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Theoretical and practical investigations on fuzzy relational systems

Investigações teóricas e práticas sobre sistemas relacionais fuzzy

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Investigações teóricas e práticas sobre sistemas relacionais fuzzy

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Resumo

O sistema relacional *fuzzy* proposto por Mamdani e Assilian em 1975 se popularizou devido à sua fácil aplicação em problemas práticos e à interpretação gráfica bastante intuitiva. Todavia, também é possível construir sistemas relacionais *fuzzy* baseados em implicações *fuzzy*, que possuem uma interpretação lógica diferente do sistema de Mamdani. Neste trabalho será feito um estudo de ambos os sistemas relacionais e duas representações de uma base de regras *fuzzy*: conjuntiva e implicativa. Uma revisão dos trabalhos de Martin Stepnicka, Bernhard Moser e Mirko Navara também será feita, onde foram estabelecidos uma série de critérios analíticos para avaliar as combinações de composições de relações *fuzzy* e bases de regras *fuzzy*. Por fim, avaliamos a aplicação dos sistemas relacionais em alguns problemas de regressão que são um referencial dentro da área.

Palavras-chave: Sistemas relacionais *fuzzy*. Aprendizado de regras *fuzzy*. Regressão.

Abstract

The fuzzy relational system proposed by Mamdani and Assilian in 1975 became popular due to its ease of application in practical problems and intuitive graphical interpretation. Nevertheless, it is also possible to build fuzzy relational systems based on fuzzy implications, which have a different logic interpretation from the one of the Mamdani system. In this work, we will study both relational systems and two representations of fuzzy rule bases: conjunctive and implicative. We will also discuss the works of Martin Stepnicka, Bernhard Moser and Mirko Navara, who established a series of analytical criteria to evaluate such combinations of compositions of fuzzy relations and fuzzy rule bases. Finally, we evaluate the performance of the relational systems applied in some benchmark regression problems.

Keywords: Fuzzy relational systems. Fuzzy rules learning. Regression.

List of abbreviations and acronyms

FRS	Fuzzy Relational System
CRI	Compositional Rule Inference
BKS	Bandler-Kohout Subproduct
RMSE	Root Mean Square Error
MOM	Mean Of Maximum

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1 Introduction

Fuzzy relational systems (FRSs) have been used in many applications during the last decades, with special emphasis on control problems. Their wide range of applications (NGUYEN; WALKER; WALKER, 2018) (BARROS; BASSANEZI; LODWICK, 2017) (PEDRYCZ; GOMIDE, 2007) (ROSS, 2005) (KLIR; YUAN, 1995) (MOURA; SUSSNER, 2018) emphasizes the importance of studying FRSs.

The fuzzy relational system proposed by Mamdani and Assilian (MAMDANI; ASSILIAN, 1975) gained popularity due to its ease of implementation and low computational cost, allowing it to be used in microcontrollers with small processing capacity. However, despite its widespread use, the FRS proposed by Mamdani-Assilian has a different interpretation than we are used to from the point of view of propositional logic, as will be seen in Chapter 3.

This work proposes to explore other relational systems, which have highly desirable characteristics such as the interpolability and significance of outputs generated. It is mostly motivated by the investigations of Martin Stepnicka (STEPNICKA; MANDAL, 2015) (STEPNICKA, 2016) (STEPNICKA; MANDAL, 2018) on the suitability of FRSs as inference systems.

In Chapter 2 we review the mathematical concepts relevant to theory of the fuzzy sets, which will be necessary for further development of the topics covered.

In Chapter 3, we define the theory of the fuzzy relational systems and develop some propositions to deepen the study of these systems. This study is accompanied by numerical examples and visualizations to aid understanding. Then, in Chapter 4 we discuss the axioms proposed by Moser and Navara (MOSER; NAVARA, 2002), as well as the developments made by Stepnicka (STEPNICKA; MANDAL, 2015) (STEPNICKA, 2016) (STEPNICKA; MANDAL, 2018), for evaluation of FRSs according to analytical criteria, and also the discussion some properties of these relational systems.

In Chapter 5 we recall the Wang-Mendel algorithm for generating fuzzy rules proposed in (WANG; MENDEL, 1992). This algorithm is applied to a series of benchmark regression problems and the rules generated are used to compose the relational systems discussed in Chapter 3. Finally, the results presented by the FRSs are compared with each other.

In Chapter 6, we make some conclusions about the discussed subjects and indications for possible future work.

2 Mathematical Concepts

In this chapter, there will be presented the basic concepts of fuzzy set theory and how they can be used to extend the operators of classical propositional logic giving rise to the known fuzzy logic.

2.1 Basic concepts of fuzzy set theory

Lofti A. Zadeh introduced the concept of a fuzzy set in 1965 (ZADEH, 1965) as an extension of the classical notion of a set, the latter being called by *crisp* sets. A fuzzy set A is defined in a universe X and represented by a membership function $\mu_A : X \to [0, 1]$. In the rest of the work, we will indistinctly adopt the notations $A(x)$ and $\mu_A(x)$ for the membership function of the set A, without risk of misunderstanding.

The membership function indicates the degree of membership of an element $x \in X$ to the set A, where $\mu_A(x) = 1$ represents the element's total membership to the set and $\mu_A(x) = 0$ represents the element not belonging to the set. The collection of all fuzzy sets defined in the universe X is represented by the symbol $\mathcal{F}(X)$. If there is at least one $x \in X$ such that $\mu_A(x) = 1$, we say that the fuzzy set A is normal. The support of a fuzzy set A, denoted by $Supp(A)$, is given by the set $\{x \in X | \mu_A(x) > 0\}$.

Thus, it is possible to view the class of *crisp* sets as a particular case of fuzzy sets. That is, given a *crisp* set A, this set can be seen as a fuzzy set where $\mu_A(x)$ only takes values in $\{0, 1\}$. Finally, we define the fuzzy sets $X \in \mathcal{F}(X)$ (universe) and $\varnothing \in \mathcal{F}(X)$ (empty), such that $\mu_X(x) = 1$ and $\mu_\varnothing(x) = 0$ for all $x \in X$.

2.2 Operations on fuzzy sets

Using this idea, Boolean logical operators can be extended giving rise to a family of operators between fuzzy sets (KLIR; YUAN, 1995).

Definition 1. *A fuzzy conjunction $C(x, y)$ is a function $C : [0, 1] \times [0, 1] \to [0, 1]$, increasing in both arguments and which satisfies the following truth table:*

x	y	$C(x, y)$
0	0	0
1	0	0
0	1	0
1	1	1

Table 1 – Truth table of a fuzzy conjunction

Definition 2. *A fuzzy conjunction C is called a t-norm if it obeys the following properties for $x, y, z \in [0, 1]$:*

1. *Commutativity:* $C(x, y) = C(y, x)$
2. *Associativity:* $C(x, C(y, z)) = C(C(x, y), z)$
3. *Neutral element:* $C(1, x) = x$
4. *Monotonicity: if* $x \leqslant y \Rightarrow C(x, z) \leqslant C(y, z)$

In the remaining of the work we will adopt the notation $x * y$ to represent a t-norm. Some examples of t-norms are:

- Mínimum: $x *_M y = x \wedge y = min\{x, y\}$
- Product: $x *_P y = x \cdot y$
- Lukasiewicz: $x *_L y = 0 \vee (x + y - 1)$
- Drastic Product: $x *_D y = \begin{cases} x, & \text{if } y = 1 \\ y, & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases}$

Lemma 1. *(KLIR; YUAN, 1995) Let $*$ be a t-norm, then for any $x, y \in [0, 1]$ we have that*

$$x *_D y \leqslant x * y \leqslant x *_M y. \tag{2.1}$$

A t-norm is said to be left continuous (KLEMENT; MESIAR; PAP, 2000) if, for a given $y \in [0, 1]$ and any non-decreasing sequence $\{x_n\}_{n \in \mathbb{N}}$, we have

$$\lim_{n \to \infty} (x_n * y) = (\lim_{n \to \infty} x_n) * y. \tag{2.2}$$

Another category of t-norms that will be important for our study are t-norms without zero divisors. A t-norm $*$ is said to be without zero divisors (KLEMENT; MESIAR; PAP, 2000) if it meets the following condition:

$$x * y = 0 \Rightarrow x = 0 \text{ or } y = 0.$$

Definition 3. *A fuzzy disjunction $D(x, y)$ is a function $D : [0, 1] \times [0, 1] \to [0, 1]$ increasing in both arguments and which satisfies the following truth table:*

x	y	$D(x,y)$
0	0	0
1	0	1
0	1	1
1	1	1

Table 2 – Truth table of a fuzzy disjunction

Definition 4. *A fuzzy disjunction is called a t-conorm (or s-norm) if it obeys the following properties for $x, y, z \in [0,1]$:*

1. *Commutativity:* $D(x,y) = D(y,x)$
2. *Associativity:* $D(x, D(y,z)) = D(D(x,y), z)$
3. *Neutral element:* $D(0,x) = x$
4. *Monotonicity: if* $x \leqslant y \Rightarrow D(x,z) \leqslant D(y,z)$

We will use the symbol xSy to denote a t-conorm. Some examples of t-conorms are:

- Maximum: $xS_My = x \vee y = max\{x,y\}$
- Probabilistic Sum: $xS_Py = x + y - x \cdot y$
- Lukasiewicz: $xS_Ly = 1 \wedge (x+y)$
- Drastic Sum: $xS_Dy = \begin{cases} x, & \text{if } y = 0 \\ y, & \text{if } x = 0 \\ 1, & \text{otherwise} \end{cases}$

Lemma 2. *(KLIR; YUAN, 1995) Let S be a t-conorm, then for any $x, y \in [0,1]$ we have that*

$$xS_My \leqslant xSy \leqslant xS_Dy. \tag{2.3}$$

Definition 5. *A fuzzy implication $I(x,y)$ is a function $I : [0,1] \times [0,1] \to [0,1]$, descending on the first argument and increasing in the second argument, which satisfies the following truth table:*

x	y	$I(x,y)$
0	0	1
1	0	0
0	1	1
1	1	1

Table 3 – Truth table of a fuzzy implication

A special class of fuzzy implications are the fuzzy residual implications (R-implications), they are built from a t-norm in the following way:

$$x \to y = \bigvee \{z \in [0,1] | x * z \leqslant y\}, \tag{2.4}$$

where $*$ is a left-continuous t-norm and $x, y \in [0,1]$. An obvious property of the fuzzy residual implications is the fact that $x \to y = 1$ always that $x \leqslant y$ since for any t-norm $x * 1 = x$ by the property of the neutral element. In this work we will focus exclusively on this class of fuzzy implications. Some examples of fuzzy residual implications are:

- Gödel: $x \to_M y = \begin{cases} 1, & \text{if } x \leqslant y \\ y, & \text{if } x > y \end{cases}$
- Goguen: $x \to_P y = \begin{cases} 1, & \text{if } x \leqslant y \\ \frac{y}{x}, & \text{if } x > y \end{cases}$
- Lukasiewicz: $x \to_L y = 1 \wedge (y - x + 1)$

Lemma 3. *(KLIR; YUAN, 1995) Let $\to$ be a fuzzy residual implication, then for any $x, y \in [0,1]$ we have that*

$$x \to_M y \leqslant x \to y. \tag{2.5}$$

2.3 Crisp and fuzzy relations

The classical concept of relation between sets is given by a subset of the Cartesian product $X \times Y$, where X and Y are any universe sets. In other words, a relation $R(x, y)$ with $x \in X$ and $y \in Y$ can be seen as a map $R : X \times Y \to \{0,1\}$. More formally, we will use the following definition for a crisp binary relation over fuzzy sets:

Definition 6. *Given two fuzzy sets $A \in \mathcal{F}(X)$ and $B \in \mathcal{F}(Y)$ and a binary operator $\sim: [0,1] \times [0,1] \to \{0,1\}$, we define a binary relation $R_{A\sim B}$ as the set of ordered pairs (x, y) so that*

$$(x, y) \in R_{A\sim B} \Leftrightarrow A(x) \sim B(y). \tag{2.6}$$

Using this definition of crisp binary relation, we will introduce a notation for selecting subsets of the space X.

Definition 7. *Let $A \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$, a binary operator $\sim: [0,1] \times [0,1] \to \{0,1\}$ and $y \in Y$ be arbitrary. The set $X_{A\sim B(y)}$ is given by*

$$X_{A\sim B(y)} = \{x \in X | (x, y) \in R_{A\sim B}\}. \tag{2.7}$$

Let us look at one example of set generated according to Definition 7.

Example 1. *Let $A \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$, and $y \in Y$ be arbitrary. The set $X_{A \geqslant B(y)} \subseteq X$ is given by*

$$X_{A \geqslant B(y)} = \{x \in X | A(x) \geqslant B(y)\}. \tag{2.8}$$

It is interesting to note that, for a given fixed y, the set $X_{A > B(y)}$ is equivalent to the α-cut of A with $\alpha = B(y)$.

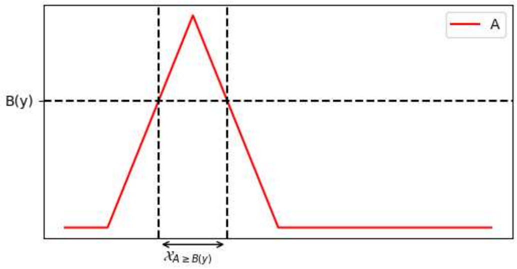


Figure 1 – Subset $X_{A \geqslant B(y)}$ of X.

Now considering two fuzzy sets defined in the same domain, that is $Y = X$, we define the following notation:

Definition 8. *Let $A, B \in \mathcal{F}(X)$ and a binary operator $\sim: [0,1] \times [0,1] \to \{0,1\}$. The set $X_{A \sim B}$ is given by*

$$X_{A \sim B} = \{x \in X | (x, x) \in R_{A \sim B}\}. \tag{2.9}$$

One example of application of this notation is the following:

Example 2. *Let $A, B \in \mathcal{F}(X)$. The set $X_{A<B} \subseteq X$ is given by $X_{A<B} = \{x \in X | A(x) < B(x)\}$.*

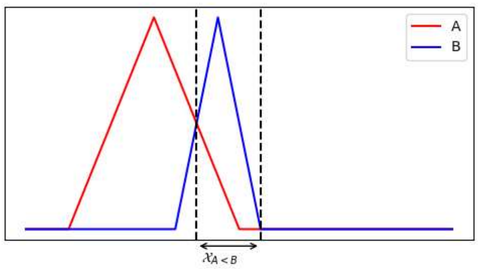


Figure 2 – Subset $X_{A<B}$ of X.

Naturally this concept of classical (or crisp) relations can be extended using the notion of fuzzy sets defined earlier, so a fuzzy relation is given by $R : X \times Y \to [0,1]$, where $R(x,y)$ can be interpreted as the degree of relation between x and y (Di NOLA et al., 1989). The operations between fuzzy sets seen on the last section are examples of fuzzy relations.

We can also define compositions of fuzzy relations (PEDRYCZ; GOMIDE, 2007). In particular, we will use two kinds of composition operators in this work: Sup-t and Inf-I compositions.

Definition 9. *Let $G : X \times Z \to [0,1]$ and $W : Z \times Y \to [0,1]$ be fuzzy relations and $*$ a t-norm. The Sup-t composition $\check{R} = G \circ W$ is a fuzzy relation with membership function given by*

$$\check{R}(x,y) = \bigvee_{z \in Z} (G(x,z) * W(z,y)), \quad \forall x \in X \text{ and } \forall y \in Y.$$

Definition 10. *Let $G : X \times Z \to [0,1]$ and $W : Z \times Y \to [0,1]$ be fuzzy relations and $\to$ a residual implication. The Inf-I composition $\hat{R} = G \lhd W$ is a fuzzy relation with membership function given by*

$$\hat{R}(x,y) = \bigwedge_{z \in Z} (G(x,z) \to W(z,y)), \quad \forall x \in X \text{ and } \forall y \in Y.$$

These definitions of compositions between relations will be used in Chapter 3 to build fuzzy relational systems.

2.4 Residuated lattices

Since we are dealing with residual implications in this work, it is important to introduce the concept of a residuated lattice. This algebraic structure that has some properties that will be very useful for developing fuzzy relational systems in the next chapter.

Definition 11. *(BIRKHOFF, 1940) A partially ordered set U is a set in which a binary relation $x \leqslant y$ is defined, which satisfies for all $x, y, z \in U$ the following conditions:*

- *Reflexive: $x \leqslant x$*
- *Antisymmetric: If $x \leqslant y$ and $y \leqslant x$, then $x = y$*
- *Transitive: If $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$*

If $x \leqslant y$ it is said that x "is less than" or "is contained in" y. On the other way, it can also be read as y "contains" x.

The *sup* between two elements x and y from a partially ordered set P, denoted by $x \vee y$, is the is the smallest element of P that is greater than or equal to both x and y. (BIRKHOFF, 1940).

Similarly, the inf between two elements x and y from a partially ordered set P, denoted by $x \wedge y$, is the greatest element of P that is smaller than or equal to both x and y (BIRKHOFF, 1940).

Definition 12. *(BIRKHOFF, 1940) A lattice is a partially ordered set P in which any two of its elements have a sup $(x \vee y)$ and an inf $(x \wedge y)$.*

A lattice P is called complete if $\bigwedge X$ and $\bigvee X$ exists in P for any subset $X \subseteq P$ (GRATZER, 1971). One special kind of lattice is the residuated lattice, that is defined as follows:

Definition 13. *(PERFILIEVA, 2005) A residuated lattice is an algebra $\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, 0, 1\rangle$ with four binary operations and two constants such that:*

- *$\langle L, \vee, \wedge, 0, 1\rangle$ is a lattice with the ordering $\leqslant$ defined using the operations $\vee$, $\wedge$ as usual and 0, 1 are the smallest and largest elements, respectively;*
- *$\langle L, *, 1\rangle$ is a commutative monoid, that is, $*$ is a commutative and associative operation with the identity $x * 1 = x$;*
- *the $\rightarrow$ is a residuation operation with respect to $*$, that is*

$$x * y \leqslant z \quad \Leftrightarrow \quad x \leqslant y \rightarrow z. \tag{2.10}$$

This algebraic structure has some properties that will be very useful in the next chapter, when we present the fuzzy relational systems. Now, we will look at some of these properties found on the literature.

Lemma 4. *(NOVAK; PERFILJEVA; MOCKOR, 1999) Let $\mathcal{L}$ be a residuated lattice. Then for every $a, b, c \in L$ the following holds true.*

1. *$a * b \leqslant a$ and $a * b \leqslant b$*
2. *$b \leqslant a \rightarrow b$*
3. *$a * (a \rightarrow b) \leqslant b$ and $b \leqslant a \rightarrow (a * b)$*
4. *if $a \leqslant b$ then*
    a) *$c \rightarrow a \leqslant c \rightarrow b$*
    b) *$a \rightarrow c \geqslant b \rightarrow c$*
5. *$a * (a \rightarrow 0) = 0$*
6. *$a \rightarrow (b \rightarrow c) = (a * b) \rightarrow c$*
7. *$a \leqslant b$ iff $a \rightarrow b = 1$*
8. *$(a \vee b) * c = (a * c) \vee (b * c)$*
9. *$a \vee b \leqslant ((a \rightarrow b) \rightarrow b) \wedge ((b \rightarrow a) \rightarrow a)$*

Proposition 1. *(GALATOS et al., 2007) Let x, y and z be elements of a residuated lattice with the operations $*$, $\wedge$, $\vee$ and $\rightarrow$. The following properties hold true:*

1. $x * (y \vee z) = (x * y) \vee (x * z)$
2. $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$
3. $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$
4. $x \rightarrow (y \rightarrow z) = (x * y) \rightarrow z$
5. $1 \rightarrow x = x$

If a residuated lattice $\mathcal{L}$ is complete, then we speak of a complete residuated lattice (SUSSNER, 2015) (BELOHLÁVEK, 2012). A well-known example of a complete residuated lattice is the algebra $\mathcal{L}_* = \langle [0,1], \vee, \wedge, *, \rightarrow, 0, 1\rangle$, where $*$ is a left-continuous t-norm and $\rightarrow$ is its adjoint residual implication (HOLČAPEK; QUOC; FERBAS, 2022) (BELOHLÁVEK, 2012). Using this well-known example, we will state two last propositions that will be very useful for developing fuzzy relational systems in the next chapter.

Proposition 2. *Let $\mathcal{L}_*$ be a complete residuated lattice and $a, b, c \in L$. If $a > b$ then we have that*

$$a \rightarrow (b * c) = (a \rightarrow b) * c. \tag{2.11}$$

Proof. Using the definition of residual implication

$$a \rightarrow (b * c) = \bigvee \{z \in [0,1] | a * z \leqslant b * c\} = \bigvee Z = z' \tag{2.12}$$

and also

$$a \rightarrow b = \bigvee \{y \in [0,1] | a * y \leqslant b\} = \bigvee Y = y'. \tag{2.13}$$

Let it be a $y \in Y$, by the definitions above we have

$$\begin{aligned} & y * c \leqslant z' \\ \Rightarrow & Y * c \subseteq Z \\ \Rightarrow & (a \rightarrow b) * c \leqslant a \rightarrow (b * c). \end{aligned} \tag{2.14}$$

If $a > b \Rightarrow a > b * c$. As $*$ is left-continuous and increasing, $z' = a \rightarrow (b * c)$ satisfies $a * z' = b * c$. In the same way, $y' = a \rightarrow b$ satisfies $a * y' = b$. Since $*$ is continuous, then $\exists y_z$ such that $z' = y_z * c$. As y' is the maximal element that satisfies $a * (y' * c) = (a * y') * c = b * c$, so we have that

$$z' = y_z * c \leqslant y' * c \Leftrightarrow a \to (b * c) \leqslant (a \to b) * c. \tag{2.15}$$

Therefore, from 2.14 and 2.15 we conclude the demonstration. □

Proposition 3. *Let $\mathcal{L}_*$ be a complete residuated lattice and $a, b, c \in L$. If $b > c$ then we have that*

$$a * c \leqslant a * (b \to c) \leqslant a \wedge (b \to c). \tag{2.16}$$

Proof. Using the fact that the minimum is the greatest t-norm from Lemma 1 we have

$$a * (b \to c) \leqslant a \wedge (b \to c). \tag{2.17}$$

On the other hand, using the fact that the Godel implication is the smallest residual implication from Lemma 3 we have

$$a * (b \to c) \geqslant a * c. \tag{2.18}$$

□

Note that this dual case is undetermined unlike the first case in Proposition 2. It depends on the specific pair of t-norm and fuzzy residual implication considered.

With these concepts of fuzzy set theory defined and the algebraic properties derived from the residuated lattices, we now present fuzzy relational systems, which are the main topic of this work.

3 Fuzzy Relational Systems

3.1 Rule bases

In general, inference is a process to obtain new information using existing knowledge (LEE, 2004). One type of deductive procedure to perform inferences is the classical *modus ponens* (NGUYEN; WALKER; WALKER, 2018). The general form of an inference following the *modus ponens* procedure is as follows:

$$\begin{aligned} &\textit{premise}\text{: If } x \text{ is } A \text{ then } y \text{ is } B\\ &\textit{fact}\text{: } x \text{ is } A\\ &\textit{conclusion}\text{: } y \text{ is } B \end{aligned}$$

Approximate or fuzzy reasoning refers to processes by which imprecise conclusions are inferred from imprecise premises (NGUYEN; WALKER; WALKER, 2018). To this end, one can use a generalized version of the *modus ponens* procedure:

$$\begin{aligned} &\textit{premise}\text{: If } x \text{ is } A \text{ then } y \text{ is } B\\ &\textit{fact}\text{: } x \text{ is } A'\\ &\textit{conclusion}\text{: } y \text{ is } B' \end{aligned}$$

It is possible to observe that, even in a scenario with uncertainties, it is still possible to reach a conclusion in light of the facts. Thus, modeling the generalized *modus ponens* using the concepts of fuzzy set theory we obtain a fuzzy inference mechanism.

The set of premises in a fuzzy inference system is called the fuzzy rule base. A fuzzy rule base can be represented using the concepts of fuzzy relations and sets seen in the previous section. In this work we will use two approaches to this modeling (PEDRYCZ; GOMIDE, 2007):

Definition 14. *Given a finite set of rules of the form*

$$\textit{If } x \textit{ is } A_i \textit{ then } y \textit{ is } B_i \quad i = 1, \ldots, n,$$

we define a fuzzy rule base as follows:

1. *Conjunctive rule base:* $\check{R}(x,y) = \bigvee_{i=1}^{n} A_i(x) * B_i(y)$.
2. *Implicative rule base:* $\hat{R}(x,y) = \bigwedge_{i=1}^{n} A_i(x) \to B_i(y)$.

Dubois and Prade made an important study (DUBOIS; PRADE, 1996) about the semantics of a conjunctive rule base, that we will quote here:

> "It seems that fuzzy rules modeled in this way are not seen as restrictions, but rather pieces of information. So, the aggregation by the maximum expresses the accumulation of information."

It is interesting to note that this approach does not directly correspond to the IF-THEN rule model, so the notation most suitable for a rule base modeled in this way would be the following:

$$x \text{ is } A_i \text{ and } y \text{ is } B_i \qquad i = 1, \ldots, n$$

However, this form of modeling has gained much popularity thanks to the pioneering work of Mamdani and Assilian (MAMDANI; ASSILIAN, 1975) demonstrating its use in practical applications. For this reason it is very common to find in the literature the nomenclature "Mamdani-Assilian rule base" referring to conjunctive rule bases using the minimum t-norm $\wedge$ as conjunction.

Dubois and Prade also studied the semantics of an implicative rule base (DUBOIS; PRADE, 1996). Quoting the authors again:

> "In this view, each piece of information (fuzzy rule) is seen as a restriction. This view naturally leads to a conjunctive way of aggregating the individual pieces of information since the more information, the more constraints and fewer possible values that satisfies them."

In the work of Klawonn and Novák (KLAWONN; NOVáK, 1996) there is an interesting discussion about the role of these two kinds of rules, especially about the role of the conjunctive rule base. According to the authors, this model works more like an interpolation, mapping inputs to outputs through a similarity relation. On the other hand, the implicative rule base is appropriate to model the generalized *modus ponens* inference process due to its implicative nature.

Let us use a numerical example to better visualize graphically the differences between these two types of rule bases.

Example 3. *Consider the following fuzzy sets:*

$$A_1(x) = \begin{cases} \dfrac{x-1}{2}, & \text{if } 1 < x \leqslant 3, \\ \dfrac{5-x}{2}, & \text{if } 3 < x \leqslant 5, \\ 0, & \text{otherwise.} \end{cases}; \quad A_2(x) = \begin{cases} \dfrac{x-3}{2}, & \text{if } 3 < x \leqslant 5, \\ \dfrac{7-x}{2}, & \text{if } 5 < x \leqslant 7, \\ 0, & \text{otherwise.} \end{cases}; \quad A_3(x) = \begin{cases} \dfrac{x-5}{2}, & \text{if } 5 < x \leqslant 7, \\ \dfrac{9-x}{2}, & \text{if } 7 < x \leqslant 9, \\ 0, & \text{otherwise.} \end{cases};$$

$$B_1(y) = \begin{cases} y-1, & \text{if } 1 < y \leqslant 2, \\ 3-y, & \text{if } 2 < y \leqslant 3, \\ 0, & \text{otherwise.} \end{cases}; \quad B_2(y) = \begin{cases} y-3, & \text{if } 3 < y \leqslant 4, \\ 5-y, & \text{if } 4 < y \leqslant 5, \\ 0, & \text{otherwise.} \end{cases}; \quad B_3(y) = \begin{cases} y-5, & \text{if } 5 < y \leqslant 6, \\ 7-y, & \text{if } 6 < y \leqslant 7, \\ 0, & \text{otherwise.} \end{cases}.$$

Also consider the two types of rule bases (conjunctive and implicative) from Definition 14, expressed as fuzzy relations:

$$\check{R}(x,y) = \bigvee_{i=1}^{3} A_i(x) * B_i(y); \qquad \hat{R}(x,y) = \bigwedge_{i=1}^{3} A_i(x) \to B_i(y).$$

Taking the Minimum t-norm (Conjunctive rule base) and the Gödel implication (Implicative rule base), we have the following visualizations of these fuzzy relations in a three-dimensional space:

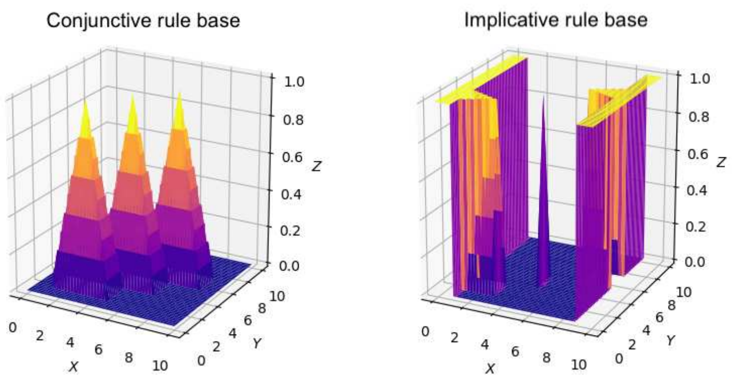


Figure 3 – Representation of the Conjunctive and Implicative fuzzy rule bases.

This graphical example clarifies the meaning of "snippets of information" (conjunctive rule base) and "constraints" (implicative rule base) cited in the work of D. Dubois and H. Prade.

3.2 Fuzzy Relational Systems and combinations

Given a fuzzy rule base represented by a fuzzy relation $R \in \mathcal{F}(X \times Y)$, and an input represented by a fuzzy set $A' \in \mathcal{F}(X)$, we can obtain an output $B' \in \mathcal{F}(Y)$ from the composition of fuzzy relations using Definitions 9 and 10:

1. Sup-t composition: $B'(y) = A'(x) \circ R(x,y) = \bigvee_{x \in X} A'(x) * R(x,y)$.
2. Inf-I composition: $B'(y) = A'(x) \lhd R(x,y) = \bigwedge_{x \in X} A'(x) \to R(x,y)$.

The Sup-t composition described here is known in the literature as the Compositional Rule of Inference or CRI (ZADEH, 1973); the Inf-I composition is also known as the Bandler-Kohout Subproduct or BKS (BANDLER; KOHOUT, 1980). Therefore, we will refer to them as CRI and BKS compositions respectively.

Thus, we have 4 types of fuzzy relational systems formed by the combinations of compositions and rule bases (STEPNICKA; MANDAL, 2018) (STEPNICKA, 2016) (STEPNICKA; MANDAL, 2015):

1. Conjunctive rules and CRI: $B'(y) = A'(x) \circ \check{R}(x, y)$
2. Implicative rules and CRI: $B'(y) = A'(x) \circ \hat{R}(x, y)$
3. Conjunctive rules and BKS: $B'(y) = A'(x) \lhd \check{R}(x, y)$
4. Implicative rules and BKS: $B'(y) = A'(x) \lhd \hat{R}(x, y)$

In the next section we will study these fuzzy relational systems in more details applying the concepts and properties seen on Chapter 2.

3.3 Characterization of the outputs of Fuzzy Relational Systems

To facilitate the understanding of the study of the FRSs, we will divide our analysis in three cases with increasing levels of complexity.

3.3.1 Crisp input

In the first case, let us consider a crisp input $A'_{x'}(x)$, that is, there is a $x' \in X$ such that

$$A'_{x'}(x) = crisp(x, x') = \begin{cases} 1, & \text{if } x = x', \\ 0, & \text{otherwise,} \end{cases} \tag{3.1}$$

and a finite rule base $R = (A_i, B_i)$ for $i = 1, \ldots, n$, where $A_i \in \mathcal{F}(X)$ e $B_i \in \mathcal{F}(Y)$. This crisp input $A'_{x'}(x)$ is graphically represented in Figure 4.

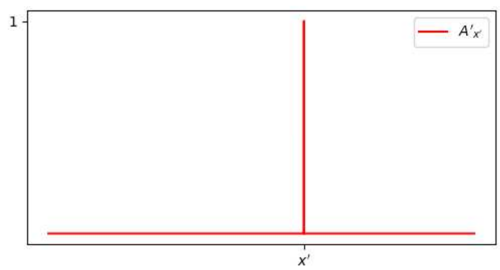


Figure 4 – Crisp set $A'_{x'}(x)$

For the combination of CRI composition with conjunctive rules, we have that

$$B'(y) = A'_{x'}(x) \circ \check{R}(x,y) = \bigvee_{x \in X} A'_{x'}(x) * \bigvee_{i=1}^{n} (A_i(x) * B_i(y)). \tag{3.2}$$

Separating the domain X in $\{x'\}$ and $X \backslash \{x'\}$,

$$\begin{aligned} B'(y) &= [A'_{x'}(x') * \bigvee_{i=1}^{n} (A_i(x') * B_i(y))] \vee \bigvee_{x \in X \backslash \{x'\}} [A'_{x'}(x) * \bigvee_{i=1}^{n} (A_i(x) * B_i(y))] \\ &= [1 * \bigvee_{i=1}^{n} (A_i(x') * B_i(y))] \vee \bigvee_{x \in X \backslash \{x'\}} [0 * \bigvee_{i=1}^{n} (A_i(x) * B_i(y))] \\ &= \bigvee_{i=1}^{n} (A_i(x') * B_i(y)) \vee 0 \\ &= \bigvee_{i=1}^{n} (A_i(x') * B_i(y)). \end{aligned} \tag{3.3}$$

As for the combination of BKS composition with conjunctive rules, we have

$$B'(y) = A'_{x'}(x) \lhd \check{R}(x,y) = \bigwedge_{x \in X} A'_{x'}(x) \to \bigvee_{i=1}^{n} (A_i(x) * B_i(y)). \tag{3.4}$$

Separating the domain X in $\{x'\}$ and $X \backslash \{x'\}$,

$$\begin{aligned} B'(y) &= [A'_{x'}(x') \to \bigvee_{i=1}^{n} (A_i(x') * B_i(y))] \wedge \bigwedge_{x \in X \backslash \{x'\}} [A'_{x'}(x) \to \bigvee_{i=1}^{n} (A_i(x) * B_i(y))] \\ &= [1 \to \bigvee_{i=1}^{n} (A_i(x') * B_i(y))] \wedge \bigwedge_{x \in X \backslash \{x'\}} [0 \to \bigvee_{i=1}^{n} (A_i(x) * B_i(y))] \\ &= \bigvee_{i=1}^{n} (A_i(x') * B_i(y)) \wedge 1 \\ &= \bigvee_{i=1}^{n} (A_i(x') * B_i(y)). \end{aligned} \tag{3.5}$$

By developing Equations 3.3 and 3.5 we can conclude the following proposition:

Proposition 4. *Given a crisp input $A'_{x'}(x)$, the combinations of CRI composition with conjunctive rules and BKS composition with conjunctive rules are equal, which means*

$$A'_{x'}(x) \circ \check{R}(x,y) = A'_{x'}(x) \lhd \check{R}(x,y) = \bigvee_{i=1}^{n} (A_i(x') * B_i(y)). \tag{3.6}$$

On the other hand, for the combination of CRI compostion with implicative rules, we have that

$$B'(y) = A'_{x'}(x) \circ \hat{R}(x,y) = \bigvee_{x \in X} A'_{x'}(x) * \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y)). \tag{3.7}$$

Separating the domain X in $\{x'\}$ and $X\backslash\{x'\}$,

$$\begin{aligned} B'(y) &= [A'_{x'}(x') * \bigwedge_{i=1}^{n}(A_i(x') \to B_i(y))] \vee \bigvee_{x\in X\backslash\{x'\}} [A'_{x'}(x) * \bigwedge_{i=1}^{n}(A_i(x) \to B_i(y))] \\ &= [1 * \bigwedge_{i=1}^{n}(A_i(x') \to B_i(y))] \vee \bigvee_{x\in X\backslash\{x'\}} [0 * \bigwedge_{i=1}^{n}(A_i(x) \to B_i(y))] \\ &= \bigwedge_{i=1}^{n}(A_i(x') \to B_i(y)) \vee 0 \\ &= \bigwedge_{i=1}^{n}(A_i(x') \to B_i(y)). \end{aligned} \tag{3.8}$$

Finally, in the combination of BKS compostition with implicative rules, we have that

$$B'(y) = A'_{x'}(x) \lhd \hat{R}(x,y) = \bigwedge_{x\in X} A'_{x'}(x) \to \bigwedge_{i=1}^{n}(A_i(x) \to B_i(y)). \tag{3.9}$$

Separating the domain X in $\{x'\}$ and $X\backslash\{x'\}$,

$$\begin{aligned} B'(y) &= [A'_{x'}(x') \to \bigwedge_{i=1}^{n}(A_i(x') \to B_i(y))] \wedge \bigwedge_{x\in X\backslash\{x'\}} [A'_{x'}(x) \to \bigwedge_{i=1}^{n}(A_i(x) \to B_i(y))] \\ &= [1 \to \bigwedge_{i=1}^{n}(A_i(x') \to B_i(y))] \wedge \bigwedge_{x\in X\backslash\{x'\}} [0 \to \bigwedge_{i=1}^{n}(A_i(x) \to B_i(y))] \\ &= \bigwedge_{i=1}^{n}(A_i(x') \to B_i(y)) \wedge 1 \\ &= \bigwedge_{i=1}^{n}(A_i(x') \to B_i(y)). \end{aligned} \tag{3.10}$$

By developing Equations 3.8 and 3.10 we can conclude the following proposition:

Proposition 5. *Given a crisp input $A'_{x'}(x)$, the combinations of CRI composition with implicative rules and BKS composition with implicative rules are equal, which means*

$$A'_{x'}(x) \circ \hat{R}(x,y) = A'_{x'}(x) \lhd \hat{R}(x,y) = \bigwedge_{i=1}^{n}(A_i(x') \to B_i(y)). \tag{3.11}$$

To finish this case, let us observe the behavior of these different combinations in a numerical example.

Example 4. *Consider the sets $X, Y = [0, 10] \subset \mathcal{R}$, an input $A'_{3.5}(x) = crisp(x, 3.5)$ and the following rule base:*

Antecedents	Consequents
$A_1 = triang(x, 1, 3, 5)$	$B_1 = triang(y, 1, 2, 3)$
$A_2 = triang(x, 3, 5, 7)$	$B_2 = triang(y, 3, 4, 5)$
$A_3 = triang(x, 5, 7, 9)$	$B_3 = triang(y, 5, 6, 7)$

where $triang(x, a, b, c)$ *represents the triangular membership function*

$$triang(x, a, b, c) = \begin{cases} \dfrac{x-a}{b-a}, & \text{if } a < x \leqslant b, \\ \dfrac{c-x}{c-b}, & \text{if } b < x \leqslant c, \\ 0, & \text{otherwise.} \end{cases}$$

Also consider the minimum t-norm $\wedge$ *and its adjoint Gödel implication* $\to_M$*. Propositions 4 and 5 imply that for crisp inputs, the output depends only on the type of rules (conjunctive or implicative). Graphically, we have the solutions represented in the Figures 5 and 6:*

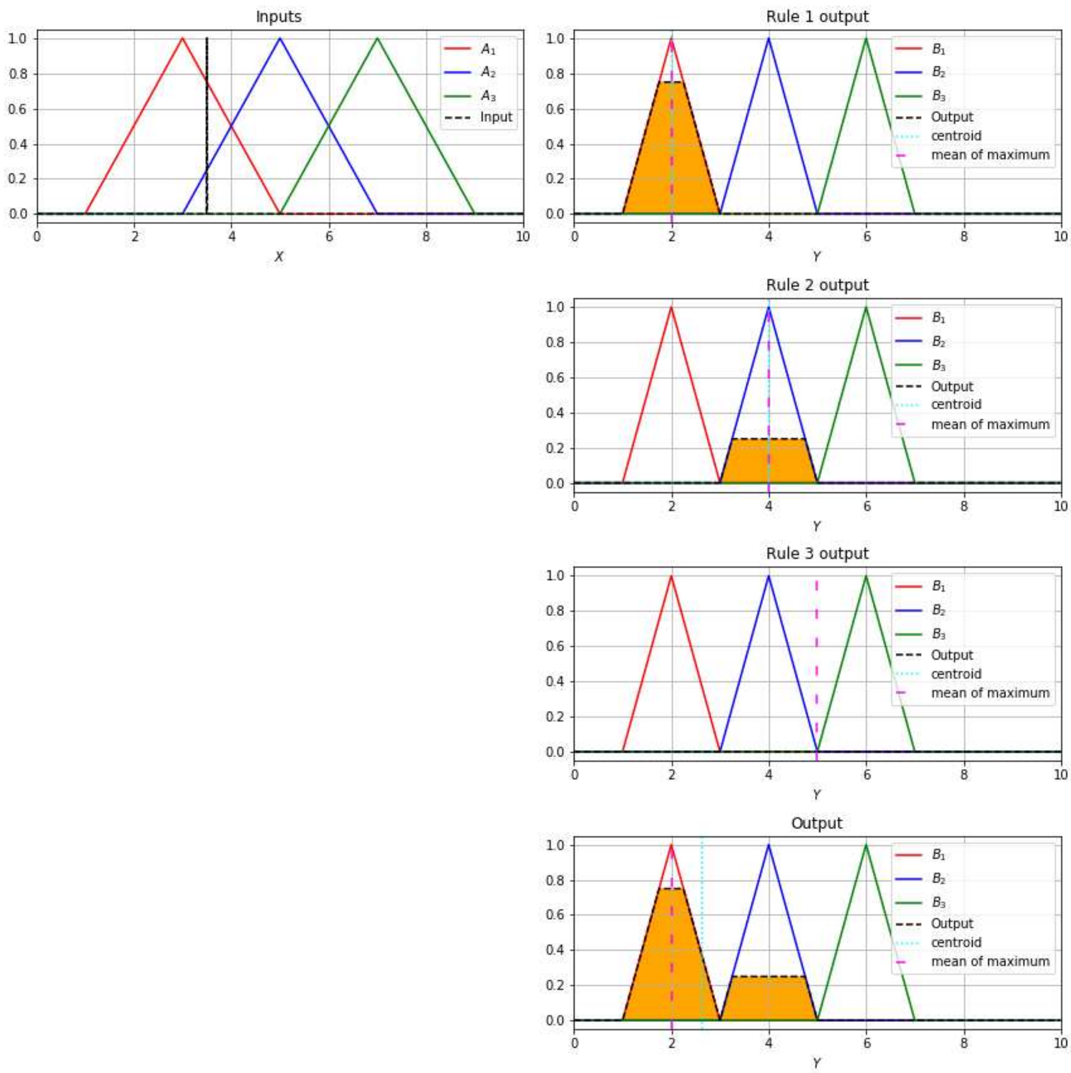


Figure 5 – Output for conjunctive rules

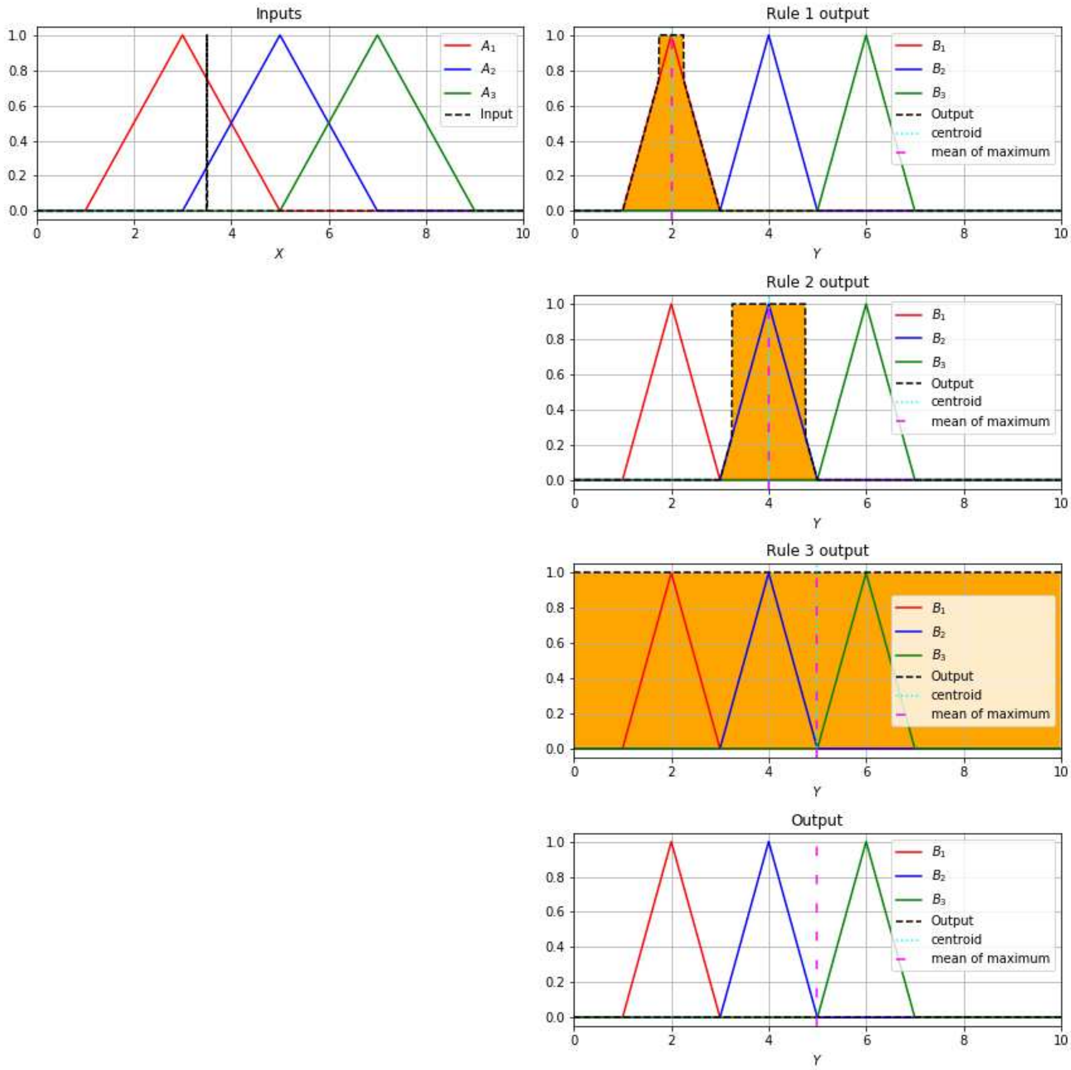


Figure 6 – Output for implicative rules

It is interesting to note that even with $A' \cap (A_1 \cup A_2) \neq \varnothing$, we have that the output is equal to 0 for the combinations using implicative rules. Let us look at another similar example, but with consequents that have a non-empty intersection.

Example 5. *Consider the sets* $X, Y = [0, 10] \subset \mathcal{R}$*, an input* $A'_{3.5}(x) = crisp(x, 3.5)$ *and the following rule base:*

Antecedents	**Consequents**
$A_1 = triang(x, 1, 3, 5)$	$B_1 = triang(y, 1, 3, 5)$
$A_2 = triang(x, 3, 5, 7)$	$B_2 = triang(y, 3.5, 5.5, 7.5)$
$A_3 = triang(x, 5, 7, 9)$	$B_3 = triang(y, 7, 8, 9)$

Also consider the minimum t-norm $\wedge$ *and its adjoint Gödel implication* $\rightarrow_M$*. Again, Propositions 4 and 5 imply that for a given type of rules, the output is equivalent for CRI and BKS compositions on crisp inputs. Graphically, we have the solutions represented in the Figures 7 and 8:*

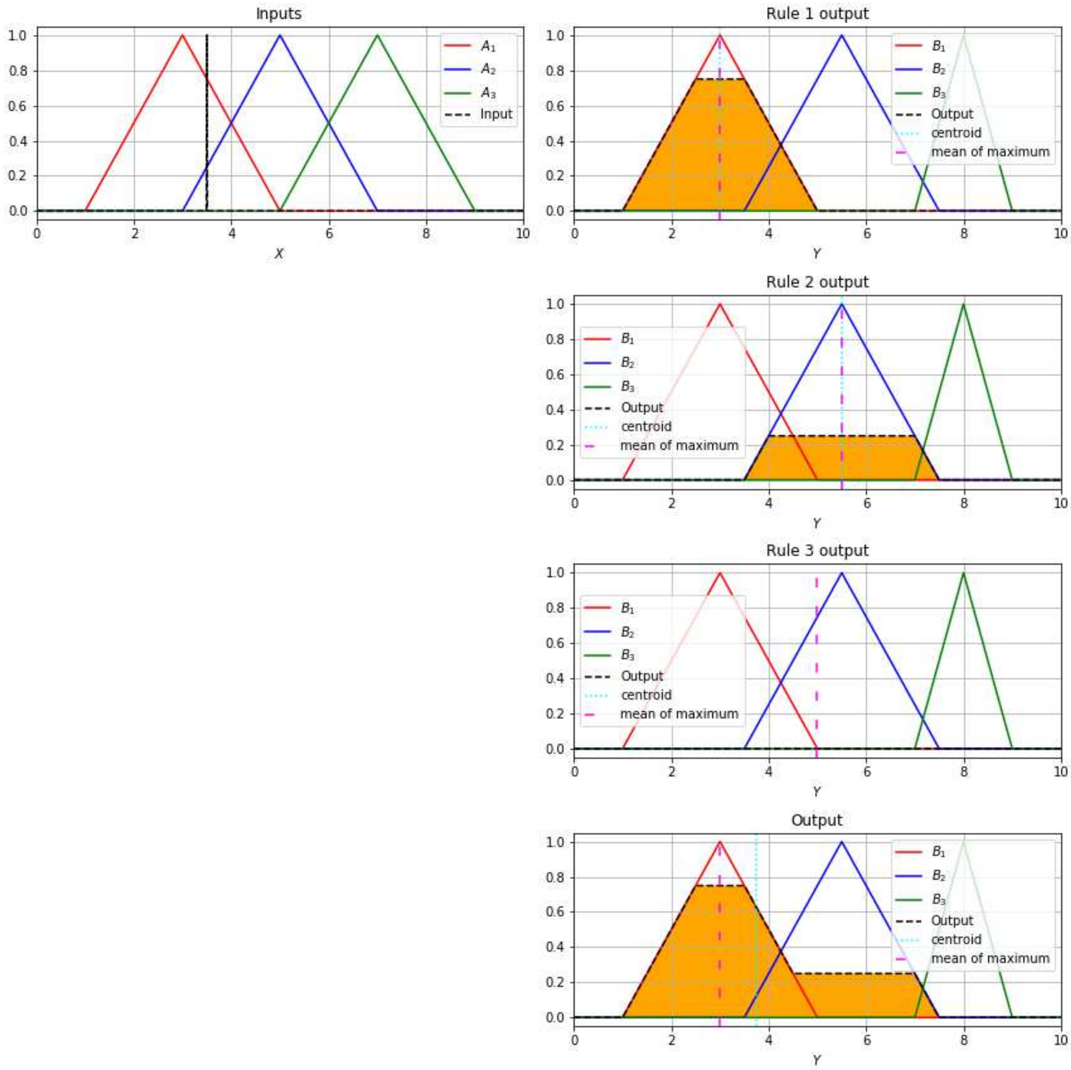


Figure 7 – Output for conjunctive rules

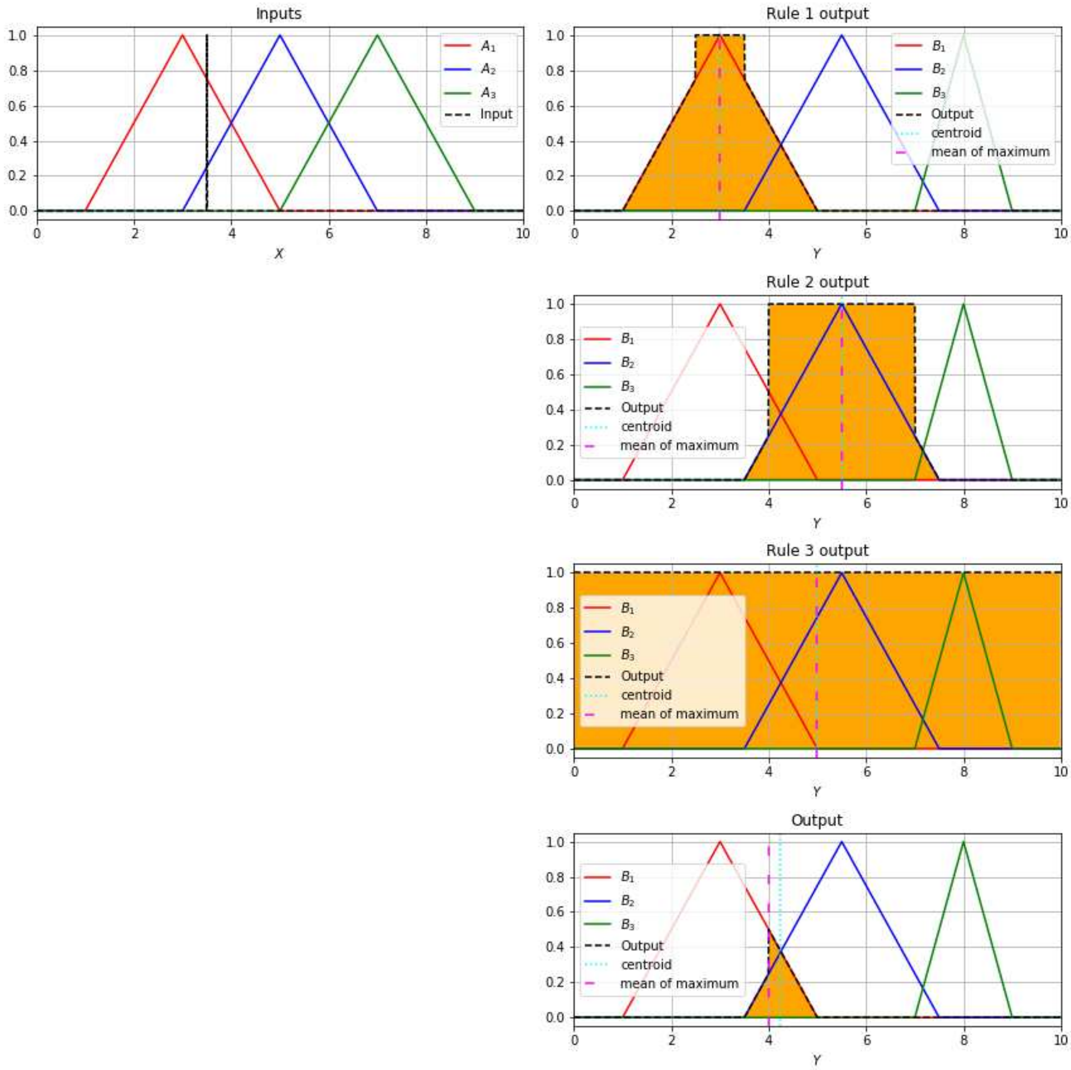


Figure 8 – Output for implicative rules

In this case, we obtained a significative output (different from 0) for implicative rules.

3.3.2 Fuzzy input and one rule

Now let us look at a slightly more complex case, where the input A' is a normal fuzzy set and the rule base has only a single rule $R = (A_1, B_1)$.

For the combination of CRI composition with a single conjunctive rule, applying the associative property of the t-norm we have that

$$\begin{aligned} B'(y) &= \bigvee_{x \in X} A'(x) * (A_1(x) * B_1(y)) \\ &= \bigvee_{x \in X} (A'(x) * A_1(x)) * B_1(y) \\ &= \alpha * B_1(y), \end{aligned} \tag{3.12}$$

where $\alpha = \bigvee_{x\in X}(A'(x) * A_1(x))$ is defined as the "degree of activation" of the rule. The value of α is fixed and independent of y.

In the combination of the BKS composition with a single conjunctive rule, we have that $B'(y) = \bigwedge_{x\in X} A'(x) \to (A_1(x) * B_1(y))$. Using Definition 8 to partition the domain X as in Example 2 leads to

$$B'(y) = \bigwedge_{x\in X_{A'\leqslant A_1}} A'(x) \to (A_1(x) * B_1(y)) \wedge \bigwedge_{x\in X_{A'>A_1}} A'(x) \to (A_1(x) * B_1(y)). \quad (3.13)$$

As we are working with normal fuzzy sets, it is guaranteed that $X_{A'\leqslant A_1} \neq \varnothing$. However, the same cannot be said about the set $X_{A'>A_1}$.

Proposition 6. *Let $A', A_1 \in \mathcal{F}(X)$ be normal and $b \in [0,1]$. If $X_{A'>A_1} \neq \varnothing$, then*

$$\bigwedge_{x\in X} A'(x) \to (A_1(x) * b) = \bigwedge_{x\in X_{A'>A_1}} A'(x) \to (A_1(x) * b). \quad (3.14)$$

Proof. Since $X = X_{A'\leqslant A_1} \cup X_{A'>A_1}$, we have

$$\bigwedge_{x\in X} A'(x) \to (A_1(x) * b) = \bigwedge_{x\in X_{A'\leqslant A_1}} A'(x) \to (A_1(x) * b) \wedge \bigwedge_{x\in X_{A'>A_1}} A'(x) \to (A_1(x) * b). \quad (3.15)$$

On the one hand, using the fact that the implication is decreasing in the first argument, we have that

$$\bigwedge_{x\in X_{A'\leqslant A_1}} A'(x) \to (A_1(x) * b) \geqslant \bigwedge_{x\in X_{A'\leqslant A_1}} A_1(x) \to (A_1(x) * b). \quad (3.16)$$

Property 3 from Lemma 4 ($b \leqslant a \to (a * b)$) implies that

$$\bigwedge_{x\in X_{A'\leqslant A_1}} A_1(x) \to (A_1(x) * b) \geqslant b. \quad (3.17)$$

On the other hand, from the definition of residual implication we have that

$$A'(x) \to (A_1(x) * b) = \bigvee \{z \in [0,1] : A'(x) * z \leqslant A_1(x) * b\}. \quad (3.18)$$

Let $z' = A'(x) \to (A_1(x) * b)$. For any $x \in X_{A'>A_1}$, we have that $A'(x) > A_1(x)$. By the monotonicity property of the t-norm, we have

$$A'(x) * z' \leqslant A_1(x) * b \Rightarrow \bigwedge_{x\in X_{A'>A_1}} A'(x) \to (A_1(x) * b) \leqslant b \quad (3.19)$$

$$\Rightarrow \bigwedge_{x\in X} A'(x) \to (A_1(x) * b) = \bigwedge_{x\in X_{A'>A_1}} A'(x) \to (A_1(x) * b). \quad (3.20)$$

□

Corollary 1. *Let $A', A_1 \in \mathcal{F}(X)$ be normal and $b \in [0,1]$. If $X_{A'>A_1} \neq \varnothing$, then*

$$\bigwedge_{x\in X} A'(x) \to (A_1(x) * b) = \beta * b, \tag{3.21}$$

where $\beta = \bigwedge_{x\in X_{A'>A_1}} (A'(x) \to A_1(x))$.

Proof. If $0 \leqslant d_i < c_i \leqslant 1$ such that $i \in I$ for some arbitrary index set I and $e \in [0,1]$, then Proposition 2 implies that $c_i \to (d_i * e) = (c_i \to d_i) * e$. Taking the infimum over the set I we have

$$\bigwedge_{i\in I}[c_i \to (d_i * e)] = \bigwedge_{i\in I}[(c_i \to d_i) * e] = \bigwedge_{i\in I}[c_i \to d_i] * e \tag{3.22}$$

$$\Rightarrow \bigwedge_{x\in X_{A'>A_1}} A'(x) \to (A_1(x) * b) = \bigwedge_{x\in X_{A'>A_1}} (A'(x) \to A_1(x)) * b = \beta * b, \tag{3.23}$$

where $\beta = \bigwedge_{x\in X_{A'>A_1}} (A'(x) \to A_1(x))$. A brief glance at Proposition 6 suffices to conclude that the claim of Corollary 1 is satisfied. □

If $X_{A'>A_1} = \varnothing$, then $A' \subseteq A_1$ and we obtain the following result:

Corollary 2. *Let $A', A_1 \in \mathcal{F}(X)$ be normal and $b \in [0,1]$. If $X_{A'>A_1} = \varnothing$, then*

$$\bigwedge_{x\in X} A'(x) \to (A_1(x) * b) = b. \tag{3.24}$$

Proof. Since $X_{A'>A_1} = \varnothing$, we have that

$$\begin{aligned}
\bigwedge_{x\in X} A'(x) \to (A_1(x) * b) &= \bigwedge_{x\in X_{A'\leqslant A_1}} A'(x) \to (A_1(x) * b) \wedge \bigwedge_{x\in X_{A'>A_1}} A'(x) \to (A_1(x) * b) \\
&= \bigwedge_{x\in X_{A'\leqslant A_1}} A'(x) \to (A_1(x) * b) \wedge \bigwedge_{x\in\varnothing} A'(x) \to (A_1(x) * b) \\
&= \bigwedge_{x\in X_{A'\leqslant A_1}} A'(x) \to (A_1(x) * b) \wedge \bigwedge \varnothing \\
&= \bigwedge_{x\in X_{A'\leqslant A_1}} A'(x) \to (A_1(x) * b) \wedge 1 \\
&= \bigwedge_{x\in X_{A'\leqslant A_1}} A'(x) \to (A_1(x) * b).
\end{aligned} \tag{3.25}$$

As we have seen in the demonstration of Proposition 6, using the fact that the implication is decreasing in the first argument and Property 3 from Lemma 4, we have

$$\bigwedge_{x\in X_{A'\leqslant A_1}} A'(x) \to (A_1(x) * b) \geqslant \bigwedge_{x\in X_{A'\leqslant A_1}} A_1(x) \to (A_1(x) * b) \geqslant b. \tag{3.26}$$

Now, since A' and A_1 are normal fuzzy sets, there exists a $x' \in X$ such that $A'(x') = 1$ and therefore $A_1(x') = 1$. In view of Proposition 1, this leads to

$$A'(x') \to (A_1(x') * b) = 1 \to (1 * b) = b. \tag{3.27}$$

Thus, $\bigwedge_{x \in X} A'(x) \to (A_1(x) * b) = b$.

□

Therefore, using the results from Proposition 6, Corollary 1 and Corollary 2, we can conclude that

$$B'(y) = \bigwedge_{x \in X} A'(x) \to (A_1(x) * B_1(y)) = \beta * B_1(y), \tag{3.28}$$

since if $X'_{A'>A_1} = \varnothing$ we would have $\beta = \bigwedge \varnothing = 1$.

For the combination of CRI composition with a single implicative rule, we have that $B'(y) = \bigvee_{x \in X} A'(x) * (A_1(x) \to B_1(y))$. As a generalization of Proposition 6, we obtain the following result:

Proposition 7. *Let $A', A_1 \in \mathcal{F}(X)$ be normal and $b \in [0,1]$. If $X_{A'>A_1} \neq \varnothing$ and there exists x^* such that $A'(x^*) = 1 > A_1(x^*)$, then*

$$\bigvee_{x \in X} A'(x) * (A_1(x) \to b) = \bigvee_{x \in X_{A'>A_1}} A'(x) * (A_1(x) \to b). \tag{3.29}$$

Proof. Property 3 of Lemma 4 states that $a * (a \to b) \leqslant b$. Therefore,

$$\bigvee_{x \in X_{A' \leqslant A_1}} A'(x) * (A_1(x) \to b) \leqslant \bigvee_{x \in X_{A' \leqslant A_1}} A_1(x) * (A_1(x) \to b) \leqslant b. \tag{3.30}$$

On the other hand, consider $x^* \in X$ such that $A_1(x^*) < 1 = A'(x^*)$:

$$\bigvee_{x \in X_{A'>A_1}} A'(x) * (A_1(x) \to b) \geqslant 1 * (A_1(x^*) \to b) = A_1(x) \to b \geqslant b. \tag{3.31}$$

Note that $\bigvee_{x \in X} A'(x) * (A_1(x) \to b)$ can be written as follows:

$$\bigvee_{x \in X} A'(x) * (A_1(x) \to b) = \bigvee_{x \in X_{A' \leqslant A_1}} A'(x) * (A_1(x) \to b) \vee \bigvee_{x \in X_{A'>A_1}} A'(x) * (A_1(x) \to b). \tag{3.32}$$

Using a combination of Equations 3.30 and 3.31 we can conclude the following:

$$\bigvee_{x \in X} A'(x) * (A_1(x) \to b) = \bigvee_{x \in X_{A'>A_1}} A'(x) * (A_1(x) \to b). \tag{3.33}$$

□

Corollary 3. *Let $A', A_1 \in \mathcal{F}(X)$ be normal and $b \in [0,1]$. If $X_{A'>A_1} = \varnothing$, then*

$$\bigvee_{x \in X} A'(x) * (A_1(x) \to b) = b. \tag{3.34}$$

Proof. Since $X_{A'>A_1} = \varnothing$, we have that

$$\begin{aligned}
\bigvee_{x \in X} A'(x) * (A_1(x) \to b) &= \bigvee_{x \in X_{A' \leqslant A_1}} A'(x) * (A_1(x) \to b) \vee \bigvee_{x \in X_{A'>A_1}} A'(x) * (A_1(x) \to b) \\
&= \bigvee_{x \in X_{A' \leqslant A_1}} A'(x) * (A_1(x) \to b) \vee \bigvee_{x \in \varnothing} A'(x) * (A_1(x) \to b) \\
&= \bigvee_{x \in X_{A' \leqslant A_1}} A'(x) * (A_1(x) \to b) \vee \bigvee \varnothing \\
&= \bigvee_{x \in X_{A' \leqslant A_1}} A'(x) * (A_1(x) \to b) \vee 0 \\
&= \bigvee_{x \in X_{A' \leqslant A_1}} A'(x) * (A_1(x) \to b).
\end{aligned} \tag{3.35}$$

By Equation 3.30

$$\bigvee_{x \in X_{A' \leqslant A_1}} A'(x) * (A_1(x) \to b) \leqslant \bigvee_{x \in X_{A' \leqslant A_1}} A_1(x) * (A_1(x) \to b) \leqslant b. \tag{3.36}$$

Since A' and A_1 are normal fuzzy sets, there exists $x' \in X$ such that $A'(x') = A_1(x') = 1$. This leads to

$$A'(x') * (A_1(x') \to b) = 1 * (1 \to b) = b. \tag{3.37}$$

Thus, $\bigvee_{x \in X} A'(x) * (A_1(x) \to b) = b.$ □

As we saw in Proposition 3, the scenario where $X_{A'>A_1} \neq \varnothing$ depends on the t-norm and implication considered. Let us consider some specific cases in the next corollaries.

Corollary 4. *Let $A', A_1 \in \mathcal{F}(X)$ be normal and $b \in [0,1]$. Considering a t-norm $*$ and its adjoint implication $\to$, we have that*

$$\bigvee_{x \in X_{A_1 \leqslant b}} A'(x) * (A_1(x) \to b) = \bigvee_{x \in X_{A_1 \leqslant b}} A'(x). \tag{3.38}$$

Proof. Property 7 of Lemma 4 states that $A_1(x) \to b = 1$ is equivalent to $A_1(x) \leqslant b$. Therefore,

$$\bigvee_{x \in X_{A_1 \leqslant b}} A'(x) * (A_1(x) \to b) = \bigvee_{x \in X_{A_1 \leqslant b}} A'(x) * 1 = \bigvee_{x \in X_{A_1 \leqslant b}} A'(x). \tag{3.39}$$

□

Corollary 5. *Let $A', A_1 \in \mathcal{F}(X)$ be normal and $b \in [0,1]$. Considering the minimum t-norm $\wedge$ with the Gödel implication $\rightarrow_M$, the product t-norm $\cdot$ with the Goguen implication $\rightarrow_P$ and the Lukasiewicz t-norm $*_L$ with the Lukasiewicz implication $\rightarrow_L$, we obtain the following identities:*

- $\bigvee_{x \in X} A'(x) \wedge (A_1(x) \rightarrow_M b) = \bigvee_{x \in X_{A_1 \leqslant b}} A'(x) \vee (\bigvee_{x \in X_{A_1 > b}} A'(x) \wedge b)$;
- $\bigvee_{x \in X} A'(x) \cdot (A_1(x) \rightarrow_P b) = \bigvee_{x \in X_{A_1 \leqslant b}} A'(x) \vee (\bigvee_{x \in X_{A_1 > b}} A'(x) \cdot \dfrac{b}{A_1(x)})$;
- $\bigvee_{x \in X} A'(x) *_L (A_1(x) \rightarrow_L b) = \bigvee_{x \in X_{A_1 \leqslant b}} A'(x) \vee \bigvee_{x \in X_{A_1 > b}} [0 \vee (A'(x) + b - A_1(x))]$.

Proof. Since $X = X_{A_1 \leqslant b} \cup X_{A_1 > b}$, considering a t-norm $*$ and its adjoint implication $\rightarrow$, we have

$$\bigvee_{x \in X} A'(x) * (A_1(x) \rightarrow b) = \bigvee_{x \in X_{A_1 \leqslant b}} A'(x) * (A_1(x) \rightarrow b) \vee \bigvee_{x \in X_{A_1 > b}} A'(x) * (A_1(x) \rightarrow b). \quad (3.40)$$

Corollary 4 implies that $\bigvee_{x \in X_{A_1 \leqslant b}} A'(x) * (A_1(x) \rightarrow b) = \bigvee_{x \in X_{A_1 \leqslant b}} A'(x)$. Thus,

$$\bigvee_{x \in X} A'(x) * (A_1(x) \rightarrow b) = \bigvee_{x \in X_{A_1 \leqslant b}} A'(x) \vee \bigvee_{x \in X_{A_1 > b}} A'(x) * (A_1(x) \rightarrow b) \quad (3.41)$$

Substituting the aforementioned pairs of t-norms and adjoint implications, we derive the equations of Corollary 5. □

Finally, for the combination of BKS composition with a single implicative rule, we have that $B'(y) = \bigwedge_{x \in X} A'(x) \rightarrow (A_1(x) \rightarrow B_1(y))$. Property 4 of Proposition 1 of residuated lattices states that $x \rightarrow (y \rightarrow z) = (x * y) \rightarrow z$. Then, we can use this result to obtain

$$B'(y) = \bigvee_{x \in X} (A'(x) * A_1(x)) \rightarrow B_1(y) = \alpha \rightarrow B_1(y), \quad (3.42)$$

where $\alpha = \bigvee_{x \in X} (A'(x) * A_1(x))$.

To finish this case, let us look at these results in a couple of numerical example.

Example 6. *Consider the sets $X, Y = [0, 10] \subset \mathbb{R}$, an input $A'(x) = triang(x, 2, 3, 4)$ and the following rule base with just a single rule:*

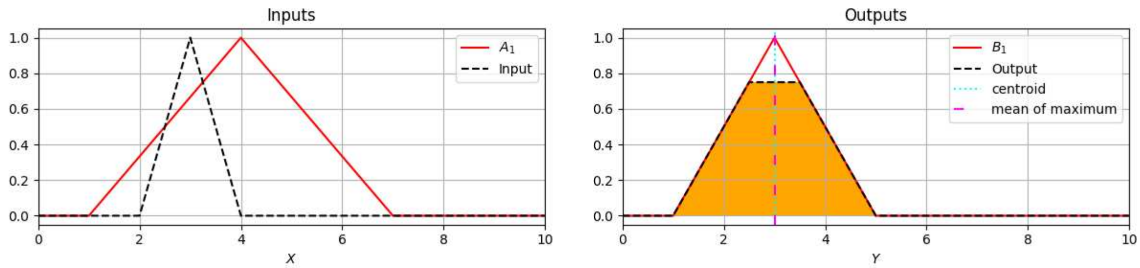
Antecedents	**Consequents**
$A_1 = triang(x, 1, 4, 7)$	$B_1 = triang(y, 1, 3, 5)$

Table 4 – Rule base with a single fuzzy rule

Also, consider the minimum t-norm $\wedge$ *and the Gödel implication* $\rightarrow_M$*. Doing the calculations we obtain the following values of* α *and* β*:*

$$\alpha = \bigvee_{x \in X} (A'(x) \wedge A_1(x)) = 0.75,$$
$$\beta = \bigwedge_{x \in X} (A'(x) \rightarrow_M A_1(x)) = 0.5.$$

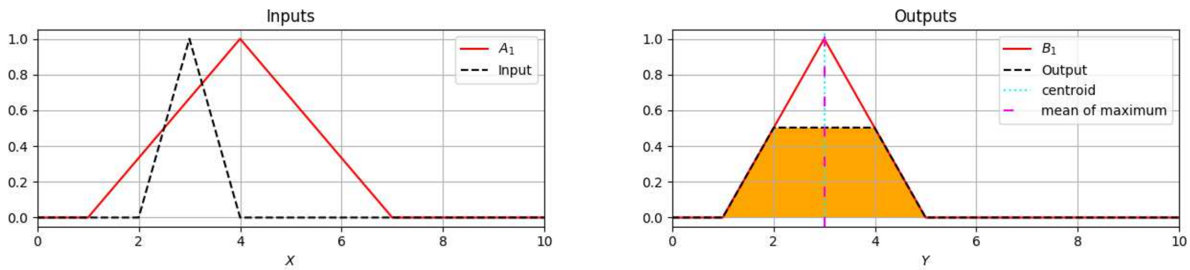
For the combination of CRI and a single conjunctive rule, we can replace the value of α *in Equation 3.12 to obtain the output* $B'(y) = 0.75 \wedge B_1(y)$*, represented in Figure 9.*



(a) Input fuzzy set and antecedent (b) Output fuzzy set and consequent

Figure 9 – Input and output for the combination of CRI and a single conjunctive rule

For the combination of BKS and a single conjunctive rule, we replace the value of β *in Equation 3.28 to obtain the output* $B'(y) = 0.5 \wedge B_1(y)$*, represented in Figure 10.*



(a) Input fuzzy set and antecedent (b) Output fuzzy set and consequent

Figure 10 – Input and output for the combination of BKS and a single conjunctive rule

For the combination of CRI and a single implicative rule, first we need to obtain *the sets* $X_{A_1 \leqslant B_1(y)}$ *and* $X_{A_1 > B_1(y)}$ *as follows:*

$$X_{A_1 \leqslant B_1(y)} = [0, 3B_1(y) + 1[\quad \cup \quad]7 - 3B_1(y), 10], \tag{3.43}$$

$$X_{A_1 > B_1(y)} = [3B_1(y) + 1, 7 - 3B_1(y)]. \tag{3.44}$$

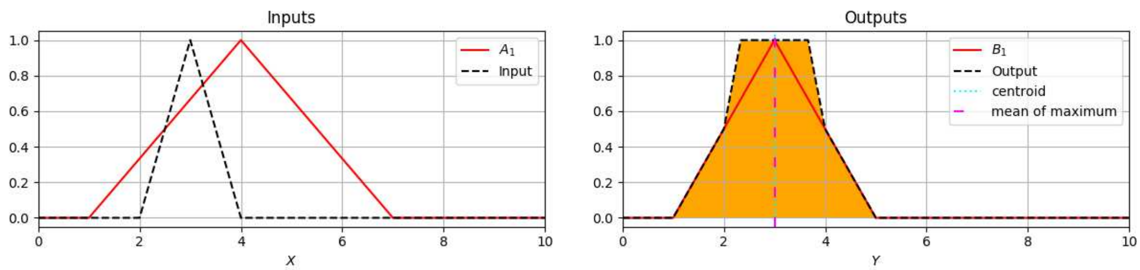
Then, doing the calculations, we have that

$$\bigvee_{x \in X_{A_1 \leqslant B_1(y)}} A'(x) = \begin{cases} 0, & \textit{if } B_1(y) \leqslant 0.333, \\ \dfrac{B_1(y) - 0.333}{0.333}, & \textit{if } 0.333 < B_1(y) \leqslant 0.666, \\ 1, & \textit{otherwise} \end{cases} \tag{3.45}$$

$$\bigvee_{x \in X_{A_1 > B_1(y)}} A'(x) \wedge B_1(y) = \begin{cases} B_1(y), & \textit{if } B_1(y) \leqslant 0.75, \\ \dfrac{1 - B_1(y)}{0.333}, & \textit{otherwise.} \end{cases} \tag{3.46}$$

Finally, we replace these results in Corollary 5 to get the following output represented in Figure 11:

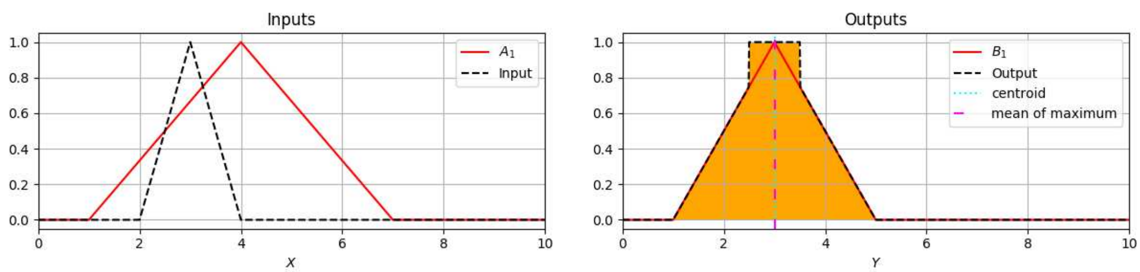
$$B'(y) = \bigvee_{x \in X_{A_1 \leqslant b}} A'(x) \vee \Big(\bigvee_{x \in X_{A_1 > B_1(y)}} A'(x) \wedge B_1(y) \Big) = \begin{cases} B_1(y), & \textit{if } B_1(y) \leqslant 0.5, \\ \dfrac{B_1(y) - 0.333}{0.333}, & \textit{if } 0.5 < B_1(y) \leqslant 0.666, \\ 1, & \textit{otherwise.} \end{cases}$$



(a) Input fuzzy set and antecedent (b) Output fuzzy set and consequent

Figure 11 – Input and output for the combination of CRI and a single implicative rule

For the combination of BKS and a single implicative rule, we replace the value of α *in Equation 3.42 to get the output* $B'(y) = 0.75 \to_M B_1(y)$ *shown in Figure 12.*



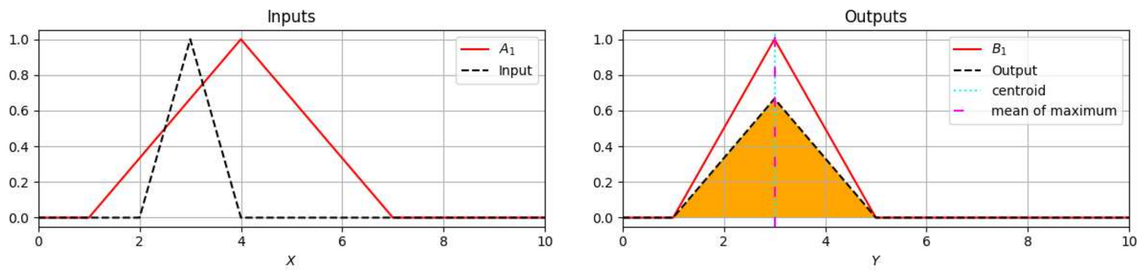
(a) Input fuzzy set and antecedent (b) Output fuzzy set and consequent

Figure 12 – Input and output for the combination of BKS and a single implicative rule

Example 7. *Considering the same rule base from Table 4 and input as before, take now the product t-norm $\cdot$ and the associated Goguen implication $\to_P$. Calculating the values of α and β we have*

$$\alpha = \bigvee_{x \in X} (A'(x) \cdot A_1(x)) \approx 0.67,$$
$$\beta = \bigwedge_{x \in X} (A'(x) \to_P A_1(x)) \approx 0.67.$$

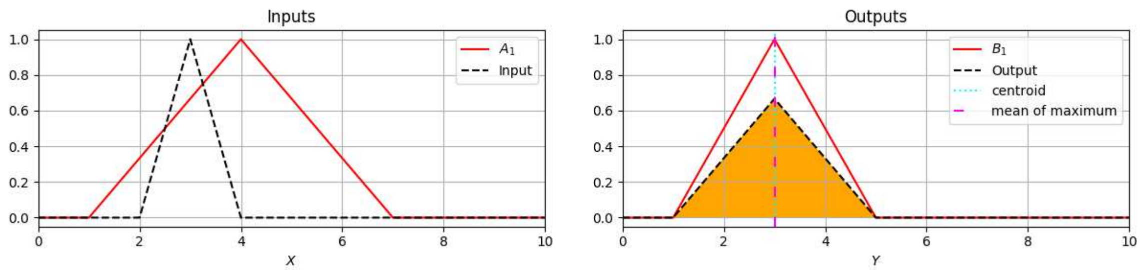
For the combination of CRI and a single conjunctive rule, we replace the value of α in Equation 3.12 to get the output $B'(y) \approx 0.67 \cdot B_1(y)$ represented in Figure 13.



(a) Input fuzzy set and antecedent (b) Output fuzzy set and consequent

Figure 13 – Input and output for the combination of CRI and a single conjunctive rule

For the combination of BKS with a single conjunctive rule, we replace the value of β in Equation 3.28 to obtain $B'(y) \approx 0.67 \cdot B_1(y)$ as shown in Figure 14.



(a) Input fuzzy set and antecedent (b) Output fuzzy set and consequent

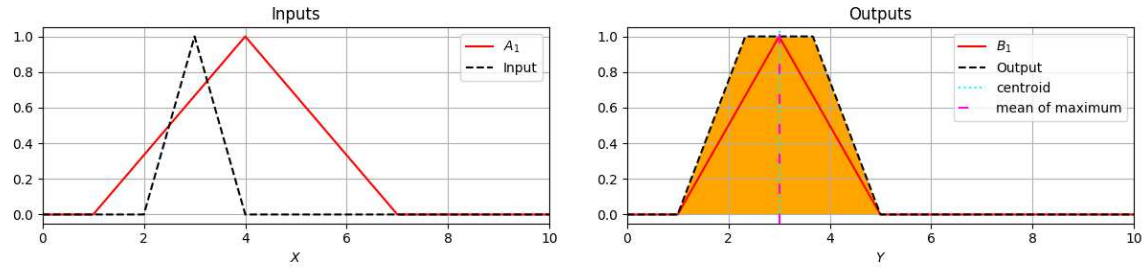
Figure 14 – Input and output for the combination of BKS and a single conjunctive rule

For the combination of CRI and a single implicative rule, we can use some results from the previous example. Specifically, the sets on Equations 3.43 and 3.44, and also the result in Equation 3.45. Then, doing the calculations we obtain

$$\bigvee_{x \in X_{A_1 > B_1(y)}} A'(x) \cdot \frac{B_1(y)}{A_1(y)} = \begin{cases} \dfrac{B_1(y)}{0.666}, & \textit{if } B_1(y) \leqslant 0.666, \\ \dfrac{1 - B_1(y)}{0.333}, & \textit{otherwise.} \end{cases}$$

Using this results in Corollary 5 we get the following output represented in Figure 15:

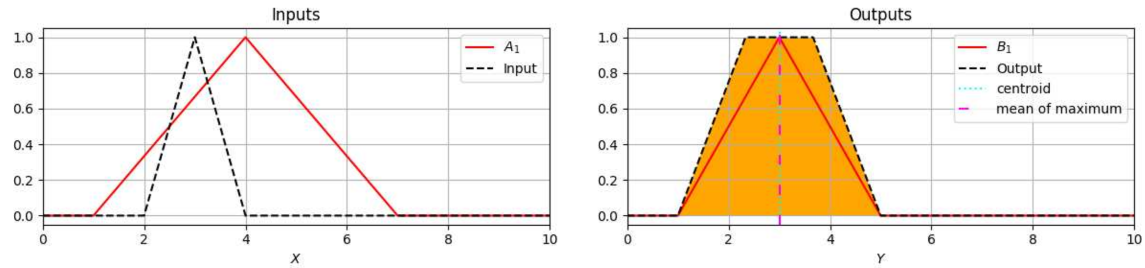
$$B'(y) = \begin{cases} \dfrac{B_1(y)}{0.666}, & \textit{if } B_1(y) \leqslant 0.666, \\ 1, & \textit{otherwise.} \end{cases} \approx 0.67 \rightarrow_P B_1(y).$$



(a) Input fuzzy set and antecedent (b) Output fuzzy set and consequent

Figure 15 – Input and output for the combination of CRI and a single implicative rule

Finally, for the combination of BKS and a single implicative rule, we Replacing the value of α *in Equation 3.42 to obtain the output* $B'(y) \approx 0.67 \rightarrow_P B_1(y)$*, shown in Figure 16.*



(a) Input fuzzy set and antecedent (b) Output fuzzy set and consequent

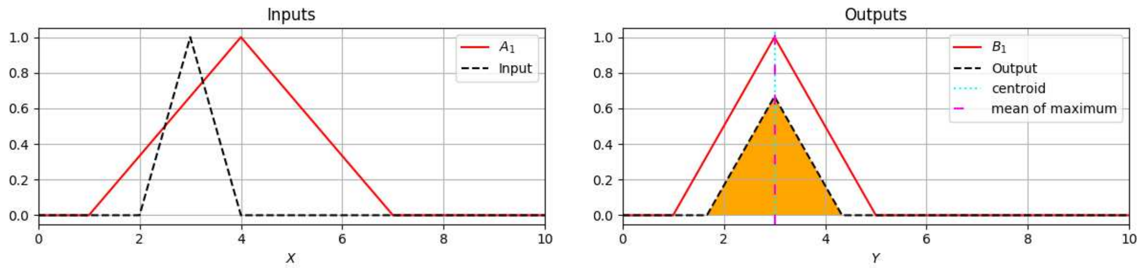
Figure 16 – Input and output for the combination of BKS and a single implicative rule

Example 8. *Finally, consider the Lukasiewicz t-norm* $*_L$ *and Lukasiewicz implication* $\rightarrow_L$*. Again, calculating the values of* α *and* β *we have*

$$\alpha = \bigvee_{x \in X} (A'(x) *_L A_1(x)) \approx 0.67,$$

$$\beta = \bigwedge_{x \in X} (A'(x) \rightarrow_L A_1(x)) \approx 0.67.$$

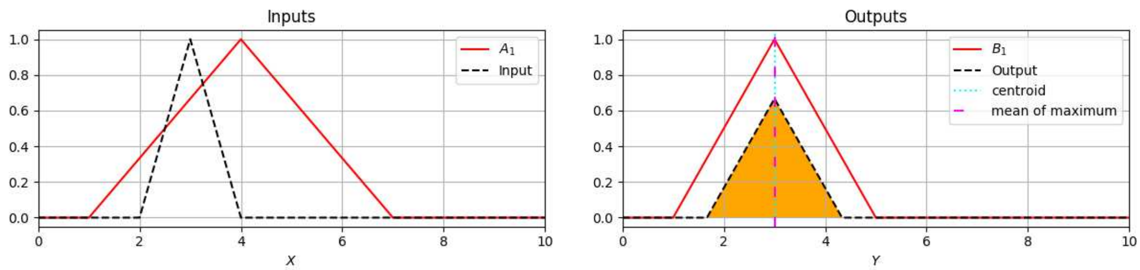
For the combination of CRI and a single conjunctive rule, we replace the value of α *in Equation 3.12 to obtain the output* $B'(y) \approx 0.67 *_L B_1(y) \approx 0 \vee (B_1(y) - 0.33)$*, that is represented in Figure 17.*



(a) Input fuzzy set and antecedent (b) Output fuzzy set and consequent

Figure 17 – Input and output for the combination of CRI and a single conjunctive rule

For the combination of BKS with a single conjunctive rule, we replace the calculated value of β *in Equation 3.28 to obtain as output* $B'(y) \approx 0.67 *_L B_1(y) \approx 0 \vee (B_1(y) - 0.33)$. *This result is equal to the one obtained from the combination of CRI and conjunctive rules, as shown in Figure 18.*



(a) Input fuzzy set and antecedent (b) Output fuzzy set and consequent

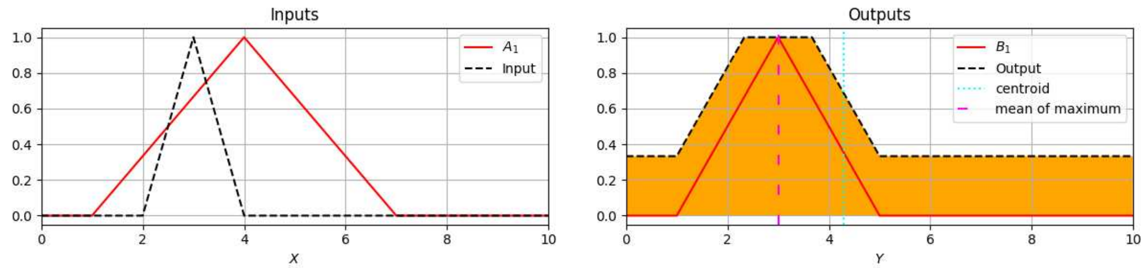
Figure 18 – Input and output for the combination of BKS and a single conjunctive rule

For the combination of CRI and a single implicative rule, we can again use the sets obtained in Equations 3.43 and 3.44, and also the result from Equation 3.45. Then, to apply Corollary 5, we do the calculations to obtain

$$\bigvee_{x \in X_{A_1 > B_1(y)}} [0 \vee (A'(x) + B_1(y) - A_1(x)] = \begin{cases} B_1(y) + 0.33, & \textit{if } B_1(y) \leqslant 0.666, \\ \dfrac{1 - B_1(y)}{0.333}, & \textit{otherwise.} \end{cases}$$

Then, we get the following output also represented in Figure 19:

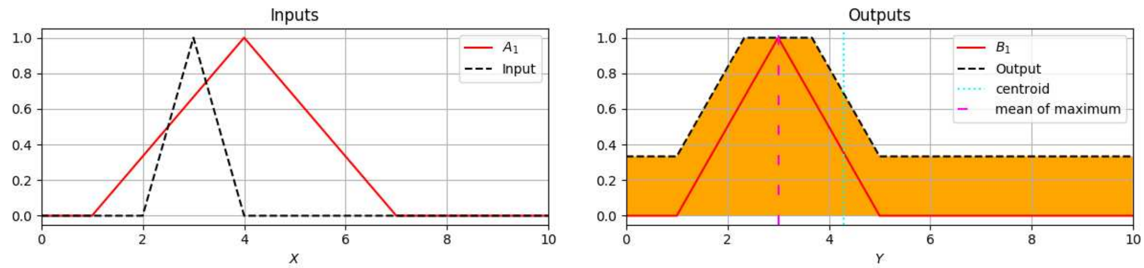
$$B'(y) = \begin{cases} B_1(y) + 0.33, & \textit{if } B_1(y) \leqslant 0.666, \\ 1, & \textit{otherwise.} \end{cases}$$
$$\approx 0.67 \to_L B_1(y) \approx 1 \wedge (B_1(y) + 0.33).$$



(a) Input fuzzy set and antecedent (b) Output fuzzy set and consequent

Figure 19 – Input and output for the combination of CRI and a single implicative rule

Finally, for the combination of BKS with a single implicative rule, we replace the value of α *in Equation 3.42 to get the output* $B'(y) \approx 0.67 \rightarrow_L B_1(y) \approx 1 \wedge (B_1(y) + 0.33)$*. Analogously, this result is equal to the combination of CRI and implicative rules, as shown in Figure 20.*



(a) Input fuzzy set and antecedent (b) Output fuzzy set and consequent

Figure 20 – Input and output for the combination of BKS and a single implicative rule

3.3.3 Fuzzy input and multiple rules

To conclude this study, let us look at the general case of a fuzzy input A' and a fuzzy rule base with a finite number of rules $R = (A_i, B_i)$ for $i = 1, \ldots, n$.

For the combination of CRI composition with conjunctive rules, we have that

$$B'(y) = \bigvee_{x \in X} [A'(x) * \bigvee_{i=1}^{n} (A_i(x) * B_i(y))]. \tag{3.47}$$

Proposition 8. *Let* $A', A_i \in \mathcal{F}(X)$*,* $B_i \in \mathcal{F}(Y)$*, and* $\alpha_i = \bigvee_{x \in X} (A'(x) * A_i(x))$

$$B'(y) = \bigvee_{x \in X} [A'(x) * \bigvee_{i=1}^{n} (A_i(x) * B_i(y))] = \bigvee_{i=1}^{n} \alpha_i * B_i(y). \tag{3.48}$$

Proof. This is a straightforward application of the previous results as

$$\bigvee_{x \in X} [A'(x) * \bigvee_{i=1}^{n} (A_i(x) * B_i(y))] = \bigvee_{i=1}^{n} \bigvee_{x \in X} [A'(x) * (A_i(x) * B_i(y))], \tag{3.49}$$

by Property 1 of Proposition 1. In addition, Equation 3.12 implies that

$$\bigvee_{i=1}^{n} \bigvee_{x \in X} [A'(x) * (A_i(x) * B_i(y))] = \bigvee_{i=1}^{n} \alpha_i * B_i(y). \tag{3.50}$$

□

For the combination of the BKS composition with conjunctive rules, we have that

$$B'(y) = \bigwedge_{x \in X} [A'(x) \to \bigvee_{i=1}^{n} (A_i(x) * B_i(y))]. \tag{3.51}$$

The next two theorems describe the values assumed by this equation for some specific cases.

Theorem 1. *(STEPNICKA; JAYARAM, 2016) Let A', A_i for all $i \in [1, n]$ be normal. If for each i there exists an $x_i \in X$ such that*

$$A_i(x_i) = 1 \text{ and } A_j(x_i) = 0, \text{ whenever } i \neq j,$$

then we have that

$$B'(y) = \bigwedge_{x \in X} [A_i(x) \to \bigvee_{i=1}^{n} (A_i(x) * B_i(y))] = B_i(y). \tag{3.52}$$

This is the interpolativity property which will be studied in more details in the next chapter. The condition that $A_i(x_i) = 1$ and $A_j(x_i) = 0$ whenever $i \neq j$, is very reasonable and can be seen in the vast majority of applications.

Theorem 2. *(STEPNICKA; MANDAL, 2018) Let A' be normal. Consider the set of activated rules given by $I = \{i \in [1, n] | Supp(A') \cap Supp(A_i) \neq \varnothing\}$. The following inequality is satisfied for every $y \in Y$:*

$$B'(y) = \bigwedge_{x \in X} [A'(x) \to \bigvee_{i=1}^{n} (A_i(x) * B_i(y))] \leqslant \bigvee_{i \in I} B_i(y). \tag{3.53}$$

This sets an upper bound for the output of this combination. More precisely, we can write the following proposition using the previous results of the single fuzzy rule case.

Proposition 9. *Let $A', A_i \in \mathcal{F}(X)$ be normal, $B_i \in \mathcal{F}(Y)$ and $y \in Y$. Let $X' = Supp(A')$ and $X'_l \subseteq X'$ be such that $\bigcup_{l=1}^{m} X'_l = X'$, and there exists $i_l \in I = \{i \in [1, n] | Supp(A') \cap$*

$Supp(A_i) \neq \varnothing\}$ for some $l = 1, ..., m$, such that $\bigvee_{i \in I}(A_i(x) * B_i(y)) = A_{i_l}(x) * B_{i_l}(y)$ for every $x \in X'_l$. The following equation holds true:

$$B'(y) = \bigwedge_{x \in X'}[A'(x) \to \bigvee_{i=1}^{n}(A_i(x) * B_i(y))] = \bigvee_{l=1}^{m}(\beta_{i_l} * B_{i_l}(y)), \tag{3.54}$$

where $\beta_{i_l} = \bigwedge_{x \in {X'_l}_{A' > A_{i_l}}} [A'(x) \to A_{i_l}(x)]$.

Proof. Let $y \in Y$ be arbitrary. Since $X = X' \cup X \backslash X'$, Equation 3.51 can be rewritten as follows:

$$B'(y) = \bigwedge_{x \in X'}[A'(x) \to \bigvee_{i=1}^{n}(A_i(x) * B_i(y))] \wedge \bigwedge_{x \in X \backslash X'}[A'(x) \to \bigvee_{i=1}^{n}(A_i(x) * B_i(y))]. \tag{3.55}$$

Note that

$$\bigwedge_{x \in X \backslash X'}[A'(x) \to \bigvee_{i=1}^{n}(A_i(x) * B_i(y))] = \bigwedge_{x \in X \backslash X'}[0 \to \bigvee_{i=1}^{n}(A_i(x) * B_i(y))] = 1 \tag{3.56}$$

holds true not only for $X \backslash X' \neq \varnothing$, but also for $X \backslash X' = \varnothing$ because $\bigwedge \varnothing = 1$. So, this implies that

$$\bigwedge_{x \in X}[A'(x) \to \bigvee_{i=1}^{n}(A_i(x) * B_i(y))] = \bigwedge_{x \in X'}[A'(x) \to \bigvee_{i=1}^{n}(A_i(x) * B_i(y))]. \tag{3.57}$$

Using the definitions of i_l and X'_l, as well as the fact that $\bigcup_{l=1}^{m} X'_l = X'$, we can write Equation 3.57 by partitioning the set X' into subsets as follows:

$$\bigwedge_{x \in X'}[A'(x) \to \bigvee_{i=1}^{n}(A_i(x) * B_i(y))] = \bigwedge_{x \in X'_1}[A'(x) \to (A_{i_1}(x) * B_{i_1}(y))]$$
$$\dots$$
$$\vee \bigwedge_{x \in X'_m}[A'(x) \to (A_{i_m}(x) * B_{i_m}(y))]. \tag{3.58}$$

Note that for each of these partitions X'_l we obtain the single fuzzy rule case studied before. Therefore we can apply the previous results. For every l, we can split the set X'_l into two smaller subsets as

$$\bigwedge_{x \in X'_l}[A'(x) \to (A_{i_l}(x) * B_{i_l}(y))] = \bigwedge_{x \in {X'_l}_{A' \leqslant A_{i_l}}}[A'(x) \to (A_{i_l}(x) * B_{i_l}(y))]$$
$$\wedge \bigwedge_{x \in {X'_l}_{A' > A_{i_l}}}[A'(x) \to (A_{i_l}(x) * B_{i_l}(y))]. \tag{3.59}$$

If $X'_{l_{A'>A_{i_l}}} \neq \varnothing$, then Corollary 1 implies that

$$\bigwedge_{x\in X'_l}[A'(x) \to (A_{i_l}(x) * B_{i_l}(y))] = \bigwedge_{x\in X'_{l_{A'>A_{i_l}}}}[A'(x) \to A_{i_l}(x)] * B_{i_l}(y) = \beta_{i_l} * B_{i_l}(y), \quad (3.60)$$

where $\beta_{i_l} = \bigwedge_{x\in X'_{l_{A'>A_{i_l}}}}[A'(x) \to A_{i_l}(x)]$.

If $X'_{l_{A'>A_{i_l}}} = \varnothing$, then Corollary 2 implies that

$$\bigwedge_{x\in X'_l}[A'(x) \to (A_{i_l}(x) * B_{i_l}(y))] = B_{i_l}(y). \quad (3.61)$$

A combination of Equations 3.60 and 3.61 yields

$$\bigwedge_{x\in X'_l}[A'(x) \to (A_{i_l}(x) * B_{i_l}(y))] = \beta_{i_l} * B_{i_l}(y). \quad (3.62)$$

Note that for $X'_{l_{A'>A_{i_l}}} = \varnothing$ we have $\beta_{i_l} = \bigwedge \varnothing = 1$. This outcome is similar to Equation 3.28 for the single rule case. Finally, combining Equations 3.62 and 3.58 yields

$$\bigwedge_{x\in X'}[A'(x) \to \bigvee_{i=1}^{n}(A_i(x) * B_i(y))] = \bigvee_{l=1}^{m}(\beta_{i_l} * B_{i_l}(y)), \quad (3.63)$$

where $\beta_{i_l} = \bigwedge_{x\in X'_{l_{A'>A_{i_l}}}}(A'(x) \to A_{i_l}(x))$.

□

We will visualize the application of this Proposition 9 in a numerical way using Examples 9, 10 and 11 afterwards in this work.

For the combination of CRI composition with implicative rules, we have that

$$B'(y) = \bigvee_{x\in X}[A'(x) * \bigwedge_{i=1}^{n}(A_i(x) \to B_i(y))]. \quad (3.64)$$

Again, we will state two theorems about the values assumed by Equation 3.64 for certain specific cases.

Theorem 3. *(STEPNICKA; JAYARAM, 2016) Let A', A_i for all $i \in [1, n]$ be normal. If for each i there exists an $x_i \in X$ such that*

$$A_i(x_i) = 1 \text{ and } A_j(x_i) = 0, \text{ whenever } i \neq j$$

then we have that

$$B'(y) = \bigvee_{x \in X} [A_i(x) * \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y))] = B_i(y). \quad (3.65)$$

Once more, we have the interpolativity property for this combination that will be studied in more details in the next chapter.

Theorem 4. *(STEPNICKA; MANDAL, 2018) Let A' be normal. Consider the set of activated rules given by $I = \{i \in [1, n] | Supp(A') \cap Supp(A_i) \neq \varnothing\}$. The following inequality is satisfied for every $y \in Y$:*

$$B'(y) = \bigvee_{x \in X} [A'(x) * \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y))] \geqslant \bigwedge_{i \in I} B_i(y). \quad (3.66)$$

This sets a lower bound for the output of this fuzzy relational equation. Likewise, we can write a general proposition using the previous results of the single fuzzy rule case.

Proposition 10. *Let $A', A_i \in \mathcal{F}(X)$ be normal, $B_i \in \mathcal{F}(Y)$ and an arbitrary $y \in Y$. Let $X' = Supp(A')$ and $X'_l \subseteq X'$ be such that $\bigcup_{l=1}^{m} X'_l = X'$, and there exists $i_l \in I = \{i \in [1, n] | Supp(A') \cap Supp(A_i) \neq \varnothing\}$ for some $l = 1, ..., m$, such that $\bigwedge_{i \in I} (A_i(x) \to B_i(y)) = A_{i_l}(x) \to B_{i_l}(y)$ for every $x \in X'_l$. The following equation holds true:*

$$\begin{aligned} B'(y) &= \bigvee_{x \in X} [A'(x) * \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y))] \\ &= \bigvee_{l=1}^{m} \left\{ \bigvee_{x \in {X'_l}_{A_{i_l} \leqslant B_{i_l}(y)}} A'(x) \vee \bigvee_{x \in {X'_l}_{A_{i_l} > B_{i_l}(y)}} [A'(x) * (A_{i_l}(x) \to B_{i_l}(y))] \right\} \end{aligned} \quad (3.67)$$

Proof. Let $y \in Y$ be arbitrary. Since $X = X' \cup X \backslash X'$, Equation 3.64 can be rewritten as follows:

$$B'(y) = \bigvee_{x \in X'} [A'(x) * \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y))] \vee \bigvee_{x \in X \backslash X'} [A'(x) * \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y))] \quad (3.68)$$

Note that

$$\bigvee_{x \in X \backslash X'} [A'(x) * \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y))] = \bigvee_{x \in X \backslash X'} [0 * \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y))] = 0 \quad (3.69)$$

holds true not only for $X \backslash X' \neq \varnothing$, but also for $X \backslash X' = \varnothing$ because $\bigvee \varnothing = 0$. So, this implies that

$$\bigvee_{x \in X} [A'(x) * \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y))] = \bigvee_{x \in X'} [A'(x) * \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y))] \quad (3.70)$$

Using the definitions of i_l and X'_l, as well as the fact that $\bigcup_{l=1}^{m} X'_l = X'$, we can write Equation 3.70 by partitioning the set X' into subsets as follows:

$$\bigvee_{x \in X'} [A'(x) * \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y))] = \bigvee_{x \in X'_1} [A'(x) * (A_{i_1}(x) \to B_{i_1}(y))] \cdots \vee \bigvee_{x \in X'_m} [A'(x) * (A_{i_m}(x) \to B_{i_m}(y))] \tag{3.71}$$

Note that for each of these partitions X'_l we obtain the single fuzzy rule case studied before. Therefore we can apply previous results. For every l, we can split the set X'_l into two smaller subsets as

$$\bigvee_{x \in X'_l} [A'(x) * (A_{i_l}(x) \to B_{i_l}(y))] = \bigvee_{x \in X'_{l_{A_{i_l} \leqslant B_{i_l}(y)}}} [A'(x) * (A_{i_l}(x) \to B_{i_l}(y))] \vee \bigvee_{x \in X'_{l_{A_{i_l} > B_{i_l}(y)}}} [A'(x) * (A_{i_l}(x) \to B_{i_l}(y))]. \tag{3.72}$$

By Corollary 4 we have

$$\bigvee_{x \in X'_{l_{A_{i_l} \leqslant B_{i_l}(y)}}} [A'(x) * (A_{i_l}(x) \to B_{i_l}(y))] = \bigvee_{x \in X'_{l_{A_{i_l} \leqslant B_{i_l}(y)}}} A'(x). \tag{3.73}$$

Using the result from Equation 3.73 we can rewrite Equation 3.72 as

$$\bigvee_{x \in X'_l} [A'(x) * (A_{i_l}(x) \to B_{i_l}(y))] = \bigvee_{x \in X'_{l_{A_{i_l} \leqslant B_{i_l}(y)}}} A'(x) \vee \bigvee_{x \in X'_{l_{A_{i_l} > B_{i_l}(y)}}} [A'(x) * (A_{i_l}(x) \to B_{i_l}(y))]. \tag{3.74}$$

Combining Equations 3.74 and 3.71 yields

$$\bigvee_{x \in X} [A'(x) * \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y))] = \bigvee_{l=1}^{m} \{ \bigvee_{x \in X'_{l_{A_{i_l} \leqslant B_{i_l}(y)}}} A'(x) \vee \bigvee_{x \in X'_{l_{A_{i_l} > B_{i_l}(y)}}} [A'(x) * (A_{i_l}(x) \to B_{i_l}(y))] \} \tag{3.75}$$

□

Once again, we can simplify this result by considering some specific pairs of t-norms and adjoint residual implications, like we did in Corollary 5 for the single rule case.

Corollary 6. *Let $A', A_i \in \mathcal{F}(X)$ be normal and $B_i \in \mathcal{F}(Y)$. Let $X' = Supp(A')$ and $X'_l \subseteq X'$ be such that $\bigcup_{l=1}^{m} X'_l = X'$ and there exists $i_l \in I = \{i \in [1,n] | Supp(A') \cap Supp(A_i) \neq \emptyset\}$ for some $l = 1,...,m$, such that $\bigwedge_{i\in I}(A_i(x) \to B_i(y)) = A_{i_l}(x) \to B_{i_l}(y)$ for every $x \in X'_l$. Considering the minimum t-norm $\wedge$ with the Gödel implication $\to_M$, the product t-norm $\cdot$ with the Goguen implication $\to_P$ and the Lukasiewicz t-norm $*_L$ with the Lukasiewicz implication $\to_L$, we obtain the following identities:*

- $\bigvee_{x\in X}[A'(x) \wedge \bigwedge_{i=1}^{n}(A_i(x) \to_M B_i(y))] = \bigvee_{l=1}^{m}[\bigvee_{x\in X'_{l_{A_{i_l} \leq B_{i_l}(y)}}} A'(x) \vee \bigvee_{x\in X'_{l_{A_{i_l} > B_{i_l}(y)}}} (A'(x) \wedge B_{i_l}(y))]$;
- $\bigvee_{x\in X}[A'(x) \cdot \bigwedge_{i=1}^{n}(A_i(x) \to_P B_i(y))] = \bigvee_{l=1}^{m}[\bigvee_{x\in X'_{l_{A_{i_l} \leq B_{i_l}(y)}}} A'(x) \vee \bigvee_{x\in X'_{l_{A_{i_l} > B_{i_l}(y)}}} (A'(x) \cdot \frac{B_{i_l}(y)}{A_{i_l}(x)})]$;
- $\bigvee_{x\in X}[A'(x) *_L \bigwedge_{i=1}^{n}(A_i(x) \to_L B_i(y))] = \bigvee_{l=1}^{m}\{\bigvee_{x\in X'_{l_{A_{i_l} \leq B_{i_l}(y)}}} A'(x) \vee \bigvee_{x\in X'_{l_{A_{i_l} > B_{i_l}(y)}}} [0 \vee (A'(x) + B_{i_l}(y) - A_{i_l}(x))]\}$.

Proof. Substituting the aforementioned pairs of t-norms and adjoint implications on Equation 3.75, we derive the equations of Corollary 6. □

We will also visualize the application of the Corollary 6 in a numerical way later on Examples 9, 10 and 11.

Finally, for the combination of BKS composition with implicative rules, we have that

$$B'(y) = \bigwedge_{x\in X}[A'(x) \to \bigwedge_{i=1}^{n}(A_i(x) \to B_i(y))]. \tag{3.76}$$

Proposition 11. *Let $A', A_i \in \mathcal{F}(X)$ and $B_i \in \mathcal{F}(y)$. Defining $\alpha_i = \bigvee_{x\in X}(A'(x) * A_i(x))$, then we have*

$$\bigwedge_{x\in X}[A'(x) \to \bigwedge_{i=1}^{n}(A_i(x) \to B_i(y))] = \bigwedge_{i=1}^{n} \alpha_i \to B_i(y). \tag{3.77}$$

Proof. Again, this is a straightforward application of the previous results as

$$\bigwedge_{x\in X}[A'(x) \to \bigwedge_{i=1}^{n}(A_i(x) \to B_i(y))] = \bigwedge_{i=1}^{n}\bigwedge_{x\in X}[A'(x) \to (A_i(x) \to B_i(y))], \tag{3.78}$$

using the property 2 of Proposition 1. Also

$$\bigwedge_{i=1}^{n}\bigwedge_{x\in X}[A'(x) \to (A_i(x) \to B_i(y))] = \bigwedge_{i=1}^{n} \alpha_i \to B_i(y), \tag{3.79}$$

using the result of Equation 3.42. □

Once again, let us finish the analysis of this case with some numerical examples.

Example 9. *Consider the sets* $X, Y = [0, 10] \subset \mathbb{R}$*, an input* $A'(x) = triang(x, 2.5, 3.5, 4.5)$ *and the following rule base:*

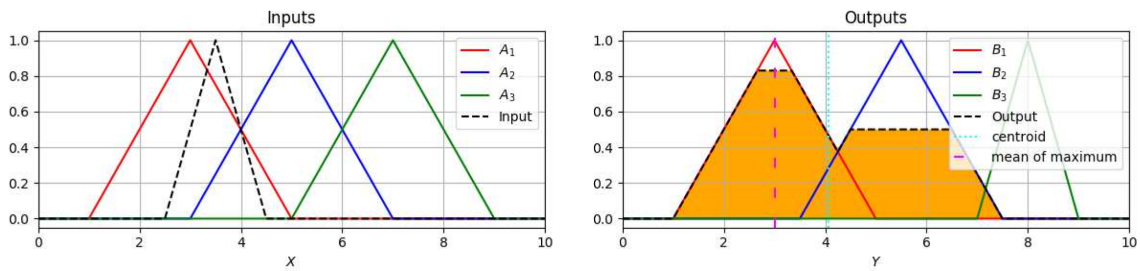
Antecedents	Consequents
$A_1 = triang(x, 1, 3, 5)$	$B_1 = triang(y, 1, 3, 5)$
$A_2 = triang(x, 3, 5, 7)$	$B_2 = triang(y, 3.5, 5.5, 7.5)$
$A_3 = triang(x, 5, 7, 9)$	$B_3 = triang(y, 7, 8, 9)$

Table 5 – Rule base with multiple fuzzy rules

Also, consider the minimum t-norm $\wedge$ *and the Gödel implication* $\rightarrow_M$*. Then, we can calculate the values of* α_i *for each rule to obtain the following values:*

$$\alpha_1 = \bigvee_{x \in X} (A'(x) \wedge A_1(x)) = 0.83;$$
$$\alpha_2 = \bigvee_{x \in X} (A'(x) \wedge A_2(x)) = 0.5;$$
$$\alpha_3 = \bigvee_{x \in X} (A'(x) \wedge A_3(x)) = 0.$$

For the combination of CRI and conjunctive rules, we replace these values in Proposition 8 to get the output $B'(y) = (0.83 \wedge B_1(y)) \vee (0.5 \wedge B_2(y))$*, that is represented in Figure 21.*



(a) Input fuzzy set and antecedents (b) Output fuzzy set and consequents

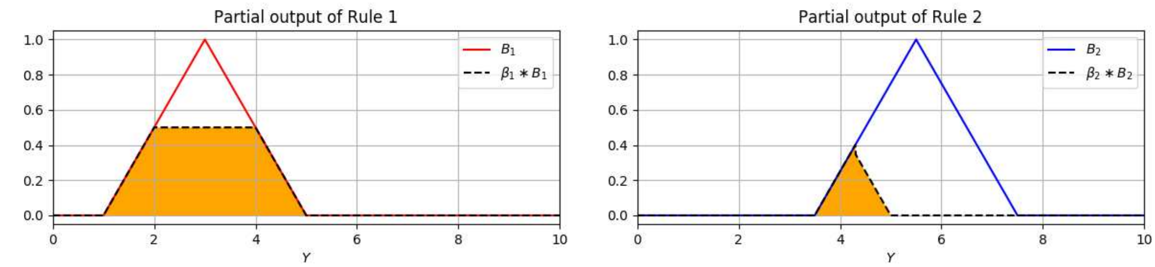
Figure 21 – Input and output for the combination of CRI and conjunctive rules

For the combination of BKS and conjunctive rules, first we need to obtain the set of activated rules, that is $I = \{i \in [1, 3] | Supp(A') \cap Supp(A_i) \neq \varnothing\} = \{1, 2\}$*, and define the set* $X' = Supp(A') = [2.5, 4.5]$*. Then, in order to apply Proposition 9, we must obtain the indexes* i_l *and the sets* X'_l *such that* $\bigcup_{l=1}^{m} X'_l = X'$ *and* $\bigvee_{i \in I} (A_i(x) * B_i(y)) = A_{i_l}(x) * B_{i_l}(y)$ *for every* $x \in X'_l$*. Evidently, these variables are dependent of* y*, so doing the calculations for each fixed value of* y *we obtain*

$$X_1' = \begin{cases} [2.5, 4.5], & \text{if } y \leqslant 4, \\ [2.5, 8.5 - y], & \text{if } 4 < y \leqslant 5, \\ \varnothing, & \text{otherwise.} \end{cases}; \qquad X_2' = \begin{cases} \varnothing, & \text{if } y \leqslant 4, \\ ]8.5 - y, 4.5], & \text{if } 4 < y \leqslant 5, \\ [2.5, 4.5], & \text{otherwise.} \end{cases}.$$

Now, we can calculate $\beta_i = \bigwedge_{x \in X'_{i_{A'>A_i}}} [A'(x) \to A_i(x)]$ *for* $i = 1, 2$, *of these regions of* y *according to Proposition 9. The obtained outputs are shown in Figure 22 and described as follows:*

$$\beta_1 = \begin{cases} 0.5, & \text{if } y \leqslant 4.3, \\ (0.5 \cdot y - 1.75), & \text{if } 4.3 < y \leqslant 5, \\ 1, & \text{otherwise.} \end{cases}; \qquad \beta_2 = \begin{cases} 1, & \text{if } y \leqslant 4.3, \\ (2.75 - 0.5 \cdot y), & \text{if } 4.3 < y \leqslant 5, \\ 0, & \text{otherwise.} \end{cases}.$$



(a) Output fuzzy set $\beta_1 * B_1$ relative to consequent B_1

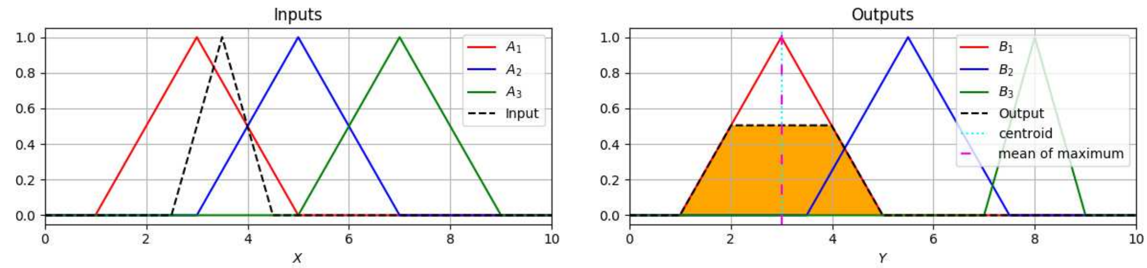
(b) Output fuzzy set $\beta_2 * B_2$ relative to consequent B_2

Figure 22 – Partial outputs for the combination of BKS and conjunctive rules

Finally, combining these partial results we obtain the following output fuzzy set that is represented in Figure 23:

$$B'(y) = \begin{cases} 0.5 \wedge B_1(y), & \text{if } y \leqslant 4, \\ (0.5 \wedge B_1(y)) \vee (B_2(y)), & \text{if } 4 < y \leqslant 4.3, \\ [(0.5 \cdot y - 1.75) \wedge B_1(y)] \vee [(-0.5 \cdot y + 2.75) \wedge B_2(y)], & \text{if } 4.3 < y \leqslant 5, \\ 0, & \text{otherwise.} \end{cases}$$

$$= 0.5 \wedge B_1(y).$$



(a) Input fuzzy set and antecedents (b) Output fuzzy set and consequents

Figure 23 – Input and output for the combination of BKS and conjunctive rules

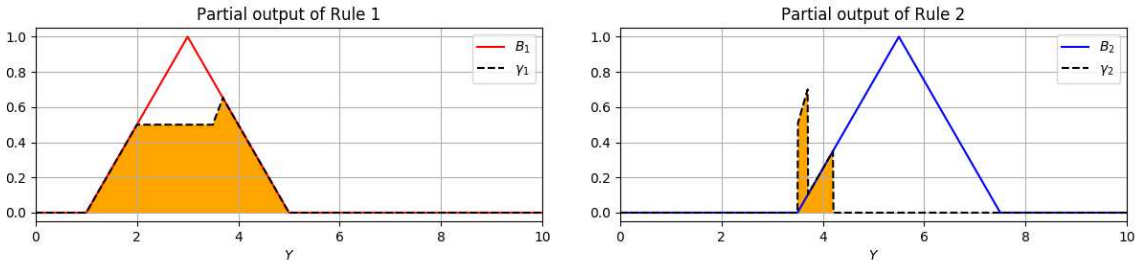
For the combination of CRI and implicative rules, we already have the sets I and $X' = Supp(A')$ defined from the previous combination. To apply Corollary 5 we must obtain the indexes i_l and the sets X'_l such that $\bigcup_{l=1}^{m} X'_l = X'$ and $\bigwedge_{i \in I}(A_i(x) \to B_i(y)) = A_{i_l}(x) \to B_{i_l}(y)$ for every $x \in X'_l$, like we did before. Doing the calculations we arrive at the following sets:

$$X'_1 = \begin{cases} [2.5, 4.5], & \textit{if } y \leqslant 1, \\ [2.5, 3], & \textit{if } 1 < y \leqslant 3.5, \\ [2.5, y - 0.5], & \textit{if } 3.5 < y \leqslant 4.5, \\ [2.5, 4.5], & \textit{otherwise.} \end{cases}; \qquad X'_2 = \begin{cases} \varnothing, & \textit{if } y \leqslant 1, \\ ]3, 4.5], & \textit{if } 1 < y \leqslant 3.5, \\ ]y - 0.5, 4.5], & \textit{if } 3.5 < y \leqslant 4.5, \\ \varnothing, & \textit{otherwise.} \end{cases}.$$

Calculating the output for each of these regions of y according to Corollary 5, we obtain the following partial outputs shown in Figure 24:

$$\gamma_1 = \bigvee_{x \in X'_{1_{A_1 \leqslant B_1(y)}}} A'(x) \vee \bigvee_{x \in X'_{1_{A_1 > B_1(y)}}} (A'(x) \wedge B_1(y)) = \begin{cases} B_1(y), & \textit{if } y \leqslant 2, \\ 0.5, & \textit{if } 2 < y \leqslant 3, \\ 0.75 \cdot y - 2.12, & \textit{if } 3.5 < y \leqslant 3.7, \\ B_1(y), & \textit{if } 3.7 < y. \end{cases};$$

$$\gamma_2 = \bigvee_{x \in X'_{2_{A_2 \leqslant B_2(y)}}} A'(x) \vee \bigvee_{x \in X'_{2_{A_2 > B_2(y)}}} (A'(x) \wedge B_2(y)) = \begin{cases} B_2(y), & \textit{if } y \leqslant 3.5, \\ y - 3, & \textit{if } 3.5 < y \leqslant 3.7, \\ B_2(y), & \textit{if } 3.7 < y \leqslant 4.2. \\ 0, & \textit{if } 4.2 < y. \end{cases}.$$

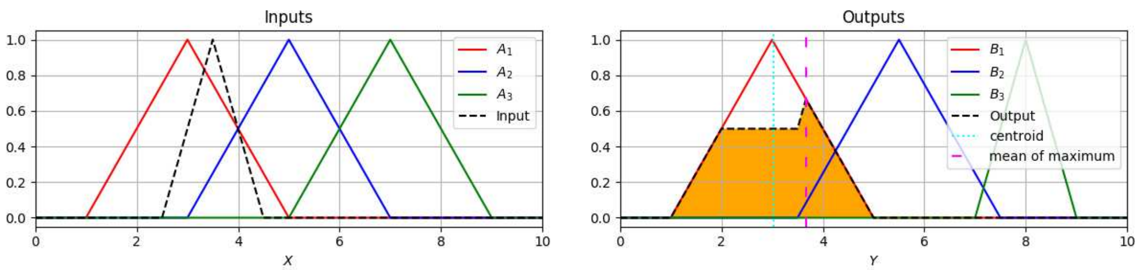


(a) Output fuzzy set γ_1 relative to consequent B_1 (b) Output fuzzy set γ_2 relative to consequent B_2

Figure 24 – Partial outputs for the combination of CRI and implicative rules

Combining these partial results we obtain the following output fuzzy set that is represented in Figure 25:

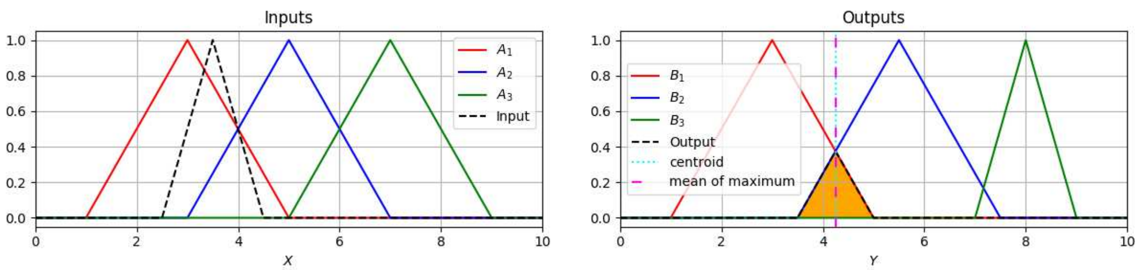
$$B'(y) = \begin{cases} 0.5 \wedge B_1(y), & \textit{if } y \leqslant 3.5, \\ y - 3, & \textit{if } 3.5 < y \leqslant 3.7, \\ B_1(y), & \textit{otherwise.} \end{cases}$$



(a) Input fuzzy set and antecedents (b) Output fuzzy set and consequents

Figure 25 – Input and output for the combination of CRI and implicative rules

For the combination of BKS with implicative rules, we replace the calculated values of α_i in Proposition 11 to obtain the output fuzzy set $B'(y) = (0.83 \rightarrow_M B_1(y)) \wedge (0.5 \rightarrow_P B_2(y))$, that is shown in Figure 26.



(a) Input fuzzy set and antecedents (b) Output fuzzy set and consequents

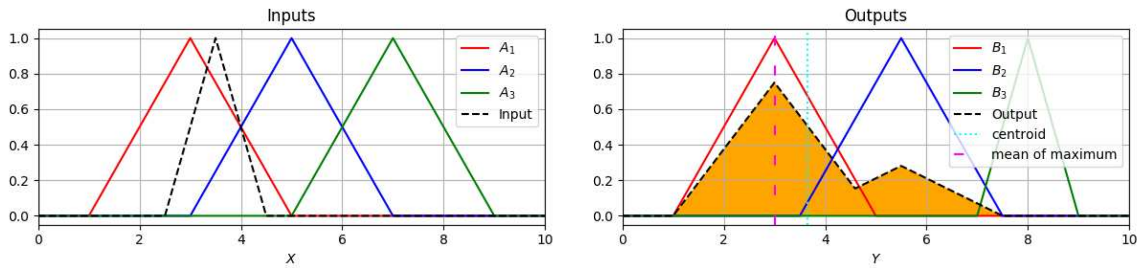
Figure 26 – Input and output for the combination of BKS and implicative rules

Example 10. *Now, consider the product t-norm $\cdot$ and its associated Goguen implication $\to_P$ with the same input $A'(x) = triang(x, 2.5, 3.5, 4.5)$ and the rule base from Table 5.*

First, we need to calculate again the values of α_i for each rule as follows:

$$\alpha_1 = \bigvee_{x\in X} (A'(x) \cdot A_1(x)) = 0.75;$$
$$\alpha_2 = \bigvee_{x\in X} (A'(x) \cdot A_2(x)) \approx 0.28;$$
$$\alpha_3 = \bigvee_{x\in X} (A'(x) \cdot A_3(x)) = 0.$$

For the combination of CRI with conjunctive rules, we replace the calculated values of α_i in Proposition 8 and obtain as output the set $B'(y) \approx (0.75 \cdot B_1(y)) \vee (0.28 \cdot B_2(y))$. This output is represented in Figure 27.



(a) Input fuzzy set and antecedents (b) Output fuzzy set and consequents

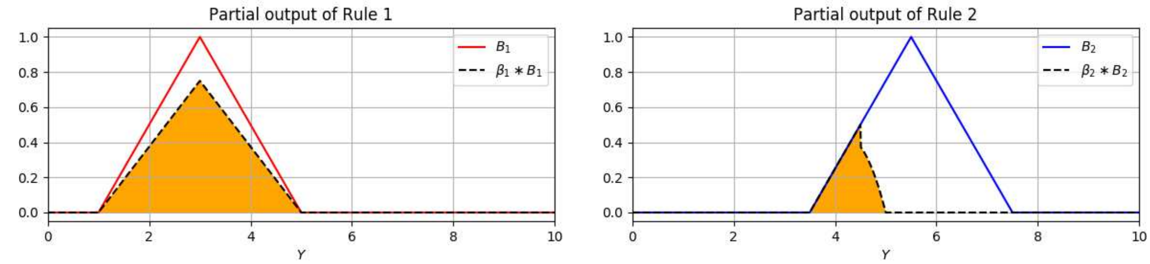
Figure 27 – Input and output for the combination of CRI and conjunctive rules

For the combination of BKS and conjunctive rules, we can use the sets $I = \{1, 2\}$ and $X' = [2.5, 4.5]$ defined in the last example. Then, to apply Proposition 9, we determine the sets X'_l as

$$X'_1 = \begin{cases} [2.5, 4.5], & \text{if } y \leqslant 4, \\ [2.5, 8.2 - y], & \text{if } 4 < y \leqslant 5, \\ \varnothing, & \text{otherwise.} \end{cases}; \qquad X'_2 = \begin{cases} \varnothing, & \text{if } y \leqslant 4, \\ ]8.2 - y, 4.5], & \text{if } 4 < y \leqslant 5, \\ [2.5, 4.5], & \text{otherwise.} \end{cases}.$$

Calculating $\beta_i = \bigwedge_{x\in X'_{i_{A'>A_i}}} [A'(x) \to A_i(x)]$ for $i = 1, 2$, for each of these regions of y according to Proposition 9, we obtain the following results represented in Figure 28:

$$\beta_1 = \begin{cases} 0.75, & \text{if } y \leqslant 5, \\ 1, & \text{if } 5 < y. \end{cases}; \qquad \beta_2 = \begin{cases} 1, & \text{if } y \leqslant 4.5, \\ (7.5 - 1.5 \cdot y), & \text{if } 4.5 < y \leqslant 5, \\ 0, & \text{if } 5 < y. \end{cases}.$$



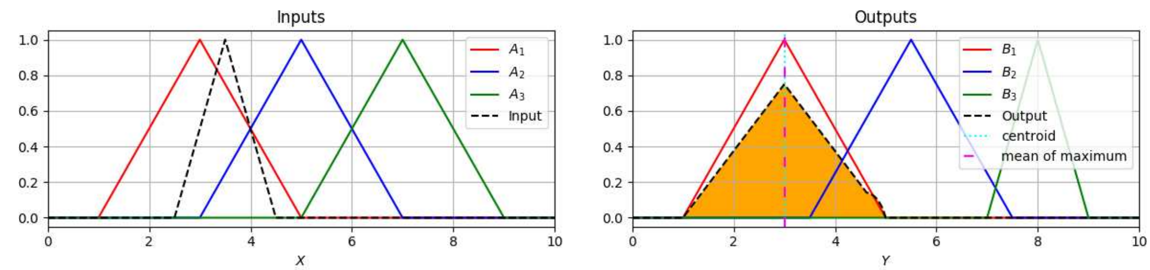
(a) Output fuzzy set $\beta_1 * B_1$ relative to consequent B_1

(b) Output fuzzy set $\beta_2 * B_2$ relative to consequent B_2

Figure 28 – Partial outputs for the combination of BKS and conjunctive rules

Combining these results we obtain the following output fuzzy set that is represented in Figure 23:

$$B'(y) \approx \begin{cases} 0.75 \cdot B_1(y), & \textit{if } y \leqslant 4, \\ (0.75 \cdot B_1(y)) \vee (B_2(y)), & \textit{if } 4 < y \leqslant 4.5, \\ (0.75 \cdot B_1(y)) \vee [(-1.5 \cdot y + 7.75) \cdot B_2(y)], & \textit{if } 4.5 < y \leqslant 5, \\ 0, & \textit{otherwise.} \end{cases}.$$



(a) Input fuzzy set and antecedents

(b) Output fuzzy set and consequents

Figure 29 – Input and output for the combination of BKS and conjunctive rules

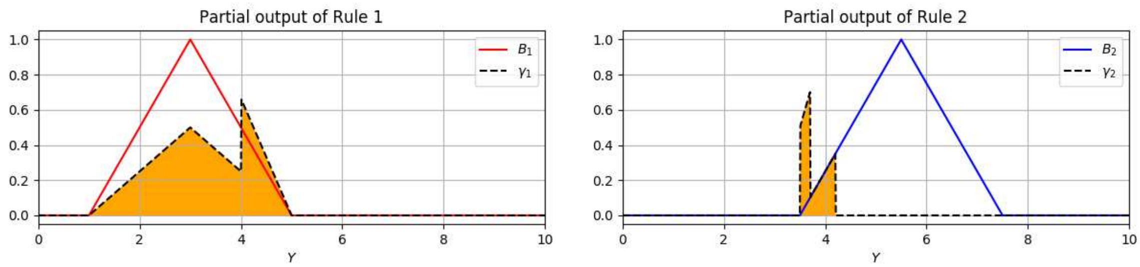
For the combination of CRI with implicative rules, we also can use the sets $I = \{1, 2\}$ *and* $X' = [2.5, 4.5]$ *defined in the last example. Then, to apply Corollary 5, we calculate the sets* X'_l *as*

$$X'_1 = \begin{cases} [2.5, 4.5], & \textit{if } y \leqslant 1, \\ [2.5, 3], & \textit{if } 1 < y \leqslant 3.5, \\ [2.5, y - 0.5], & \textit{if } 3.5 < y \leqslant 4.5, \\ [2.5, 4.5], & \textit{otherwise.} \end{cases}; \qquad X'_2 = \begin{cases} \varnothing, & \textit{if } y \leqslant 1, \\ ]3, 4.5], & \textit{if } 1 < y \leqslant 3.5, \\ ]y - 0.5, 4.5], & \textit{if } 3.5 < y \leqslant 4.5, \\ \varnothing, & \textit{otherwise.} \end{cases}.$$

Again, we calculate the output for each of these regions of y according to Corollary 5 and we obtain the following results represented in Figure 30:

$$\gamma_1 = \bigvee_{x \in X'_{1_{A_1 \leqslant B_1(y)}}} A'(x) \vee \bigvee_{x \in X'_{1_{A_1 > B_1(y)}}} (A'(x) \cdot \frac{B_1(y)}{A_1(x)}) = \begin{cases} 0.5 \cdot B_1(y), & \textit{if } y \leqslant 4, \\ 1.33 \cdot B_1(y), & \textit{if } 4 < y. \end{cases};$$

$$\gamma_2 = \bigvee_{x \in X'_{2_{A_2 \leqslant B_2(y)}}} A'(x) \vee \bigvee_{x \in X'_{2_{A_2 > B_2(y)}}} (A'(x) \cdot \frac{B_2(y)}{A_2(x)}) = \begin{cases} B_2(y), & \textit{if } y \leqslant 3.5, \\ y - 3, & \textit{if } 3.5 < y \leqslant 3.7, \\ B_2(y), & \textit{if } 3.7 < y \leqslant 4.2. \\ 0, & \textit{if } 4.2 < y. \end{cases}.$$

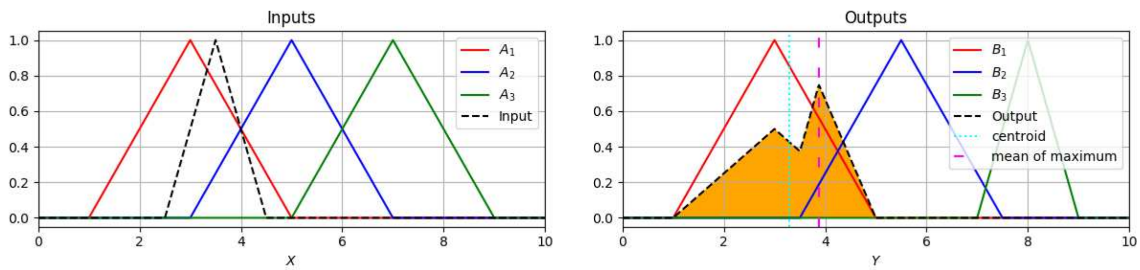


(a) Output fuzzy set γ_1 relative to consequent B_1 (b) Output fuzzy set γ_2 relative to consequent B_2

Figure 30 – Partial outputs for the combination of CRI and implicative rules

Combining these partial results we obtain the following output fuzzy set that is represented in Figure 31:

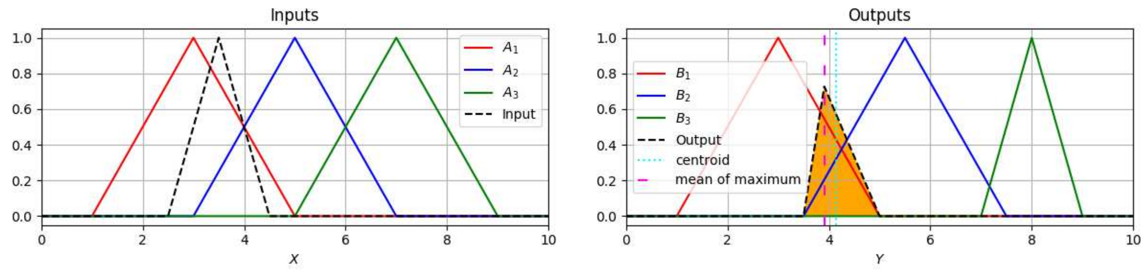
$$B'(y) = \begin{cases} 0.5 \cdot B_1(y), & \textit{if } y \leqslant 3.5, \\ y - 3, & \textit{if } 3.5 \leqslant y < 3.7, \\ 1.33 \cdot B_1(y), & \textit{otherwise.} \end{cases}$$



(a) Input fuzzy set and antecedents (b) Output fuzzy set and consequents

Figure 31 – Input and output for the combination of CRI and implicative rules

Finally, for the combination of BKS and implicative rules, we replace the values of α_i in Proposition 11 to obtain the output $B'(y) \approx (0.75 \rightarrow_P B_1(y)) \wedge (0.28 \rightarrow_P B_2(y))$, that is represented in Figure 32.



(a) Input fuzzy set and antecedents (b) Output fuzzy set and consequents

Figure 32 – Input and output for the combination of BKS and implicative rules

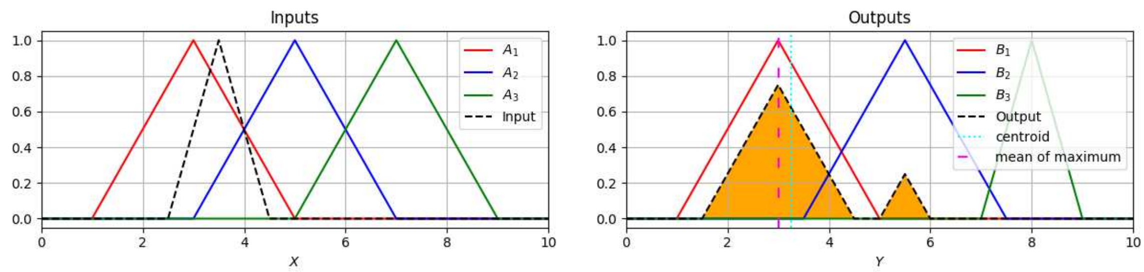
Example 11. *Finally, consider the Lukasiewicz t-norm $*_L$ and the Lukasiewicz implication $\rightarrow_L$. Also, consider the same input fuzzy set $A'(x) = triang(x, 2.5, 3.5, 4.5)$ as before and the rule base from Table 5.*

Calculating the values of α_i for each rule, we obtain the following values:

$$\alpha_1 = \bigvee_{x \in X} (A'(x) *_L A_1(x)) = 0.75;$$
$$\alpha_2 = \bigvee_{x \in X} (A'(x) *_L A_2(x)) = 0.25;$$
$$\alpha_3 = \bigvee_{x \in X} (A'(x) *_L A_3(x)) = 0.$$

For the combination of CRI and conjunctive rules, we replace these values in Proposition 8 to obtain the following output represented in Figure 33:

$$\begin{aligned} B'(y) &= (0.75 *_L B_1(y)) \vee (0.25 *_L B_2(y)) \\ &= [0 \vee (0.75 + B_1(y) - 1)] \vee [0 \vee (0.25 + B_2(y) - 1)] \\ &= [0 \vee (B_1(y) - 0.25)] \vee [0 \vee (B_2(y) - 0.75)] \\ &= [0 \vee (B_1(y) - 0.25) \vee (B_2(y) - 0.75)]. \end{aligned}$$



(a) Input fuzzy set and antecedents (b) Output fuzzy set and consequents

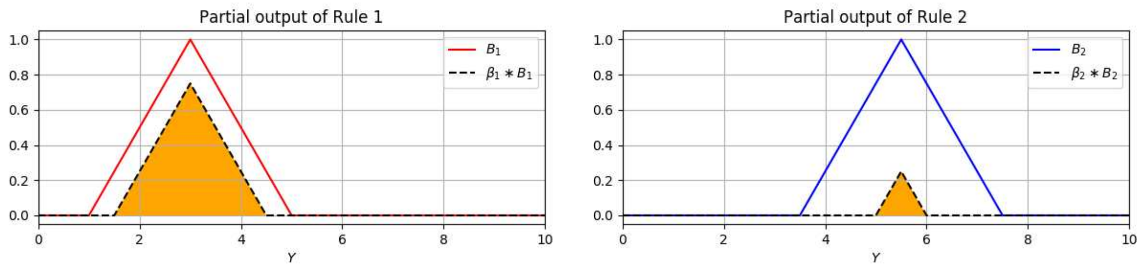
Figure 33 – Input and output for the combination of CRI and conjunctive rules

For the combination of BKS and conjunctive rules, we can use the sets $I = \{1, 2\}$ *and* $X' = [2.5, 4.5]$ *defined in the last examples. Then, to apply Proposition 9, we determine the sets* X'_l *as*

$$X'_1 = \begin{cases} [2.5, 4.5], & \text{if } y \leqslant 4, \\ [2.5, 8.5 - y], & \text{if } 4 < y \leqslant 5, ; \\ \varnothing, & \text{otherwise.} \end{cases} \qquad X'_2 = \begin{cases} \varnothing, & \text{if } y \leqslant 4, \\ ]8.5 - y, 4.5], & \text{if } 4 < y \leqslant 5, ; \\ [2.5, 4.5], & \text{otherwise.} \end{cases}$$

Again, we calculate $\beta_i = \bigwedge_{x \in X'_{i_{A' > A_i}}} [A'(x) \to A_i(x)]$ *for* $i = 1, 2$, *for each of these regions of* y *according to Proposition 9 and obtain the following partial outputs shown in Figure 34:*

$$\beta_1 = \begin{cases} 0.75, & \text{if } y \leqslant 5, \\ 1, & \text{if } 5 < y. \end{cases}; \qquad \beta_2 = \begin{cases} 0.25, & \text{if } y \leqslant 4.5, \\ (7.5 - 1.5 \cdot y), & \text{if } 4.5 < y \leqslant 5, \\ 0.25, & \text{if } 5 < y. \end{cases}.$$



(a) Output fuzzy set $\beta_1 * B_1$ relative to consequent B_1

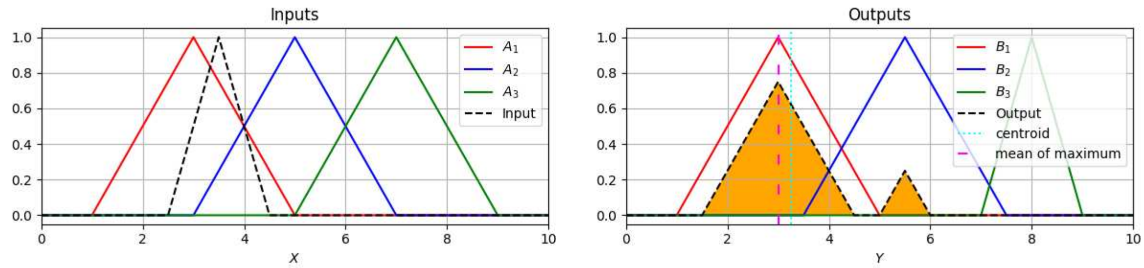
(b) Output fuzzy set $\beta_2 * B_2$ relative to consequent B_2

Figure 34 – Partial outputs for the combination of BKS and conjunctive rules

Combining these partial results we obtain the following output fuzzy set that is represented in Figure 35:

$$B'(y) \approx \begin{cases} 0.75 *_L B_1(y), & \text{if } y \leqslant 4, \\ (0.75 *_L B_1(y)) \vee (B_2(y)), & \text{if } 4 < y \leqslant 4.5, \\ (0.75 *_L B_1(y)) \vee [(-1.53 \cdot y + 7.8) *_L B_2(y)], & \text{if } 4.5 < y \leqslant 5. \\ 0.25 *_L B_2(y), & \text{otherwise.} \end{cases}$$

$$= (0.75 *_L B_1(y)) \vee (0.25 *_L B_2(y)).$$



(a) Input fuzzy set and antecedents (b) Output fuzzy set and consequents

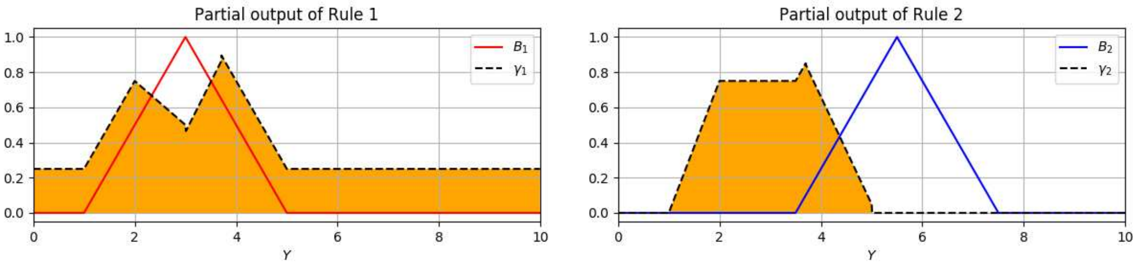
Figure 35 – Input and output for the combination of BKS and conjunctive rules

For the combination of CRI with implicative rules, we still use the sets $I = \{1, 2\}$ *and* $X' = [2.5, 4.5]$ *defined in the lasts examples. Then, to apply Corollary 5, we calculate the sets* X'_l *as*

$$X'_1 = \begin{cases} [2.5, 4], & \textit{if } y \leqslant 1, \\ [2.5, 4.5 - 0.5 \cdot y], & \textit{if } 1 < y \leqslant 3, \\ [2.5, 0.75 \cdot y + 0.75], & \textit{if } 3 < y \leqslant 5, \\ [2.5, 4.5], & \textit{otherwise}. \end{cases}; \qquad X'_2 = \begin{cases}]4, 4.5], & \textit{if } y \leqslant 1, \\ ]4.5 - 0.5 \cdot y, 4.5], & \textit{if } 1 < y \leqslant 3, \\ ]0.75 \cdot y + 0.75, 4.5], & \textit{if } 3 < y \leqslant 5, \\ \varnothing, & \textit{otherwise}. \end{cases}.$$

Calculating the output for each of these regions of y *according to Corollary 5 we obtain the following results represented in Figure 36:*

$$\begin{aligned} \gamma_1 &= \bigvee_{x \in {X'_1}_{A_1 \leqslant B_1(y)}} A'(x) \vee \bigvee_{x \in {X'_1}_{A_1 > B_1(y)}} [0 \vee (A'(x) + B_1(y) - A_1(x))] \\ &\approx \begin{cases} 0.75 \to_L B_1(y), & \textit{if } y \leqslant 2, \\ 1.25 - 0.25 \cdot y, & \textit{if } 2 < y \leqslant 3. \\ 0.57 \cdot y - 1.25, & \textit{if } 3 < y \leqslant 3.7, \\ 0.75 \to_L B_1(y), & \textit{if } 3.7 < y. \end{cases}; \\ \gamma_2 &= \bigvee_{x \in {X'_2}_{A_2 \leqslant B_2(y)}} A'(x) \vee \bigvee_{x \in {X'_2}_{A_2 > B_2(y)}} [0 \vee (A'(x) + B_2(y) - A_2(x))] \\ &\approx \begin{cases} 0, & \textit{if } y \leqslant 1, \\ 0.25 \to_L B_2(y), & \textit{if } 1 < y \leqslant 3.7, \\ 3.1 - 0.61 \cdot y, & \textit{if } 3.7 < y \leqslant 5. \\ 0, & \textit{if } 5 < y. \end{cases}. \end{aligned}$$

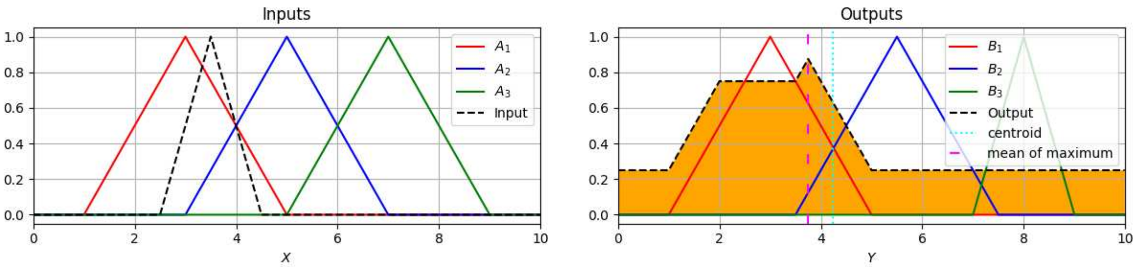


(a) Output fuzzy set γ_1 relative to consequent B_1 (b) Output fuzzy set γ_2 relative to consequent B_2

Figure 36 – Partial outputs for the combination of CRI and implicative rules

Combining these partial results we obtain the following output fuzzy set that is represented in Figure 37:

$$B'(y) = (B_1(y) + 0.25) \wedge (B_2(y) + 0.75)$$

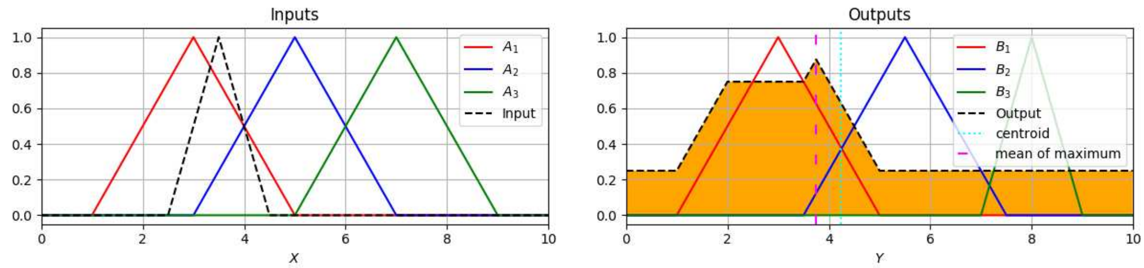


(a) Input fuzzy set and antecedents (b) Output fuzzy set and consequents

Figure 37 – Input and output for the combination of CRI and implicative rules

Finally, for the combination of BKS with implicative rules, we replace the calculated values of α_i in Proposition 11 as follows to obtain the output represented in Figure 38:

$$\begin{aligned}
B'(y) &= (0.75 \rightarrow_L B_1(y)) \wedge (0.25 \rightarrow_L B_2(y)) \\
&= [1 \wedge (B_1(y) - 0.75 + 1)] \wedge [1 \wedge (B_2(y) - 0.25 + 1)] \\
&= [1 \wedge (B_1(y) + 0.25)] \wedge [1 \wedge (B_2(y) + 0.75)] \\
&= [1 \wedge (B_1(y) + 0.25) \wedge (B_2(y) + 0.75)].
\end{aligned}$$



(a) Input fuzzy set and antecedents (b) Output fuzzy set and consequents

Figure 38 – Input and output for the combination of BKS and implicative rules

In these examples, we could see the behaviours of different pairs of t-norms and implications among the studied combinations. It is interesting to note that the combination of CRI and implicative rules in general results in more complex outputs, as it where expected from the development of equations for a single fuzzy rule. Another interesting fact is that, for this input and rule base, using the Lukasiewicz t-norm and implication the output was related only to the type of fuzzy rules chosen (conjunctive or implicative).

Example 12. *Consider the sets* $X, Y = [0, 10] \subset \mathbb{R}$*, an input* $A'(x) = triang(x, 0, 0.5, 1)$ *and the same rule base as before:*

Antecedents	**Consequents**
$A_1 = triang(x, 1, 3, 5)$	$B_1 = triang(y, 1, 3, 5)$
$A_2 = triang(x, 3, 5, 7)$	$B_2 = triang(y, 3.5, 5.5, 7.5)$
$A_3 = triang(x, 5, 7, 9)$	$B_3 = triang(y, 7, 8, 9)$

Table 6 – Rule base with multiple fuzzy rules

First, calculating the values of α_i *for each rule we have*

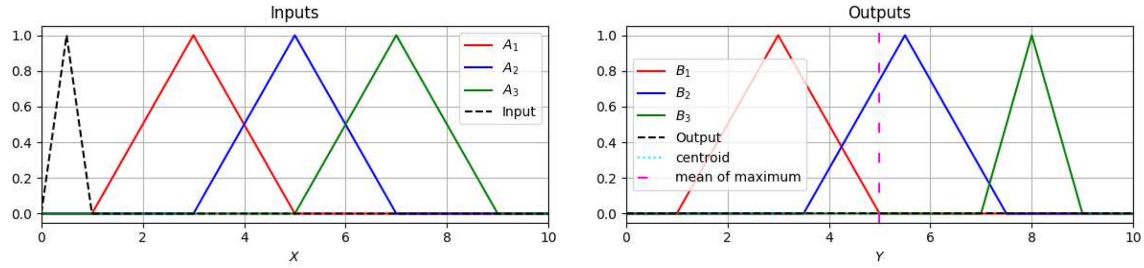
$$\alpha_i = \bigvee_{x \in X} (A'(x) * A_i(x)) = 0, \quad \forall i \in [1, 3],$$

independently of the t-norm $*$ *used.*

This implies that in the combination of CRI and conjunctive rules, from Proposition 8, we will have the output $B'(y) = 0$.

Similarly, for the combination of BKS and conjunctive rules, taking any $x \in X'$ *according to Proposition 9 we will have that* $\bigvee_{i=1}^{3} (A_i(x) * B_i(y)) = \bigvee_{i=1}^{3} (0 * B_i(y)) = 0$. *Then,* $\bigwedge_{x \in X'} (A'(x) \to 0) = A'(0.5) \to 0 = 1 \to 0 = 0$. *So, the output for this combination will also be* $B'(y) = 0$.

From this, we can conclude that independently from the t-norm and adjoint implication chosen, when we use conjunctive rules in this case we will have the output represented in Figure 39.



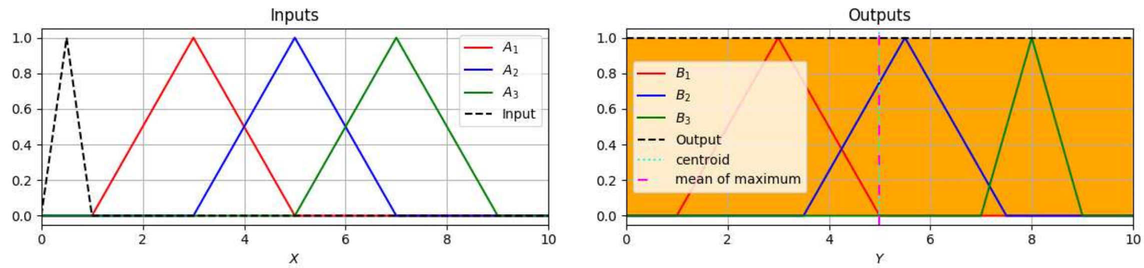
(a) Input fuzzy set and antecedents (b) Output fuzzy set and consequents

Figure 39 – Input and output for the combinations using Conjunctive rules

For the combination of CRI and implicative rules, following Proposition 10, taking any $x \in X'$ *we will have* $\bigwedge_{i=1}^{3}(A_i(x) \to B_i(y)) = \bigwedge_{i=1}^{3}(0 \to B_i(y)) = 1$. *Then,* $\bigvee_{x \in X'}(A'(x) * 1) = A'(0.5) * 1 = 1 * 1 = 1$. *So, for this combination the output will be* $B'(y) = 1$.

For the combination of BKS and implicative rules, since $\alpha_i = 0$ *as seen before, applying Proposition 11 we have that* $\bigwedge_{i=1}^{3} \alpha_i \to B_i(y) = \bigwedge_{i=1}^{3} 0 \to B_i(y) = 1$. *So, we have as output the fuzzy set* $B'(y) = 1$.

For implicative rules, we have the following results in Figure 40: Similarly, we can conclude that independently from the t-norm and adjoint implication chosen, when we use implicative rules in this case we will have the output represented in Figure 40.



(a) Input fuzzy set and antecedents (b) Output fuzzy set and consequents

Figure 40 – Input and output for the combinations using Implicative rules

In this example we have seen that for this scenario where there is no intersection between the input and the fuzzy rules, the output is related only on the type of the rules, independent of the t-norm and residual implication used.

To finish this study, we will look at three more examples where the input is totally contained in one of the antecedents.

Example 13. *Consider the sets* $X, Y = [0, 10] \subset \mathbb{R}$*, an input* $A'(x) = triang(x, 4, 5, 6)$ *and the same rule base as before:*

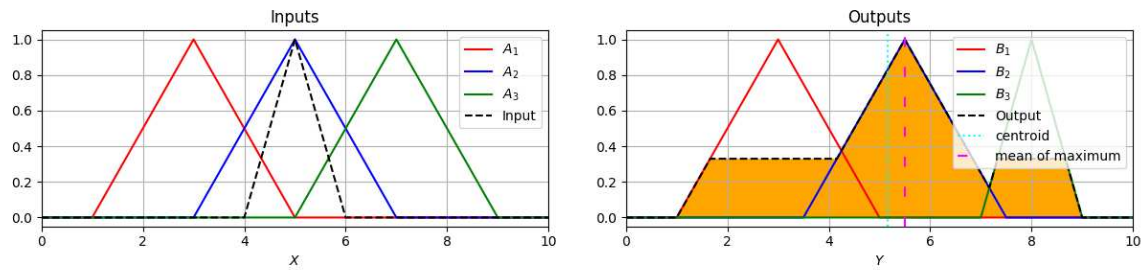
Antecedents	**Consequents**
$A_1 = triang(x, 1, 3, 5)$	$B_1 = triang(y, 1, 3, 5)$
$A_2 = triang(x, 3, 5, 7)$	$B_2 = triang(y, 3.5, 5.5, 7.5)$
$A_3 = triang(x, 5, 7, 9)$	$B_3 = triang(y, 7, 8, 9)$

Table 7 – Rule base with multiple fuzzy rules

Also, consider the minimum t-norm $\wedge$ *and the Gödel implication* $\rightarrow_M$*. With this, we can calculate the following values of* α_i *for each rule:*

$$\alpha_1 = \bigvee_{x \in X} (A'(x) \wedge A_1(x)) \approx 0.33;$$
$$\alpha_2 = \bigvee_{x \in X} (A'(x) \wedge A_2(x)) = 1;$$
$$\alpha_3 = \bigvee_{x \in X} (A'(x) \wedge A_3(x)) \approx 0.33.$$

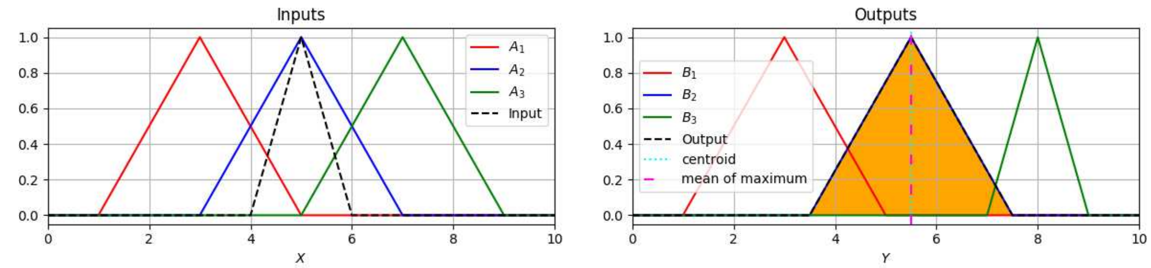
For the combination of CRI and conjunctive rules, we replace these values in Proposition 8 to obtain the output fuzzy set $B'(y) \approx (0.33 \wedge B_1(y)) \vee B_2(y) \vee (0.33 \wedge B_3(y))$ *represented in Figure 41.*



(a) Input fuzzy set and antecedents (b) Output fuzzy set and consequents

Figure 41 – Input and output for the combination of CRI and conjunctive rules

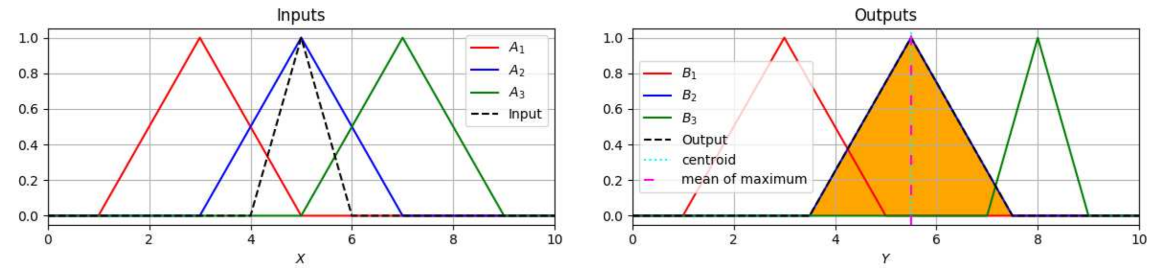
For the combination of BKS and conjunctive rules, we apply Theorem 1 we obtain $B'(y) = B_2(y)$ *as output, that is shown in Figure 42.*



(a) Input fuzzy set and antecedents (b) Output fuzzy set and consequents

Figure 42 – Input and output for the combination of BKS and conjunctive rules

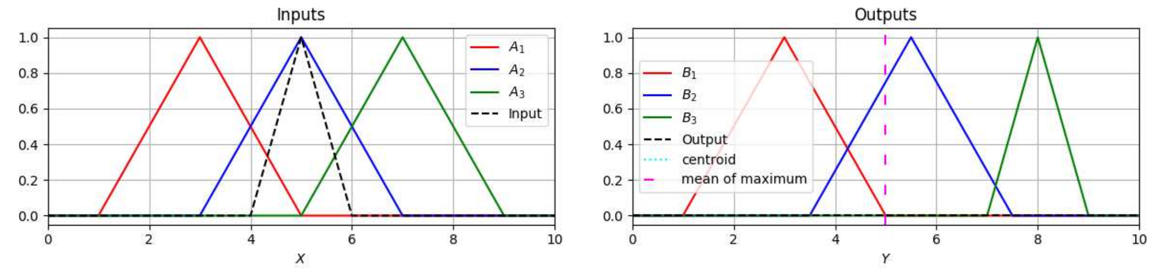
Likewise, for the combination of CRI and implicative rules, we apply Theorem 3 to also get $B'(y) = B_2(y)$ as output, that is shown in Figure 43.



(a) Input fuzzy set and antecedents (b) Output fuzzy set and consequents

Figure 43 – Input and output for the combination of CRI and implicative rules

Finally, for the combination of BKS with implicative rules, we use the calculated values of α_i on Proposition 11 to obtain the output $B'(y) = (0.33 \rightarrow_M B_1(y)) \wedge B_2(y) \wedge (0.33 \rightarrow_M B_3(y)) = 0$, that is represented in Figure 44.



(a) Input fuzzy set and antecedents (b) Output fuzzy set and consequents

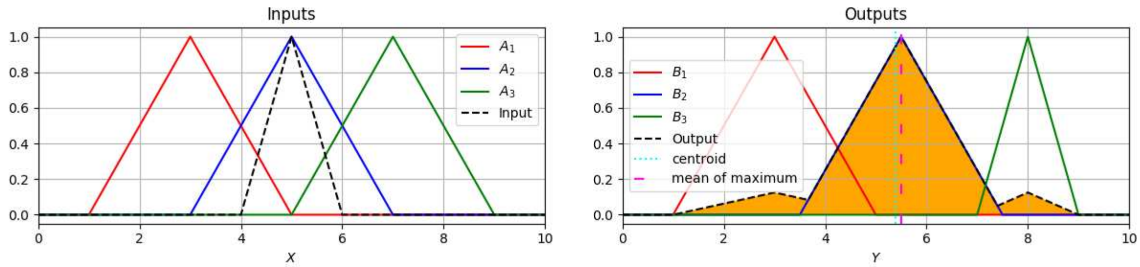
Figure 44 – Input and output for the combination of BKS and implicative rules

Example 14. *Considering now the product t-norm $\cdot$, the Goguen implication $\rightarrow_P$, the same input $A'(x) = triang(x, 4, 5, 6)$ and the same rule base from Table 7.*

Firstly, calculating the values of α_i for each rule, we have

$$\alpha_1 = \bigvee_{x \in X} (A'(x) \cdot A_1(x)) = 0.125;$$
$$\alpha_2 = \bigvee_{x \in X} (A'(x) \cdot A_2(x)) = 1;$$
$$\alpha_3 = \bigvee_{x \in X} (A'(x) \cdot A_3(x)) = 0.125.$$

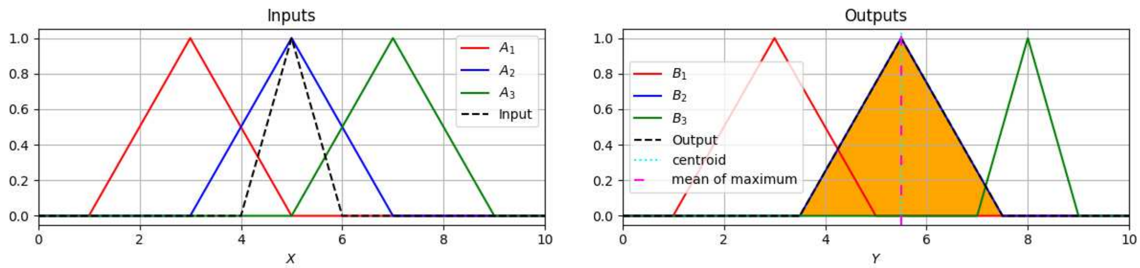
For the combination of CRI with conjunctive rules, we replace these values in Proposition 8 to obtain the output $B'(y) = (0.125 \cdot B_1(y)) \vee B_2(y) \vee (0.125 \cdot B_3(y))$, that is shown in Figure 45.



(a) Input fuzzy set and antecedents (b) Output fuzzy set and consequents

Figure 45 – Input and output for the combination of CRI and conjunctive rules

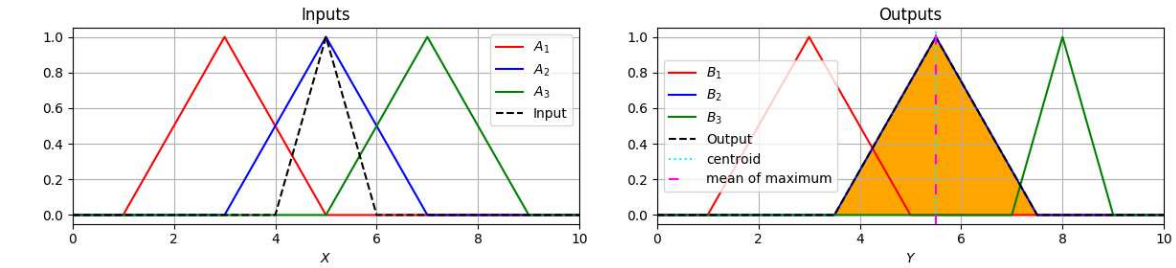
For the combination of BKS with conjunctive rules, we just apply Theorem 1 to obtain the output $B'(y) = B_2(y)$, that is represented in Figure 46.



(a) Input fuzzy set and antecedents (b) Output fuzzy set and consequents

Figure 46 – Input and output for the combination of BKS and conjunctive rules

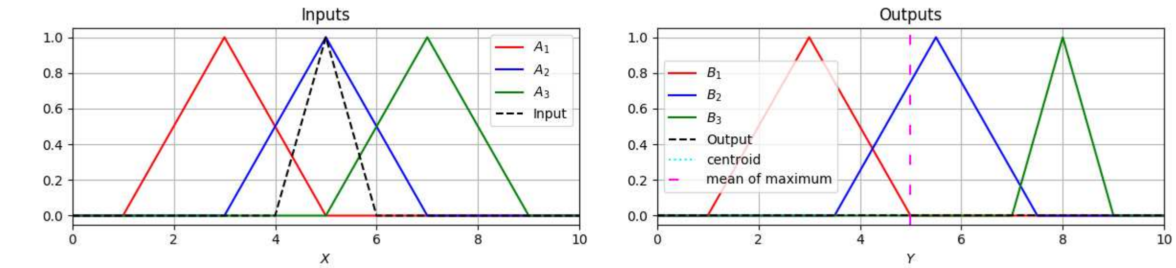
Similarly, for the combination of CRI and implicative rules, we just apply Theorem 3 to obtain as output $B'(y) = B_2(y)$, shown in Figure 47.



(a) Input fuzzy set and antecedents (b) Output fuzzy set and consequents

Figure 47 – Input and output for the combination of CRI and implicative rules

For the combination of BKS and implicative rules, we use the calculated values of α_i *on Proposition 11 to obtain the output* $B'(y) = (0.125 \to_P B_1(y)) \wedge B_2(y) \wedge (0.125 \to_P B_3(y)) = 0$*, that is represented in Figure 48.*



(a) Input fuzzy set and antecedents (b) Output fuzzy set and consequents

Figure 48 – Input and output for the combination of BKS and implicative rules

Example 15. *Considering the Lukasiewicz t-norm* $*_L$*, the Lukasiewicz implication* $\to_L$*, the same input* $A'(x) = triang(x, 4, 5, 6)$ *and the same rule base from Table 7.*

Again, calculating the values of α_i *for each rule, we have*

$$\alpha_1 = \bigvee_{x \in X} (A'(x) *_L A_1(x)) = 0;$$
$$\alpha_2 = \bigvee_{x \in X} (A'(x) *_L A_2(x)) = 1;$$
$$\alpha_3 = \bigvee_{x \in X} (A'(x) *_L A_3(x)) = 0.$$

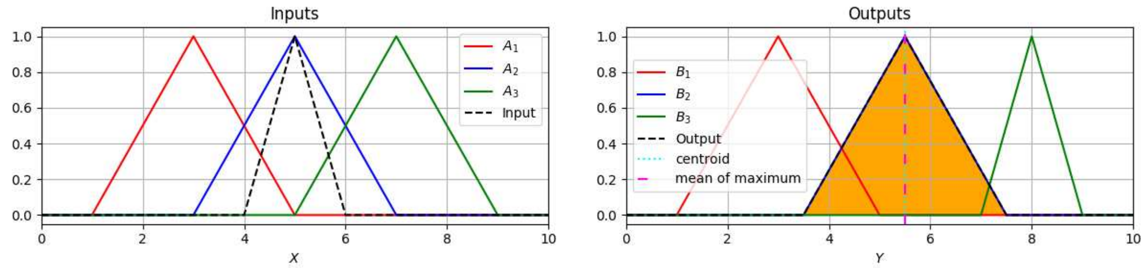
For the combination of CRI and conjunctive rules, we replace these values in Proposition 8 to obtain the output $B'(y) = B_2(y)$*.*

For the combination of BKS with conjunctive rules, using Theorem 1 we also obtain $B'(y) = B_2(y)$ *as output.*

Similarly, for the combination of CRI and implicative rules, just applying Theorem 3 we obtain the output $B'(y) = B_2(y)$*.*

Finally, for the combination of BKS and implicative rules, we use the values of α_i *in Proposition 11 to also get the output* $B'(y) = B_2(y)$.

As we can see, in this scenario where the input is completely contained in one of the antecedents and we are using the Lukasiewicz t-norm and implication, the output is equal to the corresponding consequent, independently of the combination. This is represented in Figure 49.



(a) Input fuzzy set and antecedents (b) Output fuzzy set and consequents

Figure 49 – Input and output for the combinations using Lukasiewicz t-norm and implication

In these examples we could see an interesting property of the combinations BKS with conjunctive rules and CRI with implicative rules. For these combinations, when the input is totally contained in one of the antecedents, the output is equal to the correpondent consequent. This property is known as interpolativity and it will be seen again in the next chapter. Another remark can be made for the example using the Lukasiewicz t-norm and implication, in this case all combinations presented this interpolativity property.

4 Moser-Navara Axioms

4.1 Original Moser-Novara Axioms and its adaptation for implicative systems

To ensure the robustness and coherence of fuzzy relational systems, B. Moser and M. Navara (MOSER; NAVARA, 2002) proposed three axioms that these systems should obey primarily.

Conceptually the axioms are as follows:

1. Interpolation: Given an input A' equals to and antecedent A_i, the output of the system should be equal to the corresponding consequent B_i.
2. Significance of the generated outputs: For all normal input A' $(\exists x \in X | A'(x) = 1)$, the system produces a non-trivial output, which means, $\exists y \in Y | B'(y) \neq 0$ in case of conjunctive rules and $\exists y \in Y | B'(y) \neq 1$ in case of implicative rules.
3. Robustness: Given any input, the corresponding output must be contained in the union of the consequents of the activated rules $(A' \cap A_i \neq \varnothing)$ in case of conjunctive rules; or must contain the intersection of consequents of activated rules in case of implicative rules.

Stepnicka and Mandal (STEPNICKA; MANDAL, 2015) (STEPNICKA, 2016) (STEPNICKA; MANDAL, 2018) presented a formal definition of these axioms considering any composition @ (CRI $\circ$ or BKS $\lhd$) with conjunctive rules:

A_C1 For all $i \in 1, \ldots, n$

$$A_i @ \check{R} = B_i;$$

A_C2 For each normal input $A' \in \mathcal{F}(X)$ there exists an index i such that

$$A' @ \check{R} \nsubseteq B_i;$$

A_C3 The output $A' @ \check{R}$ belongs to the union of consequents B_i of activated rules, which means,

$$A' @ \check{R} \subseteq \bigcup_{i \in F} B_i$$

where $F = \{i|Supp(A_i) \cap Supp(A') \neq \varnothing\}, (B_i \cup B_j)(y) = B_i(y) \vee B_j(y)$.

And analogously for implicative rules:

A_I1 For all $i \in 1, \ldots, n$

$$A_i @ \hat{R} = B_i;$$

A_I2 For each normal input $A' \in \mathcal{F}(X)$ there exists an index i such that

$$A' @ \hat{R} \not\supseteq B_i;$$

A_I3 The output $A' @ \hat{R}$ contains the intersection of consequents B_i of activated rules, which means,

$$A' @ \hat{R} \supseteq \bigcap_{i \in F} B_i$$

where $F = \{i|Supp(A_i) \cap Supp(A') \neq \varnothing\}, (B_i \cap B_j)(y) = B_i(y) \wedge B_j(y)$.

Originally, Moser and Navara proposed these axioms for the combination of the CRI composition with conjunctive rules and demonstrated that this combination does not satisfy axioms 1 and 2 simultaneously under general conditions according to the following proposition:

Proposition 12. *(MOSER; NAVARA, 2002) Be $*$ a t-norm without zero divisors. Be the fuzzy sets $A_i, i = 1, \ldots, n$ continuous and normal; and $B_i, i = 1, \ldots, n$ with mutually distinct supports. So the combination of the conjunctive rules model $\check{R}$ and the CRI composition $\circ$ do not satisfy axioms A_C1 and A_C2 simultaneously.*

This proposition can be illustrated with a simple example.

Example 16. *Considering the minimum t-norm $\wedge$, the universes $X = Y = [0, 10] \subseteq \mathcal{R}$ and the following fuzzy rule base:*

	Antecedents	**Consequents**
R_1**:**	$A_1(x) = triang(x, 0, 2, 4)$	$B_1(y) = triang(y, 1, 2, 3)$
R_2**:**	$A_2(x) = triang(x, 2, 4, 6)$	$B_2(y) = triang(y, 4, 5, 6)$
R_3**:**	$A_3(x) = triang(x, 4, 6, 8)$	$B_3(y) = triang(y, 7, 8, 9)$

Table 8 – Fuzzy rule base.

Suppose an input $A'(x) = A_2(x) = triang(x, 2, 4, 5)$, then the output $B'(y)$ of a CRI composition with conjunctive rules is given in Figure 50.

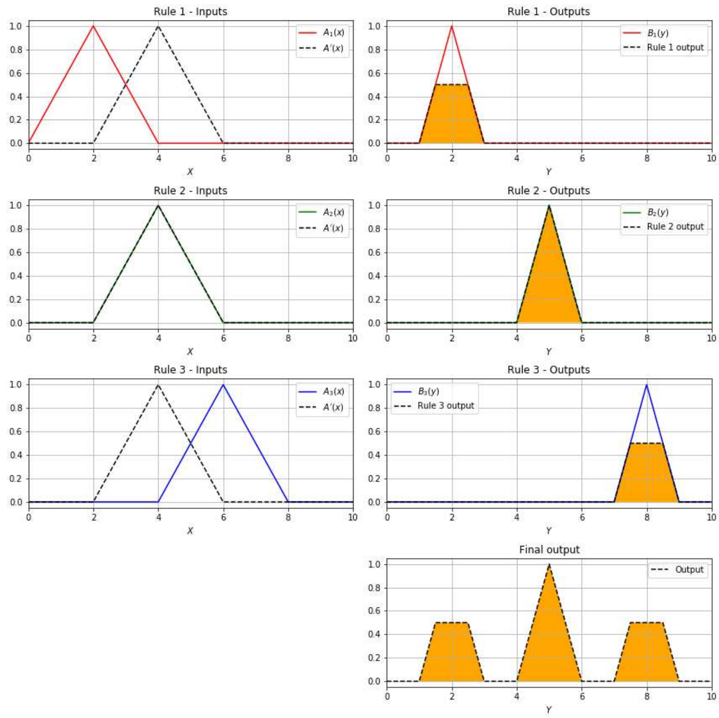


Figure 50 – Example of axioms violation.

Clearly, we have a violation of the axiom A_C1*, because given an input* $A'(x) = A_2(x)$ *we have an output* $B'(y) \neq B_2(y)$*.*

Analogously to the proposition that accompanies the original formulation of the axioms, Stepnicka and Mandal showed that the combination of implicative rules with the BKS composition does not satisfy axioms 1 and 2 simultaneously.

Proposition 13. *(STEPNICKA; MANDAL, 2015) Let* $*$ *be a left-continuous t-norm and without zero divisors. Let* $A_i, i = 1, \ldots, n$ *be continuous and normal and let* $B_i, i = 1, \ldots, n$ *fuzzy sets with mutually different supports. So the model of implicative rules* $\hat{R}$ *(with the residual implication derived from the t-norm* $*$*) and the BKS composition* $\lhd$ *do not satisfy axioms* A_I1 *and* A_I2 *simultaneously.*

Once again, let us illustrate this proposition using the last example as a basis.

Example 17. *Considering Gödel's implication* $\rightarrow_M$*, the universes* $X = Y = [0, 10] \subseteq \mathcal{R}$ *and the same fuzzy rule base as before:*

	Antecedents	**Consequents**
R_1**:**	$A_1(x) = triang(x, 0, 2, 4)$	$B_1(y) = triang(y, 1, 2, 3)$
R_2**:**	$A_2(x) = triang(x, 2, 4, 6)$	$B_2(y) = triang(y, 4, 5, 6)$
R_3**:**	$A_3(x) = triang(x, 4, 6, 8)$	$B_3(y) = triang(y, 7, 8, 9)$

Table 9 – Fuzzy rule base.

Suppose an input $A'(x) = A_2(x) = triang(x, 2, 4, 6)$, then the output $B'(y)$ of a BKS composition with implicative rules is given in Figure 51.

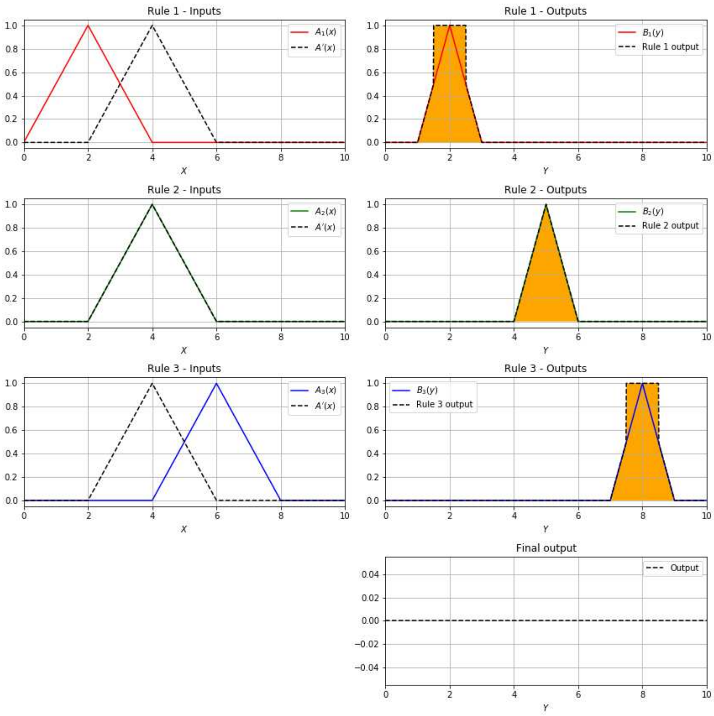


Figure 51 – Example of axioms violation.

Again we have a violation of the first axiom A_I1.

4.2 Ideal combinations

Continuing his work, Martin Stepnicka (STEPNICKA, 2016) analyzed the combinations between: CRI ($\circ$) and implicative rules ($\hat{R}$); and BKS ($\lhd$) and conjunctive rules ($\check{R}$). These combinations satisfy the axioms under some restrictions as will be seen below.

Theorem 5. *(STEPNICKA, 2016) Be $(A_i, B_i)_{N_n}$ an implicative fuzzy rule base, let A_i be normal and let*

$I1$ "covering antecedents":

$$\bigwedge_{x\in X}\bigvee_{i=1}^{n} A_i(x) \geqslant c, \quad c \in (0, 1]$$

$I2$ "disjointness of antecedents": $\forall i \in N_n$, $\exists x_i$ such that

$$A_i(x_i) = 1 \ e \ A_j(x_i) = 0, \ whenever \ i \neq j$$

$I3$ *"specificity of consequents": for each $i \in N_n$, there is a $y_i \in Y$ such that for all $j \neq i$:*

$$B_i(y_i) > c \to B_j(y_i)$$

if the input A' is normal and using the CRI composition $\circ$, then all three axioms A_I1 - A_I3 are simultaneously satisfied.

Theorem 6. *(STEPNICKA, 2016) Be $(A_i, B_i)_{N_n}$ a conjunctive fuzzy rule base, let A_i be normal and let*

$C1$ *"covering antecedents":*

$$\bigwedge_{x \in X} \bigvee_{i=1}^{n} A_i(x) \geqslant c, \quad c \in (0, 1]$$

$C2$ *"disjointness of antecedents": $\forall i \in N_n$, $\exists x_i$ such that:*

$$A_i(x_i) = 1 \text{ e } A_j(x_i) = 0, \text{ whenever } i \neq j$$

$C3$ *"specificity of consequents": for each $i \in N_n$, there is a $y_i \in Y$ such that for all $j \neq i$:*

$$B_i(y_i) < c * B_j(y_i)$$

If the input A' is normal and using the BKS composition $\lhd$, then all three axioms A_C1 - A_C3 are simultaneously satisfied.

5 Applications

In this chapter, we will apply the relational systems presented before to a series of reference problems, available in the repository *Knowledge Extraction Based on Evolutionary Learning* (ALCALA-FDEZ et al., 2011).

5.1 Wang-Mendel fuzzy rule learning algorithm

Learning the fuzzy rule base is done using the procedure proposed in (WANG; MENDEL, 1992). The learning algorithm performs the following steps:

1. Partitioning of attribute spaces and outputs into fuzzy sets:
   a) Be the attributes $x_i, i \in [1, n]$, and the output y.
   b) Taking their domain intervals $X_i = [x_i^-, x_i^+]$ and $Y = [y^-, y^+]$.
   c) Each one of these intervals is divided into N fuzzy regions (N can be different for each attribute).

   Obs: In this work, triangular membership functions will be used for each fuzzy region, so that one of the vertices is in the central position of the fuzzy region with unit value and the other two are in the centers of the neighboring regions.
2. For each training data, a new rule is defined and its "strength" is calculated:
   a) First, the degrees of membership of the inputs and outputs in each of the fuzzy regions are calculated.
   b) A fuzzy rule is composed of the fuzzy regions with the highest membership degrees for each variable.
   c) Finally, a "strength" is associated to the rule, given by the product of the membership degrees of each variable.
3. Cleaning the rules:
   a) For identical rules, the highest "strength" found prevails and duplicates are removed.
   b) For the equal rules with different consequents, the one with the greatest "strength" prevails and the inconsistent ones are removed.

After this training phase (learning the fuzzy rules), the fuzzy relational systems discussed in the previous chapters are applied and at the end the centroid and mean of maximum (MOM) methods are used for the defuzzification of the results (KLIR; YUAN, 1995)(PEDRYCZ; GOMIDE, 2007)(BARROS; BASSANEZI; LODWICK, 2017).

5.2 Regression problems

The reference regression problems considered in this work are: Diabetes, Ele-1, Plastic, Quake, Laser, Ele-2, AutoMPG6, MachineCPU, Dee and AutoMPG8. More information about these datasets can be found in the links of Appendix A.

The Table 10 contains basic information about the datasets: the number of samples, the quantities of numerical attributes and the number of rules learned by the Wang-Mendel algorithm.

	Number of samples	**Numerical attributes**	**# of rules learned**
Diabetes	43	2	15
Ele-1	495	2	13
Plastic	1650	2	15
Quake	2178	3	53
Laser	993	4	58
Ele-2	1056	4	65
AutoMPG6	392	5	115
MachineCPU	209	6	35
Dee	365	6	177
AutoMPG8	392	7	161

Table 10 – Description of the datasets

In the fuzzy relational systems, the t-norm of the minimum $\wedge$ and its adjunct implication $\rightarrow_M$ (Gödel's implication) were used. The conjunction used to aggregate the antecedents A_{ij} of an i rule was the minimum $\wedge$. Each of the domain ranges X_i and Y was divided into $N = 5$ fuzzy regions, just to be able to draw a comparison between the different combinations of compositions and fuzzy rules.

In each experiment, the root mean square error (RMSE) is calculated for each combination and the values can be seen in Tables 11 and 12. The numbers written in *italic text* are the best results in each experiment configuration (i.e. in each row of each table), and the numbers written in **bold text** are the best results for each dataset independent of the defuzzification method (i.e. best for each row in each pair of tables). In this first simulation, crisp inputs were considered for the relational systems.

	Conjunctive Rules + CRI	**Conjunctive Rules + BKS**	**Implicative Rules + CRI**	**Implicative Rules + BKS**
Diabetes	***0.47***	***0.47***	0.52	0.52
Ele-1	***715.7***	***715.7***	1167.3	1167.3
Plastic	***1.76***	***1.76***	2.37	2.37
Quake	***0.21***	***0.21***	0.28	0.28
Laser	***10.8***	***10.8***	28.4	28.4
Ele-2	***267.3***	***267.3***	644.9	644.9
AutoMPG6	***4.1***	***4.1***	6.6	6.6
MachineCPU	153.4	153.4	*119.8*	*119.8*
Dee	***0.56***	***0.56***	0.73	0.73
AutoMPG8	***4.2***	***4.2***	7.1	7.1

Table 11 – Results using centroid defuzzification

	Conjunctive Rules + CRI	**Conjunctive Rules + BKS**	**Implicative Rules + CRI**	**Implicative Rules + BKS**
Diabetes	*0.74*	*0.74*	**0.74**	**0.74**
Ele-1	*1141.02*	*1141.02*	1183.04	1183.04
Plastic	*2.68*	*2.68*	3.25	3.25
Quake	*0.30*	*0.30*	0.39	0.39
Laser	*29.95*	*29.95*	43.35	43.35
Ele-2	638.77	638.77	*509.38*	*509.38*
AutoMPG6	5.56	5.56	*4.71*	*4.71*
MachineCPU	120.47	120.47	***109.47***	***109.47***
Dee	0.71	0.71	*0.64*	*0.64*
AutoMPG8	4.91	4.91	*4.25*	*4.25*

Table 12 – Results using MOM defuzzification

As already highlighted in Chapter 3, the results of combinations with conjunctive rules are the same, considering a crisp input. Analogously for relational systems with implicative rules. Regarding the accuracy of the results, it can be noted that in most problems the relational systems with conjunctive rules obtained a better performance using the centroid defuzzification. The MOM defuzzification seems to be more adequate for FRSs that use implicative rules, specially in problems with higher number of input variables.

Next, in Tables 13 and 14 we have the results of a simulation with the same regression problems but now considering a fuzzy input. For the fuzzification of the inputs, triangular membership functions centered on the point in question x_i and with vertices at $x_i - r$ and $x_i + r$ were considered, where r is 5% of the domain interval of that variable . That is, $r = 0.05(x_i^+ - x_i^-)$ and $A_i'(x) = triang(x, x_i - r, x_i, x_i + r)$.

	Conjunctive Rules + CRI	Conjunctive Rules + BKS	Implicative Rules + CRI	Implicative Rules + BKS
Diabetes	0.69	*0.67*	1.78	1.77
Ele-1	***727.4***	975.8	1206.5	1760.6
Plastic	2.23	***2.08***	4.8	5.6
Quake	***0.28***	0.48	0.74	0.83
Laser	19.4	***15.8***	70.3	78.8
Ele-2	***367.4***	488.1	808.7	1584.9
AutoMPG6	***3.7***	6.3	15.0	22.4
MachineCPU	186.7	187.3	*137.9*	156.1
Dee	*0.64*	0.68	1.71	2.27
AutoMPG8	*5.5*	5.6	7.8	11.2

Table 13 – Results considering a fuzzy input using centroid defuzzification

	Conjunctive Rules + CRI	Conjunctive Rules + BKS	Implicative Rules + CRI	Implicative Rules + BKS
Diabetes	***0.64***	***0.64***	0.79	0.81
Ele-1	1142.38	1112.94	*1057.77*	1698.88
Plastic	*2.73*	*2.73*	3.11	3.35
Quake	*0.33*	*0.33*	0.37	0.39
Laser	26.46	*25.45*	34.04	46.29
Ele-2	663.31	640.56	*404.56*	782.84
AutoMPG6	4.7	5.02	*4.28*	5.46
MachineCPU	***108.17***	146.79	119.12	145.04
Dee	0.6	0.65	***0.55***	0.60
AutoMPG8	4.78	4.4	***3.22***	3.59

Table 14 – Results considering a fuzzy input using MOM defuzzification

As expected, the results were a little worse than in previous experiments due to the uncertainty considered in the inputs. It is also possible to conclude that performance is more related to the choice of fuzzy rules representation (conjunctive or implicative rules) and defuzzification method, than to the composition (CRI or BKS). This fact could also be deduced from the results presented in Chapter 3.

It is interesting to note that for the problems with a higher number of variables (specifically the last four), the results obtained using a fuzzy input were the best overall. This could indicate that the datasets are not large enough to learn an exhaustive rule base, and in this case an uncertainty in the inputs is beneficial for the performance.

In general, the relational systems using implicative rules had worst performance. This can be explained by the way that the Wang-Mendel algorithm works: it learns conjunctive rules by acumulating information, and not excluding possibilities, which aligns with the interpretation made by Dubois et al. (DUBOIS; UGHETTO; PRADE, 1999).

6 Final considerations

In this master's thesis, a bibliographic review of three important works related to fuzzy relational systems was made: the study of combinations of fuzzy compositions and conjunctive/implicative fuzzy rules by Martin Stepnicka et al. and, consequently, the axioms proposed by Bernhard Moser and Mirko Navara for evaluating inference systems; the interpretation of conjunctive and implicative fuzzy rules proposed by Didier Dubois et al.; and the fuzzy rule learning algorithm of Li-Xin Wang and Jerry Mendel.

The results presented do not intend to draw definitive conclusions about the fuzzy relational systems, but rather to provoke discussions on the subject and also to present different points of view, possibly new for some readers.

As mentioned throughout the text, the Mamdani-Assilian relational system (MAMDANI; ASSILIAN, 1975) has historically been the most used in practical applications. This application on a large scale is undoubtedly beneficial for the dissemination of knowledge of fuzzy set theory, but its indiscriminate use can also generate negative effects, such as the use of methods without proper knowledge of the theoretical bases. In this sense, we saw that there are alternatives to this combination of CRI composition with conjunctive rules, which can be as good or even more adequate depending on the situation.

The applications to real problems in Chapter 5, although simple, highlight an important fact: the choice of how to model the fuzzy rules directly influences the obtained results, more than the choice of the fuzzy composition. On this subject, there is a fertile field for the elaboration of new, more extensive studies, mainly in the sense of validating the hypothesis raised by Didier Dubois et al. (DUBOIS; UGHETTO; PRADE, 1999) that conjunctive rules are more suitable for modeling the knowledge acquired through data (observations), while implicative rules would be more suitable for modeling an expert's knowledge (constraints). In fact, another theme also suggested by this author is the combination of these two types of rules in a single fuzzy rule base, in order to complement each other.

Finally, the presentation of the Moser-Navara axioms and the most recent results obtained by Martin Stepnicka et al. was made in order to add more theoretical foundation to the discussed subject. However, despite violating the axioms, the combination of CRI composition with conjunctive rules performs well in real problems. This is directly related to the way that the rules are generated by the Wang-Mendel algorithm and the chosen defuzzification method. Once again, there is much room for further work exploring the possible negative implications that violating these axioms can have.

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APPENDIX A – Regression problems

In the following list are the electronic addresses for the reference problems used in the Applications chapter of this work. In these addresses it is possible to see a brief description of each dataset and the attributes that compose them.

Diabetes <http://sci2s.ugr.es/keel/dataset.php?cod=45>

Ele-1 <http://sci2s.ugr.es/keel/dataset.php?cod=39>

Plastic <http://sci2s.ugr.es/keel/dataset.php?cod=74>

Quake <http://sci2s.ugr.es/keel/dataset.php?cod=75>

Laser <http://sci2s.ugr.es/keel/dataset.php?cod=47>

Ele-2 <https://sci2s.ugr.es/keel/dataset.php?cod=40>

AutoMPG6 <https://sci2s.ugr.es/keel/dataset.php?cod=80>

MachineCPU <https://sci2s.ugr.es/keel/dataset.php?cod=82>

Dee <https://sci2s.ugr.es/keel/dataset.php?cod=46>

AutoMPG8 <https://sci2s.ugr.es/keel/dataset.php?cod=79>