem 3.12 that

$$\mathrm{d}\varphi_\varepsilon = \varepsilon\omega^2 \quad \text{and} \quad \mathrm{d}\psi_\varepsilon = 0, \tag{3.91}$$

Lemma 3.92. *[32, §2.1, Lemma 2.4] For each $\varepsilon > 0$, the G_2-structure on M^7 defined by (3.88) has torsion forms*

$$\tau_0 = \frac{6}{7}\varepsilon \qquad \tau_1 = 0,$$

$$\tau_2 = 0 \qquad \tau_3 = \frac{8}{7}\varepsilon^2\eta \wedge \omega - \frac{6}{7}\varepsilon \operatorname{Re}\Upsilon$$

Proposition 3.93. *Let φ_ε be a G_2-structure given by (3.88). Then, its full torsion tensor is*

$$T = \frac{\varepsilon}{2}g_V - \frac{3\varepsilon^3}{2}\eta^2. \tag{3.94}$$

and the full torsion tensor norm is given by $|T|^2 = \dfrac{15}{4}\varepsilon^2$.

Proof. Let ξ be the dual to η and $X, Y \in \mathcal{D}$ and let j be a linear operator given by (1.6), thus

$$\begin{aligned}
\mathrm{j}(\tau_3)(\xi,\xi) &= *_\varepsilon\Big((\xi \lrcorner \varphi_\varepsilon) \wedge (\xi \lrcorner \varphi_\varepsilon) \wedge \tau_3\Big) \\
&= *_\varepsilon(\varepsilon\eta(\xi)\omega \wedge \varepsilon\eta(\xi)\omega \wedge \frac{8}{7}\varepsilon^2\eta \wedge \omega) \\
&= *_\varepsilon(\frac{8}{7}\varepsilon^4\eta \wedge \omega^3) = \frac{48}{7}\varepsilon^3 \\
\mathrm{j}(\tau_3)(X,\xi) &= 0 \\
\mathrm{j}(\tau_3)(X,Y) &= - *_\varepsilon\Big(\varepsilon\eta \wedge (X \lrcorner \omega) \wedge (Y \lrcorner \operatorname{Re}\Upsilon) \wedge (-\frac{6\varepsilon}{7}\operatorname{Re}\Upsilon)\Big) \\
&\quad - *_\varepsilon\Big((X \lrcorner \operatorname{Re}\Upsilon) \wedge (\varepsilon\eta \wedge (Y \lrcorner \omega)) \wedge (-\frac{6\varepsilon}{7}\operatorname{Re}\Upsilon)\Big) \\
&\quad + *_\varepsilon\Big((X \lrcorner \operatorname{Re}\Upsilon) \wedge (Y \lrcorner \operatorname{Re}\Upsilon) \wedge (\frac{8}{7}\varepsilon^2\eta \wedge \omega)\Big)
\end{aligned}$$

using the identity (3.42), we obtain

$$j(\tau_3)(X,Y) = -\frac{8}{7}\varepsilon *_\varepsilon *_\varepsilon(\eta \wedge \mathrm{vol}_V) = -\frac{8}{7}\varepsilon g_V(X,Y)$$

Therefore

$$(\tau_{27})_{ij} = \frac{1}{4} j(\tau_3)_{ij} = \quad \frac{12}{7}\varepsilon^3\eta^2 - \frac{2}{7}\varepsilon g_V.$$

thus, we have

$$\begin{aligned} T &= \frac{\tau_0}{4} g_\varepsilon - (\tau_{27})_{ij} \\ &= \frac{\varepsilon}{2} g_V - \frac{3\varepsilon^3}{2}\eta^2 \end{aligned}$$

Using Lemma A.21, we compute $|T|$ which is given by

$$|T|^2 = 6\big(\frac{1}{2}\varepsilon\big)^2 + \big(-\frac{3}{2}\varepsilon^3\big)^2(\varepsilon^2)^{-2} = \frac{15}{4}\varepsilon^2 \tag{3.95}$$

□

Proposition 3.96. *Let $(M, \eta, \Phi, \Upsilon)$ be a closed contact Calabi-Yau 7-manifold, and let $\omega := \mathrm{d}\eta$. For each $\varepsilon > 0$, a (locally) S^1-invariant G_2-structure is given by*

$$\varphi_\varepsilon = \varepsilon\eta \wedge \omega + \mathrm{Re}\,\Upsilon \tag{3.97}$$

$$\psi_\varepsilon = \frac{1}{2}\omega^2 - \varepsilon\eta \wedge \mathrm{Im}\,\Upsilon. \tag{3.98}$$

Then, any solitons $(\psi_\varepsilon, X, \lambda)$ for the Laplacian coflow (4) inducing the metric g_ε on M must have $X^\flat \in \Omega^1(M)$ harmonic and

$$\lambda = \varepsilon^2. \tag{3.99}$$

Proof. Let φ_ε be a coclosed G_2-structure on a closed contact Calabi-Yau manifold. Then, using (A.26), with $\alpha = 1$ and $\beta = \varepsilon^2 - 1$, we obtain $R_\varepsilon = R - 6(\varepsilon^2 - 1) = -6\varepsilon^2$. Now if (ψ, X, λ) is a soliton, then Proposition 3.86 implies that

$$\begin{aligned} \lambda_\varepsilon &= \frac{2}{3}R_\varepsilon + \frac{4}{3}|T|^2 - \frac{4}{7}\mathrm{div}(X), \\ &= \varepsilon^2 - \frac{4}{7}\mathrm{div}(X). \end{aligned}$$

By Stokes' theorem, after integration, we conclude that $\lambda_\varepsilon = \varepsilon^2$, as claimed, which in turn also implies $\mathrm{div}(X) = 0$. But we know from Proposition 3.86 that $\mathrm{Curl}(X) = 0$, so X is actually harmonic, since $\Delta X^\flat = \mathrm{d}\mathrm{d}^* X^\flat + \mathrm{d}^*\mathrm{d}X^\flat$, but $\mathrm{d}^* X^\flat = -\mathrm{div}(X^\flat)$ and $\mathrm{d}X^\flat = \mathrm{Curl}\,X$, obtaining $\Delta X^\flat = 0$. □

4 Concluding remarks

1. The Laplacian coflow has so far received rather little attention, so there are also many open problems. Therefore, we would like to conclude the following questions for future work.

   - In view of the relation between Laplacian coflow and modified Laplacian coflow. Then, what are the critical points of the modified Laplacian coflow on Contact Calabi-Yau manifolds?
   - We consider the following family of G_2-structures

   $$\varphi_t = f_t h_t^2 \eta_0 \wedge \omega_0 + h_t^3 \operatorname{Re} \Upsilon_0, \tag{4.1}$$

   starting at the natural coclosed G_2-structure on a contact Calabi–Yau 7-manifold and we analyse the Laplacian coflow, finding a solution with a singularity. In the future work, we can consider other family of G_2-structures with deformations which stabilize the characteristic foliation $\mathcal{F}_\xi$.

2. Given a coclosed G_2-structure, the linearization at ψ of the corresponding Hodge Laplacian is an indefinite operator where the term that cause Δ_ψ to be indefinite is $\pi_7(\Delta_\psi \psi)$, which is the component of $\Delta_\psi \psi$ is the 7-dimensional representation Λ^4_7 of G_2. This term is however determined by div$T$- the divergence of the Torsion. Therefore, the condition div$T = 0$ may be thought of as another condition to make $\Delta_\psi \psi$ elliptic. Then, the future work will be analyse harmonic G-invariant G_2-structures on G/H which is div$T_\varphi = 0$ (see [36, 24, 21])

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APPENDIX A – Contact Manifolds

A.1 Contact manifolds

Before defining Sasakian manifolds it will be necessary to define contact structures. Contact transformations arose in the theory of Analytical Mechanics developed in the 19th century by Hamilton, Jacobi, Lagrange and Legendre. But its first systematic treatment was given by Sephns Lie. Consider $\mathbb{R}^{2n+1}$ with Cartesian coordinates $(x^1, \cdots, x^n, y^1, \cdots, y^n, z)$ and 1-form η given by

$$\eta = \mathrm{d}z - \sum_i y^i \mathrm{d}x^i. \tag{A.1}$$

It is easy to see that $\eta \wedge (\mathrm{d}\eta)^n \neq 0$. A 1-form on $\mathbb{R}^{2n+1}$ that satisfies this condition is called a *contact form*. Therefore, locally we have the following result:

Proposition A.2. *[4] Let η be a 1-form on $\mathbb{R}^{2n+1}$ that satisfies $\eta \wedge (\mathrm{d}\eta)^n \neq 0$. Then, there is an open set $U \subset \mathbb{R}^{2n+1}$ and local coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n, z)$ such that η has the form* (A.1) *in U.*

Definition A.3. A $2n+1$-dimensional manifold is a *contact manifold* if there exists a 1-form η, called a *contact 1-form*, on M such that

$$\eta \wedge (\mathrm{d}\eta)^n \neq 0$$

everywhere on M. A *contact structure* on M is an equivalence class of such 1-forms by the relation $\eta' \sim \eta$ if there is a nowhere vanishing function f on M such that $\eta' = f\eta$.

Definition A.4. An *almost contact structure* on a differential manifold M^{2n+1} is a quadruple (M, η, ξ, Φ) where Φ is a tensor field of type $(1,1)$ (i.e, an endomorphism of TM), ξ is a vector field, and η is a 1-form which satisfies

$$\eta(\xi) = 1, \quad \Phi\xi = 0, \quad \eta \circ \Phi = 0 \tag{A.5}$$

$$\Phi \circ \Phi = -\mathrm{id} + \xi \otimes \eta. \tag{A.6}$$

The vector field ξ is called *Characteristic vector field* or the *Reeb vector field*. A smooth manifold with such a structure is called an *almost contact manifold* . A Riemannian metric on M is said to be *compatible* with the almost contact structure if for any fields X, Y on M we have

$$g(\Phi(X), \Phi(Y)) = g(X, Y) - \eta(X)\eta(Y). \tag{A.7}$$

An almost contact structure with a compatible metric is called an *almost contact metric structure*. In case η is a contact form, then (M, η, ξ, Φ, g) is said to be a *contact metric structure* on M

and a manifold with such a structure is called *contact metric manifold*. It follows that a contact metric structure (M, η, ξ, Φ, g) satisfies

$$\omega(X, Y) = g(\Phi(X), Y) = \frac{1}{2}\mathrm{d}\eta(X, Y), \quad X, Y \in \mathcal{X}(M) \tag{A.8}$$

For a contact metric manifold (M, η, ξ, Φ, g) we take

$$\mathrm{vol}_g = \frac{\eta \wedge \omega^n}{n!} = \frac{1}{2^n n!}\eta \wedge (\mathrm{d}\eta)^n \tag{A.9}$$

as the Riemannian volume form. A contact metric structure (M, η, ξ, Φ, g) is called *K-contact* if ξ is a Killing vector field of g and we have

$$(\nabla_X \eta)(Y) = \frac{1}{2}\mathrm{d}\eta(X, Y) \tag{A.10}$$

$$r(X, \xi) = (2n)\eta(X),$$

$$g(R(X, \xi)Y, \xi) = g(X, Y) - \eta(X)\eta(Y), \tag{A.11}$$

where $X, Y \in \mathfrak{X}(M)$, ∇ is the covariant differentiation with respect g, r and R are the Ricci curvature tensor and Riemannian curvature tensor respectively.

Proposition A.12. *[4] A contact metric structure (ξ, η, Φ, g) is K-contact if and only if $\nabla\xi = -\Phi$*

We say that (M^{2n+1}, η, Φ) is called *Sasakian* if the metric cone $(C(M), dr^2 + r^2 g, d(r^2\eta))$ is Kähler and it satisfies

$$(\nabla_X \Phi)Y = g(Y, \xi)X - g(X, Y)\xi, \tag{A.13}$$

$$R(X, \xi)Y = g(Y, \xi)X - g(X, Y)\xi \tag{A.14}$$

where $Y, Z \in \mathcal{X}(M)$. A Sasakian manifold is necessarily a K-contact Riemanninan. The vector field ξ is nowhere vanishing, so there is a 1-dimensional foliation $\mathcal{F}_\xi$ associated with every Sasakian structure, called the *characteristic foliation*. Each leaf of $\mathcal{F}_\xi$ has a holonomy group associated to it. If $\mathcal{S} = (\xi, \eta, \Phi, g)$ is a Sasakian structure, then dimension of the closure of the leaves is called the *rank* of $\mathcal{S}$.

A.2 Transverse Kähler geometry

Let (M, η, ξ, Φ, g) be a Sasakian manifold, and consider the 1-form η called the *characteristic* 1-form, then it defines a $2n$-dimensional vector bundle $\mathcal{D}$, called *horizontal subbundle* of TM, where the fiber $\mathcal{D}_p$ of $\mathcal{D}$ is defined $\mathcal{D}_p := \ker \eta_p$ for every p in M. Hence we get a decomposition of the tangent bundle TM given by

$$TM = \mathcal{D} \oplus L_\xi$$

and a sequence of vector bundle

$$0 \longrightarrow L_\xi \longrightarrow TM \xrightarrow{\pi} \mathcal{V}(\mathcal{F}_\xi) \longrightarrow 0$$

where L_ξ is the trivial line bundle generated by the Reeb vector field ξ and $\mathcal{V}(\mathcal{F}_\xi) := TM/L_\xi$ which is called the *normal bundle* of the foliation $\mathcal{F}_\xi$. There is a smooth vector bundle isomorphism $\sigma : \mathcal{V}(\mathcal{F}_\xi) \longrightarrow \mathcal{D}$ such that $\pi \circ \sigma = \mathrm{id}_{\mathcal{V}(\mathcal{F}_\xi)}$. It follows that Φ induces a splitting

$$\mathcal{D} \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{D}^{1,0} \oplus \mathcal{D}^{0,1}$$

where $\mathcal{D}^{0,1}$ and $\mathcal{D}^{1,0}$ are eigenspaces of Φ with eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. It is naturally endowed with both a complex structure $J = \Phi|_{\mathcal{D}}$ and a symplectic structure $d\eta$. Hence, $(\mathcal{D}, J, \mathrm{d}\eta)$ gives M a *transverse Kähler* structure with Kähler form $\mathrm{d}\eta$ and metric $g_{\mathcal{D}}$ defined by

$$g_{\mathcal{D}}(X, Y) = d\eta(X, JY) \tag{A.15}$$

which is related to the Sasakian metric g by

$$g = g_{\mathcal{D}} + \eta \otimes \eta, \tag{A.16}$$

Beside of that, a smooth p-form α on M is called *basic* if

$$\xi \lrcorner \alpha = 0, \quad \mathcal{L}_\xi \alpha = 0$$

Therefore, let Λ^p_B denote the sheaf of germs of basic p-forms, $\Omega^p_B := \Gamma(M, \Lambda^p_B)$ the set of global sections of Λ^p_B is called *basic* on M. Let $\mathcal{D}_{\mathbb{C}}$ denote the complexification of $\mathcal{D}$, and decompose it into its eigenspaces with respect to J, that is $\mathcal{D}_{\mathbb{C}} = \mathcal{D}^{1,0} \oplus \mathcal{D}^{0,1}$. Similarly, we get a splitting of the complexification of the sheaf Ω^1_B of basic one forms on M, namely

$$\Omega^1_B \otimes \mathbb{C} = \Omega^{1,0}_B \oplus \Omega^{0,1}_B.$$

We also denote that $\Lambda^{p,q}_B := \Lambda^p(\Lambda^{1,0}_B) \otimes \Lambda^q(\Lambda^{0,1}_B)$; and

$$\Omega^{p,q}_B := \Lambda^p(\Omega^{1,0}_B) \otimes \Lambda^q(\Omega^{0,1}_B)$$

Then we have $\Lambda^p_B \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{r+s=p} = \Lambda^{r,s}_B$, and

$$\Omega^p_B \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{r+s=p} \Omega^{r,s}_B.$$

Note that $d\eta \in \Omega^{1,1}_B(\mathcal{D})$ and it determines a non-vanishing cohomology class in $H^{1,1}_B(M)$. Relatively to the Reeb foliation on a Sasakian manifold (M, η, ξ, Φ, g), the usual Hodge star induced a *transverse Hodge operator* $\widetilde{*} : \Omega^k_B(\mathcal{D}) \to \Omega^{2n-k}_B(\mathcal{D})$ given by

$$\widetilde{*}\gamma := (-1)^{2n-k} * (\gamma \wedge \eta).$$

Therefore, we have

$$*\gamma = \widetilde{*}\gamma \wedge \eta.$$

Define ∂_B and $\overline{\partial}_B$ by

$$\partial_B : \Lambda^{p,q}_B \to \Lambda^{p+1,q}_B,$$

$$\overline{\partial_B} : \Lambda_B^{p,q} \to \Lambda_B^{p,q+1},$$

which is the decomposition of d. Let $\mathrm{d}_B = \mathrm{d}|_{\Omega_B^p}$. We hace $\mathrm{d}_B = \partial_B + \overline{\partial_B}$. Let $\mathrm{d}_B^c = \frac{1}{2}\mathrm{i}(\overline{\partial_B} - \partial_B)$. it is clear that

$$\mathrm{d}_B\mathrm{d}_B^c = \mathrm{i}\partial_B\overline{\partial_B}, \quad \mathrm{d}_B^2 = (\mathrm{d}_B^c)^2 = 0 \tag{A.17}$$

Let $d_B^* : \Omega_B^{p+1} \to \Omega_B^p$ be the adjoint operator of $\mathrm{d}_B : \Omega_B^p \to \Omega_B^{p+1}$. The basic Laplacian Δ_B is defined by

$$\Delta_B = \mathrm{d}_B^*\mathrm{d}_B + \mathrm{d}_B\mathrm{d}_B^*$$

The Sasakian structure (M, η, ξ, Φ, g) also induces a natural connection $\nabla^{\mathcal{D}}$ on $\mathcal{D}$ given by

$$\nabla_X^{\mathcal{D}} Y = \begin{cases} (\nabla_X Y)^{\mathcal{D}} & \text{if} \quad X \in \mathcal{D} \\ [\xi, Y] & \text{if} \quad X = \xi \end{cases}$$

where the subscript $\mathcal{D}$ denotes the projection onto $\mathcal{D}$. Therefore, we get

$$\nabla_X^{\mathcal{D}} J = 0, \quad \nabla_X^{\mathcal{D}} g_J = 0, \quad \nabla_X^{\mathcal{D}} d\eta = 0, \quad \nabla_X^{\mathcal{D}} Y - \nabla_Y^{\mathcal{D}} X = [X, Y]^{\mathcal{D}}$$

for any $X, Y \in TM$. Moreover the *transverse Ricci tensor* Ric^T is defined as

$$\mathrm{Ric}^T(X, Y) = \sum_{i=1}^{2n} g(\nabla_X^{\mathcal{D}}\nabla_{e_i}^{\mathcal{D}} e_i - \nabla_{e_i}^{\mathcal{D}}\nabla_X^{\mathcal{D}} e_i - \nabla_{[X,e_i]}^{\mathcal{D}} e_i, Y)$$

for any $X, Y \in \mathcal{D}$, where $\{e_1, \ldots, e_{2n}\}$ is an arbitrary orthonormal frame of $\mathcal{D}$. By Tommasini and Vezzoni in [41] is know that

$$\mathrm{Ric}^T(X, Y) = \mathrm{Ric}(X, Y) + 2g(X, Y)$$

for any $X, Y \in \mathcal{D}$, where Ric denotes the Ricci tensor of the Riemannian metric $g = g_J + \eta \otimes \eta$. Let us denote by ρ^T the Ricci form of Ric^T, i.e.

$$\rho^T(X, Y) = \mathrm{Ric}^T(JX, Y) = Ric(Jx, Y) + d\eta(X, Y)$$

for any $X, Y \in \mathcal{D}$. We have that $\rho^T$is a closed form such that $(\frac{1}{2\pi})\rho$ represents the first Chern class of $(\mathcal{D}, J)$, this form is called *transverse Ricci form* of (M, η, ξ, Φ, g).

Definition A.18. The basic cohomology class

$$\mathrm{c}_B^1(M) = \frac{1}{2\pi}[\rho^T] \in \mathrm{H}_B^{1,1}(M)$$

is called the *first basic Chern class* of (M, η, ξ, Φ, g) and if it vanishes, then (M, η, ξ, Φ, g) is said to be *null-Ssakian*.

Furthermore, a Sasakian manifold is called η-Einstein if there exist $\lambda, \quad \nu \in \mathrm{C}^\infty(M, \mathbb{R})$ such that

$$\mathrm{Ric} = \lambda g + \nu\eta \otimes \eta.$$

A.3 $\mathcal{D}$-Homothetic deformations

We now address the general framework proposed by Tanno [40] on contact manifold specifically we use properties K-contact manifolds to understand the behavior of the deformation metrics called *$\mathcal{D}$-homothetic deformation*.Thus, let (M,η,ξ,Φ,g) be a K-contact manifold and $\overline{g}$ the deformation of the metric g given by

$$\overline{g} = \alpha g + \beta\eta\otimes\eta, \tag{A.19}$$

for constant α and β satisfying $\alpha > 0$ and $\alpha+\beta > 0$. The inverse matrix

$$\overline{g}^{ij} = \alpha^{-1}g^{ij} - \alpha^{-1}\beta(\alpha+\beta)^{-1}\xi^i\xi^j. \tag{A.20}$$

Lemma A.21. *Let (M,η,ξ,Φ,g)be a K-contact manifold and let $\overline{g}$ be a deformation of the metric given by (A.19). If F is a symmetric tensor of the form $F = \mathrm{s}_1 g + \mathrm{s}_2\eta\otimes\eta$, then*

$$|F|^2_{\overline{g}} = F_{ij}F_{mn}\overline{g}^{im}\overline{g}^{jn} = 6\alpha^{-2}\mathrm{s}_1^2 + (s_1+s_2)^2(\alpha+\beta)^{-2}. \tag{A.22}$$

Proof. Using (A.20) and $F = s_1 g + s_2\eta\otimes\eta$, we obtain

$$\begin{aligned}
|F|^2_{\overline{g}} =& F_{ij}F_{mn}\overline{g}^{im}\overline{g}^{jn}\\
=& s_1^2\alpha^{-2}(7 - 2\beta(\alpha+\beta)^{-1} + \beta^2(\alpha+\beta)^{-2})\\
&+ 2s_1s_2\alpha^{-2}(1-\beta(\alpha+\beta)^{-1})^2 + s_2^2\alpha^{-2}(1-\beta(\alpha+\beta)^{-1})^2\\
=& s_1^2\alpha^{-2}(6 + (1-\beta(\alpha+\beta)^{-1})^2)\\
&+ 2s_1s_2\alpha^{-2}(1-\beta(\alpha+\beta)^{-1})^2 + s_2^2\alpha^{-2}(1-\beta(\alpha+\beta)^{-1})^2\\
=& 6\alpha^{-2}\mathrm{s}_1^2 + (s_1+s_2)^2(\alpha+\beta)^{-2}.
\end{aligned}$$

□

Denoting by $A = \overline{\nabla} - \nabla$ the difference of the connections associated to $\overline{g}$ and g, which is a tensor. Therefore, the tensor A on a K-contact manifold satisfies

$$A^i_{jk} = -\alpha^{-1}\beta(\Phi^i_j\eta_k + \eta_j\Phi^i_k). \tag{A.23}$$

Putting this into

$$\overline{R}^i_{jkl} = R^i_{jkl} + \nabla_l A^i_{jk} - \nabla_k A^i_{jl} + A^i_{rl}A_{jk} - A^i_{rk}A^r_{jl},$$

we have

$$\begin{aligned}
\overline{R}^i_{jkl} =& R^i_{jkl} + \alpha^{-1}\beta(2\Phi^i_j\Phi_{kl} + \Phi^i_k\phi_{jl} - \Phi^i_l\Phi_{jk}) + \alpha^{-2}\beta^2(\delta^i_l\eta_j\eta_k - \delta^i_k\eta_j\eta_l)\\
&+ \alpha^{-1}\beta\Big(\nabla_k\Phi^i_j\eta_l + \nabla_k\Phi^i_l\eta_j - \nabla_l\Phi^i_j\eta_k - \nabla_l\Phi^i_k\eta_j\Big).
\end{aligned}$$

In the case of Sasakian manifolds, we obtain

$$\begin{aligned}
\overline{R}^i_{jkl} =& R^i_{jkl} + \alpha^{-1}\beta(\xi^i\eta_l g_{jk} - \eta_k\xi^i g_{jl} + \alpha^{-1}(2\alpha+\beta)\delta^i_l\eta_j\eta_k - \alpha^{-1}(2\alpha+\beta)\delta^i_k\eta_j\eta_k)\\
&+ \alpha^{-1}\beta(2\Phi^i_j\Phi_{kl} + \Phi^i_k\Phi_{jl} - \Phi^i_l\Phi_{jk}),
\end{aligned} \tag{A.24}$$

where we have used (A.13) and (A.14). Contracting with respect to i and l, we have

$$\overline{R}_{jk} = R_{jk} - 2\alpha^{-1}\beta g_{jk} + \alpha^{-2}\beta(2(2n+1)\alpha + (2n+1)\beta - \beta)\eta_j\eta_k, \tag{A.25}$$

Contracting the last equation with (A.20), we get

$$\overline{R} = \alpha^{-1}R - \alpha^{-2}\beta(2n). \tag{A.26}$$

where $\overline{R}$ is the scalar curvature.

Remark A.27. Note that if $X \in \mathcal{D}$, then $(\overline{\nabla}_X\eta)(Y) = (\nabla_X\eta)(Y) + A(X,Y)$, and using $(\nabla_X\eta)(Y) = \frac{1}{2}\mathrm{d}\eta(X,Y)$ and (A.23), we obatin $((\overline{\nabla}_X\eta)(Y))^\sharp \in \mathcal{D}_0$.

Lemma A.28. *Let (M,η,ξ,Φ,g) be a K-contact manifold and let $\overline{g}$ be a deformation of the metric given by* (A.19)*. If F is a symmetric tensor of the form $F = \mathrm{s}_1 g + \mathrm{s}_2\eta\otimes\eta$, then*

$$(\overline{\nabla}_X F)(Y,Z) = s_2(\nabla_X(\eta\otimes\eta))(Y,Z) - s_1 g(A(X,Y),Z) - s_1 g(A(X,Z),Y). \tag{A.29}$$

Proof. we know that $\nabla g = 0$ and $\nabla_\xi\xi = 0$. Using the Levi-Civita connection of $\overline{g}$, we have

$$\overline{\nabla}_X F(Y,Z) = X(F(Y,Z)) - F(\overline{\nabla}_X Y, Z) - F(Y, \overline{\nabla}_X Z),$$

substituting $\overline{\nabla} = \nabla + A$ where A is given by (A.23), we obtain that

$$\overline{\nabla}_X F(Y,Z) = X(F(Y,Z)) - F(\nabla_X Y + A(X,Y), Z) - F(Y, \nabla_X Z + A(X,Z)),$$

using $F = s_1 g + s_2\eta\otimes\eta$, we have

$$\begin{aligned}\overline{\nabla}_X F(Y,Z) =& X(s_1 g(Y,Z)) + X(s_2\eta(Y)\eta(Z)) - s_1 g(\nabla_X Y, Z) - s_1 g(A(X,Y),Z)\\ &- s_2\eta(\nabla_X Y)\eta(Z) - s_1 g(Y,\nabla_X Z) - s_1 g(Y, A(X,Z)) - s_2\eta(\nabla_X Z)\eta(Y)\\ =& s_1(\nabla_X g)(Y,Z) + s_2\nabla_X(\eta\otimes\eta)(Y,Z) - s_1 g(A(X,Y),Z) - S_1 g(A(X,Z),Y)\\ =& s_2\nabla_X(\eta\otimes\eta)(Y,Z) - s_1 g(A(X,Y),Z) - s_1 g(A(X,Z),Y).\end{aligned}$$

□

A.4 Contact Calabi-Yau manifold

In this section we discuss several preliminary properties of the contact-Calabi-Yau manifolds which will be important to analyse singularities of Laplacian coflow of G_2-structures on contact Calabi-Yau manifolds.

Definition A.30. A *contact Calabi-Yau manifolds* is denoted by (M,η,Φ,Υ) such that

- (M,η,ξ,Φ,g) is a $2n+1$-dimensional Sasakian manifold.

- Υ is a nowhere vanishing transversal form on $\mathcal{D} = \ker\eta$ of type $(n, 0)$:

$$\Upsilon \wedge \overline{\Upsilon} = c_n \omega^n, \quad d\Upsilon = 0,$$

where $c_n = (-1)^{\frac{n(n+1)}{2}} \mathrm{i}^n$ and $\omega = d\eta$. Furthermore, we have that

$$\mathrm{Re}\,\Upsilon := \frac{\Upsilon + \overline{\Upsilon}}{2}, \quad \mathrm{Im}\,\Upsilon := \frac{\Upsilon - \overline{\Upsilon}}{2\mathrm{i}}.$$

Proposition A.31. *[41, §3, Proposition 3.4] Let M be a $2n+1$-dimensional compact manifold. Assume that M admits a Contact Calabi-Yau structure, then the following hold*

1. *if n is even, then $\mathrm{b}_{n+1}(M) > 0$;*
2. *If n is odd, then*

$$\begin{cases} \mathrm{b}_n(M) > 2 \\ \mathrm{b}_{n+1} \geq 2, \end{cases}$$

where $\mathrm{b}_j(M)$ denotes the j^{th} Betti number of M.

Corollary A.32. *[41, §3, Corollary 3.5] A 3-dimensional compact manifold M admitting contact Calabi-Yau structure has $\mathrm{b}_1(M) > 2$. In particular, there are no compact 3-dimensional simply connected contact Calabi-Yau manifolds. Moreover, the $2n+1$-dimensional sphere has no contact Calabi-Yau structure.*

Proposition A.33. *[41, §3, Proposition 3.6] Let (M, η, ξ, Φ, g) be a $2n+1$-dimensional Sasakian manifold and $\mathcal{D} = \ker\eta$. The following facts are equivalent:*

1. $\mathrm{Hol}^0(\nabla^{\mathcal{D}}) \subset \mathrm{SU}(n)$.
2. $\mathrm{Ric}^T = 0$.

Corollary A.34. *[41, §3, Corollary 3.7] Let $(M, \eta, \Phi, \Upsilon)$ be a contact Calabi-Yau manifold. Then (M, η, ξ, Φ, g) is a null-Sasakian and the metric g induced by (η, Φ) is a η-Einstein with $\lambda = 2$ and $\nu = 2n+2$. In particular the scalar curvature of the metric g associated to (η, Φ) is equal to $2n-1$.*

We consider the following two results arising from section [22] which are the following:

Proposition A.35. *[22, §6, Proposition 6.5] Let (M, η, ξ, Φ, g) be a compact simply-connected null Sasakian η-Einstein manifold. Then $\mathrm{Hol}(\nabla)$ is contained in $\mathrm{SU}(n)$.*

Proposition A.36. *[22, §6, Proposition 6.6] Let (M, η, ξ, Φ, g) be a compact simply-connected Sasakian manifold with $\mathrm{c}^1_B(M) = 0$. Then there exist a unique Sasakian structure $(M, \eta', \xi', \Phi', g')$ and a basic 1-form ζ on M such that*

$$\xi' = \xi, \quad \eta' = \eta + \zeta, \quad \Phi' = \Phi - \xi \otimes \zeta \circ \Phi, \quad g' = d\eta' \circ (\mathrm{id} \otimes \Phi') + \eta' \otimes \eta',$$

and the transverse holonomy group of the metric g' is contained in $\mathrm{SU}(n)$.

APPENDIX B – 3-Sasakian manifolds

Definition B.1. Let M be a manifold with a family of almost contact structures $\{\xi(\mathbf{r}), \eta(\mathbf{r}), \Phi(\mathbf{r})\}$ parameterized by points $\mathbf{r} = (\mathrm{r}_1, \mathrm{r}_2, \mathrm{r}_3) \in S^2 \subset \mathbb{R}^3$ on a unit sphere. We say $\{\xi(\mathbf{r}), \eta(\mathbf{r}), \Phi(\mathbf{r})\}_{\mathbf{r}\in S^2}$ is an *almost hypercontact structure* on M if

$$\Phi(\mathbf{r}) \circ \Phi(\mathbf{r}^{'}) - \eta(\mathbf{r}) \otimes \xi(\mathbf{r}^{'}) = -\Phi(\mathbf{r} \times \mathbf{r}^{'}) - (\mathbf{r} \cdot \mathbf{r}^{'})\mathbf{1}$$

$$\Phi(\mathbf{r})\xi(\mathbf{r}^1) = -\xi(\mathbf{r} \times \mathbf{r}^{'}), \quad \eta(\mathbf{r}) \circ \Phi(\mathbf{r}^{'}) = -\eta(\mathbf{r} \times \mathbf{r}^{'})$$

For any two $\mathbf{r}, \mathbf{r}^{'} \in S^2$. Furthermore, a Riemannian metric g on M is said to be a *compatible* with (or *associated* to) the almost hypercontact structure if

$$g(\Phi(\mathbf{r})X, \Phi(\mathbf{r})Y) = g(X, Y) - \eta(\mathbf{r})(X)\eta(\mathbf{r})(Y)$$

for all $\mathbf{r} \in S^2$. In such a case $\{\xi(\mathbf{r}), \eta(\mathbf{r}), \Phi(\mathbf{r}), g\}_{\mathbf{r}\in S^2}$ is called an *almost hypercontact meteric structure* on M.

We can recover the standard definition of the almost contact (metric) 3-structure via a choice of an arbitrary orthonormal basis $\{e_1, e_2, e_3\}$ in $\mathbb{R}^3$ and setting $\Phi(e_a) = \Phi_a$, $\xi(e_a) = \xi_a$ and $\eta(e_a) = \eta_a$. Hence,

Definition B.2. An almost contact structure (metric) 3-structure $\{\xi_a, \eta_a, \Phi_a, g\}_{a=1}^3$ on M is an almost hypercontact (metric) structure together with a choice of an orthonormal frame on $\mathbb{R}^3$.

Proposition B.3. *[4, §13, Proposition 14.1.4] There is a one to one correspondence between almost hypercontact metric structure* $\{\xi(\mathbf{r}), \eta(\mathbf{r}), \Phi(\mathbf{r}), g\}_{\mathbf{r}\in S^2}$ *on M and reductions of the frame bundle to the group* $\mathrm{Sp}(n) \times \mathrm{id}_3$

Proposition B.4. *[4, §13, Proposition 13.1.6] Let M^{4n+3} be a compact manifold with an almost contact 3-structure. Then the only possible non-vanishing Stiefel-Whitney classes are $w_{4i}(M)$. In particular, M is a spin manifold. Furthermore, all the integral Stiefel-Whitney classes $W_i(M)$ must vanish.*

Proposition B.5. *[4, §13, Theorem 13.1.7] Let M be a compact smooth 7-manifold and let* $\mathrm{SU}(2) \subset \mathrm{SU}(3) \subset \mathrm{G}_2 \subset \mathrm{Spin}(7)$, *where the subgroup of* $\mathrm{Spin}(7)$ *are the stability subgroup fixing one, two, or three spinors, respectively. Then the following condition are equivalent:*

(i) M admits a reduction of the structure group to $\mathrm{SU}(2)$,

(ii) M admits a reduction of the structure group to $\mathrm{SU}(3)$,

(iii) M admits a reduction of the structure group G_2,

(iv) $w_1(M) = w_2(M) = 0$*.i.e., M is orientable and admits a spin structure.*

In particular, every compact oriented spin 7-manifold admits an almost hypercontact structure.

Proposition B.6. *[4, §13, Theorem 13.4.6] Let M^{4n+3} be a 3-Sasakian homogeneous space. Then $M = G/H$ is precisely one of the following:*

$$\frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)}, \quad \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)\times\mathbb{Z}_2}, \quad \frac{\mathrm{SU}(m)}{S(\mathrm{U}(m-2)\times\mathrm{U}(1))}, \quad \frac{\mathrm{SO}(k)}{\mathrm{SO}(k-4)\otimes\mathrm{Sp}(1)}$$
$$\frac{\mathrm{G}_2}{\mathrm{Sp}(1)}, \quad \frac{F_4}{\mathrm{Sp}(3)}, \quad \frac{E_6}{\mathrm{SU}(6)}, \quad \frac{E_7}{spin(12)}, \quad \frac{E_8}{E_7}.$$

Here $n \geq 0$, $\mathrm{Sp}(0)$ denote the trivial group, $m \geq 3$ and $k \geq 7$. Hence, there is one-to-one correspondence between the simple Lie algebra and the simply connected 3-Sasakian homogeneous manifolds.

APPENDIX C – Collapsing manifolds

The compactness of solutions to geometric and analytic equations, when it is true, is fundamental in the study of geometric analysis. In this chapter, we state Hamilton's compactness theorem for solutions of the Ricci flow assuming Cheeger and Gromov's compactness theorem for Riemannian manifolds with bounded geometry. Throughout this chapter, quantities depending on the metric g_k (or $g_k(t)$) will have a subscript k, for instance ∇_k and Rm_k denote the Riemannian connection and Riemannian curvature tensor of g_k. Quantities without a subscript depend on the background metric g.

Let (M^n, g) be a complete Riemannian manifold

Definition C.1. Given $\rho \in (0, \infty]$ and $k > 0$, we say that the metric g is k-noncollapsed below the scale ρ if for any metric ball $B(x, r)$ with $r < \rho$ satisfying $|Rm(y)| \leq r^{-2}$ for all $y \in B(x, r)$ and

$$\frac{\operatorname{vol} B(x, r)}{r^n} \geq k. \tag{C.2}$$

If g is k-noncollapsed below the scale ∞, we say that g is k-noncollapsed at all scales.

Definition C.3. We say that g is *k-collapsed at the scale r at the point x* if $|Rm| \leq r^{-2}$ for all $y \in B(x, r)$ and

$$\frac{\operatorname{vol} B(x, r)}{r^n} < k \tag{C.4}$$

The metric g is said to be k-collapsed at the scale r if there exist $x \in M$ such that g is k-collapsed at the scale r at the point x.

Thus g is not k-noncollapsed below the scale ρ if and only if there exists $r < \rho$ and $x \in M$ such that g is k-collapsed at the scale r at the point x.

Remark C.5.
1. If M is closed and flat, then g cannot be k-noncollapsed at all scales since $|Rm| = 0 \leq r^{-2}$ for all r and $\operatorname{vol} B(x, r) \leq \operatorname{vol}(M)$ so that $\lim_{r\to\infty} \frac{\operatorname{vol} B(x, r)}{r^n} = 0$ for all $x \in M$.
2. If (M, g) is a closed Riemannian manifold, then for any $\rho > 0$ there exist $k > 0$ such that g is k-noncollapsing below the scale ρ.

we have the following elementary scaling propety for k-noncollased metrics.

Lemma C.6. *[13] [*Scaling property of k-noncollapsed*] If a metric g is k-noncollapsed below the scale ρ, then for any $\alpha > 0$ the metric $\alpha^2 g$ is k-noncollapsed below the scale $\alpha\rho$.*

In [13] was showed that k-noncollapsing and a lower bound of a injectivity radius are equivalent.

Lemma C.7. *[13] Let (M^n, g) be a complete Riemannian manifold and fix $\rho \in (0, \infty]$.*

1. *If the metric g is not k-collapsed $\delta = \delta(n, k)$ which is independent of ρ and g such that for any $x \in M$ and $r < \rho$, if $|Rm| \leq r^{-2}$ in $B(x, r)$, then* $\mathrm{inj}(x) \geq \delta r$.

2. *Suppose that for any $x \in M$ and $r < \rho$ with $|Rm| \leq r^{-2}$ in $B(x, r)$ we have* $\mathrm{inj}(x) \geq \delta r$ *for some $\delta > 0$. Then there exist a constant $k = k(n, \delta)$, independent of ρ and g, such that g is not k-collapsed below the scale ρ.*

The property of being k-noncollapsed below the scale ρ is preserved under pointed Cheeger-Gromov limits. We start giving some definitions of convergence of manifolds and Cheeger-Gromov limits. Then we review the definition of C^∞-convergence on compact set in a smooth manifold M^n.

Definition C.8 (C^p-convergence)**.** Let $K \subset M$ be a compact set and $\{g_k\}_{k \in \mathbf{N}}$, g_∞ and g be a Riemannian metrics on M. For $p \in \{0\} \cup \mathbf{N}$ we say that g_k *converges in C^p to g_∞ uniformly on* K if for every $\varepsilon > 0$ there exist $k_0 = k_0(\varepsilon)$ such that for $k \geq k_0$,

$$\sup_{0 \leq \alpha \leq p} \sup_{x \in K} |\nabla^\alpha (g_k - g_\infty)|_g < \varepsilon, \tag{C.9}$$

where the covariant derivative ∇ is with respect to g.

Note that since we are on a compact set, the choice of a metric g on K does not affect the convergence. For instance, we may choose $g = g_\infty$. In regards to C^∞-convergence on mannifolds, with the noncompact case in mind, we have tge following. We say that a sequence of open sets $\{U_k\}_{k \in \mathbf{N}}$ in a manifold M^n is an *exhaustion of M by open sets* if for any compact set $K \subset M$ there exist $k_0 \in \mathbf{N}$ such that $K \subset U_k$ for all $k \geq k_0$.

Definition C.10 (C^∞-convergence uniformly on compact sets)**.** Suppose $\{U_k\}_{k \in \mathbf{N}}$ is an exhaustion of a smooth manifold M^n by open sets and g_k are Riemannian metrics on U_k. We say that (U_k, g_k) *converges in C^∞ to (M, g_∞) uniformly on compact sets in M* if for any compact set $K \subset M$ and any $p > 0$ there exist $k_0 = k_0(K, p)$ such that $\{g_k\}_{k \geq k_0}$ converges in C^p to g_∞ uniformly on K.

Definition C.11. A sequence $\{(M_k^n, g_k, O_k)\}_{k \in \mathbf{N}}$ of complete pointed Riemannian manifold *converges* to complete pointed Riemannian manifold $\{(M_\infty, g_\infty, O_\infty)\}$ if there exist

1. An exhaustion $\{U_k\}_{k \in \mathbf{N}}$ of M_∞ by open sets with $O_\infty \in U_k$ and

2. a sequence of diffeormophism $\beta_k : U_k \to V_k := \beta_k(U_k) \subset M_k$ with $\beta(O_\infty) = O_k$

such that $\left(U_k, \beta_k^*[g_k|_{V_k}]\right)$ converges in C^∞ to (M_∞, g_∞) uniformly on compact sets in M_∞.

When there is a bound on the curvature (recall that when given a singular solution and a suitable sequence of space-time points, the choice of dilation factors is chosen to guarantee this for the associated sequence of solutions) and an injectivity radius estimate for a sequence of solutions for example in the Ricci flow or our case of G_2–structures is using for Ricci-like flow, which we explain later. Then, the following compactness theorem provides a subsequence which will converge in the C^∞-Cheeger-Gromov sense. Thus, we give a definition which is related to the assumption of bounded curvature.

Definition C.12 (Bounded gemetry)**.** we say that a sequence or family of Riemannian manifolds has *bounded geometry* if there exist positive constant C_p such that

$$|\nabla^p Rm| \leq C_p$$

for all $p \in \mathbf{N} \cup \{0\}$ and for all metrics in this sequence or family. That is the curvature and their covariant derivatives of each order have uniform bounds.

Let $\text{inj}_g(\mathcal{O})$ denote the injectivity radius of the metric g at the point $\mathcal{O}$. for sequences of Riemannian manifolds we have the following convergence theorems.

Theorem C.13. *[13][Compactness for metrics] Let $\{(M_k^n, g_k, \mathcal{O}_k)\}_{k\in\mathbf{N}}$ be a sequence of complete pointed Riemannian manifolds that satisfy*

1. *(*uniformly bounded geometry*)*

$$|\nabla_k^p Rm|_k \leq C_p \quad on \quad M_k$$

for all $p \leq 0$ and k where $C_p < \infty$ is a sequence of constants independent of k and

2. *(*injectivity radius estimate*)*

$$\text{inj}_{g_k}(\mathcal{O}_k) \geq i_0 \tag{C.14}$$

for some constant $i_0 > 0$

Then there exist a subsequence $\{j_k\}_{k\in\mathbf{N}}$ such that $\{(M_{j_k}.g_{j_k}, \mathcal{O}_{j_k})\}_{k\in\mathbf{N}}$ converges to a complete pointed Riemannian manifold $(M_\infty, g_\infty, \mathcal{O}_\infty)$ as $k \to \infty$.

So, the next lemma says the property of being k-noncollapsed below the scale ρ is preserved (stable) under pointed Cheeger-Gromov limits.

Lemma C.15. *[13] Let $\{(M_k^n, g_k, \mathcal{O}_k)\}$ be a sequence of pointed complete Riemannian manifolds. Suppose that there exist $k > 0$ and $\rho > 0$ so that each (M_k, g_k) is k-noncollapsed below the scale ρ. Furthermore assume that $(M_k, g_k, \mathcal{O}_k)$ convergence to $(M_\infty, g_\infty, \mathcal{O}_\infty)$ in the pointed Cheeger-Gromov C^2-topology. Then the limit (M_∞, g_∞) is k-noncollapsed below the scale ρ.*

To apply Cheeger-Gromov-type compactness theorem to study time singularities, the no local collapsing theorem is evident useful. Before of that, we first give the following relation between volume ratios and injectivity radius in the presence of a curvature bound

Theorem C.16 (Cheeguer-Gromov-Taylor[13])**.** *For any constant $c > 0$, $r_0 > 0$, there exist a constant $i_0 > 0$ such that if (M^n, g) a n-dimensional compact manifold is a complete Riemannian manifold with $|\text{sect}| \leq 1$ and if $p \in M$ is a point with*

$$\frac{\text{vol}(B(p,r))}{r^n} \geq c$$

for all $r \in (0, r_0]$, *then*

$$\mathrm{inj}(p) \geq i_0$$

Remark C.17. Since $|\mathrm{sect}| \leq 1$, by the Bishop-Gromov comparison theorem, if $\dfrac{\mathrm{vol}(B(p, r_0))}{r_0^n} \geq c^{'} > 0$, then $\dfrac{\mathrm{vol}(B(p, r))}{r^n} \geq c$ for all $r \in (0, r_0]$, where c depends only on $c^{'}$ and n (see [13]).

Definition C.18. The Riemannian metric g on M^n is said to be k-non-collapsing relative to upper bound of scalar curvature on the scale ρ if for any $B_g(p, r) \subset M$ with $r < \rho$ such that $\sup\limits_{B_g(p,r)} R_g \leq r^{-2}$, we have $\mathrm{vol}_g B_g(p, r) \geq kr^n$.

Definition C.19. we say that $\tilde{g}$ is k-collapsed at the scale r at the point x if $|Rm(y)| \leq r^{-2}$ for all $y \in B(x, r)$ and

$$\frac{vol(B(x, r))}{r^n} \leq K$$