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Contextuality: A Topological and Geometrical Journey

Contextualidade: Uma Jornada Topológica e Geométrica

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Supervisor: Marcelo de Oliveira Terra Cunha

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"To effectively contain a civilization's development and disarm it across such a long span of time, there is only one way: kill its science."
(Liu Cixin in The Three-Body Problem)

"The incomplete is to blame for the unknown right now...
Not everything unanswered needs an answer
The harder it is, the more fun it gets
We have to be logical, I know it
Let's oppose this crisis."
(BiSH "in case..." in Godzilla Singular Point)

Resumo

Nesta tese, apresentamos uma jornada de investigação da relação entre contextualidade, topologia e geometria. Utilizamos duas abordagens para a contextualidade: a Abordagem de Feixes (Sheaf Approach) e a Abordagem de Contextualidade Generalizada, assim como representações gráficas em feixe (bundle) dos modelos empíricos. Tomando como objetos fundamentais as medições e os contextos de compatibilidade, a Abordagem de Feixes nos permite uma parcial identificação topológica da contextualidade. Utilizando a fração contextual, apresentamos a n-contextualidade, uma hierarquia na construção do cenário de medição por sua estrutura de complexo simplicial que permite verificar a influência da topologia do cenário de medição sobre o comportamento contextual. A Abordagem de Contextualidade Generalizada se volta a uma representação geométrica das representações ontológicas clássicas e da contextualidade, e possui processos como objetos fundamentais. É para esta abordagem que nos voltamos ao identificar as limitações de tratar as medições como fundamentais, e sua forma mais geral de contextualidade baseada no Princípio da Identidade dos Indiscerníveis de Leibniz nos permite construir uma abordagem baseada em geometria e topologia diferencial. Nela, a contextualidade de uma representação ôntica se expressa de uma maneira superior às probabilidades, como uma forma diferencial análoga ao tensor eletromagnético. Codificar esta forma se relaciona com a escolha do tipo de realismo escolhido. Esta abstração do gerador da contextualidade nos permite relacioná-lo à fração contextual, interferência em teoria de medida quântica, não-comutatividade, medidas com sinal e mergulho ontológico, permitindo-nos generalizar o teorema de Voroby'ev e tratar modelos perturbativos. Neste nível de abstração, nós nos voltamos ao caso específico do problema de marginalização do conhecimento de agentes, conhecido como paradoxos em cenários multi-agente, tendo como exemplo cenários de Amiga de Wigner estendidos. Generalizando o conceito de confiança entre conjuntos de agentes e identificando a construção de uma Verdade Fundamental efetiva, nós utilizamos a topologia induzida explícita na Abordagem de Feixes para construir seu equivalente na semântica topológica da lógica multi-modal do conhecimento de agentes. Munidos de tal mapa e conhecendo suas limitações, nós recuperamos o uso da lógica multi-modal para cenários não-clássicos e identificamos que a origem dos paradoxos nos principais exemplos de cenários de Amiga de Wigner estendidos são gerados pela contextualidade.

Palavras-chave: Contextualidade; Holonomia; Cohomologia; Abordagem de Feixe; Abordagem Diferencial; Lógica Modal; Cenários Multi-Agente.

Abstract

In this thesis, we present an investigative journey into the relationship between contextuality, topology, and geometry. We utilize two approaches to contextuality: the Sheaf Approach and the Generalized Contextuality Approach, as well as bundle representations of empirical models. Taking measurements and compatibility contexts as fundamental objects, the Sheaf Approach allows us to partially identify the topological nature of contextuality. Using the contextual fraction, we present n-contextuality, a hierarchy in constructing the measurement scenario through its simplicial complex structure, which allows us to verify the influence of the measurement scenario's topology on contextual behavior. The Generalized Contextuality Approach focuses on a geometric representation of classical ontological representations and contextuality, with processes as fundamental objects. We turn to this approach when identifying the limitations of treating measurements as fundamental, and its more general form of contextuality based on Leibniz's Principle of the Identity of Indiscernibles allows us to construct an approach based on differential geometry and topology. In this approach, the contextuality of an ontic representation is expressed in a manner superior to probabilities, as a differential form analogous to the electromagnetic tensor. Encoding this form relates to the choice of the type of realism selected. This abstraction of the generator of contextuality allows us to relate it to the contextual fraction, interference in quantum measure theory, non-commutativity, signed measures, and ontological embedding, enable us to generalize Voroby'ev theorem and address disturbing models. At this level of abstraction, we turn to the specific case of the marginalization problem of agents' knowledge, known as paradoxes in multi-agent scenarios, using extended Wigner's Friend scenarios as examples. By generalizing the concept of trust among sets of agents and identifying the construction of an effective Fundamental Truth, we use the explicitly induced topology in the Sheaf Approach to construct its equivalent in the topological semantics of the multi-modal logic of agents' knowledge. Armed with such a map and knowing its limitations, we recover the use of multi-modal logic for non-classical scenarios and identify that the origin of the paradoxes in the main examples of extended Wigner's Friend scenarios is generated by contextuality.

Keywords: Contextuality; Holonomy; Cohomology; Sheaf Approach; Differential Approach; Modal Logic; Multi-Agent Scenarios.

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1 Introduction

1.0.1 Justifications and Objectives

Why should one delve into learning topology, cohomology, and geometry for contextuality? The most straightforward answer is that you don't need to if your application doesn't require it. A generalization, so common in mathematics, does not focus on the application, but on the foundation of the concept. The same is true for contextuality.

Leaving a more formal concept serves its broad application in areas that are sometimes far from its origin. For example, leaving it free for semi-ring valuation is justified by its use in computer science. But the math can scale so fast that, in the end, few people will exploit such results for application: they may be too general for the original concept to be identifiable. The hope is that in the future, this level of abstraction will be useful if an application is found that needs more depth. An already evident application is the possibility of questioning what contextuality really is. Fundamentally, what are we dealing with? An example is the Kochen-Specker theorem, which is logically defined and rarely encountered by the vast majority of physicists in their careers. In any case, there is the possibility of interpreting contextuality as a violation of logic. Fundamentally clear formalism does not necessarily imply direct application, but it has a lot of strength in general.

It's the physicist's problem: they can only delve into the mathematics needed and won't go further once the objective is to come up with results. Many mathematical works do not even have a conclusion, as if the author had been lost in mathematics, perhaps for some aesthetic value. However, the path taken is already self-justified by simple exploration, leaving it to the reader to make connections with possible applications. Mathematicians seek to open the way. It's okay to dig deep enough to come back with results, but it's interesting to know that if you want or need a little more, there is something deeper. If you cannot find what you want at a certain depth, it may be interesting to delve a little further to gain perspective on what is being explored, and it is of great importance that the path is already signposted and explored.

This thesis follows a journey in seeking to understand contextuality within the intersection between topology and geometry. The idea is to follow the intuition that contextuality is a failure of a local structure to be seen as the marginalization of a global one, similar to how charts in a topological manifold may or may not be described by a single chart. Our main objective is to explore this analogy, seeking to deepen and simplify the intuition and clarify the formalization. With this, a series of applications becomes possible, where contextuality is expressed in different forms, all seeking to represent the same underlying phenomenon. With contextuality taking a deeper form, it is possible to identify it in logical paradoxes, those related to marginal problems of some kind. In our case, we focus on the problem of marginalization of agents' knowledge, which our mathematical tools can, albeit superficially, identify as contextuality, the same that is present in quantum systems, but in a different guise. This journey ends up having another objective that is expressed in a dialectical manner, that contextuality is the most important resource for non-classical technologies, and it is also a deeper phenomenon than we can ontologically capture.

1.1 Historical Perspective

To understand the objectives and justifications of an author in his/her studies of a subject, one needs to know the historical evolution of the subject and the personal reasons of the author to clearly see the forces orienting the development of the present thesis. I will provide here, with some limitations, the historical evolution of contextuality with some key references.

1.1.1 Until 1950

Something was wrong. It had been more than two hundred years since Isaac Newton presented his ideas, which gave rise to a systematic formalization of classical mechanics, with unprecedented developments in the eighteenth and nineteenth centuries. The classical realm began to face challenges, imposing modifications on how we view its components, which had been so well accepted since at least ancient Greece. But such modifications were nothing compared to the perfect storm at the beginning of the twentieth century. The newborn quantum theory was a necessary formalism to address contradictory empirical data. However, prominent figures like Erwin Schrödinger, with his famous wave description [Schrödinger 1926], attempted to keep it at the level of modifications to classical theory, as had been done with Albert Einstein's relativity. These efforts, however, were in vain. The probabilistic interpretation of Schrödinger's wave function by Max Born [Born and Jordan 1925] agreed, as shown by Schrödinger himself [Schrödinger 1926], with the point of view introduced by Werner Heisenberg [Heisenberg 1925] and developed by him and Paul Dirac [Dirac 1935], Born, Pascual Jordan, and Wolfgang Pauli, where the quantum properties can only be said to be real when observed. Apparently, reality as we understand it is just a fantasy[1].

John von Neumann formalized quantum theory mathematically [Neumann 2018] and was the first to question the impossibility of a classical representation of it. Once accepting the amazing results of quantum theory, one could ask about its completeness. The

[1] For a presentation in a scientific dissemination manner of the history of quantum theory, see Refs. [Rovelli 2020, Hürter 2021].

discussion between Niels Bohr and Einstein was exactly about accepting the consequences of the completeness of quantum theory, with the Einstein–Podolsky–Rosen paradox [Einstein, Podolsky and Rosen 1935] being an example of how strange and crazy quantum behavior was and still is to the scientific community. Einstein wished for a realistic description of quantum phenomena that preserved classical behavior, with a modification as a hidden ontic variable. The anti-realist view and the Copenhagen interpretation were defended by Bohr as a way to think about quantum phenomena. It had increased acceptance following the necessity of applications in the Second World War and the importance of those after it, to the detriment of a new journey in the foundations [Ronde 2020].

1.1.2 1950s and 1960s

A realistic description of quantum theory was first introduced by Louis de Broglie and later completed by David Bohm [Bohm 1952a, Bohm 1952b], but it did not gather much attention in the community. The anti-realistic view was too strong after the war, and some features of the pilot wave theory, like its explicit nonlocality, invalidated it as a classical hidden variable theory[2]. Also, the modification imposed by it in classical mechanics is the introduction of a new object, disconnected from our daily life, out of our control. The cost looks too high to be paid only for the belief in realism. In the same decade, Andrew M. Gleason demonstrated his famous theorem [Gleason 1957], and consequently showed the impossibility of many classical hidden variable theories. The realistic view lost even more space. Some problems with the standard Copenhagen interpretation were already evident in the second half of the last century, once Eugene Wigner proposed his friend paradox to show the problems with Heisenberg's cut between observer and observed system [Wigner 1961]. Hugh Everett III introduced in his thesis the Many-World interpretation [Everett 1957], giving a new anti-realistic way to understand quantum theory. Different from Copenhagen's, where the operation of measurement was clear as a generic process in a laboratory, Many-Worlds imposes the existence of inaccessible realities. Again, we need to believe in something beyond our reach but which would influence empirical results.

Bohm's interpretation and subsequent results inspired John Bell to propose a method to empirically test the classical reality of a model [Bell 1964]. Using a relativistic argument to define locality, he mathematically showed the nonlocal, and therefore non-classical, behavior of quantum theory. It was the renascence of research in foundations[3]. In the next year, Simon B. Kochen and Ernst Specker presented a no-go theorem without the necessity of a relativistic argument [Kochen and Specker 1967], showing that some

[2] According to John Stewart Bell, the famous originator of Bell's theorem, de Broglie and Bohm did the impossible by constructing a hidden variable theory for quantum mechanics, which had been considered non-existent since von Neumann's work [Bell 1982].

[3] Abner Shimony called "experimental metaphysics" the use of scientific experiments, such as those involving Bell's theorem, to investigate metaphysical questions [Shimony et al. 1997].

models in quantum theory described by partial Boolean algebras cannot be represented by a classical hidden variable model described by a Boolean algebra. Such non-classical behavior was called "contextuality" because of the importance of the contexts of compatible measurements involved.

In all this development on the foundation front, quantum theory was applied to an enormously large set of problems. Of our interest here is the development of quantum field theory from Dirac and his work with the electromagnetic field to the quantization methods of general relativity, as in the work by John Archibald Wheeler and Bryce DeWitt [DeWitt 1967]. The first was deeply explored and extended to gauge theory, resulting in the Standard Model of particle physics[4] The second was, in some sense, ignored as a distant possibility.

1.1.3 1970s, 1980s and 1990s

With the development of the Standard Model and its empirical validation, further theoretical investigations were proposed to incorporate gravity, such as in string theory and loop quantum gravity. The former is rooted in the mathematical framework of quantum chromodynamics prior to the discovery of quarks, while the latter involves modifications to gravitational formalism by Wheeler and DeWitt. Both are active fields that rely on quantum theory, stimulating the advancement and acceptance of interpretations of the theory, even during a period not particularly inclined towards interpretations. Unfortunately, these and other theoretical endeavors in particle physics lacked experimental validation, either due to energy constraints that are not foreseeable in the near term or due to the identification of models that render predictability unattainable. This practically resulted in the stagnation of particle physics, which persists.

In the experimental domain, a series of experiments were conducted to test Bell's theorem. John Clauser and Stuart Freedman were the first to perform such an experiment [Freedman and Clauser 1972], followed by Alain Aspect and collaborators, who used distant detectors, partially closing the locality loophole [Aspect, Dalibard and Roger 1982]. This loophole was only fully closed with more sophisticated experiments, first performed by Anton Zeilinger and his collaborators [Weihs et al. 1998][5]. Although some loopholes remain open for future experiments[6], these results already strongly indicated

[4] For an informal introduction to gauge theory and the Standard Model, see Ref. [Han 2004], while an undergraduate introduction can be found in Ref. [Zee 2013]. A more complete presentation is in Ref. [Schwartz 2013], and a more mathematical presentation is in Ref. [Ticciati 2008].

[5] Alain Aspect, John F. Clauser, and Anton Zeilinger were awarded the 2022 Nobel Prize in Physics "for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science" [The Royal Swedish Academy of Sciences 2022].

[6] The locality and detection loopholes were closed in experiments in 2015 [Hensen et al. 2015, Giustina et al. 2015, Shalm et al. 2015], showing the impossibility of a realistic description of our reality within a classical theory.

that quantum theory is much stranger than previously imagined. Something new was necessary to explain contextuality and nonlocality.

The experimental verification of nonlocality sparked renewed interest in purely quantum phenomena and their technological applications. For instance, Richard Feynman's work on quantum computing for simulating physical systems [Feynman and Hey 1996] and the encryption method utilizing a quantum protocol devised by Charles Bennett and Gilles Brassard [Bennett and Brassard 2014] are notable examples.

In the theoretical domain, Arthur Fine's theorem [Fine 1982] is particularly important for modern approaches to the phenomenon of non-classicality. This theorem establishes an equivalence between a deterministic hidden-variables model, a factorizable stochastic model, a joint distribution for the observables, and the validity of Bell inequalities. Paving the way for future advancements in exploring the non-classicality of quantum theory, Artur Ekert's famous work applied Bell's theorem to key distribution processes in cryptography [Ekert 1991]. It was through this work that a series of applications for previously metaphysical questions in quantum theory were developed.

Famous quantum protocols were developed, such as teleportation [Bennett et al. 1993], integer factorization by Peter Shor [Shor 1994], and Lov Grover's quantum search algorithm [Grover 1996]. The potential for utilizing quantum behavior in advanced technologies compelled physicists to delve into quantum information theory. Entanglement, perhaps the most prominent resource to date, has been utilized in non-classicality theorems like Bell's and has been present since the Einstein-Bohr debates. Investigating its role as a resource and delving into its properties gave rise to a thriving field that remains robust today [Chitambar and Gour 2019].

1.1.4 New Millennium

Evidence of serious flaws in the formalism of quantum field theory has already been acknowledged, such as causality violations [Anastopoulos and Savvidou 2021], problems with the quantization of gravity, and a lack of evidence to determine if quantization is necessary [Jacobson 1995], along with issues regarding its mathematical formalization itself. Alongside the promise of technological applications, one potential solution to these problems lies in understanding the fundamentals of quantum theory, deciphering its true implications. A crucial initial step involves grasping what makes quantum theory non-classical, and this is where contextuality takes center stage. Robert W. Spekkens' approach [Spekkens 2005] emerges as particularly intriguing, delving into the philosophical inquiry of contextual definitions for all aspects of an operation, while also employing the concept of process as the subject of study.

The profound mathematical formalism of topos theory by Christopher J. Isham, Jeremy Butterfield, and Andreas Döring [Isham and Butterfield 2000, Döring and Isham 2010] has advanced with the aim of understanding the logical underpinnings of quantum theory, seeking to revive lost realism. They have identified contextuality as the root cause of the peculiar behaviors exhibited by quantum systems [Döring and Frembs 2020]. This has spurred a wave of various axiomatizations, beginning with Hardy's work [Hardy 2001], aiming to create the quantum equivalent of Einstein's simple axioms for special relativity.

The exploration of interpretations, now gaining prominence in the revitalized field of theoretical foundations, has been dubbed by Cabello as the "Map of Madness" [Cabello 2017]. However, consensus seems distant on the horizon. New robust interpretations have been proposed, driven by a philosophical inclination towards realism and the necessity in quantum cosmology, where there exists no observer [Bojowald 2015]. Yet, no-go results, such as those related to the Wigner's friend scenario, seem to lean towards anti-realism, suggesting a possible inherent inconsistency in reality itself [Frauchiger and Renner 2018, Brukner 2018]. To address paradoxes of this kind, new interpretations have been developed. Notably, Quantum Bayesianism, also known as Qbism [Caves, Fuchs and Schack 2002], stands out. In this interpretation, the agent's perspective is taken as central, making many aspects of quantum theory subjective, as it deals not with elements of reality but with the agent's belief about the outcomes of experiments.

With the realization that nonlocality is merely a form of contextuality [Abramsky and Brandenburger 2011], it becomes apparent that contextuality is not only the wellspring of the technological advancements we seek [Howard et al. 2014, Shahandeh 2021], but also the source of perplexity in understanding the foundational aspects of the theory. Different approaches to contextuality have emerged depending on the intended application. For technological purposes, measurement contextuality suffices in most cases, following the research of Kochen and Specker, rendering generalized contextuality overly broad. In the realm of theoretical computing foundations, Samson Abramsky and Adam Brandenburger have proposed the sheaf approach [Abramsky and Brandenburger 2011]; for computational application, Cihan Okay and Robert Raussendorf's homological approach [Okay and Raussendorf 2020] and Abramsky and Bob Coecke's categorical formalism [Abramsky and Coecke 2004] alongside diagrammatic calculations derived from it [Coecke and Kissinger 2017] are viable options; for studying correlations, the exclusivity graph approach [Cabello, Severini and Winter 2010, Cabello, Severini and Winter 2014, Vandré and Terra Cunha 2022] remains highly developed; rooted in probability theory, Dzavarov's approach [Dzhafarov, Kujala and Cervantes 2015] stands out; philosophically grounded, the generalized contextuality approach [Spekkens 2005] has evolved to be studied within generalized probabilistic theories [Janotta and Hinrichsen 2014, Schmid et al. 2021]. There exist numerous formalisms for contextuality, intricately interconnected, prompting the consideration of unification as a primary challenge to tackle.

1.1.5 Some Historical Examples

To demonstrate how this idea of contextuality manifests in the literature, we'll examine a few specific examples of frameworks. They are presented in historical order and in an informal manner. For a more precise exposition, refer to Ref. [Varadarajan 1968, Selleri 1990, Budroni et al. 2022, Masse 2021]. Let's begin with the earliest works on non-classicality.

Example 1 (Birkhoff and von Neumann)**.** *In 1936, Birkhoff and von Neumann [Birkhoff and Neumann 1936] presented a semi-lattice structure to explore the logic of propositions in quantum theory. The quantum object* $\mathcal{M}$ *is identified as the Hilbert lattice* $L(\mathcal{H})$*, the lattice of closed linear subspaces of a Hilbert space* $\mathcal{H}$*, or equivalently the lattice of projective measurements of* $\mathcal{H}$*. They questioned the properties satisfied and found the violation of the distributive identity, indicating the impossibility of explaining it via classical, Boolean logic. As we now know, one can use Boolean valuation and get a contradiction. Therefore, the following diagram generally does not commute:*

$$\begin{array}{ccc} L(\mathcal{H}) & \longrightarrow & \{0,1\} \\ & {}_{i}\searrow \quad \nearrow & \\ & \mathbb{P}(A) & \end{array} \quad . \tag{1.1}$$

There is the well-known use of such a formalism to define quantum logic. More importantly to us, this formalism has the limitation of dealing only with projective measurements, which imposes outcome-determinism. There is also the restriction to quantum theory of the phenomenon of non-classicality.

Example 2 (Gleason)**.** *In 1957, Gleason proved his famous theorem in Ref. [Gleason 1957]. He used the unital* C^**-algebra* $\mathcal{B}(\mathcal{H})$ *of bounded operators on a Hilbert space* $\mathcal{H}$*, the usual mathematical framework of quantum theory, to derive the states represented by density operators and the Born rule for dim*$(\mathcal{H}) \geqslant 3$*: any function* $f : \mathcal{B}(\mathcal{H}) \mapsto [0,1]$ *has the form* $f(x_i) = \mathrm{Tr}\{(\Pi_i \rho)\}$*, with state* ρ *and* Π_i *the projection operator onto the basis vector corresponding to the measurement outcome* i *of a observable. This result can be used to find inconsistencies [Budroni et al. 2022] using an infinite set of measurements. One can informally describe them as that impossibility of a measure on the rays of a Hilbert space with dim*$\mathcal{H} \geqslant 3$ *that describes "yes/no" questions. The point of Gleason's theorem is that if we have a hidden variable explanation of quantum theory, we could map subjectively the* C^**-algebra in a commutative sub-algebra. As a commutative* C^**-algebra is equivalent to a complete Boolean algebra, and any sub-algebra of it is also a Boolean algebra, these inconsistencies are showing the non-commutativity of the diagram*

$$\begin{array}{ccc} \mathcal{B}(\mathcal{H}) & \longrightarrow & [0,1] \\ & {}_{i}\searrow \quad \nearrow & \\ & \mathbb{P}(A) & \end{array} \quad . \tag{1.2}$$

Two limitations follow from this framework: the restriction to outcome-deterministic models and the restriction to quantum theory. The latter follows from the Gelfand–Naimark theorem [Gelfand 1943]. This result is related to an already proposed (but not formally proved) result by von Neumann in his famous book where he formalized quantum theory [Neumann 2018], as explained in [Bell 1966].

The rebirth of quantum foundations and the quantum explanation of hidden variables started with the work by Bell. The important change is the passage from a formal, mathematical description of non-classicality to an empirical, but not less formal, point of view. To achieve this, the set of measurements necessary to detect violations becomes finite.

Example 3 (Bell)**.** *The work by Bell [Bell 1964] of 1964 marked a change in the way hidden variables are viewed. Here we have the use of a relativistic argument to isolate quantum systems $\mathcal{A}$ and $\mathcal{B}$, and the formalization of what a hidden variable is with well-defined properties. The result was not a theorem on the mathematical structure of quantum theory, but a formal empirical challenge. Therefore, one could test if quantum theory is explained or not by hidden variables Λ. In the diagram*

$$\begin{array}{ccc} \mathcal{A}\otimes\mathcal{B} & \longrightarrow & [0,1] \\ & {\scriptstyle i}\searrow \quad \nearrow & \\ & \mathcal{A}\times\mathcal{B}\times\Lambda & \end{array} \quad . \tag{1.3}$$

The use of inequalities, and thus a convex set of models that satisfy the imposed condition of nonlocality on probabilities, allowed the expansion of this framework to more general theories than quantum. This condition lies on inequalities of the accessible (marginal) probability distributions, and such a violation results from logical inconsistencies [Abramsky 2020, Boole 1862]. The use of relativity is a limitation in Bell's work, and outcome-determinism is implicit by the use of projective measurements.

Example 4 (Kochen and Specker)**.** *In 1967, the usual notion of contextuality was introduced. Using Gleason's theorem, one can rewrite the problem back to a possibilistic problem. Kochen and Specker [Kochen and Specker 1967] were inspired by this idea to codify a quantum system as a partial Boolean algebra* $\mathrm{P}\mathbb{B}$*, but with a finite number of measurements. The main idea is to identify the contexts: sets of jointly measurable measurements, here understood as measurements that commute [Heinosaari and Wolf 2010]. Contexts are the fundamental elements, as the parts of the system that are accessible by their joint measurement, and so they define Boolean algebras. Contextuality here is the impossibility of embedding the combination of the set of Boolean algebras satisfying the condition to be a partial Boolean algebra into a Boolean algebra, translated as the non-commutativity of* *the diagram*

$$\begin{array}{ccc} P\mathbb{B} & \xrightarrow{\qquad\qquad} & \{0,1\} \\ & {\searrow}^{i} \quad \nearrow & \\ & \mathbb{B} & \end{array} \quad . \tag{1.4}$$

Quantum theory presents sets of measurements and states with a non-commutative diagram. In contrast to the Bell framework, there is no need for relativistic arguments to achieve non-classical behavior; generally, one explores logical contradictions. Again, the use of projective measurements and the initial restriction to quantum theory are evident. The initial framework uses possibilistic models, but one can also use probabilities, inequalities, and convex theory to deal with general models. A beautiful example is in Ref. [Araújo et al. 2013].

A lot has been done to explore the limits of quantum and classical models. Bell scenarios were extended to more complex systems, and new nonlocality inequalities were systematically explored. The same was done for contextuality and scenarios without the necessity of local systems. Applications of quantum systems, such as in quantum computation and quantum technologies [Cabello 2000], bring up questions about what causes the increase in performance in relation to the methods already used. At the same time, and not independently, the increasing interest in the foundations of quantum theory led to the construction of new methods, some even oriented to specific tasks, in the last two decades. An important achievement is the identification of nonlocality as the contextuality of multiple, isolated systems, by the Fine–Abramsky–Brandenburger theorem [Abramsky and Brandenburger 2011].

Example 5 (Graph Approach)**.** *Continuing on the same path as Kochen and Specker's work, still dealing with inequalities, convex theory, and now with computational tools, a systematic framework to construct and explore scenarios was developed without the logical formalism [Cabello, Severini and Winter 2010, Amaral and Terra Cunha 2018]. The point is to define the object on which the (usually probabilistic) valuation will be made. As a mathematical relation, belonging to a context can be codified in a hypergraph [Montanhano 2021]. A different construction is by exclusivity [Cabello, Severini and Winter 2014, Amaral 2015], that can emerge from general theories [Chiribella et al. 2020], where the object is a graph where outcomes that do not happen together (they are exclusive) are linked by an edge. Amazing results are the identification of the noncontextual, quantum, and non-disturbing* *convex sets as graph invariants, and that the sub-diagrams of the diagram*

$$\begin{array}{ccc} NC & \longrightarrow & [0,1] \\ {\scriptstyle i}\big\downarrow & & \big\| \\ Q & \longrightarrow & [0,1] \\ {\scriptstyle i}\big\downarrow & & \big\| \\ ND & \longrightarrow & [0,1] \end{array} \tag{1.5}$$

commutes, but the sub-diagrams of the diagram

$$\begin{array}{ccc} ND & \longrightarrow & [0,1] \\ {\scriptstyle i}\big\downarrow & & \big\| \\ Q & \longrightarrow & [0,1] \\ {\scriptstyle i}\big\downarrow & & \big\| \\ NC & \longrightarrow & [0,1] \end{array} \tag{1.6}$$

do not. There is also the identification of the quantum correlations from first axioms in certain scenarios [Cabello 2019, Cabello 2019]. Non-disturbance and outcome-determinism are usual restrictions, and the origin of the main limitations to empirical exploration of this formalism, due to the problems these restrictions cause in dealing with experimental applications.

Example 6 (Contextuality by Default)**.** *In the Contextuality by Default framework [Dzhafarov, Kujala and Cervantes 2015, Dzhafarov 2021], the origin of contextuality in quantum theory isn't the main point: contextuality is seen as the nonexistence of a maximal coupling of random variables. Contexts are seen at first as independent, and contextual behavior appears as a violation of concordance in the marginals, which can be interpreted as the non-existence of a coupling satisfying concordance in relation to all the contexts (for more general cases, as in disturbing models, the coupling must be modified). In the diagram*

$$\begin{array}{ccc} R & \longrightarrow & [0,1] \\ & {\scriptstyle i}\searrow \quad \nearrow & \\ & S & \end{array} , \tag{1.7}$$

the set of random variables R is coupled in S by a map i, with the non-existence of a coupling captured by non-commutativity of the diagram. This framework deals naturally with disturbing models, allowing experimental exploration and interesting applications, like in human behavior where non-disturbance doesn't hold [Wang et al. 2021].

Example 7 (Homotopical Approach)**.** *The homotopical approach [Okay et al. 2017, Okay and Raussendorf 2020] has as its main idea to codify a group element for each measurement, and with it an orientation codified by a negative or positive signal. Quantum* *theory is described with transformations on each context depending on each measurement contributing individually to the overall transformation. In contrast, classical theory is described by transformations that depend solely on the context, independent of the individual measurements. This fundamental distinction results in path dependence in the quantum case, which can be effectively analyzed using topological tools. This approach is particularly suited for computational applications of quantum theory, especially measurement-based quantum computation (MBQC), due to its natural ability to identify state-independent proofs of measurement contextuality. One can represent a model as a topological space, and the possibility of a classical representation is equivalent to this space being topologically trivial. Diagrammatically,*

$$\begin{array}{ccc} \pi_1(X) & \longrightarrow & U(n)/\langle\omega\rangle \\ & {\scriptstyle i}\searrow \quad \nearrow & \\ & U(n) & \end{array} , \tag{1.8}$$

where the objects involved are groups, with the fundamental group being said to be classically realizable if the fundamental representation of the fundamental group lifts to a linear representation. It is interesting to see the interpretation of Mermin's square and star by representing them as tori and using non-trivial topology to detect contextual behavior, something similar to the non-trivial field in electromagnetism. But the inclusion of a group structure for each measurement is a representation of a model that takes its toll by allowing a breach of the contextual behavior detection tool. Also, the homotopical approach is constructed to deal with quantum models and is limited by them.

Example 8 (Sheaf Approach)**.** *The sheaf approach [Abramsky and Brandenburger 2011] is articulated in a categorical language of presheaves, starting from the hypergraph of compatibility C with contexts as hyperedges and the set of measurements X as vertices. A sheaf $\mathcal{E}$ attaches the outcomes to each measurement, and a presheaf attaches the distribution with a semi-ring (usually probabilistic, possibilistic, or the real numbers). Contextuality is identified by the Fine–Abramsky–Brandenburger theorem as the failure of a local section to have an extension to a global section. In the diagram*

$$\begin{array}{ccc} \mathcal{E}C & \longrightarrow & [0,1] \\ & {\scriptstyle i}\searrow \quad \nearrow & \\ & \mathcal{E}X & \end{array} . \tag{1.9}$$

It also has a cohomological tool to examine contextual behavior, using Čech cohomology of a presheaf [Abramsky, Mansfield and Barbosa 2012, Abramsky et al. 2015]. The idea is to codify the model with a presheaf of Abelian groups $\mathcal{F}$ and identify non-trivial obstructions as a sufficient condition for contextual behavior. The diagram is now represented by groups

$$\begin{array}{ccc} C & \xrightarrow{\mathcal{F}} & \mathcal{F}C \\ & {}_{i}\searrow \quad \nearrow & \\ & X & \end{array} \quad . \tag{1.10}$$

But since this group structure is a representation of the model, it can have contextual models without cohomological characterization [Carù 2017]. One can overcome this failure, but much of the cohomological structure is lost [Montanhano 2021]. The sheaf approach is related to the previously mentioned ones:

- *The models used to prove non-classicality with finite measurements presented by Bell and Kochen-Specker, and others like Hardy's model [Hardy 1992, Hardy 1993], can be constructed in this formalism.*
- *The graph approach can be related to the sheaf approach, and the notion of strong contextuality in the former is identified as the cause of contextual models [Silva 2017]. In particular, inequalities obtained in the graph approach can be explained by the contextual fraction [Abramsky, Barbosa and Mansfield 2017], a measure of contextuality constructed in the sheaf approach.*
- *In the restriction to non-disturbing models, the sheaf approach and Contextuality by Default can both detect the same contextual behavior [Dzhafarov 2020]. Contextual fraction can also be explored in this framework [Kujala and Dzhafarov 2019].*
- *The relation with the homotopical approach was explored in relation to their cohomological tools to identify contextual behavior [Aasnæss 2020]. It concludes that Čech cohomology can capture any contextual behavior that the cohomology in the homotopical approach can identify.*

The restrictions of this approach are the imposition of outcome-determinism to link contextuality and factorizability, and non-disturbance to fix the intersection between contexts.

Example 9 (Bundle Approach)**.** *A way to visualize a model, particularly the possibilistic ones, is through bundle diagrams [Beer and Osborne 2018]. Logical contradictions present in some models can be seen as the failure of a trivial fiber bundle [Terra Cunha 2019], constructed with the hypergraph of compatibility as the base and finite outcomes and measurements as fibers. To deal with more general fibers and measures on a semi-ring R, one must formalize the R-measure bundle [Montanhano 2021]. Again, the notion of a local section to be extended to a global section is important, and it is linked to the triviality of the bundle, which is initially trivial only locally in a context. The Fine–Abramsky–Brandenburger theorem can be generalized to this level of generality, formalizing the intuition of a trivial* *bundle being noncontextual. In a diagram*

$$\begin{array}{ccc} \mathcal{R} & \longrightarrow & R \\ & {\scriptstyle i}\searrow \quad \nearrow & \\ & \mathcal{T} & \end{array}, \tag{1.11}$$

with the measure bundle $\mathcal{R}$ representing the model and $\mathcal{T}$ its trivial equivalent. This framework is interesting because it allows us to understand the influence that the topology of the base has on a model and the relationship between contextual behavior and the geometry of the bundle, by drawing a parallel with differential geometry. It gives rise to non-trivial holonomy as a signal of contextuality in some models, but again the group structure is imposed as a representation, resulting in violations of the characterization.

Example 10 (Generalized Approach)**.** *The generalized approach to contextuality [Spekkens 2005] was initially developed using more philosophical concepts to construct an ontological representation of physical processes, which differentiates it from other approaches right from the start. It is based on the idea that an ontology must capture all empirical information of a physical process, and that being classical means it should not admit more than one representation for the same process. This is an application of Leibniz's principle of the identity of indiscernibles to physical theories. This approach can be naturally formalized and constructed in categorical language, albeit more generally than the sheaf approach, using process theory and generalized probabilistic theories [Amaral 2014]. By encompassing all parts of a process within its framework, it can be applied to various scenarios depending on the imposed structure. A process in a given scenario is divided into three types: a set of preparations $\mathcal{P}$, a set of transformations $\mathcal{T}$, and a set of effects $\mathcal{E}$. Noncontextuality implies that the processes can be represented as classical processes: a simplex of preparations of classical probability distributions $\mathcal{S}$, a set of transformations of these distributions $\mathcal{M}$, and a set of classical effects dual to the simplex, represented by a hypercube $\mathcal{H}$. This can be depicted as commutativity of the following diagram:*

$$\begin{array}{ccc} \{\mathcal{P}, \mathcal{T}, \mathcal{E}\} & \longrightarrow & [0,1] \\ & {\scriptstyle i}\searrow \quad \nearrow & \\ & \{\mathcal{S}, \mathcal{M}, \mathcal{H}\} & \end{array}, \tag{1.12}$$

Since it is constructed at the level of effects, this approach is inherently limited by the non-disturbance condition, which presents challenges in addressing the inevitable uncertainties in experimental implementations. When this condition is satisfied, and under the assumptions of outcome-determinism, no transformations, and a single preparation, the set of effects can be encoded as an empirical model within the sheaf approach.

All these examples and developments are still active and yielding interesting results. They are all present in the structure of contextuality discussed earlier. As a probabilistic concept, the natural generalization is to define each context as a σ-algebra where a measure can be defined, and even outcome-determinism can be violated (by working not with subsets of the set of outcomes, but with functions on it). Naturally, the σ-algebra can be identified as a Boolean algebra, justifying its use as a fundamental object in Kochen and Specker's work and the logic behind contextuality.

1.2 Structure of the Thesis

This thesis is structured in three parts. The first part presents the approaches to contextuality used and the initial results that motivated the study of such a phenomenon in a topological and geometric language. The following two parts are independent, each following the papers that originate them [Montanhano 2022, Montanhano 2023]. The chapters are not disconnected: they describe the journey that follows from the exploration of the intersection between contextuality, geometry, and topology.

Starting with the definition of contextuality in the Sheaf Approach and the Generalized Contextuality Approach, we explore some results that explicitly show the influence of topology on contextual behavior. From these results and what we have learned from them, we can construct a differential representation to capture contextuality in its many forms. It is through the connection that topology, already explicitly linked to the phenomenon of contextuality, has with logic that we can identify it in paradoxes emerging from the combination of agents' knowledge and non-classical theories.

1.2.1 Part 1: Topology and Contextuality

Part I of the thesis initially serves a dual function: presenting the notations and objects to introduce contextuality, and showing how topology can influence the extent of this non-classical behavior. In chapter 2, we present the fundamental objects of most contextuality approaches (section 2.1): measurements and contexts. With these, we form measurement scenarios by linking contexts to their respective sets of outcomes, such that the marginalization of these outcomes in sub-contexts is consistent. We then introduce the intuition of contextuality as a marginal problem and an analogy between empirical models and manifolds. To formalize the concept of contextuality, we present two approaches that will be used in the other two parts of the thesis, the Sheaf Approach and the Generalized Contextuality Approach (sections 2.2 and 2.3).

In chapter 3 we first explore examples and their bundle diagrams, which are very natural in the Bundle Approach, and we see how contextuality appears to be topological, but reality is a bit more complicated 3.1. The cohomology in the Sheaf Approach also fails to get a topological characterization of contextuality; the imposition of the use of algebraic groups allows for violations, although certain cases can be detected through sheaf cohomology (section 3.2). Restricting ourselves to the topology of measurement scenarios, which apparently has no direct influence on contextual behavior, we present a hierarchy constructed by applying the contextual fraction at each step of the inductive construction of the simplicial complex (section 3.3), showing that there is indeed some influence of topological defects on contextual behavior, even for defects larger than those detected by the first homology group.

We conclude this first part with chapter 4, where we discuss the limitations of the described approaches and methods. Considering these limitations and the similarities to a marginal problem of distributions, we seek the change of fundamental objects proposed by the Generalized Contextuality Approach and then apply the topological and geometric ideas suggested by the Sheaf Approach. This sets the stage for the second part of the thesis.

1.2.2 Part 2: Differential Geometry of Contextuality

In Part II we focus on the use of geometrical and topological ideas of differential geometry for Generalized Contextuality. This approach deals directly with processes to define sets of effects, states, and transformations, allowing us to explore the contextual behavior of other parts of a process. Contextuality now depends on an ontic representation, which in the Sheaf Approach follows directly from the global sections due to the outcome-determinism condition. It is also defined differently, based on Leibniz's Principle of the Identity of Indiscernibles, and can be described as the existence of a phase in the valuation when applied to a discrete loop in the real vector space where the set of processes of the ontic representation is encoded. In this version, we can use discrete differential geometry (section 5.1) and generalize to the continuous case the encoding of contextual behavior.

We present a differential approach to the contextuality of an ontic representation (section 5.2), drawing an analogy with manifolds, and identifying violations of classicality with two possible causes: topological and geometric. We name these causes the Heisenberg and Schrödinger views, respectively. Both causes are realizations of contextuality, which is now at a higher level than valuations as a differential form that corrects them, analogous to how the electromagnetic tensor corrects the movement of charges. The difference is that we can choose whether contextuality presents itself as a direct correction of valuation by holonomy or if it identifies by monodromy the topological defects that represent "holes in reality."

This freedom of choice allows us to explore other concepts of non-classicality and their relationship to contextual behavior (section 5.3). For the restriction to the Sheaf Approach, the ontic representation and the Schrödinger view are fixed, and the contextual form presents itself explicitly in the contextual fraction. Still in the Schrödinger view, the interference of quantum measure theory places the contextual form as the non-classical contribution of the term that violates Kolmogorov's third axiom for probabilities. The need for signed measures results from this non-classical part that follows from contextuality, while the impossibility of embedding in a classical theory follows from how in any of the views, the contextuality of the representation cannot be classically encoded without dependence on contexts. The Differential Approach has a direct way of dealing with disturbance through non-trivial transition maps, realizing this intuition of the analogy with manifolds. The Voroby'ev theorem, which when thought of in measurement scenarios is not topological, in the Differential Approach becomes explicitly topological when viewed through the Heisenberg view, being merely a statement about the triviality of the first cohomology group. We end this second part with a verification of how quantum interpretations choose which view of quantum contextuality they use, and the relationship between the two views and the choice of which type of realism of the interpretations.

1.2.3 Part 3: Wigner and Friends, a Map is Not the Territory! Contextuality in Multi-Agent Paradoxes

Part III is an indirect application of the ideas developed so far. We start with chapter 7, with a review of multi-modal logic (section 7.1). Of special importance to us is the construction of the S4 system, which is the weakest and therefore most general system for dealing with epistemic logic, the logic of knowledge and belief. Its Kripke and topological semantics allow for an almost immediate relationship with the topological content present in the Sheaf Approach. The definition of knowledge operators and the concepts of soundness and completeness will be essential to understanding how the choice of one of the knowledge operators and its respective topology can violate the relationship between syntax and semantics.

The axioms of the S4 system impose an absolute truth, which does not align with a relational treatment of knowledge (section 7.2). We review the concept of trust, which defines a relationship between agents' knowledge operators, and which replaces an underlying absolute truth. We present generalizations of the trust relationship that deal with sets of agents and different knowledge operators of these sets. It is through the relationship between these different operators, viewed through their respective topologies, that we can construct an effective absolute truth for a given set of agents, explaining the preservation of multi-agent paradoxes even when considering only trust.

The construction of effective absolute truths is the first step in constructing the map between empirical models and multi-agent scenarios that we obtain in chapter 8. We start by identifying agents and measurements, then impose conditions for the generalized versions of the trust relationship to define compatibility contexts, and finally take the main step of the construction and analyze its consequences (section 8.1). The step is to change the way Kripke semantics is defined, inspired by pointless topology applied to topological semantics, where the possible worlds are defined by the opens of the topology of a knowledge operator of the set of agents, and not by assuming the possible worlds beforehand. With this step, and respecting the limitations imposed by the Sheaf Approach, we identify the sections of the sheaf of events as given by the basis topology, obtaining the desired map.

An immediate consequence of constructing the map that turns any empirical model into a multi-agent scenario is the identification of contextuality as being generated by the implicit use of a knowledge operator that limits possible worlds to those that have no dependence on contexts, which is encoded as a violation of soundness. If we switch the operator to the one that generates an effective fundamental truth, we can use multi-modal logic in the usual way. This choice of where to place contextuality, either in the context-dependence of possible worlds or in the violation of the soundness condition, is the realization of the Schrödinger and Heisenberg views presented in Part II, respectively. Despite the limitations, in the main examples of multi-agent paradoxes, there is an inverse map that allows transforming multi-agent scenarios into empirical models and expressing the paradoxes as contextual behavior (section 8.2). We conclude this part of the thesis by analyzing the contextuality of three examples: the famous Wigner's Friend scenario in quantum theory and two generalizations consisting of two correlated copies of this scenario, the Frauchiger-Renner scenario based on Hardy's paradox, and the Vilasini-Nurgalieva-del Rio scenario that uses a Popescu-Rohrlich box and exemplifies paradox beyond quantum.

1.2.4 Final Considerations

In the last part of the thesis, chapter 10, we discuss the presented results and their connections. The journey undertaken points to a path that possibly generalizes the concept of contextuality by defining it in a modal approach, with topology and geometry being explicitly used, and with the Differential Approach as a possible representation. Naturally, future research paths are presented, making the end of this journey the beginning of many others.

Part I

Topology and Contextuality

2 Approaches to Contextuality

2.1 Basic Objects

Contextuality is, informally, the property of a physical system that cannot be explained classically, where this classicality is thought of as an ontological reality that is coarse-grained to the system[1]. The most common approaches choose as the fundamental objects the measurements and their joint measurability. The set of outcomes of each measurement, which consistently extends to the outcomes of the contexts, completes what we call a measurement scenario. Contextuality appears as a behavior of the probabilistic distributions on the outcomes, defining what we call an empirical model. Let's define these objects.

2.1.1 Measurements

As in Ref. [Terra Cunha 2019] we will define measurements in a generic manner by their outcomes, also called events. For simplicity, we will keep all the sets with a finite number of elements.

Definition 1. *A measurement M is a set of labels $\{s^i\}_{i=1}^n$ for the possible n events. We will denote by $M[s^i]$ the event s^i of the measurement M.*

The formal definition of a measurement in a physical theory depends on the theory in which it is constructed. A well-known example of measurement is given by a positive operator-valued measure in quantum theory. Its events are given by its outcomes, the effects that sum to the identity.

2.1.2 Compatibility Between Measurements

The measurements are organized as a covering through compatibility, or joint measurability. One can understand compatibility as classicality at the ontological level. It is a property between measurements and their respective outcomes given by the theory, not dealing with distributions over outcomes. It is a condition of classicality stronger than the concept of noncontextuality, which is epistemic in nature and arises from classicality at the level of distributions over outcomes, as we will see later. Compatibility imposes the existence of a "mother" measurement, such that our accessible measurements have origin by classical post-processing[2].

[1] See Ref. [Budroni et al. 2022, Masse 2021] for a general revision of contextuality.
[2] Both notions are equivalent, see Ref. [Filippov, Heinosaari and Leppäjärvi 2017].

Definition 2. *Let $\{M_k\}_{k=1}^{m}$ be a set of measurements with a respective set of events $O^{(k)} = \{s^{(k)}\}$. They are jointly measurable if there exists a measurement G with a set of events $O^{(1)} \times ... \times O^{(m)}$ satisfying*

$$M_k[s^{(k)}] = \sum_{s^{(j)}:j\neq k} G[s^{(1)}, ..., s^{(k)}, ..., s^{(m)}] \tag{2.1}$$

for all k.

Therefore, a set of compatible measurements allows the existence of a measurement that can recover the original measurements when appropriately marginalized. As shown in Ref. [Heinosaari and Wolf 2010], in quantum theory commuting implies jointly measurable, and the inverse holds if the measurements are sharp.

2.1.3 Measurement Scenarios

The covering of measurements given by the jointly measurable measurements has as elements the contexts. Its most general version has a hypergraph structure, and along with the events, they form what we call a measurement scenario [Abramsky 2018].

Definition 3. *A measurement scenario $\langle X, \mathcal{U}, (O_x)_{x\in X}\rangle$ is a hypergraph*[3] *$\langle X, \mathcal{U}\rangle$, where X is the set of measurements and $\mathcal{U}$ a covering of contexts (a family of sets of compatible measurements), plus the sets $(O_x)_{x\in X}$ for each $x \in X$ are called outcome sets, with their elements the possible events of each measurement.*

For simplicity, let's suppose that outcome sets are finite, and therefore one can define an outcome set O for all the measurements x[4]. We will also work with a measurement scenario with a simplicial complex structure of contexts since in physical systems the sub-contexts can reconstruct a context of the covering.

2.1.4 Contextuality as a Marginal Problem

Based on the objects already defined, we can define an empirical model and its contextuality. Here we will limit ourselves to seeing contextuality as a marginal problem. Contextuality can appear when one deals with the collection $\mathcal{U}$ of contexts that can be understood as a covering of a more fundamental set, usually of measurements. Each context is accessible in the sense of a given data defined on it by a map, which we will call a valuation map, which defines distributions on the events. We need to fix where this data

[3] Usually one imposes the hypergraph has some additional structure, usually enough to identify it as a simplicial complex. See Ref. [Montanhano 2021] for a justified construction of the measurement scenario.

[4] We can codify any O_x through an injective function $f_x : O_x \to O$; we just need to ignore elements that aren't in the image of f_x, such that these elements aren't in the distribution's support.

is codified, usually an algebraic object R as a semi-ring or a group, such as the Boolean semiring $\mathbb{B}$, the reals $\mathbb{R}$, or the probability semiring $\mathbb{R}^+$. We can define R-empirical models, or just empirical models when R is implied. The choice of an R defines a way to probe the model.

Definition 4. *An empirical model is given by a set of distributions* $\left\{\mu_R^{O^U}\right\}$ *over the outcomes* $(O_U)_{U\in\mathcal{M}}$ *of a measurement scenario* $\langle X, \mathcal{U}, (O_x)_{x\in X}\rangle$*.*

We can write the valuation map μ that defines an empirical model $\mathcal{M}$ acting on each context diagrammatically as:

$$\mathcal{M} \xrightarrow{\mu} R \ . \tag{2.2}$$

One can ask if we can understand $\mathcal{M}$ and its valuation without the use of contexts. If so, the local data we can access can be explained as the marginalization of a global object $\mathcal{X}$, where $\mathcal{M}$ is embedded by a map i, and where clearly the structure of $\mathcal{X}$ imposes a limit on its valuation. Contextuality, in a generic manner, is the noncommutativity of the following diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\quad\mu\quad} & R \\ & {\scriptstyle i}\searrow \qquad \nearrow & \\ & \mathcal{X} & \end{array} \ . \tag{2.3}$$

But one can map such a diagram to other objects, like in some examples where contextuality is explored through non-trivial topology.

Once assuming that the events given by the outcomes can be extended to global events, that is, events over the entire set of measurements X and not only limited to the contexts, we have that all contextual behavior can only appear in the distributions, which by definition are limited to the contexts. Conditions that will be important to us are no-disturbance and outcome-determinism. The first is defined in the intersection of contexts: if the valuation of an intersection (when one defines it) $\mu(C \cap C')$ is independent of C and C' for all pairs of contexts, then $\mathcal{M}$ is said to be non-disturbing. It is related to parameter-independence [Brandenburger and Yanofsky 2008, Barbosa et al. 2019]. The second is defined in the valuation: the outcomes defined on the contexts where the distribution is defined can be explained in a deterministic way. It is equivalent to restricting our interest to ideal measurements, which one can easily criticize when thinking of empirical applications [Spekkens 2014].

2.1.5 An Analogy with Manifolds

The definition of a manifold M is given by charts $\{\phi_i\}_{i\in I}$, homeomorphisms $\phi_i : C_i \mapsto U \subseteq \mathbb{R}^n$ of open sets of M to an open set of $\mathbb{R}^n$, with a set of charts that covers the manifold being an atlas, describing locally the data of the manifold. The passage of a chart C to another chart C' is codified by a transition map $t_{C'C} : C|_{C\cap C'} \mapsto C'|_{C\cap C'}$. The non-trivial topology can be understood by the notion of contractibility, when one can continuously deform the manifold into a point. Every chart is contractible, and topological failures result in the failure of this property. In diagram,

$$\begin{array}{ccc} M & \xrightarrow{\qquad\qquad} & \mathbb{R}^n \\ & {\scriptstyle i}\searrow \qquad \nearrow & \\ & U \in \mathbb{R}^n & \end{array}, \tag{2.4}$$

where being contractible is codified as a deformation to an open set $U \in \mathbb{R}^n$, and the valuation is the charts.

The analogy between manifolds and empirical models is intriguing: contexts as charts; transition maps being the local data, that when no-disturbance holds, implies identity maps; contractibility as being described by hidden variables. The analogy goes even further. The use of cohomology in both areas, contextuality and topology, is an example of how far it can go. In models, we are not doing our valuation in $\mathbb{R}^n$, but rather working with measurements and outcomes. Precisely, locally, we have σ-algebras of outcomes that are given by the events of each context. We can define trivial transitions by imposing no-disturbance and define valuation as a probabilistic map. Contextuality is interpreted as the possibility of embedding this object described by local σ-algebras into a σ-algebra such that by marginalization we obtain the local valuations in the beginning

$$\begin{array}{ccc} \{\Sigma_i, t_{ij}\}_{i,j\in I} & \xrightarrow{\qquad\qquad} & R \\ & {\scriptstyle i}\searrow \qquad \nearrow & \\ & \Lambda & \end{array} \tag{2.5}$$

2.2 Sheaf Approach to Contextuality

Before presenting the approaches to contextuality that we will see in this thesis, we will review some basic concepts of category theory.

2.2.1 Category Theory

Introduced by Samuel Eilenberg and Saunders Mac Lane, in some sense following the ideas of Emmy Noether and others about the necessity of formalizing abstract processes. The idea behind it is to understand a mathematical structure by the processes that preserve it, like the use of Abelian groups to understand non-trivial topological spaces by a process that relates them, as for example, homologies and cohomologies. Interestingly, the mathematical objects they wanted to explore were those that related to such processes, known today as natural transformations, not the processes themselves. Later, when applications of formalism appeared, the systematization of category theory was identified with other abstract objects. An example of this is the Grothendieck topoi used in studies of topological spaces by algebraic geometry, which were generalized by the concept of elementary topoi, which is closely related to higher-order logic and the foundations of mathematics itself.

The sheaf approach is defined by the identification of the compatibility hypergraph as a category. The first chapter of this thesis already has some commentaries about such categorical properties, but it is in the second chapter that the concept is highly necessary. Here I will present some introductory concepts about category theory, just enough to follow the main text and provide references for the interested reader. Let's start with a definition.

Definition 5. *A category $\mathcal{C}$ consists of objects $A, B, C, \ldots \in ob(\mathcal{C})$ and morphisms $f, g, h, \ldots \in \mathcal{C}(A, B)$ between each pair of objects A and B, such that there is composition between consecutive morphisms, there are identity morphisms $\mathbb{1}_A \in \mathcal{C}(A, A)$ satisfying $f \circ \mathbb{1}_A = \mathbb{1}_B \circ f = f$ for all $f \in \mathcal{C}(A, B)$, and the composition is associative.*

The collection of objects and morphisms may not be sets, in the mathematical sense, which gives absolute strength to category theory. If the collections of objects and morphisms are sets, then the category is called small (there are more details in the definition of sets that I won't enter). If each $\mathcal{C}(A, B)$ is a set, the category is said to be locally small.

Examples of categories are plentiful and common. Taking sets as objects and functions as morphisms, we have the category of sets **Set**; groups as objects and homomorphisms as morphisms, we have the category of groups **Grp**; topological spaces as objects and continuous functions as morphisms, we have the category of topological spaces **Top**. Any group can be understood as a category with just one object, where the morphisms are identified as the elements of the group.

A functor is a process between two categories, mapping not just the objects, but the data of the morphisms as well. As we can use the morphisms of an object as a way to define it, a functor is a way to translate this data to another category with easier understanding. It is the case of cohomology, as used in the main text, where groups and homomorphisms have useful algebraic properties for working than topological spaces and continuous functions, or hypergraphs and inclusions.

Definition 6. *A functor $F : \mathcal{C} \to \mathcal{C}'$ between two categories $\mathcal{C}$ and $\mathcal{C}'$ is given by an object map $F : ob(\mathcal{C}) \to ob(\mathcal{C}') :: A \mapsto FA$, a morphism map $F : \mathcal{C}(A, B) \to \mathcal{C}'(FA, FB) :: f \mapsto Ff$, satisfying preservation of identity $F(\mathbb{1}_A) = \mathbb{1}_{FA}$ and preservation of composition $F(f \circ g) = Ff \circ Fg$.*

For a first read on more topics in category theory applied to quantum theory, see Ref. [Coecke 2008], while a formal book-length option is Ref. [Heunen and Vicary 2019].

A more mathematical introduction is in the book of one of its creators, Ref. [Lane 2010].

2.2.2 Presheaves and Sheaves

Definition 7. *A presheaf is a functor $F : C^{op} \to \textbf{Set}$ of a category C to the category of sets.*

Let (C, J) be a site, a small category C equipped with a coverage J. In other words, any object $U \in C$ admits a collection of families of morphisms $\{f_i : U_i \to U\}_{i\in I}$ called covering families.

Definition 8. *A presheaf on (C, J) is a sheaf if it satisfies the following axioms*

- *Gluing: if for all $i \in I$ we have $s_i \in F(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there is $s \in F(U)$ satisfying $s_i = s|_{U_i}$;*
- *Locality: if $s, t \in F(U)$ such that $s|_{U_i} = t|_{U_i}$ for all U_i, then $s = t$.*

Definition 9. *Elements $s \in F(U)$ of the image of a presheaf are called local sections if $U \neq X$, and global sections if $U = X$.*

Definition 10. *A compatible family is a family of sections $\{s_i \in F(U_i)\}_{i\in I}$ such that for all $j, k \in I$ holds $s_j|_{U_j \cap U_k} = s_k|_{U_j \cap U_k}$ in $F(U_j \cap U_k)$.*

2.2.3 Sheaf of Events

The covering $\mathcal{U}$ can be restricted to maximal contexts, which will also be denoted by $\mathcal{U}$, so that $\langle X,\mathcal{U}\rangle$ can be understood as a set X with a covering $\mathcal{U}$ of maximal contexts $U_{j\in I}$ indexed by an ordered set I[5]. Since the intersection of contexts is a context, we can define the inclusion morphism $\rho(jk, j) : U_j \cap U_k \to U_j$, which turns the set of contexts and the inclusion morphisms into a small category[6].

Definition 11. *The outcome sets are defined by a functor $\mathcal{E} : \langle X,\mathcal{U}\rangle^{op} \to \textbf{Set}$, with $\mathcal{E} :: U \mapsto O^U = \bigtimes_{x\in U} O_x$ and $\mathcal{E} :: \rho \mapsto \rho'$, such that for each element $U \in \mathcal{U}$ we have an outcome set O^U of the context and ρ' is the restriction to the outcome sets, $\rho'(j, kj) : O^j \to O^{kj} = \mathcal{E}(U_j \cap U_k) :: s_j \mapsto s_j|_{kj}$.*

Proposition 1. *The functor $\mathcal{E}$ is a sheaf in the site of measurements and contexts, called the sheaf of events of a given measurement scenario.*

5 Given a covering, one can construct a locale, a pointless space, using unions and intersections. This means that the measurements are not the fundamental objects, but rather the minimal contexts become the effective measurements of the scenario, depending on how one chooses the covering. A physical example of refinement is spin degeneration, where refinement occurs by applying a suitable magnetic field.

6 From Ref. [Johnstone 2002], we can see that the category of contexts with the inclusion is a site.

2.2.4 Empirical Models in Sheaf Approach

To define R-empirical models in Sheaf Approach, we use another functor $\mathcal{D}_R : \textbf{Set} \to \textbf{Set} :: O^U \mapsto \left\{\mu_R^{O^U}\right\}$, taking a set of local events to the set of R-distributions defined on it $\mu_R^{O^U} : \mathbb{P}\left(O^U\right) \to R$ that satisfies $\mu_R^{O^U}(O^U) = 1_R$, in analogy with probabilistic distributions. We will denote by $\mu_R :: U \in \mathcal{U} \mapsto \mu_R^{O^U}$ a set of R-distribution defined in each element of $\mathcal{U}$, and call it a state. In the morphisms, $\mathcal{D}_R :: \rho'(j, kj) \mapsto \rho''(j, kj)$, with $\rho''(j, kj) :: \mu_R^{O^j} \mapsto \mu_R^{O^j|_{kj}} = \mu_R^{O^j}|_{kj}$ the marginalization of the R-distribution j on the intersection kj.

Definition 12. *The tuple $(X, \mathcal{U}, \mathcal{E}, \mu_R) = e_R$ is called an R-empirical model over the measurement scenario $\langle X, \mathcal{U}, (O_x)_{x\in X}\rangle = (\langle X, \mathcal{U}\rangle, \mathcal{E})$ given by the state μ_R, defining a set of local sections $\left\{\mu_R^{O^U} \in \mathcal{D}_R\mathcal{E}(U); U \in \mathcal{U}\right\}$.*

2.2.5 No-Disturbance

As already mentioned, the no-disturbance condition says that $\mu_R^{O^j}|_{kj} = \mu_R^{O^k}|_{kj}$ for all k and j, which means there is local agreement between contexts. This condition is equivalent to the existence of a compatible family to $\mathcal{D}_R\mathcal{E}$, but it doesn't imply $\mathcal{D}_R\mathcal{E}$ to be a sheaf. Since we can only have access to contexts, it is possible to define the functor $\mathcal{D}_R\mathcal{E}$ through a state that can't be extended to a distribution in the global events.

No-disturbance is equivalent to the notion of parameter-independence, as explained in Ref. [Barbosa et al. 2019], a property that, if violated, means the existence of non-trivial data between contexts. As stated in Ref. [Dzhafarov and Kujala 2018], where disturbance is called inconsistent connectedness: "Intuitively, inconsistent connectedness is a manifestation of direct causal action of experimental set-up upon the variables measured in it". We will work with non-disturbing models.

2.2.6 Contextuality

Contextuality is the impossibility of describing a given R-empirical model in classical terms, but one must first define which classical notion to use. We will call it R-contextuality to make explicit the chosen semiring. First, we know that any distribution can be described as the marginalization of another one,

$$\mu_R^{O^U}(A) = \sum_{\lambda\in\Lambda} k^{O^U}(\lambda, A), \tag{2.6}$$

for all $A \in \mathbb{P}(O^U)$, where $k^{O^U} : \Lambda \times \mathbb{P}(O^U) \to R$ is an R-distribution that satisfies $\sum_{\lambda\in\Lambda} k^{O^U}\left(\lambda, O^U\right) = 1_R$. In the literature of contextuality and nonlocality, Λ is called the set of hidden variables, which is statistically taken into account but is empirically inaccessible.

To impose a classical behavior, the hidden variables must be independent of the contexts, a property called lambda-independence[7]. To reflect such behavior of independence, our model must show independence between measurements, in other words, be factorizable. Such independence allows us to write

$$\mu_R^{O^U}(A) = \sum_{\Lambda} p(\lambda, A) \prod_{x \in U} \mu_R^{O^x}(\rho'(U,x)(A)), \tag{2.7}$$

with the assistance of the set of hidden variables Λ being statistically taken into account by a distribution $p : \Lambda \times \mathbb{P}(O^U) \to R$. Summing it up with lambda-independence implies

$$\mu_R^{O^U}(A) = \sum_{\Lambda} p(\lambda) \prod_{x \in U} \mu_R^{O^x}(\rho'(U,x)(A)), \tag{2.8}$$

with $p(\Lambda) = 1_R$, closing the representation of an R-empirical model as a classical system.

Definition 13. *An R-empirical model is said to be R-noncontextual if there is an R-distribution p and a set of hidden variables Λ such that equation 2.8 holds for all $U \in \mathcal{U}$.*

Another property we can impose is outcome-determinism, which is the property of logically distinguishing between outcomes.

Definition 14. *Outcome-determinism in an R-distribution of an empirical model is defined as for all $\lambda \in \Lambda$ there is an outcome $o \in O^U$ such that $k^{O^U}(\lambda, A) = \delta_o(A)$. Equivalently, an empirical model is outcome-deterministic if it satisfies $\prod_{x \in U} \mu_R^{O^x}(\rho'(U,x)(A)) \in \{0,1\}$.*

In combination with no-disturbance, we get the following result [Abramsky and Brandenburger 2011, Abramsky, Barbosa and Mansfield 2017].

Proposition 2. *A non-disturbing R-empirical model that satisfies the outcome-determinism condition has as its hidden variables exactly its global events.*

With this result, one can prove the Fine–Abramsky–Brandenburger Theorem [Abramsky and Brandenburger 2011], where R-contextuality can be understood as the non-extendability of a local section to a global section of $\mathcal{D}_R\mathcal{E}$, or in other words, as the nonexistence of a global R-distribution with marginalization to a context $U \in \mathcal{U}$.

Theorem 1 (Fine-Abramsky-Brandenburger)**.** *For an empirical model satisfying no-disturbance and outcome-determinism, the following are equivalent:*

- *to be described by deterministic hidden variables described by equation 2.8;*

[7] Lambda-independence is related to the concept of free choice in nonlocality [Cavalcanti 2018, Abramsky, Brandenburger and Savochkin 2014]. It can be understood as a dependence of the hidden variables, sometimes called ontic variables, on the contexts. Such dependence can store contextuality, as free choice can be understood as storing nonlocality [Blasiak et al. 2021]. For more details on the classification of hidden variables in the subject of nonlocality, see Ref. [Brandenburger and Yanofsky 2008].

- *all local sections extending to global sections;*
- *a distribution $\mu_R^{O^X}$ that marginalizes to $\mu_R^{O^U}$.*

We can graphically describe noncontextual behavior as the commutation of the diagram

$$\begin{array}{ccc} \mathcal{E}(\mathcal{U}) & \xrightarrow{\mu_R} & R \\ & {\scriptstyle i'} \searrow \quad \nearrow {\scriptstyle \nu_R} & \\ & \mathcal{E}(X) & \end{array} \tag{2.9}$$

The global events define a global R-distribution $\mu_R^{O^X}$, and the commutation implies the realization of the R-empirical model by it. Here, i' is the inclusion of local events in global events. As explored in Ref. [Montanhano 2022] and in part II of this thesis, the failure of commutativity can be seen in two independent ways: the first due to i', which is linked to anti-realist interpretations, and the second due to ν_R, linked to realist interpretations. With the previous results, the Sheaf Approach chooses to attribute the failure to ν_R by imposing the sheaf condition on events.

2.2.7 A Hierarchy of Contextuality

The Sheaf Approach generally restricts itself to models satisfying outcome-determinism and no-disturbance to be able to utilize the strong prior results. Under such conditions, we will cite some facts that follow from the choice of a semi-ring R.

Every noncontextual non-disturbing $\mathbb{R}$-empirical model is equivalent to a non-disturbing $[0, 1]$-empirical model. If the $\mathbb{R}$-empirical model allows a non-negative description, then it is $[0, 1]$-noncontextual. $[0, 1]$-contextuality is referred to as probabilistic contextuality. An example is the Bell-Clauser-Horne-Shimony-Holt model [Bell and Aspect 2004, Clauser et al. 1969], which exhibits contextuality only when dealing with probabilities since it does not feature any local event that does not allow an extension to a global event. In other words, contextuality is only observable through probability distributions and not logically.

$\mathbb{B}$-contextuality is called logical or possibilistic. A $\mathbb{B}$-contextual empirical model has at least one local event that does not allow extension to a global event, and therefore the model allows verification of non-classicality through logical methods. An example of this case is the Hardy model [Hardy 1992, Hardy 1993, Cabello et al. 2013]. If the $\mathbb{B}$-contextual empirical model does not exhibit any local event that can be extended to a global event, then it is said to exhibit strong contextuality, the highest level of contextuality a model can exhibit. Examples of this case include the Popescu-Rohrlich boxes model [Popescu and Rohrlich 1994], the Klyachko-Can-Binicioğlu-Shumovsky pentagram model [Klyachko et al. 2008], and the Greenberger-Horne-Zeilinger model [Greenberger, Horne and Zeilinger

2007]. Every $[0,1]$-empirical model can be decomposed into a noncontextual part and a strong contextual part.

A $\mathbb{B}$-empirical model uniquely defines a $[0,1]$-model induced by the standard map $[0,1] \to \mathbb{B}$ defined by $0 \mapsto 0$ and $(0,1] \mapsto 1_{\mathbb{B}}$. This determines that logical contextuality is stronger than probabilistic contextuality, as there exist models that do not exhibit logical contextuality even when they are contextual. On the other hand, strong contextuality is a special case of logical contextuality, since no local event admits an extension. We then have a hierarchy relating the contextuality of non-disturbing models:

$$\text{strongly contextual} > \text{logical contextual} > \text{probabilistic contextual} > \text{noncontextual} \tag{2.10}$$

2.2.8 Contextual Fraction

The contextual fraction is a quantification of contextuality, based on the fact that a noncontextual model can be written as a convex combination of global events. The finite version was present in Ref. [Abramsky, Barbosa and Mansfield 2017], and the more general case of non-finite fibers in Ref. [Barbosa et al. 2019].

Formally, each measurable bundle defines an incidence matrix

$$M\left(\sigma_U, \sigma_{\mathcal{B}}\right) = \begin{cases} 1 \text{ if } \sigma_{\mathcal{B}}|_U = \sigma_U; \\ 0 \text{ otherwise.} \end{cases} \tag{2.11}$$

It has the possible global events indexing the columns and the possible local events of maximal contexts indexing the rows, such that the entry will only be non-null if the local event is a restriction to the context of the global event. A model to be noncontextual per section is to be a convex combination of global events, and therefore we will have a weight b_i for each of them, which should add 1. It is equivalent to

$$M\vec{b} := \vec{p} \tag{2.12}$$

where $\vec{p}$ is the vector of the probabilities of the outcome in each context (certain care must be taken so that the vector, which originates in the usual probability table, is in the correct position in the incidence matrix).

Definition 15. *The noncontextual fraction NCF is defined as the maximum value of $\sum_i b_i$ such that $b_i \geqslant 0$ and $\sum_j M_{ij} b_j \leqslant p_i$, i.e.,*

$$NCF := \max_{\vec{b}} \left\{ \sum_i b_i ; b_i \geqslant 0, \sum_j M_{ij} b_j \leqslant p_i \right\}. \tag{2.13}$$

The contextual fraction is then defined as $CF = 1 - NCF$.

Because of its linear definition, one can easily define NCF as a linear programming problem, that presents a well explored literature. The contextual fraction is related to the contextuality inequalities in the literature, given the maximal violation of inequalities of the measurable bundle, and has the necessary properties to be considered a "good" quantification of contextuality. In the case of interest in such properties, the references already mentioned have more details, and Ref. [Amaral and Cunha 2017] presents a good review.

2.3 Generalized Contextuality and Generalized Probability Theories

Let's review here some concepts regarding generalized probability theories. Such theories will be important for formalizing sets of operational objects that are accessible for probabilistic valuation. They will serve as a substrate for subsequent analyses.

The objective of operational probability theories is to provide an operational description of physical theories, serving as an initial construction for their purely operational depiction [Schmid et al. 2020]. We will work with standard probabilities and with only one system, so it is unnecessary to define how to combine systems together. This restriction categorizes operational probability theories as generalized probability theories, or GPTs.

2.3.1 Generalized Probability Theories

A GPT can be described as a category. A category is given by a class of objects and a class of morphisms between each pair of objects, with one as the source and the other as the target, where there exists composition of morphisms, and such composition is associative. In the case of GPTs, the morphisms define operations that represent physical operations between the objects, and we have a trivial object that serves as a fundamental element for the construction of processes and their valuation. For a presentation of category theory aimed at applications in quantum theory, see Refs. [Coecke 2008, Coecke and Paquette 2011].

The operations from the trivial object, which we denote by $\perp$, to any other object are called states, denoted generally by P or categorically by $f \leftarrow \perp$, and we will denote them as $|P\rangle$ for reasons that will become clear later on. The set of operations from objects other than the trivial one to it are called effects, denoted generally by E or categorically by $\perp \leftarrow f$, and we will denote them as $\langle E|$.

The other morphisms between two non-trivial objects can be understood as representations of transformations. Any transformation $f \leftarrow g$ can be thought of as a transformation that takes the state $f \leftarrow \perp$ to the state $g \leftarrow \perp$, which we can denote as $T|P_f\rangle = |P_g\rangle$, with T denoting the transformation in question. The same reasoning applies analogously to effects. This identifies physically interesting transformations as functions of the set of states to itself, or equivalently as functions of the set of effects to itself.

The automorphisms of the trivial object possess a structure of scalars, usually taken as a semiring or semifield, such as the Boolean semiring $\mathbb{B}$ given by $0, 1$, and the probabilistic semifield $\mathbb{R}^+$ given by $[0, 1]$. Here, we will focus on the probabilistic semifield $\mathbb{R}^+$. The sets of states $\mathcal{P}$, transformations $\mathcal{T}$, and effects $\mathcal{E}$ provide us with the probabilities of a process in the system they represent through a function:

$$p : \mathcal{P} \times \mathcal{T} \times \mathcal{E} \to [0,1] :: (P, T, E) \mapsto p(E|T, P) \tag{2.14}$$

interpreted as the probability of obtaining the outcome E when starting with a state P that underwent a transformation T. The composition of operations used in the argument of the function $p(E|T, P)$ can be understood as a path from the trivial object to itself, passing through the operations that define P first, then the operations that define T, and finally the operation of E, closing the loop in the category of operations. This identifies p as a function of loops passing through $\perp$ to the set of scalars, in our case, the probabilistic one.

Usually, we can use the bracket notation that we have already introduced to write

$$p(E|T, P) = \langle E| \, T \, |P\rangle, \tag{2.15}$$

indicating that this is an identification with a loop of processes and in analogy with quantum theory and linear algebra. To turn this analogy into an identification, one imposes that the function p preserves mixtures of operations, i.e., the convex combination with scalar coefficients. From the preservation of mixtures and their identification as valid operations, the convexity of the sets of operations naturally follows. This preservation of mixtures can be further extended to establish the linearity of p, enabling the representation of states and effects in a vector space, with transformations described as linear maps acting on them while preserving the sets of states and effects. Naturally, the identification of the bracket notation arises from associating p with an inner product in this vector space.

Example 11. *Let's explicitly illustrate, as an example, the GPT structure of a qubit, the two-dimensional quantum system. Naturally, we can begin with the Bloch sphere representation, where a state ρ is encoded as a vector $\vec{a}$ on a unit sphere in $\mathbb{R}^3$, thus $|\vec{a}| \leqslant 1$. Here, we have*

$$\rho = \frac{1}{2}(I + \vec{a} \cdot \vec{\sigma}), \tag{2.16}$$

where I is the identity matrix and $\vec{\sigma}$ is the vector given by the Pauli matrices basis. The origin of the space can be identified with $\perp$, and every vector from the origin to a point within the sphere determines the process defining the state given by that point.

To define the effects, let us use the fact that quantum theory satisfies the condition of strong duality [Müller and Ududec 2012]. This property allows the set of states *and effects to be represented in the same way, with the difference that the process occurs in the opposite direction, from the point on the sphere to the origin, but also uniquely defining a vector to represent the effect.*

Another property of quantum theory is that its set of physical transformations is maximal, meaning that all possible transformations are physically feasible. Due to linearity, we have that a transformation will be represented by a linear map $\overline{T}$ *with induced norm* $|\overline{T}| \leqslant 1$.

With all this representation in a space $\mathbb{R}^3$*, it is natural that probabilistic valuation be given by an inner product, which indeed occurs by the Born rule. We have that the trace can be calculated as*

$$p(E|T, P) = \mathrm{Tr}\{eT\rho\} = \frac{1}{4}(1 + \vec{a}e^{\intercal}\overline{T}\vec{a}\rho). \tag{2.17}$$

There are further intriguing mathematical intricacies regarding this construction. For a more formal exposition, see Refs. [Amaral 2014, Janotta and Hinrichsen 2014, Müller 2021, Selby, Scandolo and Coecke 2021].

2.3.2 Ontic Representation

A (classical) ontic representation of a (physical) theory involves embedding its vector space representation into classical probability theory. In such a theory, a simplex is defined as a set of states, and its dual as the set of effects. The primary aim of an ontic representation is to expand upon the original theory by refining the involved variables, thus enabling an explanation of the statistics of the GPT as a sub-model of classical theory.

Example 12. *As a second example of GPT, which is important for addressing ontic representation, we have a theory of classical probabilities. In it, the states already directly represent the probabilities. In a classical system generated by* n *possible outcomes, we have a vector in* $\mathbb{R}^n$ *whose elements are nonnegative real numbers that sum to 1 as a state. In other words, the set of states* $\mathcal{P}$ *is given by a simplex in* $\mathbb{R}^n$.

Probabilistic valuation is nothing more than the entries of the vector, and by taking the maximal set of effects, we have that the set of effects $\mathcal{E}$ *will be given by the dual set of vectors. This defines a hypercube in* $\mathbb{R}^n$*, with vertices on each of the coordinate axes, at the origin, and on the vector with entries being* 1*, which acts as the identity vector.*

Transformations are also taken as the maximal set, given by linear maps that preserve probabilities. These are stochastic maps, or Markov maps.

We can always do an embedding into a classical GPT, and the probability of the model will be given by the chain rule

$$p(E_r|T_t, P_s) = \sum_{\lambda,\lambda'} \xi(E_r|\lambda')\Gamma(\lambda', T, \lambda)\mu(\lambda|P_s), \tag{2.18}$$

the valuation functions ξ, Γ, μ, and the set of ontic variables denoted by Λ. However, this does not guarantee classicality, as one needs to impose conditions on the valuation functions to ensure they do not violate any classical behavior.

We need to impose the independence of the measurements to which the effects belong. But note that such representation is independent of measurements, once the function ξ has the outcomes as its domain, in the form of a set of effects. Thus, it is restricted to non-disturbing models[8], or in other words, for measurements m and n, it holds

$$\xi(E_r|\lambda, m) = \xi(E_r|\lambda, n) = \xi(E_r|\lambda), \tag{2.19}$$

fixing the conditions for embedding to preserve classical GPT.

At this level of generality, outcome-determinism[9] is not required, unless one wants to use factorizability as a condition for noncontextuality, as in the Sheaf Approach [Wester 2018][10]. Once the outcomes are fundamental in this framework, represented in the events, and deal with the non-classicality of all the steps in a physical process, it is natural that this is the strongest framework to construct new generalized models up to the limitation by no-disturbance.

2.3.3 Generalized Contextuality

Generalized contextuality [Spekkens 2005] deals with preparations, transformations, and unsharp measurements (the latter equivalent to effect algebras defined by the set of effects in a GPT). It is a method to investigate the classicality of a system through operational equivalences. This approach is based on Leibniz's principle of indiscernibles, which states that if two objects possess the same properties, and therefore are indiscernible, then they are identical.

Generalized contextuality [Spekkens 2005] deals with preparations, transformations, and unsharp measurements (the latter equivalent to effect algebras defined by the set of effects in a GPT). It is a method to investigate the classicality of a system through operational equivalences. This approach is based on Leibniz's principle of indiscernibles, which states that if two objects possess the same properties, and therefore are indiscernible, then they are identical.

[8] No-disturbance is defined in the intersection of contexts: if the valuation of an intersection (when one defines it) $\xi(C \cap C')$ is independent of C and C' for all pairs of contexts, then $\mathcal{M}$ is said to be non-disturbing. It is related to parameter-independence [Brandenburger and Yanofsky 2008, Barbosa et al. 2019].

[9] Outcome-determinism is defined in the valuation: the outcomes defined on the contexts where the distribution is defined can be explained in a deterministic way. It is equivalent to restricting our interest to ideal measurements, which one can easily criticize when thinking in empirical applications [Spekkens 2014].

[10] Outcome-determinism implies that $\xi : \mathcal{E} \times \Lambda \to \{0, 1\}$, thus codifying the determinism of this valuation.

Using linearity to represent the construction of the final process from basic processes by their summation with certain coefficients, and rearranging the equation to make the linear dependence explicit, we find that for the states, the effects, and the transformations, operational equivalences can be expressed as linear conditions [Selby et al. 2021]

$$\sum_s a_s^{(\alpha)} P_s = 0, \tag{2.20}$$

$$\sum_r b_r^{(\beta)} E_r = 0, \tag{2.21}$$

$$\sum_t c_t^{(\tau)} T_t = 0, \tag{2.22}$$

indexed by α, β, and τ. This form will be important for what follows in the next sections. Note that although the initial construction imposes that the processes remain within the set of processes, and therefore limits the values of the coefficients, the reorganization in the linear conditions above allows the coefficients to turn the vectors into objects outside the set of processes.

Definition 16. *A theory is noncontextual if for a classical ontic representation the operational equivalences are preserved in the probabilities given by the valuation maps*

$$\sum_s a_s^{(\alpha)} \mu(\lambda|P_s) = 0, \tag{2.23}$$

$$\sum_r b_r^{(\beta)} \xi(E_r|\lambda') = 0, \tag{2.24}$$

$$\sum_t c_t^{(\tau)} \Gamma(\lambda', T_t, \lambda) = 0, \tag{2.25}$$

for all λ *and* λ'[11].

As the ontic representation is an embedding in a classical GPT, the ontic space Λ is a simplicial set. The conditions of noncontextuality stated above assert that the original theory, its states, effects, and transformations, can be embedded in Λ, and its probabilities encoded in it as a coarse graining without violating classical probability theory in the sense of the Kolmogorov axioms [Schmid et al. 2020, Schmid et al. 2021]. Interestingly, this is equivalent to there being no need for negative values for the functions ξ, Γ, and μ when represented in an embedding as described above [Spekkens 2008].

[11] This approach is more refined than other frameworks because of its operational interpretation, which allows the exploration of non-classicality beyond measurement and effects. The outcome-determinism is not imposed, but when imposed this approach is equivalent to the Sheaf Approach to contextuality [Staton and Uijlen 2015].

A property of valuation functions is their linearity within the convex set of objects in the domain. As an example, let's confine ourselves to the set of effects $\mathcal{E}$, though the same argument holds for states and transformations. Let $A, B \in \mathcal{E}$, and $A + B \in \mathcal{E}$, then

$$p(A + B) = p(A) + p(B) \tag{2.26}$$

if and only if

$$\xi(A + B|\lambda') = \xi(A|\lambda') + \xi(B|\lambda'), \tag{2.27}$$

which follows from the definition of ontic representation. Another way to see this is to note that if $A + B \in \mathcal{E}$, then $\{A, B, \mathbb{1} - (A + B)\}$ is a valid measurement, thus

$$\begin{aligned} 1 =& \xi(\mathbb{1}|\lambda') \\ =& \xi(\mathbb{1} - (A + B)|\lambda') + \xi((A + B)|\lambda') \\ =& \xi(\mathbb{1} - (A + B)|\lambda') + \xi(A|\lambda') + \xi(B|\lambda') \end{aligned} \tag{2.28}$$

and linearity follows, since $\xi(C|\lambda') + \xi(\mathbb{1} - C|\lambda') = 1$ for every effect C. But there is no guarantee that such linearity holds outside $\mathcal{E}$, as the following example shows.

Example 13. *The Wigner's representation of quantum mechanics is an ontic representation of quantum theory, and it is linear for mixed states. However, as explained in Ref. [Case 2008], $W_\psi = W_\alpha + W_\beta$ with $\psi = \psi_\alpha + \psi_\beta$ generally does not hold. On the other hand, this holds for classical theories, which follow from the Kolmogorov axioms for probabilities.*

3 The Influence of Topology

3.1 Empirical Models and Bundles

We will study some well-known examples of empirical models through their bundle diagrams and expose their contextual or noncontextual behaviors. Our goal is to see how an initial intuition of the relationship between contextuality and topology fades as we advance to more complicated examples. Let's begin with two emblematic examples that have a representation in bundle diagrams illustrating their behavior.

Example 14 (Trivial)**.** *The standard trivial example of a non-disturbing empirical model, analogous to the informal description of a trivial bundle, is given by Table 1. The contexts of this example are* $\mathcal{U} = \{ab, bc, cd, de, ea, a, b, c, d, e\}$*, with measurements* $X = \{a, b, c, d, e\}$*. All the sets of outcomes for measurements are the same,* $O^i = \{0, 1\}$*. Its table gives the probabilities for each event of each maximal context. Examples with finite outcomes can be described by the values of each element of* O^U*, using the discrete* σ*-algebra. The bundle diagram is the possibilistic coarse-graining of the model, showing only the non-null events. All events at all intersections have a probability of* $\frac{1}{2}$*. It can be described as two global events each with a probability of* $\frac{1}{2}$*. Its non-disturbance follows from the fact that all marginalizations over the probabilities of measurement outcomes also yield* $\frac{1}{2}$ *each. It is noncontextual, as it can be described as the combination of two global events,*

$$s_X^{trivial} = \left\{ \frac{1}{2}(abcde \to 00000) + \frac{1}{2}(abcde \to 11111) \right\}, \tag{3.1}$$

in agreement with noncontextual fraction $NCF = 1$*.*

Example 15 (Liar cycle)**.** *An example of non-disturbing empirical model with no description by global events is the liar cycle, Table 2, here with five measurements. It could be understood as a set of individuals saying that the next one will say the truth, cyclically, but the last one saying that the first one lied, such that a paradox occurs. The structure of the measurement scenario is the same as the previous example, it only differs in the valuation. The model is non-disturbing. It can't be described as a combination of global events. The cause is the swap in the context ea, giving a contextual behavior for this model, with noncontextual fraction* $NCF = 0$*.*

The liar n-cycles is an important example of n-cycle scenarios, i.e. scenarios with n measurements organized as a cycle, once one can construct any paradoxical behavior on n-cycle scenarios by them [Santos and Amaral 2021]. It is also important as an example

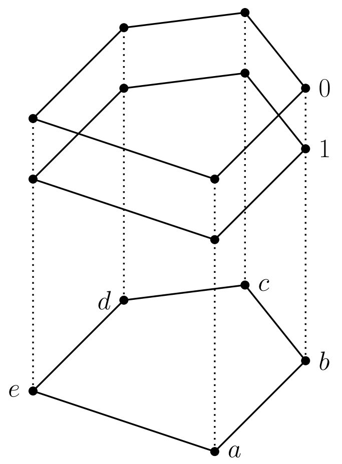

Figure 1 – The visualization of the trivial example of an empirical model through its bundle diagram.

	00	01	10	11
ab	$\frac{1}{2}$	0	0	$\frac{1}{2}$
bc	$\frac{1}{2}$	0	0	$\frac{1}{2}$
cd	$\frac{1}{2}$	0	0	$\frac{1}{2}$
de	$\frac{1}{2}$	0	0	$\frac{1}{2}$
ea	$\frac{1}{2}$	0	0	$\frac{1}{2}$

Table 1 – Table of outcome probabilities of each context of the trivial example.

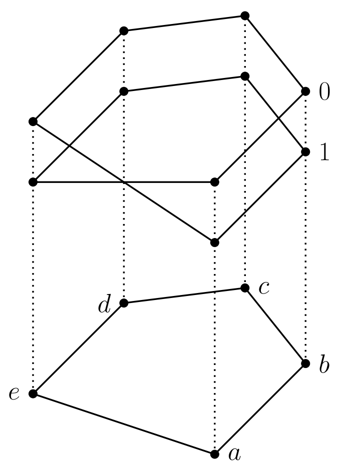

Figure 2 – The visualization of the liar cycle example through its bundle diagram. There isn't any global event, caused by the swap of elements in the path *ea*.

	00	01	10	11
ab	$\frac{1}{2}$	0	0	$\frac{1}{2}$
bc	$\frac{1}{2}$	0	0	$\frac{1}{2}$
cd	$\frac{1}{2}$	0	0	$\frac{1}{2}$
de	$\frac{1}{2}$	0	0	$\frac{1}{2}$
ea	0	$\frac{1}{2}$	$\frac{1}{2}$	0

Table 2 – Table of outcome probabilities of each context of the liar cycle example.

of nonlocal "superquantum" correlations genarated by the well-known Popescu-Rohrlich boxes [Popescu and Rohrlich 1994].

Example 16 (KCBS)**.** *The next example appears to be an extreme version of the liar cycle, but it simply repeats the permutation of elements of the outcome fiber in every context. In logical terms, each participant claims that the subsequent one is lying, as shown in Table 3. The measurement scenario remains unchanged; the difference lies in the distributions, which defines the maximal violation of the Klyachko-Can-Binicioglu-Shumovsky inequality [Klyachko et al. 2008]. The example in Table 3 follows similar* *reasoning to the previous one. It cannot be described by a combination of global events due to an odd number of swaps, hence it exhibits contextuality, consistent with a noncontextual fraction* $NCF = 0$.

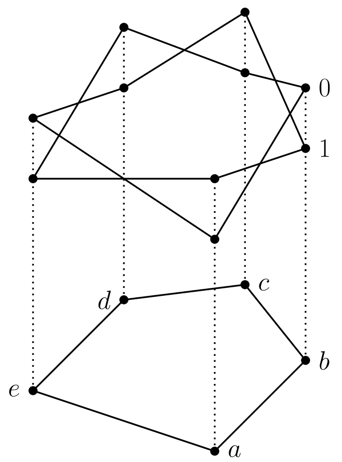

Figure 3 – The visualization of the KCBS example through its bundle diagram. Every local section is a swap, implying that for an odd number of sides there isn't any well defined global event.

	00	01	10	11
ab	0	$\frac{1}{2}$	$\frac{1}{2}$	0
bc	0	$\frac{1}{2}$	$\frac{1}{2}$	0
cd	0	$\frac{1}{2}$	$\frac{1}{2}$	0
de	0	$\frac{1}{2}$	$\frac{1}{2}$	0
ea	0	$\frac{1}{2}$	$\frac{1}{2}$	0

Table 3 – Table of outcome probabilities of each context of the KCBS example.

Previous examples can be represented as fiber bundles, and contextuality appears as a consequence of the non-trivial topology of the fiber bundle. At this point, the reader must be alerted that this reasoning only works because of the triviality of these examples, where the global events are distinguishable and there is a clear topological translation. Future examples will lose such a direct topological interpretation.

Example 17 (Hardy)**.** *The example in Table 4 is more complicated, but one can still determine its contextuality just by examining the events. It was first introduced in Ref. [Cabello et al. 2013] as a Hardy-like model [Hardy 1992, Hardy 1993] exhibiting contextual behavior in quantum theory, with a noncontextual fraction* $NCF = \frac{7}{9}$. *The model is non-disturbing and contextual. It includes global events such as* $(ab \to 00)$, $(cd \to 00)$, *and* $(ea \to 00)$, *but there are events that cannot be extended globally, such as* $(ab \to 01)$, $(ab \to 10)$, $(cd \to 01)$, $(cd \to 10)$, $(ea \to 01)$, *and* $(ea \to 10)$.

Example 18 (Bell)**.** *The model of Table 5 is a well-known example of a non-disturbing contextual model in quantum mechanics. It is also known as the Bell-CHSH model, due to its application in the Clauser-Horne-Shimony-Holt inequality and its importance as an example of bipartite scenarios exhibiting nonlocal behavior [Bell and Aspect 2004, Clauser et al. 1969]. The* ab *local section is trivial, but the others have probabilities that reflect the*

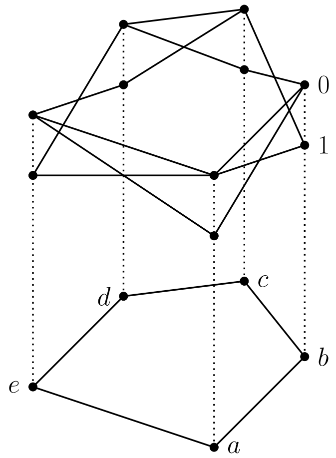

Figure 4 – The visualization of the Hardy example through its bundle diagram. This model has global events, but some local events can't be extended to a global one.

	00	01	10	11
ab	$\frac{2}{9}$	$\frac{2}{3}$	$\frac{1}{9}$	0
bc	0	$\frac{1}{3}$	$\frac{2}{3}$	0
cd	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
de	0	$\frac{2}{3}$	$\frac{1}{3}$	0
ea	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{2}{3}$	0

Table 4 – Table of outcome probabilities of each context of the Hardy example.

liar cycle. In this sense, this model can be understood as a combination of a trivial example and three liars, which agrees with the noncontextual fraction $NCF = \frac{3}{4}$. *As can be seen from its bundle diagram, all events are defined as restrictions of global events. Therefore, its contextual behavior only manifests when considering the values of distributions.*

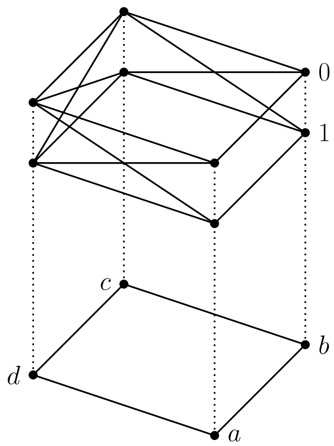

Figure 5 – The visualization of the Bell example through its bundle diagram. This model has only global events, and the contextual behavior only appears because of the impossibility to explain it with positive real numbers.

	00	01	10	11
ab	$\frac{1}{2}$	0	0	$\frac{1}{2}$
bc	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
cd	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
da	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Table 5 – Table of outcome probabilities of each context of the Bell example.

In ref. [Montanhano 2021], the topological structure of bundles generated by empirical models in the Bundle Approach is studied. The use of holonomy in the corresponding frame bundle is proposed, formalizing the intuition that contextuality generates logical paradoxes that can be seen through bundle topology. However, the holonomy applied in this structure does not allow for a characterization. This limitation arises from the need to use group structures that are not natural to empirical models, as we will discuss in the next section.

3.2 Sheaf Cohomology to Detect Contextuality

To define the Čech cohomology, we utilize the covering $\mathcal{U}$, disregarding individual measurements. We define the nerve $\mathrm{N}(\mathcal{U})$ as the collection of contexts and their non-empty intersections, with elements $\sigma = (U_{j_0}, ..., U_{j_q})$ given by $|\sigma| = \bigcap_{k=0}^{q} U_{j_k} \neq \varnothing$, where $j_k \in I$ are ordered. The nerve of a covering defines an abstract simplicial complex, where the q-simplices form a collection $\mathrm{N}(\mathcal{U})^q$. Note that for this to hold, the contexts forming σ must be distinct[1]. We define the map

$$\partial_{j_k} : \mathrm{N}(\mathcal{U})^q \to \mathrm{N}(\mathcal{U})^{q-1} \tag{3.2}$$

as $\partial_{j_k}(\sigma) = (U_{j_0}, ..., \widehat{U_{j_k}}, ..., U_{j_q})$, where the hat denotes the omission of the context.

3.2.1 Cohomological Contextuality

The Čech cohomology for contextuality is defined not by $\mathcal{D}_R\mathcal{E}$, but by a functor $\mathcal{F} : \mathrm{N}(\mathcal{U}) \to \mathbf{AbGrp}$ from the nerve to the category of Abelian groups, representing the presheaf $\mathcal{D}_R\mathcal{E}$. Typically, $\mathcal{F} = F_S\mathcal{L}$, where $\mathcal{L}$ is at least a subsheaf of $\mathcal{E}$ and $F_S : \mathbf{Set} \to \mathbf{AbGrp}$ assigns to a set O the free Abelian group $F_S(O)$ generated by it related to the ring S. Specifically, it is defined as a free module on a ring S (see Ref. [Abramsky et al. 2015]). In other words, it is a representation of $\mathcal{E}$. Such a presheaf must satisfy:

- $\mathcal{F}(U) \neq \varnothing$ for all $U \in \mathcal{U}$
- $\mathcal{F}$ is flasque beneath the cover (the restriction map $\rho^{\mathcal{F}}(U', U)$ on $\mathcal{F}$ is surjective whenever $U \subseteq U' \subseteq V$, $V \in \mathcal{U}$)
- any compatible family given by just one of the events for each context, here thought of as an element of the basis, induces a unique global section.

The first condition implies that a context must have a non-trivial image by F, and the second one codifies that the information given by F in a subcontext is already in the context that contains it. The third condition follows from the imposition of Gluing and Locality axioms on the basis and is induced by the sheaf structure of the events.

1 There is an equivalence between ordered and unordered Čech cohomology, where in the latter, contexts can be repeated. For clarity, I work with the ordered version.

We can define an augmented Čech cochain complex:

$$\mathbf{0} \longrightarrow C^0(\mathcal{U},\mathcal{F}) \xrightarrow{d^0} C^1(\mathcal{U},\mathcal{F}) \xrightarrow{d^1} C^2(\mathcal{U},\mathcal{F}) \xrightarrow{d^2} \dots \tag{3.3}$$

where the Abelian group of q-cochains is:

$$C^q(\mathcal{U},\mathcal{F}) = \prod_{\sigma\in \mathrm{N}(\mathcal{U})^q} \mathcal{F}\left(|\sigma|\right) \tag{3.4}$$

and the coboundary map $d^q : C^q(\mathcal{U},\mathcal{F}) \to C^{q+1}(\mathcal{U},\mathcal{F})$ is the group homomorphism given by:

$$d^q(\omega)(\sigma) = \sum_{k=0}^{q+1}(-1)^k\rho'(|\partial_{j_k}\sigma|\,,|\sigma|)\omega(\partial_{j_k}\sigma) \tag{3.5}$$

with $\omega \in C^q(\mathcal{U},\mathcal{F})$ and $\sigma \in \mathrm{N}(\mathcal{U})^{q+1}$. One can show that $d^{q+1}d^q = 0$, allowing construction of the cohomology of this cochain complex.

The Abelian group of q-cocycles is defined by:

$$Z^q(\mathcal{U},\mathcal{F}) = \{c \in C^q(\mathcal{U},\mathcal{F})|d^q c = 0\} = \ker(d^q). \tag{3.6}$$

The Abelian group of q-coboundaries is defined by:

$$B^q(\mathcal{U},\mathcal{F}) = \left\{c \in C^q(\mathcal{U},\mathcal{F})|c = d^{q-1}z,\ z \in C^{q-1}(\mathcal{U},\mathcal{F})\right\} = \mathrm{Im}(d^{q-1}). \tag{3.7}$$

Clearly, $B^q(\mathcal{U},\mathcal{F}) \subseteq Z^q(\mathcal{U},\mathcal{F})$, and we define the q-th Čech cohomological group as the quotient $\check{H}^q(\mathcal{U},\mathcal{F}) = Z^q(\mathcal{U},\mathcal{F})/B^q(\mathcal{U},\mathcal{F})$.

3.2.2 Cohomological Contextuality

Let's explore two well-known results in the literature of Čech cohomology and cohomological contextuality [Abramsky, Mansfield and Barbosa 2012]. The first result establishes a bijection between compatible families and elements of the zeroth cohomology group $\check{H}^0(\mathcal{U},\mathcal{F})$.

Proposition 3. *There exists a bijection between compatible families and elements of the zeroth cohomology group $\check{H}^0(\mathcal{U},\mathcal{F})$.*

This result follows from the observation that, in the augmented cochain complex, the coboundary group $B^0(\mathcal{U},\mathcal{F})$ is trivial. This allows the use of elements from $\check{H}^0(\mathcal{U},\mathcal{F})$ in the search for an extension to a compatible family.

The second result characterizes what is known as cohomological contextuality. It involves constructing an obstruction on an initial 0-cochain that codifies the local sections of $\mathcal{F}$ on the covering $\mathcal{U}$.

To establish this, we introduce an auxiliary presheaf $\mathcal{F}|_U(V) = \mathcal{F}(U \cap V)$, and the canonical presheaf map $p : \mathcal{F} \to \mathcal{F}|_U :: p_V : r \mapsto r|_{U\cap V}$. Another auxiliary presheaf is $\mathcal{F}_{\bar{U}}(V) = \ker(p_V)$, defining the exact sequence of presheaves:

$$0 \longrightarrow \mathcal{F}_{\bar{U}} \longrightarrow \mathcal{F} \xrightarrow{\ p\ } \mathcal{F}|_U \tag{3.8}$$

The U-relative cohomology is defined as the Čech cohomology of the presheaf $\mathcal{F}_{\bar{U}}$.

Elements of $\check{H}^0(\mathcal{U}, \mathcal{F}_{\bar{U}_j})$ are in bijection with compatible families $\{r_k\}$ where $r_j = 0$. Starting with a local section s_{j_0} of U_{j_0}, the no-disturbance condition implies the existence of a family $\{s_{j_k}\}$ such that $s_{j_0}|_{j_0 j_k} = s_{j_k}|_{j_0 j_k}$ for all $k \neq 0$. Defining $c = \{s_{j_k}\}_{0\leqslant k} \in C^0(\mathcal{U}, \mathcal{F})$, a trivial computation shows that $z = dc \in Z^1(\mathcal{U}, \mathcal{F}_{\bar{U}_{j_0}})$, and we define the obstruction $\gamma(s_{j_0})$ as the cohomology class $[z] \in \check{H}^1(\mathcal{U}, \mathcal{F}_{\bar{U}_{j_0}})$[2].

Proposition 4. *Let $\mathcal{U}$ be connected, $U_{j_0} \in \mathcal{U}$, and $s_{j_0} \in \mathcal{F}(U_{j_0})$. Then $\gamma(s_{j_0}) = 0$ if and only if there exists a compatible family $\{r_{j_k} \in \mathcal{F}(U_{j_k})\}_{U_{j_k}\in\mathcal{U}}$ such that $r_{j_0} = s_{j_0}$.*

The idea behind this construction is to take a local section and investigate the possibility of extending it to a global section, examining the triviality of the obstruction. It provides a sufficient condition for contextual behavior, indicating a non-trivial obstruction in some local sections, but it is not a necessary one. A model presenting at least one non-trivial obstruction will be termed logically cohomological contextual, distinguishing it from the usual possibilistic contextual models.

3.2.3 The Failure of Cohomological Characterization

The intuition behind using cohomology in contextuality is as a way to encode, in an Abelian group, the data in each context, outcome set, and possible measures, and use them to study their behavior. It is natural to use the free Abelian group generated by the outcome set given by a functor over the category of contexts, thereby encoding contexts and outcomes. We can understand it in the following way. Assume enough conditions are presented such that contextual behavior appears as the non-factorizability of the model. Graphically:

$$\begin{array}{ccc} \mathcal{E}(\mathrm{N}(\mathcal{U})^0) & \longrightarrow & S \\ & {\scriptstyle i'} \searrow \quad \nearrow & \\ & \mathcal{E}(X) & \end{array} \tag{3.9}$$

where we start with the data of $\mathcal{E}(\mathrm{N}(\mathcal{U})^0) \to S$, and ask for the existence of the data of $\mathcal{E}(X) \to S$ plus a map i' such that the diagram commutes. We are asking for a factorization of the local events to global events when using F_S as the tool to quantify the measurement scenario. These diagrams are an informal way to show the data in $\check{H}^0(\mathcal{U}, \mathcal{F}) \to C^0(\mathcal{U}, \mathcal{F})$

2 Another way to show this is by using the snake lemma [Abramsky et al. 2015].

for $\mathcal{F} = F_S\mathcal{E}$. The previous construction of cohomology asks for the data that can't be encoded in a context, and cohomological contextuality is the existence of more data than the context can store. However, the choice of the ring S for the definition of the group does not come from the measure. This is the reason for the impossibility of the characterization with standard Čech cohomology.

The categories of rings, Abelian groups, and modules are canonically related, as are the categories of semi-rings, Abelian monoids, and semi-modules. So, the relation between these algebraic structures is simply encoded by the canonical forgetful functor $F : \mathbf{Ring} \to \mathbf{Rig}$, which forgets the negatives of the ring category. To study such a functor, let's review what we can forget with it [Baez and Shulman 2009].

Definition 17. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. It*

- *is called full if, for all pairs of objects A, A' of $\mathcal{A}$, the map $F : hom(A, A') \to hom(F(A), F(A'))$ induced by F is surjective; it forgets structure if it is not full;*
- *is called faithful if the induced map is injective for all pairs of objects; it forgets stuff if it is not faithful;*
- *is called essentially surjective if, for any object B of $\mathcal{B}$, there is an object A of $\mathcal{A}$ such that $F(A)$ is isomorphic to B; it forgets properties if it is not essentially surjective;*
- *forgets nothing if it is full, faithful, and essentially surjective, or equivalently, F admits an inverse, and the categories are called equivalent.*

A known result is that a semi-ring can be completed into a ring if and only if the semi-ring is cancellative[3]. As an example, the Boolean ring $\mathbb{B}$ isn't cancellative since

$$1_{\mathbb{B}} + 1_{\mathbb{B}} = 1_{\mathbb{B}} = 1_{\mathbb{B}} + 0_{\mathbb{B}} \tag{3.10}$$

implies that $1_{\mathbb{B}} = 0_{\mathbb{B}}$ if the cancellative property holds, which is absurd. Therefore, the functor $F : \mathbf{Ring} \to \mathbf{Rig}$ forgets properties. Also, there are more homomorphisms between objects of $\mathbf{Rig}$ than in objects of $\mathbf{Ring}$, implying that F forgets structure. However, it doesn't forget stuff since every ring in $\mathbf{Ring}$ is a semi-ring in $\mathbf{Rig}$, with the morphisms being preserved.

In conclusion, there is too much structure and property in $\mathbf{Ring}$, the category we use to define the cohomology with Abelian groups. Extra property plus structure allows the violation of the cohomological characterization of contextual behavior, providing too many ways to justify noncontextuality[4]. A characterization of contextuality can be

[3] The left adjoint of the forgetful functor $F : \mathbf{Ring} \to \mathbf{Rig}$ is not monic if the additive monoid of R is not cancellative.

[4] Categorically, we can construct a natural transformation between the functors $\mathcal{D}_R$ and F_S. With the forgetful map, one can show that usually there isn't a natural equivalence between them, allowing for violations even for strong contextual models, see Ref. [Carù 2018, Carù 2017].

obtained with sheaf cohomology of semi-modules [Montanhano 2021] that do not modify the semiring used. The setback occurs in the loss of the cochain complex structure, which depends on the group structure. The complexity of the construction makes it impractical even for simple models.

3.3 Topology of Measurement Scenarios and Vorob'ev Theorem

In Ref. [Montanhano 2021], we study the topology of measurement scenarios and its influence on contextuality, which we explore in this section. The argument about this influence begins with the Vorob'ev theorem [Vorob'ev 1959], linked with the notion of cyclicity, in the sense of Graham's reduction.

Definition 18. *A hypergraph $(X, \mathcal{C})$ is acyclic if it can be reduced to the empty set through Graham's reduction, an algorithm achieved by repeated application of the following operations:*

- *If $m \in X$ belongs to a single hyperedge, then delete m from this hyperedge.*
- *If $V \subsetneq U$, with $V, U \in \mathcal{C}$, then delete V from $\mathcal{C}$.*

The Graham's reduction algorithm can be interpreted as a coarse-graining of contexts, "forgetting" contexts that can be described by marginalization. Contextuality depends on maximal contexts and their intersections, as seen in the contextual fraction algorithm. Graham's reduction removes measurements that do not intersect and contexts that are not maximal, also erasing the minimal context of each measurement from the hypergraph. Hence, acyclic scenarios allow arbitrary simplification, while cyclic scenarios do not. Importantly, Graham's reduction does not preserve the simplicial complex structure of the scenario, as shown in Ref. [Barbosa 2015]. The property of being cyclic, although appearing to be a topological property in capturing "holes" in the hypergraph, is not a topological invariant, as illustrated in the following example.

Example 19. *The first triangle in the Graham process in Fig. 6 is simply connected when equipped with the topology of its geometric representation, and therefore collapsible. It is also acyclic when considered as a hypergraph, as depicted. In the case of the triangle with*

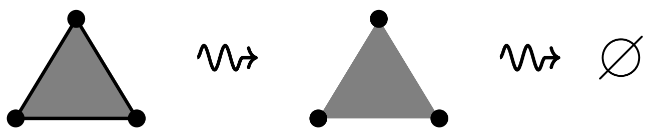

Figure 6 – Graham reduction of a filled triangle. The steps are: to exclude the edges; to exclude the isolated vertices. The result is the empty set, therefore the hypergraph is acyclic.

barycentric subdivision, the first of the Graham process in Fig. 7, despite being simply connected when equipped with the topology of its geometrical representation and therefore collapsible, being homeomorphic to the previous case, it is not an acyclic hypergraph and can be a measurable scenario for an empirical model as we will see in a following example.

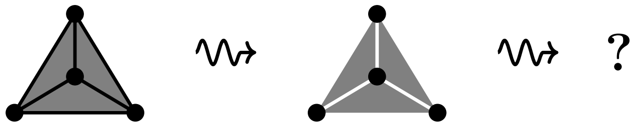

Figure 7 – Graham reduction of a triangle with barycentric subdivision. The first and unique allowed step is to exclude the edges. There isn't any other possible step, therefore the hypergraph is cyclic.

Having a topologically trivial base doesn't imply a noncontextual empirical model, a marked difference from fiber bundle theory, where having a topologically trivial base (when equipped with a group structure) implies a topologically trivial total space [Schwarz 1994]. This example also shows that the role of topology in contextuality, if any, should be more subtle.

We can rewrite Vorob'ev theorem as follows:

Theorem 2 (Vorob'ev's theorem)**.** *Given a measurement scenario, any non-disturbing empirical model defined on it is noncontextual if and only if its base, seen as a hypergraph, is acyclic.*

In short, contextuality does not directly follow from the standard topology of the scenario, even with acyclicity structurally linked with noncontextuality [Karvonen 2019]. The examples in Figures 6 and 7 demonstrate the independence in terms of topology by dissociating the topology of the simplicial complex from its suitability as a measurement scenario for an empirical model.

3.3.1 A Hierarchy in the Simplicial Complex: n-contextuality

Given an empirical model and once the simplicial complex structure of scenarios is identified, one can investigate the relation between the inductive definition of a simplicial complex (by incrementing the dimension of the included simplices at each step) and the contextuality demonstrated at each step of such induction in the measurement scenario.

Definition 19. *Given an empirical model, its* 0*-skeleton consists of the set of measurements* X *and their sections; its* n*-skeleton for* $n \geqslant 1$ *consists of the set of measurements* X *and the contexts involving up to* $n+1$ *measurements, and their sections.*

One can also define this construction as the marginalization of the empirical model on the set of contexts in the n-skeleton. The following definition addresses the question of contextuality induced by the n-skeleton using the contextual fraction.

Definition 20. *Let CF_n be the contextual fraction of the empirical model induced by the marginalization of a given empirical model on the n-skeleton of its base. The n-contextuality is defined as the difference $CF_n - CF_{n-1}$ between two steps in the inductive definition of the scenario.*

The concept of n-contextuality is well-defined once for vertices the scenario is acyclic, and therefore noncontextual ($CF_0 = 0$), and also due to its monotonicity concerning the free operations in a resource theory of contextuality [Amaral 2019]. Specifically, the sum of the contextual fractions of disjoint sub-models is less than the contextual fraction of the entire model: for any empirical model represented by two disjoint parts $\mathcal{M} = \mathcal{M}_1 \sqcup \mathcal{M}_2$,

$$CF(\mathcal{M}) \geqslant CF(\mathcal{M}_1) + CF(\mathcal{M}_2) \tag{3.11}$$

This generalizes to any finite disjoint union of sub-models. By construction of the contextual fraction, the following holds:

$$1 \geqslant CF(\mathcal{M}) \geqslant \sum_{k=1}^{n} CF(\mathcal{M}_k) \tag{3.12}$$

for any empirical model $\mathcal{M} = \bigsqcup_{k=1}^{n} \mathcal{M}_k$. Therefore, adding edges can induce contextuality, and we can define the sequence:

$$0 = CF_0 \leqslant CF_1 \leqslant ... \leqslant CF_n = CF. \tag{3.13}$$

n-contextuality offers a means to explore the influence of topological properties of the scenario on the empirical model's contextual behavior, isolating topological properties by the dimension of the n-skeleton. A question in the literature [Terra Cunha 2019] regarding the relationship between scenario topology and acyclicity is whether all contextual behavior arises due to topological flaws captured by the first homology group, or equivalently, whether there exists n-contextuality for $n > 1$, as the 1-skeleton captures any property of the scenario's first homology group. While an example previously illustrated the separation of standard topology and cyclicity, the question remains: for $n > 1$, can contextuality arise solely due to n-dimensional objects? The next example shows that it can.

Example 20. *Consider a scenario defined by the boundary of a tetrahedron, exhibiting non-trivial cyclicity with only four vertices, six edges, and four faces. Table 6 presents the probabilities of obtaining specific outcomes within maximal contexts, where outcomes*

Table 6 – Table of outcome probabilities of each context for the tetrahedron model.

	000	001	010	011	100	101	110	111
abc	$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$	0
abd	$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$	0
acd	0	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$
bcd	0	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$

$\{0, 1\}$ *are associated with each measurement, and the base has contexts represented by the tetrahedron's boundary with measurements* $\{a, b, c, d\}$*. This construction originated from Svetlichny's box [Barrett et al. 2005] as detailed in Ref. [Barbosa 2014], and is described in Table 8.* [5] *As one can see, the probabilities in any subcontext represented by the edges are maximally random, implying noncontextuality when restricted to the* 1*-skeleton. There is concordance between the faces; thus, it is a non-disturbing empirical model. It is also a contextual empirical model once it has a non-null contextual fraction [Abramsky, Barbosa and Mansfield 2017],* $CF = \frac{3}{4}$*. In conclusion, the contextual behavior of this model does not originate from the* 1*-skeleton but from the* 2*-skeleton; thus, this is an example of* 2*-contextuality and a counter-example to the assertion that contextuality follows, even indirectly, from the first holonomy group of the scenario with the standard topology of its geometric representation.*

Another question one can ask is whether quantum theory has any examples of n-contextuality for $n > 1$. The next example shows quantum theory has a well-known example for all $n \geqslant 1$.

Example 21. *The* n*-dimensional GHZ model is a scenario with* $n+1$ *parts, each with two measurements and fibers with two elements. For* $n = 1$ *we have the Bell scenario, a graph with a square form and an example of* 1*-contextuality. For* $n = 2$*, the scenario has an octahedron shape. The model is defined when choosing the quantum state*

$$|GHZ\rangle = \frac{|0\rangle^{\otimes(n+1)} + |1\rangle^{\otimes(n+1)}}{\sqrt{2}} \tag{3.14}$$

which is maximally entangled. By theorem 1 and the structure of measurements in quantum theory, it is always possible to find a set of measurements that exhibit contextuality for this model. The marginalized measures are maximally uniform (a reflection of the state's

5 This example was independently constructed by the author of this thesis, but was previously reported in Ref. [Dzhafarov, Kujala and Cervantes 2020].

Table 7 – Table of outcome probabilities of each context for the GHZ model.

	000	001	010	011	100	101	110	111
ABC	0	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$
ABc	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
AbC	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
Abc	$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$	0
aBC	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
aBc	$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$	0
abC	$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$	0
abc	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

non-biseparability), and therefore the contextuality does not appear in simplices of smaller dimension than n.

The case of the GHZ model in an octahedral shape is well-known, and its standard version presents the joint probability table given in Table 7. In this model, each of the three parties can perform Pauli X or Y measurements on the state

$$|GHZ\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}, \tag{3.15}$$

which can be verified as a strongly contextual model ($NCF = 0$). In Table 7, we have A, B, C as X, and a, b, c as Y.

Its importance as a model has direct applications even in the foundations [Lawrence, Markiewicz and Żukowski 2023], placing n-contextuality as an important phenomenon to validate interpretations of quantum theory. Another example, also exhibiting strong contextuality but being super-quantum, which inspired the tetrahedron model in example 20, is Svetlichny's box from Table 8. which also has maximally uniform marginals, and therefore only exhibits contextuality in dimension $n = 2$.

3.3.2 An Example of Topological Influence

Armed with n-contextuality, we can explore whether the topology of the scenario interferes, even indirectly, with the degree of contextuality. Here, we will construct an

Table 8 – Table of outcome probabilities of each context for the Svetlichny's box.

	000	001	010	011	100	101	110	111
ABC	$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$	0
ABc	$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$	0
AbC	$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$	0
Abc	0	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$
aBC	$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$	0
aBc	0	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$
abC	0	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$
abc	0	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$

example that allows us to observe this influence. Let's start with an empirical model in a barycentric triangle, which ends up being a sub-model of the tetrahedron presented in example 20. In this model, the face abc is ignored, resulting in a noncontextual fraction of $NCF = \frac{1}{2}$, while the complete tetrahedron has $NCF = \frac{1}{4}$. We have two possible explanations:

- either the face adds $\frac{1}{4}$ of contextuality to the model,
- or it occurs due to a change in topology.

To test these possibilities, we can study the local sections on the face, since the marginals are maximally uniform. Observing that local sections in the triangle satisfy the system of linear equations:

$$\sum_x p(x, y, z|a, b, c) = p(y, z|b, c) \tag{3.16}$$

$$\sum_y p(x, y, z|a, b, c) = p(x, z|a, c) \tag{3.17}$$

$$\sum_z p(x, y, z|a, b, c) = p(x, y|a, b) \tag{3.18}$$

Table 9 – Table of outcome probabilities of each context for the tetrahedron model with generic context *abc*.

	000	001	010	011	100	101	110	111
abc	$\frac{1}{4}-\eta$	η	η	$\frac{1}{4}-\eta$	η	$\frac{1}{4}-\eta$	$\frac{1}{4}-\eta$	η
abd	$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$	0
acd	0	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$
bcd	0	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$

$$\sum_{x,y,z} p(x,y,z|a,b,c) = 1, \tag{3.19}$$

which are valid due to no-disturbance and the probability condition, we observe that the triangle *abc* has a generic section on the face represented by the vector:

$$\left(\frac{1}{4}-\eta,\eta,\eta,\frac{1}{4}-\eta,\eta,\frac{1}{4}-\eta,\frac{1}{4}-\eta,\eta\right), \tag{3.20}$$

with $\eta \in [0, \frac{1}{4}]$, and the maximally uniform case occurring at $\eta = \frac{1}{8}$. However, when adding the face to the model, one obtains the generic case as shown in Table 9. Contextuality remains the same, with $NCF = \frac{1}{4}$, corroborating the option of topological dependency. Therefore, the simple addition of a face, regardless of which one, generates more contextuality, in line with the topological influence option. A formalization of this dependency must be constructed so that something more can be said about this result.

4 The Learned Path

Topology expresses itself more subtly in contextuality than contexts allow us to see. The Bundle Approach and its diagrams fail as soon as determinism disappears from the model. Even its refinement through holonomy presents serious flaws in capturing the contextual behavior of simple models. At this point, a more comprehensive mathematical approach seems to be suggested.

The Sheaf Approach appears to be such an approach. It naturally allows the use of cohomology to attempt to capture contextuality, but even strong contextual behaviors escape. One characterization causes the mathematical structure itself to break and become somewhat useless. In addition, the approach has limitations in the face of the different parts in which contextual behavior presents itself in physical processes. What both approaches have in common is the use of measurements as fundamental, forcing mathematical tools to use groups. Here we identify that the previous construction, with a description of contexts and measurements, should take a back seat. The influence of topology appears imperfectly, a sign that if it does indeed play a role, then we need to base models on other objects.

A description of processes seems to be the most natural path, which operational theories and their representation in generalized probabilistic theories allow us. The Generalized Contextuality Approach generalizes and puts a philosophical basis on contextuality. In it, contexts and measurements become questions that can be asked, propositions. It is in this approach that we will explore contextuality and its geometric and topological roots.

With this lesson, we can tread the path that will give us the main fruits of our journey.

Part II

Differential Geometry of Contextuality

Contextuality, the ultimate form of non-classicality, has many diverse mathematical approaches. Each approach was built for a specific purpose or strategy to exploit its characteristics, and some were developed long before being identified as contextuality. They all start from the encoding of physical systems into some mathematical structure that cannot be represented by another structure called classical. There are already examples of topological representations [Abramsky and Brandenburger 2011, Abramsky, Mansfield and Barbosa 2012, Abramsky et al. 2015, Okay et al. 2017, Okay and Raussendorf 2020, Montanhano 2021], and algebraic representations [Birkhoff and Neumann 1936, Bell 1964, Kochen and Specker 1967, Gleason 1957]. Some have evolved into a geometrical representation due to the relationship between inequalities and convex sets [Cabello, Severini and Winter 2014, Amaral and Cunha 2017], and others seek a foundation in measure theory [Dzhafarov, Kujala and Cervantes 2015]. Other notions [SORKIN 1994, Spekkens 2008, Schmid et al. 2021] are known to be related to standard contextuality, and they are more or less explored in the literature.

Non-classicality has an incredible number of applications, and more are being presented each day. Contextuality is the fuel for this revolution. It is known that contextuality is the origin of quantum behavior [Döring and Frembs 2020], and it is the generalization of the famous notion of nonlocality [Abramsky and Brandenburger 2011]. It is necessary for any computational advantage over classical computers [Shahandeh 2021], and it is explicitly the "magic" required for some types of quantum computers [Howard et al. 2014]. But this phenomenon is not just a resource for technological applications. Understanding contextuality in a more general formulation is essential to understand why and how we live in a quantum reality, and whether we need to search for more general theories than quantum theory itself. This fundamental exploration ultimately aims to establish the framework where future theories and technologies will be built.

In this Part, we will explore the geometric or topological origins of contextuality. We will use its more generic version, generalized contextuality [Spekkens 2005], restricting as necessary to treat tools from other approaches. The general strategy will be to rethink the operational equivalences of the ontic representation as loops with discrete parts in the tangent space of a suitable manifold, usually piecewise linear given by the elements of a process (for example, an extension of the set of effects). The noncontextual condition becomes the preservation of probabilistic valuation maps for these loops, thus, without the presence of non-trivial phases.

A relevant ontic representation will be the one present in generalized probability theories [Janotta and Hinrichsen 2014]. Generally, generalized probability theories are constructed with a finite set of extremal effects, and demonstrations of contextual behaviors use them. But there are theories where the set of such effects is infinite and, like quantum theory, even continuous. This implies the possibility of different types of operational equivalences. The framework we build explores all operational equivalences on equal footing. The condition of contextual behavior, described through discrete differential geometry, ends up generalizing to the domain of differential geometry for the continuous case, again as non-trivial phases of probabilistic valuation maps.

With this framework, we present two ways to interpret contextuality, depending on the choice of how to encode the physical system. The violation of phase triviality in valuation results from holonomy or monodromy, respectively, linked to the intrinsic-realistic and participatory-realistic views of the theory. Holonomy follows from a geometric cause, by imposing that the ontological set be classically complete, which imposes on valuations the correction of contextual behavior, expressed by a curvature term. Monodromy follows from a topological cause, by imposing that the valuation must be classical in the sense of measure theory, expressed by topological defects in the ontological set. They are equivalent and have dual interpretations of the fact that we have lost classicality.

We will use these two equivalent interpretations to explore the relationship between contextual behavior and different notions of non-classicality. Using the geometric version, called "Schrödinger", we start by imposing outcome determinism, making the contextual fraction, a quantifier of contextuality [Abramsky, Barbosa and Mansfield 2017], to be defined in its usual form. We show that the resource it is quantifying is nothing more than the curvature term that corrects valuation. The same occurs with the interference term in the sense of quantum measure theory [SORKIN 1994,Sorkin 1995], where the interference part that cannot be explained classically also arises from the curvature term. The need for signed measures for valuation maps [Spekkens 2008, Abramsky and Brandenburger 2011] follows analogously as we force the corrected valuation to be treated in the same way as a classical measure, even though we have already imposed the non-existence of topological defects. Using the topological version, called "Heisenberg", we identify it as the cause of the impossibility of incorporation into a classical mathematical structure [Schmid et al. 2021, Schmid et al. 2020]. The topological view also offers a generalization of the famous Vorob'ev theorem [Vorob'ev 1959], which characterizes the inevitability of noncontextuality, while the geometric view establishes a relationship between transition maps and disturbance [Montanhano 2021, Amaral and Duarte 2019]. They also allow for a clearer understanding of how quantum interpretations deal with contextuality, in a way mapping the madness of interpretations [Cabello 2017].

This Part is divided as follows. A brief presentation of differential geometry (5.1.1) and discrete differential geometry (5.1.2) is given in section 5.1. In section 5.2, contextuality is identified as a phase of the valuation functions. Beginning with the formal identification (5.2.1), we construct its two interpretations. In the geometric view (5.2.2), we have the notion of contextual curvature for a generalized probability theory. In the topological view (5.2.3), we have the relationship between contextuality and non-trivial topology. As equivalent interpretations, they can be translated into each other (5.2.4). The tour of already known non-classical phenomena and their relations with contextuality begins in section 5.3. We start with the identification of contextual fraction (5.3.1), the interference in quantum measure theory (5.3.2), and with the special example of quantum theory and its dependence on Planck's constant. We then explore the necessity of signed measures and the impossibility of embedding the process in a classical mathematical object (5.3.3), and generalize the Vorob'ev theorem with the topological view (5.3.4). We propose a generalized framework for generalized probabilistic theories, incorporating non-trivial transition functions to introduce zeroth-order processes that codify disturbance. This adds a new term to the valuation decomposition, referred to as the disturbance fraction (5.3.5). We finalize this section with an exposition of the main interpretations of quantum theory and its relation to the interpretations of the origin of contextuality (5.3.6). Chapter 6 follows, where we discuss the results and potential future avenues.

5 A Differential Approach to Contextuality

In the previous Part, we introduced the concepts of GPT, ontic representation, and how noncontextuality is expressed in such a formalism. In this chapter, we will present tools to glimpse what contextuality is actually doing in a physical model.

5.1 Concepts of Differential Geometry and Topology

5.1.1 Differential Geometry

Let's quickly introduce the main concepts of differential geometry that we will use in the upcoming sections. The connection with what we have already presented will occur in the second part of this section, where we discretize what we will present here. With this, we will be able to present a characterization of contextuality in differential geometry in Section 5.2 that can be immediately generalized to the continuous case. This will have important implications for the applications presented in Section 5.3.

We begin with smooth manifolds, which are locally similar to a vector space so that we can utilize calculus. We define a smooth manifold $\mathcal{M}$ as a topological space with topology τ that is covered by domains of homeomorphisms called charts $\varphi : U \in \tau \to \mathbb{R}^n$, such that for every pair of charts φ_a, φ_b, the transition map $\varphi_a \circ \varphi_b^{-1}$ is smooth. A set of charts that cover $\mathcal{M}$ is called an atlas, and it fully describes $\mathcal{M}$.

With this structure, we can work with the tangent vector space at a point $p \in \mathcal{M}$, denoted by $T_p\mathcal{M}$, which is given by a chart containing it that serves as a neighborhood. As in $\mathbb{R}^n$, we can define directional derivative $\partial_k = \dfrac{\partial}{\partial x_k}$. We are interested in studying how infinitesimal objects behave in this local environment and what properties they exhibit when seeking to extend them globally throughout $\mathcal{M}$. Since each point in $\mathcal{M}$ has its tangent vector space, we have a tangent bundle $T\mathcal{M}$.

The dual vector space of a tangent space is called the cotangent space, denoted by $T_p^*\mathcal{M}$, which is also defined at each point and generates a bundle $T^*\mathcal{M}$ called the cotangent bundle. It is important to note that this duality depends on which value we want to obtain from the application of an object from the cotangent space to the tangent space. Here we will deal with values in $\mathbb{R}$, meaning that if $\langle\pi| \in T_p^*\mathcal{M}$ and $|\omega\rangle \in T_p\mathcal{M}$, then $\langle\pi|\omega\rangle \in \mathbb{R}$. The objects of $T_p^*\mathcal{M}$ are the covectors, and we can also define infinitesimal elements dx_k dual to differentials ∂_k, known as differential 1-forms.

The next step will be to think of ∂ and d as operators by themselves. Continuing with the informality level of this presentation, ∂ acts as a boundary operator on a region, effectively reducing its dimension by one unit. Notice that each piece has its orientation, which is captured by the differential. We can define $\mathcal{C}_n(\mathcal{M})$ as the entire finite set of n-dimensional pieces of $\mathcal{M}$. The boundary operator is nothing more than a map $\partial : \mathcal{C}_n \to \mathcal{C}_{n-1}$, and since the boundary of a boundary is always an empty set, we have $\partial\partial = 0$, and we obtain the chain complex

$$0 \xleftarrow[\partial_0]{} \mathcal{C}_0(\mathcal{M}) \xleftarrow[\partial_1]{} \mathcal{C}_1(\mathcal{M}) \xleftarrow[\partial_2]{} \mathcal{C}_2(\mathcal{M}) \xleftarrow[\partial_3]{} \dots \tag{5.1}$$

We can explore the topology of $\mathcal{M}$ through its pieces and the natural group structure of these pieces via homology groups. First, we define the kernel of ∂_n denoted by $Z_n(\mathcal{M})$, whose elements are called n-cycles. Meanwhile, the image of $\partial_{(n+1)}$ denoted by $B_n(\mathcal{M})$, consists of n-boundaries. The group given by $H_n(\mathcal{M}) = Z_n(\mathcal{M})/B_n(\mathcal{M})$ is the n-homology group, which intuitively captures topological failures of dimension $n-1$ in $\mathcal{M}$. For instance, $Z_1(\mathcal{M})$ will consist of all one-dimensional objects in $\mathcal{M}$ that have no boundary, while $B_1(\mathcal{M})$ will consist of all boundaries of two-dimensional objects. Since the latter necessarily have no boundary themselves, $H_n(\mathcal{M})$ is capturing one-dimensional objects without boundaries that are not boundaries of any two-dimensional piece, identifying a failure in having such a piece, a "missing" point.

With d, referred to as the coboundary operator, we can utilize its duality with ∂ to show that $dd = 0$, obtaining the cochain complex

$$0 \longrightarrow \mathcal{C}^0(\mathcal{M}) \xrightarrow{d_0} \mathcal{C}^1(\mathcal{M}) \xrightarrow{d_1} \mathcal{C}^2(\mathcal{M}) \xrightarrow{d_2} \dots \tag{5.2}$$

Where $\mathcal{C}^n$ are the duals of $\mathcal{C}_n$, which are infinitesimally generated by n-differential forms. The n-forms possess a beautiful mathematical structure which unfortunately will not be presented here. What will be important for us is that we can once again explore the topology of $\mathcal{M}$ with the n-forms, but this time not directly with pieces of $\mathcal{M}$ but rather with the functions that act upon these pieces. The kernel of d_n is denoted by $Z^n(\mathcal{M})$, with elements called n-cocycles or closed n-forms, while the image of $d_{(n-1)}$ is denoted by $B^n(\mathcal{M})$, with elements called n-coboundaries or exact n-forms. The resulting algebraic structure $H^n(\mathcal{M}) = Z^n(\mathcal{M})/B^n(\mathcal{M})$ is called the de Rham n-cohomology.

To conclude this brief presentation on elements of differential geometry, there are three theorems that will appear during this differential approach to contextuality and its immediate applications. The first is the generalized version of Stokes' theorem.

Theorem 3. *Let ω be a smooth $(n-1)$-form with compact support on an oriented, n-dimensional manifold-with-boundary S, where ∂S is given the induced orientation. Then*

$$\int_{\partial S} \omega = \langle \omega | \partial S \rangle = \langle d\omega | S \rangle = \int_S d\omega. \tag{5.3}$$

This theorem is the direct verification that the region S can be seen as analogous to a region in $\mathbb{R}^n$, with the n-form $d\omega$ being in some way extended throughout the entire region.

Another important theorem is the Ambrose-Singer theorem, which relates the holonomy of a principal bundle to the curvature in the region where the holonomy is found. It identifies that the holonomy is the expression of curvature, and that curvature generates holonomy.

Theorem 4. *In a smooth manifold M with a principal bundle P over M and a connection 1-form ω, the holonomy algebra at a point $p \in P$ is generated by the curvature form F derived from ω and evaluated along loops based at p. Specifically, the holonomy algebra is determined by the curvature form F and its covariant derivatives evaluated on all possible pairs of horizontal vector fields at p.*

The relationship between holonomy and curvature allows us to identify geometry through the phases of transport in loops. The curvature associated with the 1-form ω is nonzero at a point in M if and only if there exists a nontrivial closed curve passing through that point whose holonomy phase along it is nontrivial.

Lastly, we have the Poincaré Lemma, which states that in a contractible manifold isomorphic to a region of $\mathbb{R}^n$, all closed forms are exact.

Theorem 5. *Let M be a smooth, orientable manifold of dimension n that is isomorphic to $\mathbb{R}^n$. Then for every closed differential form ω of degree k on M (i.e., $d\omega = 0$), there exists a differential form η of degree $k - 1$ on M such that $d\eta = \omega$.*

5.1.2 Discrete Differential Geometry

As is customary in foundational studies, we initially confine ourselves to dealing with finite sets to elucidate non-classical behavior. This means that contextuality is initially addressed in finite structures, as are the usual versions of generalized contextuality. Furthermore, many GPTs, such as the classical one, feature edges that are obviously not smooth. Therefore, we need to take a step back and seek a way to incorporate contextuality into a formulation that is compatible with these conditions. This is the role of discrete differential geometry, which we will now briefly review.

An operational equivalence, as defined through a linear condition as above, encodes a discrete loop in its respective space. Contextuality is expressed in how the functions ξ, Γ, and μ deal with such loops. The noncontextual conditions can be thought of as discrete parallel transport of the probability functions that present no phase in a closed loop. Contextuality, as the violation of such conditions, is the discrete phase in each set. To formalize it, we will employ discrete differential geometry, or DDG [Crane et al. 2013, Grady and Polimeni 2010].

The formalism of DDG arises from the need for discrete methods to describe approximately smooth manifolds, as in computer graphics and geometry processing. While it's always possible to triangulate a smooth manifold, DDG, without imposing the usual differential structure, enables the study of more general topological manifolds known as piecewise linear manifolds.

We begin with a piecewise linear manifold $\mathcal{M} = \bigcup_n \mathcal{C}_n$, composed of sets of n-simplices $\mathcal{C}_n$. An n-simplex is treated as an n-dimensional "unit of space", and the topology is derived from the topology of the simplicial complex. To be valid as an approximation, each simplex is regarded as the tangent space of a point in a hypothetical smooth manifold. For calculus operations on this simplicial complex, we can introduce the formalism of discrete differential forms, which can be loosely understood as a method to quantify the "size" of the simplices. Discrete differential forms are defined as linear duals of the simplices, denoted by $\mathcal{C}^n$ representing the set of n-forms. If $\omega \in \mathcal{C}^n$, then we have

$$\begin{aligned} \omega : \mathcal{C}_n &\to R \\ |S\rangle &\mapsto \langle \omega | S \rangle = \int_S \omega. \end{aligned} \tag{5.4}$$

The first operator in DDG is the boundary $\partial : \mathcal{C}_n \to \mathcal{C}_{n-1}$, defined as usual by the orientation defined in the simplicial complex

$$\partial\{a_1 a_2 ... a_n\} = \{a_2 ... a_n\} - \{a_1 a_3 ... a_n\} + ... \pm \{a_1 a_2 ... a_{n-1}\}. \tag{5.5}$$

As an example, a tetrahedron $\{abcd\}$ has boundary

$$\partial\{abcd\} = \{bcd\} - \{acd\} + \{abd\} - \{abc\} \tag{5.6}$$

in an alternate way. The second one is the coboundary $d : \mathcal{C}^n \to \mathcal{C}^{n+1}$. It is defined as the unique linear map that satisfies the generalized Stokes theorem for DDG

$$\int_{\partial S} \omega = \langle \omega | \partial S \rangle = \langle d\omega | S \rangle = \int_S d\omega, \tag{5.7}$$

where the bracket notation will be used to denote the action of a n-form on a m-dimensional region, $n \geqslant m$, both for the discrete and continuum cases. See that the integral gives a $(n - m)$-form as expected, and 0-forms are identified as scalar.

The homology of the manifold $\mathcal{M}$ follows the simplicial homology [Hatcher 2000], and explores the topological structure of $\mathcal{M}$ through its simplicial complex structure and the boundary operator. Once the boundary of a boundary is empty, we have $\partial\partial = 0$, and we also obtain the chain complex

$$0 \xleftarrow[\partial_0]{} \mathcal{C}_0(\mathcal{M}) \xleftarrow[\partial_1]{} \mathcal{C}_1(\mathcal{M}) \xleftarrow[\partial_2]{} \mathcal{C}_2(\mathcal{M}) \xleftarrow[\partial_3]{} \dots \tag{5.8}$$

The kernel of ∂_n, denoted by $Z_n(\mathcal{M})$, will have as its elements the n-cycles, while the image of $\partial_{(n+1)}$, denoted by $B_n(\mathcal{M})$, will have as its elements the n-boundaries. The algebraic structure $H_n(\mathcal{M}) = Z_n(\mathcal{M})/B_n(\mathcal{M})$ is the n-homology. It explores the shape of $\mathcal{M}$ by directly studying the "quanta of space", or equivalently, the tangent space of $\mathcal{M}$ that is locally isomorphic to itself. Non-trivial n-holonomy implies an n-dimensional topological failure in the simplicial complex.

On the other hand, cohomology deals with the dual of the simplices, the discrete differential forms, and the coboundary operator. By the property $dd = 0$, which follows from the definition of the coboundary using the generalized Stokes theorem, we have the cochain complex

$$0 \longrightarrow \mathcal{C}^0(\mathcal{M}) \xrightarrow{d_0} \mathcal{C}^1(\mathcal{M}) \xrightarrow{d_1} \mathcal{C}^2(\mathcal{M}) \xrightarrow{d_2} \dots \tag{5.9}$$

The kernel of d_n is denoted by $Z^n(\mathcal{M})$, and it comprises the n-cocycles, also known as closed n-forms. The image of $d_{(n-1)}$, denoted by $B^n(\mathcal{M})$, comprises the n-coboundaries, also called exact n-forms. The algebraic structure $H^n(\mathcal{M}) = Z^n(\mathcal{M})/B^n(\mathcal{M})$ is the de Rham n-cohomology. It involves studying what we integrate on $\mathcal{M}$ and how it relates to the shape of $\mathcal{M}$.

5.2 Differential Geometry of Contextuality

Cohomology studies the failure of solutions to equations of the form $d\omega = \sigma$, which reside in the cotangent space. Generally, the equation $d\omega = \sigma$ seeks a global cause for the local relationship it describes. Our aim here is to identify that contextual behavior is the inability to obtain such a global solution.

To this end, we demonstrate that it is possible to represent it in its usual form in discrete differential geometry, with the possibility of using differential geometry as a generalization. Such a description allows us to have a more natural view of the phenomenon of contextuality in ontic representations, both because it is mathematically akin to the well-known mathematics in physical applications and because it allows for the direct interpretation of mathematical objects regarding what is happening with the physical model.

5.2.1 Operational Equivalences and Contextuality

Effects, states, and transformations live in a real vector space by construction, which is isomorphic to its own tangent space. They form convex subsets through the imposition of convex combinations. This naturally gives rise to piecewise linear manifolds embedded in real vector spaces.

Each operational object is represented as an element of these manifolds. In the tangent space, they are represented not by points, but by vectors. To differentiate the objects from their vector representation, we will use the bracket notation to the former. Vectors are oriented pieces, quantities, of the manifold, and therefore reside in $\mathcal{C}^1$. The equations 2.20, 2.21 and 2.22 are just saying that an operational equivalence is a closed discrete loop γ in the tangent space,

$$\sum_s a_s^{(\alpha)} |P_s\rangle = \gamma^{(\alpha)}, \tag{5.10}$$

$$\sum_r b_r^{(\beta)} |E_r\rangle = \gamma^{(\beta)}, \tag{5.11}$$

$$\sum_t c_t^{(\tau)} |T_t\rangle = \gamma^{(\tau)}. \tag{5.12}$$

Operational equivalences and closed loops generated by elements of each subset encode the same information. What was described as two processes generating the same final process here is depicted as two distinct paths departing and arriving at the same point. The rearrangement under linear conditions turns the paths into a loop.

The noncontextual conditions presented in equations 2.23, 2.24, and 2.25 are defined by probabilistic functions ξ, Γ, and μ, indexed by ontic variables $\lambda \in \Lambda$. They live in the cotangent space as differential forms, acting on vectors and giving us a probability. Rewriting equations 2.23, 2.24, and 2.25, we obtain

$$\phi^{(\alpha)} = \sum_s a_s^{(\alpha)} \mu_\lambda(P_s) = \langle \mu_\lambda | \left(\sum_s a_s^{(\alpha)} |P_s\rangle \right) = 0, \tag{5.13}$$

$$\phi^{(\beta)} = \sum_r b_r^{(\beta)} \xi_{\lambda'}(E_r) = \langle \xi_{\lambda'} | \left(\sum_r b_r^{(\beta)} |E_r\rangle \right) = 0, \tag{5.14}$$

$$\phi^{(\tau)} = \sum_t c_t^{(\tau)} \Gamma_{\lambda'\lambda}(T_t) = \langle \Gamma_{\lambda'\lambda} | \left(\sum_t c_t^{(\tau)} |T_t\rangle \right) = 0, \tag{5.15}$$

for all λ and λ'. The noncontextual conditions become just the valuation of the 1-forms given by the functions ξ, Γ and μ in each space given by the ontic representation to preserve the flat behavior of the vector spaces involved. In other words, we can understand such functions as potential vector fields in our discrete space, and ask for the preservation of the convex combination in the sense that the phase $\phi = 0$ when evaluated on a loop γ.

A comment on linearity in the forms. The map that defines the vectors is $E \mapsto |E\rangle$, but generally, a vector $\sum_r c_r |E_r\rangle$ is different from $\left| \sum_r c_r E_r \right\rangle$, since the latter can lie outside $\mathcal{E}$. It can also include negative elements, so even the linear operations in equations 2.20, 2.21, and 2.22 generally lie outside of $\mathcal{E}$. This is necessary to deal with noncontextuality: the objective is to classically complete the theory, thus embedding it into a classical one, where $\sum_r c_r |E_r\rangle = \left|\sum_r c_r E_r\right\rangle$.

An important part of this construction is the immediate application in non-discrete loops. For any loop γ, we can integrate the differential form representing the valuation function of the ontic representation, obtaining a phase ϕ. As we will see, in quantum theory this phase is already a well-studied non-classical phenomenon.

In summary, the present approach captures the sets of operational objects — preparations, transformations, and effects — and represents them as piecewise linear manifolds. A straightforward way to achieve this is by utilizing the real linear space representation of the generalized approach and restricting it to convex sets. The tangent space of each manifold corresponds to the respective real linear space itself, but now unrestricted to the convex sets. In this space, operational objects are represented as vectors, and an operational equivalence corresponds to a loop. Given a classical ontological representation, we have a probabilistic valuation function, which resides in the cotangent space as a differential form. It is the failure of this form to respect the loop, by presenting a non-trivial phase, that characterizes contextuality, consistent with the generalized approach.

5.2.2 Schrödinger's View: Geometric Contextual Behavior

Let's keep our model in a flat space, such as understanding it as a submodel of a classical theory. This assumption is the trivial extension of the convex set to all vector spaces without any topological failure. Without such failures, all loops are just boundaries, $\gamma = \partial S$, and noncontextuality conditions can be rewritten as

$$\langle \mu_\lambda | \partial S_\alpha \rangle = 0, \tag{5.16}$$

$$\langle \xi_{\lambda'} | \partial S_\beta \rangle = 0, \tag{5.17}$$

$$\langle \Gamma_{\lambda'\lambda} | \partial S_\tau \rangle = 0, \tag{5.18}$$

in the language of differential forms. Here we can use Stokes theorem to define the coboundary operator and get

$$\langle d\mu_\lambda | S_\alpha \rangle = 0, \tag{5.19}$$

$$\langle d\xi_{\lambda'} | S_\beta \rangle = 0, \tag{5.20}$$

$$\langle d\Gamma_{\lambda'\lambda} | S_\tau \rangle = 0. \tag{5.21}$$

Again, this is possible because we are in $\mathbb{R}^n$, with states, effects, and transformations represented by vectors in its tangent space, making sense of S_α, S_β, and S_τ as regions in $\mathbb{R}^n$.

In these conditions, every closed differential form is exact: if $\langle d\xi_{\lambda'}|S\rangle = 0$ for all regions S, then $d\xi'_\lambda = 0$, which means it is closed and thus exact, $\xi_{\lambda'} = dc_{\lambda'}$ with $c_{\lambda'}$ a function. The failure of noncontextual conditions implies that $\xi_{\lambda'} = dc_{\lambda'} + \omega_{\lambda'}$, where $d\omega_{\lambda'} \neq 0$, and by

$$\begin{aligned} \langle d\xi_{\lambda'}|S_\beta\rangle &= \langle ddc_{\lambda'}|S_\beta\rangle + \langle d\omega_{\lambda'}|S_\beta\rangle \\ &= \langle d\omega_{\lambda'}|S_\beta\rangle \end{aligned} \tag{5.22}$$

we see that $\omega_{\lambda'}$ is the connection that generates contextual behavior, and $F_{\lambda'} = d\omega_{\lambda'}$ as the curvature 2-form. The same holds for states and transformations.

Theorem 6. *Noncontextuality for measurements (transformations; states) of an ontic representation is equivalent to a null contextual curvature* $0 = F_{\lambda'} = d\xi_{\lambda'}$ *(respectively* $0 = F_{\lambda'\lambda} = d\Gamma_{\lambda'\lambda}$*;* $0 = F_\lambda = d\mu_\lambda$*) for all hidden variables that index it.*

Geometrically, we can view each valuation and set of objects as defining a fiber bundle, with $\mathbb{R}$ regarded as a commutative group. As an $\mathbb{R}$-bundle, it is isomorphic to the trivial bundle $\mathbb{R}^n \times \mathbb{R}$, and with the restriction $\mathcal{E} \times [0,1]$ well defined (and analogously for $\mathcal{T}$ and $\mathcal{P}$). The curvature F is in the lifting by the valuation function of the set of objects in the fiber bundle.

In this view there is no topological failure; it is a geometrical question. It is analogous to electromagnetism, with an electromagnetic tensor F that can be written through holonomic loops [Rosenstock and Weatherall 2016, Weatherall 2016]. The geometrical view identifies contextuality with non-trivial holonomy of the contextual connection ω.

Example 22. *The Sheaf approach [Abramsky and Brandenburger 2011], as well as the Kochen-Specker contextuality [Kochen and Specker 1967], impose a classical structure on local events by forcing them to be elements of their Boolean completion. In the Sheaf approach, this is done by imposing the sheaf properties on the presheaf of events that encode the outcomes. Since events are treated deterministically, they are identified as vertices in a classical GPT. Once completed in a Boolean structure given by the classical GPT, any contextual behavior is expressed in the valuations. For example, it is the presheaf that defines the distributions that need to be studied to verify the model's contextuality. Therefore, in its construction, they use Schrödinger's view.*

5.2.3 Heisenberg's View: Topological Contextual Behavior

Let's reject the use of curvature to explain contextuality. This means that we want a valuation to satisfy the properties of a classical probability distribution, satisfying Kolmogorov's axioms. Thus, $F = 0$, and contextuality is not a correction in the valuation but lies in a different part of the model.

Theorem 7. *If $F = 0$, then contextuality of an ontic representation is equivalent to monodromy.*

Proof. Contextuality implies a correction in the valuation, once that by construction the form dc satisfies the noncontextuality conditions. Thus the valuation must be $dc + \omega$ with $\phi = \langle \omega | \gamma \rangle$. As $F = d\omega = 0$, the 1-form must be closed but not exact to show any non-trivial phase ϕ, which only happens when the loop γ is not the boundary ∂S of a region S when seen by the valuation. In other words, the valuation cannot be defined in S, implying that the ontic representation in $\mathbb{R}^n$ does not preserve the topology induced by the set of objects $\mathcal{E}$, $\mathcal{P}$ or $\mathcal{T}$. The form ω capts such topological failures through monodromy ϕ, once we cannot access these topological failures. □

Without curvature, we still need to define a correction $\xi_{\lambda'} = dc_{\lambda'} + \omega_{\lambda'}$, with $F = d\omega_{\lambda'} = 0$. But now closed forms cannot be exact, which means $\omega_{\lambda'}$ is a representation of a topological failure. Specifically, it represents a non-trivial element of the first cohomological group $[\omega_{\lambda'}] \in H^1$ defined on the set of objects. In the topological view, even with the fiber $\mathbb{R}$ and with the restriction $\mathcal{E}$ well defined (and analogously for $\mathcal{T}$ and $\mathcal{P}$), the fiber bundle is not trivial. The basis is not topologically trivial, and so is the fiber bundle. And this is what the valuation detects.

The topological view allows us to generalize results from the standard contextuality framework to the generalized one [Montanhano 2021].

Theorem 8. *The $\mathbb{R}$-fiber bundle described by a model on an ontic representation is trivial in the topological view and so noncontextual if and only if any local section admits an extension to a global section.*

This result follows from the equivalence of the extendability of local sections and triviality for the $\mathbb{R}$-fiber bundle. The ontic representation is noncontextual for a given valuation if and only if the fiber bundle presents no phase, which is equivalent to present extensions to global sections for any ontic variable λ (or the pair λ and λ' for transformations), implying the fiber bundle being trivial.

Por ser fiel a contextualidade generalizada, na visão topológica temos a fiel identificação da contextualidade como um fenômeno topológico. A abordagem aqui descrita refina o uso de ferramentas cohomologicas para identificar contextualidade na abordagem de sheaf [Abramsky, Mansfield and Barbosa 2012, Abramsky et al. 2015], que sofre com modelos contextuais não sendo identificados como tais. Consequentemente, ele também generaliza a identificação de uma topologia não-trivial com o comportamento contextual da abordagem homotópica [Okay et al. 2017, Okay and Raussendorf 2020].

Being faithful to generalized contextuality, the topological view of this approach provides an accurate identification of contextuality as a topological phenomenon. It refines the use of cohomological tools to identify contextuality within the sheaf approach [Abramsky, Mansfield and Barbosa 2012, Abramsky et al. 2015], which struggles with contextual models not being properly recognized as such. Consequently, it also generalizes the identification of non-trivial topology with the contextual behavior in the homotopical approach [Okay et al. 2017, Okay and Raussendorf 2020].

Example 23. *In quantum theory, Gleason's theorem relates the properties of the set of effects of a quantum system of dimension 3 or more with the quantum probabilistic valuation given by the Born rule. To do this, it imposes certain conditions.*

The first is that the set of effects will be studied in its representation as rays of a Hilbert space. The second is that all states with probabilistic valuations over the set of effects are valid, thus defining the physical states. The third condition imposes the continuity of such valuation, thus relating the topology of the set to what the valuation is capturing. The last condition is the noncontextuality of the valuation, which is nothing more than the imposition that the valuation does not carry the contextuality of the effects, i.e., there is no curvature.

When making the ontic representation, the condition of no curvature is maintained, but the representation cannot capture all the details of quantum theory. We conclude that the Born rule in the standard representation of a quantum system arises from Heisenberg's view, imposing all the contextuality in the topology of the set of effects. In such a topology, the contextuality is encoded, and it is by exploring its expressions that fundamentally quantum phenomena can be identified.

5.2.4 The Nature of Contextuality: A Choice Between Topology and Geometry

Contextuality is a property presented by an ontic representation. A theory will be contextual if and only if no ontic representation can describe it. A representation of a given set of objects, be it effects, states, or transformations, occurs in two ways: in the encoding of the processes; and in the encoding of the valuation function that acts on these processes to give us the probabilities. Taking all processes and levels of encoding into account, there are many different ways contextuality can be encoded in the representation. This shows its lack of empirical fundamentality and guides us to what is truly fundamental in the model. That's what we'll discuss here.

Contextual behavior can be encoded in each part of the process depending on the ontic representation, but that's not what we'll address here. What we aim to demonstrate here is the structure that the process/valuation levels possess, and the freedom of representation that it allows. We can codify what is going on with a diagram (here I will use $\mathcal{E}$, but the same can be said about $\mathcal{P}$ and $\mathcal{T}$), , where contextuality arises from the non-commutativity of the diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\quad p \quad} & [0,1] \\ & {\scriptstyle i} \searrow \quad \nearrow {\scriptstyle \xi_{\lambda'}} & \\ & \mathcal{S} & \end{array} . \tag{5.23}$$

The three elements, the system $\mathcal{E}$, the classical representation $\mathcal{S}$, and the target for valuation $[0,1]$, are all fixed, as is the map p. As contextuality is the failure of one of the maps to keep the data of the system, the inclusion i or the valuation $\xi_{\lambda'}$ of the representation fail.

The first case, failure of inclusion, is the usual interpretation in terms of contextuality. This is the topological view, where we interpret that it's the inclusion of the system that causes the problem. The noncommutativity of the diagram is fundamental, and it cannot be understood in any other way than as a departure from our notion of reality, as in the participatory-realistic interpretations of quantum theory. A justification for its use is that a loop could not be immediately written as the boundary of a region since an inner region is not supposed to exist in the first place, at least not for every loop. Thus, the curvature could not be defined. To avoid such a problem, the extreme is to suppose $F = 0$ everywhere, which one can interpret as the non-existence of a correction in the valuation changing the classical behavior. It is an intrinsic description, whose contextuality depends on the set of objects itself. Therefore, we have failures in reality itself, which defines the topological description as an participatory-realistic point of view.

The second case, the failure of valuation, imposes embedding. It's the inadequate valuation that causes problems. All the properties of the original system can be captured once one uses a modified valuation. It's not just hidden variables of the ontic representation; it also needs new rules to extract the probabilities. This is what happens in intrinsic-realistic interpretations of quantum theory, and this is the point of the geometrical view. The trivial fiber bundle is imposed, but curvature in the connection creates the phases by holonomy, following the Ambrose–Singer theorem. The geometrical view changes the valuation function by a generator of non-classicality. It can be thought of as curvature of the valuation on a set of classical objects. It's a modification of our classical laws by a hidden nature.

Both notions are equivalent, and one can argue that both causes can coexist. Choosing to what extent the topological and geometric causes generate non-classicality is just a matter of representing a deeper phenomenon: contextuality. While there are no ways to differentiate between different representations by verifying that they aren't faithful to some level of reality yet to be explored, for example, empirically verifying in which ontic representation the model works, it doesn't make much difference what is actually going on at the ontic level.

5.3 Applications of the Differential Approach

We are in a position to apply the understanding of the contextuality that the description by differential geometry provides us. The strategy in all of them is simple: to look for where the 1-form that carries the contextual behavior is in the respective formalism and, respecting its constraints, explore where it is expressed.

Each application has a preference for one of the views we described earlier. It's important to keep in mind that this is a choice for building each application, but it's not the natural description in which contextual behavior is encoded. Such a description is indifferent to interpretations, and as we'll see, it presents itself differently for each of the interpretations of quantum theory.

5.3.1 Contextual Fraction

The contextual fraction [Abramsky, Barbosa and Mansfield 2017, Barbosa et al. 2019] was introduced in the Sheaf approach [Abramsky and Brandenburger 2011]. Usually, it is applied in response to the limitations of this approach, only exploring the contextuality of effects, known as measurement contextuality, and imposing outcome-determinism. To achieve this, we fix a state and do not apply any transformation. An ontic representation will have this form

$$p(E) = \int_\Lambda \mu(\lambda)\xi(E|\lambda), \tag{5.24}$$

with μ a measure on the set of ontic variables Λ. Due to the condition of outcome-determinism, the valuation function ξ will assume only the values 0 or 1, and, for simplicity, we assume the finiteness of the sets.

With these constraints, the contextual fraction can be used in its usual form, and we can express the probability p as a decomposition

$$p(E) = (\mathrm{NCF})p_{NC}(E) + (\mathrm{CF})p_{SC}(E), \tag{5.25}$$

with the noncontextual fraction (NCF) and the contextual fraction (CF), where p_{NC} represents the probabilistic distribution of a noncontextual model, and p_{SC} represents the probabilistic distribution of a strong contextual model, the part without any noncontextual contribution. For the definition of CF and NCF, we seek to maximize the quantity of NCF among the set of distributions in the ontic variables μ that can be used. Even with the maximization of NCF, the distributions p_{NC} and p_{SC} are not uniquely defined. The fractions also satisfy the sum property of probability

$$\begin{aligned} 1 &= \sum_r p(E_r) \\ &= (\text{NCF}) \sum_r p_{NC}(E) + (\text{CF}) \sum_r p_{SC}(E) \\ &= \text{NCF} + \text{CF}, \end{aligned} \tag{5.26}$$

exposing the meaning of CF and NCF being referred to as fractions.

Just like the Sheaf approach, contextual fraction follows Schrödinger's view, embedding the effects in a classical GPT. The outcome-determinism condition implies that the valuation function must have Boolean values, thus forcing the effects to be fixed on the vertices of the effect hypercube. To expose contextuality, we need to choose the probabilistic distribution over ontic variables that maximizes NCF.

The maximization is in terms of μ, and it pertains to how the valuation of the ontic representation encodes the behavior of the effects, given by the probabilistic weight of μ when embedding the set of effects in the ontic representation. For the valuation function ξ, we have that the previous conditions fix the set Λ and its relation with $\mathcal{E}$, thus also fixing ξ. This means that the 1-forms are not subject to maximization, as they are intrinsic to the ontic representation previously fixed.

We can do the decomposition of the valuation $\xi = dc + \omega$

$$p(E) = \int_\Lambda \mu(\lambda) \langle dc_\lambda | E \rangle + \int_\Lambda \mu(\lambda) \langle \omega_\lambda | E \rangle , \tag{5.27}$$

that also satisfies the sum property of probability

$$1 = \sum_r p(E_r) = \sum_r \int_\Lambda \mu(\lambda) \langle dc_\lambda | E_r \rangle + \sum_r \int_\Lambda \mu(\lambda) \langle \omega_\lambda | E_r \rangle . \tag{5.28}$$

Let's assume that the maximization of NCF has already been done. Identifying the contextual and noncontextual parts of the previous equation as those containing ω and dc, respectively, we get the fractions

$$\text{NCF} = \sum_r \int_\Lambda \mu(\lambda) \langle dc_\lambda | E_r \rangle , \tag{5.29}$$

$$\text{CF} = \sum_r \int_\Lambda \mu(\lambda) \langle \omega_\lambda | E_r \rangle , \tag{5.30}$$

and the probabilistic distributions

$$p_{NC}(E) = \frac{\int_\Lambda \mu(\lambda) \langle dc_\lambda | E \rangle}{\sum_r \int_\Lambda \mu(\lambda) \langle dc_\lambda | E_r \rangle}, \tag{5.31}$$

$$p_{SC}(E) = \frac{\int_\Lambda \mu(\lambda) \langle \omega_\lambda | E \rangle}{\sum_r \int_\Lambda \mu(\lambda) \langle \omega_\lambda | E_r \rangle}, \tag{5.32}$$

which are not unique since they depend on the choice of μ in the set that maximizes NCF.

In particular, a property of the contextual fraction is that the maximal violation of a generalized Bell inequality of the model is given by CF [Abramsky, Barbosa and Mansfield 2017]. Since CF $= \sum_r \int_\Lambda \mu(\lambda) \langle \omega_\lambda | E_r \rangle$, we have the explicit dependence on the contextual correction of the valuation.

5.3.2 Interference in Quantum Measure Theory and Quantum Theory

Interference is a natural property of quantum theory when described by wave functions. Richard Feynman said that interference "contains the only mystery" of quantum theory [Feynman, Leighton and Sands 1965]. Rafael Sorkin [SORKIN 1994, Sorkin 1995] introduced a generalized notion of interference as a correction to the standard measure theory based on Kolmogorov axioms, by modifying the disjoint rule to

$$p(A \sqcup B) = p(A) + p(B) + I_2(A, B) \tag{5.33}$$

for two disjoint sets,

$$\begin{aligned} p(A \sqcup B \sqcup C) = &p(A) + p(B) + p(C) \\ &- I_2(A, B) - I_2(B, C) - I_2(A, C) \\ &+ I_3(A, B, C) \end{aligned} \tag{5.34}$$

for three disjoint sets, and so on, following analogously to the inclusion-exclusion principle in combinatorics.

With valuation functions that allow interference terms I_n, we have a generalization of the usual measure theory, which is called quantum measure theory. The set on which the probability measure acts is taken as a measurable set, and since we are dealing with the finite case, it has a natural Boolean structure. This implies that we are embedding the processes into a classical GPT and imposing that any correction must be expressed in the valuation function, specifically in the interference terms I_n. Therefore, quantum measure theory follows Schrödinger's view.

Here our interest will be in how such formalism can be used to capture contextuality, as already explored in Refs. [Craig et al. 2006, Dowker and Ghazi-Tabatabai 2008, Dowker, Henson and Wallden 2014].

Example 24. *Quantum theory presents only the correction I_2 fundamentally non-trivial, which can be seen clearly in the path integral approach. The form of I_2 follows directly from the calculation of quantum probabilities from the Born rule. For sharp effects e and e',* *which we assume are not in the same subspace to be disjoint, and a pure state* $\rho = |\rho\rangle\langle\rho|$*, we can write*

$$\begin{aligned} |\langle e \vee e'|\rho\rangle|^2 &= \langle e \vee e'|\rho\rangle + \langle\rho|e \vee e'\rangle \\ &= |\langle e|\rho\rangle|^2 + |\langle e'|\rho\rangle|^2 + I_2^\rho(e,e') \end{aligned} \tag{5.35}$$

with

$$I_2^\rho(e,e') = \langle e|\rho\rangle\langle\rho|e'\rangle + \langle e'|\rho\rangle\langle\rho|e\rangle \tag{5.36}$$

the symmetric interference function.

The relationship with the interference of a double-slit experiment follows from treating each effect as a distinct path from the slit, which is explored by a state ρ*. In this case, we can identify the isolated terms* $|\langle e|\rho\rangle|^2$ *and* $|\langle e'|\rho\rangle|^2$ *as closed paths that loop back on themselves. As for the term* $I_2^\rho(e,e')$*, it comprises two terms accounting for the two directions in which we can loop through the two paths. Interpreting them as "passing through both paths simultaneously" is what generates the notion of interference, as well as the non-classicality that such a term may carry.*

For sharp effects, only I_2 *is non-trivial, which follows from Specker's Principle [Specker 2009, Cabello 2012]. For unsharp ones, high-order interference appears, but they are non-fundamental once they can be rewritten from the intersections* ($\wedge$) *and* I_2*.*

Interference terms are necessary but not sufficient for contextualilty in a model. For instance, classical systems can exhibit interference, imposing a correction term on distributions. This term is given by classical correlations that are generated when marginalizing a GPT that captures all classical details into a smaller GPT through a marginalization process that acts as coarse graining. The difference is that the terms I_n need not be non-negative, which, for the case of outcome-determinism and restricted to effects, is a sign of contextualilty [Abramsky and Brandenburger 2011, Spekkens 2008].

For the ontic representation given by quantum measure theory, the set of objects in the ontic representation is such that any correction to Kolmogorov's axioms follows from the connection ω. This follows from the property of dc satisfying the axioms in the ontic representation when subjected to the condition of outcome-determinism, as it arises from a probabilistic distribution over a set of effects of a Boolean GPT. Any part of ω that ends up being classical follows from the non-refinement of the ontic representation, as discussed in the previous paragraph.

Aware of this limitation, we can see how ω appears as the generator of the interference term. Suppose the effects E, E', and $E \vee E'$ are not in a Boolean sub-algebra.

Since $dc_{\lambda'}$ satisfies Kolmogorov axioms, we have that

$$\begin{aligned} p(E \vee E') &= \int d\mu \, \langle \xi | E \vee E' \rangle \\ &= \int d\mu \, (\langle dc | E \vee E' \rangle + \langle \omega | E \vee E' \rangle) \\ &= \int d\mu \, (\langle dc | E \rangle + \langle dc | E' \rangle + \langle \omega | E \vee E' \rangle) \\ &= \int d\mu \, \langle \xi | E \rangle + \int d\mu \, \langle \xi | E' \rangle \\ &\quad + \int d\mu \, (\langle \omega | E \vee E' \rangle - \langle \omega | E \rangle - \langle \omega | E' \rangle) \\ &= p(E) + p(E') + \\ &\quad \int d\mu \, (\langle \omega | E \vee E' \rangle - \langle \omega | E \rangle - \langle \omega | E' \rangle) \, . \end{aligned} \tag{5.37}$$

So, for disjoint effects,

$$I_2(E, E') = \int d\mu \, (\langle \omega | E \vee E' \rangle - \langle \omega | E \rangle - \langle \omega | E' \rangle) \tag{5.38}$$

showing that the failure of ω to satisfy the Kolmogorov disjoint axiom is the cause of interference. Note that what is measured in the geometrical view is the failure of the parallelogram law of the valuation function captured by the distribution μ that represents the relationship between the state and the ontic representation. The analogous construction can be made with high-order interference.

As shown in [Anastopoulos and Savvidou 2002], in quantum theory, the decoherence functional [Dowker and Halliwell 1992] is determined by geometric phases. However, only its real part possesses “reality” due to being Hermitian, a property that arises from strong positivity [Wilkes 2019]. This real part induces interference, which is the primary focus of quantum measure theory [SORKIN 1994, Sorkin 1995, Surya and Wallden 2008, Gudder 2009]. Therefore, the connection between interference and geometric phase is profound in quantum theory, and it can be utilized to detect non-classicality [Asadian et al. 2015].

Example 25. *Noncommutativity in sharp quantum theory is where contextual behavior on effects is hidden. This follows from the capacity of using unitary transformations in relation to a fixed effect to define any other effect. As incompatibility of sharp effects is necessary for contextuality, and it is equivalent to noncommutativity [Heinosaari, Reitzner and Stano 2008, Heinosaari and Wolf 2010], the non-exact part ω of the valuation depends on the non-trivial commutator.*

For two noncommutative unitaries, U and U', the structure constant depends on $\hbar$, and defines a loop $U^{-1}U'^{-1}UU'$. Noncommutativity implies a geometric phase that can generate an interference correction, thus given by the non-exact term ω. The limit

$\hbar \to 0$ cancels the non-classical behavior, which means we can write $\omega = \hbar\tilde{\omega}$, to explicitly denote its dependence on $\hbar$,

$$\xi_{\lambda'} = dc_{\lambda'} + \hbar\tilde{\omega}_{\lambda'}. \tag{5.39}$$

This also holds for states and transformations, which can exhibit non-classicality, as exemplified by the Bargmann invariants [Bargmann 1964] and the Aharonov–Bohm effect [Popescu 2010]. For this, one can use unitary transformations on them, analogous to what has already been done in addressing the relationship between contextuality and the Wigner function representation of quantum theory [Kocia, Huang and Love 2017, Kocia and Love 2017, Kocia and Love 2018].

5.3.3 Signed Measures and Embedding of GPTs

The relationship between contextuality and negativity has already been explored in Refs. [Abramsky and Brandenburger 2011, Spekkens 2008], as well as the relationship between embedding and contextuality in Refs. [Shahandeh 2021, Schmid et al. 2021]. The unification of contextuality, embedding, and negativity was achieved in Ref. [Schmid et al. 2020], within the formalism of GPTs. Our goal here is to discuss how the term ω and the two views on its nature allow us to understand such relationships, given that a classical ontic representation is defined by the commutation of diagram 5.23, and each view of noncommutativity explains the origin of these non-classical notions.

As shown in Ref. [Spekkens 2008], the necessity of signed measures for all ontic representations is equivalent to the violation of the noncontextual conditions, thus implying the existence of non-trivial phases in the valuation maps. It is in the correction term, which arises from the curvature in Schrödinger's view, that codifies the negative part of the valuation function. Heisenberg's view explains why we cannot access negative probabilities. When translated into Heisenberg's view, the negative values of the valuation become topological failures; therefore, they are not within the set of physical processes. In the geometrical view, a classical model cannot have non-trivial curvature to correct the valuation, and any theory that exhibits such curvature for all ontic representation cannot be represented by a classical model. Since the necessity of negative values implies the existence of curvature that does not preserve the valuation function, we conclude that it characterizes the contextuality of the theory.

The embedding of the model into a classical GPT is equivalent to an ontic representation. In GPTs, such an embedding is a set of linear maps, each mapping to a set of objects that form the processes, such that the valuation is preserved. One way to determine if a theory is contextual is to verify that for every embedding in any classical GPT, a valuation map is not preserved. Another way is to show that if the map preserves the valuation functions, then for any embedding, at least one set of objects will be larger than the respective set of objects in the classical model. In the topological view, a classical model exhibits no topological failure, thus monodromy is impossible, and a theory with monodromy for all ontic representations cannot be represented by a classical model. The impossibility of embedding while preserving the valuation function implies the expression of these failures by imposing the valuation on them, which also characterizes contextual behavior.

All these non-classical notions are just different ways to explain to our classical eyes what contextuality is. They are just different representations of this phenomenon.

5.3.4 Generalizing Vorob'ev Theorem

For measurement contextuality satisfying outcome-determinism and no-disturbance, the measurements can be represented by a hypergraph of compatibility, where measurements are vertices and the contexts with mutually compatible measurements are the hyperedges connecting them[1]. Vorob'ev theorem [Vorob'ev 1959] imposes conditions for the hypergraph of compatibility to only describe noncontextual models. In the language of the hypergraph of compatibility, it takes the following form[2]:

Theorem 9. *A hypergraph of compatibility* $(\mathbf{M}, \mathcal{C})$ *has only noncontextual behavior if and only if it is acyclic, which means it can be reduced to the empty set by the operations known as Graham's reduction:*

- *if* $m \in C$ *belongs to only one hyperedge, then delete* m*;*
- *if* $C \subsetneq C'$*, with* $C, C' \in \mathcal{C}$*, then delete* C*.*

We can rethink what these operations mean for the structure of the measurement's effects. Remembering that we are considering each context with a finite set of outcomes. Let us also assume, without loss of generality, that the set of outcomes is the same for each the measurement. A context is identified by its σ-algebra of local events, with the elements of the σ-algebra having the structure of a set of deterministic effects of a classical GPT, thus a Boolean structure. In other words, each context can be defined as a classical sub-set of effects.

The separation of an effect into its context and the outcome it codifies turns the set of effects in a fiber bundle with a base set given by the compatibility hypergraph and the fibers as a finite σ-algebra. Vorob'ev theorem being restricted to the compatibility

[1] See Ref. [Montanhano 2021] and references therein for the construction of these scenarios.

[2] Here, the Graham reduction acts on a hypergraph, allowing the deletion of a sub-hyperedge $\sigma \subsetneq C$ without requiring the deletion of the hyperedge C. This is not possible in simplicial complexes, as in Ref. [Barbosa 2015], which restricts the Graham reduction to its first step. For the construction that follows, we will use the full version of the Graham reduction on hypergraphs, as already applied in Ref. [Budroni, Miklin and Chaves 2016].

hypergraph is equivalent to identifying that the Boolean structure of the fibers in the bundle does not interfere with a necessary and sufficient condition for always having noncontextuality. Even as a property projected onto the base set, it is still a property of the total set, the set of effects.

As a fiber bundle, we can define the projection $\pi : \mathcal{E} \to \mathcal{C}$ projecting the set of effects into its contexts, where $\mathcal{E} = O \times \mathcal{C}$ means the effects can be rewrite as a pair of outcome and context. The projection maps the σ-algebra of outcomes of a context to the σ-algebra given by the context and its subcontexts. The measurements are nothing more than the atoms of this resulting structure. With this fact in mind, it is easy to rewrite Graham's reduction:

- if an atom is just in one σ-algebra, we can ignore it.
- if a sub-σ-algebra is proper, we can simplify it as a trivial σ-algebra, consequently coarse-graining the σ-algebra which contains it.

We can initiate this version of Graham's reduction without performing the projection, by considering the effects as the atoms. The projection follows from the reduction itself, specifically from the second item, where the proper sub-σ-algebra of the outcomes O is erased. Therefore, Graham's reduction and Vorob'ev theorem can be extended to the level of effects, and are not restrict as a property of contexts.

Measurement contextuality follows from a loop γ in the set of effects. As any loop in a Boolean structure has a trivial contextual connection when viewed as a classical GPT, $\omega = 0$, only loops defined through different Boolean structures can show any contextuality. Vorob'ev theorem identifies the fact that without such loops, no contextual behavior appears. Graham's reduction simplifies the set of effects by erasing the internal structures of the σ-algebras. For the argument to work such that we can recover the standard Vorob'ev theorem, we need to make the dependence on contexts explicit in the set of effects, placing us in Heisenberg's view, therefore $F = d\omega = 0$.

Theorem 10 (Generalized Vorob'ev)**.** *A ontic representation is noncontextual if its first de Rham cohomological group is trivial and* $F = 0$*.*

Proof. Once $F = 0$, we need to use the topological view, and contextuality will appear as topological failures that cause the monodromy of the probabilistic valuation. If the first cohomological group is trivial, then there are no differential forms that capture these failures, thus $\omega = 0$. Therefore, the ontic representation satisfies the noncontextuality condition. □

To identify the unavoidable contextual behavior of a model, we must exhaust all possible ontic representations. In other words, all possible valuation 1-forms indexed by the ontic variables must present a cohomological obstruction. For the case of outcome-determinism, we have a fixed ontic representation, as we discussed earlier, which allows us to apply the previous theorem and obtain the noncontextuality of the model even before knowing the valuation functions. Acyclicity is a special case in which, still in the compatibility hypergraph, we can identify that there is no possibility of there being a loop that allows the expression of contextual behavior.

An important point is that H^1 is defined by the effects, not measurements. Thus, the intuition that the first homology groups of the compatible hypergraph must be non-trivial to show contextuality is false, even though the topological view holds that $\phi = \langle \xi_{\lambda'} | \gamma \rangle \neq 0$ is a failure also detected by H_1. For discussion and counterexamples, see Ref. [Montanhano 2021]. A study on the topological expression of Vorob'ev's theorem will be conducted in a future work.

5.3.5 Disturbance and Transition Functions

Disturbing models for measurement contextuality are not standard in almost all frameworks. A well-known exception is the Contextuality by Default approach [Dzhafarov, Kujala and Cervantes 2015]. Disturbance is necessary for describing experimental applications. One way to indirectly address them is to modify the scenario [Amaral and Duarte 2019], which seems to reflect William James's point of view on contradiction: when encountering a contradiction, make a distinction. The idea is to make the contradiction that disturbance represents explicit in the scenario by adding new maximal contexts representing the disturbing intersections. Due to its nature of being a discordance in the intersections between contexts, it is natural to relate disturbance to transition maps in a suitable approach where the latter are defined, such as in the bundle approach [Montanhano 2021].

A measurement is given by its effects, seen as possible events, not necessarily deterministic. In the deterministic case, the set of effects has a natural σ-algebra structure. Seen in this way, no-disturbance is exactly the triviality of the transition maps on intersections of σ-algebras. The set of measurements covers the set of effects and can be seen as charts of an atlas. Each deterministic measurement is a classical GPT in itself, or in other words, there exists an embedding of it as an entire classical GPT.

The geometrical view, where the contextual phases follow from holonomy, provides a direct approach to deal with disturbance, given that holonomy can be encoded in an element of a commutative group, the group of automorphisms of $\mathbb{R}$. For the 1-form ω, one can express such a phase as stated in [Waldorf 2020]

$$Hol(\partial S) = exp\left(\langle \omega | \partial S \rangle\right) \tag{5.40}$$

where we identify $\langle \omega | \partial S \rangle$ as an element of the Lie algebra of the Lie group of transformations of $\mathbb{R}$. For discrete cases, one can embed the group in the Lie group of transformations of $\mathbb{R}$ and the same argument follows. As a commutative group, one only needs to track each chart, here σ-algebra,

$$Hol(\partial S) = exp\left[\sum_r \langle\omega| (b_r |E_r\rangle)\right]. \tag{5.41}$$

Let's consider the disturbing case now. Due to the presence of disturbance, it is not possible to have a global ontic representation, as it erases disturbance since classical GPT is non-disturbing. It can only be done locally for each chart. Similar to in differential geometry, the non-triviality between intersections defines the transition maps $t_{r,r'}$, with the indices r and r' denoting the charts, which lead from one chart to another. The holonomy transformation can then be calculated, and it will be given by

$$Hol(\partial S) = \prod_r exp\left[\langle\omega| (b_r |E_r\rangle)\right] \prod_r t_{r,r-1}, \tag{5.42}$$

where commutativity was used to rearrange and combine the transition maps. They can be put in the Lie algebra form, $t_{r,r-1} = exp(\eta_{r,r-1})$, to rewrite the holonomy term as

$$\sum_r \langle\omega| (b_r |E_r\rangle) + \sum_r \eta_{r,r-1}. \tag{5.43}$$

What the transition map is doing in the geometrical view is taking one classical GPT into another, carrying the effects on themselves, but changing the valuation. This means that just as in contextual behavior, a correction term that is sensitive to chart changes should be included in the valuation. For this adjustment, we can define a 1-form $|\eta\rangle$ that satisfies $\eta_{r,r-1} = \langle\eta| (b_r |E_r\rangle)$, and rewrite the valuation function as

$$\xi = dc + \omega + \eta. \tag{5.44}$$

The first term values the global contribution, the second term values dependencies on parallel paths, while the third term values changes in an effect when transitioning from one chart to another.

The disturbance form η is on the same footing as contextuality, which is not surprising, as charts are nothing more than contexts. In a certain sense, we have that it is the same phenomenon, the dependence on contexts of the valuation. The difference is that ω deals with paths, while η is pointwise on the effects. For this reason, just as the 1-form ω depends on the ontic representation to be defined, the 1-form η depends on the atlas to be defined.

The explicit construction of examples of this formalism to deal with disturbance as transition maps will be done in a subsequent work. However, we can already identify its relationship with extended contextuality [Amaral and Duarte 2019]. Adding contexts and duplicating measurements that show disturbance is nothing more than turning the form η into an effective form ω of an ontic representation without disturbance by making effects in the intersection into paths. Thus, the disturbance becomes contextuality. It can explain the fact that disturbance consumes contextual behavior, as already noted in the nonlocality literature [Abramsky, Brandenburger and Savochkin 2014, Blasiak et al. 2021].

In the same way as in contextual fraction, we can think of an analogous quantifier for disturbance. Let's suppose here that we already have the valuation ξ defined in a fixed ontic representation, just as in the case of contextual fraction, which implies that we already have c, ω, and η. The disturbance fraction (DF) will be given by

$$\mathrm{DF} = \sum_r \int_\Lambda d\mu(\lambda) \langle \eta_\lambda | E_r \rangle \tag{5.45}$$

through reasoning analogous to that of contextual fraction. Similarly, an induced maximally disturbing model is given by

$$p_D(E) = \frac{\int_\Lambda d\mu(\lambda) \langle \eta_\lambda | E \rangle}{\sum_r \int_\Lambda d\mu(\lambda) \langle \eta_\lambda | E_r \rangle}, \tag{5.46}$$

with $p = (\mathrm{NCF})p_{NC} + (\mathrm{CF})p_{SC} + (\mathrm{DF})p_D$, where $p_D(E)$ concentrates all the disturbance. The conditions for the explicit construction of this disturbance fraction, as well as its relation to other proposals in the literature [Vallée et al. 2023], will be addressed in a subsequent work.

5.3.6 Contextuality in Interpretations of Quantum Theory

An interpretation of quantum theory does not aim solely to create models for isolated processes, but rather to create a consistent framework for the entire quantum theory. The goal is to present an ontic structure and probabilistic valuation functions capable of explaining quantum phenomena, even if some properties understood as classical need to be violated.

At the core of all interpretations lies the measurement problem, the fact that a measurement does not follow from the standard evolution of the theory. This problem arises from the incompatibility of measurements and the impossibilities they generate. Contextuality is at the root of the measurement problem; indeed, it is the phenomenon that prevents a classical ontology, and it is the phenomenon that interpretations indirectly deal with.

The number of interpretations grows exponentially, and it is beyond the scope of this work to address them individually. We will use the classification by Cabello [Cabello 2017], exploring some specific examples a bit more deeply. Our goal is to try to identify how the interpretations exhibit the correction of contextuality ω. Their incompatibilities stem from choosing different ontic representations, or even where contextuality appears, following one of the views presented in section 5.2.

In [Cabello 2017], we have two types of interpretations, denoted as Type-I and Type-II. The Type-I interpretations presume an intrinsic realism, that is, they seek an ontological foundation. Within Type-I, there is a further distinction concerning the ontological value of the quantum state. The ψ-Ontic interpretations assign ontological value to the states. We will explore two well-known examples.

Example 26. *Bohmian mechanics [Bohm 1952a, Bohm 1952b, Goldstein 2021] generates an interesting interpretation in which we add a purely non-local term to classical deterministic dynamics. The quantum state is taken as ontology and broken down into its local classical and purely non-local quantum parts, which "pilot" classical objects.*

The embedding of the theory into a classical framework is exact, with the correction given by the influence that the global part of the ontology exerts on the local classical part. This correction alters the valuation that an agent has access to, which is the restriction of classical determinism with dependence on global properties.

The encoding in classical ontology shows that Bohmian mechanics utilizes Schrödinger's view. Describing an extra term to correct classical behavior, and that this term is non-local, embodies the correction that quantum contextual behavior imposes.

Example 27. *The Many Worlds Interpretation [Everett 1957, Vaidman 2021] takes the quantum state as the ontological object. Reality would be entirely described by a quantum state that evolves unitarily. Each state is possible as a classical reality, each possibility is a reality in some world.*

We immerse quantum theory in a classical theory, since we give reality to all states in a multiverse. The difficulty lies in retrieving the quantum valuation rule once the worlds are taken as real and can be treated as classical. This imposes restrictions on the classical distribution that introduce correlations between worlds, thus modifying the valuation function to fit the Born rule.

The Many Worlds Interpretation is expressed through Schrödinger's view, which is natural since this interpretation follows from assuming the absolute reality of Schrödinger's formalism.

In Type-I interpretations, those that are not ψ-Ontic are ψ-Epistemic. Such interpretations remove the ontological value of the quantum state, proposing an ontology that completes this state, resulting in an epistemic value for it. Let's explore an example for this case.

Example 28. *Consistent Histories Interpretation [Griffiths 1984, Griffiths 2019] treats events in a classical manner. Quantum theory would be given through a stochastic process over events, forming histories. It is in the valuation that we have the expression of purely quantum behavior, with the existence of incompatible histories. To make a measurement,* *the agent needs to choose a set of compatible histories, called a framework, where the Kolmogorov axioms for classical probability hold.*

Thought of as an ontic representation of quantum theory, Consistent Histories Interpretation embeds the objects of processes in a classical structure. Classical valuation undergoes modification akin to that presented in quantum measure theory, altering stochastic behavior beyond the classical. A study of quantum theory in this interpretation seeks means of recovering quantum statistics through physical principles in valuation functions, since the events are already fixed.

This is an interpretation that explicitly employs Schrödinger's view, with contextuality expressed in the geometric part given by the valuation of the ontic representation.

Type-II interpretations deal with a participatory realism, where the information extracted from a system is not intrinsic to it, but rather a result of its relationship with the observer. They also have a distinction into two types, but of a more epistemological nature. The Type-II interpretations that deal with knowledge are those that treat the quantum state as an object that encodes the observer's knowledge. Let's look at an example.

Example 29. *Relational interpretation [Rovelli 1996, Laudisa and Rovelli 2024] proposes that ontology lies not in states, but in a relational structure between agents and between processes. It assumes that state concerns the knowledge of an agent and is no longer fundamental but rather local. Consistency in this relational structure arises from an agent having restricted knowledge of the process when positioned as an observer within a system. For a third party, the system's evolution alongside the agent would still be the usual unitary evolution of quantum theory.*

There is no completion of the theory, and probabilities are treated as classical. What is lost is the absoluteness of the quantum state for each process as seen by each agent, yet with agreement enforced by an underlying fundamental structure. There are no local contradictions since the measurement is classical, but only when comparing such local measurements.

This identifies this interpretation as utilizing Heisenberg's view, but concealing within the breakdown of the global into local the topological flaws that contextuality identifies.

The Type-II interpretations that deal with belief treat the quantum state not as knowledge but as an object that encodes the agent's expectation. Let's explore an example of this type of interpretation.

Example 30. *Quantum Bayesianism [Caves, Fuchs and Schack 2002, Healey 2023] takes a step further and presents an interpretation that avoids ontology. Agents' beliefs are optimized by imposing the Born rule valuation on the quantum state that represents those beliefs. This explicit non-realism positions the Born rule merely as a function that selects* *certain events, which only hold if they can be experimentally accessible. This highlights complete freedom regarding the structure of events.*

It is within events that the expression of contextuality occurs, as valuations are merely classical mechanisms that an agent employs. We are adopting Heisenberg's view, in a position where no ontology holds any meaning other than that of mere representation.

The examples in this section make explicit the relationship that the views presented in section 5.2 have with the interpretations of a non-classical theory. Schrödinger's view is the intrinsic-realistic treatment of the model, with ontology being represented classically, and with all modifications occurring in the valuation of such ontic variables. It is in the geometry of this valuation that contextual behavior is found, usually as a correction term when forcing measurements outside the classical scope. Heisenberg's view is the participatory-realistic treatment, presupposing that ontology is not classical due to flaws in accessible propositions. These are "holes" in reality, and it can only be seen in pieces, by a covering of classical pieces. It is the identification of these "holes" as topological failures that allows contextuality to appear in valuation even if it is classical.

6 Discussion of Part II

In this Part, we looked for an alternative description of contextuality in the generalized approach by constructing a representation of contextuality as a differential geometry problem.

The approach we used involved identifying the operational equivalences as discrete loops in the vector space where the objects are represented, and contextuality for them as the non-preservation of such discrete loops by the valuation functions. This identification allows the use of discrete differential geometry to deal with contextuality and naturally extends to non-discrete cases.

There are two different ways in this approach to understand contextual behavior, different views of contextuality. In the first case, the classical ontic representation is imposed, which implies the existence of a correction to the valuation function. In the second case, the classical probabilistic valuation is imposed, thus forbidding a correction without a fundamental cause, the non-triviality of the topology of the set of objects itself. Both notions are equivalent, and they are expressed in other concepts of non-classicality: contextual fraction, interference, signed measures, and non-embeddability. Rethinking contextuality of an ontic representation allows us to rethink and generalize the Vorob'ev theorem, and also gives us a natural way to deal with models that violate non-disturbance.

Contextuality is more than just topological [Mansfield 2020], even if it can be expressed as such if we assume Heisenberg's view. It is a higher-level phenomenon than that expressed in representations and interpretations. The choice of how to bring such a phenomenon to this level can be related to notions of realism. For example [Myers 2021], if we consider Fixed Realism, where there is one model, and the real is what is true in it, then we impose that contextuality is no longer topological, but geometric as in Schrödinger's view. Even if we consider Covariant Realism, where we have equivalent models, and the real is how things change when one changes the model, still the global realism would impose a view with correction in the valuation, and not in the events. Now, if we consider Local Realism, where there are nonequivalent models, and the real is how to handle disagreement, then we can use this disagreement to bring contextuality to be encoded at the level of events, allowing the valuation not to be modified.

In the first two cases of realism, Fixed and Covariant, we have an intrinsic-realistic view, where what matters is the reality itself and how we see it. Examples of interpretations of quantum theory that are of this type are Many Worlds, Bohmian Mechanics, and Consistent Histories, with the latter being explicitly of the second case. The third case, Local Realism, is a participatory-realistic view, where the most important thing is a pragmatic description of what is observed. Contextuality is generally presented in this way, in its nonlocality version. The topological view explains that non-trivial cohomology is a signal of disagreement, in this case, contextual behavior. It is a matter of pragmatism, where if we accept the existence of new hidden features, we can recover the intrinsic-realistic [Biagio and Rovelli 2021].

Some open questions for future exploration naturally follow. The exploration of the relationship between the curvature of the sets of states, effects and transformations that naturally appear in quantum theory, and the curvature in Schrödinger's view is an important step to make quantum contextuality explicit. More generally, a deep exploration of models and interpretations, with the explicit construction of their connection and curvature, or their topological failures. The importance of this exploration goes beyond the foundations of non-classical theories and their applications, as it would elucidate the main resource to be explored in emerging technologies. To make contact with such technologies, a next step would be to explore how the classical limit would be expressed with the approach presented here, and its relationship with mechanisms that erase the contextual connection. Furthermore, as disturbance is natural in experimental issues, the identification of transition maps as representations of disturbance is a way to bring the formalism of contextuality to such issues. And the identification also points to the exploration of higher holonomies and their possible relationship with the contextuality of higher-level processes. An explicit construction of the bundles, especially with the same language used in other areas of physics like field theory, would enable a greater understanding of the phenomenon of contextuality. In contact with areas of computation, an explicit construction of a topological characterization of the Vorob'ev theorem would also provide a more intuitive view of it.

If the contextuality presented by certain processes causes discomfort when thought of as something about the questions we can empirically ask, questions that we unjustifiably assume exist, we can change the point of view and faithfully represent the same phenomenon with all the questions we assume but with a correction in the answers. It's another lesson in humility that nature gives us, but also of our inventiveness. We can't ask the questions we want, we don't have the power to force it into an interrogation. We only receive answers that we are ready to receive, and only what it allows us to access. However, we can interpret the answers and represent the reality in which we live. Our confusion over quantum theory and its contextual behavior maybe stems more from our arrogance in forcing ourselves into the first case than our wisdom in adapting to the second case.

Part III

Wigner and Friends, a Map is Not the Territory! Contextuality in Multi-Agent Paradoxes

Multi-agent paradoxes [Frauchiger and Renner 2018, Nurgalieva and Renner 2020, Vilasini, Nurgalieva and Rio 2019] are violations of agreement among agents about some global information. They appear in generalizations of the Wigner's friend (thought experimental) scenario [Wigner 1961, Deutsch 1985], which itself extends Schrödinger's famous thought experiment with his cat [Schrödinger 1935]. The exploration of such paradoxes and other related phenomena has proven to be significant for the fundamental understanding of quantum theory and its interpretations [Brukner 2015, Brukner 2018, Bong et al. 2020, Cavalcanti and Wiseman 2021, Haddara and Cavalcanti 2023, Relaño 2020, Rossi and Soares-Pinto 2021, Guérin et al. 2021, Schmid, Yīng and Leifer 2023]. The formal construction of these paradoxes employs the language of modal logic [Garson 2021, Smets and Velázquez-Quesada 2019, Rendsvig and Symons 2021] to show how (thought experimental) scenarios, found both in quantum theory and in other non-classical theories beyond quantum theory, presents a violation in the structure of classical logic.

Contextuality in its standard definition [Kochen and Specker 1967] deals with global inconsistencies in measurements even when there is local consistency, that is, even if the model is non-disturbing, generalizing the famous phenomena of nonlocality [Bell 1964] and the condition of no-signaling between observers [Abramsky and Brandenburger 2011]. This concept has been formalized in various ways in the literature, including the topological [Abramsky and Brandenburger 2011, Abramsky, Mansfield and Barbosa 2012, Abramsky et al. 2015, Okay et al. 2017, Okay and Raussendorf 2020, Montanhano 2021], the algebraic [Birkhoff and Neumann 1936, Bell 1964, Kochen and Specker 1967, Gleason 1957], the geometrical [Cabello, Severini and Winter 2014, Amaral and Cunha 2017] and generalizations [Dzhafarov, Kujala and Cervantes 2015, SORKIN 1994, Spekkens 2008, Schmid et al. 2021, Montanhano 2022]. This phenomenon has a significant impact on the fundamental properties of quantum theory [Döring and Frembs 2020], and is necessary for any potential computational advantage over classical computers [Schmid et al. 2022, Shahandeh 2021, Howard et al. 2014]. Its formal description employs the categorical language of sheaves and presheaves [Abramsky and Brandenburger 2011], enabling the construction of bundle diagrams for each model [Beer and Osborne 2018]. Equivalently, a contextual model implies the inability to describe it classically, in the sense of an embedding of propositions into a set of Boolean propositions with probabilistic valuation satisfying Kolmogorov's probability axioms, even with additional propositions in the set of Boolean propositions serving as hidden variables.

Our objective in this part is to construct a map between empirical model and multi-agent scenarios such that the contextuality of the first and the multi-agent paradox of the second could be identified as the same phenomenon. To achieve this objective, we will need to rewrite the Sheaf Approach [Abramsky and Brandenburger 2011], the formalism of contextuality closest to multi-modal logic, in such a way that we can understand how each part of the construction of empirical models can be identified in a multi-agent scenario. This will require a more refined treatment of multi-modal logic, as concepts like fundamental truth become elusive when non-classicality is present. Such refinement, which explicitly utilizes knowledge operators and the trust relation defined between sets and agents, can be understood as the logical formalization of Alfred Korzybski's statement "A map is not the territory" [Korzybski 1933] applied to the knowledge that agents can access. Consequently, our main finding is that, contrary to what is claimed in the literature [Nurgalieva and Rio 2019], modal logic is suitable for quantum and other non-classical settings. To illustrate this, we analyze three famous examples that accept the inverse in the constructed map.

The part is structured as follows. The section 7.1 serves to fix the notation used and give the basics of modal logic. We employ the topological semantics of multi-modal logic (specifically the $S4$ system) to initially investigate the application of trust [Nurgalieva and Rio 2019, Vilasini, Nurgalieva and Rio 2019] when the knowledge operators are explicitly utilized. This exploration is grounded in the idea that knowledge is inherently relational—something is not merely known; it must be known by someone. Trust can be understood as a relational way to define the Truth Axiom; in fact, they are equivalent when seen by the topology induced by distributed knowledge of the agents, as shown in section 7.2. We can thus use the knowledge operators and trust to create a translation between multi-agent scenarios and empirical models up to restrictions. In section 8.1, we systematically identify the elements of an empirical model as elements of a multi-agent scenario in multi-modal logic, exploring the implications of this identification, and discussing the limitations of empirical models in handling generic multi-agent scenarios. The violation of soundness described in Ref. [Nurgalieva and Rio 2019] that appears as the failure of classical logic to deal with quantum theory is identified as the hidden imposition of mutual knowledge on the agents. This implies the conclusion that modal logic fails to deal with multi-agent paradoxes. Interestingly, this issue disappears when distributed knowledge is imposed. A similar resolution occurs when translating the contextuality conditions to multi-modal language, albeit with the cost of lambda-dependence. Next, in section 8.2, we work out the three examples of multi-agent paradoxes: the Wigner's friend scenario, the Frauchiger-Renner scenario, and the Vilasini-Nurgalieva-del Rio scenario, in topological semantics and translate them to their sheaf representation. We then identify contextuality as the origin of their paradoxes when they appear. In chapter 9, we provide insights into the results and explore future research possibilities.

7 Modal Logic

7.1 Multi-Modal Logic

A modal logic is defined with a set Ω of propositional variables and the usual set of connectives $\neg$ ("not"), $\wedge$ ("and"), $\vee$ ("or"), $\leftrightarrow$ ("if and only if"), $\rightarrow$ ("if . . . then"), besides the use of parentheses.

In addition to the usual connectives, a modal logic has a modal operator called "possibility" $\Diamond$. When combined with $\neg$ one can define the modal operator "necessity" $\Box$ as $\neg\Diamond\neg$.[1]

When dealing with a set of agents indexed by a finite set $I \ni i$, one can define $\Diamond_i$ (and consequently $\Box_i$) as the necessity modal operator from the point of view of agent i. This defines a multi-modal logic, with one modal logic for each agent, but all of them agreeing on the usual propositional logic structure.

Once the set of propositional variables Ω and symbols are defined, one can define the formulas as follows:

- All the propositional variables are formulas.
- If A is a formula, then $\neg A$, $\Diamond A$, and $\Box A$ are formulas.
- If A and B are formulas, then $(A \wedge B)$, $(A \vee B)$, $(A \leftrightarrow B)$, and $(A \rightarrow B)$ are also formulas.
- There are no other formulas.

The collection of propositions Φ is defined by the possible formulas.

7.1.1 Kripke Semantics of Multi-Modal Logic

A Kripke frame $\langle \Sigma, R \rangle$ is a pair consisting of a non-empty set of states or worlds Σ and a binary relation R on Σ, called the accessibility relation, such that aRb means "b is possible given a" or "b is accessible by a".

A relational structure $\langle \Sigma, \{R_i\}_{i \in I} \rangle$ is a finite set of Kripke frames with the same Σ, where each R_i is given by an agent i. In other words, aR_ib is understood as "b is possible given a in the point of view of agent i" or "b is accessible by a in the point of view of agent i".

[1] One can also start with $\Box$ and to define $\Diamond$ as $\neg\Box\neg$, but certain care must be taken when defining the modal operator in this dual manner [Smets and Velázquez-Quesada 2019].

A Kripke structure $M = \langle \Sigma, \{R_i\}_{i \in I}, \nu \rangle$ is a relational structure $\langle \Sigma, \{R_i\}_{i \in I} \rangle$ equipped with a Boolean valuation $\nu : \Omega \to \mathcal{P}(\Sigma)$, with $\mathcal{P}(\Sigma)$ the power set of Σ, that indicate the worlds where a proposition variable is true: given $A \in \Omega$, $\nu(A) \in \mathcal{P}(\Sigma)$ is the set of worlds where A is true. The valuation of a generic proposition in Φ obeys the ordinary rules of propositional logic for each world, plus rules to the modal operators as we will see.

7.1.2 Rules, Soundness and Completeness

Given a Kripke structure $M = \langle \Sigma, \{R_i\}_{i \in I}, \nu \rangle$ with possible worlds $w \in \Sigma$ we define $M, w \vDash \phi$ as the proposition ϕ being true for the world $w \in \Sigma$ of the Kripke structure M. Equivalently, we write $M, w \vDash \phi$ if $w \in \nu(\phi)$ in the Kripke structure M.

Sentences, also known as closed formulas, are formulas without free variables. Let Q be a sentence. The symbol $Q \vDash \phi$, where $\phi \in \Phi$, can be read as "Q semantically entails ϕ", meaning that if Q is true, then ϕ is also true. We can have a finite set of sentences $Q_1, ..., Q_n$ semantically entailing ϕ, $Q_1, ..., Q_n \vDash \phi$, which reads as "if the sentences $Q_1, ..., Q_n$ are true, then ϕ is true."

Another symbol $\vdash$ can be read as "Q syntactically entails ϕ", meaning Q proves ϕ. Again, we can have a finite set of sentences $Q_1, ..., Q_n$ syntactically entailing ϕ, in symbols $Q_1, ..., Q_n \vdash \phi$, which is read as "the sentences $Q_1, ..., Q_n$ prove ϕ."

The ordinary rules of propositional logic hold here for each world, and additional rules for the modal operators in Kripke semantics are added:

- $(M, w \vDash \Box\phi) \leftrightarrow \forall u(wRu \to (M, u \vDash \phi))$.
- $(M, w \vDash \Diamond\phi) \leftrightarrow \exists u(wRu \wedge (M, u \vDash \phi))$.

A system satisfies completeness (also called semantic completeness) if $Q \vDash \phi$ implies $Q \vdash \phi$, and a system satisfies soundness if $Q \vdash \phi$ implies $Q \vDash \phi$.

7.1.3 Knowledge Operators

The valuation ν being unique for all agents reflects the philosophical statement that truth is independent of any agent; it is absolute. This can be understood as a strong axiom to determine the distinction between knowledge and belief, with the former being a direct consequence of truth and the latter not needing any relation to it. This definition of knowledge is Plato's "justified true belief." However, as one can readily see, different agents have different knowledge, which is a coarse-graining of the fundamental truth. Therefore, for multi-agent scenarios, we must use the knowledge of each agent to valuate propositions.

One can define, for an agent, the basic modal operator of epistemic logic K that means "it is known that". Let $R(w) = \{u|wRu\}$, and for $A \subseteq \Sigma$ denote $M, A \vDash \phi$ as $M, u \vDash \phi$ for all $u \in A$. Them, in Kripke semantics, one add a new rule to define knowledge:

- $(M, w \vDash K\phi) \leftrightarrow (M, R(w) \vDash \phi)$.

In the case of multiple agents indexed by a set I, one can define an operator K_i for each agent i, where $K_i\phi$ can be read as "agent i knows that ϕ". We need to add a new item to the list of formulas:

- If A is a formula, then K_iA for all $i \in I$ is a formula.

To preserve the truth by the knowledge operators, one imposes the Knowledge generalization rule, also known as **N** and Necessitation Rule, that says for a Kripke structure M and any $\phi \in \Phi$, we have

$$(M, w \vDash \phi)\forall w \rightarrow (M, w \vDash K_i\phi)\forall i. \tag{7.1}$$

This rule can be written as well for modal operators,

$$(M, w \vDash \phi)\forall w \rightarrow (M, w \vDash \Box\phi). \tag{7.2}$$

There are two more modal operators dealing with knowledge of a subset of agents $U \subset I$ that are interesting to us. Mutual or common knowledge E_G means "every agent in G knows". Formally, for all ϕ, we define the mutual knowledge operator as follows:

$$E_G\phi = \bigwedge_{i \in U} K_i\phi, \tag{7.3}$$

which defines a relation

$$R_{E_G} = \bigcup_{i \in G} R_i \tag{7.4}$$

that allows the addition of the following rule in the Kripke semantics:

- $(M, w \vDash E_G\phi) \leftrightarrow (M, R_{E_G}(w) \vDash \phi)$.

Distributed knowledge D_G means "it is distributed knowledge to the whole U", not just describing the knowledge of individual agents but all knowledge combined of U as an entity itself. Formally, for all ϕ, we define mutual knowledge operator by its relation

$$R_{D_G} = \bigcap_{i \in G} R_i \tag{7.5}$$

which defines the operator D_G by the addition of the following rule in the Kripke semantics:

- $(M, w \vDash D_G\phi) \leftrightarrow (M, R_{D_G}(w) \vDash \phi)$.

7.1.4 Axioms of System **S4**

Different axioms can be imposed on the accessibility relation of a frame (Frame Conditions) that equivalently[2] result in properties of modal (Modal Axioms) and knowledge (Axioms of Knowledge) operators, thus defining different systems of modal logic [Garson 2021, Rendsvig and Symons 2021].

Axiom 1 (Distribution Axiom or **K**)**.** *It holds true for any frame. For modal operators, we have that for any* $\psi, \phi \in \Phi$ *it holds that*

$$(\Box(\psi \to \phi)) \to (\Box\psi \to \Box\phi) \tag{7.6}$$

while for knowledge operators, for any $\psi, \phi \in \Phi$*, we have*

$$(K_i\phi \wedge K_i(\phi \to \psi)) \to K_i\psi. \tag{7.7}$$

System **K** is the simplest kind of logic described by Kripke semantics and establishes modus ponens for each world. An equivalent way to write it as a Modal Axiom is

$$\Box(\phi \wedge (\phi \to \psi)) \to \Box\psi, \tag{7.8}$$

in a similar format to the respective Axiom of Knowledge. Normal Modal System is defined as a system **K** satisfying Rule **N**.

Axiom 2 (Truth Axiom, or **T**, or **M**)**.** *For any frame and* $\phi \in \Phi$*:*

- *(Frame Condition) The accessibility relation is reflexive.*
- *(Modal Axiom)* $\Box\phi \to \phi$.
- *(Axiom of Knowledge)* $K_i\phi \to \phi$.

As a result of this axiom, one can show that $\phi \to \Diamond\phi$ holds. System **T** (also known as System **M**) is defined as a System **K** satisfying the Truth Axiom.

Axiom 3 (Positive Introspection Axiom or **4**)**.** *For any frame and* $\phi \in \Phi$*:*

- *(Frame Condition) Accessibility relation is transitive.*
- *(Modal Axiom)* $\Box\phi \to \Box\Box\phi$.
- *(Axiom of Knowledge)* $K_i\phi \to K_iK_i\phi$.

[2] They follow from the preservation of such properties on the accessible worlds of each world.

A result of this Axiom is that holds $\Diamond\Diamond\phi \to \Diamond\phi$. System **S4** is defined as a System **T** satisfying Axiom **4**.[3]

7.1.5 Topological Semantics of Multi-Modal Logic

A natural semantics for the system **S4** is the topological [AWODEY and KISHIDA 2008, Coniglio and Prieto-Sanabria 2017, Sustretov 2007, Awodey and Kishida 2012].

Definition 21. *A topological model is a pair (T, ν) where $T = (X, \tau)$ is a topological space and a function $\nu : \Phi \to \mathcal{P}(X)$, called interpretation, that satisfies for any $\phi, \psi \in \Phi$*

$$\begin{aligned}
\nu(\phi \wedge \psi) &= \nu(\phi) \cap \nu(\psi) \\
\nu(\phi \vee \psi) &= \nu(\phi) \cup \nu(\psi) \\
\nu(\neg\phi) &= \nu(\phi)^{\complement} \\
\nu(\Box\phi) &= \mathring{\nu(\phi)} \\
\nu(\Diamond\phi) &= \overline{\nu(\phi)},
\end{aligned}$$

with $\mathring{A}$, $\overline{A}$ and $A^{\complement}$ respectively the topological interior, closure and complement of $A \in \mathcal{P}(X)$.

The elements of X are the worlds, and ν can be understood as the valuation in the topological semantics, by giving the set of worlds where a formula is true, $M, w \vDash \phi$ if and only if $w \in \nu(\phi)$. Also, for any two formulas $\phi, \psi \in \Phi$ one can prove $\phi \vdash \psi$ if and only if $\nu(\phi) \subseteq \nu(\psi)$. One can show that this semantics imposes system **S4** to the logic. In this sense, the system **S4** is said to be the logic of topological spaces.

An Alexandrov topological space is a topological space where every point of the space has a minimal neighborhood. Alexandrov topologies can also be defined as topological spaces where arbitrary intersections of open sets are open sets. In particular, any finite topology, i.e., only finitely many open sets, is an Alexandrov topology. A well-known result is the equivalence between Kripke and topological semantics for Alexandrov topological spaces [McKinsey and Tarski 1944]:

Theorem 11. *For any Alexandrov topological space (X, τ) there exists a binary relation R such that for any Boolean valuation ν and for any formula $\phi \in \Phi$, $(X, \tau, \nu), w \vDash \phi$ if and only $(X, R, \nu), w \vDash \phi$. For any transitive reflexive frame (Σ, R), equivalently a frame satisfying* **S4**, *there exists a topology τ on Σ such that for any valuation ν and for any formula $\phi \in \Phi$, $(\Sigma, R, \nu), w \vDash \phi$ if and only $(\Sigma, \tau, \nu), w \vDash \phi$.*

[3] Another important axiom known as Negative Introspection Axiom or **5** is the imposition of symmetry of the accessibility relation, resulting for any $\phi \in \Phi$ the validity of $\neg K_i\phi \implies K_i\neg K_i\phi$ for knowledge operators and $\Diamond\phi \to \Box\Diamond\phi$ for modal operators. System **S5** is defined as a System **S4** satisfying Axiom **5**, and is exactly the system where the accessibility relation is an equivalence relation. Usually one drops **5** once when an agent does not know something, it is hard to such agent judge its own lack of knowledge.

This result [Davoren 2007] follows from the identification of the accessibility relation R with the specialization pre-order $\leqslant$, $x \leqslant y$ if and only if $\forall U \in \tau$ we have $(x \in U) \to (y \in U)$, which turns $(X, \leqslant)$ into a poset. Such a relation defines a topology generated by the basis of open sets $U^x = \{y|y \leqslant x\}$, which is an equivalent definition of the Alexandrov topology[4], and any Alexandrov topology has such natural pre-order that defines the semantics satisfying **S4**. Also, the system **S4** satisfies completeness and soundness in relation to the topological semantics of Alexandrov topological spaces.

7.2 Knowledge, Trust, and the Construction of Fundamental Truth

In this section, we will address the concept of truth in modal logic and its relationship with different types of knowledge operators. The Truth Axiom **T**, which states that if a proposition is known by an agent, then it is true, appears to be too strong to handle multi-agent scenarios. Hence, as is conventional in the literature of multi-agent scenarios, we will not directly assume the Truth Axiom **T**. Instead, we will use trust between agents, as suggested by [Nurgalieva and Rio 2019], to weaken the classical notion of fundamental truth. Even with this more relational approach to multi-agent scenarios, the same paradoxes arise. The reason for this, as we will see, is that we can use the more refined knowledge operator in the hierarchy of operators, distributed knowledge, to reconstruct a fundamental truth, which would be equivalent to the Truth Axiom.

We will work with the system **S4**, which imposes Kripke semantics (Distribution Axiom) of possible worlds and accessibility relations, the transitivity (Positive Introspection Axiom) and the reflexive (Truth Axiom) properties of the accessibility relations, standard when dealing with epistemic logic [Rendsvig and Symons 2021]. In particular we will mostly work with the topological semantics of the system **S4**, naturally related to a topological view of the Sheaf Approach. In such semantics, we can think of the propositions of the system **S4** as open sets in a set of possible worlds, with the usual operations between open sets (union, intersection, complement, interior, closure, complement) playing the role of logical operations between propositions (or, and, negation, necessity, possibility).

7.2.1 Trust Instead of Fundamental Truth

Let's introduce and justify the concept of trust between agents as an alternative to a fundamental truth. Multi-agent scenarios defined with modal logic in the literature use only trust between agents. To construct the map we aim for between empirical models and multi-agent scenarios in the next section, we will define here a generalization of the

[4] This is the upper Alexandrov topology, and one can think of it as defining open sets as generated by the causal past cones of points. The lower Alexandrov topology, with the basis $U_y = \{x|y \leqslant x\}$, is given by the future causal cones [Bennequin et al. 2020].

concept of trust that also applies to sets of agents. With such trust relation, we will formally define multi-agent scenarios.

Knowledge, mutual knowledge, and distributed knowledge operators are important for writing formulas when one imposes the following principle: there is no knowledge without an agent. This principle can be understood as the embodiment of the obvious idea that fundamental truth, in the sense of being absolute to all agents, following Plato's "justified true belief" definition of knowledge, is a philosophical position rather than an empirical fact. One can only assume something is true for all agents, but cannot test such a thing.

As we will see in the examples in Section 8.2, quantum theory imposes limitations on this Platonic view of knowledge aforementioned. One way to attempt to circumvent such limitations is to weaken the concept of knowledge to the "justified belief" definition, which does not presuppose any fundamental truth, but rather relies on justification based on the inevitably incomplete data accessible to the agent. Therefore, every formula must be evaluated through a knowledge operator. Axioms **K** and **4** do not present any issues once the operator is introduced. However, **T** relies on the notion of fundamental truth, leading to certain philosophical complications. Let's ignore them by allowing beliefs to be on the same level as knowledge. To simplify matters, we will assume any further mechanism beyond the scope of this thesis to distinguish them. Given the absence of an absolute notion of knowledge, we must find knowledge by trust between agents [Vilasini, Nurgalieva and Rio 2019, Nurgalieva and Rio 2019].

Axiom 4 (Trust)**.** *The trust relation $\rightsquigarrow$ between agents i and j is given by*

$$(j \rightsquigarrow i) \leftrightarrow (K_i K_j \phi \rightarrow K_i \phi) \forall \phi, \tag{7.9}$$

meaning "i trusts j".

This notion of trust between agents must be generalized to deal with sets of agents, as it will become important when we discuss the relationship between the trust relation and contexts. An agent i could not trust the agents of a set G separately, but only when seen as a collective entity. In other words, the agent i trusts the distributed knowledge of G. In this sense, "i trusts G" if and only if, for all propositions, the knowledge of i that the distributed knowledge of G implies the knowledge of i. This is the weakest way to describe such a relation, where all agents in G could not know ϕ individually[5]. Since for an agent i, we have that K_i, D_i, and E_i are equivalent, we can generalize agent i

[5] A simple example of applying this concept of trust arises in a presidential election within a presidential system with direct voting. Agent i wishes to determine which candidate has been elected. Immediately before the election results are disclosed and the winner becomes mutual knowledge, agent i cannot rely solely on individual agents to ascertain the victor, as knowledge is distributed across the electorate. It is agent i's trust in the electorate as a whole, denoted as G, that leads them to accept the election outcome.

to a set G' of agents with the mutual knowledge operator $E_{G'}$ in the previous argument with minimal modifications, or even restrict trust in G to the mutual knowledge of its agents. The two trust relations can thus be defined as follows.

Axiom 5 (Trust (sets of agents)). *The trust relation* $\overset{D}{\rightsquigarrow}$ *between a set of agents* G' *and a set of agents* G *is given by*

$$(G \overset{D}{\rightsquigarrow} G') \leftrightarrow (E_{G'} D_G \phi \rightarrow E_{G'} \phi) \forall \phi, \tag{7.10}$$

meaning "G' *trusts* G*", and the trust relation* $\overset{E}{\rightsquigarrow}$

$$(G \overset{E}{\rightsquigarrow} G') \leftrightarrow (E_{G'} E_G \phi \rightarrow E_{G'} \phi) \forall \phi, \tag{7.11}$$

meaning "G' *trusts the consensus of* G*".*

We can formally define multi-agent scenarios through Kripke semantics, knowledge operators, and the generalization of the concept of trust relation.

Definition 22. *A multi-agent scenario is given by a tuple*

$$(\Sigma, I, \{K_i\}_{i \in I}, \{E_G\}_{G \in \mathcal{P}(I)}, \{D_G\}_{G \in \mathcal{P}(I)}, \overset{D}{\rightsquigarrow}, \overset{E}{\rightsquigarrow}), \tag{7.12}$$

where each element $i \in I$ *is an agent,* Σ *is the set of possible worlds the agents explore,* $\{K_i\}_{i \in I}$ *are the knowledge operators of each agent,* $\{E_G\}_{G \in \mathcal{P}(I)}$ *and* $\{D_G\}_{G \in \mathcal{P}(I)}$ *are the operators of mutual and distributed knowledge among sets of agents,* $\overset{E}{\rightsquigarrow}$ *and* $\overset{D}{\rightsquigarrow}$*) are the trust relations between sets of agents.*

7.2.2 The Relation between the Topology of Different Kinds of Knowledge

The topological semantics of the system **S4** is deeply related to knowledge. The definition of the knowledge operator K in Kripke semantics can be rewritten as:

$$(M, w \vDash K\phi) \leftrightarrow (M, U^w \vDash \phi) \tag{7.13}$$

In other words, in the world w, the agent knows something if and only if for all worlds in the element U^w of the topological basis of the Alexandrov topology, that something is true. Here again, we have the problem of fundamental truth, with the important property by **T** that $w \in U^w$, which allows one to interpret U^w as the natural neighborhood of w. In this sense, an agent knows something in a world if it is true in a neighborhood of such a world.

Epistemic logic with more than one agent defines an Alexandrov topology for each accessibility relation, which can be interpreted as different ways the agents perceive the worlds. We have the following relationship:

$$(K_i \phi \rightarrow K_j \phi) \forall \phi \leftrightarrow (R_j \subseteq R_i) \leftrightarrow (\tau_j \subseteq \tau_i) \tag{7.14}$$

between the knowledge operators, the induced relation, and the topology, respectively, in Kripke and topological semantics. In particular, one can show that the relationships

$$(K_i\phi \to D_I\phi)\forall\phi \leftrightarrow (R_{D_I} \subseteq R_i) \leftrightarrow (\tau_{D_I} \subseteq \tau_i) \tag{7.15}$$

and

$$(E_I\phi \to K_i\phi)\forall\phi \leftrightarrow (R_i \subseteq R_{E_I}) \leftrightarrow (\tau_i \subseteq \tau_{E_I}) \tag{7.16}$$

hold. They state that something known to an agent $i \in I$ is also known distributively, and that something mutually known is known to any agent in $i \in I$, respectively. Additionally, one can show that a fundamental property of distributed knowledge is

$$(D_I\phi \to \phi)\forall\phi, \tag{7.17}$$

meaning the distributed knowledge of something implies its truth, which follows from the Truth Axiom **T**.

The hierarchy of knowledges presented in the previous paragraph (mutual knowledge implies individual knowledge from 7.16; individual knowledge implies distributed knowledge from 7.15; distributed knowledge implies fundamental truth from 7.17) will be important in which follows. Each relationship between the knowledge operators, usually represented by its topological incarnation in topological semantics, will be explored in the translation of an empirical model. But first, as said before, we need to address fundamental truth.

7.2.3 Recovering Fundamental Truth from Trust

By definition, the finest topology generated by all agents is τ_{D_I}, the topology given by the distributed knowledge operator of the set of all agents, which implies that the most refined knowledge that this set of agents can construct is given by the distributed knowledge. Therefore, there is no way to construct any other knowledge operator that captures more knowledge of fundamental truth than D_I. Once we need to understand knowledge in a relational manner, always explicitly specifying the knowledge operator, and both $(D_I\phi \to \phi)\forall\phi$ and $(E_I\phi \to K_i\phi)\forall\phi$ hold, we obtain the following proposition.

Proposition 5. *The following statements are valid:*

- *Axiom* **T** *turns trust relations vacuous.*
- *The trust relation* $\overset{D}{\rightsquigarrow}$*, along with the condition that* $(\phi \leftrightarrow D_I\phi)\forall\phi$*, induces a fundamental truth.*

Proof. According to Axiom **T**, we have $(K_j\phi \to \phi)\forall\phi$, which implies $(K_i(K_j\phi) \to K_i\phi)\forall\phi$. In other words, the existence of fundamental truth makes the knowledge of every agent reliable. This allows us to generalize to sets of agents and trust relations given by D_G and E_G by means of $(D_I\phi \to \phi)\forall\phi$ and $(E_I\phi \to \phi)\forall\phi$, which follow from their definitions. Hence, we have $(K_i(D_I\phi) \to K_i\phi)\forall\phi$ and $(K_i(E_I\phi) \to K_i\phi)\forall\phi$, meaning they are also rendered reliable. Therefore, we obtain vacuity.

Suppose a set I of agents and their trust relation $\overset{D}{\rightsquigarrow}$. The construction of D_I, even more explicitly in its topology τ_{D_I}, depends on the trust relation $\overset{D}{\rightsquigarrow}$ between sets of agents. We can define an effective fundamental truth by imposing that $(\phi \leftrightarrow D_I\phi)\forall\phi$, which is saying that the distributed knowledge is equivalent to a fundamental truth. In fact, as it has the finest topology, there is no other knowledge that the agents can access beyond what is captured by D_I. Furthermore, $(K_iD_I\phi \to K_i\phi)\forall\phi$, meaning all agents trust the distributed knowledge. □

What the above proposition intuitively means is that one can say there is a limit to the knowledge a set of agents can access, which is the distributed knowledge, and there is no way to distinguish this limit from a fundamental limit of reality[6,7].

[6] In this sense, it is not surprising that an isolated population in an approximately stable environment that comes into contact with another civilization can suffer a significant impact on their culture. If they survive, their distributed knowledge usually loses its fundamentality.

[7] One could use a Bayesian vision to justify the existence of absolute truth through an induction argument, which holds in a classical description of reality, but it must be limited by Kant's epistemology.

8 Multi-Agent Scenarios

8.1 The Modal Logic of an Empirical Model and Contextuality

In this section, we will construct the map that turns a generic empirical model into a multi-agent scenario in such a way that we can identify the paradoxes of the former with the contextuality of the latter. For simplicity, we will deal with finite objects.

As we will see, assumptions about the nature of a scenario, such as classicality manifested in the form of global concordance between agents, impose restrictions on the possible worlds presumed to be accessible to agents through their topologies. We will demonstrate that paradoxes arise when an agent discovers that these worlds are insufficient to explain the scenario, potentially leading to the mistaken conclusion that modal logic is incapable of handling non-classical cases. We adopted a strategy of explicitly defining the knowledge operators, with a particular focus on identifying limits on the knowledge of a set of agents. This approach leads to another interesting consequence—the possibility of rethinking how possible worlds are chosen. We will use this when dealing with events in an empirical model.

Another important point is that we need to make it clear that the agents' knowledge will not be distorted when transferred between agents. In order for the trust relationship to truly be trustworthy, we need to define what makes someone trustworthy, even if there is trust.

Definition 23. *Given that $j \rightsquigarrow i$, an agent j is trustworthy to the agent i if $(K_i K_j \phi \rightarrow K_j \phi) \forall \phi$, i.e. if any information that agent i knows from agent j must also be known by agent j.*

This condition is essential to prevent any hidden information; therefore, trust implies that the topology of the trusting party is finer than the trustworthy part. With trust, an agent can reconstruct all the information provided by its trustworthy part, which encompasses all of its information. An agent that is terminal in the network generated by the trust relation can, under this condition, reconstruct the information of all the agents, obtaining a global perspective of the knowledge.

8.1.1 A Measurement to each Agent

To construct the map we seek, we will follow the steps involved in building an empirical model. First, we start with a set X containing elements referred to as measurements. Subsequently, we define a cover of contexts $\mathcal{M} \subset \mathcal{P}(X)$ that satisfies

$\bigcup \mathcal{M} = X$, and $C' \subset C \in \mathcal{M}$ implies $C' \in \mathcal{M}$. Next, we apply a sheaf of events $\mathcal{E} : \langle X, \mathcal{M} \rangle^{op} \to \mathbf{Set}$, associating outcomes O^U with each context $U \in \mathcal{U}$ — the local events — thus defining a measurement scenario. The global events follow from the sheaf property of $\mathcal{E}$. An empirical model is then characterized by a presheaf $\mathcal{D}_R : \mathbf{Set} \to \mathbf{Set} :: O^U \mapsto \left\{\mu_R^{O^U}\right\}$, with R a semiring, typically of the probabilistic or boolean type. Such a presheaf specifies distributions with values in R for the outcomes of each context, usually imposing the no-disturbance condition $\mu_R^{O^j}|_{kj} = \mu_R^{O^k}|_{kj}$. Finally, contextuality arises when it becomes impossible to explain these distributions as marginalizations of a distribution on the global events.

The natural mapping of the measurements of an empirical model to a multi-agent scenario is achieved through the agents' measurements, as each agent in a multi-agent scenario is restricted to a measurement. Our first identification is as follows:

- The measurements of an empirical model are mapped to the agents of its corresponding multi-agent scenario.

It's important to note that what we are identifying here is each measurement of an empirical model with a measurement of an agent in a multi-agent scenario. This constraint differs from scenarios in which agents have multiple measurements and the free will to choose among them, as in standard nonlocality scenarios. Through distributed knowledge, we have:

$$(K_i(K_i\phi) \to D_I(K_i\phi) \to K_i\phi)\forall\phi, \tag{8.1}$$

which means that an agent trusts itself. This is in contrast to scenarios where an agent can choose between incompatible measurements. Therefore, agents in a multi-agent scenario cannot choose their measurements. If they could, each measurement would define different agents who cannot trust each other due to the incompatibility of their measurements.

8.1.2 Contexts Come from Trust

Contexts can be understood as an island of classicality at the measurement level. Every measurement is nothing more than the marginalization of the "mother" measurement of the entire context. Such a definition of a context allows the construction of stochastic maps between all subcontexts, and these maps can define the probabilities of the context given the marginals.

Once the measurements are identified with the agents in the multi-agent scenario, the covering of contexts locally define agents who are in a classical environment, with the maps transmitting knowledge from one subcontext to another. Since the subcontexts, and therefore the agents, are terminal in the network of maps between subcontexts, and by imposing that they are all trustworthy, a condition called "flasque beneath the cover" in the literature of the Sheaf Approach, they all have a global view of the context U in which they are. In logical notation, for any subcontexts $G, G' \in U$ we have the following:

$$(E_G E_{G'} \phi \to E_G \phi) \forall \phi. \tag{8.2}$$

Thus, in a context its classicality imposes that for any two subcontexts $G, G' \in U$ the trust relation $G \overset{E}{\rightsquigarrow} G'$ holds, with $\overset{E}{\rightsquigarrow}$ being an equivalence relation between the subcontexts of U (reflexivity follows from trust itself, symmetry follows from the symmetry in the choice of subcontexts of U, and transitivity follows from the transitivity of maps between subcontexts), identified with the stochastic maps. Our second identification is as follows:

- Contexts are sets of agents in which the trust relation $\overset{E}{\rightsquigarrow}$ is an equivalence relation between subcontexts.

With these two identifications, we can rewrite the hypergraph of compatibility $\langle X, \mathcal{U} \rangle$ of any empirical model as parts of a multi-agent scenario with the property of being covered by sets of agents with an equivalence trust relation $\overset{E}{\rightsquigarrow}$ between their subsets.

8.1.3 Global Events Follow from the Topology of Mutual Knowledge

The next step is to identify the events of the empirical model given by the Sheaf of events. As elements of reality to which we have empirical access, such events must naturally be associated with some structure involving the possible worlds. However, worlds are defined globally, while events do not need to be. Here, we will see that such a distinction is related to the topology we are dealing with.

To overcome this apparent obstacle, the strategy is to use pointless topology [Johnstone 1983]. In this formalism, we start with the topology and its elements — the open sets — as the primitives of the topological space. In the topological semantics of modal logic, the use of pointless topology implies that we will take propositions, represented by the open sets, as the primitives, which is equivalent to the standard formalism where possible worlds are the primitive objects. When propositions are considered as primitives, the focus shifts to the operator of knowledge, its topology, and the set of agents that define it, with possible worlds emerging as consequent constructions derived from the propositions.

One cannot know the fundamental possible worlds, even if they exist, but only the propositions one can access. In other words, the worlds are defined by the propositions, and not the other way around. Therefore, possible worlds must be defined by the topology to which an agent has access, as the elements of a basis of such topology. Even with the most refined set of propositions, the questions that agents pose about the world cannot be taken as refined as the fundamental truth. Thus, any construction must begin with the limitation of the agents in “mapping the territory.” In an empirical model, this situation becomes more complicated, as we also have the local events organized into contexts as the propositions attempting to find out the possible world in which the scenario is in a locally classical way. Generally, there isn't a “global map” of the “territory” when dealing with empirical models.

Let $\langle X, \mathcal{U} \rangle$ be the hypergraph of compatibility of a connected measurement scenario. Since it is connected and $\overset{E}{\rightsquigarrow}$ satisfies the conditions that determine a context, every element of $\mathcal{U}$ is a terminal object in the trust between sets of agents. In logical terms, for any $G \in \mathcal{P}(I)$ representing a element of $\mathcal{U}$, with $\mathcal{P}(I)$ being the power set of I, we have $(E_G\phi \to E_I\phi)\forall\phi$, implying $\tau_{E_G} = \tau_{E_I}$ for all G representing a context, as $G \subseteq I$ implies $\tau_{E_I} \subseteq \tau_{E_G}$ and $(E_G\phi \to E_I\phi)\forall\phi$ implies $\tau_{E_G} \subseteq \tau_{E_I}$. Let's call $\mathcal{B}_{E_I}$ the basis of τ_{E_I}. We define the possible worlds as $\Sigma = \mathcal{B}_{E_I}$, and the accessible relations R_i as induced by $\tau_i = \tau_{E_I}$. The elements of $\mathcal{B}_{E_I}$ are global and atomic objects, such as global events. Our next identification is as follows:

- Global events correspond to the basis topology of the mutual knowledge operator.

Therefore, any global description of an empirical model is given by the possible worlds $\Sigma = \mathcal{B}_{E_I}$ induced by the mutual knowledge. Therefore, according to Fine-Abramsky-Brandenburger Theorem 1, we can conclude the following.

Result 1. *Mutual knowledge is the knowledge that logically explains non-disturbing outcome-deterministic noncontextual empirical models.*

8.1.4 Local Events Follows from the Topology of Distributed Knowledge

In an analogous way to the previous argument, we can identify local sections as the elements of the basis of the topology induced by the mutual knowledge of their respective context. Since in a context G, every subcontext trusts each other, we have $D_G = E_G$: all distributed knowledge is described by mutual knowledge between the agents, with each of them having the information of all G. This is an example of how the trust relation influences the definition of distributed knowledge. Local events are the most refined propositions that can be made in the empirical model while respecting the contexts and the non-disturbing condition, and thus generate the most refined topology. On the other hand, the topology generated by the distributed knowledge operator is the most refined topology possible among a set of agents. Therefore, we can identify $\mathcal{B}_{D_I}$ as given by the local events. Our final identification of an element of a measurement scenario is as follows:

- Local events correspond to the basis topology of the distributed knowledge operator.

The mapping of a measurement scenario to its respective multi-agent scenario is summarized in the following dictionary:

Result 2. *Given a measurement scenario* $\langle X, \mathcal{U}, (O_x)_{x\in X}\rangle$ *with the sheaf of events* $\mathcal{E}$*, the identification in the table below defines a multi-agent scenario with a set of agents* I*, trust relation* $\overset{E}{\rightsquigarrow}$ *and knowledge operators* D_I *and* E_I *induced by the basis topologies* $\mathcal{B}_{D_I}$ *and* $\mathcal{B}_{E_I}$*, respectively.*

Measurement scenario	**Multi-agent scenario**
X	I
$\mathcal{U}$	$G \subset I$ with $\overset{E}{\rightsquigarrow}$ an equivalence relation
$\mathcal{E}(X)$	$\mathcal{B}_{E_I}$
$\mathcal{E}(U)_{U\in\mathcal{U}}$	$\mathcal{B}_{D_I}$

8.1.5 Logic Contextuality is the Failure of Soundness

How does contextuality manifest when viewed from a modal perspective and how does it relate to multi-agent paradoxes? The map presented in Result 2 allows us to answer this question, as we will do below. For simplicity, we will limit ourselves to Boolean valuation functions, while making it clear that the probabilistic case with outcome-determinism follows from probabilistic distribution over the logical events identified by such Boolean valuation functions.

Previously, we saw that if an agent is terminal in the trust relationship, it has access to all the information of the other agents and sets of agents. Therefore, it can reconstruct the global view of the multi-agent scenario, and every other terminal agent will also agree with this description. What happens if the agents cannot agree on their global description? Well, one can argue that trust between agents and the sharing of information are not enough to access all the information of a scenario; $D_I \neq E_I$. In other words, the fundamental truth cannot be accessed by any agent individually. This is the key to characterizing contextual behavior, as we will see.

In an empirical model with Boolean valuation, the equation that represents noncontextuality remains as follows:

$$\mu_R^{O^U}(A) = \sum_{\Lambda} p(\lambda) \prod_{x\in U} \mu_R^{O^x}(\rho'(U,x)(A)), \tag{8.3}$$

but now every function is a Boolean function. Let's translate this equation into logical terms. In the Sheaf Approach we are dealing with, the hidden variables $\lambda \in \Lambda$ are identified as the deterministic global events. It evaluates a formula ϕ, which encodes a local event given by the context U and the outcome A, represented by the left-hand side of the equation. It asks whether, given all possible worlds, one can semantically evaluate that there is at least one deterministic global event λ such that it marginalizes to ϕ in the empirical model, $\lambda \to \phi$. On the other hand, this always holds syntactically, since in a measurement scenario, a local event is always the marginalization of at least one global event. This can be expressed as:

Result 3 (Logical contextuality)**.** *The definition of logical noncontextuality condition in logical notation takes the form of*

$$\phi \vDash \bigvee_{\lambda \in \Lambda} (\lambda \wedge (\lambda \to \phi)), \tag{8.4}$$

where due to the **S4** *system always holds that*

$$\phi \vdash \bigvee_{\lambda \in \Lambda} (\lambda \wedge (\lambda \to \phi)). \tag{8.5}$$

Therefore, it is the violation of soundness which gives contextuality in the logical form.

In Ref. [Nurgalieva and Rio 2019], it is stated that we have an "inadequacy of modal logic in quantum settings" precisely because we have multi-agent scenarios that exhibit logical paradoxes resulting from violations of soundness. Without the violation of soundness, multi-agent paradoxes do not arise[1]. However, the identification we made in Result 2 allows us to have an insight into what is actually happening, but which is left implicit in the literature. We can rewrite Result 3 as follows:

Result 4 (Logical contextuality with knowledge operators)**.** *The definition of logical noncontextuality condition in logical notation and with explicit knowledge operators as identified in Result 2 takes the form of*

$$K_i\phi \vDash \bigvee_{E_I\lambda \in \mathcal{B}_{E_I}} (E_I\lambda \wedge (E_I\lambda \to K_i\phi)). \tag{8.6}$$

This semantic equation does not always hold even if

$$K_i\phi \vdash \bigvee_{E_I\lambda \in \mathcal{B}_{E_I}} (E_I\lambda \wedge (E_I\lambda \to K_i\phi)) \tag{8.7}$$

holds syntactically. In other words, the logical form of contextuality follows from the violation of soundness when we define possible worlds as the elements of $\mathcal{B}_{E_I}$.

The last equation of Result 4 states that if one can describe ϕ using elements from $\mathcal{B}_{E_I}$ that are true, then the agents know it. This differs from the semantic equation, where all ϕ must be described by it. This time, contextuality is not the failure of soundness

[1] Since we trivially have completeness of the valuation in a multi-agent scenario, it is inconsistencies that give rise to multi-agent paradoxes. Soundness implies consistency, thereby avoiding the paradoxes.

that renders modal logic inadequate to deal with paradoxical behavior. Instead, it is the choice of the set of possible worlds as $\Sigma = \mathcal{B}_{E_I}$ that forces the logical violation to manifest in this way. There are insufficient possible worlds to adequately describe non-classical settings.

8.1.6 Modal Logic is Suitable for Non-Classical Settings

Paradoxes do not imply that modal logic is inadequate, as it is sound and complete in topological semantics. The problem is that we assume all descriptions must be consistent and global, which is not true. In other words, the worlds we are constructing in our scenario are too simplistic; they are defined by E_I, thereby disregarding any information beyond mutual knowledge.

We need to encode the information from all the empirical models, describing every detail of the agents, including their trust relationships, into the possible worlds. This is the case where the model exhibits lambda-dependence, where the worlds depend on the contexts. The elements of the basis must be the set of local events with their respective contexts, which, according to Result 2, is exactly given by τ_{D_I}. With this new set of possible worlds, contextuality ceases to be the failure of soundness and becomes a matter of the empirical model not being described by classical worlds.

Result 5 (Logical contextuality as $E_I \neq D_I$)**.** *Due to soundness and completeness of the topological semantics,*

$$K_i\phi \vDash \bigvee_{D_I\lambda \in \mathcal{B}_{D_I}} (D_I\lambda \wedge (D_I\lambda \to K_i\phi)). \tag{8.8}$$

if and only if

$$K_i\phi \vdash \bigvee_{D_I\lambda \in \mathcal{B}_{D_I}} (D_I\lambda \wedge (D_I\lambda \to K_i\phi)). \tag{8.9}$$

Therefore, contextuality is the difference between E_I and D_I.

The fundamental set of worlds is $\Sigma = \mathcal{B}_{D_I}$, while for a classical description, we are assuming that $\mathcal{B}_{E_I}$ is the set of words, a coarse-graining of the fundamental truth. In fact, Result 5 allows us to reach the following:

Corolary 1. *Modal logic is able to deal with the apparent violations if we do not restrict the knowledge to a mutual one, which we usually implicitly do.*

8.1.7 Limitations of the Map for Multi-Agent Scenarios

The map constructed here has limitations in handling multi-agent scenarios with the contextual toolkit, and the cause of these limitations lies in the stringent constraints of the Sheaf Approach. To meet the criteria for analysis using the Sheaf Approach to contextuality, certain conditions must be fulfilled.

Firstly, the agents should have only one measurement each, and these measurements must satisfy outcome-determinism, i.e., in quantum theory, they must be projection-valued measures[2]. The trust relation defined between agents, and more generally between elements of the power set of the set of agents, must conform to the structure of contexts, specifically the equivalence trust relation. In particular, it must be symmetric, which prohibits the use of the map to address non-classicality in causal structures, a significant kind of generalization of the Wigner's friend scenario. Once these conditions are met, the measurement scenario becomes well-defined.

To establish an empirical model, the events must satisfy the sheaf conditions, while the valuation must satisfy the no-disturbance condition. When these conditions are met, equivalence becomes possible, allowing one to explore the multi-agent paradox as contextuality using the tools of the Sheaf Approach.

8.2 Three Examples of Contextuality in Multi-Agent Scenarios

We will apply the previous results to analyze three well-known examples of multi-agent scenarios: Wigner's friend scenario, Frauchiger-Renner scenario, and Vilasini-Nurgalieva-del Rio scenario. Before delving into the actual examples, let's discuss common properties of these three scenarios.

The scenarios are formed by a set I of agents. We will use names for Wigner, his friend (Alice), and their duplicated versions (Ursula and Bob respectively).

The trust relation in all examples occurs between individual agents, simplifying the trust relations of Proposition 5 to the usual definition in Axiom 4. Furthermore, the trust relation is symmetric in all examples. Therefore, we can represent the multi-agent scenarios as empirical models. They have contexts with two measurements defining a covering of contexts as a graph, where the measurements are identified as the vertices and the maximal contexts as the edges. All measurements have two outcomes, defining four local events.

All examples begin with a system in a certain initial state (Wigner's friend and Frauchiger-Renner scenarios a quantum states, Vilasini-Nurgalieva-del Rio scenario a Popescu-Rohrlich box). As shown in Ref. [Vilasini, Nurgalieva and Rio 2019], the act of an agent measuring the state defines an isomorphism between the system and the agent, allowing us to ignore the system and deal only with agents. In the same reference, it is also shown that the scenarios satisfy the property called "information-preserving

[2] One can generalize the Sheaf Approach to deal with outcome-indeterminism [Wester 2018], but that is outside the scope of this part since the examples satisfy outcome-determinism.

memory update," which implies the same data being accessed by all the agents, i.e., there is trustworthiness among the agents who trust each other.

8.2.1 Wigner's Friend Scenario Is Noncontextual

The standard Wigner's Friend scenario is defined with Alice A performing a measurement on the system R, and with Wigner W describing R and A in an entangled state due to her previous measurement. It asks for the different points of view between Alice and Wigner in the fundamental description of the nature of the probabilities involved.

The scenario deals with an initial state $|\phi\rangle = \alpha |0\rangle + \beta |1\rangle$, with Alice's measurement in the basis $\{|0\rangle, |1\rangle\}$. The problem here is where to put the Heisenberg's cut, before or after Alice. From Alice's point of view, after her measurement, the state is in a classical probability distribution $p_R(0) = \alpha^2$ and $p_R(1) = \beta^2$, and if she has already observed the result, it is certain to be one given eigenvalue. However, from Wigner's point of view, R and A defines a system $R \otimes A$ in a superposition being described by $|\phi\rangle$, thus the system and therefore Alice are described by a quantum superposition of states.

There is no empirical contradiction here, as the classical probability distribution and the quantum state will give the same probabilities, and no discordance appears between Alice and Wigner. The problem that the Wigner's Friend scenario brings up is of an ontological nature: what is really happening with Alice?

Let's construct the empirical model of this scenario. Let Wigner perform a measurement in the system given by $R \otimes A$. We identify Alice and Wigner as the agents and we ignore the system R. There are two possibilities. The first one deals with Wigner's measurement being compatible with Alice's, which results in both of them trusting each other and defining a context

$$A \leftrightsquigarrow W. \tag{8.10}$$

The second possibility changes the basis in which Wigner performs his measurement to an incompatible one, for example $|+\rangle = \sqrt{\frac{1}{2}}\left(|0\rangle + |1\rangle\right)$ and $|-\rangle = \sqrt{\frac{1}{2}}\left(|0\rangle - |1\rangle\right)$. To Wigner, Alice's measurement is represented as a unitary transformation on $R \otimes A$ that changes Alice's state to a superposition. To him, the probability will be $p_{R\otimes A}(+) = \frac{(\alpha+\beta)^2}{2}$ and $p_{R\otimes A}(-) = \frac{(\alpha-\beta)^2}{2}$. To Alice, there is no probability at all if she already saw the measurement result and Wigner's measurement will just project the reduced state to his new basis. The problem here is that she knows her result, and Wigner erased it with his measurement, allowing no contradiction once the measurement erased Alice's memory as well[3].

3 There is the problem of how to do it with a macroscopic entity, but this is not the point here. Our objective is not to address the possibility of implementing the scenario but rather to identify the source of paradoxical behavior in a generic and formal manner.

Both possibilities allow analysis by considering only the measurement scenario. The first one has only one context, thus it must be noncontextual. The second one differs from the first by displaying two nonconnected contexts, making it a noncontextual empirical model. We can conclude:

Result 6. *Wigner's friend scenario is represented by an empirical model with nonconnected contexts, therefore it is noncontextual and, consequently, shows no multi-agent paradox.*

A realization of this result can be found in Ref. [Lostaglio and Bowles 2021], where the authors construct a noncontextual model for Wigner's friend scenario.

8.2.2 Frauchiger-Renner Scenario is Logic Contextual

The Frauchiger-Renner scenario [Frauchiger and Renner 2018] starts with an entangled state

$$|\phi\rangle = \sqrt{\frac{1}{3}}\,|0\rangle \otimes |0\rangle + \sqrt{\frac{2}{3}}\,|1\rangle \otimes \sqrt{\frac{1}{2}}\,(|0\rangle + |1\rangle)\,. \tag{8.11}$$

between two systems, R and S, measured in the basis $\{|0\rangle, |1\rangle\}$ by a respective friend, Alice A and Bob B. After measurement, $R \otimes A$ and $S \otimes B$ are isomorphic to agents A and B, respectively. As mentioned before, we can thus ignore the systems R and S. The system A measured by Ursula U and the system B measured by Wigner W are measured in the basis $\{|+\rangle, |-\rangle\}$, with $|+\rangle = \sqrt{\frac{1}{2}}\,(|0\rangle + |1\rangle)$ and $|-\rangle = \sqrt{\frac{1}{2}}\,(|0\rangle - |1\rangle)$. A locality argument can be used to describe who trusts whom. As we can ignore R and S, the agents are Alice, Bob, Ursula, and Wigner. Trust is symmetric, and Alice's (Bob's) measurement is incompatible with Ursula's (Wigner's) measurement. Thus we get $A \leftrightsquigarrow W$, $U \leftrightsquigarrow B$, $A \leftrightsquigarrow B$, and $U \leftrightsquigarrow W$.

Once we are given the outcomes of the measurements, we can define the possible worlds using the knowledge operators of each agent. The topology induced by the mutual knowledge E_I is generated by the elements of the basis, which consist of all 2^4 combinations of the outcomes from the four agents. The outcome of a single agent is represented by the union of all the elements of this basis that contains such an outcome. The valuation is given by the initial state, but can only be calculated for the set of agents which mutually trust. $|\phi\rangle_{A\leftrightsquigarrow B}$ can be written exactly like equation 8.11, while the state that will be measured by $U \leftrightsquigarrow W$ will be

$$|\phi\rangle_{U\leftrightsquigarrow W} = \sqrt{\frac{1}{12}}\,(|+\rangle + |-\rangle) \otimes (|+\rangle + |-\rangle) + \sqrt{\frac{1}{3}}\,(|+\rangle - |-\rangle) \otimes |+\rangle\,, \tag{8.12}$$

and for $U \leftrightsquigarrow B$

$$|\phi\rangle_{U\leftrightsquigarrow B} = \sqrt{\frac{1}{6}}\,(|+\rangle + |-\rangle) \otimes |0\rangle + \sqrt{\frac{1}{6}}\,(|+\rangle - |-\rangle) \otimes (|0\rangle + |1\rangle)\,, \tag{8.13}$$

and finally for $A \leftrightsquigarrow W$

$$|\phi\rangle_{A\leftrightsquigarrow W} = \sqrt{\frac{1}{6}}\,|0\rangle \otimes (|+\rangle + |-\rangle) + \sqrt{\frac{2}{3}}\,|1\rangle \otimes |+\rangle\,. \tag{8.14}$$

Calling the outcomes + and − of Ursula and Wigner respectively as 0 and 1, we can construct the table of the probabilities as shown in Table 10.

	00	01	10	11
$A \leftrightsquigarrow B$	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$
$A \leftrightsquigarrow W$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$	0
$U \leftrightsquigarrow W$	$\frac{3}{4}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$
$U \leftrightsquigarrow B$	$\frac{2}{3}$	$\frac{1}{6}$	0	$\frac{1}{6}$

Table 10 – Probabilities of the Frauchiger-Renner scenario. We call the outcomes + and − of Ursula and Wigner respectively as 0 and 1. Section 11 of the context $U \leftrightsquigarrow W$ does not have a global event. Like Hardy's model, it shows possibilistic contextuality.

Let's follow the sequence of trust presented in Ref. [Nurgalieva and Rio 2019]:

$$A \rightsquigarrow B \rightsquigarrow U \rightsquigarrow W \rightsquigarrow A. \tag{8.15}$$

If Ursula measures $|-\rangle$, then Bob must measure $|1\rangle$ since $p(10|U \leftrightsquigarrow B) = 0$. Consequently, Alice must measure $|1\rangle$ since $p(01|A \leftrightsquigarrow B) = 0$, and Wigner must measure $|+\rangle$ since $p(11|A \leftrightsquigarrow W) = 0$. However, as shown in Table 10, Wigner can measure $|-\rangle$ since $p(11|U \leftrightsquigarrow W) = \frac{1}{12}$, contradicting Ursula's conclusion of $p(11|U \leftrightsquigarrow W) = 0$. This is the violation presented in Ref. [Frauchiger and Renner 2018]. The assumptions in Ref. [Frauchiger and Renner 2018] are as follows:

- (Q) All agents use quantum theory.
- (C) Agents can use the results from another agent.
- (S) A measurement by an agent has an output defined for that agent.

The assumptions are an informal description of the definition of a multi-agent scenario, with assumption (S) defining the local events of each agent, assumption (C) connecting these events in global events, and assumption (Q) saying the valuation will be calculated by quantum mechanics. Such use of colloquial language was avoided by the formal construction of multi-modal logic multi-agent scenarios, but the conclusion remains: something in the assumptions must be weakened to explain the paradoxical behavior.

The empirical model can be constructed directly from Table 10, which is equivalent to the previous equivalence. The possible worlds are defined as the basis of the topology generated by the mutual knowledge E_I and identified as the global events. The empirical model that results from the valuation is non-disturbing, as one can directly verify, and contextual[4]. The possibilistic bundle diagram of Table 10 is given by Figure 8. The section 11 of the context $U \leftrightsquigarrow W$ does not have a possibilistic global event, and by imposing Ursula's conclusion, $p(11|U \leftrightsquigarrow W) = 0$, the induced possibilistic empirical model becomes noncontextual, showing that it is the cause of the possibilistic contextuality and equivalently the cause of the multi-agent paradox in the Frauchiger-Renner scenario. Thus we have the following conclusion:

Result 7. *Frauchiger-Renner scenario is mapped as an empirical model presenting logic contextuality, the result of its multi-agent paradox with quantum origin.*

As mentioned earlier, from the Result 5, we conclude that if we extend the possible worlds to encompass all local sections indexed by their contexts, the paradox also disappears. However, indexing leads to lambda-dependence, a non-classical property that ultimately embodies contextuality. This affirms a claim made in Ref. [Nurgalieva and Rio 2019] stating that the inclusion of contexts as data in propositions avoids logical contradictions in the Frauchiger-Renner scenario.

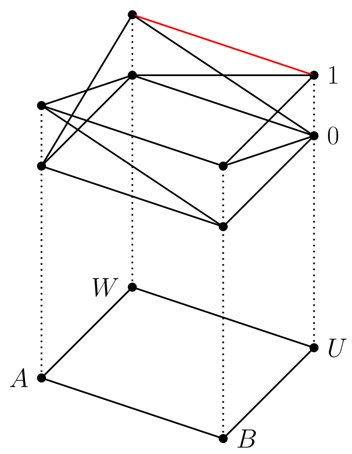

Figure 8 – Possibilistic bundle of the Frauchiger-Renner scenario.

8.2.3 Vilasini-Nurgalieva-del Rio Scenario is Strongly Contextual

Another example is the Vilasini-Nurgalieva-del Rio scenario [Vilasini, Nurgalieva and Rio 2019]. It generalizes the conditions for multi-agent paradoxes in generalized

[4] Using the noncontextual fraction [Abramsky, Barbosa and Mansfield 2017], one can find $NCF = \frac{5}{12}$.

probability theories using modal logic and explicitly constructs a paradox for the box world. The construction of the multi-agent scenario in Ref. [Vilasini, Nurgalieva and Rio 2019] has equivalent assumptions to those described at the beginning of the section, in addition to requiring treatment of states, effects, and the channels that the trust relation defines between the agents, all using the formalism of Generalized Probabilistic Theories [Janotta and Hinrichsen 2014, Müller 2021, Selby, Scandolo and Coecke 2021] applied in the box world defined with Popescu-Rohrlich boxes [Popescu and Rohrlich 1994]. Here, we will limit ourselves to what is necessary for the calculation of valuations and the construction of the empirical model, leaving the original article as a reference for the detailed construction of the multi-agent scenario.

The structure of the agents, trust relation, and the possible worlds is identical to the one presented in the Frauchiger-Renner scenario. We have two systems R and S, two friends Alice A and Bob B, Wigner W and Ursula U, with a symmetric trust relation given by $A \leftrightsquigarrow W$, $U \leftrightsquigarrow B$, $A \leftrightsquigarrow B$, and $U \leftrightsquigarrow W$, while the systems can be ignored due to their isomorphism with the friends. Each measurement will have two outcomes, defining the 2^4 possible worlds given by E_I.

The valuation follows the initial state given by the sharing of a Popescu-Rohrlich box between R and S, thus satisfying $X_i X_j = x_i \oplus_{mod2} x_j$ with X_i measurements and x_i outcomes. The authors of [Vilasini, Nurgalieva and Rio 2019] show that all pairs of agents trusting each other can be understood as being correlated by Popescu-Rohrlich boxes. By using trustworthy and fixing the conditions $X_U = X_A \oplus_{mod2} 1$, $X_W = X_B \oplus_{mod2} 1$, the measurements $X_A = X_B = 0$ and the outcomes $x_i \in \{0, 1\}$, we can propagate the initial correlation between R and S to obtain the possibilistic values presented in Table 11.

	00	01	10	11
$A \leftrightsquigarrow B$	1	0	0	1
$A \leftrightsquigarrow W$	1	0	0	1
$U \leftrightsquigarrow W$	0	1	1	0
$U \leftrightsquigarrow B$	1	0	0	1

Table 11 – Prossibilities of the Vilasini-Nurgalieva-del Rio scenario. It defines the well-known Popescu-Rohrlich box empirical model, showing the Liar Cycle paradox with four agents. It is strong contextual once all local sections show violations, thus making it stronger than the previous example.

As shown in Ref. [Vilasini, Nurgalieva and Rio 2019], all agents find a contradiction in any chosen sequence of agents, presenting a stronger violation than the one presented by the Frauchiger-Renner scenario. Using the same assumptions as the Frauchiger-Renner scenario, with the necessary modification that in (Q) the agents use the box world, at least one of them would need to be violated to explain the paradoxical behavior.

The identification with an empirical model follows the exact same construction to the one for the Frauchiger-Renner scenario, but now we are dealing with possibilistic values, thus allowing a faithful representation of Table 11 as the bundle diagram in Figure 9. It defines the well-known Popescu-Rohrlich box empirical model, showing the Liar Cycle paradox with four agents. Once all local sections show violations,thus making it stronger than the empirical model of the previous example, we have the following conclusion:

Result 8. *Vilasini-Nurgalieva-del Rio scenario is mapped as an empirical model known as Popescu-Rohrlich box empirical model, a main example of strong contextuality, the result of its multi-agent paradox with post-quantum origin.*

Since they share the same measurement scenario, both the Frauchiger-Renner scenario and the Vilasini-Nurgalieva-del Rio scenario have the same set of possible worlds given by $\mathcal{B}_{D_I}$. Similarly to the previous example, we can use Result 5 to rectify the paradoxical behavior at the expense of lambda-dependence.

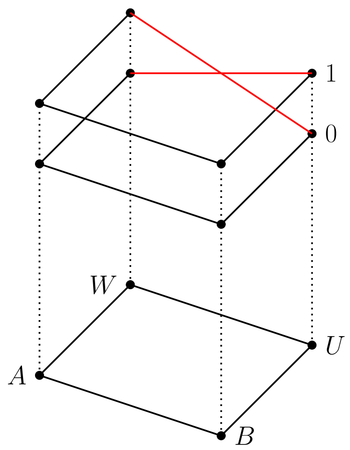

Figure 9 – Possibilistic bundle of the Vilasini-Nurgalieva-del Rio scenario.

9 Discussion of Part III

Multi-agent paradoxes and the phenomenon of contextuality are examples of systems with local, marginal access to a given global whole that cannot be adequately explained in a classical manner. Alfred Korzybski's statement, "a map is not the territory" [Korzybski 1933], serves as a reminder that even when dealing with the classical world, we need to keep in mind that we do not have the capacity to discern all the details of the "territory"; we are only capable of constructing "maps." An agent's knowledge only extracts parts of a "map" of the "territory" we call physical reality. This aligns with Refs. [Santo and Gisin 2019, Gisin 2019, Gisin 2021] regarding the ontological assumptions made in physical theories when using mathematical objects that do not align with a purely empiricist viewpoint and how such assumptions can be understood as the hidden variables of classical theory. The non-classical world imposes an even greater limitation on us by isolating us in classical islands, which, much like the concept of charts in the theory of manifolds, can only be connected through an atlas but lose something essential that a map of a single chart possesses. In a non-classical world, we not only have to remember that a map is not the territory, but also that the map is merely a chart in an atlas. There is something beyond what we can individually perceive with our classically limited perspective, making Korzybski's statement even more imperative.

The main point we obtained was that multi-modal logic can be used to handle non-classical scenarios, provided that due care is taken with the agents' knowledge. During our journey to build the map between empirical models and multi-agent scenarios, we encountered other results. Here are the results we have obtained in this part:

- We generalized the concept of trust to also apply to sets of agents.
- We have identified a construction of a fundamental truth from the trust relation between agents, which is used in the literature to weaken the Truth Axiom, thus recovering the Truth Axiom from trust.
- We have translated the components of an empirical model (measurements, contexts, events) into the components of a multi-agent scenario (measurement by an agent, equivalence trust relation, elements of the basis topology of a knowledge operator), exposing the limitation of such a mapping to describe a generic multi-agent scenario as an empirical model.
- We have shown that contextuality is the violation of soundness, precisely the violation that causes multi-agent paradoxes.

- We have demonstrated that contextuality only appears because of the imposition of the mutual knowledge operator as the generator of events, imposing exactly the classicality captured by noncontextuality.
- By allowing the operator that generates events to be that of distributed knowledge, we have shown that we recover soundness at the cost of lambda-dependence, proving a generalized version of the conjecture in Ref. [Nurgalieva and Rio 2019] that says the inclusion of contexts as data of propositions avoids logical contradictions.
- We have translated the three main examples of multi-agent scenarios, Wigner's Friend, Frauchiger-Renner, and Vilasini-Nurgalieva-del Rio scenarios, into empirical models and identified their types of contextuality that generate their multi-agent paradoxes.

These results enable the creation of new multi-agent paradoxes and the application of mathematical tools from the Sheaf Approach to contextuality in these scenarios. The discussed examples in section 8.2 illustrate how the map can identify paradoxes in a clear manner, with the possibility of even quantifying them using tools already known in the literature on contextuality. It's interesting to highlight the types of contextuality that emerge from the explored example scenarios. In the Wigner's Friend scenario, no contextuality appears, clearly indicating its non-empirical nature. On the other hand, the Frauchiger-Renner scenario exhibits logical contextuality, demonstrating its already noted similarity to the well-known Hardy's paradox. The Vilasini-Nurgalieva-del Rio scenario demonstrates strong contextuality, as expected since it was constructed with Popescu-Rohrlich boxes. The results also facilitate the translation of contextual empirical models into multi-agent scenarios with probabilistic paradoxes, and possibilistic contextual empirical models into logical multi-agent paradoxes.

The examples show us that paradoxes and contextuality are the same phenomenon, at least in scenarios that accept the inverse map, and through their valuations, it is evident that there is more knowledge than the mutual one. They point to the argument we use to construct events, which is that we cannot define the worlds our logic will work out, but by the knowledge we can explore and refine the worlds we can have access to in a more empiricist and relational sense. Distributed knowledge is the finest way to understand what is happening, as it encodes all the data in the propositions, safeguarding our classical logic from the non-classical nature of phenomena. It also demonstrates that we have more data than classical mutual knowledge, more possible worlds, and, as we can observe today with quantum technology, more resources to explore.

An immediate path for future research would be to seek generalizations of the Sheaf Approach, as explored in Ref. [Gogioso and Pinzani 2021, Gogioso and Pinzani 2023, Abramsky, Barbosa and Searle 2023], in order to expand the scope of application of the map constructed here. The ultimate goal is to enable the representation of every multi-agent scenario with empirical models, allowing the use of sheaf theory tools to investigate their non-classicality. Another avenue is to leverage the relationship that the Sheaf Approach has with other approaches to the phenomenon of contextuality, as organized in Ref. [Masse 2021] and references therein, to describe multi-agent paradoxes in these languages. In particular, utilizing their different domains of application to further extend the map constructed here. Both of these aforementioned paths would be highly valuable for formalizing the language of protocols that present extended versions of the Wigner's friend scenario, especially those with a causal structure [Schmid, Yīng and Leifer 2023]. While an extension of the map between empirical models and multi-agent scenarios is not yet constructed, it would be interesting to identify and develop examples that allow analysis through the standard Sheaf Approach, with ideal candidates already existing in Ref. [Leegwater 2022, Schmid, Yīng and Leifer 2023, Walleghem et al. 2023, Walleghem et al. 2024]. Furthermore, the use of multi-modal logic and the construction done here using topological semantics and pointless topology may shed light on the formalization of interpretations of quantum theory, such as the relational interpretation in Ref. [Lawrence, Markiewicz and Żukowski 2023, Laudisa and Rovelli 2024].

10 Final Considerations

As the title of this thesis suggests, it presents a journey. The initial proposal to use topology and geometry to find generators of quantum contextuality began with the Bundle Approach, its diagrammatic representation, and the use of holonomy to detect contextuality. This evolved through a transitional phase in the Sheaf Approach and cohomological detection methods, culminating in the development of a differential approach based on the Generalized Contextuality Approach. This approach identifies generators of contextuality for an ontic representation and enables a more controlled exploration of applications. The journey was inspired by the similarity between topological manifolds and empirical models, both problems of marginalization seeking to extend from local to global, dealing with the phenomenon where such extension is not always possible.

In this thesis, we have seen several results, starting with the concept of n-contextuality and the construction of examples of empirical models with contextuality that depend on the dimension of topological failures. We not only constructed an example with a tetrahedron scenario, but also identified a generic model based on the Greenberger-Horne-Zeilinger state for all levels of the hierarchy within quantum theory. It would be interesting to further explore this hierarchy, although the usual topology of a measurement scenario is not suitable to advance the endeavor of using topology to handle contextuality, leaving all this construction only as an indication of the influence that the topology of the measurement scenario has on the contextuality of an empirical model.

The difficulty in studying topology is not limited to the influence mentioned above, but also in identifying indicators of contextual behavior. All suffer from the problem of imposing group structure, which leads to violations of characterization. This indicates that the fundamental objects chosen to construct the model need to be changed, not just the topology on them. Hence the shift to an approach where events become fundamental and measurements are merely special sets of events. With this change, which generalizes contextuality to deal with more parts of a process and admits the study of situations that violate the outcome-determinism condition, we can ground contextuality in a clear philosophical principle and represent it, now from a certain ontic perspective, as nothing more than the non-preservation of loops by probabilistic valuations.

Contextuality, now viewed through a generator, the contextual form, allows us to explicitly articulate choices made both in interpretations of quantum theory and in approaches to contextuality, finding it in other non-classical phenomena. The most important lesson from the Differential Approach is not its construction itself, but where it situates contextuality, as the corrective term of a description with classical ontology. This privileged position places contextuality as an object beyond interpretations, capable of taking many forms. It places this phenomenon at the heart of what makes classical models so different from non-classical ones. It is contextuality that prevents completing empirical access with hidden variables, a natural occurrence in any theory allowing classical ontic representation. On the other hand, this gives us the freedom to treat contextual behavior in fundamentally different ways, choosing the most suitable approach to address the process in question. Topological characterization is regained, with cohomology highlighted, while geometry can be explicitly expressed through holonomy, depending on the chosen viewpoint. The group structure that previously caused problems becomes natural in the Differential Approach, clearly exposing the influence of topology identified in measurements and contexts.

It is in the description as a logical paradox that contextuality takes its original form, and it is natural for its study to develop in the logical formalism of knowledge and in the relationship of trust between sets of agents. Even more natural is the topological structure of propositions, maintaining the fundamental objects used in the developed approach. It is in this type of mathematical structure that contextuality presents itself masterfully, translating measurements, contexts, and outcomes into propositions that express their topological contextuality. Manifesting as a failure in the relationship between semantics and syntax, it suggests that reality has gaps in propositions that do not exist, true "holes" in a classical view of reality. If we want to close these gaps and restore classical sanity to knowledge, we must pay the price that contexts are expressed into the probabilistic valuations. It is our choice to expand possible worlds so that there are no gaps, or to accept living with these gaps for empirical purity. Either way, there is no escape; contextuality will appear.

The end of this journey is only the beginning of many others. Paths for future research are diverse, and we will describe some of them below:

- A topological Vorob'ev theorem was described in this thesis. Detailing its construction would be a first step to understand the influence that the topology of measurement scenarios seems to exert. The general idea is quite simple: just identify Graham's reduction in processes with a single state and no transformations. Effectively, we will extract from the valuation function the generators of contextuality, i.e., the contextual form of the scenario.

- The Bundle Approach has great elegance due to the use of bundle theory to model physical theories. Its encoding in operational structure, even if restricted to effects, has interesting potential. The fibers will no longer be made of outcomes, as these are now at the base of the bundle as effects, but will be given by valuation. Contextuality will ultimately be expressed in a principal bundle, with direct influence of the base's topology on valuation functions.

- The failure of the Sheaf Approach already indicates a path to be pursued, expanding its formalism to more complicated situations that include different kinds of relations between its constituents. Much work is already being done in this direction. The contexts themselves must be modified, and even the sheaf of events will end up changing entirely in meaning. Once in the hands of such a simple approach, we can explore contextuality in a much larger set of already studied phenomena.

- Quantifiers for contextuality are important to know how much of this resource we have in a model. Their axiomatic construction would be of great importance in future technological applications. Especially if quantifiers enable their use in situations where disturbance, outcome-indeterminism, and even context dependence occur. Such situations occur to a greater or lesser extent in the laboratory and need to be controlled. A formally constructed quantifier of contextuality would be an excellent tool, but it needs a theoretical background like those described above to be built.

Research in contextuality and its topological and geometric representations can bring new tools to deal with this behavior, which, in our classical view, is strange. It will allow us to apply this knowledge to technological tools that are still being imagined and contribute to the debate on how best to deal with non-classicality, essential for the development of our future understanding of reality.

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