UNIVERSIDADE ESTADUAL DE CAMPINAS
Instituto de Matemática, Estatística
e Computação Científica

YGOR ARTHUR CESAR DE JESUS

CONTRIBUTIONS TO THE THEORY OF PARTIALLY HYPERBOLIC FLOWS: CONSTRUCTION OF GEODESIC FLOWS AND ERGODICITY CRITERION

CONTRIBUIÇÕES À TEORIA DE FLUXOS PARCIALMENTE HIPERBÓLICOS: CONSTRUÇÃO DE FLUXOS GEODÉSICOS E CRITÉRIO DE ERGODICIDADE

CAMPINAS
2025

YGOR ARTHUR CESAR DE JESUS

CONTRIBUTIONS TO THE THEORY OF PARTIALLY HYPERBOLIC FLOWS: CONSTRUCTION OF GEODESIC FLOWS AND ERGODICITY CRITERION

CONTRIBUIÇÕES À TEORIA DE FLUXOS PARCIALMENTE HIPERBÓLICOS: CONSTRUÇÃO DE FLUXOS GEODÉSICOS E CRITÉRIO DE ERGODICIDADE

Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas (UNICAMP) como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

Thesis presented to the Institute of Mathematics, Statistics and Scientific Computing of the Universidade Estadual de Campinas (UNICAMP) in partial fulfillment of the requirements for the degree of Doctor in Mathematics.

Advisor: Gabriel Ponce

Coadvisor: Ali Tahzibi

Este trabalho corresponde à versão final da tese defendida pelo aluno YGOR ARTHUR CESAR DE JESUS, e orientada pelo Prof. Dr. Gabriel Ponce

Campinas

2025

Ficha catalográfica
Universidade Estadual de Campinas (UNICAMP)
Biblioteca do Instituto de Matemática, Estatística e Computação Científica
Ana Regina Machado - CRB 8/5467

	Jesus, Ygor Arthur Cesar de, 1994-
J499c	Contributions to the theory of partially hyperbolic flows : construction of geodesic flows and ergodicity criterion / Ygor Arthur Cesar de Jesus. – Campinas, SP : [s.n.], 2025.
	Orientador: Gabriel Ponce. Coorientador: Ali Tahzibi. Tese (doutorado) – Universidade Estadual de Campinas (UNICAMP), Instituto de Matemática, Estatística e Computação Científica.
	1. Fluxos geodésicos. 2. Hiperbolicidade parcial. 3. Teoria de Pesin. 4. Deformação de métricas riemannianas. I. Ponce, Gabriel, 1989-. II. Tahzibi, Ali. III. Universidade Estadual de Campinas (UNICAMP). Instituto de Matemática, Estatística e Computação Científica. IV. Título.

Informações complementares

Título em outro idioma: Contribuições à teoria de fluxos parcialmente hiperbólicos : construção de fluxos geodésicos e critério de ergodicidade
Palavras-chave em inglês:
Geodesic flows
Partial hyperbolicity
Pesin theory
Deformation of Riemannian metrics
Área de concentração: Matemática
Titulação: Doutor em Matemática
Banca examinadora:
Gabriel Ponce [Orientador]
Boris Hasselblatt
Paulo Regis Caron Ruffino
Pedro Jose Catuogno
Daniel Smania Brandão
Data de defesa: 08-07-2025
Programa de Pós-Graduação: Matemática

Objetivos de Desenvolvimento Sustentável (ODS)
ODS: 4. Educação de qualidade
ODS: 9. Inovação e infraestrutura
ODS: 17. Parcerias e meios de implementação

Identificação e informações acadêmicas do(a) aluno(a)
- ORCID do autor: https://orcid.org/0000-0003-4117-1212
- Currículo Lattes do autor: http://lattes.cnpq.br/8848953914394407

Prof. Dr. GABRIEL PONCE

Prof. Dr. BORIS HASSELBLATT

Prof. Dr. PAULO REGIS CARON RUFFINO

Prof. Dr. PEDRO JOSE CATUOGNO

Prof. Dr. DANIEL SMANIA BRANDÃO

A Ata da Defesa, assinada pelos membros da Comissão Examinadora, consta no SIGA/Sistema de Fluxo de Dissertação/Tese e na Secretaria de Pós-Graduação do Instituto de Matemática, Estatística e Computação Científica.

A todos aqueles que foram podados de boas oportunidades.

Agradecimentos

Primeiro agradeço a minha mãe, Josineide Lopes da Silva. A mulher mais sábia, forte, bondosa que conheço e que me mostrou o poder da educação. Obrigado por cozinhar condições para que eu pudesse trilhar este caminho.

Ao meu irmão mais velho, protetor e cuidador, Daniel Cesar de Jesus. Obrigado por ser a minha referência intelectual desde a infância.

A minha querida esposa Tami. Obrigado por tanto amor, companheirismo, fé e incentivo durante todo o processo. Obrigado por ser um lar e porto seguro durante tempos tão sombrios.

À toda minha família que sempre me deu muito apoio, amor e carinho. Minha querida avó Edileide, minhas tias Renata, Kelly e Tatiana, meus primos Kety, Kauã, Geovana, Bruno e Julio.

A todos os meus professores desde o ensino básico até a conclusão do doutorado. Em particular à professora Lusia que, na escola municipal Brigadeiro Haroldo Veloso de Itaquera, zona leste de São Paulo, me ensinou a ler e com isso me abriu para um mundo novo de conhecimentos. Obrigado por ter sido a primeira a enxergar algum potêncial em mim.

Ao meu grande amigo e especialista em deformação de métricas Riemannianas Leonardo Cavenaghi. Obrigado pelas conversas, recomendações e conselhos para melhorar este trabalho. Em particular, agradeço por ter me apresentado deformações conformes, pois essa foi a técnica que ajudou a produzir muitos dos meus resultados.

Aos meus grande amigos dinâmicos Pedro Mattos e Daniel Gomes pela parceria nesses mais de 10 anos de Unicamp. Foi um prazer tê-los como colegas de turma e fico muito agradecido por diversas conversas sobre matemática, ciencia, conhecimento e bobeiras.

Aos meus orientadores Prof. Gabriel Ponce e Prof. Ali Tahzibi pelo apoio, ajuda, paciência e confiança no meu potencial. Obrigado por iterarem a minha órbita até conjuntos abertos de oportunidades. Agradeço muito pelo incentivo a explorar, aprender e me maravilhar com dinâmica.

A todos os funcionários da UNICAMP que a mantêm operando e gerando conhecimento, desenvolvimento e serviço à sociedade.

Ao Prof. Federico Rodriguez-Hertz por ser meu anfitrião durante meu estágio de pesquisa na Penn State University. Agradeço por sua extrema gentileza e apoio, bem como seus conselhos e sugestões. Estendo meus agradecimentos a todo grupo de sistemas dinâmicos da Penn State, em particular, meu queridos colegas alunos latinos Sebastian Burgos, Sebastian Molina, Nicolás Ramos e Angela Flores por todo incentivo e conversas frutíferas.

Ao Prof. Enrique Pujals pela disposição e empenho em me receber em Nova York para discutirmos meu trabalho. Foi um momento muito significativo na minha trajetória acadêmica ter esta oportunidade.

Aos meus colaboradores Marcielis Espítia, Sergio Romaña, Luis Piñeyrua e Gabriel Ponce por me ensinarem tanto e pela oportunidade de desenvolvermos pesquisa juntos. Isso tornou todo o processo muito mais divertido.

O presente trabalho foi realizado com apoio do Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), processo número 140518/2021-5, e Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) via os processos de número 2021/02913-0 e 2023/05100-5.

“Que no meio do caminho da educação havia uma pedra.
E havia uma pedra no meio do caminho.”
— Criolo, *Duas de Cinco* (2014)

“Lembra a vó, ó, dá mó dó.
Criança na periferia vive sem estudo e só.
A mercê da mó’, two, three, sabó’.
[...]
Destino indica a correria de um homem.
Alternativa pra criança aprender basta quem ensina.
Essa é a verdade, criança aprende cedo a ter caráter.”
— Sabotage, *Canão Foi Tão Bom* (2016)

Resumo

Esta tese investiga as propriedades ergódicas de fluxos que exibem diversas formas de comportamento hiperbólico, com enfoque em dinâmica parcialmente hiperbólica e não uniformemente hiperbólica. Uma classe central de exemplos considerados é a de fluxos geodésicos. Neste contexto, a tese apresenta a construção de diversos exemplos de fluxos geodésicos parcialmente hiperbólicos, mas não Anosov, com propriedades geométricas e ergódicas ricas, bem como propriedade misturadora, unicidade de medidas de máxima entropia e expansividade. Além disso, estuda como a variação da dinâmica do fluxo geodésico se relaciona com deformações da métrica riemanniana. Além de fluxos geodésicos, a tese também explora fluxos mais gerais analizando a relação entre classes homoclínicas, ergodicidade e unicidade de medidas de Sinai-Ruelle-Bowen (SRB).

Keywords: Fluxos geodésicos, dinâmica parcialmente hiperbólica, Teoria de Pesin, Deformação de métricas riemannianas.

Abstract

This dissertation investigates the ergodic properties of flows exhibiting various forms of hyperbolic behavior, with a focus on partially hyperbolic and non-uniformly hyperbolic dynamics. A central class of examples considered is that of geodesic flows. In this context, the dissertation presents the construction of several non-Anosov, partially hyperbolic geodesic flows exhibiting rich ergodic and geometric features, including mixing, uniqueness of the measure of maximal entropy, and expansivity. Furthermore, it examines how the dynamics of geodesic flows relates to deformations of the underlying Riemannian metric. Beyond geodesic flows, the dissertation also explores more general flows, analyzing the interplay between homoclinic classes, ergodicity, and the uniqueness of Sinai–Ruelle–Bowen (SRB) measures.

Keywords: Geodesic flows, partial hyperbolicity, Pesin Theory, deformation of Riemannian metrics.

Contents

Introduction **13**

1 Preliminaries **25**

1.1 Riemannian Geometry . . . 26

1.1.1 Basic concepts . . . 26

1.1.2 Conformal metrics . . . 38

1.1.3 Locally symmetric metrics . . . 41

1.2 Geodesic Flow . . . 45

1.3 Partial Hyperbolicity . . . 57

1.4 Nonuniform hyperbolicity . . . 62

1.4.1 Sinai-Ruelle-Bowen measures . . . 70

2 Partially hyperbolic geodesic flows **73**

2.1 Setting up the deformation . . . 76

2.2 For which h the Anosov Property is broken? . . . 78

2.3 The new geodesic flow is partially hyperbolic . . . 82

2.3.1 Cone invariance for the initial metric . . . 84

2.3.2 Cone invariance for the deformed metric . . . 88

2.3.2.1 Cone invariance for parallel geodesics . . . 91

2.3.2.2 Cone invariance for s-almost parallel geodesics . . . 95

2.3.2.3 Cone invariance for s-transversal geodesics . . . 96

2.3.3 Proof of Theorem A . . . 97

2.4 Proof of Theorem B . . . 99

2.4.1 Curvature over the central geodesic . . . 101

2.4.2 Curvature outside the central geodesic . . . 102

2.5 Further results and questions 105

3 Ergodic Homoclinic Classes for Flows **115**

3.1 Typical points for smooth and SRB measures 117

3.2 Proof of Theorem C 119

3.2.1 Proof of the second statement 119

3.2.2 Proof of the first Statement 121

3.3 Proof of Theorems D and E 123

3.4 Proof of Theorem F 124

3.5 Proof of Theorem G 125

3.5.1 Further questions 125

Bibliography **127**

Introduction

A dynamical system can be understood as the process of iteration of certain rules on a fixed phase space. Typically, the study of dynamical systems focus in understanding the behavior of the possible states of a particle after a large number of iterations of these rules. As an illustration, let us consider the following fanciful situation: consider that T is a ghost train traveling along 5 ghost cities with stations named C_i, for $i = 1, ..., 5$, placed as a circular ghost railway. Our imaginary ghost train never stops, but at the ghost stations C_i. It moves all the way from station C_1 to station C_5 stopping at each station in between and then it goes from C_5 to C_1. We can think this motion of the ghost train as a dynamical system for which the "rule of motion" is simply moving to the next station. In this case, the phase space is the set $M = \{C_1, C_2, C_3, C_4, C_5\}$ and if we call the application of the rule "next station" by f, this dynamical system has a single orbit, i.e. for each $x \in M$ the set of iteration $\{f^n(x)\}_{n=1}^{\infty}$ coincides with M. Of course, this is a very simple model and we are interested in more general examples of dynamical systems, but it illustrates well this notion of "movement" and "moving to the next step" associated to dynamical systems. In fact, during this dissertation we will be mostly interested in study a particular family of dynamical systems called *Geodesic flows*.

The *Geodesic flow* is a dynamical systems naturally associated to any Riemmannian metric and is a very important model for several properties and techniques known nowadays in the class of *Smooth Dynamical Systems*. Given a Riemannian manifold (M, g), the *Geodesic flow* is dynamical system with phase space the tangent bundle of M (or the unit tangent bundle) that associates for each tangent vector v the velocity of the geodesic with starting velocity v (check Section 1.2 for the formal definition). The study of the dynamics of the geodesic flow lead to the understanding of several more general chaotic systems and their behavior from the statistical point of view, mainly studied as *Ergodic theory*.

The starting point of *Ergodic theory* is commonly understood as the formulation of the so-called *Boltzmann's Ergodic Hypothesis* and is dated back to the 19th century's last decades. In modern language (although it was not exactly formulated like this), if $f : M \to M$ is a diffeomorphism, $U \subset M$ is a measurable set and μ a $f-$invariant probability measure ($f^*\mu = \mu$ and $\mu(M) = 1$) then the mean sojourn time

$$\tau(x, U) := \lim_{n\to\infty} \frac{\#\{j = 0, ..., n-1 : f^j(x) \in U\}}{n}$$

should coincide with the measure of the set U for almost every point x (check [VP91] for a more complete historical exposition of the *Boltzmann's Ergodic Hypothesis*). If this hypothesis holds for every measurable set U, then we say that the dynamical system (f, μ) is *Ergodic*. Sometimes, in the literature, ergodicity of a dynamical system can be stated as "f is ergodic with respect to the measure μ" or "μ is an ergodic measure for the diffeomorphism f", thus let us fix it here that all this possible formulations mean the same definition as above. Beyond several interesting equivalent conditions to the definition above is the notion that ergodic systems can not be "broken" into subsystems which do not interact with each other. More precisely, ergodicity is equivalent to the statement "any measurable set A that is invariant by the dynamics ($f^{-1}(A) = A$) necessarily satisfies $\mu(A) \in \{0, 1\}$".

Given a closed and connected Riemannian manifold (M, g), one of the oldest open problems in the area of dynamical systems and ergodic theory is determining conditions that guarantee ergodicity of the geodesic flow with respect to the *Liouville measure*. A classical result states that the geodesic flow is of *Anosov* type when all the sectional curvatures are strictly negative. This connection is indeed the first model of Anosov systems and the origin of the well-known (and still standard) technique called *Hopf Argument*, which is very useful for obtaining ergodicity. It is still not a completely solved question whether the geodesic flow for a surface of genus $g \geq 2$ and non-positive curvature is ergodic with respect to the *Liouville measure*. However, Pesin obtained great advances in [Pes77b]. In this work, Pesin proves that if the action of the geodesic flows is restricted to vectors for which the curvature is negative along its underlying geodesic, then this flow is conjugated to a Bernoulli flow, and in particular, it is ergodic.

The analogous question in higher dimensions is determining the ergodicity for non-positively curved Riemannian metrics that are rank one. The rank of a tangent vector $v \in T_xM$ is the dimension of the space of parallel orthogonal *Jacobi Fields* along the geodesic $\gamma_{(x,v)}(t)$ and the rank of the manifold (M, g) is defined as the infimum of the rank of the unitary tangent vector. Celebrated works from the 80's, such as [BB82] and [Bur83], explored the relation between the ergodicity of the geodesic flow and the rank of the metric. They concluded, for example, that the restriction of the geodesic flow to the set of rank one vectors is also Bernoulli.

It is natural to ask if non-positivity of the curvature is essential to obtain ergodicity for the geodesic flow. In [BG89] and [Don06] a negative answer to this question is given. They obtain examples of ergodic geodesic flows in surfaces with some positive curvature. It is worth mentioning that, in their case, the presence of negative curvature is still essential to obtaining ergodicity. It is also not true that the Anosov property for the geodesic flow must imply negative curvature, indeed in [Ebe73] Eberlein states several equivalent conditions to the Anosov property. Essentially negative curvature must appear along any geodesic, but it does not prevent positive curvature from existing. In fact, in [Gul75], Gulliver constructs examples of Anosov geodesic flows such that the underlying metric has some positive sectional curvatures.

The time one map (g_1) of an Anosov geodesic flow (g_t) is one of the most classical examples of partially hyperbolic systems. By partial hyperbolicity we mean that there is a continuous splitting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$ into three non-trivial invariant subbundles, where E^s is uniformly contracted for the future, E^u is uniformly contract for the past and E^c has an intermediate behavior, i.e. it does not contracts nor expands as much as the other two. It is a famous conjecture by Pugh and Shub the genericity of ergodic systems inside this class (check [RHRHTU11a], [BRHRH$^+$08] and [BW10]). According to the Pugh and Shub program, this main conjecture may be a consequence of another two conjectures: accessibility is dense among partially hyperbolic systems and accessibility implies ergodicity.

It is well known that geodesic flows are examples of contact flows, i.e. the geodesic vector field is the Reeb flow of the contact form induced by the Riemannian metric. In [CP14] Carneiro and Pujals constructed the first class of examples of partially hyperbolic geodesic flows and in [FH22] Fisher and Hasselblatt proved that is possible to perturb any contact form for which the Reeb flow is partially hyperbolic to make it accessible. However, there is a lack of examples in this class of systems and one of the consequences of the present work is to present new examples and the study of new techniques to produce them. Besides the genericity of the ergodic property in some classes inside the partially hyperbolic setting, it is a delicate problem to produce examples of ergodic systems of some predetermined class that are not of Anosov type. It is even more delicate to produce such examples in the class of geodesic flows. In the two-dimensional setting, there are simple constructions one can think of, for example, the surface of revolution obtained by rotating the graph of $f(x) = x^4 + 1$, defined in some interval $[-a, a]$, around the $x-$axis and then gluing two negatively curved surfaces to this neck. The obtained surface is non-positively curved, and moreover it has negative Gaussian curvature besides a central closed geodesic at $x = 0$ for which the Gaussian curvature is identically zero. Such an example has a Bernoulli geodesic flow that is not of Anosov type by the previously mentioned works of Pesin and Eberlein. This example also presents more interesting dynamical properties, for example the decay of correlations studied in [LMM24]. One realizes very fast that this construction of examples by rotating the graph of a function cannot be generalized to higher dimensions to obtain non-positively curved manifolds with a single geodesic with vanishing sectional curvatures, hence different techniques and approaches are needed.

In [Rug91a] Ruggiero proved that the C^2-interior of the set of Riemannian metrics without conjugate points, say $\mathcal{NC}(M)$, coincides with the set of Riemannian metrics for which the geodesic flow is of Anosov type, say $\mathcal{A}(M)$. It is still not well understood the set of metrics on the boundary of $\mathcal{NC}(M)$ nor the relation with the expansivity of the geodesic flow. We say that a continuous flow is expansive if two different orbits eventually grow apart. It is known in the two-dimensional setting that expansivity of the geodesic flow implies the absence of conjugate points (see [Pat93]), however, it is a *Conjecture* attributed to Ricardo Mañe that this result should be true for higher dimension as well. Notice that the converse is not true, since we can consider a piece of flat cylinder and glue two negatively curved surfaces to the boundary of this flat neck. For this construction we have that for every $\varepsilon > 0$, there exist two closed geodesics $\gamma_1(t)$ and $\gamma_2(t)$ in the cylinder that are $\varepsilon-$close for every $t \in \mathbb{R}$. This phenomenon is detected by the *Flat Strip Theorem* which states that if the geodesic flow is not expansive, then for every $\varepsilon > 0$ there exist two distinct geodesics γ_1 and γ_2 such that $d(\gamma_1(t), \gamma_2(t)) < \varepsilon$, for all $t \in \mathbb{R}$, thus we can find a a flat strip on the universal cover, i.e. an isometric copy of an $[-\delta, \delta] \times \mathbb{R}^n$. Therefore, to produce expansive geodesic flows that are not of Anosov type it is necessary to have some control on the amount of zero curvature in the non-positively curved setting. It is interesting to remark that in [Rug97] Ruggiero proves that expansive geodesic flows with no conjugated points have density of periodic points and a local product structure, which implies they are topologically transitive, that is they admit a dense forward orbit. Hence, examples of such geodesic flows are rich from the geometric and dynamical point of view.

Just like the ergodicity in the non-Anosov case, there is a shortage of higher dimensional examples of expansive geodesic flows admitting some zero sectional curvatures. A natural attempt would be starting with two negatively curved manifolds, say (M_1, g_1) and (M_2, g_2), and then considering the Riemannian product manifold $(M_1 \times M_2, g_1 + g_2)$. In this case, the product metric admits several vanishing sectional curvatures and it is not even partially hyperbolic as proved in [CP14]. Therefore it is crucial to develop techniques of metric deformation. By this, we mean the construction of examples by changing the curvature of a negatively curved metric in a controllable way. In this dissertation, we explore the use of conformal deformation in order to produce the above-mentioned examples. We say that two Riemannian metrics g and g^* are C^k-conformally related if there exists a positive function $\phi : M \to \mathbb{R}_{>0}$ of class C^k such that $g^* = \phi g$. It is clear that the conformality of Riemannian metrics is an equivalence relation, thus we can consider the conformal class of a metric. In [Kat82], [Kat88] and [BE21], Katok, Barthelmé, and Erchenko study rigidity and flexibility on the dynamics inside the conformal class of a Riemannian metric, namely they study the variation of the entropy and length spectra for some conformally related metrics. This technique of deformation was also used by Ruggiero in the previously mentioned paper [Rug91a].

Understanding ergodicity often involves decomposing the system into "fundamental pieces" where detailed analysis can be carried out. Identifying these pieces is a challenging problem but significantly enhances our understanding of ergodicity. While one might expect a version of the Hopf Argument to apply within these "fundamental pieces," extending Hopf's ideas to systems lacking uniform rates of contraction or expansion is neither straightforward nor trivial. In [RHRHTU11a], the authors introduced a new notion of *ergodic homoclinic classes* in the context of non-uniform diffeomorphisms, providing an elegant description of the potential "fundamental pieces". Aiming to explore and developing tools to understand the ergodic phenomena of the geodesic flow, this dissertation studies an analogous decomposition for a more general class of flows. The description of *ergodic homoclinic classes* is as follows (see Chapter 1 for details): consider the flow $\varphi_t : M \to M$

generated by a vector field X and γ a closed periodic hyperbolic orbit for φ_t, i.e. there exists $T > 0$ such that $\varphi_T(p) = p$ for all $p \in \gamma$ and a decomposition $T_\gamma M = E^s \oplus \mathbb{R}X \oplus E^u$, with E^s contracted by $D\varphi_t$ and E^u expanded by $D\varphi_t$. In this context we can consider the stable and unstable Pesin's manifolds of the orbit γ, denoted by $W^\tau(\gamma)$ $\tau = s, u$. Then, the stable (resp. unstable) homoclinic class of γ, which we are going to denote by $\Lambda^s(\gamma)$ (resp. $\Lambda^s(\gamma)$), is constitutes by the points x with well defined stable (resp. unstable) Pesin's manifold which present transverse intersection to $W^u(\gamma)$ (resp. $W^s(\gamma)$). Finally, the *ergodic homoclinic class* is defined by $\Lambda(\gamma) = \Lambda^s(\gamma) \cap \Lambda^u(\gamma)$.(see Section 1.4 for further details). Notice that *apriori* there is no reason to believe that this intersection is non-empty. Our results (Theorems C, D and E) presented below give sufficient conditions to get non-empty intersection. These sets are the objects we study in Chapter 3.

An essential property of smooth measures for the *Hopf Argument* is *Absolute Continuity* (check Definition 1.4.11). This is a property also satisfied by the so called *SRB measures* introduced by Sinai, Ruelle and Bowen in the 70s. *SRB measures* are known as the invariant measures that are most compatible with a volume measure for non conservative systems and play an important role in the general ergodic theory. Such measures had been studied in several different context and many questions about them remain open, we mention existence, uniqueness and ergodicity. It is also not well understood which properties of the systems are beneficial to these questions. Inspired in [RHRHTU11b], we study how the presence of the previously defined homoclinic classes relates to ergodicity and uniqueness of SRB measures in the flow setting.

Within the context given by the paragraphs above, the main goals of this dissertation are the following:

(1) Produce examples of ergodic geodesic flows which are not Anosov in higher dimension.

(2) Develop a theory of "control of the dynamics" of the geodesic flow via deformations of Riemannian metrics, in the sense of flexibility of the hyperbolic behavior.

(3) Study rigidity of the dynamics of geodesic flows for metrics in the same conformal class.

(4) Construction of partially hyperbolic geodesic flows.

(5) Explore the possible dynamical properties of metrics on the boundary of $\mathcal{NC}(M)$.

(6) Construct other examples of partially hyperbolic contact flows.

(7) Study general properties of partially hyperbolic contact flows.

(8) Investigate the existence of partially hyperbolic contact flows in dimension 5.

(9) Develop a new approach to ergodicity of flows that present some kind of hyperbolicity.

(10) Understand the relations between homoclinic classes of flows and SRB measures.

The development of the next result encompass the above objectives from (1) to (4):

Theorem A. Let (M, g) be a compact Kähler manifold of holomorphic curvature -1 or a compact locally symmetric quaternionic Kähler manifold of negative curvature. Then there exists a conformal metric $\widetilde{g} = \phi g$ such that the geodesic flow $\widetilde{g}_t$ is partially hyperbolic and not Anosov.

The Riemannian manifolds considered in this result are contained in a larger class of interesting Riemannian manifolds called *Locally Symmetric Manifolds* (it is also common to just say locally symmetric metric since it is as metric property). By definition, a metric is said to be *Locally Symmetric* if its curvature tensor is parallel with respect to the *Levi-Civita connection*, i.e. $\nabla R \equiv 0$. One of the interesting properties of this class of metrics, from the geometry point of view, is the fact that the sectional curvatures K are not affected by parallel transport. This means that given two tangent vectors X, Y, then if $\tilde{X}(t)$ and $\tilde{Y}(t)$ denote their parallel transport along a geodesic, then $K(\tilde{X}, \tilde{Y})(t) = K(\tilde{X}, \tilde{Y})(0) = K(X, Y)$. From the dynamical point of view, since we are considering negatively curved metrics, the correspondent geodesic flow is of Anosov type. However, the action of the derivative of the geodesic flow on the canonical contact structure also has a partially hyperbolic structure, i.e. the splitting occurs not only into two invariant subbundles but in four invariant subbundles in a way that we have stronger and weaker behavior (see Lemma 2.3.2). Besides this splitting property, the class of locally symmetric Riemannian metrics also has an interesting behavior with respect to the topological entropy of its geodesic flow. In [Kat82], Katok proved that for surfaces of genus bigger or equal than two we have a fascinating comparison between the measure entropy for the Liouville measure and topological entropy of different Riemannian metrics. More precisely, let g be any negatively curved Riemannian metric and g^* be a Riemannian metric with constant negative curvature. Suppose in addition that g and g^* determine the same area, then $h_{\text{Liou}}(g) \leq h_{\text{Liou}}(g^*)$ and $h_{\text{top}}(g) \geq h_{\text{top}}(g^*)$. The inequalities are strict when g has non-constant negative curvature. This result led Katok to state the following conjecture for negatively curved Riemannian metrics

Conjecture 1 ([Kat82])**.** One has $h_{\text{top}}(g) = h_{\text{Liou}}(g)$ on a compact manifold if, and only if, g is a locally symmetric Riemannian metric.

Conjecture 1 was proved by Katok himself in [Kat82] for metrics that are conformal to a locally symmetric one. In particular, it implies that the result is true for surfaces. It is known by [KKW91] that locally symmetric metrics are critical points for both h_{top} and h_{Liou} when seen as functions from the space of Riemannian metrics. More about this conjecture can be found in [BCG95]. We are unable to relate the above conjecture to the conformal deformation construction in Theorem A since we do not know if the resulting metric is even non-positively curved, the best we can state is that if some positive curvature is created, then it can be made as small as we want. However, working in a slightly different and less rigid kind of deformation, we are able to produce an example with non-positive curvature and this encompass the above objectives from (1) to (5). More specifically,

Theorem B. Let (M, g) be a compact Kähler manifold of holomorphic curvature -1 or a compact locally symmetric quaternionic Kähler manifold of negative curvature, then there exists a C^2-deformation $\widetilde{g}$ of the metric g with the following properties

(1) $\widetilde{g}_t$ is partially hyperbolic.

(2) There exists a closed geodesic γ with a parallel Jacobi field along it.

(3) The sectional curvatures $\widetilde{K}$ are all negative outside γ.

As a consequence of the construction of Theorem B we get the following corollary:

Corollary B.1. There exists a Riemannian manifold (M, g) with no conjugate points such that its geodesic flow is partially hyperbolic, non-Anosov, ergodic for Liouville, mixing, expansive and has a unique measure of maximal entropy.

We highlight the ergodicity for the Liouville measure, which was a question stated in [CP14]. It is somehow possible to relate the previous result to the mentioned Katok's conjecture in the sense that we obtain a unique measure of maximal entropy. However, the metrics are non-positively curved and they are not negatively curved. Besides that, we are unable to say if the measure of maximal entropy is the Liouville measure or not.

In [CP14] the authors do not guarantee the existence nor absence of conjugate points in their examples, indeed they use a curve (a straight line) of metrics to prove the existence of a metric with partially hyperbolic geodesic flows and non-conjugate points. We were unable to find a reference that guarantees that the set of metrics for which the geodesic flow is partially hyperbolic is convex or at least path-connected. Thus the previous corollary guarantees by other arguments the absence of conjugate points for the resulting metric. Furthermore, we also can prove the existence of examples with conjugate points in the following way: recall that Ruggiero proved in [Rug91a] that the C^2-interior of the set of Riemannian metrics without conjugate points coincides with the set of Riemannian metrics for which the geodesic flow is of Anosov type. The metrics obtained via Theorem B are on the boundary of $\mathcal{NC}(M)$ and their geodesic flows are partially hyperbolic. Since partial hyperbolicity is an open condition, it implies that there exist Riemannian metrics with conjugate points and partially hyperbolic, non-Anosov, geodesic flows. In other words, we obtain

Corollary B.2. There exists a metric C^2-close to $\tilde{g}$ that has conjugate points and a partially hyperbolic geodesic flow. Moreover, there exists a C^2-open set of Riemannian metrics with conjugate points for which the geodesic flow is partially hyperbolic.

Preliminary results were obtained in the direction of objectives from (6) to (8). They are presented in Section 2.5.

A few results were obtained about the general class of flows. The results obtained here are analogous to those obtained in [RHRHTU11a] and [RHRHTU11b], but in the flow setting. Although they are interesting by themselves, we believe they are going to be useful in future works. It is worth mentioning that some new treatments were needed to adapt some of the proofs and deal with the flow direction, which does not appear for diffeomorphism. The definitions of the stable and unstable homoclinic classes, respectively, $\Lambda^s(\gamma)$ and $\Lambda^u(\gamma)$, are given more explicitly by (1.4.1) and (1.4.2). Recall also that we have defined the *ergodic homoclinic class of* γ as the intersection $\Lambda(\gamma) =$

$\Lambda^s(\gamma) \cap \Lambda^u(\gamma)$. The results stated bellow complete the final objectives (9) and (10) of this dissertation:

Theorem C. Let $\varphi_t : M \to M$ be a C^2-flow on a closed connected Riemannian manifold M and let m be a smooth φ_t-invariant measure. If $m(\Lambda^s(\gamma)) \cdot m(\Lambda^u(\gamma)) > 0$ for a certain hyperbolic periodic orbit γ, then:

i) $\Lambda(\gamma) = \Lambda^u(\gamma) = \Lambda^s(\gamma) \mod 0$,

ii) $\Lambda(\gamma)$ is a hyperbolic ergodic component for φ_t.

Theorem D. Let $\varphi_t : M \to M$ be a C^2-flow on a closed connected manifold and μ a SRB measure. If $\mu(\Lambda^s(\gamma)) \cdot \mu(\Lambda^u(\gamma)) > 0$, then $\Lambda^u(\gamma) \subset \Lambda^s(\gamma) \mod 0$. In addition, μ restricted to $\Lambda(\gamma)$ is ergodic, non-uniformly hyperbolic and physical measure.

Theorem E. Let $\varphi_t : M \to M$ be a C^2-flow on a closed connected manifold, m a smooth measure and μ a SRB measure. If $m(\Lambda^s(\gamma)) \cdot \mu(\Lambda^u(\gamma)) > 0$, then $\Lambda^u(\gamma) \subset \Lambda^s(\gamma) \mod 0$. Furthermore, μ restricted to $\Lambda(\gamma)$ is a hyperbolic ergodic measure.

These results show a close relation between the ergodic phenomena and the existence of "significant" homoclinic classes from the point of view of the measure itself. Next results deal with the existence of a homoclinic class related to a SRB measure in some cases and also provide us a criterion for obtaining uniqueness of SRB measures from homoclinic classes.

Theorem F. Let $\varphi_t : M \to M$ be a C^2-flow on a compact manifold and μ a regular, hyperbolic and SRB invariant measure. Then for every ergodic component ν of μ there exist a hyperbolic closed orbit γ such that $\nu(\Lambda(\gamma)) = 1$.

Theorem G. Let $\varphi_t : M \to M$ be a C^2-flow on a compact manifold and μ and ν two ergodic SRB measures. Suppose that there exist a periodic hyperbolic orbit γ such that $\mu(\Lambda(\gamma)) = \nu(\Lambda(\gamma)) = 1$. Then, $\mu = \nu$.

Notice that because of Theorem D, the ergodicity assumption in the previous theorem is somehow redundant. We choose to state the theorem in this manner since ergodicity is a key property for the proof, which is basically showing that the basins of the two measures must intersect.

In general, it is highly non-trivial to obtain known results for diffeomorphisms in the flow setting by two main reasons: perturbations results for diffeomorphisms are not, in general, applicable for flows and in hyperbolic dynamics there is no contraction nor expansion in the flow direction, hence many of the arguments which depend on this kind of structure are not adaptable. In the same direction of our propose we may cite the recent work by Boris Hasselblatt and Todd Fisher [FH22], where the authors need to overcome many difficulties to obtain analogous results from [ACW22] in the flow setting. The other way around is also tremendously difficult in many cases, as an example we cite the celebrated work by Artur Avila [Avi10] on the regularization of conservative diffeomorphisms. It is worth mentioning that flows with some non-uniform hyperbolic behavior appear naturally in dynamical systems, in particular in the context of geodesic flows for non-positively curved Riemannian metrics. Therefore, it is extremely important to develop tools to study such systems and it illustrates well the power of the results presented by this dissertation.

This dissertation is structured as follows: in Chapter 1 we present the necessary background from Riemannian geometry, partially hyperbolic dynamics, non-uniform hyperbolic systems and the dynamics for the geodesic flow. For some elementary computations within Riemannian geometry, we were unable to find a reference, so we provided short proofs for completeness. Chapter 2 encompass the content of the *preprint* [dJPR24]. In Section 2.2, we show how to use conformal deformations in order to destroy the Anosov property of the geodesic flow. The construction consists of deforming the initial metric g by multiplying it by a conformal factor supported in a tubular neighborhood of closed geodesic, which we will mention as *central geodesic*. We investigate the necessary conditions on the multiplying factor and show how to construct it in general. The construction can be made with a multiplying function as regular as needed, at least of class C^4. It is also possible to make it smooth, however, with this technique, we can not control how large is the positive curvature that appears. In section 2.3, we show that the resulting flow is partially hyperbolic by using the *cones criterion* (check Section 1.3.2). We begin by explicitly exhibiting the invariant splitting along the *central geodesic*, which gives us the candidates for the invariant cone families. We compute the variation of the cones' openness along a general orbit of the flow, then the proof splits into some cases: for geodesics that are parallel to the central geodesic or almost parallel, we show that openness variation is approximated by a positive quantity and this implies cone invariance. We then show that we can shrink the deformed region in order to prevent transversal geodesics of losing hyperbolicity, and then we also obtain invariance of the cone family. The proof is completed by using the symplectic behavior of the derivative of the geodesic flow acting on the contact structure (check Lemma 1.3.1). This strategy is also used in [CP14], however the computations that appear are completely different and allow us to control better the deformation on the curvature. In Section 2.4, we prove Theorem B and Corollary B.1. The proof of partial hyperbolicity with a different deformation works in the same way, thus we are left to analyze the curvature. We show that every sectional curvature can be bounded by a non-positive function, which vanishes just along the *central geodesic*. In Section 2.5, are presented extra properties for the constructed geodesic flows, partial results on partially hyperbolic contact flows and some derived questions from this work. Chapter 3 is devoted to present results from the *preprint* [dJEP25] on *Homoclinic classes for flows*. The development of this chapter is essentially straightforward in the sense that no extra tools are needed, each section is composed by the proof of one of the above theorems. Theorems C, D and E are proved in Sections 3.2 and 3.3. Section 3.4 consists of the proof of Theorem F and finally we prove Theorem G in Section 3.5.

Chapter 1

Preliminaries

For this dissertation, we will assume that (M, g) is a compact smooth Riemannian manifold without boundary and dimension $n \geq 2$, and we will denote by TM the tangent bundle of M. We will see that, associated with a Riemannian metric g, it is important for the study of the *geodesic flow* considering the so-called *unit tangent bundle* defined by:

$$T^1M := \{v \in TM : g(v, v) = 1\}.$$

Remark 1.0.1. Many different notations, such as UM and SM, can appear in the literature. To avoid some possible confusion of notation with other structures we are going to define later, we have chosen the above notation for this space.

Remark 1.0.2. It is clear that the unit tangent bundle depends on the fixed Riemannian metric. However, we are not going to explicitly state its dependence on the notation for a matter of simplicity. If any confusion can appear when dealing with different metrics, then we are going to be careful with the notation.

Let us fix some usual ergodic theory definitions:

Definition 1.0.1. we are going to use: a set $A \subset M$ will be called φ-*invariant* if $\varphi_t(A) = A$, for every $t \in \mathbb{R}$.

Definition 1.0.2. A Borel probability measure μ is called φ-invariant if $\mu(\varphi_t(A)) = \mu(A)$, for every $t \in \mathbb{R}$ and every Borel set A.

Definition 1.0.3. A φ-invariant measure is called *ergodic* if $\mu(A) = 1$ or $\mu(A) = 0$, for every invariant measurable set A.

Definition 1.0.4. Given a C^1 vector field X on M, an invariant measure μ is said to be *regular* if $\mu(Sing(X)) = 0$, where $\mathrm{Sing}(X)$ denotes the set of points in M where X vanishes.

1.1 Riemannian Geometry

In this section we are going to discuss some of the important concepts of Riemannian geometry that are necessary to understand the dynamics of the geodesic flow. Although we tried our best to make this dissertation as self-contained as possible, many interesting and useful concepts are left out of our exposition on this topic. Therefore, for a more complete approach of the theory of Riemannian manifolds, we suggest the books [Lee18], [DCF92] and [Wal04].

Many computations in Riemannian Geometry can become very complicated to read when summing with different indices, for example, when tensors are expressed in coordinates. For simplicity of notation, and to avoid writing too many summation symbols, we are going to use the Einstein summation convention: we are going to write $x^i E_i$ to mean $\sum_i x^i E_i$ whenever the index i appears once as an upper index and once as a lower index and the summation is considered among all possible values of i. It can seem to be a silly simplification when summing only on one symbol, but the above situation is very common when making computations with tensors:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} f_l^{i,j,k} E_{i,j,k,l} := f^{i,j,k,l} E_{i,j,k}$$

We are going to use the summation sign whenever any confusion with indices is possible.

1.1.1 Basic concepts

A classical problem that arises when studying manifolds is defining a notion of derivative for vector fields; this is solved by the notion of *connections*. Although this notion can be studied in general situations such as fiber bundles, we are going to restrict ourselves to the tangent bundle of a manifold M. Let us denote by $\mathcal{X}(M)$ the set of smooth vector fields on M and by $C^\infty(M)$ the set of real-valued smooth functions on M. We make the following definition:

Definition 1.1.1. A *smooth connection* is a map $\nabla \colon \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$, denoted by $\nabla_X Y$ and satisfying the following properties:

(1) Given $\alpha, \beta \in \mathbb{R}$, and X, Y and $Z \in \mathcal{X}(M)$, then

$$\nabla_X(\alpha Y + \beta X) = \alpha \nabla_X Y + \beta \nabla_X Z.$$

(2) Given $f, g \in C^\infty(M)$ and X, Y and $Z \in \mathcal{X}(M)$, then

$$\nabla_{fX+gY} Z = f \nabla_X Z + g \nabla_Y Z.$$

(3) Given $f \in C^\infty(M)$, and X and $Y \in \mathcal{X}(M)$, then

$$\nabla_X(fY) = Xf \cdot Y + f \cdot \nabla_X Y.$$

In particular, a smooth connection gives rise to the notion of derivative of vector fields along curves: given a smooth curve $\alpha \colon I \to M$ we say that X is a vector filed along α if $X : I \to TM$ and $X(t) \in T_{\alpha(t)}M$ for all $t \in I$. Of course, $\alpha'(t)$ is a vector field along α. Given a smooth connection ∇, it induces a map $\nabla \colon \mathcal{X}(\alpha) \times \mathcal{X}(\alpha) \to \mathcal{X}(\alpha)$, where $\mathcal{X}(\alpha)$ is the space of vector fields along α. A vector field X along α will be called *parallel* if $\nabla_{\alpha'} X(t) = 0$, for all $t \in I$. For completeness, we are going to state a proposition on the existence of parallel vector fields along curves given initial conditions. The proof can be found in [Lee18, Theorem 4.32].

Proposition 1.1.1. *Given a smooth curve $\alpha \colon I \to M$, with $t_0 \in I$, a smooth connection ∇ and $v \in T_{\alpha(t_0)}M$, then there exists an unique parallel vector field V along α such that $V(t_0) = v$.*

Last Proposition implies the existence of a well-defined map $\mathcal{P}_{t_0 t} \colon T_{\alpha(t_0)}M \to T_{\alpha(t)}M$ given by

$$\mathcal{P}_{t_0 t}(v) = V(t).$$

The map $\mathcal{P}_{t_0 t}$ is called the *Parallel Transport along α from t_0 to t* and it will be important for setting up the constructions made in the next Chapter.

In this context, we can also define one of the most important concepts for the study developed by this dissertation, the *geodesics*:

Definition 1.1.2. A curve α will be called a *geodesic* for the connection ∇ if α' is a parallel vector field along α, i.e. $\nabla_{\alpha'} \alpha'(t) = 0$, for all $t \in I$.

A smooth connection $\nabla\colon \mathcal{X}(M)\times\mathcal{X}(M)\to\mathcal{X}(M)$ is totally determined by its values on an local frame in the following sense: given a local frame $\{E_i\}$, we know that for each pair of indices (i,j) $\nabla_{E_i}E_j \in \mathcal{X}(M)$, therefore it can be written in terms of the local frame $\{E_i\}$, i.e. there exit smooth functions Γ_{ij}^k (locally defined wherever the local frame is defined) such that

$$\nabla_{E_i}E_j = \Gamma_{ij}^k E_k.$$

Now, suppose that X and Y are any smooth vector fields written locally as $X = X^iE_i$ and $Y = Y^jE_j$, then by the properties in Definition 1.1.1 we get

$$\begin{aligned}\nabla_X Y = \nabla_{X^iE_i}(Y^jE_j) = X^i\nabla_{E_i}(Y^jE_j) &= X^i(E_iY^j\cdot E_j + Y^j\cdot\nabla_{E_i}E_j)\\ &= X^i(E_iY^j\cdot E_j + Y^j\cdot\Gamma_{ij}^kE_k) = (X^i\cdot E_iY^k + X^iY^j\cdot\Gamma_{ij}^k)E_k.\end{aligned}$$

Thus, the vector field $\nabla_X Y$ can be determined by the coordinates of X and Y in the given local frame and the functions Γ_{ij}^k. These functions are called the *Christoffel Symbols* of the connection ∇ with respect to the local frame $\{E_i\}$.

So far, we have not assumed any relation between connections and the Riemannian metric g. Indeed, the theory of connections is general and does not depend on any notion of metric. For the purpose of this dissertation, it is important to relate both theories, i.e. we are going to be interested in connections that are defined in terms of the Riemannian metric. A classical result of Riemannian geometry guarantees the existence of a unique smooth connection with the following two properties:

(1) Given X, Y and $Z \in \mathcal{X}(M)$, then

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

(2) Given X and $Y \in \mathcal{X}(M)$, then

$$\nabla_X Y - \nabla_Y X = [X,Y].$$

The unique smooth connection satisfying 1. and 2. is called the *Levi-Civita connection.* Essentially its existence and uniqueness is proved by exhibiting its Christoffel Symbols for any local charts: let $(U,(x^i))$ be a local chart and $g_{ij} := g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$. Denote by g^{ij}

the entries of the inverse matrix of (g_{ij}), then the Christoffel Symbols of the Levi-Civita connection are given by:

$$\Gamma_{ij}^k = \frac{g^{kl}}{2}(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}).$$

Although the Levi-Civita connection theory is very rich and interesting, we are not going to explore it here. The important concept in this context is *geodesics*:

Definition 1.1.3. We are going to call a curve α a geodesic for the metric g if it is a geodesic for the Levi-Civita connection.

The theory of *Ordinary Differential Equations* guarantee that given $p \in M$ and $v \in T_pM$ there exists a unique geodesic γ satisfying

$$\begin{cases} \gamma(0) = p \\ \gamma'(0) = v \end{cases}$$

Let us also fix some notation: the geodesic with initial conditions (p, v) will be denoted by $\gamma_{(p,v)}$. Given a curve α we will use the notation $D_t := \nabla_{\alpha'}$ if there is no confusion on the curve we are considering, so α is a geodesic if, and only if, $D_t\alpha' = 0$. We are also going call the operator D_t the "covariant derivative in the direction of α'".

We now introduce an important concept for the study of the dynamics of the *Geodesic Flow*: the *Riemann curvature tensor*.

First, let us define the *curvature endomorphism* as the map $R\colon \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

for X, Y and $Z \in \mathcal{X}(M)$. It is worth mentioning that there is a sign convention in the expression above; it can be found in the literature (check [DCF92]), the curvature endomorphism tensor with opposite sign to ours, so the reader may find some expressions that we are going to use also with different signs. In general, this convention will not impact our analysis.

Another very important tool in Riemannian geometry is the exponential map: given $p \in M$ we can define a map $\exp_p\colon T_pM \to M$ by

$$\exp_p(v) := \gamma_{(p,v)}(1).$$

This map is well-defined in a small neighborhood of $0 \in T_pM$ and it is in fact a local diffeomorphism. Therefore, the collection of exponential maps can also be used to define a smooth atlas for M. Each chart of this atlas is called *Normal Coordinates*. A particularly interesting property of *Normal Coordinates* is the fact that Christoffel Symbols vanish on the base point, i.e. if $(U, \exp_p)$ is a *Normal Chart* around p, then $\Gamma^k_{ij}(p) = 0$.

Now, given two tangent vectors $X, Y \in T_pM$ we can consider the plane spanned by them $\Pi(X,Y) = span\{X,Y\}$ and its image under the exponential map $S := \exp_p(\Pi)$, which is a 2-dimensional embedded submanifold of M. Therefore, the metric g induces a metric on S and since it is 2-dimensional, we can consider its *Gaussian Curvature* at p, say K_p. It can be proved that with this construction, the Gaussian Curvature coincides with the following quantity:

$$K(\Pi) := K(X,Y) = \frac{g(R(X,Y)Y,X)}{|X \wedge Y|^2}, \tag{1.1.1}$$

where $|X \wedge Y| = \sqrt{g(X,X)g(Y,Y) - g(X,Y)}$. This makes clear that the whole construction does not depend on X and Y in the sense that if X' and Y' form another basis for Π, then we get the same quantity in (1.1.1).

To organize this text, we are going to prove two Lemmas, which are just elementary computations with tensors. It will be useful to have these computations done, so we can refer to them in future computations. To simplify the notation, we are going to use $\partial_s := \frac{\partial}{\partial x^s}$.

Lemma 1.1.1. *If g_{ij} are the components of a Riemannian metric in coordinates and g^{ij} of its inverse, then the following expressions hold*

(1)

$$\partial_s g^{ik} g_{kj} = -g^{ik} \partial_s g_{kj}. \tag{1.1.2}$$

(2)

$$\partial_p g_{ms} = g_{ns}\Gamma^n_{pm} + g_{nm}\Gamma^n_{ps}. \tag{1.1.3}$$

Proof. To prove (*1*), remember that $g^{ik}g_{kj} = \delta^i_j$, where

$$\delta^i_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Then, taking the derivative ∂_s on both sides of the above expression, we get from one side

$$\partial_s(g^{ik}g_{kj}) = \partial_s g^{ik} \cdot g_{kj} + g^{ik} \cdot \partial_s g_{kj}.$$

On the other hand, we have

$$\partial_s \delta^i_j = 0.$$

From both equalities we get (*1*). To prove (*2*) we apply property (*1*) of the Levi-Civita connection:

$$\begin{aligned}\partial_p g_{ms} &= \frac{\partial}{\partial x^p} g\left(\frac{\partial}{\partial x^m}, \frac{\partial}{\partial x^s}\right) = g\left(\nabla_{\frac{\partial}{\partial x^p}} \frac{\partial}{\partial x^m}, \frac{\partial}{\partial x^s}\right) + g\left(\frac{\partial}{\partial x^m}, \nabla_{\frac{\partial}{\partial x^p}} \frac{\partial}{\partial x^s}\right) \\ &= g\left(\Gamma^n_{pm}\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^m}\right) + g\left(\frac{\partial}{\partial x^m}, \Gamma^n_{ps}\frac{\partial}{\partial x^n}\right) = g_{ns}\Gamma^n_{pm} + g_{nm}\Gamma^n_{ps}.\end{aligned}$$

■

Similarly, as we have done to define the Christoffel symbols as the components of the connection in coordinates, let us express the curvature tensor in coordinates. Given any chart $(U, (x^i))$ we know that $R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} \in \mathcal{X}(M)$, so there exists a family of smooth functions R^l_{ijk} defined on U such that $R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = R^l_{ijk}\frac{\partial}{\partial x^l}$. In the next Lemma, we express R^l_{ijk} in terms of the Christoffel symbols and their partial derivatives.

Lemma 1.1.2. *The components of the curvature endomorphism R^l_{ijk} can be expressed as*

$$R^l_{ijk} = \partial_j\Gamma^l_{ik} + \Gamma^r_{ik}\Gamma^l_{jr} - \partial_i\Gamma^l_{jk} - \Gamma^r_{jk}\Gamma^l_{ir}$$

Proof. Consider a coordinate system $(U, (x^i))$ and denote by $X_i := \dfrac{\partial}{\partial x^i}$. We know that the Christoffel Symbols satisfy

$$\nabla_{X_i}X_j = \Gamma^k_{ij}X_k$$

Since $[X_i, X_j]$ (because these are coordinate vectors), it follows that

$$\begin{aligned}R(X_i, X_j)X_k &= \nabla_{X_j}\nabla_{X_i}X_k - \nabla_{X_i}\nabla_{X_j}X_k \\ &= \nabla_{X_j}(\Gamma^r_{ik}X_r) - \nabla_{X_i}(\Gamma^r_{jk}X_r) \\ &= \partial_j\Gamma^r_{ik}X_r + \Gamma^r_{ik}\nabla_{X_j}X_r - \partial_i\Gamma^r_{jk}X_r - \Gamma^r_{jk}\nabla_{X_i}X_r \\ &= \partial_j\Gamma^r_{ik}X_r + \Gamma^r_{ik}\Gamma^l_{jr}X_l - \partial_i\Gamma^r_{jk}X_r - \Gamma^r_{jk}\Gamma^l_{ir}X_l \\ &= (\partial_j\Gamma^l_{ik} + \Gamma^r_{ik}\Gamma^l_{jr} - \partial_i\Gamma^l_{jk} - \Gamma^r_{jk}\Gamma^l_{ir})X_l \\ &= R^l_{ijk}X_l\end{aligned}$$

■

It will also be useful to express the following components of the curvature tensor

$$g(R(X_i, X_j)X_k, X_s) = R^l_{ijk} g_{ls} =: R_{ijks}.$$

in terms of the components of the metric g and its Christoffel symbols:

Lemma 1.1.3. *In any coordinates, the curvature tensor can be expressed as*

$$R_{ijks} := R^l_{ijk} g_{ls} = -\frac{1}{2}(\partial^2_{js} g_{ik} + \partial^2_{ik} g_{js} - \partial^2_{is} g_{jk} - \partial^2_{jk} g_{is}) - g_{mn}(\Gamma^m_{ki}\Gamma^n_{js} - \Gamma^m_{si}\Gamma^n_{jk}). \quad (1.1.4)$$

Proof. To prove the above equality, we expand the expression from the previous lemma. To be careful, we will analyze each part of the sum we are interested in. From the previous lemmas, we get

$$\begin{aligned} g_{ls}\partial_j\Gamma^l_{ik} &= \frac{1}{2}(g_{ls}\partial_j g^{ml}(\partial_k g_{mi} + \partial_i g_{mk} - \partial_m g_{ik}) + g_{ls}g^{ml}(\partial^2_{jk} g_{mi} + \partial^2_{ji} g_{mk} - \partial^2_{jm} g_{ik})) \\ &= \frac{1}{2}(-g^{ml}\partial_j g_{ls}(\partial_k g_{mi} + \partial_i g_{mk} - \partial_m g_{ik}) + \delta^m_s(\partial^2_{jk} g_{mi} + \partial^2_{ji} g_{mk} - \partial^2_{jm} g_{ik})) \\ &= -\partial_j g_{ls}\Gamma^l_{ik} + \frac{1}{2}(\partial^2_{jk} g_{si} + \partial^2_{ji} g_{sk} - \partial^2_{js} g_{ik}) \\ &= -(g_{ns}\Gamma^n_{jl} + g_{nl}\Gamma^n_{js})\Gamma^l_{ik} + \frac{1}{2}(\partial^2_{jk} g_{si} + \partial^2_{ji} g_{sk} - \partial^2_{js} g_{ik}) \\ &= \frac{1}{2}(\partial^2_{jk} g_{si} + \partial^2_{ji} g_{sk} - \partial^2_{js} g_{ik}) - g_{ns}\Gamma^n_{jl}\Gamma^l_{ik} - g_{nl}\Gamma^n_{js}\Gamma^l_{ik} \end{aligned}$$

By an analogous calculation, we have

$$g_{ls}\partial_i\Gamma^l_{jk} = \frac{1}{2}(\partial^2_{ik} g_{sj} + \partial^2_{ij} g_{sk} - \partial^2_{is} g_{jk}) - g_{ns}\Gamma^n_{il}\Gamma^l_{jk} - g_{nl}\Gamma^n_{is}\Gamma^l_{jk}.$$

Subtracting the last two equations, we get,

$$\begin{aligned} g_{ls}\partial_j\Gamma^l_{ik} - g_{ls}\partial_i\Gamma^l_{jk} =& \frac{1}{2}(\partial^2_{jk} g_{si} + \partial^2_{is} g_{jk} - \partial^2_{js} g_{ik} - \partial^2_{ik} g_{sj}) \\ &+ g_{ns}\Gamma^n_{il}\Gamma^l_{jk} + g_{nl}\Gamma^n_{is}\Gamma^l_{jk} - g_{ns}\Gamma^n_{jl}\Gamma^l_{ik} - g_{nl}\Gamma^n_{js}\Gamma^l_{ik} \end{aligned}$$

Finally, R_{ijks} will be given by

$$\begin{aligned} R_{ijks} &= g_{ls}\partial_j\Gamma^l_{ik} - g_{ls}\partial_i\Gamma^l_{jk} + g_{ls}\Gamma^r_{ik}\Gamma^l_{jr} - g_{ls}\Gamma^r_{jk}\Gamma^l_{ir} \\ &= \frac{1}{2}(\partial^2_{jk} g_{si} + \partial^2_{is} g_{jk} - \partial^2_{js} g_{ik} - \partial^2_{ik} g_{sj}) \\ &\cancel{+g_{ns}\Gamma^n_{il}\Gamma^l_{jk}} + g_{nl}\Gamma^n_{is}\Gamma^l_{jk} \cancel{-g_{ns}\Gamma^n_{jl}\Gamma^l_{ik}} - g_{nl}\Gamma^n_{js}\Gamma^l_{ik} \\ &\cancel{+g_{ls}\Gamma^r_{ik}\Gamma^l_{jr}} - \cancel{g_{ls}\Gamma^r_{jk}\Gamma^l_{ir}} \end{aligned}$$

■

From the above lemma, we see that in order to change sectional curvatures of a given metric via deformations, we need to deform the metric at least at order 2, i.e. if two metrics are C^2-close, then their curvatures will also be close to each other. The key point of our results presented in Chapter 2 is the development of a theory of metric deformations to control the behavior of the *Geodesic Flow*. Several deformations can be used to study how to change and control the dynamics of the *Geodesic Flow*, but in this work, we have chosen to study conformal deformations. There are several advantages to this approach, especially it is not difficult to relate some of the Riemannian objects we have defined for conformal metrics. In the next subsection, we describe all the relations that are going to be useful for our purposes.

We introduce now a key object in the study of the dynamics of geodesic flows known as *Jacobi Fields*. The intuition is the following: suppose γ is a geodesic defined on an interval I containing 0 and consider $F\colon I \times (-\varepsilon, \varepsilon) \to M$ a variation of γ by geodesics, i.e. $F(t, 0) = F_0(t) = \gamma(t)$ and each curve $\gamma_{s_0}(t) := F(t, s_0)$, with $s_0 \in (-\varepsilon, \varepsilon)$, is a geodesic. In this context we can also consider the curves $\alpha_{t_0}(s) := F(t_0, s)$, with $t_0 \in I$, then we have a well-defined vector field along γ given by

$$J(t) := \frac{d}{ds}\alpha_t(s)|_{s=0}.$$

A vector field $J(t)$ constructed as above is called a *Jacobi Field*. It can be shown that a

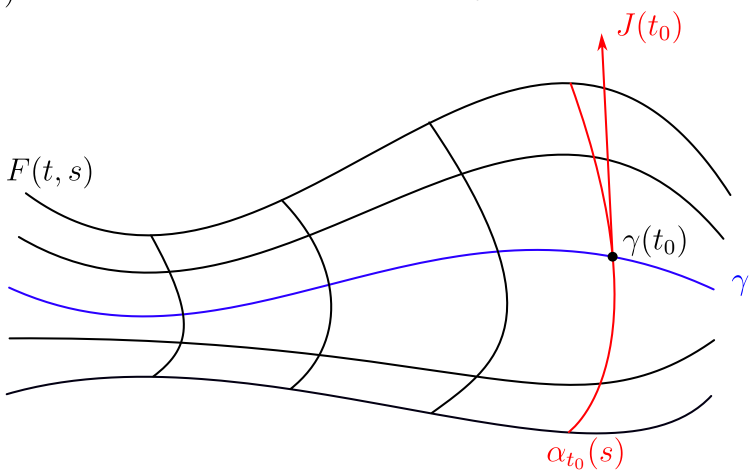


Figure 1.1: Definition of Jacobi Fields by geodesic variation

Jacobi Field defined as above satisfies the following equation:

$$J'' + R(J, \gamma')\gamma' = 0, \tag{1.1.5}$$

where $J'' = D_t^2 J$. In fact, since the curves $\gamma_s(t) = F(t, s)$ is a geodesic, then

$$D_t \frac{d}{dt}\gamma_s(t) \equiv 0.$$

Therefore, $D_sD_t\dfrac{d}{dt}\gamma_s(t) \equiv 0$ and by the defintion of R and using the symmetric property of the connection we get

$$\begin{aligned}0 = D_sD_t\frac{d}{dt}\gamma_s(t) &= D_tD_s\frac{d}{dt}\gamma_s(t) + R\left(\frac{d}{ds}\alpha_t(s), \gamma_s'(t)\right)\gamma_s'(t)\\ &= D_tD_t\frac{d}{ds}\alpha_t(s) + R(\alpha_t'(s), \gamma_s'(t))\gamma_s'(t)\end{aligned}$$

Applying for $s = 0$, we get $\alpha_t'(0) = J(t)$ and $\gamma_0'(t) = \gamma'(t)$, which gives us (1.1.5). Again, the theory of ordinary differential equations guarantees the following result as a direct application:

Proposition 1.1.2. *Let $\gamma : I \to M$ be a geodesic and $v, w \in T_{\gamma(0)}M$. Then there exists a unique Jacobi Field $J\colon I \to TM$ such that*

$$\begin{cases} J(0) = v, \\ J'(0) = w \end{cases}$$

The description of Jacobi Fields by Equation (1.1.5) is interesting as it gives us an idea of how the geodesics behave around each other in the presence of curvature. Indeed, we are going to see below how the sign of the curvature can interfere on the behavior of the function $f(t) := \|J(t)\|$, where J is a Jacobi Field along some geodesic γ and as a consequence it will give us information on the behavior of geodesics.

Lemma 1.1.4. *Let γ be a unit geodesic with perpendicular Jacobi Field J. Suppose the sectional curvatures are all uniformly bounded from above by a constant C, that is $K(X, Y) \le C$, for all $X, Y \in TM$. Then, $f''(t) \ge -Cf(t)$, for all $t \in R$ for which $f(t) = \|J(t)\| \ne 0$.*

Proof. This will be followed by direct computation. First, notice that

$$f' = \frac{g(J', J)}{\sqrt{g(J, J)}}.$$

Then, using the Cauchy-Schwarz inequality

$$\begin{aligned}
f'' &= \frac{\sqrt{g(J,J)}(g(J'',J)+g(J',J')) - g(J',J)\frac{g(J',J)}{\sqrt{g(J,J)}}}{g(J,J)} \\
&= \frac{1}{\sqrt{g(J,J)}}(-g(R(J,\gamma')\gamma',J) + \|J'\|^2) - \frac{g(J',J)^2}{\|J\|^3} \\
&= \frac{1}{\|J\|^3}(-g(R(J,\gamma')\gamma',J)\,\|J\|^2 + \|J'\|^2\,\|J\|^2 - g(J',J)^2) \\
&\geq -\frac{K(J,\gamma')\,\|J\|^4}{\|J\|^3} \\
&\geq -C\,\|J\|
\end{aligned}$$

■

This Lemma shows us that if the sectional curvatures are non-positive, then $f(t)$ is a convex function, indeed, this is true without assuming $g(J,\gamma') = 0$. This fact is interesting because of the following definition

Definition 1.1.4. Let $\gamma\colon I \to M$ be a unit speed geodesic. We say that two points $\gamma(a)$ and $\gamma(b)$ are conjugate if there exists a non-trivial Jacobi Field along γ satisfying $J(a) = 0$ and $J(b) = 0$.

Intuitively, two points are conjugate if there exists a variation of γ by geodesics emanating from $\gamma(a)$ and meeting again in $\gamma(b)$ (see Figure 1.2 below). A simple example is given by the sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ with the induced metric. It is well known that geodesics in this case are great circles, then any two distinct geodesics emanating from the north pole will meet again at the south pole. Therefore, the north and south poles are conjugate points. The previous Lemma gives us that $f(t)$ is a convex function; thus, if it vanishes at two different points, it must be identically zero. Consequently we have

Corollary 1.1.1. *If (M,g) is a Riemannian metric with non-positive sectional curvatures, then it has no conjugate points.*

Below, in Figure 1.3, we see an illustration of how the norm of a Jacobi Field vanishing at $t = 0$ behaves when the sectional curvature is constant. Although it is easy to analyze the norm of Jacobi Fields in constant curvature, in general, it is very difficult to understand it when the curvature varies. One may wonder if conjugate points should

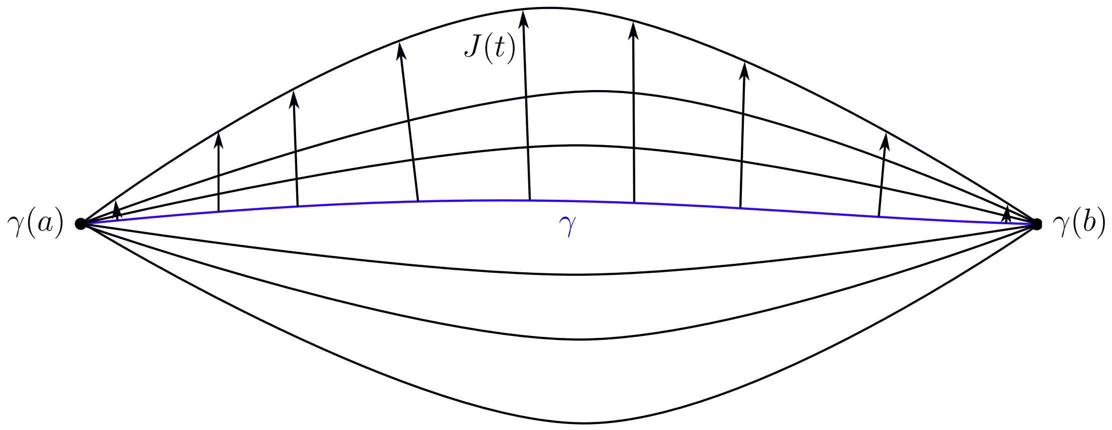


Figure 1.2: Illustration of two conjugate points

imply positive curvature. However, it was proved by Robert Gulliver in [Gul75] the existence of Riemannian metrics admitting planes with both positive and negative sectional curvatures but not presenting conjugate points. In light of subsections 1.2 and 1.3, it is interesting to point out that Gulliver also proves that the examples satisfy the Anosov property for the geodesic flow.

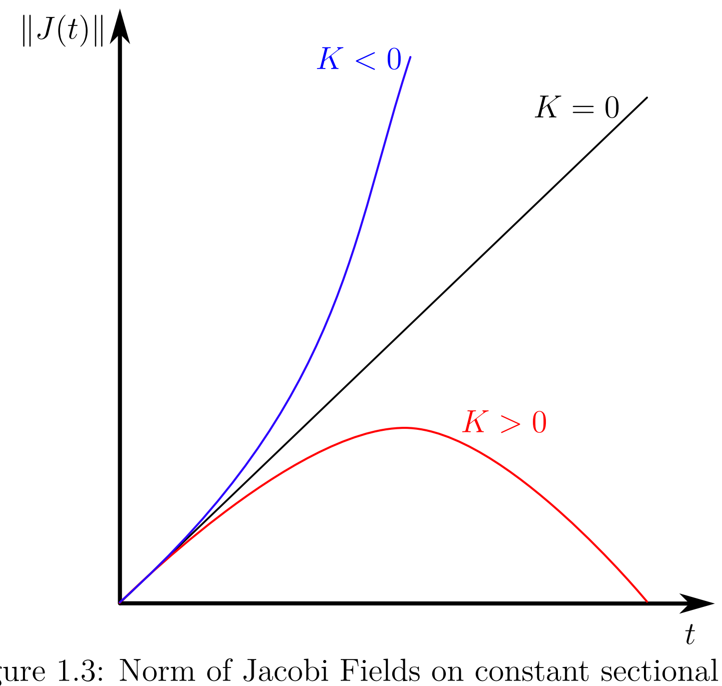


Figure 1.3: Norm of Jacobi Fields on constant sectional curvature.

The following results will not be directly used, but they are interesting to enrich our analysis and presentation.

Lemma 1.1.5. *Let (M, g) be a compact Riemannian manifold with non-positive sectional curvatures. Suppose $\gamma\colon \mathbb{R} \to M$ is a unit speed geodesic with J a Jacobi Field, then the following claims are equivalent:*

(1) J *is a parallel vector field.*

(2) $\|J(t)\| = \|J(0)\|$, *for all* $t \in \mathbb{R}$.

(3) There exists a constant $C > 0$ *such that* $\|J(t)\| \le C$, *for all* $t \in \mathbb{R}$.

Furthermore, $(1), (2)$ *or* (3) *implies* $g(R(J(t), \gamma'(t))\gamma'(t), J(t)) = 0$, *for all* $t \in \mathbb{R}$.

Proof. To see $(1) \implies (2)$ just observe that $\frac{d}{dt}\|J(t)\|^2 = 2g(J'(t), J(t)) \equiv 0$. $(2) \implies (3)$ is obvious. Now, $(3) \implies (2)$ follows by the convex property of $f(t) = \|J(t)\|$. Finally we prove $(2) \implies (1)$: since $\|J(t)\|^2$ is constant, then

$$0 = \frac{d}{dt}g(J(t), J(t)) = 2g(J'(t), J(t)).$$

Therefore, by the Jacobi Equation

$$\begin{aligned} 0 &= \frac{d}{dt}(2g(J'(t), J(t))) = 2(g(J'(t), J'(t)) + g(J''(t), J)) \\ &= 2(g(J'(t), J'(t)) - g(R(J(t), \gamma'(t))\gamma'(t), J)). \end{aligned}$$

Since $-g(R(J(t), \gamma'(t))\gamma'(t), J) \ge 0$, last equation implies $J'(t) \equiv 0$. ■

We end this subsection by presenting, without proving, some results about the comparison of Jacobi Fields norms. For the proofs check, respectively, [Bal95, 2.9 Proposition] and [Lee18, Theorem 11.9].

Proposition 1.1.3. (2.9 Proposition in [Bal95]) *Let* (M, g) *be a compact Riemannian manifold with non-positive sectional curvatures. Suppose* $\gamma\colon \mathbb{R} \to M$ *is a unit speed geodesic with* J *a Jacobi Field satisfying* $\|J(t)\| \le C$ *for some positive constant* $C \in \mathbb{R}$ *and all* $t \ge 0$. *Then the following claims hold:*

(1) If there exists a constant $a \in \mathbb{R}$ *such that* $K_{\gamma(t)}(X, Y) \le -a^2 \le 0$, *for all* $t \in \mathbb{R}$ *and* $X, Y \in T_{\gamma(t)}M$, *then*

$$\|J(t)\| \le \|J(0)\| \, e^{-at} \ , \ \|J'(t)\| \ge a \|J(t)\| \, , \ \forall \ t \ge 0$$

(2) If there exists a constant $b \in \mathbb{R}$ *such that* $K_{\gamma(t)}(X, Y) \ge -b^2 \ge 0$, *for all* $t \in \mathbb{R}$ *and* $X, Y \in T_{\gamma(t)}M$, *then*

$$\|J(t)\| \ge \|J(0)\| \, e^{-bt} \ , \ \|J'(t)\| \le b \|J(t)\| \, , \ \forall \ t \ge 0$$

Define the following family of functions $s_c(t)$:

$$s_c(t) := \begin{cases} t, \text{ if } c = 0 \\ R \sin \frac{t}{R}, \text{ if } c = \frac{1}{R^2} \\ R \sinh \frac{t}{R}, \text{ if } c = -\frac{1}{R^2} \end{cases}$$

Theorem 1.1.5. (Theorem 11.9 in [Lee18]) *Let (M, g) be a compact Riemannian manifold, with sectional curvatures K, $\gamma\colon [0, b] \to M$ a unit-speed geodesic segment, and J an orthogonal Jacobi Field. Then we get*

(1) If $K \leq c$, then

$$\|J(t)\| \geq s_c(t) \|J'(0)\|,$$

for all $t \in [0, b_1]$, where $b_1 = b$ if $c \leq 0$ and $b_1 = \min(b, \pi R)$ if $c = \frac{1}{R^2}$.

(2) If $K \geq c$, then

$$\|J(t)\| \leq s_c(t) \|J'(0)\|,$$

for all $t \in [0, b_2]$, where $b_2 \in [0, b]$ is such that $\gamma(b_2)$ is a conjugate point to $\gamma(0)$ and there no conjugate point to $\gamma(0)$ for $t \in [0, b_2)$. Otherwise, if there is no conjugate point along γ, then $b_2 = b$.

1.1.2 Conformal metrics

In this section we are going to present the main technique of deformation we are going to use to prove Theorem A. Essentially, two Riemannian metrics are said to be conformal if one is obtained by multiplying the other by a positive function. It is easy to see from the definition bellow that conformal deformations preserve angles between vectors and "being conformal" is an equivalence relation between Riemannian metrics, therefore it make sense to talk about the conformal class of a Riemannian metric. This two trivial observations together with relations we can see between geometric quantities of two conformally related metrics justify our choice for this technique. We are going to see in the next chapter that some of the computations to produce our examples can be verified very explicitly and without to much effort. Therefore, the development is somehow "clean".

Let us start with the formal definition:

Definition 1.1.6. We say that two Riemannian metrics g and $\tilde{g}$ are conformally related if there exists a smooth positive function $\phi \in C^\infty(M)$ such that $\tilde{g} = \phi g$.

Since $\phi > 0$, it will be very useful for our computations to write $\phi = \mathrm{e}^h$ and write the formulas in terms of the function $h \in C^\infty(M)$. Let us start by relating their Christoffel symbols:

Lemma 1.1.6. *If $\tilde{g} = \mathrm{e}^h g$ are two Riemannian metrics conformally related, then their Christoffel symbols are related by the following formula:*

$$\widetilde{\Gamma}^k_{ij} = \Gamma^l_{ij} + \frac{1}{2}(\partial_i h \delta^k_j + \partial_j h \delta^k_i - \partial_l h g^{lk} g_{ij}), \tag{1.1.6}$$

Proof. First, notice that $\tilde{g}^{ij} = \mathrm{e}^{-h} g^{ij}$. Then, by the expression of Christoffel symbols in coordinates, we get

$$\begin{aligned}
\tilde{\Gamma}^k_{ij} &= \frac{\tilde{g}^{kl}}{2}(\partial_i \tilde{g}_{lj} + \partial_j \tilde{g}_{li} - \partial_l \tilde{g}_{ij}) \\
&= \frac{\mathrm{e}^{-h} g^{kl}}{2}(\partial_i(\mathrm{e}^h g_{lj}) + \partial_j(\mathrm{e}^h g_{li}) - \partial_l(\mathrm{e}^h g_{ij})) \\
&= \frac{\mathrm{e}^{-h} g^{kl}}{2}(\mathrm{e}^h(\partial_i h g_{lj} + \partial_i g_{lj} + \partial_j h g_{li} + \partial_j g_{li} - \partial_l h g_{ij} - \partial_l g_{ij})) \\
&= \frac{g^{kl}}{2}(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}) + \frac{1}{2}(\partial_i h g^{kl} g_{lj} + \partial_j h g^{kl} g_{li} - \partial_l h g^{kl} g_{ij}) \\
&= \Gamma^k_{ij} + \frac{1}{2}(\partial_i h \delta^k_j + \partial_j h \delta^k_i - \partial_l h g^{lk} g_{ij})
\end{aligned}$$

■

We can now give a relation between the Levi-Civita connection for conformal metrics. Remember that the *Riemannian Gradient* of a smooth function is defined as the only vector field $\nabla_g h$ which satisfies the relation

$$X(h) = g(\nabla_g h, X)$$

In coordinates, it can be expressed as

$$\nabla_g h = g^{ij} \partial_j h \frac{\partial}{\partial x^i}.$$

When the metric g is clear by the context or there is no need to be precise, we are going to use the notation ∇h instead of $\nabla_g h$ to simplify the notation.

Lemma 1.1.7. *If $\tilde{g} = \mathrm{e}^h g$ are two Riemannian metrics conformally related, then their Levi-Civita connections are related by the following formula:*

$$\widetilde{\nabla}_V W = \nabla_V W + \frac{1}{2}(V(h)W + V(h)W - g(V, W)\nabla_g h), \tag{1.1.7}$$

Proof. We only need to check the above relation locally. So, let us fix a chart $(U, (x^i))$ and suppose we can write $V = V^i X_i$ and $W = W^j X_j$, with $X_i := \dfrac{\partial}{\partial x^i}$, $V^i, W^j \in C^\infty(U)$. Remember that the connection applied to the pair (V, W) is totally determined by the functions V^i, W^j and the Christoffel symbols:

$$\widetilde{\nabla}_V W = (V^i \cdot X_i W^k + V^i W^j \cdot \widetilde{\Gamma}^k_{ij}) X_k.$$

Using the previous Lemma, we get

$$\begin{aligned}
\widetilde{\nabla}_V W &= (V^i \cdot X_i W^k + V^i W^j \cdot \Gamma^k_{ij}) X_k + \frac{V^i W^j}{2}(\partial_i h \delta^k_j + \partial_j h \delta^k_i - \partial_l h g^{lk} g_{ij}) X_k \\
&= \nabla_V W + \frac{1}{2}((V^i \partial_i h)(W^j \delta^k_j) X_k + (W^j \partial_j h)(V^i \delta^k_i) X_k - (V^i W^j g_{ij})(g^{lk} \partial_l h) X_k) \\
&= \nabla_V W + \frac{1}{2}(V(h) W^k X_k + W(h) V^k X_k - g(V, W) \nabla_g h) \\
&= \nabla_V W + \frac{1}{2}(V(h) W + W(h) V - g(V, W) \nabla_g h)
\end{aligned}$$

■

It is interesting to observe that if two metrics are conformally related, it does not imply that they have the same geodesics. Assume for example that γ is a geodesic for the metric g, then $D_t \gamma' = 0$ and by the previous Lemma we get

$$\widetilde{D_t} \gamma' = D_t \gamma' + \gamma'(h) \gamma' - \frac{1}{2} \|\gamma'\|^2 \nabla h = g(\nabla h, \gamma') \gamma' - \frac{1}{2} \|\gamma'\|^2 \nabla h.$$

Thus, γ is also a geodesic by the metric $\widetilde{g}$ if $\nabla h|_\gamma = 0$, i.e. γ is a critical set for the function h.

Another straightforward computation (but way longer, so it is going to be omitted. Check [Wal04].) gives us the relation between the curvature endomorphisms:

Lemma 1.1.8. *If $\tilde{g} = \mathrm{e}^h g$ are two Riemannian metrics conformally related, then their curvature endomorphisms are related by the following formula:*

$$\begin{aligned}
\widetilde{R}(X, Y)Z = R(X, Y)Z &+ \frac{1}{2}\{g(\nabla_X \nabla h, Z)Y - g(\nabla_Y \nabla h, Z)X \\
&+ g(X, Z)\nabla_Y \nabla h - g(Y, Z)\nabla_X \nabla h\} \\
&+ \frac{1}{4}\{((Yh)(Zh) - g(Y, Z)|\nabla h|^2)X - ((Xh)(Zh) - g(X, Z)|\nabla h|^2)Y \\
&+ ((Xh)g(Y, Z) - (Yh)g(X, Z))\nabla h\}.
\end{aligned} \tag{1.1.8}$$

We are finally able to relate the sectional curvatures for conformal metrics. This is the key relation we are going to use to "break" the Anosov property (check Section 1.3) of some class of Riemannian metrics.

First, observe that $|\widetilde{X \wedge Y}|^2 = \tilde{g}(X,X)\tilde{g}(Y,Y) - \tilde{g}(X,Y)^2 = e^{2h}|X \wedge Y|^2$. Then, using the previous Lemma for any pair of linearly independent vectors X, Y, the sectional curvature of the plane $\Pi = span\{X,Y\}$ is given by

$$\begin{aligned} e^h \widetilde{K}(X,Y) = K(X,Y) &+ \frac{1}{2|X \wedge Y|}\{g(\nabla_X \nabla h, Y)g(Y,X) - g(\nabla_Y \nabla h, Y)g(X,X) \\ &+ g(X,Y)g(\nabla_Y \nabla h, X) - g(Y,Y)g(\nabla_X \nabla h, X)\} \\ &+ \frac{1}{4}\{((Yh)(Yh) - g(Y,Y)|\nabla h|^2)g(X,X) \\ &- ((Xh)(Yh) - g(X,Y)|\nabla h|^2)g(Y,X) \\ &+ ((Xh)g(Y,Y) - (Yh)g(X,Y))g(\nabla h, X)\}. \end{aligned}$$

An interesting property that holds for any two metrics in the same conformal class is orthogonality of any two tangent vectors, i.e. if $\tilde{=} e^h g$, then X and Y are orthogonal for g if, and only if, $\tilde{g}$. Then we can state the following Lemma

Lemma 1.1.9. *If $\tilde{g} = e^h g$ are two Riemannian metrics conformally related, then their sectional curvature of a plane Π with orthogornal basis $\{X, Y\}$, with $g(X,X) = g(Y,Y) = 1$, are related by the following formula:*

$$e^h \widetilde{K}(X,Y) = K(X,Y) - \frac{1}{2}(g(\nabla_X \nabla h, X) + g(\nabla_Y \nabla h, Y)) - \frac{1}{4}(|\nabla h|^2 - (Xh)^2 - (Yh)^2). \tag{1.1.9}$$

The term $g(\nabla_X \nabla h, Y)$ in the previous Lemma is called *Riemannian Hessian of the function* h and is also typically denoted by $\mathrm{Hess}(h)(X,Y)$.

1.1.3 Locally symmetric metrics

Our constructions made in Chapter 2 work for a particular class of Riemannian manifolds called *Locally Symmetric*. We are going to introduce the definition and present a classification of such spaces. We also going to explore some of the interesting properties we are going to need for our constructions and fix some notation. For further results about such spaces, we refer to [Mos73] and [Jos08].

Definition 1.1.7. A Riemannian metric g is called locally symmetric if, and only if, the curvature endomorphism R is parallel with respect to the Levi-Civita connection, i.e. $\nabla R \equiv 0$.

Remember that $\nabla R\colon \mathcal{X}(M)^4 \to \mathcal{X}(M)$ is defined by

$$\begin{aligned}\nabla R(X,Y,Z,W) &:= (\nabla_W R)(X,Y,Z)\\ &= \nabla_W(R(X,Y)Z) - R(\nabla_W X,Y)Z - R(X,\nabla_W Y)Z - R(X,Y)\nabla_W Z.\end{aligned}$$

Classical examples of locally symmetric manifolds are Kähler manifolds [Mor07].

It is known that compact locally symmetric Riemannian manifolds are obtained by compact quotients of hyperbolic spaces:

Proposition 1.1.4. *A Riemannian closed manifold (M,g) is a compact locally symmetric manifold of negative sectional curvatures if, and only if, it is a compact quotient of the following hyperbolic spaces:*

(1) $\mathbb{RH}^n$*: real hyperbolic space.*

(2) $\mathbb{CH}^{\frac{n}{2}}$*: complex hyperbolic space.*

(3) $\mathbb{HH}^{\frac{n}{4}}$*: quaternionic hyperbolic space.*

(4) $\mathbb{OH}^2$*: octonionic hyperbolic space.*

The sectional curvatures are bounded by an interval depending on the holomorphic curvature of each space in the following way: let us call the holomorphic curvature by $-H^2$, then

(1) $\mathbb{RH}^n$: $K = -H^2$.

(2) $\mathbb{CH}^{\frac{n}{2}}$: $-4H^2 \leq K \leq -H^2$.

(3) $\mathbb{HH}^{\frac{n}{4}}$: $-4H^2 \leq K \leq -H^2$.

(4) $\mathbb{OH}^2$: $-4H^2 \leq K \leq -H^2$.

From now on, for our computations, we are going to consider $H = \frac{1}{2}$, but everything can be adapted modulo changing some constants in the computations.

The first interesting property we highlight is the invariance of sectional curvatures by parallel transport, that is suppose $\gamma(t)$ is a geodesic and $X(t), Y(t)$ are two parallel vector fields along $\gamma(t)$ we claim that $K(X(t), Y(t)) \equiv K(X(0), Y(0))$. Indeed, suppose with no lost of generality that $\{X(t), Y(t)\}$ are orthonormal, then

$$\begin{aligned}\frac{d}{dt}g(R(X,Y)Y,X) &= g(D_tR(X,Y)Y,X) + g(R(X,Y)Y,D_tX)\\ &= g(R(D_tX,Y)Y,X) + g(R(X,D_tY)Y,X) + g(R(X,Y)D_tY,X)\\ &= 0\end{aligned}$$

Another property that is going to be very useful for us is a characterization of the sectional curvatures: let $\gamma(t)$ be a geodesic, then there exists an orthonormal parallel frame along γ, say $\{e_0(t), e_1(t), \dots, e_r(t), e_{r+1}(t), \dots, e_{n-1}(t)\}$ such that for, $i \neq 0$, we have $R(e_i(t), e_0(t))e_0(t) = -\lambda_i^2 e_i(t)$, where $\lambda_i = 1, \frac{1}{4}$ depending on i. Therefore, the possible sectional curvatures are $K(e_i(t), e_0(t)) = -\lambda_i^2 = -1$ or $-\frac{1}{4}$. Here we make a distinction on the index r above because depending on each case of Proposition 1.1.4 we consider, there will be a fixed number of directions for which $-\lambda_i^2 = -1$ and $-\lambda_i^2 = -\frac{1}{4}$, then let us fix that

$$-\lambda_i^2 = \begin{cases} -1, \text{ if } \; i = 1, ..., r \\ -\frac{1}{4}, \text{ if } \; i = r+1, ..., n-1 \end{cases}$$

For example, for $\mathbb{RH}^n$ r is equal to $n-1$, then we have a manifold of constant curvature -1. On the other hand, for $\mathbb{CH}^{\frac{n}{2}}$ r is equal to $n-2$, that is there is only one direction $i = n-1$ such that $K(e_{n-1}(t), e_0(t)) = -\frac{1}{4}$.

The examples of compact locally symmetric Riemannian manifolds from Proposition 1.1.4 we are going to consider are the following:

(1) Compact Kähler manifolds of holomorphic curvature -1 (cf. [Gol99]).

(2) Compact locally symmetric quaternionic Kähler manifolds of negative curvature (cf. [Bes07]).

As we observed above, for a fixed tangent vector $v \in T_xM$, the tangent space (minus the direction $\mathbb{R}v$) splits into two distinct subspaces for which the sectional curvatures are -1 and $-\frac{1}{4}$. We are going to denote these spaces by $A(x, v)$ and $B(x, v)$, respectively, rather than indicating in the notation the numbers -1 and $-\frac{1}{4}$ so we can simplify the notation (see (1.1.10)) when dealing with the computations. Let us make this notation explicit to avoid any confusion: for every $v \in T_xM$, let us define the following spaces

$$A(x,v) := \{w \in T_xM : K(v,w) = -1\}$$
$$B(x,v) := \left\{w \in T_xM : K(v,w) = -\frac{1}{4}\right\}$$

Again, for locally symmetric manifolds (check [Jos08]) we get that

$$\{v\}^{\perp} = A(x,v) \oplus B(x,v).$$

Denote by Pr_C the projection to the space $C(x,v)$, with $C = A, B$. For any vector field X we will use the following notations: if α is any geodesic and $\frac{D}{dt} = \nabla_{\alpha'}$ is the covariant derivative along α, then

$$\begin{cases} X_C := & Pr_C X \\ X'_C := & Pr_C\left(\frac{DX}{dt}\right) \\ (X_C)' := & \frac{D}{dt}(Pr_C X) \\ X_{C'} := & \left(\frac{D}{dt}Pr_C\right)X \end{cases} \tag{1.1.10}$$

Since $\nabla R \equiv 0$, the subbundles $C(x,v)$ are parallel, i.e. the parallel transport of vectors in $C(x,v)$ along a smooth curve α remain in the correspondent space $C(\alpha,\alpha')$. Then for any vector field X along a geodesic α we have

$$\frac{D}{dt}(Pr_C X) = Pr_C\left(\frac{DX}{dt}\right).$$

Another way to state this is the following: saying that C is parallel also means that the covariant derivative of its projection is parallel. Remember that for any endomorphism $F\colon TM \to TM$ (also $(1,1)$-tensor) its covariant derivative is given by

$$\nabla_X F = \nabla_X \circ F - F \circ \nabla_X.$$

So, if Pr_C is parallel along every geodesic γ, then

$$(\nabla_{\gamma'} Pr_C) = \nabla_{\gamma'} \circ Pr_C - Pr_C \circ \nabla_{\gamma'} = 0.$$

Therefore, the above notation gives us $X_{C'} = 0$ and $(X_C)' = X'_C$. This remark will be important when we deform the metric since the deformed metric will not be locally symmetric, thus, the computations must consider the term "$X_{C'}$". In addition, the parallel transport of the spaces A and B along a closed prime geodesic also preserves orientation.

With our notation, the property we mentioned before about the behavior of $R(X,v)v$ will be written as follows (check [Jos08]):

$$R(X,v)v = -\frac{1}{4}X_{B(x,v)} - X_{A(x,v)}. \tag{1.1.11}$$

1.2 Geodesic Flow

One of the most natural dynamical systems arising from Riemannian geometry is the *geodesic flow*. It describes the motion of a tangent vector to the trajectory of a free particle traveling along geodesics at constant speed on a Riemannian manifold. More precisely, the *geodesic flow* is a smooth flow on the unit tangent bundle of the manifold, associating to each unit tangent vector the velocity of the unit-speed geodesic determined by that initial condition. This flow not only encodes the geometry of the manifold but also exhibits rich dynamical behavior, particularly in the presence of negative curvature.

Beyond its geometric origin, the geodesic flow fits into several broader mathematical frameworks. It is a special case of a *Hamiltonian system*, where the dynamics are governed by the kinetic energy $H(x,v) = \frac{1}{2}g_x(v,v)$ and the canonical symplectic structure on T^*M. This places a relation between the *geodesic flow* and symplectic and contact geometry.

Because of this interplay between geometry, dynamics, and analysis, the geodesic flow serves as a central object in several areas of mathematics, including spectral geometry, microlocal analysis, and topological dynamics.

In this section, we are going to define and explore the geometric and dynamical properties of the *geodesic flow*. In here, we intend to restrict ourselves to the properties that are going to be useful for the understanding of our results and we have no intention to completely develop the general theory. For a more complete exposition we refer the great book by Gabriel Paternain [Pat99].

We are going to denote a general element of TM by $\theta = (p,v)$ or just v when convenient.

Definition 1.2.1. Given a Riemannian manifold (M,g) we define the *Geodesic Flow* $g_t \colon TM \to TM$ as the family of diffeomorphisms given by

$$g_t(p,v) = (\gamma_{(p,v)}(t), \gamma'_{(p,v)}(t)).$$

The image below illustrates some possible behaviors of orbits of the geodesic flow. Notice that even though the geodesic with $\gamma'(0) = 0$ on the image has self-intersection,

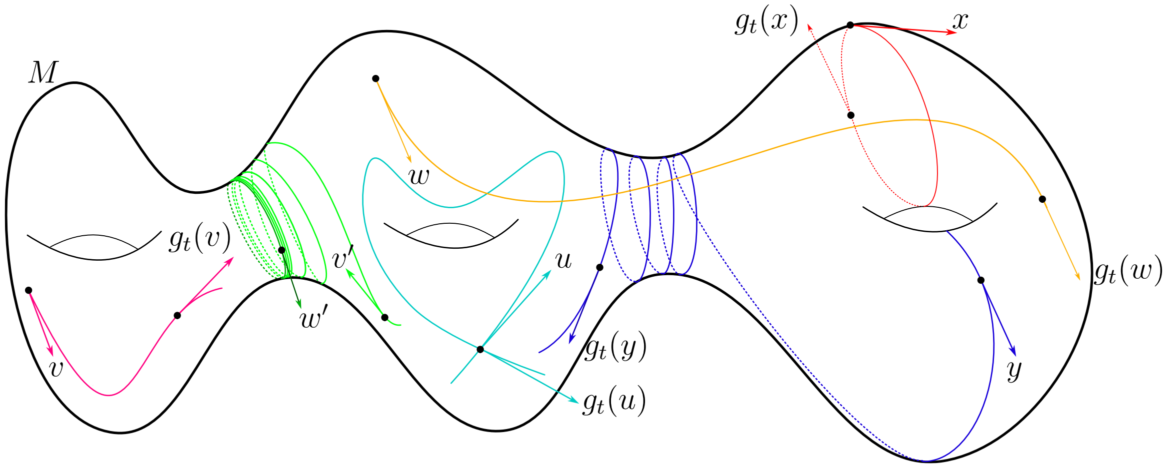


Figure 1.4: The geodesic flow on a surface of genus 3.

it does not represent a closed orbit of the geodesic flow once $u \neq g_t(u)$. Differently, we observe that the geodesic with $\gamma'(0) = x$ represents a closed orbit for the geodesic flow. It is known that there are plenty of closed orbits for the geodesic flow in negatively curved manifolds, and also, there exists at least one dense orbit.

There are many interesting questions about the dynamical system (TM, g_t). However, we are particularly interested in the Anosov property, and generally the partially hyperbolic property, which occurs for some classes of geodesic flows (check Section 1.3). We are going to properly define these properties later, but let us mention that partial hyperbolicity of a diffeomorphism $f\colon M \to M$ concerns the behavior of its derivative acting on TM. Essentially, a diffeomorphism has the partial hyperbolicity property if TM decomposes into three Df-invariant subbundles E^s, E^u and E^c such that vectors in E^s are uniformly contracted by the action of Df, vectors in E^u are uniformly contracted by the action of Df^{-1} and vectors in E^c have an intermediate behavior. For us now it is important to understand the phase space of Df in the context of the geodesic flow, i.e. the geodesic flow acts on TM, thus Dg_t acts on the tangent bundle of TM, say $T(TM)$. Initially, it is not clear what the space $T(TM)$ is and how to study it, but there are some strategies from Riemannian Geometry which are very useful.

Let us start by introducing special coordinates in $T(TM)$: denote by π the canonical projection of the vector bundle $\pi\colon TM \to M$. For $\theta \in TM$ we define the *Vertical Space at* θ as the space $V(\theta) := \ker D\pi_\theta$. Intuitively, we can see this space as *the space of tangent vectors to curves inside a single fiber of* $\pi\colon TM \to M$. Suppose $\alpha\colon (-\varepsilon, \varepsilon) \to TM$ is a curve inside some fiber of $\pi\colon TM \to M$, i.e. $\pi \circ \alpha(t) = \pi(\alpha(0))$ for all $t \in (-\varepsilon, \varepsilon)$. Taking the derivative on t, we get

$$D\pi_{\alpha(t)}\alpha'(t) = 0.$$

Thus, $\alpha'(t) \in V(\alpha(t))$. Notice that $V(\theta)$ is a subspace of $T_\theta TM$ of dimension n because π is a submersion.

For each $\theta \in TM$ one could chose any complementary space $H(\theta) \subset T_\theta TM$ of dimension n such that $T_\theta TM = V(\theta) \oplus H(\theta)$. However, there is no canonical way to choose such a space $H(\theta)$ so that it defines a subbundle structure in TTM from the bundle structure. Here, the notion of connection plays an important role in defining the spaces H in a coherent way. The construction is general, but let us assume that ∇ is the Levi-Civita connection for some Riemannian metric g. Let us show how to use the connection to define a bundle map $\mathcal{K}\colon TTM \to TM$: let $\theta \in TM$ and $\xi \in T_\theta TM$ and choose a curve $\alpha\colon (-\varepsilon, \varepsilon) \to TM$ such that

$$\begin{cases} \alpha(0) = \theta \\ \alpha'(0) = \xi \end{cases}$$

Since $\alpha(t) \in TM$, we can write $\alpha(t) = (\widetilde{\alpha}(t), W(t))$, where $\widetilde{\alpha}(t) = \pi \circ \alpha(t)$ and $W(t)$ is a vector field along $\widetilde{\alpha}$. Therefore, we can define $\mathcal{K}$ as

$$\mathcal{K}_\theta(\xi) := (\nabla_{\widetilde{\alpha}} W)(0).$$

Since $\mathcal{K}_\theta(\xi) := (\nabla_{\widetilde{\alpha}} W)(0)$ depends only on the coordinates of $\widetilde{\alpha}(0)$, $W(0)$ and Christoffel symbols, the map $\mathcal{K}$ does not depend on the choice of α, so it is well-defined. By the properties of smooth connection (remember Definition 1.1.1), it is not difficult to see that $\mathcal{K}_\theta$ is a linear map. Finally we define the *Horizontal Space* as $H(\theta) := \ker \mathcal{K}_\theta$.

Lemma 1.2.1. *Given $\theta = (p, v)$, then the following maps are linear isomorphisms:*

(1) $D\pi_\theta|_{H(\theta)}\colon H(\theta) \to T_pM$.

(2) $\mathcal{K}_\theta|_{V(\theta)}\colon V(\theta) \to T_pM$.

Proof. We prove (*2*) first. Since $\dim V(\theta) = n$ it is enough to prove that $\mathcal{K}_\theta$ is injective. In fact, suppose $\mathcal{K}_\theta(\xi) = 0$ and $\xi \in V(\theta)$. Let $\alpha\colon (-\varepsilon, \varepsilon) \to TM$ such that

$$\begin{cases} \alpha(0) = \theta, \\ \alpha'(0) = \xi \end{cases}$$

Since $\xi \in V(\theta)$ it implies that $\pi \circ \alpha(t) = \pi \circ \alpha(0) = p$. As before, write $\alpha(t) = (\tilde{\alpha}(t), W(t)) = (\tilde{\alpha}(0), W(t))$ and we have $\mathcal{K}_\theta(\xi) = (\nabla_{\tilde{\alpha}'} W)(0) = 0$. Working in a co-ordinate system we get $(\tilde{\alpha}^i)'(t) = 0$ and from the last equality that

$$(W^k)'(t) = -W^j(t)(\tilde{\alpha}^i)'(t)\Gamma_{ij}^k = 0.$$

Therefore, $W(t) \equiv W(0) = v$. We conclude that $\alpha(t) \equiv (p, v)$, so $\alpha'(t) \equiv 0$. In particular, $\alpha'(0) = \xi = 0$. To prove *(1)* it is enough to observe that $\dim H(\theta) = \dim T_\theta TM - \dim \mathrm{Im}\mathcal{K}_\theta = 2n - n = n$ and the proof is the same as in *(2)*. ∎

The previous Lemma gives us that $T_\theta TM = V(\theta) \oplus H(\theta)$. Moreover, we have that the map $j_\theta \colon T_\theta TM \to T_pM \times T_pM$ given by

$$j_\theta(\xi) = (D\pi_\theta(\xi), \mathcal{K}_\theta(\xi))$$

defines a linear isomorphism. For simplicity, we are going to use the following notation based on the previous result

$$\begin{cases} \xi_h = D\pi_\theta(\xi), \\ \xi_v = \mathcal{K}_\theta(\xi) \end{cases}$$

So, an element ξ of $T_\theta TM$ will be written as $\xi = (\xi_h, \xi_v)$.

The first interesting application of this construction we are going to see is a description of the behavior of the *Geodesic Vector Field*, i.e, the vector field $G \colon TM \to TTM$ that generates the geodesic flow g_t. In other words, the *Geodesic Vector Field* is given by

$$G(\theta) = \frac{d}{dt} g_t(\theta)|_{t=0} = \frac{d}{dt}(\gamma_{(p,v)}(t), \gamma'_{(p,v)}(t))|_{t=0}$$

Lemma 1.2.2. *Let $\theta = (p, v)$, then the Geodesic Vector Field G satisfies $G(\theta)_h = v$ and $G(\theta)_v = 0$.*

Proof. The proof is essentially two elementary computations:

$$G(\theta)_h = D\pi_\theta G(\theta) = \frac{d}{dt}\pi \circ (\gamma_{(p,v)}(t), \gamma'_{(p,v)}(t))|_{t=0} = \frac{d}{dt}\gamma_{(p,v)}(t)|_{t=0} = v.$$

By definition, we have

$$G(\theta)_v = \mathcal{K}_\theta(G(\theta)) = (\nabla_{\gamma'_{(p,v)}} \gamma'_{(p,v)})(0) = 0$$

. ∎

Our second application of the decomposition $T_\theta TM = V(\theta) \oplus H(\theta)$ is the definition of a symplectic structure on TM invariant by the geodesic flow: first, notice that we have naturally a well-defined complex structure on $T_\theta TM$ given by

$$\mathcal{J}_\theta(\xi_h, \xi_v) := (-\xi_v, \xi_h),$$

i.e. mcJ_θ is a linear isomorphism satisfying $\mathcal{J}_\theta^2 = -Id_{T_\theta TM}$. Now, let us define a Riemannian metric on TM called *Sasaki Metric* by

$$\widehat{g}((\xi_h, \xi_v), (\eta_h, \eta_v)) := g(\xi_h, \eta_h) + g(\xi_v, \eta_v).$$

Thus, $H(\theta)$ and $V(\theta)$ are orthogonal with respect to the *Sasaki Metric*. Finally, define the following 2-form on TM:

$$\Omega_\theta((\xi_h, \xi_v), (\eta_h, \eta_v)) := \widehat{g}(\mathcal{J}_\theta(\xi_h, \xi_v), (\eta_h, \eta_v)) = g(\xi_h, \eta_v) - g(\xi_v, \eta_h)$$

Lemma 1.2.3. *The 2-form Ω defined above satisfies the following properties:*

(1) Ω is anti-symmetric.

(2) Ω is non-degenerated.

(3) Ω is invariant by the geodesic flow.

Proof. To see (*1*), we compute $\Omega_\theta((\xi_h, \xi_v), (\eta_h, \eta_v))$:

$$\begin{aligned}\Omega_\theta((\xi_h, \xi_v), (\eta_h, \eta_v)) &= \widehat{g}(\mathcal{J}_\theta(\xi_h, \xi_v), (\eta_h, \eta_v)) = g(\xi_h, \eta_v) - g(\xi_v, \eta_h) \\ &= -(g(\xi_h, -\eta_v) + g(\xi_v, \eta_h)) = -\widehat{g}(\mathcal{J}_\theta(\eta_h, \eta_v), (\xi_h, \xi_v)) \\ &= -\Omega_\theta((\eta_h, \eta_v), (\xi_h, \xi_v))\end{aligned}$$

To see (*2*), let $(\xi_h, \xi_v) \in T_\theta TM$ and consider $(-\xi_v, \xi_h) \in T_\theta TM$, then

$$\Omega_\theta((\xi_h, \xi_v), (-\xi_v, \xi_h)) = g(\xi_h, \xi_h) + g(\xi_v, \xi_v) \geq 0$$

then $\Omega_\theta((\xi_h, \xi_v), (-\xi_v, \xi_h)) = 0$ if, and only if, $(\xi_h, \xi_v) = (0, 0)$. Finally, let us prove (*3*) by showing that $G(\theta)$ is the *Hamiltonian Vector Field* of $H(v) := \frac{1}{2} g(v, v)$ with respect to Ω, i.e. $\mathrm{d}H(\cdot) = \Omega(G, \cdot)$: let $\xi \in T_\theta TM$ and $\alpha\colon (-\varepsilon, \varepsilon) \to TM$ be a curve $\alpha(t) = (\widetilde{\alpha}(t), v(t))$, with $\alpha(0) = \theta = (p, v)$ and $\alpha'(0) = \xi$, then

$$\mathrm{d}H_\theta(\xi) = \frac{\mathrm{d}}{\mathrm{d}t} H \circ \alpha(t)|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} g(v(t), v(t))|_{t=0} = g(\nabla_{\widetilde{\alpha}'} v(0), v(0)) = g(\xi_v, v)$$

On the other hand, by Lemma 1.2.2

$$\Omega_\theta(G(\theta), \xi) = g(G(\theta)_h, \xi_v) - g(G(\theta)_v, \xi_h) = g(v, \xi_h)$$

■

The previous Lemma together with the following Proposition gives us that Ω is a symplectic form for TM:

Proposition 1.2.1. (Proposition 1.24 in [Pat99]) *The* 2*-form* Ω *satisfies*

$$\Omega = -\mathrm{d}\alpha,$$

where α *is the* 1*-form*

$$\alpha_\theta(\xi) := \widehat{g}(G(\theta), \xi) = g(v, \xi_h)$$

Proof. See [Pat99] Proposition 1.24 for the proof. ■

The last proposition is a key step to define a structure we will be very interested in when studying the dynamics of the geodesic flow known as *Contact Structure*.

Definition 1.2.2. For a compact manifold M^{2n+1}, a smooth non-vanishing 1-form α is called a *Contact Form* if $\mathrm{d}\alpha|_{\ker\alpha}$ is non-degenerated. Alternatively, α is called a *Contact form* if $\alpha \wedge (\mathrm{d}\alpha)^n$ is a volume form.

Definition 1.2.3. Given a contact form α, we will call the distribution $\ker\alpha$ a *Contact Structure*.

The following result is elementary from the theory of contact geometry:

Proposition 1.2.2. *Given a contact form* α*, there exists a unique vector field* R_α*, called The Reeb Vector Field of* α*, determined by* $\alpha(R_\alpha) \equiv 0$ *and* $\mathrm{d}\alpha(R_\alpha, \cdot) \equiv 0$.

Observe that contact forms can only be defined in odd-dimensional manifolds, then there is no way we can define such a structure on TTM. However, remember that TM is not a compact manifold and we are interested in studying the dynamics in compact manifolds so we are going to restrict its action to the *Unit Tangent bundle*

$$T^1M := \{(x, v) \in TM : g(v, v) = 1\},$$

which is a compact manifold whenever M is. Since geodesics are curves with constant speed $\left(\frac{\mathrm{d}}{\mathrm{d}t}g(\gamma',\gamma')\equiv 0\right)$, then the geodesic flow leaves T^1M invariant. Therefore, it makes sense to study the dynamical systems $g_t\colon T^1M\to T^1M$ with the restricted Sasaki metric $\widehat{g}$. Next Lemma shows us that $\alpha|_{T^1M}$ defines a contact form with contact structure $S(\theta):=\ker\alpha(\theta)$. Notice that $S(\theta)$ consists of the vectors in T^1M orthogonal to $\mathbb{R}\cdot G(\theta)$ with respect to the Sasaki metric.

Lemma 1.2.4. *Given $\theta=(p,v)$ the following properties hold*

(1) An element $\xi\in T_\theta TM$ is an element of $T_\theta T^1M$ if, and only if, $g(\xi_v,v)=0$.

(2) $\Omega|_{S(\theta)}$ is non-degenerate.

Proof. A tangent vector $\xi\in T_\theta T^1M$ if, and only if, it is the tangent vector to a curve in T^1M, say α. It means that $g(\alpha(t),\alpha(t))\equiv 1$ and taking the derivative on both sides and using the definition of ξ_v we get (1). Now, notice that $\xi\in S(\theta)\subset T_\theta T^1M$ if, and only if, $g(\xi_v,v)=g(\xi_h,v)=0$. Thus, as before, given $\xi\in S(\theta)\subset T_\theta T^1M$ consider $(-\xi_v,\xi_h)\in S(\theta)\subset T_\theta T^1M$ to get $\Omega_\theta((\xi_h,\xi_v),(-\xi_v,\xi_h))\geq 0$. ■

Corollary 1.2.1. *$\alpha|_{T^1M}$ defines a contact form with Reeb Vector field G.*

Since M is a compact manifold, any contact form induces a probability measure on T^1M.

Definition 1.2.4. Let α be the contact form defined above on T^1M, then the induced volume form $\alpha\wedge(\mathrm{d}\alpha)^n$ induces a probability measure called *Liouville Measure* and denoted by Liou.

One of the most remarkable results in the modern theory of dynamical systems is the ergodicity of the Liouville Measure for the geodesic flow when the metric is negatively curved. This result was proved by Anosov by using the known Hopf's Argument (which will be discussed in Chapter 3) by proving that in this case the geodesic flow has the *Uniform Hyperbolic Property* or *Anosov Property* (which will be defined below and explored more closely in Section 1.3). Remember that we have already introduced this notion above, and it refers to the behavior of Dg_t, so let us explore how to connect the sign of the curvature with Dg_t. We will see that the relation is due the *Jacobi Equation* (1.1.5)

Proposition 1.2.3. *The action of the derivative of the geodesic flow* $(Dg_t)_\theta \colon T_pM \times T_pM \to T_{\gamma_{(p,v)}(t)}M \times T_{\gamma_{(p,v)}(t)}M$ *is given by*

$$(Dg_t)_\theta(\xi_h, \xi_v) = (J_\xi(t), J'_\xi(t)),$$

where J_ξ *is the Jacobi Field along* $\gamma_{(p,v)}$ *with initial conditions* $J_\xi(0) = \xi_h$ *and* $J'_\xi(0) = \xi_v$.

Proof. Let $\alpha \colon (-\varepsilon, \varepsilon) \to TM$ be a curve of the form $\alpha(s) = (\pi \circ \alpha(s), W(s))$ such that

$$\begin{cases} \alpha(0) = \theta \\ \alpha'(0) = \xi \end{cases}$$

Consider the variation of the geodesic $\gamma_{(p,v)(t)}$ by geodesics given by

$$\begin{aligned} F \colon \mathbb{R} \times (-\varepsilon, \varepsilon) &\to TM \\ (t, s) &\mapsto \pi \circ g_t(\alpha(s)) \end{aligned}$$

It implies that $J(t) := \frac{\partial F}{\partial s}|_{s=0}(t)$ is a Jacobi Field along $\gamma_{(p,v)}$ with initial conditions

$$J(0) = \frac{\partial}{\partial s}\pi \circ g_t(\alpha(s))|_{s=0} = (D\pi)_\theta \xi$$

and

$$\begin{aligned} J'(0) &= \frac{D}{dt}\frac{\partial F}{\partial s}|_{t,s=0} = \frac{D}{dt}\frac{\partial}{\partial s}\pi \circ g_t(\alpha(s))|_{t,s=0} = \frac{D}{ds}\frac{\partial}{\partial t}\pi \circ g_t(\alpha(s))|_{t,s=0} \\ &= \frac{D}{ds}W(s)|_{s=0} = \mathcal{K}_\theta(\xi) \end{aligned}$$

On the other hand, we have

$$J(t) = \frac{\partial F}{\partial s}(t)|_{s=0} = \frac{\partial}{\partial s}\pi \circ g_t(\alpha(s))|_{s=0} = (D\pi \circ g_t)_\theta(\xi) = D\pi_{g_t(\theta)}(Dg_t)_\theta(\xi)$$

and the symmetry of the Levi-Civita connection, we get

$$\begin{aligned} J'(t) &= \frac{D}{dt}\frac{\partial F}{\partial s}(t)|_{s=0} = \frac{D}{dt}\frac{\partial}{\partial s}\pi \circ g_t(\alpha(s))|_{s=0} = \frac{D}{ds}\frac{\partial}{\partial t}\pi \circ g_t(\alpha(s))|_{s=0} \\ &= \frac{D}{ds}\gamma'_{\alpha(s)}(t)|_{s=0} = \mathcal{K}_{g_t(\theta)}((Dg_t)_\theta(\xi)) \end{aligned}$$

■

Last Proposition combined with the comparison results for Jacobi Fields we have presented in 1.1 indicates that Dg_t should exponentially contract and expand vectors somehow in the presence of negative curvature. Let us define *Anosov Geodesic Flows* and explore this property through the lens of the celebrated work by Patrick Eberlein [Ebe73]. We are going to study this property in a more general setting in Section 1.3.

Remark 1.2.1. We denote the space generated by a vector V by $\mathbb{R}V$.

Definition 1.2.5. The geodesic flow $g_t \colon T^1M \to T^1M$ is called an *Anosov flow* (with respect to the Sasaki metric on T^1M) if $T(T^1M)$ has a splitting $T(T^1M) = E^{ss} \oplus \mathbb{R}G \oplus E^{uu}$ such that there exist constants $C > 0$ and $\lambda < 0$ satisfying

$$(Dg_t)_\theta(E^{ss}(\theta)) = E^{ss}(g_t(\theta)),$$

$$(Dg_t)_\theta(E^{uu}(\theta)) = E^{uu}(g_t(\theta)),$$

$$||(Dg_t)_\theta\big|_{E^{ss}}|| \leq Ce^{\lambda t},$$

$$||(Dg_{-t})_\theta\big|_{E^{uu}}|| \leq Ce^{\lambda t},$$

for all $t \geq 0$.

We now give an intuition on the meaning of this property in terms of the dynamics of g_t: the general theory of Anosov systems guarantees that E^{ss} and E^{uu} are integrable distributions by submanifolds $\mathcal{W}^{ss}$ and $\mathcal{W}^{uu}$ satisfying that for $\eta \in \mathcal{W}^{ss}(\theta)$ (resp. $\mathcal{W}^{uu}(\theta)$), we have $d(g_t(\theta), g_t(\eta)) \to 0$ as $t \to \infty$ (resp. $t \to -\infty$).

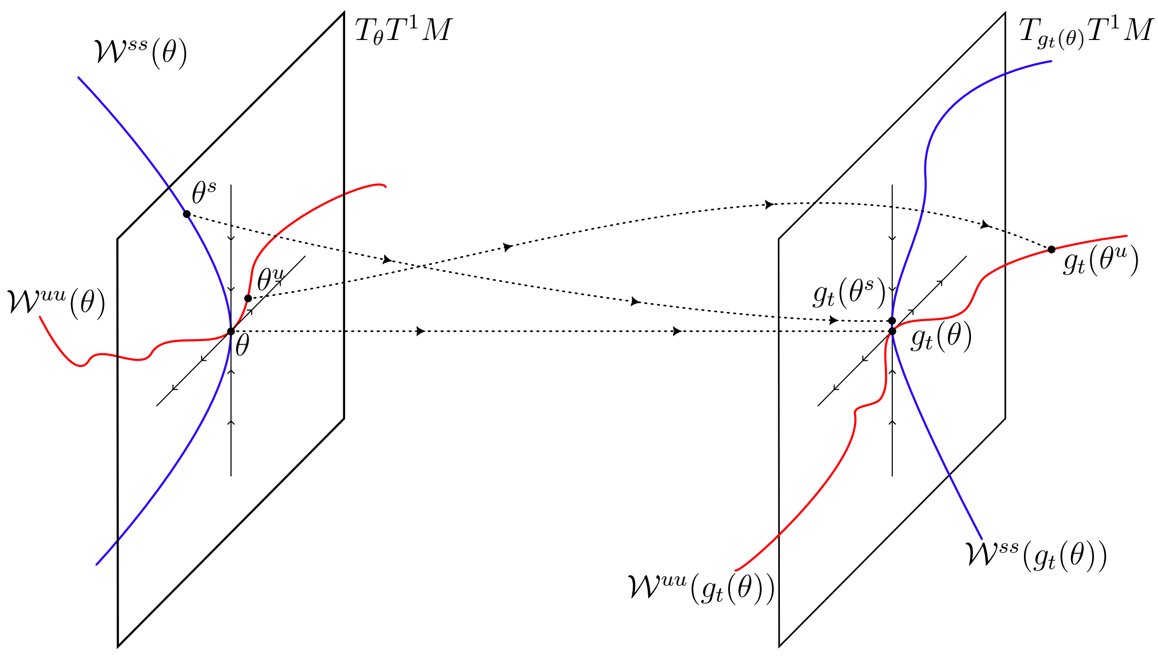


Figure 1.5: Illustration of an Anosov geodesic flow

Essentially, a geodesic flow is of Anosov type if there exist complementary directions to the flow lines such that we have contraction or expansion of distances. Previously, we mentioned that the Anosov property for the geodesic flow is closely related to the sign of the sectional curvatures. This relation is not trivial, but comparing Theorem 1.1.5 and Proposition 1.2.3 can give us an idea of the behavior of Dg_t acting on vectors. To get a better idea, let us explore some simple examples.

Example 1.2.6. Suppose $K \equiv -1$ a consider $\gamma\colon \mathbb{R} \to M$ an unit-speed geodesic. To evaluate the Jacobi Fields in this case we can proceed as follows: let $\{e_0(t) := \gamma'(t), e_1(t), \ldots, e_{n-1}(t)\}$ be an orthonormal parallel frame along γ, i.e. the following properties are satisfied

(1) $g(e_i(t), e_j(t)) = \delta_{ij}$.

(2) $D_t e_i(t) = 0$, for all $i = 0, ..., n-1$.

(3) $span\{e_0(t), ..., e_{n-1}(t)\} = T_{\gamma(t)}M$.

A Jacobi field along γ can be written as $J(t) = J^i(t)e_i(t)$, with $J^i\colon \mathbb{R} \to \mathbb{R}$, and it satisfies $J''(t) = (J^i)''e_i(t)$. Then $g(J'', e_i) = (J^i)'' = -g(R(J,\gamma')\gamma', e_i) = -J^i K(e_i, \gamma') = J^i$. Then, $J^i = (J^i)''$ and solving this ODE we get

$$J^i(t) = \left(\frac{J^i(0) - (J^i)'(0)}{2}\right) e^{-t} + \left(\frac{J^i(0) + (J^i)'(0)}{2}\right) e^t.$$

Since Jacobi Fields are determined by the initial conditions, we can see that if $J(0) = -J'(0)$, then $\|Dg_t(J(0), -J(0))\| \leq Ce^{-t}$ for some big enough constant $C > 0$. In this case is easy to see that for Dg_t exponentially contracts vectors of the form $(v, -v)$ and $t > 0$. The same analysis works for vectors of the form (v, v). ♦

Example 1.2.7. Suppose $K \equiv 0$ and proceeding as in the previous example, we get that Jacobi Fields have coordinates satisfying $(J^i)'' = 0$. Therefore, $J^i(t) = (J^i)'(0)t + J^i(0)$ and there is no exponential contraction nor expansion of the norm of a Jacobi field along time. In this case, the geodesic flow does not satisfy the Anosov property. ♦

Example 1.2.8. Suppose $K \equiv 1$. By the work of Wilhelm Klingenberg [Kli74], we know that if the geodesic flow is of Anosov type, then (M, g) has no conjugate points. It implies that if $K \equiv 1$, then the geodesic flow can not present the Anosov property. ♦

Beyond constant curvature examples, it is nontrivial to determine if the geodesic flow presents the Anosov property. It is worth remembering that exploring the Anosov property for the geodesic flow was crucial in the study of ergodicity for the Liouville measure. First results on this direction were obtained by Eberhard Hopf in [Hop39] and [Hop40] for surfaces and constant curvatures. Hopf developed a very beautiful idea to obtain ergodicity, now known as the *Hopf's Argument*. We will extensively explore this idea in Chapter 3 when proving ergodicity of the restriction of a flow to ergodic homoclinic classes. Later, Dmitry Anosov and Yakov Sinai generalized the results by Hopf for the general setting of any dimension and varying negative curvature in [Ano67] and [AS67]. At this point, Anosov formalized the proof of the property geodesic flows satisfy for negatively curved metrics, the previously defined *Anosov property*:

Theorem 1.2.9 ([Ano67])**.** *If (M, g) is a compact Riemannian manifold with negative sectional curvatures, then g_t is of Anosov type.*

A natural question is whether the only Anosov geodesic flows are those for which the sectional curvatures are all negative. However, there are also examples of compact manifolds with regions of zero and positive curvature whose geodesic flow is Anosov ([Gul75], [DP03], and [Ebe73]). Nevertheless, the Anosov condition is inherently connected to the presence of negative curvature as explored in the celebrated paper by Patrick Eberlein [Ebe73]. The classification made by Eberlein is one of the key points for our work in order to break the Anosov property of some class of metrics. We are going to present his results not in full generality since we have not defined many of the objects that appear in his paper, but we are going to explicitly indicate how we are going to use them in Chapter 2.

Theorem 1.2.10. (Theorem 3.2 in [Ebe73]) *The following are equivalent:*

(1) The geodesic flow in T^1M is of Anosov type.

(2) There exists no nonzero perpendicular Jacobi vector field $J(t)$ along a unit-speed geodesic $\gamma(t)$ such that $\|J(t)\|$ is bounded for all $t \in \mathbb{R}$.

The above theorem has the following consequence:

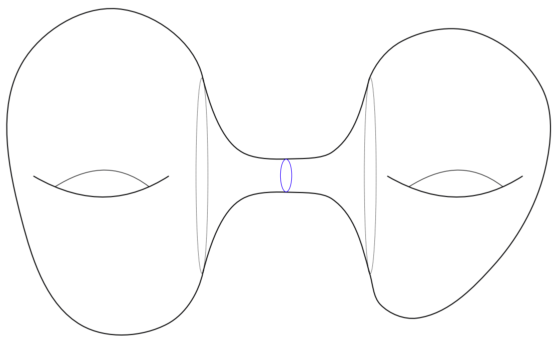
Proposition 1.2.4. (Corollary 3.4 in [Ebe73]) *Let (M, g) be a Riemannian manifold. If the geodesic flow g_t is of Anosov type, the following holds: Let γ be any unit speed geodesic of M, and $X(t)$ be any nonzero perpendicular parallel vector field along α. Then the sectional curvature $K(X, \alpha')(t) < 0$ for some $t \in \mathbb{R}$.*

Last proposition can be understood as an obstruction for a Jacobi field J to be parallel along a geodesic γ once a Jacobi field is parallel if, and only if, the planes $\Pi(t) =$

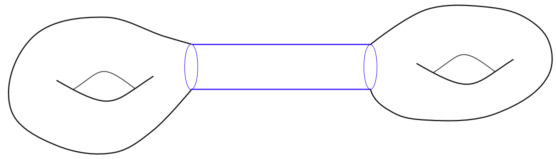
$span\{J(t), \gamma'(t)\}$ all have zero sectional curvature. It also indicates how to break the Anosov property in the sense that if we start with a metric g for which the geodesic flow is of Anosov type, then we can obtain a non-Anosov geodesic flow by performing a C^2-deformation (remember Lemma 1.1.3) on g to produce a parallel Jacobi Field. This is done in Subsection 2.2. This result also makes it trivial to check if some surfaces have Anosov geodesic flows. Let us consider the following examples (check the Figure below):

Example 1.2.11. Let S be the surface of revolution obtained by rotating the graph of $f(x) = x^4 + 1$, defined in some interval $[-a, a]$, around the x-axis and then gluing two negatively curved surfaces to this neck. The obtained surface is non-positively curved, and moreover, it has negative Gaussian curvature besides a central closed geodesic γ at $x = 0$ for which the Gaussian curvature is identically zero. Since the dimension is 2, the Gaussian curvature determines the only possible sectional curvature. Then, any parallel vector field along γ is a Jacobi Field, thus, the geodesic flow is not of Anosov type. ♦

Example 1.2.12. Another easy example is obtained by considering a piece of cylinder C and then gluing to negatively curved surfaces to this neck. The obtained surface has a large set of closed geodesics with null Gaussian curvature (the whole C), thus by the same arguments, the geodesic flow is not Anosov. ♦



(a) Surface of Example 1.2.11



(b) Surface of Example 1.2.12

We are not going to prove ergodicity of Anosov geodesic flows in this dissertation but we refer to the beautiful book by Werner Ballmann [Bal95] where the reader can find many nice results about the geodesic flow for the general setting of non-positively curved metrics and an Appendix by Misha Brin about the ergodicity of the geodesic flow. Ergodicity for the Liouville measure is not exclusive of Anosov geodesic flow. Indeed, our results show the existence of ergodic geodesic flows with a weaker notion of hyperbolicity called *Partial*

Hyperbolicity (check Section 1.3 for the definition). We get ergodicity by using more general results obtained back in the 80s as [BBE85], [BB82], and [Bur83]. Their results concern metrics with *rank* one: the rank of a tangent vector $v \in T^1M$ is the dimension of the vector space of parallel Jacobi fields along the geodesic with initial condition $\gamma'(0) = v$ and the rank of the Riemannian metric is $\text{rank}(M, g) = \inf_{v \in T^1M} \text{rank}(v)$. The only issue with their results is the assumption that rank one implies that the set of vectors with rank bigger than one has measure zero. If this condition holds and the curvature is non-positive, then the geodesic flow has even more chaotic behavior, such as mixing and the Bernoulli property. Combining their results we can state the following theoem

Theorem 1.2.13 ([BB82],[BBE85], [Bur83])**.** *Let (M, g) be a Riemannian metric with non-positive sectional curvatures and rank one. If the set of vectors with rank bigger than one has measure zero, then the geodesic flow is a Bernoulli flow. In particular, the geodesic flow is ergodic for the Liouville measure.*

The setting of the theorem above is precisely the setting of our Theorem B and also Example 1.2.11. Non-positive curvature is also not a necessary condition to get hyperbolicity, some results by Burns, Gerber, Donnay, and Gulliver in [BG89], [DP03], [Don06], and [Gul75].

1.3 Partial Hyperbolicity

In this section, we are going to present all the definitions in the particular setting of the geodesic flow, however, the definitions are the same for the general case of flows and diffeomorphisms. We are not going to use many of the interesting properties of a partially hyperbolic system, but the definition and equivalent relations. Therefore, we are going to make this section concise and direct to the results we need. The classical definition of partial hyperbolicity is the following:

Definition 1.3.1. A geodesic flow $g_t \colon T^1M \to T^1M$ is called *Partially Hyperbolic* if there exists a nontrivial Dg_t-invariant splitting $T(T^1M) = E^{ss} \oplus E^c \oplus \mathbb{R}G \oplus E^{uu}$ such that

$$\|(Dg_t)_\theta|_{E^{ss}}\| < Ce^{\lambda t},$$

$$\|(Dg_t)_\theta|_{E^{uu}}\| \leq Ce^{-\lambda t},$$

$$Ce^{\mu t} \leq \|(Dg_t)_\theta|_{E^c}\| \leq Ce^{-\mu t},$$

for some $\lambda < \mu \leq 0 < C$ and for all $\theta \in T^1M$.

In other words, it means that the dynamics decomposes in directions for which distances are exponentially contracting for positive time, exponentially contracting for negative time, and an intermediate behavior. The action of Dg_t on E^c may be contracting or expanding the size of vectors as well as be an isometry on some vectors, the important property that we should have in mind is that if there are expansion (resp. contraction) it is not as strong as the contraction (resp. expansion) of E^{ss} (resp. E^{uu}). It is worth clarifying the meaning of "nontrivial" in Definition 1.3.1: by "nontrivial" we mean that $E^{ss} \neq \{0\}$ and $E^{uu} \neq \{0\}$ so the system always presents some exponential contraction and expansion. However, we do not require $E^c \neq \{0\}$. Indeed, it leads us to the first example:

Example 1.3.2. From Definition 1.2.5 we have that Anosov geodesic flows are examples of *Partially hyperbolic* geodesic flows with $E^c = \{0\}$. Then, from the discussion at the end of the previous Section, we see that negatively curved compact Riemannian manifolds present partially hyperbolic geodesic flows. Our results, Theorem A and Theorem B, are precisely the construction of *Partially hyperbolic* geodesic flows which are not Anosov. ♦

Notice that no surface can present a partially hyperbolic geodesic flow that is not Anosov. This is just a matter of dimension since $\dim T^1S = 3$. So, for our construction, we must consider higher-dimensional manifolds. Indeed, we can not visualize partially hyperbolic flows which are not Anosov because this is a phenomenon of dimension at least 4. However, we can analogously define partial hyperbolicity for diffeomorphisms:

Definition 1.3.3. A diffeomorphism $f\colon M \to M$ is called *Partially Hyperbolic* if there exists a nontrivial Df-invariant splitting $TM = E^{ss} \oplus E^c \oplus E^{uu}$ such that

$$\|(Df^n)_x|_{E^{ss}}\| < Ce^{\lambda n},$$

$$\|(Df^n)_x|_{E^{uu}}\| \leq Ce^{-\lambda n},$$

$$Ce^{\mu n} \leq \|(Df^n)_x|_{E^c}\| \leq Ce^{-\mu n},$$

for some $\lambda < \mu \leq 0 < C$ and for all $x \in M$.

Examples in dimension 3 are now possible and not difficult to produce:

Example 1.3.4. Consider the following matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix A induces a partially hyperbolic diffeomorphism f_A on $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$. ♦

Example 1.3.5. Let $\varphi_t \colon M^3 \to M^3$ be an Anosov flow (for example, the geodesic flow for a negatively curved Riemannian metric), then for each $T > 0$ the diffeomorphism $f = \varphi_T$ as the time$-T$ map is a partially hyperbolic diffeomorphism. In this case $E^c = \mathbb{R}X$, where X is the vector field which generates the flow φ_t. ♦

On the other hand, there is a simple construction to produce a flow example from a diffeomorphism.

Example 1.3.6. Suppose $f \colon M \to M$ is a partially hyperbolic diffeomorphism, then any suspension flow φ_t inherits a partially hyperbolic structure. ♦

Another way to present some kind of partial hyperbolicity appears as the so-called *Dominated Splitting*, which we present below

Definition 1.3.7. An Dg_t-invariant splitting $E \oplus F$ of $T(T^1M)/\mathbb{R}G$ is called a dominated splitting if

(1) E and F are Dg_t-invariant: $Dg_tE(\theta) = E(g_t(\theta))$ and $Dg_tF(\theta) = F(g_t(\theta))$, for all $\theta \in T^1M$ and $i = 1, ..., k$.

(2) The following inequality holds

$$\|(Dg_t)_\theta|_{E(\theta)}\| \cdot \|(Dg_{-t})_{g_t(\theta)}|_{F(g_t(\theta))}\| < Ce^{-\lambda t},$$

for some $C, \lambda > 0$ and for all $\theta \in T^1M$.

In general, we say that a splitting $T(T^1M)/\mathbb{R}G = E_1 \oplus \cdots \oplus E_k$ is dominated if

(1) Each E_i is Dg_t-invariant: $Dg_tE_i(\theta) = E_i(\theta)$, for all $\theta \in T^1M$ and $i = 1, ..., k$.

(2) The following inequality holds

$$\|(Dg_t)_\theta|_{E_i(\theta)}\| \cdot \|(Dg_{-t})_{g_t(\theta)}|_{E_{i+1}(g_t(\theta))}\| < Ce^{-\lambda t},$$

for some $C, \lambda > 0$ and for all $\theta \in T^1M$.

The concepts of splitting domination and partial hyperbolicity are related by the following result

Proposition 1.3.1. *A geodesic flow $g_t\colon T^1M \to T^1M$ is partially hyperbolic if, and only if, there exists a dominated splitting $T(T^1M)/\mathbb{R}G = E^{ss} \oplus E^c \oplus E^{uu}$ such that vectors in E^{ss} are exponentially contracted by Dg_t and $t \geq 0$ and vectors in E^{uu} are exponentially expanded by Dg_t and $t \geq 0$.*

Dominated splitting and the uniform contraction expansion in E^{ss} and E^{uu} in the definition of partial hyperbolicity are, in general, different. However, in the special case of symplectic dynamics (as the case of geodesic flows), we have a beautiful connection from the works [Con02] and [Rug91b] (see also [CP14, Lemma 2.8]).

Lemma 1.3.1. *Let (N, ω) be a symplectic manifold, ω its symplectic 2-form and $\phi_t\colon N \to N$ a flow on N generated by a vector field X such that $\mathcal{L}_X(\omega) = 0$,* i.e., *the flow preserves the symplectic structure of N. If there is a dominated splitting $TN/\mathbb{R}X = E \oplus E^c \oplus F$ such that $dim(E) = dim(F)$, then for all $x \in N$ there exist $C > 0$ and $\lambda < 0$ such that*

$$\|(D\phi_t)_x|_E\| \leq Ce^{\lambda t}, \qquad \|(D\phi_{-t})_x|_F\| \leq Ce^{\lambda t}.$$

Essentially, the previous lemma says that in symplectic dynamics, if there is a dominated splitting with extremal subbundles of the same dimension, then the dynamics is partially hyperbolic. This connection will be important in our case because our proof of partially hyperbolicity in Chapter 2 is based on proving the existence of a dominated splitting for the constructed geodesic flow, then concluding partial hyperbolicity via this lemma.

To prove the existence of dominated splitting in the next chapter, we are going to use a classical result on hyperbolicity and partial hyperbolicity called *the Cone Criteria*. Let us start by defining what a cone is in our context:

Definition 1.3.8. Given $\theta \in T^1M$, a subspace $E(\theta) \subset T_\theta T^1M$, and $\delta > 0$, we define the cone at θ centered around $E(\theta)$ with angle δ (or opening of the cone) as

$$\mathcal{C}(\theta, E(\theta), \delta) := \{\xi \in T_\theta T^1M : \angle(\xi, E(\theta)) < \delta\},$$

where $\angle(\xi, E(\theta))$ is the angle between ξ and $E(\theta)$. Alternatively, one may define a cone by any non-degenerate quadratic form Q on $T_\theta(T^1M)$ as

$$\mathcal{C}_\theta^Q = \{\xi \in T_\theta(T^1M) : Q(\xi) \geq 0\}.$$

In any of the above cases, a cone field is a continuous choice of cones $\mathcal{C}_\theta$, for each $\theta \in T^1M$. It is possible to consider variations on the metric which are being considered here but it can be proved that partial hyperbolicity does not depend on the considered metric. Therefore, we are going to always assume the results for any previously fixed metric. So, for example, when considering the angle between two tangent vectors $\xi, \xi' \in T_\theta(T^1M)$ it is the number $\Theta \in [0, \pi)$ such that

$$\cos\Theta = \frac{\widehat{g}(\xi, \xi')}{\|\xi\| \, \|\xi'\|},$$

where $\widehat{g}$ is the Sasaki metric. Then, a cone as defined above can be written as

$$\mathcal{C}(\theta, E(\theta), \delta) = \mathcal{C}^*(\theta, E(\theta), c) = \{\xi \in T_\theta(T^1M) : \cos\Theta(\xi, E(\theta)) \geq c\}$$

The next proposition gives us an equivalent definition for partially hyperbolicity by using cones:

Proposition 1.3.2 (Cone criterion)**.** *The geodesic flow g_t is partially hyperbolic if there are $\delta > 0$, $T > 0$, and two continuous families of cones $\mathcal{C}(\theta, E_1(\theta), \delta)$ and $\mathcal{C}(\theta, E_2(\theta), \delta)$ such that:*

(1) $(Dg_t)_\theta(\mathcal{C}(\theta, E_1(\theta), \delta)) \subsetneq \mathcal{C}(\theta, E_1(g_t(\theta)), \delta)$

(2) $(Dg_{-t})_\theta(\mathcal{C}(\theta, E_2(\theta), \delta)) \subsetneq \mathcal{C}(\theta, E_2(g_{-t}(\theta)), \delta)$

(3) $\|(Dg_t)_\theta(\xi_1)\| < Ke^{\lambda t}$

(4) $\|(Dg_{-t})_\theta(\xi_2)\| < Ke^{\lambda t}$

for all $t > 0$, $\xi_1 \in \mathcal{C}(\theta, E_1(\theta), \delta)$, $\xi_2 \in \mathcal{C}(\theta, E_2(\theta), \delta)$, and some constants $K > 0$, $\lambda < 0$.

We highlight that conditions (*1*) and (*2*) guarantee the existence of a dominated splitting $T(T^1M)/\mathbb{R}G = E_1 \oplus E^c \oplus E_2$ and conditions (*3*) and (4) imply exponential contraction and expansion, then partial hyperbolicity is satisfied as state in Proposition 1.3.1.

Remark 1.3.1. Wrapping everything together, we see that in the case of symplectic dynamics (see Lemma 1.3.1) it is enough to check the invariance of the cone families, i.e. conditions (*1*) and (*2*). This is indeed the strategy used in Section 2.3.2. We also are able to prove in Section 2.3.1 the presence of dominated splitting for the initial locally symmetric Riemannian metrics.

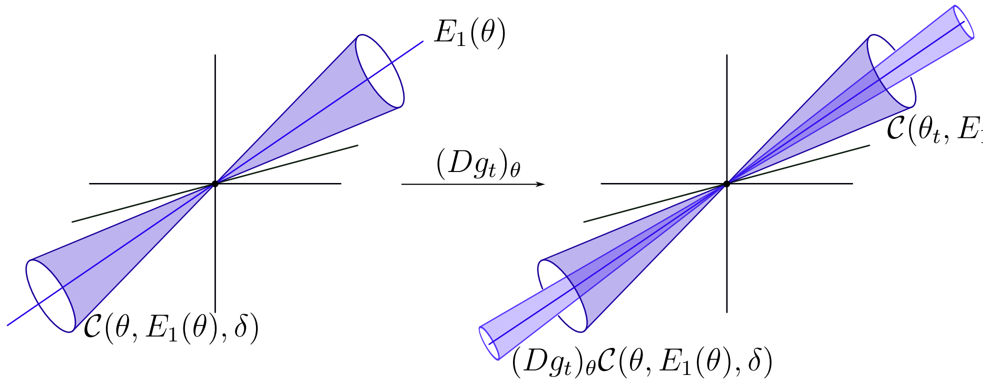


Figure 1.7: Cone criteria for partial hyperbolicity for $\theta_t = g_t(\theta)$.

In the following, we present a strategy to prove that a family of cones is invariant along any orbit of the geodesic flow, and also that it is enough to check it for the boundary of the cone. Let E be a vector bundle over T^1M which is a subbundle of TT^1M and $\pi_E \colon E \to T^1M$ its canonical projection. Let $\mathrm{Pr}_E \colon TT^1M \to E$ be the orthogonal projection to E. We define the real function $\Theta_E \colon TT^1M \to \mathbb{R}$ as

$$\Theta_E(\xi) := \frac{g(\mathrm{Pr}_E(\xi), \mathrm{Pr}_E(\xi))}{g(\xi, \xi)}. \tag{1.3.1}$$

Lemma 1.3.2. [CP14, Lemma 2.10] *For a $\delta > 0$, and a fixed vector bundle E on T^1M, if*

$$\frac{d}{dt}\Theta_{E(g_t(\theta))}((Dg_t)_\theta(\xi)) > 0, \tag{1.3.2}$$

for $\xi \in \partial C(\theta, E(\theta), \delta) := \{\xi \in T_\theta T^1M : \angle(\xi, E(\theta)) = \delta\}$, then the family of cones $C(\theta, E(\theta), \delta)$ is invariant for he geodesic flow.

1.4 Nonuniform hyperbolicity

In this section, we present another notion that generalizes the Anosov property called *Nonuniform hyperbolicity.* The theory of *Nonuniform Hyperbolic systems* are also known as *Pesin's Theory* due the work of Yakov Pesin, who introduced and formalized this weaker notion of hyperbolicity in [Pes74], [Pes76], [Pes77a] and [Pes77b]. In the last one, Pesin proves stronger results about the chaotic behavior of the geodesic flow of surfaces (and also some cases in higher dimensions). He proves that for a surface of genus bigger than one and without focal points, the geodesic flow is conjugate to a Bernoulli flow, which implies ergodicity in particular.

Here we are not going to restrict ourselves to the case of geodesic flow, although many of the techniques here were developed in order to understand this kind of dynamical system, and also this class of systems presents many important examples, such as Example 1.2.11.

Remember that the notion of Anosov systems and, more generally, partially hyperbolic systems requires some exponential rate of expansion and contraction. In some sense, this quantity can be "almost always" calculated as a notion called *Lyapunov Exponents* (check definition below). In the nonuniform hyperbolic case, we will see that these quantities are neither constant nor can be continuously bounded away from 0 as in Anosov systems. We also point out that for the general scenario of *Nonuniform hyperbolicity*, the constant C in Definition 1.3.1 will be substituted by a measurable function,n and this is one of the important differences of the generalization.

Once again, to make this dissertation as self-contained as possible, but still concise we chose to present the definitions and results we are going to use and give the appropriate references for the proofs. Among several nice references available on this topic, we indicate the books by Yakov Pesin and Luis Barreira [BP02] and [BP07].

Let us start by defining a key concept to the study of Nonuniform hyperbolic systems called *Lyapunov Exponents*. Indeed, *Lyapunov Exponents* are fundamental tools in the study of dynamical systems as they quantify the exponential rates at which nearby trajectories diverge or converge over time. Let us properly define this concept for the general setting of a flow with no restriction whatsoever to geodesic flows: let $\varphi_t \colon M \to M$ be a C^1-flow

Definition 1.4.1. For each $x \in M$ and $v \in T_xM$ we define the *Lyapunov Exponent* associated to (x, v) as

$$\chi(x, v) = \limsup_{|t|\to\infty} \frac{1}{|t|} \log \|(D\varphi_t)_x v\|,$$

when the above limit exists.

For each $x \in M$, it can be proved that $\chi(x, \cdot)$ attains finitely many distinct values on the tangent space at x, so we denote them by

$$\chi_1(x) < \cdots < \chi_{l(x)}(x).$$

At any point $x \in M$, the Lyapunov Exponents define a filtration of the tangent space at

x, i.e. a sequence of subspaces $\{W_i(x)\}_{i=0}^{l(x)}$, defined by

$$W_i(x) = \{v \in T_xM : \chi(x, v) \leq \chi_i(x)\},$$

which satisfies $\{0\} = W_0(x) \subsetneq W_1(x) \subsetneq \cdots \subsetneq W_{l(x)}(x) = T_xM$. The multiplicity of each Lyapunov Exponent is the number $n_i(x) = \dim W_{i+1}(x) - \dim W_i(x)$.

The first natural question that arises about Lyapunov Exponents is why or when the limit in Definition 1.4.1 exists. This question is answered by one of the most fundamental theorems in the theory of nonuniform hyperbolic systems, and it is a consequence of the Multiplicative Ergodic Theorem by Valery Oseledets [Ose68]:

Theorem 1.4.2 (Oseledets' Decomposition). *Let $\varphi_t\colon M \to M$ be a C^1-flow on a closed Riemannian manifold M. There exists an invariant set $\mathcal{R} \subset M$ of full measure with respect to any invariant Borel probability measure μ, such that for every $x \in \mathcal{R}$:*

(1) The tangent space T_xM admits a splitting:

$$T_xM = \bigoplus_{i=1}^{l(x)} E_i(x),$$

where $E_i(x)$ are called the Oseledets subspaces and $l(x)$ is the number of distinct Lyapunov exponents at x.

(2) There exist real numbers $\chi_1(x) < \cdots < \chi_{l(x)}(x)$ such that for any $v \in E_i(x) \setminus \{0\}$:

$$\lim_{|t|\to\infty} \frac{1}{t} \log \|D\varphi_t(x)v\| = \chi_i(x).$$

(3) The subspaces $E_i(x)$ are invariant under the derivative of the flow:

$$D\varphi_t(x)(E_i(x)) = E_i(\varphi_t(x)).$$

(4) For any disjoint subsets $I, J \subset \{1, \ldots, l(x)\}$, let $E_I(x) = \bigoplus_{i\in I} E_i(x)$ and $E_J(x) = \bigoplus_{j\in J} E_j(x)$. Then:

$$\lim_{|t|\to\infty} \frac{1}{t} \log \angle(D\varphi_t(x)E_I(x), D\varphi_t(x)E_J(x)) = 0.$$

(5) The growth rate of the determinant of the derivative of the flow is given by:

$$\lim_{|t|\to\infty} \frac{1}{t} \log\det(D\varphi_t(x)) = \sum_{i=1}^{l(x)} \chi_i(x) \dim E_i(x).$$

(6) The functions $\chi_i(x)$ and $\dim E_i(x)$ are φ_t-invariant. In particular, if μ is ergodic, then $\chi_i(x)$ and $\dim E_i(x)$ are μ-almost everywhere constant.

Definition 1.4.3. The set $\mathcal{R}$ is called the set of *Regular Points.*

We now outline essential results from Pesin's Theory, focusing on their application to flows, which is our main interest in this dissertation. All the presented results have analogous for diffeomorphisms. We are not going to present the proofs here since they are mostly technical and go beyond the purpose of this dissertation. For the complete development of the theory, we refer to [BP02] and [BP07].

Definition 1.4.4. Let μ be an ergodic measure. If, there exists a μ-full measure set $\widetilde{\mathcal{R}} \subset \mathcal{R}$ such that for $x \in \widetilde{\mathcal{R}}$ we have: the subspace $E_0(x)$, generated by the vectors with zero Lyapunov exponents, satisfies $E_0(x) = \mathbb{R}X$, where X is the vector field that generates the flow φ_t. Then the flow is said to be *nonuniformly hyperbolic* on $\widetilde{\mathcal{R}}$, and μ is called a *hyperbolic measure* on $\widetilde{\mathcal{R}}$.

For the points $x \in \widetilde{\mathcal{R}}$ notice that there exists $k \in \mathbb{N}$ such that $\chi_k(x) < 0 < \chi_{k+1}(x)$ and we can define

$$E^s(x) = \bigoplus_{\chi_i(x)<0} E_i(x) \quad \text{and} \quad E^u(x) = \bigoplus_{\chi_i(x)>0} E_i(x).$$

We have the following results that summarize the properties of these spaces

Theorem 1.4.5. [BP02, Theorem 2.1.3] *The following properties hold for $x \in \widetilde{\mathcal{R}}$:*

(1) $E^s(x)$ and $E^u(x)$ depend measurably on $x \in \widetilde{\mathcal{R}}$.

(2) We have the splitting $T_xM = E^s(x) \oplus \mathbb{R}X \oplus E^u(x)$.

(3) $(D\varphi_t)_x E^{s,u}(x) = E^{s,u}(\varphi_t(x))$, for all $t \in \mathbb{R}$.

There exists $\varepsilon_0 > 0$ and Borel functions $C(x,\varepsilon) > 0$ and $K(x,\varepsilon) > 0$ such that for all $x \in \widetilde{\mathcal{R}}$ and $0 < \varepsilon \leq \varepsilon_0$

(1) For all $v \in E^s(x)$ and $t > 0$ it holds that

$$\|(D\varphi_t)_x v\| \leq C(x,\varepsilon)\mathrm{e}^{(\chi_k+\varepsilon)t} \|v\| .$$

(2) For all $v \in E^u(x)$ and $t < 0$ it holds that

$$\|(D\varphi_t)_x v\| \leq C(x,\varepsilon)\mathrm{e}^{(\chi_k-\varepsilon)t}\|v\|.$$

(3) The angle between the spaces $E^s(x)$ and $E^u(x)$ is bounded from below by the function $K(x,\varepsilon)$.

(4) The functions $C(x,\varepsilon)$ and $K(x,\varepsilon)$ are not φ_t-invariant in general but they satisfy

$$C(\varphi_t(x),\varepsilon) \leq C(x,\varepsilon)\mathrm{e}^{\varepsilon|t|}$$

and

$$K(\varphi_t(x),\varepsilon) \geq K(x,\varepsilon)\mathrm{e}^{-\varepsilon|t|}$$

From now on, let us denote $\widetilde{\mathcal{R}}$ by $\mathcal{R}$, just to simplify our notation. We can provide a more detailed description of the structure of a nonuniform hyperbolic set $\mathcal{R}$ of Regular Points. Let $\varepsilon > 0$ and $l > 0$, we define the *Pesin block* (of level l) as the following set:

$$\mathcal{R}^{i,j}_{\varepsilon,l} = \left\{x \in \mathcal{R} \,:\, C(x,\varepsilon) \leq l,\ K(x,\varepsilon) \geq \frac{1}{l}, \dim E^s(x) = i \text{ and } \dim E^u(x) = j\right\}.$$

Sometime we will simply consider

$$\mathcal{R}^l := \left\{x \in \mathcal{R} \,:\, C(x,\varepsilon) \leq l,\ K(x,\varepsilon) \geq \frac{1}{l}\right\}.$$

This set has the following fundamental properties:

(1) $T_xM = \bigoplus_{\lambda<0} E_\lambda(x) \oplus E^c(x) \bigoplus_{\lambda>0} E_\lambda$, where $E^c(x) = E_0(x) \oplus X(x)$.

(2) $\mathcal{R}^{i,j}_{\varepsilon,l} \subset \mathcal{R}^{i,j}_{\varepsilon,l+1}$;

(3) for any $t \in \mathbb{R}$, $\varphi_t(\mathcal{R}^{i,j}_{\varepsilon,l}) \subset \mathcal{R}^{i,j}_{\varepsilon,l'}$, where $l' = l\exp(|t|\varepsilon)$;

(4) the subspaces $E^s(x)$ and $E^u(x)$ depend continuously on $x \in \mathcal{R}^{i,j}_{\varepsilon,l}$.

(5) There exists $\delta := \delta(i,j,\varepsilon,l) > 0$ such that for any $x \in \mathcal{R}^{i,j}_{\varepsilon,l}$, $W^s(x)$ and $W^u(x)$ contain open disks containing x of dimension i and j, respectively, and uniform diameter δ. They are called, respectively, local stable and unstable manifolds of x and are denoted, respectively, by $W^s_{\mathrm{loc}}(x)$ and $W^u_{\mathrm{loc}}(x)$.

Now we present the Stable Manifold Theorem of Pesin, a fundamental result in the theory of nonuniformly hyperbolic dynamical systems, particularly in the context of flows.

Theorem 1.4.6. [BP02, Stable manifold for flows] *Let $\mathcal{R}$ be a nonuniformly hyperbolic set for a smooth flow φ_t. Then, for every $x \in \mathcal{R}$, there exist a local stable manifold $W^s_{loc}(x)$ and a local unstable manifold $W^u_{loc}(x)$ such that:*

(1) $x \in W^s_{loc}(x)$, $T_xW^s_{loc}(x) = E^s(x)$, and for any $y \in W^s_{loc}(x)$ and $t > 0$,

$$d(\varphi_t(x), \varphi_t(y)) \leq T(x)\lambda^t e^{\varepsilon t} d(x, y),$$

where $T : \mathcal{R} \to (0, \infty)$ is a Borel function satisfying, for all $s \in \mathbb{R}$,

$$T(\varphi_s(x)) \leq T(x)e^{10\varepsilon|s|}.$$

(2) $x \in W^u_{loc}(x)$, $T_xW^u_{loc}(x) = E^u(x)$, and for any $y \in W^u_{loc}(x)$ and $t < 0$,

$$d(\varphi_t(x), \varphi_t(y)) \leq T(x)\lambda^{-t} e^{\varepsilon|t|} d(x, y).$$

Definition 1.4.7. The manifolds $W^{s,u}_{loc}(x)$ are called the (un)stable Pesin's manifolds.

For a regular point $x \in M$, denote its orbit by $\gamma := \{\varphi_t(x)\}_{t\in\mathbb{R}}$. So we can define its *global (un)stable manifold* as

$$W^s(x) = \bigcup_{t\geq 0} \varphi_{-t}(W^s_{loc}(\varphi_t(x)), \quad W^u(x) = \bigcup_{t\geq 0} \varphi_t(W^s_{loc}(\varphi_{-t}(x)).$$

We also define for every $x \in \mathcal{R}$ its *global weakly stable manifolds* and *global weakly unstable manifolds* at x by

$$W^{ws}(x) = \bigcup_{t\in\mathbb{R}} W^s(\varphi_t(x)), \quad W^{wu}(x) = \bigcup_{t\in\mathbb{R}} W^u(\varphi_t(x))$$

For a hyperbolic orbit $\gamma = \{\varphi_t(x)\}_{t\in\mathbb{R}}$, we denote also the global weakly stable and unstable manifolds in x by $W^s(\gamma)$ and $W^u(\gamma)$, respectively.

We are now finally able to define the main object for our study in Chapter 3, the *Ergodic Homoclinic classes for flows*: given a periodic hyperbolic orbit γ, we define the *stable and unstable homoclinic class of* γ as follows:

$$\Lambda^s(\gamma) = \{x \in M : x \text{ is a regular point and } W^s(x) \pitchfork W^u(\gamma) \neq \emptyset\}, \tag{1.4.1}$$

and

$$\Lambda^u(\gamma) = \{x \in M : x \text{ is a regular point and } W^u(x) \pitchfork W^s(\gamma) \neq \emptyset\}. \tag{1.4.2}$$

Definition 1.4.8. Here, the symbol "$\pitchfork$" means the following: given two submanifolds of M, say N_1 and N_2, then $N_1 \pitchfork N_2 \neq \emptyset$ if

(1) $N_1 \cap N_2 \neq \emptyset$.

(2) For every $z \in N_1 \cap N_2$, one has $T_zN_1 + T_zN_2 = T_zM$.

In this case, we say that N_1 and N_2 have a transverse intersection.

For the above definitions, we are not assuming that $Sing(X) = \emptyset$ nor that γ is not a singular orbit; it could be that $X|_\gamma = 0$. We also highlight that $\Lambda^s(\gamma)$ is s-saturated, $\Lambda^u(\gamma)$ is u-saturated, i.e. for every $x \in \Lambda^{s,u}(\gamma)$ we have $W^{s,u}(x) \subset \Lambda^{s,u}(\gamma)$. We also have that both sets are φ_t-invariant. The *Ergodic homoclinic class of* γ is then defined as the s,u-saturated and φ_t-invariant set:

$$\Lambda(\gamma) = \Lambda^s(\gamma) \cap \Lambda^u(\gamma).$$

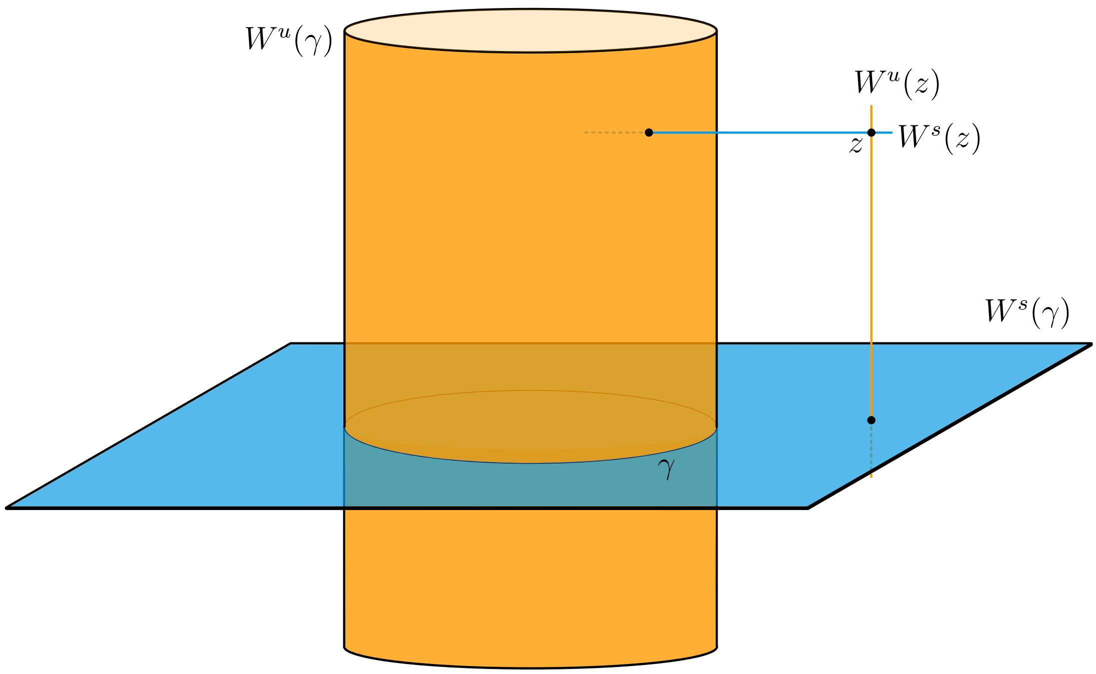


Figure 1.8: Illustration of the local dynamics around a point $z \in \Lambda(\gamma)$.

A prior, there is no reason to believe that the above set is not empty. Indeed, it is precisely the result obtained by Theorem C to state conditions that guarantee that $\Lambda(\gamma)$ is nonempty.

We now discuss the main property we are going to use to prove the results in Chapter 3 the *Absolute Continuity* of the stable and unstable partitions by W^s and W^u. Essentially, it says that we can transfer information from a lamination $W^u(x)$ to another $W^u(y)$ by flowing through $W^s(x)$ without losing too much information from the measure theoretical point of view. Below, we are going to make precise the expression "flowing through $W^s(x)$" as the holonomy maps. *Absolute Continuity* is the core property of the Hopf's Argument.

Definition 1.4.9. A partition ξ of M is called a *Measurable Partition* if the quotient space M/ξ can be generated by a countable collection of measurable sets.

When M is a Lebesgue space, the quotient space M/ξ obtained from a measurable partition ξ is also a Lebesgue space [Roh52] (see also the chapter 15 of [Cou16]). Measure partitions are important from the measure-theoretical point of view since they allow us to decompose a measure into measures along the "pieces" of the partition:

Proposition 1.4.1. *For any measurable partition ξ of a Lebesgue space $(M, \mathcal{B}, m)$, there exists a canonical system of Conditional Measures m_x^ξ with the following properties:*

(1) The conditional measures m_x^ξ are defined in $\xi(x)$, the partition element containing x.

(2) For any $A \in \mathcal{B}$, the set $A \cap \xi(x)$ is measurable in $\xi(x)$ for almost all $\xi(x) \in M/\xi$.

(3) The mapping $x \mapsto m_x^\xi(A \cap \xi(x))$ is measurable, and the measure m satisfies:

$$m(A) = \int_{M/\xi} m_x^\xi(A \cap \xi(x))\, dm_T,$$

where m_T denotes the quotient measure in M/ξ.

This canonical system of conditional measures is unique (mod 0) for any measurable partition. Conversely, if a canonical system of conditional measures exists for a partition, the partition must be measurable.

Proof. See the Chapter 5 of [VO16]. ■

Definition 1.4.10. A measurable partition ξ is said to be *subordinate* to the unstable partition W^u if, for m-almost every $x \in M$, the following conditions hold:

(1) $\xi(x) \subset W^u(x)$

(2) $\xi(x)$ contains a neighborhood of x that is open in the topology of $W^u(x)$.

Given a Riemannian metric g for M, it induces Riemannian measures on each $W^\tau(x)$ which we are going to denote by λ_x^τ, $\tau = s, u$.

Definition 1.4.11. We say that a measure ν has absolutely continuous conditional measures with respect to the unstable (resp. stable) manifolds if for any measurable partition $\mathcal{P}$ subordinated to W^u (resp. W^s) we have $\nu_x^{\mathcal{P}} << \lambda_x^u$ (resp. λ_x^s) for ν-a.e. x.

We are now able to make sense to the previously used expression "transfer information from a lamination $W^u(x)$ to another $W^u(y)$" which is the notion of *Holonomy Map*: let $\mathcal{R}_{\varepsilon,l}^{i,j}$ be a Pesin block. For each point $x \in \mathcal{R}_{\varepsilon,l}^{i,j}$ that admits a negative (resp. positive) Lyapunov exponent, there exists the local Pesin stable (resp. unstable) manifold, denoted by $W_{\mathrm{loc}}^s(x)$ (resp. $W_{\mathrm{loc}}^u(x)$), with diameter at least $\delta > 0$. For such $x \in \mathcal{R}_{\varepsilon,l}^{i,j}$ and given two transversal disks D_1 and D_2 to $W_{\mathrm{loc}}^u(x)$ (resp. $W_{\mathrm{loc}}^s(x)$) and close to each other, the stable (resp. unstable) holonomy map $h^{s,u} \colon D_1' \subset D_1 \to D_2$ is defined as:

$$h^{s,u}(z) := W^{s,u}(z) \cap D_2,$$

where

$$D_1' := \{z \in D_1 \cap R_{\varepsilon,l}^{i,j} : W_{loc}^{s,u}(z) \pitchfork D_{1,2} \neq \emptyset\}$$

Essentially, for a point on D_1 we consider (when it exists) its stable (resp. unstable) Pesin manifold and its intersection with D_2. Analogously, we can define the weak-stable and weak-unstable holonomy maps by switching $W_{loc}^{s,u}$ by $W_{loc}^{ws,wu}$.

Theorem 1.4.12. [BP02, Theorem 4.3.1] *The (weak) holonomy maps ($h^{ws,wu}$) $h^{s,u}$ are measurable and absolutely continuous with respect to the Lesbegue measures induced on D_1 and D_2, that is they send sets of zero Lesbegue measure into sets of zero Lesbegue measure.*

1.4.1 Sinai-Ruelle-Bowen measures

In this subsection, we will recall a particular class of measures called *Sinai-Ruelle-Bowen measures* or only *SRB measures*. These are the measures treated in Theorems D, E, F, and G. We do not intend to develop the whole theory about these measures since we are going to use mostly their definition. For a broader treatment, we refer to [You02]. SRB measures are frequently introduced as "the invariant measures most compatible with

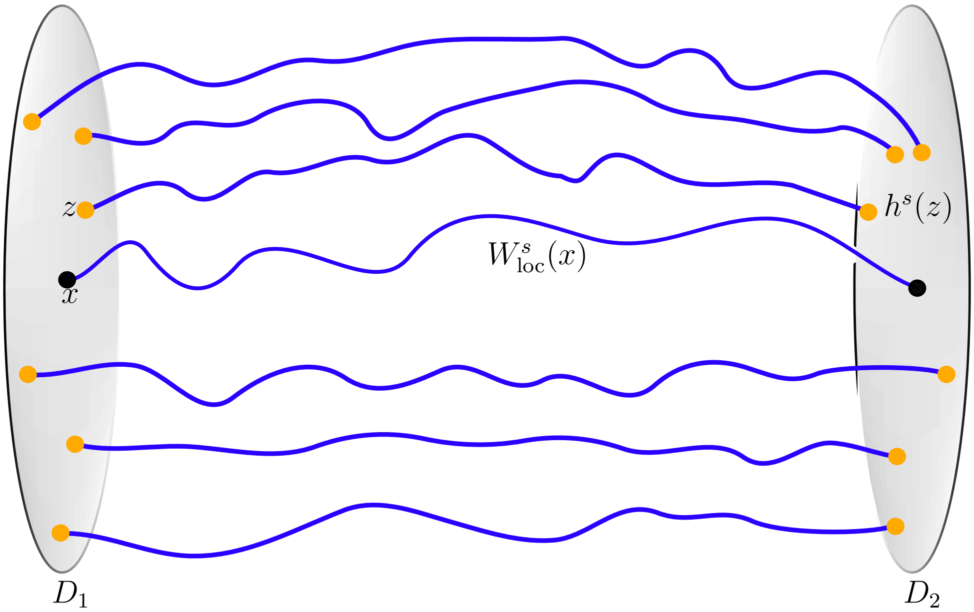


Figure 1.9: Representation of the stable holonomy map

volume when volume is not preserved" (see [You02]). The most important property for us is going to be absolute continuity of conditional measures:

Definition 1.4.13. A measure ν is called an SRB (Sinai-Ruelle-Bowen) measure if it has a positive Lyapunov Exponent at ν-almost every point x and absolutely continuous conditional measures with respect to the unstable manifolds.

Although other equivalent definitions are possible, the above definition presents precisely the important property that allow us to perform a version of the Hopf Argument in our context. Remember that our results deal with the relations between SRB measures and homoclinic classes for flows, and no further directions are considered in this dissertation. The techniques we use to obtain Theorem C are extremely similar to those for SRB measure because of the absolute continuity property. Besides that, in some cases it is easier to work with the equivalent definition after the work of Ledrappier-Young [LY85]: absolute continuity with respect to the unstable manifolds is equivalent to Pesin's formula, i.e., it holds that

$$h_\nu(\varphi_t) = \int \sum_{\lambda(x)>0} \lambda(x) d\nu.$$

Recall that Theorem F refers to ergodic components of a metric. This notion is due the classical result *Ergodic decomposition theorem*, which we state below for completeness in the manifold setting. For the proof, we recommend the book of Marcelo Viana and

Krerley Oliveira [VO16] Chapter 5. Remember that given a partition $\mathcal{P}$ of a probability space (M, μ) into measurable sets, then there exists a canonical structure of probability space $(\mathcal{P}, \widehat{\mu})$.

Theorem 1.4.14 (Ergodic decomposition)**.** *If M is a compact manifold, $f\colon M \to M$ a measurable transformation and μ a probability measure. Then, there exist a measurable set M_0 with full measure, a partition $\mathcal{P}$ of M_0 into measurable subsets and a collection of probability measures in M, say $\{\mu_P : P \in \mathcal{P}\}$, such that the following hold:*

(1) $\mu_P(P) = 1$, for $\widehat{\mu}$ almost every $P \in \mathcal{P}$.

(2) For any measurable subset $E \subset M$, the map $P \mapsto \mu_P(E)$ is measurable.

(3) For $\widehat{\mu}$almost every $P \in \mathcal{P}$, μ_P is f-invariant and ergodic.

(4) For any measurable subset $E \subset M$, it holds:

$$\mu(E) = \int_{\mathcal{P}} \mu_P(E) \mathrm{d}\widehat{\mu}(P).$$

For the proof of Theorem F, we are going to use the following flow version of a result by Katok in [Kat80] whose proof can be found in [LLL24]:

Theorem 1.4.15 ([Kat80, LLL24])**.** *Let μ be a regular hyperbolic ergodic measure of a C^2 flow φ_t. Then there exists a hyperbolic periodic orbit γ such that supp$(\mu) \subset \overline{\Lambda(\gamma)}$ and μ is homoclinically related with γ. In particular, for μ-almost every point x, $\varphi_t(W^u(x))$ accumulates on $W^u(\gamma)$ as t goes to infinity.*

Chapter 2

Partially hyperbolic geodesic flows

This chapter encompasses the content of the preprint [dJPR24], which is a work in collaboration with Luis Piñeyrúa (Udelar - Uruguay) and Sergio Romaña (Sun Yat-sen Univ. - China), about constructions of partially hyperbolic geodesic flows. In particular, we discuss the use of conformal deformations of Riemannian metrics with Anosov geodesic flow. As mentioned in the introduction, it is not known whether there exists a Riemannian metric constructed by standard techniques that presents a non-Anosov partially hyperbolic geodesic flow. As an illustration, we mention the following results by Fernando Carneiro and Enrique Pujals [CP14], which is the paper that motivates our work here:

Theorem 2.0.1. (Theorem 3.2 in [CP14]) *Let (M_1, g_1) and $(M_2, g_2 2)$ be two Riemannian manifolds whose geodesic flows are Anosov. Then the geodesic flow of the Riemannian manifold $(M_1 \times M_2, g_1 + g_2)$ is not Anosov.*

Theorem 2.0.2. (Theorem 3.3 in [CP14]) *Let (M_1, g_1) and $(M_2, g_2 2)$ be two Riemannian manifolds whose geodesic flows are Anosov. Then the geodesic flow of the Riemannian manifold $(M_1 \times M_2, g_1 + g_2)$ is not partially hyperbolic.*

Thus, we can state the following general questions:

Question 2.1. *How can examples of partially hyperbolic geodesic flows that are not Anosov be constructed?*

Question 2.2. *Which dynamic and ergodic properties can partially hyperbolic geodesic flows satisfy?*

Since geodesic flows are, in particular, contact flows, we may also ask:

Question 2.3. *How can examples of partially hyperbolic contact flows that are not Anosov be constructed? Are there examples that are not geodesic flows? (see Section 2.5)*

This dissertation aims to give some answers to the above questions, in particular, Theorems A and B answer the first two questions.

From subsections 1.1 and 1.2, in particular Theorem 1.1.5, Proposition 1.2.3 and Theorem 1.2.4 one realizes that if the geodesic flow is not of Anosov type, then some zero or positive curvature must appear. Since the sectional curvature is a function of points on the manifolds and 2-planes in its tangent bundle, it is highly non-trivial to control its behavior by changing the metric itself. Furthermore, the curvature must be controlled specially for planes generated by geodesics' velocities and Jacobi Fields. Namely, to obtain a partially hyperbolic geodesic flow the Riemannian metric must present, for each geodesic γ, Jacobi Fields fields such that the planes $\{\gamma', J\}$ present "only mostly negative curvature" so that $\|J(t)\|$ is a function with exponential contraction or expansion. On the other hand, if this is the only behavior we observe, then the geodesic flow may indeed be Anosov. Thus, for some Jacobi fields directions, we must observe "only mostly nonnegative curvature".

Our strategy is based on deformations of Riemannian metrics, which means we start with a particular Riemannian metric with some nice properties (local symmetry), then we modify it somehow in order to obtain a new Riemannian metric which is not close to the initial one in some topology. The technique used to deform the metrics is a *conformal deformation*, thus we can make good use of the several formulas presented in Subsection 1.1.2. This kind of technique was also used by Rafael Ruggiero in [Rug91a] to prove the following interesting result

Theorem 2.0.3. (Theorem A in [Rug91a]) *The C^2-interior of the set of Riemannian metrics with no conjugate points coincides with the set or Riemannian metrics for which the geodesic flow is Anosov.*

Besides the result by Ruggiero, there are still several open questions on the topology of the set of Riemannian metrics with no conjugate points. For example, it is not known if this set is convex or even path-connected. The metrics obtained by Theorem A are not known to be inside this set or if it is true that partial hyperbolicity should imply no conjugate points as the Anosov property. We also state the following question:

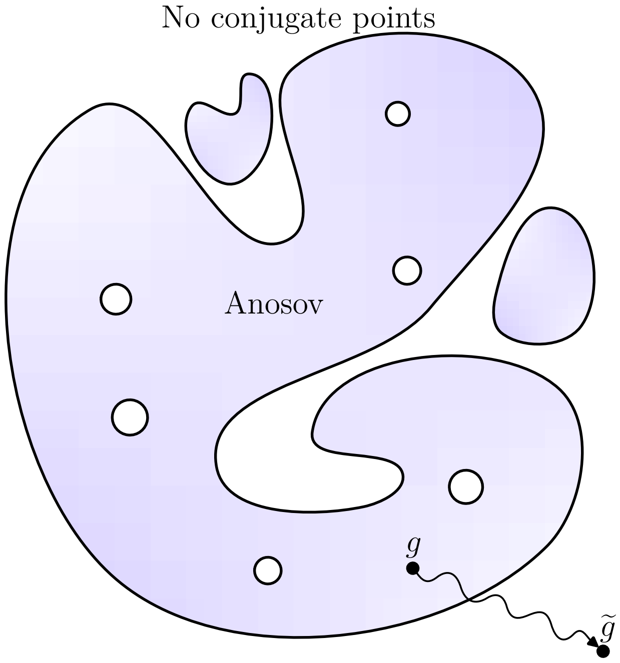


Figure 2.1: The C^2-interior of the metric with no conjugate points

Question 2.4. *How are the metrics on the boundary of the set of metrics with no conjugate points? What dynamical and ergodic properties do they present?*

The metrics obtained via Theorem B lie on the boundary of the set of metrics with no conjugate points since all the sectional curvatures are non-positive and their geodesic flows are not Anosov. Corollary B.1 lists several interesting properties for those metrics, so it would be interesting to know which properties could be generalized for the general setting. Ergodicity is not known in this context, for example.

Let us describe the steps of the construction:

(1) Start with a locally symmetric Riemannian metric g with spaces A and B as in Subsection 1.1.3.

(2) Break the Anosov property with conformal deformations. This is done in Section 2.2.

(2.1) Consider a closed geodesic γ with $\gamma'(0) = \gamma'(T)$.

(2.2) Define a open tubular neighborhood U around γ with coordinates (t, x).

(2.3) Multiply the metric g by a conformal factor e^h, with h supported on U.

(2.4) Find conditions on h so that $\widetilde{g} = \mathrm{e}^h g$ admits a parallel Jacobi Field. Equivalently, a geodesic with some direction with zero curvature along the whole geodesic.

(3) Control the effect on the curvature tensor so that there exist two families of invariant cones. This is done in Section 2.3 by analyzing the angle variation for different classes of geodesics.

(4) Use Theorem 1.3.1 to conclude.

This strategy works for Theorem A and B. However, to have more control over the behavior of the curvature endomorphism, we need to use a different deformation than a conformal one. This allows us to show that the new metric is non-positively curved with a single closed geodesic presenting some zero curvature. Thus, all the interesting ergodic properties follow directly from the known results by Eberlein, Burns, Brin, Ballman, and Knieper.

2.1 Setting up the deformation

We are going to perform a perturbation of the metric g in a tubular neighborhood of a closed geodesic. Let γ be a closed geodesic with period T and $\gamma'(0) = \gamma'(T)$ just as in [CP14]. For each $x \in M$ and $v \in T_xM$ remember that we have defined

$$A(x, v) := \{w \in T_xM : K(v, w) = -1\}$$

$$B(x, v) := \left\{w \in T_xM : K(v, w) = -\frac{1}{4}\right\}$$

We consider an orthonormal frame field along γ, say $\{e_0(t) := \gamma'(t), ..., e_{n-1}(t)\}$, such that $\{e_1(t), ..., e_r(t)\}$ is a basis for $A(\gamma(t), \gamma'(t))$ and $\{e_{r+1}(t), ..., e_{n-1}(t)\}$ is basis for $B(\gamma(t), \gamma'(t))$. This can be done by choosing such a basis for $T_{\gamma(0)}M$ and then considering its parallel transport, since the metric is locally symmetric, the respective vectors are still in the respective spaces A and B. Define the Fermi coordinates $\Psi : [0, T] \times (-\varepsilon_0, \varepsilon_0)^{n-1} \to M$ by

$$\Psi(t, x) = \exp_{\gamma(t)}(x_1 e_1(t) + ... + x_{n-1} e_{n-1}(t)),$$

with ε_0 less than the injective radius of the exponential map. In particular, for Fermi coordinates we have that $g_{ij}(t, 0) = \delta_{ij}$ and $\Gamma^k_{ij}(t, 0) = 0$. For each $\varepsilon < \varepsilon_0$ we define the following sets

- $U := [0, T] \times (-\varepsilon_0, \varepsilon_0)^{n-1}$.

- $U(\varepsilon) := [0, T] \times (-\varepsilon, \varepsilon)^{n-1}$.
- $B(\gamma, \varepsilon) = \Psi(U(\varepsilon))$.

The choice of ε small enough will be important to control the deformation of the metric up to the first derivatives of its component functions.

We consider a conformal deformation of the initial metric given by $\widetilde{g} = \phi g$, with $\phi = \mathrm{e}^h$ and h a function of class C^2 supported in $B(\gamma, \varepsilon)$ to be determined. We will construct the function h to maintain γ as a geodesic for the new metric, and that $\widetilde{g}$ is C^1-close to g and C^2-far. It means that we will make a small perturbation on the norms and Christoffel symbols of g, but a deformation on the curvature. Notice that if h is C^1-small, then $\widetilde{g}$ is C^1 close to g. Besides that, all pairs of orthogonal vectors for the metric g are still orthogonal for the metric $\widetilde{g}$. Furthermore, if we set $h(t, 0) = 0$, then $\{e_0(t), ..., e_{n-1}(t)\}$ is still an orthonormal basis for $T_{\gamma(t)}M$ considering the metric $\tilde{g}$.

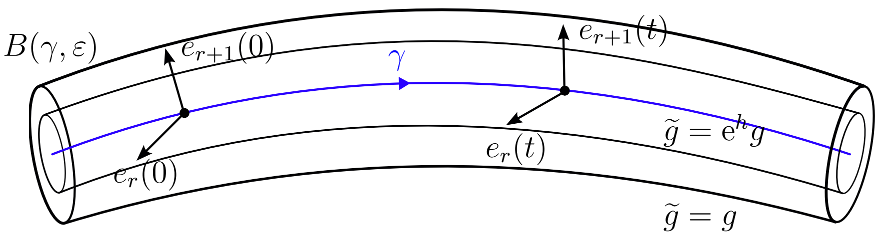


Figure 2.2: Tubular neighborhoods $B(\gamma, \varepsilon)$ for ε small, where the metric is deformed.

The first property we need to obtain for the new metric is that γ is still a $\widetilde{g}$-geodesic. For any curve α we will denote by $\widetilde{D}_t = \widetilde{\nabla}_{\alpha'}(\cdot)$ and $D_t = \nabla_{\alpha'}(\cdot)$ the covariant derivative along α gave by the Levi-Civita connection of the metrics $\widetilde{g}$ and g, respectively. By the equation (1.1.7) we get for γ

$$\widetilde{D}_t\gamma' = D_t\gamma' + \gamma'(h)\gamma' - \nabla h|_\gamma = g(\nabla h, \gamma')\gamma' - \nabla h|_\gamma$$

It is sufficient to have that γ is a critical set for h, i.e. $\nabla h|_{\gamma'} = 0$. Furthermore, in this case, by the same calculation every parallel vector field for the metric g along γ is still parallel for the metric $\widetilde{g}$, in particular for every $k = 1, ..., n-1$ the vector field $e_k(t)$ is parallel along γ for the metric $\widetilde{g}$.

2.2 For which h the Anosov Property is broken?

We are going to investigate which properties we need for the function h such that the Anosov property is broken. We list some sufficient conditions to have a non-Anosov geodesic flow. However, for a general function with those properties, it is not clear for us whether the geodesic flow should be partially hyperbolic or present any other interesting dynamical property. We can define explicitly an example of h for which our results hold. We believe that techniques "from the PDE worl" should be useful for a different proof, but this goes beyond our expertise.

The key phenomenon that guarantees the breaking of the Anosov property is essentially the creation of a parallel 2-plane along γ and containing γ'. This is also a manifestation of 0 sectional curvature. Remember that a parallel vector field X along a geodesic γ is a Jacobi Field if, and only if $\widetilde{K}(X, \gamma') \equiv 0$. By Proposition 1.2.4, this is precisely the situation we seek to break the Anosov property.

Remember from the previous section that if $\nabla h|_\gamma = 0$, then each $e_k(t)$ is still a parallel vector field along γ, thus we must guarantee that some of them can be turned into Jacobi Fields for the new metric. This is done by making $\widetilde{K}(e_k(t), \gamma'(t))$ to vanish. Let us investigate the new curvature in Fermi coordinates. We are interested in $\widetilde{K}\left(\frac{\partial}{\partial x^k}, \gamma'\right)$ given by the equation (1.1.9). In coordinates, we have

$$\nabla h = g^{ij}\frac{\partial h}{\partial x^j}\frac{\partial}{\partial x^i}$$

and

$$\begin{aligned}
\nabla_X(\nabla h) &= X\left(g^{ij}\frac{\partial h}{\partial x^j}\right)\frac{\partial}{\partial x^i} + g^{ij}\frac{\partial h}{\partial x^j}\nabla_X\left(\frac{\partial}{\partial x^i}\right) \\
&= \left(X(g^{ij})\frac{\partial h}{\partial x^j} + g^{ij}X\left(\frac{\partial h}{\partial x^j}\right)\right)\frac{\partial}{\partial x^i} + g^{ij}\frac{\partial h}{\partial x^j}\nabla_X\left(\frac{\partial}{\partial x^i}\right)
\end{aligned}$$

If $X_k := \dfrac{\partial}{\partial x^k}$, then

$$\begin{aligned}
\nabla_{X_k}(\nabla h) &= \left(X_k(g^{ij})\frac{\partial h}{\partial x^j} + g^{ij}X_k\left(\frac{\partial h}{\partial x^j}\right)\right)\frac{\partial}{\partial x^i} + g^{ij}\frac{\partial h}{\partial x^j}\nabla_{X_k}X_i \\
&= \left(X_k(g^{ij})\frac{\partial h}{\partial x^j} + g^{ij}\frac{\partial^2 h}{\partial x^k \partial x^j}\right)X_i + g^{ij}\frac{\partial h}{\partial x^j}\Gamma^l_{ki}X_l \\
&= \left(X_k(g^{ij})\frac{\partial h}{\partial x^j} + g^{ij}\frac{\partial^2 h}{\partial x^k \partial x^j}\right)X_i + g^{lj}\frac{\partial h}{\partial x^j}\Gamma^i_{kl}X_i \\
&= \left(X_k(g^{ij})\frac{\partial h}{\partial x^j} + g^{ij}\frac{\partial^2 h}{\partial x^k \partial x^j} + g^{lj}\frac{\partial h}{\partial x^j}\Gamma^i_{kl}\right)X_i
\end{aligned}$$

Now, using Lemma 1.1.1 we get

$$\begin{aligned}
g(\nabla_{X_k}\nabla h, X_k) &= \left(X_k(g^{ij})\frac{\partial h}{\partial x^j} + g^{ij}\frac{\partial^2 h}{\partial x^k \partial x^j} + g^{lj}\frac{\partial h}{\partial x^j}\Gamma^i_{kl}\right) g_{ik} \\
&= X_k(g^{ij})g_{ik}\frac{\partial h}{\partial x^j} + g^{ij}g_{ik}\frac{\partial^2 h}{\partial x^k \partial x^j} + g^{lj}g_{ik}\frac{\partial h}{\partial x^j}\Gamma^i_{kl} \\
&= \partial_k(g^{ij})g_{ik}\partial_j h + \partial^2_{kk}h + g^{lj}g_{ik}\partial_j h\Gamma^i_{kl} \\
&= -g^{ij}\partial_k g_{ik}\partial_j h + \partial^2_{kk}h + g^{lj}g_{ik}\partial_j h\Gamma^i_{kl} \\
&= -g^{ij}(\Gamma^l_{ki}g_{lk} + \Gamma^l_{kk}g_{li})\partial_j h + \partial^2_{kk}h + g^{lj}g_{ik}\partial_j h\Gamma^i_{kl} \\
&= -g^{ij}g_{li}\Gamma^l_{kk}\partial_j h + \partial^2_{kk}h \\
&= -\delta^j_l\Gamma^l_{kk}\partial_j h + \partial^2_{kk}h \\
&= -\Gamma^j_{kk}\partial_j h + \partial^2_{kk}h.
\end{aligned}$$

In particular, since the Christoffel symbols of g vanish along γ, we get

$$g(D_t\nabla h, \gamma') = \partial^2_{00}h - \Gamma^k_{00}\partial_k h = \partial^2_{00}h(t,x).$$

Then the new sectional curvatures along γ are given by

$$\phi\widetilde{K}(X_k, \gamma') = K(X_k, \gamma') - \frac{1}{2}\partial^2_{kk}h(t,0) - \frac{1}{2}\partial^2_{00}h(t,0).$$

If h does not depend on t, we get

(1) $\phi\widetilde{K}(X_k, \gamma') = -1 - \frac{1}{2}\partial^2_{kk}h(0)$, for $k = 1, ..., r$.

(2) $\phi\widetilde{K}(X_k, \gamma') = -\frac{1}{4} - \frac{1}{2}\partial^2_{kk}h(0)$, for $k = r+1, ..., n-1$.

We summarize our discussion as the following proposition:

Proposition 2.2.1. *In the context above, the geodesic flow $\widetilde{g}_t$ is not Anosov for any function h supported in $B(\gamma, \varepsilon)$ satisfying the following properties*

(1) h is of class at least C^2.

(2) h does not depend on t.

(3) $h(0) = 0$

(4) $\nabla h|_\gamma = 0$.

(5) There exists $s \in \{r+1, ..., n-1\}$ such that $\partial^2_{ss}h(0) = -\frac{1}{2}$.

Proof. By properties (*3*) and (*4*), γ is a unit speed geodesic for $\widetilde{g}$ and $e_k(t)$ is an unitary parallel vector field along γ for $\widetilde{g}$, for $k = 1, ..., n-1$. By property (*2*) and the computations above, we get for $k = r+1, ..., n-1$

$$\phi\widetilde{K}(X_k, \gamma') = -\frac{1}{4} - \frac{1}{2}\partial^2_{kk}h(0).$$

Thus, for $k = s$, we get that $\widetilde{K}(e_s(t), \gamma'(t)) \equiv 0$. By Proposition 1.2.4, the geodesic flow $\widetilde{g}_t$ is not Anosov. ∎

Concretely, we have

$$D\widetilde{g}_t(e_s(0), 0) = (e_s(t), 0)$$

and so, for every $t \in \mathbb{R}$,

$$\|D\widetilde{g}_t(e_s(0), 0)\| = \|e_s(t)\| = \|e_s(0)\|.$$

The last equality is since $e_s(t)$ is parallel. It means that the derivative of the geodesic flow $\widetilde{g}_t$ acts as an isometry on the vector $(e_s(t), 0)$.

We will now provide an example of such a function h. We are going to make some choices for our construction to work, and also to find an example as close as we can to the set of metrics with no conjugate points. For example, to avoid the creation of planes with positive curvature along γ we are going to choose h such that the property (5) in Proposition 2.2.1 is satisfied by a unique index s and such that $\partial^2_{kk}h(0) = 0$ for all other indices. This guarantees that there are no conjugate points along the central geodesic γ, but we were not able to verify if some geodesics with conjugate points were created in the process. The best we can expect is that all sectional curvatures are bounded from above by a small constant depending on the size of the tubular neighborhood ε.

To construct the function h with the above properties, for each natural number $n \geq 2$, let s_n be the following function $s_n : \mathbb{R} \to \mathbb{R}$

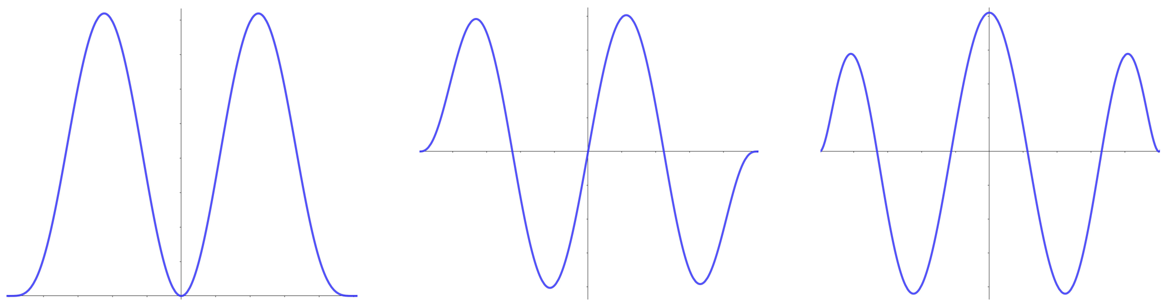
$$s_n(x) = \begin{cases} 1/8(x+1)^{2n}(x-1)^{2n}, & x \in [-1, 1] \\ 0, & \text{otherwise} \end{cases}$$

Define also the functions $r_n(x) = x^2 s_n(x)$. The family $\{r_n\}_n$ is smooth in $\mathbb{R} \setminus \{\pm 1\}$ and is of class C^{2n} in $\{\pm 1\}$. It is easy to see that

- $r_n(0) = 0$.

- $r'_n(0) = 0$.
- $r''_n(0) = \frac{1}{4}$.

It is also not difficult to see that $|r''_n(x)| < \frac{1}{4}$ if $x \neq 0$. Now, let $f_n : \mathbb{R} \to \mathbb{R}$ be the function $f_n(x) = x^2 s_n\left(\frac{x}{\varepsilon^2}\right)$ which is smooth in $\mathbb{R} \setminus \{\pm\varepsilon^2\}$ and of class C^{2n} in $\{\pm\varepsilon^2\}$. The family $\{f_n\}_n$ satisfies the same properties as r_n, we just changed the support of it to be $[-\varepsilon^2, \varepsilon^2]$ and we also have $f_n(x) \leq \varepsilon^4$. The graphs of r_n, r'_n, and r''_n for $n = 2$ are shown below in Figure 2.3.



(a) Graph of r_2 (b) Graph of r'_2 (c) Graph of r''_2

Figure 2.3: Behavior of r_2

Finally, for a fixed $s \in \{r+1, ..., n-1\}$ we define h_k as the following product

$$h_k(x_0, x_1, ..., x_{n-1}) := \phi\left(\frac{x_1}{\varepsilon}\right) \cdots (-2f_k(x_s)) \cdots \phi\left(\frac{x_{n-1}}{\varepsilon}\right),$$

where $\phi(x)$ are bump functions such that $\phi(0) = 1$ and supported in $[-1, 1]$. For simplicity, let us denote by $\Phi(x) = \phi\left(\frac{x_1}{\varepsilon}\right) \cdots \phi\left(\frac{x_{n-1}}{\varepsilon}\right)$ the product of bump functions. Notice that h_k is smooth outside the region $x_s = \pm\varepsilon^2$, where it is of class C^{2k}. Constructing h_k like this, we get:

Lemma 2.2.1. *h_k satisfies the following properties*

(1) $h_k(0) = 0$.

(2) *h_k does not depend on* $t = x_0$.

(3) $\partial_j h_k(0) = 0$, *for all* j.

(4) $\partial^2_{ij} h_k(0) = 0$, *if* $i \neq j$ *or* $i = j \neq s$.

(5) $\partial^2_{ss} h_k(0) = -\frac{1}{2}$.

Proof. (*1*) and (*2*) are clear by definition. For (*3*), see that $-2f_k'(0) = 0$ and $\phi'(0) = 0$. For $(i,j) \neq (s,s)$, each derivative $\partial^2_{ij} h_k(x)$ contains some term x_s multiplying it, thus $\partial^2_{ij} h_k(0) = 0$. Finally, $\partial^2_{ss} h_k(0) = -2f_k''(0) = -2 \cdot 2s_n(0) = -4 \cdot \frac{1}{8} = -\frac{1}{2}$, so we get (*5*). ■

Observe that h_k is C^1 close to zero, since

$$|-2f_k(x)| \leq 2\varepsilon^4 \left| r_k\left(\frac{x}{\varepsilon^2}\right)\right| \leq \frac{2}{100}\varepsilon^4 < \varepsilon^4,$$

$$\begin{aligned} |-2f_k'(x)| &= 2\left| 2x r_k\left(\frac{x}{\varepsilon^2}\right) + x^2 r_k'\left(\frac{x}{\varepsilon^2}\right)\frac{1}{\varepsilon^2}\right| \\ &\leq 4\varepsilon^2 \frac{1}{100} + \varepsilon^2 \frac{4}{100} < \varepsilon^2 \end{aligned}$$

We also have $(\phi\left(\frac{x}{\varepsilon}\right))' \leq \frac{2}{\varepsilon}$ and $(\phi\left(\frac{x}{\varepsilon}\right))'' \leq \frac{M}{\varepsilon^2}$. Once $|\phi(x)| \leq 1$, it follows that

Lemma 2.2.2. *The following inequalities hold*

(*1*) $|h_k(x)| \leq \varepsilon^4$.

(*2*) $|\partial_j h_k| \leq \frac{2}{\varepsilon}\varepsilon^4 = 2\varepsilon^3$, *if* $j \neq s$.

(*3*) $|\partial_s h_k| \leq \varepsilon^2$.

(*4*) $|\partial^2_{ij} h_k| \leq \frac{2}{\varepsilon}\frac{2}{\varepsilon}\varepsilon^4 = 4\varepsilon^2$, *if* $i \neq j$ *and non of them are equal to* s.

(*5*) $|\partial^2_{sj} h_k| \leq \varepsilon^2 \frac{2}{\varepsilon} = 2\varepsilon$, *if* $i \neq s$.

(*6*) $|\partial^2_{ii} h_k| \leq \frac{M}{\varepsilon^2}\varepsilon^4 = M\varepsilon^2$, *if* $i \neq s$.

(*7*) $|\partial^2_{ss} h_k| \leq 2|f_k''(x_s)| \leq 2\left| r_k''\left(\frac{x_s}{\varepsilon^2}\right)\right| \leq \frac{1}{2}$.

Now fix some $k_0 \geq 2$ and call $h = h_{k_0}$. This will imply that we work with a function of class C^∞ outside $x_s = \pm x_s$ and of class C^{2k_0} in $\{\pm x_s\}$. Since our construction works for any $k_0 \geq 2$, the resulting Riemannian metric can be made as regular as needed, but not globally smooth.

2.3 The new geodesic flow is partially hyperbolic

We start this section by illustrating the behavior of the new flow. In the next Lemma, we see that the orbit $\{\widetilde{g}_t(\gamma'(0))\}$ is not hyperbolic, but it is still partially hyperbolic. Our construction allows us to verify precisely what the invariant splitting for the deformed flow is. It will lead to the analysis of other trajectories $\widetilde{g}_t$.

Lemma 2.3.1. *The orbit* $\{\widetilde{g}_t(\gamma'(0))\}$ *is partially hyperbolic.*

Proof. We will analyze the Jacobi equation for the new metric along γ. Define by $\mathcal{J}(\gamma)$ the space of Jacobi fields orthogonal to γ'. Since $\{e_1(t), ..., e_{n-1}(t)\}$ is an orthonormal parallel frame along γ we can write $J \in \mathcal{J}(\gamma)$ as $J = J^i(t)e_i(t)$, then denoting $\widetilde{D}_t^2 J = J''$ (the second covariant derivative) we get

$$J'' = (J^i)''e_i = -J^i\widetilde{R}(e_i, \gamma')\gamma'.$$

Then, we get

$$\widetilde{g}(J'', e_j) = (J^i)''\widetilde{g}_{ij} = -J^i\widetilde{K}_{ij},$$

with $\widetilde{K}_{ij} = \widetilde{g}(\widetilde{R}(e_i, \gamma')\gamma', e_j) =: \widetilde{R}(e_i, \gamma', \gamma', e_j)$. From the computations above, we have the following relation

$$\tilde{R}(e_i, \gamma', \gamma', e_j) = R(e_i, \gamma', \gamma', e_j) - \frac{1}{2}\partial^2_{ij}h.$$

We get

(1) $\widetilde{R}(e_i, \gamma', \gamma', e_j) = R(e_i, \gamma', \gamma', e_j)$, for $i \neq j$ and $i = j \neq s$.

(2) $\widetilde{R}(e_s, \gamma', \gamma', e_s) = R(e_s, \gamma', \gamma', e_s) + \frac{1}{4} = 0$.

Since the metric g is locally symmetric of nonconstant negative curvature, we can apply the relation (1.1.11). Notice that, $e_i = (e_i)_A$ and $(e_i)_B = 0$ if $i = 1, ..., r$ and if $i = r+1, ..., n-1$, then $e_i = (e_i)_B$ and $(e_i)_A = 0$. Hence if $i \neq j$

$$R(e_i, \gamma', \gamma', e_j) = \lambda g(e_i, e_j) = 0,$$

where $\lambda \in \left\{\frac{1}{4}, 1\right\}$ depending on i and j. For $i = j \in \{1, ..., r\}$

$$\tilde{R}(e_i, \gamma', \gamma', e_i) = R(e_i, \gamma', \gamma', e_i) = -1.$$

For $i = j \in \{r+1, ..., n-1\} \setminus \{s\}$

$$\tilde{R}(e_i, \gamma', \gamma', e_i) = R(e_i, \gamma', \gamma', e_i) = -\frac{1}{4}.$$

We conclude that the matrix $\tilde{K} = (\tilde{R}(e_i, \gamma', \gamma', e_j))_{ij}$ is

$$\tilde{K} = \begin{bmatrix} -Id_r & 0 \\ 0 & I_{n-r-1} \end{bmatrix}$$

with I_{n-r-1} the diagonal matrix with entries $-\frac{1}{4}$ but the entry $(I_{n-r-1})_{s^*s^*} = 0$, with $s^* = s - r$. This implies that

$$\begin{cases} (J^i)'' = J^i, & \text{for } i = 1, ..., r \\ (J^i)'' = \frac{1}{4}J^i, & \text{for } i = r+1, ..., n-1 \text{ and } i \neq s \\ (J^s)'' = 0 \end{cases}$$

it implies that the ODE solutions are

$$\begin{cases} J^i = \left(\frac{J^i(0)-(J^i)'(0)}{2}\right)e^{-t} + \left(\frac{J^i(0)+(J^i)'(0)}{2}\right)e^{t}, & \text{for } i = 1, ..., r \\ J^i = \left(\frac{J^i(0)-2(J^i)'(0)}{2}\right)e^{-\frac{t}{2}} + \left(\frac{J^i(0)+2(J^i)'(0)}{2}\right)e^{\frac{t}{2}}, & \text{for } i = r+1, ..., n-1 \text{ and } i \neq s \\ J^s = (J^s)'(0)t + J^s(0) \end{cases}$$

From this, we can see that the splitting along γ' is given by

$$\begin{aligned} E^{ss}(\gamma') =& \{(J, J') : J \in \mathcal{J}(\gamma), J^i(0) = -(J^i)'(0), \text{for } i = 1, ..., r \\ & \text{and } J^i(0) = (J^i)'(0) = 0 \text{ otherwise}\}, \\ E^{uu}(\gamma') =& \{(J, J') : J \in \mathcal{J}(\gamma), J^i(0) = (J^i)'(0), \text{for } i = 1, ..., r \\ & \text{and } J^i(0) = (J^i)'(0) = 0 \text{ otherwise}\}, \\ E^{s}(\gamma') =& \{(J, J') : J \in \mathcal{J}(\gamma), J^i(0) = -2(J^i)'(0), \text{for } i = r+1, ..., n-1, i \neq s \\ & \text{and } J^i(0) = (J^i)'(0) = 0 \text{ otherwise}\}, \\ E^{u}(\gamma') =& \{(J, J') : J \in \mathcal{J}(\gamma), J^i(0) = 2(J^i)'(0), \text{for } i = r+1, ..., n-1, i \neq s \\ & \text{and } J^i(0) = (J^i)'(0) = 0 \text{ otherwise}\}, \\ E^{c}(\gamma) =& \operatorname{span}\{(e^s(t), 0)\} \end{aligned}$$

This means that we have strong stable and unstable bundles E^{ss} and E^{uu}, weak stable and weak unstable bundles E^s and E^u with weaker but still exponential contraction and expansion, and finally a center bundle E^c with no exponential contraction nor expansion. Of course we also can consider $E^s(\gamma') \oplus E^c(\gamma') \oplus E^u(\gamma')$ as a center bundle. ■

2.3.1 Cone invariance for the initial metric

In this section, we will analyze the behavior of other obits of the geodesic flow $\tilde{g}_t$. The strategy is similar to the one used by Carneiro and Pujals in [CP14]. For the sake of completeness and to build intuition, we first see that the contact structure for the initial metric splits in four Dg_t-invariant subbundles. From this analysis and Lemma 2.3.1, it will be made clear what the real effect of our deformation is in terms of the dynamics. Remember that in Lemma 2.3.1 we have identified explicitly the strong stable and unstable bundles. The key property for that was given by equation (1.1.11), which works for the initial metric, and the relations found in Subsection 2.2. What we will see is that the strong stable and unstable bundles for the initial metric coincide (or at least they are arbitrarily close) to the candidates for strong stable and unstable bundles for $\widetilde{g}_t$.

Remember from Section 1.2 that the contact structure $S(T^1M) \to T^1M$, given by $S(\theta) = \ker_\theta \alpha$, has naturally the following identification: $S(\theta) = (A(\theta) \oplus B(\theta))^2$. As mentioned before, the locally symmetric property implies that the spaces $A(\theta)$ and $B(\theta)$ are parallel, i.e. if $V_0 \in A(\theta)$ and $V(t)$ is the vector field obtained by parallel transport along a smooth curve γ, then $V(t) \in A(\gamma(t), \gamma'(t))$. Analogously for $B(\theta)$. *(Proposition 1.24 in* [Pat99]*)*

Lemma 2.3.2. (Lemma 4.1 in [CP14]*) The geodesic flow of the locally symmetric spaces of non-constant negative curvature induces a hyperbolic splitting of the contact structure defined on T^1M:*

$$S(T^1M) = E^{ss} \oplus E^s \oplus E^u \oplus E^{uu}.$$

Proof. Define the invariant subbundles

$$\begin{aligned}
P^u_A(\theta) &= \{(w, w) \in S(\theta) : w \in A(\theta)\},\\
P^u_B(\theta) &= \left\{\left(w, \frac{1}{2}w\right) \in S(\theta) : w \in B(\theta)\right\},\\
P^s_A(\theta) &= \{(w, -w) \in S(\theta) : w \in A(\theta)\},\\
P^s_B(\theta) &= \left\{\left(w, -\frac{1}{2}w\right) \in S(\theta) : w \in B(\theta)\right\}.
\end{aligned}$$

Proceeding as in Lemma 2.3.1 one can conclude that

$$E^{uu}(\theta) = P^u_A(\theta),\ E^{ss}(\theta) = P^s_A(\theta),$$
$$E^u(\theta) = P^u_B(\theta),\ E^s(\theta) = P^s_B(\theta).$$

■

Previous Lemma together with Lemma 2.3.1 give us the candidates to spaces E_1 and E_2 in Proposition 1.3.2.

We are now going to define the family of cones that are invariant under the geodesic flow g_t. In essence, the cones we are considering are cones around the strong stable and strong unstable spaces given by Lemma 2.3.2. In the next section, we show that the same family of cones is invariant under the geodesic flow of the deformed metric, which means that our deformation is not changing the dynamics in these directions. Although our deformation does not make a big change on the spaces with smaller curvature, the computations to show the cone invariance are much more complicated, and it will be useful to have some of the computations done for the initial metric.

Remark 2.3.1. Remember that if $E \subset TM$ is subbundle, if we denote by Pr_E the projection to the subspace E, then we can define the quantity $\Theta_E(v) = \frac{g(P_E(v),P_E(v))}{g(v,v)}$ which is equal to the square of the cosine of the angle between the vector v and the space E.

The family of invariant cone fields we are going to consider is the following

$$C(v, P_A^{u,s}(\theta), c) = \left\{(\xi,\eta) \in S_\theta T^1M : \Theta_A^{u,s}(\xi,\eta) = \frac{\widehat{g}(Pr_{P_A^{u,s}(\theta)}(\xi,\eta), Pr_{P_A^{u,s}(\theta)}(\xi,\eta))}{\widehat{g}((\xi,\eta),(\xi,\eta))} \geq c\right\},$$

here $\widehat{g}$ is the Sasaki metric and $P_A^{u,s}(\theta)$ are the spaces given in Lemma 2.3.2. It is enough to see that this amount is increasing to guarantee the invariance. For the geodesic flow g_t we want to compute

$$\frac{d}{dt}\Theta_A^{u,s}((Dg_t)_\theta(\xi,\eta)).$$

By Proposition 1.2.3, it holds that

$$(Dg_t)_\theta(\xi,\eta) = (J(t), J'(t)),$$

where J is the Jacobi Field along the geodesic $\gamma := \pi(g_t(\theta))$ with initial conditions $J(0) = \xi$ and $J'(0) = \eta$. For simplicity of notation, let us denote $J(t) = \xi$ and $J'(t) = \eta$, so they are related by the equations (see also Proposition 1.2.3)

$$\begin{cases} \xi' &= \eta, \\ \eta' &= -R(\xi,\gamma')\gamma'. \end{cases}$$

Hence, we can write

$$\Theta_A^{u,s}(\xi,\eta) = \frac{g(\xi_A \pm \eta_A, \xi_A \pm \eta_A)}{g(\xi,\xi) + g(\eta,\eta)}.$$

We calculate only $\frac{d}{dt}\Theta_A^u(\xi,\eta)$, because for $\Theta_A^s(\xi,\eta)$ it is analogous. The next Lemma is also proved in Section 4.1.1 of [CP14]. We present the proof here because the computations will be used in the next sections.

Lemma 2.3.3. *With the notation above,*

$$\frac{d}{dt}\Theta^u_A(\xi,\eta) > 0.$$

Proof. The straightforward computation of the derivative gives us the following equality

$$\begin{aligned}\frac{d}{dt}\Theta^u_A(\xi,\eta) =& \frac{2g((\xi_A)' + (\eta_A)', \xi_A + \eta_A)(g(\xi,\xi) + g(\eta,\eta))}{(g(\xi,\xi) + g(\eta,\eta))^2} \\ & - 2\frac{(g(\xi',\xi) + g(\eta',\eta))g(\xi_A + \eta_A, \xi_A + \eta_A)}{(g(\xi,\xi) + g(\eta,\eta))^2}\end{aligned}$$

A priori $(\xi_A)' = \xi_{A'} + \xi'_A$, but $\xi_{A'} = 0$ since A is parallel. Then we write

$$\begin{aligned}\frac{d}{dt}\Theta^u_A(\xi,\eta) =& 2\frac{g(\xi'_A + \eta'_A, \xi_A + \eta_A)}{g(\xi,\xi) + g(\eta,\eta)} \\ & - 2\frac{(g(\xi',\xi) + g(\eta',\eta))g(\xi_A + \eta_A, \xi_A + \eta_A)}{(g(\xi,\xi) + g(\eta,\eta))^2}\end{aligned}$$

Because ξ and η satisfies the following equations (remember that ξ is a Jacobi field)

$$\begin{cases} \xi' &= \eta \\ \xi'_A &= Pr_A(\xi') = Pr_A(\eta) = \eta_A \\ \eta' &= -R(\xi,\gamma')\gamma' \\ \eta'_A &= Pr_A(\eta') = Pr_A(-R(\xi,\gamma')\gamma') = -(R(\xi,\gamma')\gamma')_A \end{cases}$$

then

$$\begin{aligned}\frac{d}{dt}\Theta^u_A(\xi,\eta) =& 2\frac{g(\eta_A - (R(\xi,\gamma')\gamma')_A, \xi_A + \eta_A)}{g(\xi,\xi) + g(\eta,\eta)} \\ & - 2\frac{g(\xi_A + \eta_A, \xi_A + \eta_A)}{(g(\xi,\xi) + g(\eta,\eta))^2}(g(\eta,\xi) + g(-R(\xi,\gamma')\gamma',\eta))\end{aligned} \tag{2.3.1}$$

Because of the properties $R(X,v)v = -\frac{1}{4}X_B - X_A$ and A orthogonal to B, we have that

$$(R(\xi,\gamma')\gamma')_A = -\xi_A$$

and

$$g(-R(\xi,\gamma')\gamma',\eta) = \frac{1}{4}g(\eta_B,\xi_B) + g(\eta_A,\xi_A)$$

So we get

$$\begin{aligned}\frac{d}{dt}\Theta^u_A(\xi,\eta) =& 2\frac{g(\eta_A + \xi_A, \xi_A + \eta_A)}{g(\xi,\xi) + g(\eta,\eta)} \\ & - 2\frac{g(\xi_A + \eta_A, \xi_A + \eta_A)}{(g(\xi,\xi) + g(\eta,\eta))^2}\Big(g(\eta,\xi) + \frac{1}{4}g(\eta_B,\xi_B) + g(\eta_A,\xi_A)\Big) \\ =& 2\frac{g(\eta_A + \xi_A, \xi_A + \eta_A)}{(g(\xi,\xi) + g(\eta,\eta))^2}\Big(g(\xi,\xi) + g(\eta,\eta) - g(\eta,\xi) - \frac{1}{4}g(\eta_B,\xi_B) - g(\eta_A,\xi_A)\Big)\end{aligned} \tag{2.3.2}$$

Define $M := 2\frac{g(\eta_A+\xi_A,\xi_A+\eta_A)}{(g(\xi,\xi)+g(\eta,\eta))^2} > 0$. Then,

$$\begin{aligned}\frac{d}{dt}\Theta^u_A(\xi,\eta) =&M\left(g(\xi,\xi)+g(\eta,\eta)-g(\eta,\xi)-\frac{1}{4}g(\eta_B,\xi_B)-g(\eta_A,\xi_A)\right)\\ =&M\Bigg(g(\xi_A,\xi_A)+g(\xi_B,\xi_B)+g(\eta_A,\eta_A)+g(\eta_B,\eta_B)\\ &-g(\xi_A,\eta_A)-g(\xi_B,\eta_B)-\frac{1}{4}g(\xi_B,\eta_B)-g(\xi_A,\eta_A)\Bigg)\\ =&M\Bigg(g(\xi_A,\xi_A)-2g(\xi_A,\eta_A)+g(\eta_A,\eta_A)+g(\xi_B,\xi_B)\\ &-\frac{5}{4}g(\xi_B,\eta_B)+g(\eta_B,\eta_B)\Bigg)\\ =&M\left(g(\xi_A-\eta_A,\xi_A-\eta_A)+g\left(\xi_B-\frac{5}{8}\eta_B,\xi_B-\frac{5}{8}\eta_B\right)+\frac{39}{64}g(\eta_B,\eta_B)\right)\end{aligned}$$

The last expression is trivially positive. ■

2.3.2 Cone invariance for the deformed metric

To prove that the geodesic flow $\widetilde{g}_t$ is partially hyperbolic, we follow the same idea as in [CP14]. We prove that there is a family of invariant cones around the candidates for strong stable and unstable bundles. The strategy is to prove the invariance of the cones for geodesics that are parallel to the central geodesic γ and geodesics that are "s-almost parallel" by computing the angle variation. If a geodesic α enters the deformed region, we can write it in coordinates as $\alpha' = (\alpha^0, \cdots, \alpha^{n-1})$. Then α will be called a *parallel geodesic* if $\alpha^1 = \cdots \alpha^{n-1} = 0$ and $\alpha^0 \neq 0$. The geodesic α with controlled s-component, that is $|\alpha^s| < \theta$, for some small value of θ, will be called a *s-almost parallel geodesic*. For transversal geodesics, we prove that by shrinking the deformed region, we can guarantee that these geodesics spend just a small time in the deformed region in comparison to the time they spend outside, thus we can conclude the cone invariance for those geodesics too. Here we just need to do the computations for parallel and almost parallel geodesics with controlled components in the s-direction once their arguments for the transversal geodesics do not depend on the type of deformation made, so it automatically works in our case.

As before, we are interested in the angle variation between the orbit of the derivative of the geodesic flow and a subbundle of the contact structure for the new metric. Again, it is given by the angle of the corresponding Jacobi field and the subbundle. For this subsection, we will use the same notation for the Jacobi Fields of the new metric, but we need to keep in mind that it is not a Jacobi Field for the initial metric.

Let us again fix some notation. Remember that the angle we are interested in is given by $\tilde{\Theta}_{A^{u,s}}((D\tilde{g}_t)_\theta(\xi,\eta))$. Write $\alpha(t)=\pi\circ\tilde{g}_t(\theta)$ and $(D\tilde{g}_t)_\theta(\xi,\eta)=(\xi(t),\eta(t))$, where $\xi(t)$ and $\eta(t)$ satisfy

$$\begin{cases}\xi'(t)=\eta(t),\\ \eta'(t)=-\tilde{R}(\xi(t),\alpha'(t))\alpha'(t).\end{cases}$$

Once again, we write

$$\tilde{\Theta}_A^{u,s}(\xi,\eta)=\frac{\tilde{g}(\xi_A\pm\eta_A,\xi_A\pm\eta_A)}{\tilde{g}(\xi,\xi)+\tilde{g}(\eta,\eta)}$$

Remark 2.3.2. We need to be careful with the differences in the metric phenomena. For instance, we will deal with a geodesic α for the deformed metric. Then, from equation (1.1.7) we get

$$0=\frac{\tilde{D}}{dt}\alpha'=\frac{D}{dt}\alpha'+\alpha'(h)\alpha'-\frac{1}{2}g(\alpha',\alpha')\nabla h.$$

So, if the geodesic α lies outside the deformed region, then it is a geodesic for the initial metric, but inside this region, it may not be a geodesic for the initial metric. The same remark can be made about Jacobi Fields. For simplicity and to differentiate the operators $\frac{\tilde{D}}{dt}$ and $\frac{D}{dt}$, we will use the notation $\tilde{D}_t:=\frac{\tilde{D}}{dt}$ and $D_t:=\frac{D}{dt}$. With this notation

$$\begin{cases}\tilde{D}_t\xi &=\eta,\\ \tilde{D}_t\eta &=-\tilde{R}(\xi,\alpha')\alpha'.\end{cases}$$

Of course, if the geodesic α does not cross the deformed region, then the angle variation is the same as before.

The Figure 2.4 represents the region we are interested in, that is the region of the unit tangent bundle over the deformed region, say $T^1M|_{B(\gamma,\varepsilon)}$. In the figure, γ is the central geodesic, and the small strip around it represents $B(\gamma,\varepsilon)$, where we are deforming the metric. The green small block represents the region of $T^1M|_{B(\gamma,\varepsilon)}$, where the vectors have small components in the s-direction. For example, α represents a geodesic inside $B(\gamma,\varepsilon)$ which is s-transversal to the deformed region. Notice that on the fiber $\pi^{-1}(\gamma(0))$, we have that $\alpha'(0)$ is a point outside the green region. On the other side, β represents a geodesic s-almost parallel to γ, then $\beta'(0)$ lies inside the green region. Our argument is basically proving that for geodesics with velocity vectors inside the green region, the

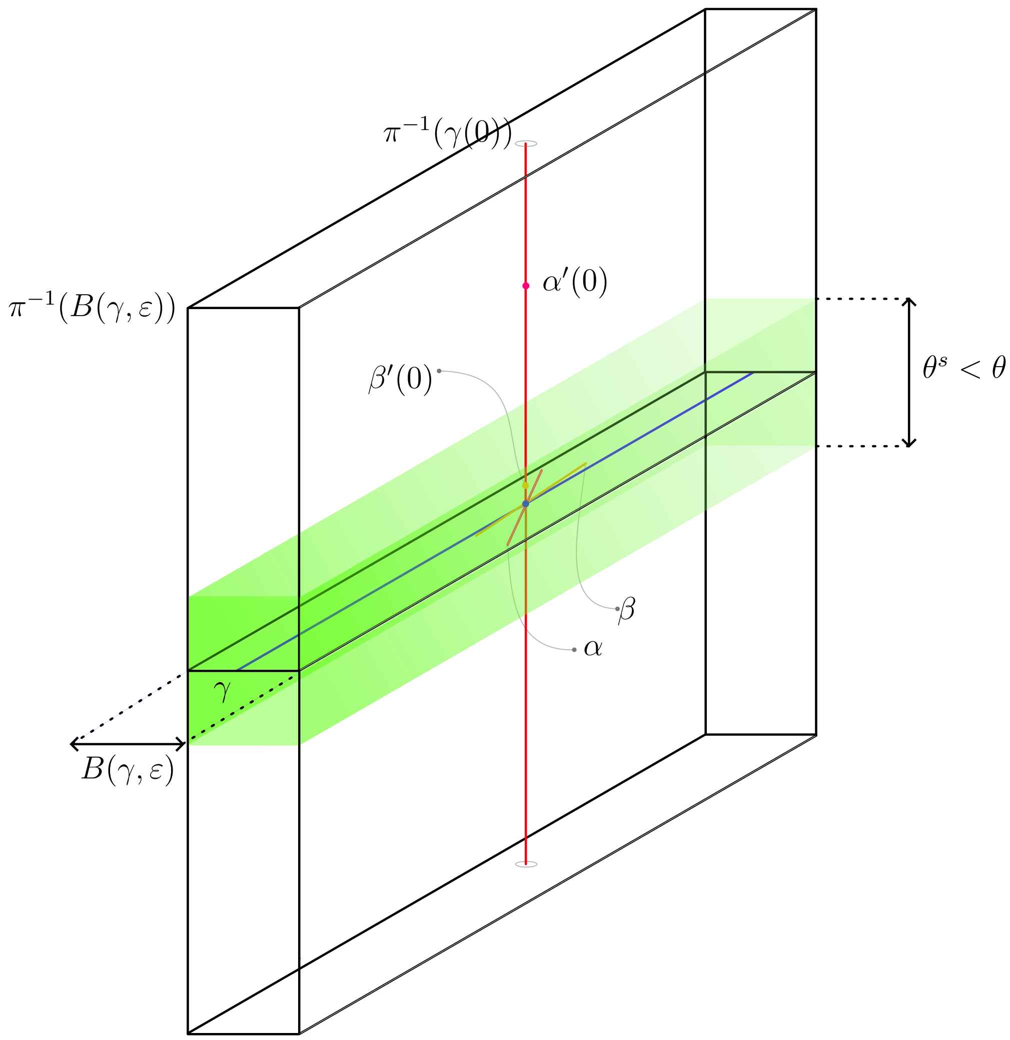


Figure 2.4: Region of interest in T^1M over the deformed region.

angle variation can be approximated by an expression close to that obtained in Lemma 2.3.3 for the initial metric plus a controlled term, thus invariance of the cone fields holds for the new metric in this case. We proceed to observe that if a geodesic has velocity outside the green region, then it must cross the region $B(\gamma, \varepsilon)$ with time comparable to ε and it does not enter again the deformed region for a while due to negative curvature outside. Thus, the new cone fields inherit the contraction from the initial metric.

2.3.2.1 Cone invariance for parallel geodesics

First, we will evaluate the variation of the angle for geodesics that point in the same direction as the central geodesic, i.e. $\alpha' = (\alpha^0, 0, \dots, 0)$. In general, we can write

$$\begin{aligned}\frac{d}{dt}\widetilde{\Theta}^u_A(\xi,\eta) =& 2\frac{\widetilde{g}((\xi_A)' + (\eta_A)', \xi_A + \eta_A)}{\widetilde{g}(\xi,\xi) + \widetilde{g}(\eta,\eta)} \\ &- 2\frac{(\widetilde{g}(\xi',\xi) + \widetilde{g}(\eta',\eta))\widetilde{g}(\xi_A + \eta_A, \xi_A + \eta_A)}{(\widetilde{g}(\xi,\xi) + \widetilde{g}(\eta,\eta))^2} \\ =& 2\frac{\widetilde{g}(\xi_{A'} + \xi'_A + \eta_{A'} + \eta'_A, \xi_A + \eta_A)}{\widetilde{g}(\xi,\xi) + \widetilde{g}(\eta,\eta)} \\ &- 2\frac{(\widetilde{g}(\xi',\xi) + \widetilde{g}(\eta',\eta))\widetilde{g}(\xi_A + \eta_A, \xi_A + \eta_A)}{(\widetilde{g}(\xi,\xi) + \widetilde{g}(\eta,\eta))^2} \\ =& 2\frac{\widetilde{g}(\xi'_A + \eta'_A, \xi_A + \eta_A)}{\widetilde{g}(\xi,\xi) + \widetilde{g}(\eta,\eta)} + 2\frac{\widetilde{g}(\xi_{A'} + \eta_{A'}, \xi_A + \eta_A)}{\widetilde{g}(\xi,\xi) + \widetilde{g}(\eta,\eta)} \\ &- 2\frac{(\widetilde{g}(\xi',\xi) + \widetilde{g}(\eta',\eta))\widetilde{g}(\xi_A + \eta_A, \xi_A + \eta_A)}{(\widetilde{g}(\xi,\xi) + \widetilde{g}(\eta,\eta))^2}\end{aligned}$$

Write $N := \widetilde{g}(\xi,\xi) + \widetilde{g}(\eta,\eta)$ to make our notation simpler.

$$\begin{aligned}\frac{d}{dt}\widetilde{\Theta}^u_A(\xi,\eta) =& 2\frac{\widetilde{g}(\xi'_A + \eta'_A, \xi_A + \eta_A)}{N} + 2\frac{\widetilde{g}(\xi_{A'} + \eta_{A'}, \xi_A + \eta_A)}{N} \\ &- 2\frac{(\widetilde{g}(\xi',\xi) + \widetilde{g}(\eta',\eta))\widetilde{g}(\xi_A + \eta_A, \xi_A + \eta_A)}{N^2} \\ =& 2\frac{\widetilde{g}(\eta_A - (\widetilde{R}(\xi,\alpha')\alpha')_A, \xi_A + \eta_A)}{N} + 2\frac{\widetilde{g}(\xi_{A'} + \eta_{A'}, \xi_A + \eta_A)}{N} \\ &- 2\frac{(\widetilde{g}(\eta,\xi) + \widetilde{g}(-\widetilde{R}(\xi,\alpha')\alpha', \eta))\widetilde{g}(\xi_A + \eta_A, \xi_A + \eta_A)}{N^2}\end{aligned}$$

The strategy to show that this quantity is positive is to approximate it by (2.3.1) plus terms that can be made small or do not change its sign. Since h does not depend on x_0, we have that $\alpha'(h) = 0$. We first deal with the term $\widetilde{g}(\eta_A - (\widetilde{R}(\xi,\alpha')\alpha')_A, \xi_A + \eta_A)$: from the equation (1.1.8) and by the definition of the space A we get the following:

$$\begin{aligned}-(\widetilde{R}(\xi,\alpha')\alpha')_A =& -(R(\xi,\alpha')\alpha')_A + \frac{1}{2}Hess(h)(\alpha',\alpha')\xi_A \\ &+ \frac{1}{2}\|\alpha'\|^2 (\nabla_\xi \nabla h)_A + \frac{1}{4}\|\alpha'\|^2 \|\nabla h\|^2 \xi_A - (\xi h)\|\alpha'\|^2 (\nabla h)_A\end{aligned}$$

Notice that by the estimates in Lemma 2.2.2

- There exists $C_1 > 0$ such that $|Hess(h)(\alpha',\alpha')| \leq C_1\varepsilon^2$.

- There exists $C_2 > 0$ such that $\frac{1}{4} \|\alpha'\| \|\nabla h\|^2 \leq C_2 \varepsilon^2$.
- There exists $C_3 > 0$ such that $|(\xi h) \|\alpha'\|^2 \|(\nabla h)_A\| \leq C_3 \|\xi\| \varepsilon^2$.

It implies that there exists $C > 0$ such that we can write

$$-(\widetilde{R}(\xi, \alpha')\alpha')_A = -(R(\xi, \alpha')\alpha')_A + \frac{1}{2} \|\alpha'\|^2 (\nabla_\xi \nabla h)_A + v_A + w_A, \tag{2.3.3}$$

with $v_A, w_A \in A$ such that $\|v_A\| \leq C\varepsilon^2$ and $\|w_A\| \leq C \|\xi\| \varepsilon^2$. We claim that we can also control the norm of $(\nabla_\xi \nabla h)_A$. First, remember that since A is parallel to the initial metric, we can write

$$(\nabla_\xi \nabla h)_A = \nabla_\xi (\nabla h)_A.$$

Now,

$$\nabla_\xi (\nabla h)_A = \nabla_\xi \left(\partial_i h \left(\frac{\partial}{\partial x^i} \right)_A \right) = \xi(\partial_i h) \left(\frac{\partial}{\partial x^i} \right)_A + \partial_i h \nabla_\xi \left(\frac{\partial}{\partial x^i} \right)_A .$$

Because the Christoffel symbols vanish at $x = 0$, there exists a constant $D_1 > 0$ such that the second term has norm less than or equal to $D_1 \|\xi\| \varepsilon^2$. For the first term also by Lemma 2.2.2 we get that there exists $D_2 > 0$ such that for all $i \neq s$

$$\left\| \xi(\partial_i h) \left(\frac{\partial}{\partial x^i} \right)_A \right\| \leq D_2 \|\xi\| \varepsilon.$$

For $i = s$

$$\left\| \xi(\partial_s h) \left(\frac{\partial}{\partial x^s} \right)_A \right\| = \left\| \sum_j \xi_j \partial^2_{js} h \left(\frac{\partial}{\partial x^s} \right)_A \right\| \leq \left\| \sum_{j \neq s} \xi_j \partial^2_{js} h \left(\frac{\partial}{\partial x^s} \right)_A \right\| + \left\| \xi_s \partial^2_{ss} h \left(\frac{\partial}{\partial x^s} \right)_A \right\| . \tag{2.3.4}$$

Then, there exists $D_3 > 0$ such that

$$\left\| \sum_{j \neq s} \xi_j \partial^2_{js} h \left(\frac{\partial}{\partial x^s} \right)_A \right\| \leq D_3 \|\xi\| \varepsilon.$$

For the last term of (2.3.4) notice that $\left(\frac{\partial}{\partial x^s} \right)_{A(\gamma, \gamma')} = 0$, then, since α is parallel, by continuity of the subbundle A, there exists $D_4 > 0$ such that

$$\left\| \xi_s \partial^2_{ss} h \left(\frac{\partial}{\partial x^s} \right)_A \right\| \leq D_4 |\xi_s| \varepsilon.$$

We conclude that there exists $D > 0$ such that

$$\|(\nabla_\xi \nabla h)_A\| \leq D \|\xi\| \varepsilon.$$

Back to equality (2.3.3), there exists a constant $\widetilde{C} > 0$ such that

$$-(\widetilde{R}(\xi,\alpha')\alpha')_A = -(R(\xi,\alpha')\alpha')_A + v_A + w_A, \tag{2.3.5}$$

with $v_A, w_A \in A$ such that $\|v_A\| \leq \widetilde{C}\varepsilon^2$ and $\|w_A\| \leq \widetilde{C}\,\|\xi\|\,\varepsilon$. Finally, using the equality $(R(\xi,\alpha')\alpha')_A = -\xi_A$, we can write

$$\widetilde{g}(\eta_A - (\widetilde{R}(\xi,\alpha')\alpha')_A, \xi_A + \eta_A) = \widetilde{g}(\xi_A + \eta_A, \xi_A + \eta_A) + \widetilde{g}(v_A, \xi_A + \eta_A) + \widetilde{g}(w_A, \xi_A + \eta_A).$$

We have shown that $2^{\frac{\widetilde{g}(\eta_A - (\widetilde{R}(\xi,\alpha')\alpha')_A, \xi_A + \eta_A)}{N}}$ can be made arbitrarily close to $2^{\frac{\widetilde{g}(\xi_A + \eta_A, \xi_A + \eta_A)}{N}}$, which is similar to the first term that appears in (2.3.2). We now deal with $\widetilde{g}(-\widetilde{R}(\xi,\alpha')\alpha', \eta)$: from the relation (1.1.8) and the fact that η is orthogonal to α' we get

$$\begin{aligned}\widetilde{g}(-\widetilde{R}(\xi,\alpha')\alpha',\eta) =& \widetilde{g}(-R(\xi,\alpha')\alpha',\eta) + \frac{1}{2}Hess(h)(\alpha',\alpha')\widetilde{g}(\xi,\eta) \\ &+ \frac{1}{2}\|\alpha'\|^2\, Hess(h)(\xi,\eta) + \frac{1}{4}\|\alpha'\|^2\,\|\nabla h\|^2\,\widetilde{g}(\xi,\eta) \\ &+ \frac{1}{4}(\xi h)\,\|\alpha'\|^2\,(\eta h)\end{aligned} \tag{2.3.6}$$

Proceeding similarly as before and applying the estimates from Lemma 2.2.2, we have the following:

- There exists $C_1 > 0$ such that $\frac{1}{2}|Hess(h)(\alpha',\alpha')| \leq C_1\varepsilon^2$.
- There exists $C_2 > 0$ such that $\frac{1}{4}\|\alpha'\|\,\|\nabla h\|^2 \leq C_2\varepsilon^2$.
- There exists $C_3 > 0$ such that $\frac{1}{4}\|\alpha'\|^2\,|(\xi h)(\eta h)| \leq C_3\,\|\xi\|\,\|\eta\|\,\varepsilon^2$.

We also can have some control on $Hess(h)(\xi,\eta)$: there exists C_4 such that

$$|Hess(h)(\xi,\eta) - \xi_s\eta_s\partial^2_{ss}h| \leq \sum_{(i,j)\neq(s,s)} |\xi^i\eta^j||\partial^2_{ij}h - \Gamma^k_{ij}\partial_k h| \leq C_4\,\|\xi\|\,\|\eta\|\,\varepsilon.$$

Putting everything together, we guarantee that there exists $C > 0$ such that we can rewrite equation (2.3.6) as

$$\widetilde{g}(-\widetilde{R}(\xi,\alpha')\alpha',\eta) = \widetilde{g}(-R(\xi,\alpha')\alpha',\eta) - \frac{1}{2}\|\alpha'\|^2\,\xi_s\eta_s\partial^2_{ss}h + K_1 + K_2, \tag{2.3.7}$$

with $|K_1| \leq C\varepsilon^2$ and $|K_2| \leq C\,\|\xi\|\,\|\eta\|\,\varepsilon$. The last term we need to deal with is $\widetilde{g}(\xi_{A'} + \eta_{A'}, \xi_A + \eta_A)$, but record that from our notation $\xi_{A'} = (\widetilde{D}_t Pr_A)\eta$ and $\widetilde{D}_t$ is ε-close to D_t for which we had $D_t Pr_A = 0$ by the locally symmetric assumption. It implies that there exists

$E > 0$ such that $\widetilde{g}(\xi_{A'} + \eta_{A'}, \xi_A + \eta_A) \leq E \left\| \xi_A + \eta_A \right\| \varepsilon$. All the above estimates imply, by mimicking what we have done in equation (2.3.2) and using [CP14, Lemma 2.10], that there exist constants $K > 0$ and $K' > 0$ such that we can write the angle variation as

$$\frac{d}{dt}\widetilde{\Theta}^u_A(\xi,\eta) = 2\frac{\widetilde{g}(\eta_A + \xi_A, \xi_A + \eta_A)}{(\widetilde{g}(\xi,\xi) + \widetilde{g}(\eta,\eta))^2}\Big(\widetilde{g}(\xi,\xi) + \widetilde{g}(\eta,\eta) - \widetilde{g}(\eta,\xi) - \frac{1}{4}\widetilde{g}(\eta_B,\xi_B) - \widetilde{g}(\eta_A,\xi_A) - \frac{1}{2}\left\|\alpha'\right\|^2 \xi_s\eta_s\partial^2_{ss}h\Big) + U_1 + U_2 + U_3 + U_4, \tag{2.3.8}$$

with

- $|U_1| \leq K\frac{\|\xi_A+\eta_A\|}{N}\varepsilon^2 \leq K'\varepsilon^2$,
- $|U_2| \leq K\frac{\|\xi_A+\eta_A\|\|\xi\|}{N}\varepsilon \leq K'\varepsilon$,
- $|U_3| \leq K\frac{\|\xi_A+\eta_A\|^2}{N^2}\varepsilon^2 \leq K'\varepsilon^2$,
- $|U_4| \leq K\frac{\|\xi_A+\eta_A\|\|\xi\|\|\eta\|}{N^2}\varepsilon \leq K'\varepsilon$.

Thus, it is sufficient to prove that the quantity inside the parentheses is positive and then choose $\varepsilon > 0$ sufficiently small such that the expression (2.3.8) is positive. To see that, notice that since α' is a unit geodesic for the metric $\widetilde{g}$, then $\left\|\alpha'\right\|^2 = g(\alpha',\alpha') = \mathrm{e}^{-h}$ does not exceed $\mathrm{e}^{2\varepsilon^4}$. It follows from the construction of the function h that

$$-\frac{1}{2}\left\|\alpha'\right\|^2\partial^2_{ss}h \leq \frac{\mathrm{e}^{2\varepsilon^4}}{4} \leq \frac{1}{2}.$$

If $\xi_s\eta_s \geq 0$, then by the same computations for the initial metric, we can conclude that the expression (2.3.8) is positive. If $\xi_s\eta_s < 0$ then we must have

$$\frac{1}{2}\xi_s\eta_s \leq -\frac{1}{2}\left\|\alpha'\right\|^2\partial^2_{ss}h\xi_s\eta_s.$$

So, it is sufficient to show that the following function is positive

$$f(x) := \widetilde{g}(\xi,\xi) + \widetilde{g}(\eta,\eta) - \widetilde{g}(\eta,\xi) - \frac{1}{4}\widetilde{g}(\eta_B,\xi_B) - \widetilde{g}(\eta_A,\xi_A) - \frac{1}{2}\xi_s\eta_s.$$

Notice that f does not depend on ε, so we prove that $f(0) > 0$, and by shrinking the deformed neighborhood if necessary, we guarantee that $f(x) > 0$.

$$\begin{aligned} f(0) &= g(\xi,\xi) + g(\eta,\eta) - g(\eta,\xi) - \frac{1}{4}g(\eta_B,\xi_B) - g(\eta_A,\xi_A) - \frac{1}{2}\xi_s\eta_s \\ &= \sum_{i=1}^{n-1}(\xi_i^2+\eta_i^2) - \sum_{i=1}^{n-1}\xi_i\eta_i - \frac{1}{4}\sum_{i=r+1}^{n-1}\xi_i\eta_i - \sum_{i=1}^{r}\xi_i\eta_i - \frac{1}{2}\xi_s\eta_s \\ &= \sum_{i=1}^{r}(\xi_i^2 - 2\xi_i\eta_i + \eta_i^2) + \sum_{\substack{i=r+1\\ i\neq s}}^{n-1}\left(\xi_i^2 - \frac{5}{4}\xi_i\eta_i + \eta_i^2\right) + \left(\xi_s^2 - \frac{7}{4}\xi_s\eta_s + \eta_s^2\right) \\ &= \sum_{i=1}^{r}(\xi_i-\eta_i)^2 + \sum_{\substack{i=r+1\\ i\neq s}}^{n-1}\left(\xi_i - \frac{5}{8}\eta_i\right)^2 + \frac{39}{64}\sum_{\substack{i=r+1\\ i\neq s}}^{n-1}\eta_i^2 + \left(\xi_s - \frac{7}{8}\eta_s\right)^2 + \frac{15}{64}\eta_s^2 \\ &> 0 \end{aligned}$$

We have proven the following proposition:

Proposition 2.3.1. *There exists $\varepsilon > 0$ such that the family of cones $C(v, P_A^{u,s}(x,v), c)$ is invariant along parallel geodesics that cross $U(\varepsilon)$.*

2.3.2.2 Cone invariance for s-almost parallel geodesics

Suppose that $\alpha' = \alpha^i \dfrac{\partial}{\partial x^i}$ is such that $|\alpha^s| \leq \theta$. We will prove that the cone family is invariant along such geodesics if θ is small enough. The proof follows the same lines as the proof for parallel geodesics, the difference here is that for some estimates we can not assume that α' has no components in other directions, but it is possible to obtain essentially the same estimates for the first derivatives of h in the α' direction and θ small will control the Hessian of h applied to α'. Indeed, for a s-almost parallel geodesic, the same analysis holds but for terms that are controlled by θ: by equation (1.1.8) we can see that

$$\begin{aligned} -(\widetilde{R}(\xi,\alpha')\alpha')_A = -(R(\xi,\alpha')\alpha')_A + v_A + w_A - \frac{1}{4}(\alpha'(h))^2\xi_A + \frac{1}{4}(\xi h)(\alpha'(h))(\nabla h)_A \\ -\frac{1}{2}Hess(h)(\alpha',\alpha')(h)\xi_A, \end{aligned}$$

With v_A and w_A as before. However, there exists $C_1 > 0$ and $C_2 > 0$ such that

$$\left\| -\frac{1}{4}(\alpha'(h))^2\xi_A - \frac{1}{4}(\xi h)(\alpha'(h))(\nabla h)_A \right\| \leq C_1 \|\xi\| \varepsilon^2$$

and

$$|Hess(h)(\alpha',\alpha')| \leq C_1 \|\alpha'\|^2 \varepsilon + C_2\theta^2.$$

So, without losing generality, there exists a constant $C > 0$ such that we still can write the relation (2.3.5) for s-almost parallel geodesics plus a term that is controlled by θ, i.e.

$$-(\widetilde{R}(\xi, \alpha')\alpha')_A = -(R(\xi, \alpha')\alpha')_A + v_A + w_A + u_A, \tag{2.3.9}$$

where $\|u_A\| \leq C\|\xi\|\theta^2$. Analogously, there exists a constant $D > 0$ such that we can rewrite the equation (2.3.7) for s-almost parallel geodesics as follows:

$$\begin{aligned}\widetilde{g}(-\widetilde{R}(\xi, \alpha')\alpha', \eta) =& \widetilde{g}(-R(\xi, \alpha')\alpha', \eta) - \frac{1}{2}\|\alpha'\|^2 \xi_s \eta_s \partial^2_{ss} h + K_1 + K_2 + K_3 \\ & - \frac{1}{4}(\alpha'(h))^2 \widetilde{g}(\xi, \eta) + \frac{1}{4}(\xi h)(\alpha'(h))(\eta h).\end{aligned}$$

Again, the extra terms satisfy the same properties as K_1 and K_2 and $|K_3| \leq D\|\xi\|\|\eta\|\theta^2$. Then we can write, without losing generality, the equation (2.3.8) for s-almost parallel geodesics plus a term controlled by θ^2. Since we already proved that this expression is positive, next Proposition follows:

Proposition 2.3.2. *There exist numbers $\varepsilon > 0$ and $\theta > 0$ such that the family of cones $C(v, P^{u,s}_A(x, v), c)$ is invariant along any geodesic α with $|\alpha^s| < \theta$ that crosses $U(\varepsilon)$.*

Notice that the above proposition works for any $\varepsilon^* \leq \varepsilon$ and $\theta^* \leq \theta$.

2.3.2.3 Cone invariance for s-transversal geodesics

Here we show how to control the time that a s-transversal geodesic spends inside the deformed region.

Fix $\varepsilon_1 > 0$ and $\theta_1 > 0$ given by Proposition 2.3.1 and Proposition 2.3.2. Since $\Gamma^k_{ij}(t, 0) = 0$, then by relation (1.1.6) we get that $\widetilde{\Gamma}^k_{ij}(t, 0) = 0$ and also by the same relation we can find a constant $C > 0$ such that $|\widetilde{\Gamma}^k_{ij}(t, x)| \leq C\varepsilon_1$. Then, denoting by $\widetilde{G}$ the geodesic vector field for the metric $\widetilde{g}$ we can find a constant $D > 0$ such that

$$\left\|\widetilde{G}(v) - (v_1, ..., v_n, 0, ..., 0)\right\| \leq D\varepsilon_1.$$

Therefore, we have that any geodesic $\alpha'(0) = (v_0, ..., v_{n-1})$ in $U(\varepsilon_1)$ is ε_1-close to the curve $\beta(s) = (sv_0, ..., sv_{n-1})$. If $|v_s| > \theta_1$, then for $s \geq \frac{2\varepsilon_1}{\theta_1}$ the curve $\beta(s)$ escapes the neighborhood $U(2\varepsilon_1)$, so the geodesic α escapes the neighborhood $U(\varepsilon_1)$. Since θ_1 is fixed, let $\varepsilon_2 < \varepsilon_1$ such that we have $\frac{2\varepsilon_2}{\theta_1} < \varepsilon_1$.

2.3.3 Proof of Theorem A

From Sections 2.3.1 2.3.2.1, 2.3.2.2, 2.3.2.3 the same arguments in [CP14] Section 5.3.6. imply the following proposition:

Proposition 2.3.3. *There exists $\varepsilon > 0$ and $b > 0$ such that if h is supported in $U(\varepsilon)$, the family of cones $C(v, P_A^{u,s}(x,v), b)$ is invariant by the action of $d\tilde{g}_t$.*

For completeness we explain here the main idea. Let $\tilde{g}_t(v)$ be an orbit of the new geodesic flow. If the geodesic $\pi(\tilde{g}_t(v))$ never crosses the deformed region, then the computations for the initial metric shows that there is invariance of the cone field. Now, suppose it crosses the region. Then it could be a parallel, s-almost parallel or s-transverse geodesic. In the first two cases we have shown that once a geodesic points in the parallel or s-almost parallel direction, then the family of cone fields is invariant. The last case to deal with is the s-transverse one. In this case, the last section shows that the time a s-transverse geodesic can spend in the deformed region is bounded from above by the size of the deformed region. The time a s-transversal geodesic spends outside is bounded from below by a positive constant, therefore it must spend enough time outside the deformed region to get invariance by the cone field. Therefore, we can shrink once again the deformed region if necessary to get invariance of the cone field. Hence, we obtain the following Corollary:

Corollary 2.3.1. *The geodesic flow $\tilde{g}_t : T^1M \to T^1M$ admits a dominated splitting of the form $ST^1M = E^{uu} \oplus E^c \oplus E^{ss}$.*

To conclude the proof of Theorem A, see that we have proved in Corollary 2.3.1 that the geodesic flow $D\tilde{g}_t : ST^1M \to ST^1M$ is a symplectic flow with dominated splitting of the form $ST^1M = E^s \oplus E^c \oplus E^u$. Now Lemma 1.3.1 implies that the vectors in E^u and E^s are uniformly expanding and contracting, respectively, thus $\tilde{g}_t$ is partially hyperbolic by the cone criterion.

We finish this section with some remarks:

Remark 2.3.3. The key property of h to guarantee the invariance of the cone fields is $-\frac{1}{2} \leq \partial^2_{ss} h \leq \frac{1}{2}$ and it attains its minimum at $x = 0$. No other property is needed, hence with Proposition 2.2 we can summarize the properties on h so that $\tilde{g}$ has a partially hyperbolic and non-Anosov geodesic flow:

(1) h is supported on $B(\gamma, \varepsilon)$, for sufficiently small $\varepsilon > 0$.

(2) h is of class at least C^2.

(3) h does not depend on t.

(4) $h(0) = 0$

(5) $\nabla h|_\gamma = 0$.

(6) There exists $s \in \{r+1, ..., n-1\}$ such that $\partial^2_{ss} h(0) = -\frac{1}{2}$.

(7) $-\frac{1}{2} \leq \partial^2_{ss} h \leq \frac{1}{2}$ and it attains its minimum at $x = 0$.

Remark 2.3.4. We do not know a priori if the new metric has conjugate points. The best we can expect to answer in this direction is the following: since $-\frac{1}{2}\partial^2_{ss} h \leq \frac{1}{4}$, $K(X,Y) \leq -\frac{1}{4}$, the Lemma 2.2.2 and the estimates on the Christoffel Symbols for the initial metric, it follows that $\tilde{K}(X,Y) \leq M\varepsilon$. So, the resultant sectional curvatures could be positive, but it is still controlled by ε and can be made as small as we want. If we have uniform control over the amount of time a geodesic can spend in a region of possible positive curvature, then we can conclude the non-existence of conjugate points by the arguments in [Gul75].

Remark 2.3.5. For our construction, we fixed the s-direction for which we perform the largest curvature deformation. However, under small adaptations to the arguments, it is possible to produce partially hyperbolic examples by deforming the curvature in several directions. To do this, consider some subcollection of indices $\mathcal{I} \subset \{r+1, ..., n-1\}$ and let h be given by

$$h(x) = \sum_{s \in \mathcal{I}} h^s(x),$$

where h^s is the function constructed in Section 2.2 for the s-direction. However, by doing so, we certainly create planes of positive curvature. More precisely, in the Kähler setting we have that $A(x,v) = \text{span}\{Jv\}$ and for some $s, l \in \mathcal{I}$ we may get

$$\widetilde{K}(e_s(t), e_l(t)) = K(e_s(t), e_l(t)) - \frac{1}{2}\partial^2_{ss} h^s(0) - \frac{1}{2}\partial^2_{ll} h^l(0) = -\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Although some positive curvature does not imply immediately the existence of conjugate points, this was something we wanted to avoid in this work.

2.4 Proof of Theorem B

The proof of Theorem B is similar to Theorem A. We work in the same context and with the same tubular neighborhood of a closed geodesic. The difference lies in the different deformations we make to have better control of the resulting sectional curvatures. Here we present only the estimates on the curvature once the proof of partial hyperbolicity is the same as presented in the previous section, since the curvature tensor estimates needed still hold. The deformation goes as follows: let $g_{ij}(t,x)$ and $g^{ij}(t,x)$ be the components of the initial locally symmetric metric and its inverse in Fermi coordinates defined in the previous section. Considering h as the same function as before, the new metric $\widetilde{g}$ is given by

$$\widetilde{g}_{00}(t,x) = e^h g_{00}(t,x)$$

$$\widetilde{g}_{ij}(t,x) = g_{ij}(t,x), \ \ (i,j) \neq (0,0).$$

The new deformation is almost a conformal deformation of the initial metric, however, we only multiply by a conformal factor the components of the metric in the geodesic direction. By doing so, we prevent not only the curvature from becoming positive but also we do not create other planes with zero curvature as before (remember that the planes $\{e_s(t), e_i(t)\}$, with $i = r+1, ..., n-1$ also have zero curvature). So, suppose some sectional curvature is slightly growing when we move apart from γ, then we can prevent it from becoming non-negative by pushing it to be negative along γ.

For this deformation we have $\tilde{g}^{ij} = g^{ij}$ for $(i,j) \neq (0,0)$ and $\tilde{g}^{00} = e^{-h} g_{00}$. Remember that from the formula given by Lemma 1.1.3, the deformation we made is not changing many of the components R_{ijkl}, and it has just a small effect on a few of them, as we are going to see.

Differently from a "complete" conformal deformation, the relation between some geometric quantities is less clear. For instance, we do not have an explicitly formula relating the covariant derivatives of the two metrics, however we are still able to show that γ is still a geodesic by calculating the new Christoffel symbols (indeed any unitary parallel vector field along γ is still unitary parallel with respect to the new metric).

Remark 2.4.1. The following identities for the derivatives of the new metric will be useful in our calculations:

(1) $\partial_i(\widetilde{g}_{00}) = \partial_i h \widetilde{g}_{00} + e^h \partial_i g_{00}$.

(2) $\partial^2_{ij}(\widetilde{g}_{00}) = \partial^2_{ij}h\widetilde{g}_{00} + \partial_i h\partial_j h\widetilde{g}_{00} + e^h\partial_i h\partial_j g_{00} + e^h\partial_j h\partial_i g_{00} + e^h\partial^2_{ij}g_{00}$

We will first evaluate the Christoffel Symbols of the new metric. A straightforward computation proves:

Lemma 2.4.1. *The Christoffel symbols for $\widetilde{g}$ satisfy the following relations in $B(\gamma, \varepsilon)$:*

(1) $i, j \neq 0$

(1.1) $k = 0$

$$\begin{aligned}\widetilde{\Gamma^0_{ij}} &= \frac{\widetilde{g}^{l0}}{2}(\partial_i\widetilde{g}_{jl} + \partial_j\widetilde{g}_{il} - \partial_l\widetilde{g}_{ij}) \\ &= \sum_{l\neq 0}\frac{\widetilde{g}^{l0}}{2}(\partial_i\widetilde{g}_{jl} + \partial_j\widetilde{g}_{il} - \partial_l\widetilde{g}_{ij}) + \frac{\widetilde{g}^{00}}{2}(\partial_i\widetilde{g}_{j0} + \partial_j\widetilde{g}_{i0} - \partial_0\widetilde{g}_{ij}) \\ &= \sum_{l\neq 0}\frac{g^{l0}}{2}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) + \frac{e^{-h}g^{00}}{2}(\partial_i g_{j0} + \partial_j g_{i0} - \partial_0 g_{ij}).\end{aligned}$$

(1.2) $k \neq 0$

$$\widetilde{\Gamma^k_{ij}} = \Gamma^k_{ij}.$$

(2) $i = 0,\ j \neq 0$

(2.1) $k = 0$

$$\widetilde{\Gamma}^0_{0j} = \sum_{l\neq 0}\frac{g^{0l}}{2}(\partial_0 g_{jl} + \partial_j g_{0l} - \partial_l g_{0j}) + e^{-h}\frac{g^{00}}{2}(\partial_0 g_{j0} + e^h\partial_j g_{00} - \partial_0 g_{0j}) + \frac{g^{00}}{2}g_{00}\partial_j h.$$

(2.2) $k \neq 0$

$$\widetilde{\Gamma}^k_{0j} = \sum_{l\neq 0}\frac{g^{kl}}{2}(\partial_0 g_{jl} + \partial_j g_{0l} - \partial_l g_{0j}) + \frac{g^{k0}}{2}(\partial_0 g_{j0} + e^h\partial_j g_{00} - \partial_0 g_{0j}) + \frac{g^{k0}}{2}g_{00}e^h\partial_j h.$$

(3) $i = j = 0$

(3.1) $k = 0$

$$\widetilde{\Gamma}^0_{00} = \sum_{l\neq 0}\left(\frac{g^{0l}}{2}(\partial_0 g_{0l} + \partial_0 g_{0l} - e^h\partial_l g_{00}) - e^h\frac{g^{0l}}{2}g_{00}\partial_l h\right) + \frac{g^{00}}{2}\partial_0 g_{00}.$$

(3.2) $k \neq 0$

$$\widetilde{\Gamma}^k_{00} = \sum_{l\neq 0}\left(\frac{g^{kl}}{2}(\partial_0 g_{0l} + \partial_0 g_{0l} - e^h\partial_l g_{00}) - e^h\frac{g^{kl}}{2}g_{00}\partial_l h\right) + \frac{g^{k0}}{2}\partial_0 g_{00}.$$

The properties of h and the computations above prove that $\widetilde{\Gamma}^k_{ij}(t,0) = \Gamma^k_{ij}(t,0) = 0$, so γ is still a geodesic for the metric $\widetilde{g}$. Furthermore, we also have by the properties of h that the contribution of the new Christoffel symbols to the curvature $\widetilde{R}_{s00s}$ that appears in the equation (2.4.1) below is close to the contribution of the initial Christoffel symbols and with the same sign inside the deformed region. These terms are slightly smaller and grow slightly slower. To see this, consider for instance the functions $\widetilde{f}_{nm}(x) = \widetilde{\Gamma}^m_{0s}\widetilde{\Gamma}^n_{0s}$ and $f_{nm}(x) = \Gamma^m_{0s}\Gamma^n_{0s}$. A straightforward computation shows that at $x = 0$ we get that $\widetilde{f}_{nm}$ and f_{nm} are equal up to the second derivative. Indeed we have $\widetilde{\Gamma}^m_{0s}\widetilde{\Gamma}^n_{0s} \leq \Gamma^m_{0s}\Gamma^n_{0s}$ by computing higher order derivatives of the same functions for some indices (m,n), using the properties of h and shrinking the deformed region if necessary. By doing the same kind of comparison with the terms $\widetilde{\Gamma}^m_{00}\widetilde{\Gamma}^n_{ss}$ we get that $-\widetilde{\Gamma}^m_{ss}\widetilde{\Gamma}^n_{00} \leq -\Gamma^m_{ss}\Gamma^n_{00}$.

2.4.1 Curvature over the central geodesic

In this Subsection, we show how to deal with the new geometric quantities since we have no "nice" formula relating them to the previous one. We are still able to relate them with some more work, especially along γ. Let us start by showing that the new geodesic flow is not Anosov. We use the same type of argument with Proposition 1.2.4. Remember that from Lemma 1.1.3 we can write

$$\widetilde{R}_{ijkr} = \widetilde{R}^l_{ijk}\widetilde{g}_{lr} = \frac{1}{2}(\partial^2_{jr}\widetilde{g}_{ik} + \partial^2_{ik}\widetilde{g}_{jr} - \partial^2_{ir}\widetilde{g}_{jk} - \partial^2_{jk}\widetilde{g}_{ir}) + \widetilde{g}_{mn}(\widetilde{\Gamma}^m_{ki}\widetilde{\Gamma}^n_{jr} - \widetilde{\Gamma}^m_{ri}\widetilde{\Gamma}^n_{jk}). \tag{2.4.1}$$

In particular, for $j = k$ and $i = r$ we get

$$\widetilde{R}_{ikki} = \frac{1}{2}(\partial^2_{ki}\widetilde{g}_{ik} + \partial^2_{ik}\widetilde{g}_{ki} - \partial^2_{ii}\widetilde{g}_{kk} - \partial^2_{kk}\widetilde{g}_{ii}) + \widetilde{g}_{mn}(\widetilde{\Gamma}^m_{ki}\widetilde{\Gamma}^n_{ki} - \widetilde{\Gamma}^m_{ii}\widetilde{\Gamma}^n_{kk}).$$

For $k = 0$ at $x = 0$, we get by Remark 2.4.1

$$\begin{aligned}
\widetilde{R}_{i00i} &= \frac{1}{2}(\partial^2_{0i}\widetilde{g}_{i0} + \partial^2_{i0}\widetilde{g}_{0i} - \partial^2_{ii}\widetilde{g}_{00} - \partial^2_{00}\widetilde{g}_{ii}) \\
&= \frac{1}{2}(\partial^2_{0i}g_{i0} + \partial^2_{i0}g_{0i} - e^h\partial^2_{ii}g_{00} - \partial^2_{00}g_{ii}) \\
&- \frac{1}{2}\partial^2_{ii}he^h g_{00} - \frac{1}{2}(\partial_i h)^2 e^h g_{00} - e^h\partial_i h\partial_i g_{00} \\
&= R_{i00i} - \frac{1}{2}\partial^2_{ii}h
\end{aligned}$$

Then, for $i = s$ we get $\widetilde{R}_{s00s}(t,0) = 0$ and for $i \neq s$ $\widetilde{R}_{i00i}(t,0) = R_{i00i}(t,0) < 0$. For $Y = Y^i X_i$ a vector field orthonormal to X_s at $x = 0$ such that $Y \neq X_0$ we get

$$\begin{aligned}
\widetilde{R}(X_s, Y, Y, X_s) = Y^i Y^j \widetilde{R}_{sijs} &= \frac{1}{2} \sum_{(i,j)\neq(0,0)} Y^i Y^j (\partial^2_{is} g_{sj} + \partial^2_{sj} g_{is} - \partial^2_{ss} g_{ij} - \partial^2_{ij} g_{ss}) \\
&+ \frac{1}{2} Y_0^2 (\partial^2_{0s} g_{s0} + \partial^2_{s0} g_{0s} - \partial^2_{ss} \widetilde{g}_{00} - \partial^2_{00} g_{ss}) \\
&= R(X_s, Y, Y, X_s) - \frac{1}{2} \partial^2_{ss} h Y_0^2 \\
&\leq -\frac{1}{4} + \frac{1}{4} Y_0^2 \\
&< 0
\end{aligned}$$

The inequality above shows that every sectional curvature is negative over the central geodesic, but the sectional curvature of the plane $\Pi = \mathrm{span}\{\gamma', e_s\}$, which is zero.

2.4.2 Curvature outside the central geodesic

Notice that outside the central geodesic, we can write

$$\widetilde{R}_{s00s} = R^*_{s00s} - \frac{1}{2} \partial^2_{ss} h e^h g_{00} - \frac{1}{2} (\partial_s h)^2 e^h g_{00} - e^h \partial_s h \partial_s g_{00},$$

where

$$R^*_{s00s} = \frac{1}{2} (\partial^2_{0s} g_{s0} + \partial^2_{s0} g_{0s} - e^h \partial^2_{ss} g_{00} - \partial^2_{00} g_{ss}) + \widetilde{g}_{mn} (\widetilde{\Gamma}^m_{0s} \widetilde{\Gamma}^n_{0s} - \widetilde{\Gamma}^m_{ss} \widetilde{\Gamma}^n_{00}),$$

which is similar to the expression for R_{s00s} but for the terms $-e^h \partial^2_{ss} g_{00}$ and the small perturbation of the Christoffel Symbols, which does not change the sign of the contribution of this part as mentioned before. In Fermi coordinates we have that $g_{s0} \neq 0$ and $g_{ss} \neq 1$ for $x \neq 0$, therefore we can write

$$\widetilde{K}(X_s, X_0) = \widetilde{K}\left(\frac{X_s}{g_{ss}}, Y\right)$$

where Y is the orthonormal (to X_s) vector field we obtain by applying Gram-Schmidt process for the metric $\widetilde{g}$:

$$Y = \left(\frac{g_{ss}}{e^h g_{00} g_{ss} - g_{s0}^2}\right) X_0 - \left(\frac{g_{s0}}{e^h g_{00} g_{ss} - g_{s0}^2}\right) X_s = \widetilde{\alpha} X_0 + \widetilde{\beta} X_s.$$

Notice that since Y is a normal vector, we get $\widetilde{\alpha} < 1$ (for $x \neq 0$). Besides that, since $h \leq 0$ an easy computation shows that $\widetilde{\alpha} \geq \alpha$, where α is the component we would obtain by the Gram-Schmidt process for the metric g, namely $\alpha = g_{ss}(g_{00}g_{ss} - g_{s0}^2)^{-1}$. Then, we can write

$$\begin{aligned}\widetilde{K}(X_s, X_0) &= \widetilde{K}\left(\frac{X_s}{g_{ss}}, Y\right) = \widetilde{R}\left(\frac{X_s}{g_{ss}}, Y, Y, \frac{X_s}{g_{ss}}\right)\\ &= \frac{\widetilde{\alpha}^2}{g_{ss}^2}\widetilde{R}(X_s, X_0, X_0, X_s) = \frac{\widetilde{\alpha}^2}{g_{ss}^2}R^*_{s00s}\\ &+ \frac{\widetilde{\alpha}^2}{g_{ss}^2}\left(-\frac{1}{2}\partial^2_{ss}he^hg_{00} - \frac{1}{2}(\partial_s h)^2e^hg_{00} - e^h\partial_s h\partial_s g_{00}\right).\end{aligned}$$

By the observations above and since $R^*_{s00s} \leq R_{s00s} < 0$, we get

$$\frac{\widetilde{\alpha}^2}{g_{ss}^2}R^*_{s00s} \leq \frac{\alpha^2}{g_{ss}^2}R_{s00s} = K(X_s, X_0) \leq -\frac{1}{4}$$

On the other hand, one has

$$-\frac{1}{2}\partial^2_{ss}he^hg_{00} - \frac{1}{2}(\partial_s h)^2e^hg_{00} - e^h\partial_s h\partial_s g_{00} \leq -\frac{1}{2}\partial^2_{ss}h - e^h\partial_s h\partial_s g_{00},$$

and for the above inequality, we used the Taylor expansion of g_{00} in Fermi coordinates (check [MM63] and adjust the indices), in our case, is given by

$$g_{00} = 1 - 2\sum_k R_{k00k}|_\gamma x_k^2 + \mathcal{O}(x^3),$$

so, for $\varepsilon > 0$ small enough we guarantee that $e^hg_{00} \leq 1$. This expansion also gives us that $\partial_s g_{00} = \frac{1}{2}x_s + \mathcal{O}(x_s^2)$, thus the term of order 1 determines the sign of $\partial_s g_{00}$. Hence, we need to study the sign of the function

$$g(x) := -\frac{1}{2}\partial^2_{ss}h - e^h\partial_s h\partial_s g_{00} = \left(f''(x_s) + x_sf'(x_s)e^h\right)\Phi(x)$$

In fact, it is enough to analyze the behavior of $p(x_s) = f''(x_s) + x_sf'(x_s)$, once the delicate part is analyzing for x_s around 0 where $x_sf'(x_s) > 0$, so since $e^h \leq 1$ the worst scenario is given by p. Notice that

(1) $p(0) = \frac{1}{4}$.

(2) $p'(0) = 0$.

(3) $p''(0) = -\frac{12}{\varepsilon^4} + \frac{1}{2} < 0$.

We conclude that $x = 0$ is a point of maximum value for p, indeed, it is a global maximum by checking other critical points. See its graph in Figure 2.5 below.

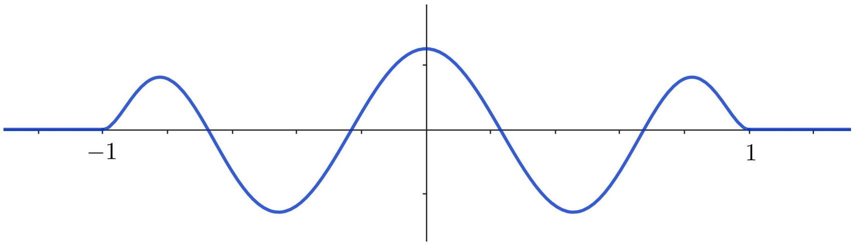


Figure 2.5: Graph of the function p for $\varepsilon = 1$

We conclude that $|g(x)| \leq \frac{1}{4}$ and equality holds if, and only if, $x = 0$. Finally, $\frac{\widetilde{\alpha}^2}{g_{ss}^2}|g(x)| < \frac{1}{4}$. Then, $\widetilde{K}(X_s, X_0) \leq 0$ with equality if, and only if, $x = 0$.
More generally, for any other orthonormal (to X_s) vector field Y not equal to X_0 we write for $x \neq 0$

$$\begin{aligned}
\widetilde{K}\left(\frac{X_s}{g_{ss}}, Y\right) &= \frac{1}{g_{ss}^2}\widetilde{R}(X_s, Y, Y, X_s) \\
&= \frac{1}{g_{ss}^2}R^*(X_s, Y, Y, X_s) + Y_0^2\left(-\frac{1}{2}\partial_{ss}^2 h e^h g_{00} - \frac{1}{2}(\partial_s h)^2 e^h g_{00} - e^h \partial_s h \partial_s g_{00}\right) \\
&< K(X_s, Y) + \frac{Y_0^2}{4} \\
&< 0
\end{aligned}$$

We have proved the following proposition:

Proposition 2.4.1. *All sectional curvatures for $\widetilde{g}$ are negative but the sectional curvature $\widetilde{K}(\gamma', e_s) \equiv 0$.*

From this proposition, we conclude that $(M, \widetilde{g})$ has non-positive sectional curvatures and just one plane with zero curvature along a single closed geodesic. We prove the following corollaries that imply Corollary B.1:

Corollary 2.4.1. *The metric $\widetilde{g}$ has no conjugate points.*

Proof. This is immediate from non-positive curvature. ■

Corollary 2.4.2. *The geodesic flow $\widetilde{g}_t$ is expansive.*

Proof. If the geodesic flow was not expansive, then by the well-known Flat Strip Theorem [EO73] there would exist flat strips on the universal cover of $(M, \widetilde{g})$, which is impossible since $(M, \widetilde{g})$ has only one closed geodesic with zero curvature and negative curvature elsewhere. ■

Corollary 2.4.3. *The dynamical system* $(S(T^1M), \tilde{g}_t, \widetilde{\text{Liou}})$ *is ergodic.*

Proof. The set of vectors with a rank bigger than one has measure zero, so it follows from the arguments in [BB82] and [Bur83] that the geodesic flow is a Bernoulli flow; thus, it is ergodic and also mixing. ∎

Corollary 2.4.4. *The geodesic flow* $\tilde{g}_t$ *has a unique measure of maximal entropy.*

Proof. By Theorem 7.2 of [CP14], we know that the metric $\tilde{g}$ is rank 1 (indeed, this is immediate in our case). Now, since $\tilde{K} \leq 0$, then the main theorem from [Kni98] implies that $\tilde{g}_t$ admits a unique measure of maximal entropy. ∎

2.5 Further results and questions

Previously, we have mentioned that the Anosov property implies the non-existence of conjugate points, and indeed by Theorem 2.0.3 the C^2-interior of the set of metrics with no conjugate points coincides with the set of metrics with Anosov geodesic flows. One may naturally ask:

Question 2.5. *What about partially hyperbolic geodesic flows? Does partial hyperbolicity imply the nonexistence of conjugate points?*

We were unable to check if the examples given by Theorem A present any geodesic with conjugate points, even though our construction "tries to avoid" such phenomena. On the other side, examples given by Theorem B do not present conjugate points and are not Anosov, thus they must lie on the topological boundary of the set of metrics with no conjugate points. As a consequence of our construction,n we prove the following:

Theorem H. There exists a C^2-open set $\mathcal{U}$ of Riemannian metrics such that if $g \in \mathcal{U}$, then g_t is partially hyperbolic, non-Anosov, and g has conjugate points.

Proof. Since partial hyperbolicity is a C^1-open property, there exists a C^2-open set $\mathcal{U}_1$ of Riemannian metrics containing the metric $\tilde{g}$ obtained via Theorem B. Since $\tilde{g}$ lies on the boundary of the set of Riemannian metrics with no conjugate points, we can find $\mathcal{U} \subset \mathcal{U}_1$ with the desired property. ∎

In [CP14], the authors have a different approach to produce an example on the boundary of the metrics with no conjugate points. Their strategy is to consider a straight line of Riemannian metrics from the initial metric g to a metric with partially hyperbolic geodesic flow non-Anosov with conjugate points, then the line of metrics must cross the boundary, producing the existence of such an example. The issue with this argument is due to the fact that it is unknown if the set of metrics with partially hyperbolic geodesic flow is convex or even path-connected. Hence, our approach overcomes this difficulty.

Question 2.6. *Do the metrics obtained via Theorem A have conjugate points?*

If they do not, then their geodesic flows are topologically transitive, i.e., there exists a dense orbit. It follows from [Ebe72] once the manifold admits a background metric for which its geodesic flow is of Anosov type.

On the other side, Corollary B.1 states that $\tilde{g}_t$ are mixing and, in particular, they are all topologically transitive. We can ask whether this property persists under a small perturbation of the metric $\tilde{g}$. In other words:

Question 2.7. *Is the geodesic flow $\tilde{g}_t$ robustly transitive inside the class of geodesic flows?*

Although we do not have the necessary technology to approach this question yet, we have made some advances by showing our examples present the so-called "SH-saddle" property that can be helpful to approach this problem. SH is a short abbreviation to "Some Hyperbolicity". Essentially, it says that given a sufficiently large piece of unstable leaf, there is some point in it for which $D\varphi_t$ presents some expansion in the center direction. Let us properly define this property and prove that our examples satisfy it:

Let f be a partially hyperbolic diffeomorphism with a splitting of the form $TM = E^{ss} \oplus E^c \oplus E^{uu}$ and invariant foliations $\mathcal{W}^{ss}$ and $\mathcal{W}^{uu}$. We are going to denote by $k = \dim E^c$. Then a *d-center cone* in $x \in M$ is simply a cone $\mathcal{C}(x)$ in $E^c(x)$ of dimension $d \leq k$. Recall that

$$\mathcal{W}_f^{\tau}(x,\varepsilon) := \{y \in \mathcal{W}_f^{\tau}(x) : d_{\mathcal{W}_f^{\tau}}(x,y) < \varepsilon\}$$

is the ε-ball in $\mathcal{W}_f^{\tau}$ of center x and radius ε for $\tau = ss, uu$.

Definition 2.5.1. Given a partially hyperbolic diffeomorphism f, we say that the strong unstable foliation $\mathcal{W}^{uu}$ has the SH-Saddle property of index $d \leq k$ if there are constants $L > 0$, $\theta > 0$, $\lambda_0 > 1$ and $C > 0$ such that the following holds for every point $x \in M$: there exists a point $x^u \in \mathcal{W}_f^{uu}(x, L)$ such that:

(1) There is a d-center cone field of opening θ along the forward orbit of x^u which is Df-invariant, i.e. there exist $\mathcal{C}^u_\theta(f^l(x^u)) \subset E^c_f(f^l(x^u))$ such that $Df(\mathcal{C}^u_\theta(f^l(x^u))) \subset \mathcal{C}^u_\theta(f^{l+1}(x^u))$ for every $l \geq 0$.

(2) $\left\| Df^n_{f^l(x^u)}(v) \right\| \geq C\lambda_0^n \|v\|$ for every $v \in \mathcal{C}^u_\theta(f^l(x^u))$ and every $l, n \geq 0$.

Analogously, we define it for the stable foliation by considering f^{-1}.

Definition 2.5.2. Given f a partially hyperbolic diffeomorphism, it has (d_1, d_2) SH-Saddle property if the following conditions hold:

(1) $\mathcal{W}^{ss}$ has the SH-Saddle property of index d_1.

(2) $\mathcal{W}^{uu}$ has the SH-Saddle property of index d_2.

Definition 2.5.3. Given a partially hyperbolic flow $\varphi_t : M \to M$, we say that the strong unstable foliation $\mathcal{W}^{uu}$ has SH-Saddle property of index $d \leq c$ if there exist $T \in \mathbb{R}$ such that the induced partially hyperbolic diffeomorphism $f := \varphi_T$ has strong unstable foliation $\mathcal{W}^{uu}$ with SH-Saddle property of index d. Analogously, for the strong stable foliation.

It is possible to prove that this property is open in the C^1-topology (check [Piñ23] and [PS06] for more information).

Before proceeding, let us fix some notation to simplify our development. Fix $\varepsilon > 0$ such that our construction works for a closed geodesic, and consider that we have done the construction for $\frac{\varepsilon}{2}$. Remember that we have defined the ε-tubular-neighborhood by $B(\gamma, \varepsilon)$. It means that the dynamics outside $V_{\frac{\varepsilon}{2}} := \pi^{-1}\left(B\left(\gamma, \frac{\varepsilon}{2}\right)\right)$ is hyperbolic. We will use the notation $V_{\varepsilon,\theta}$ for the subset of V_ε of θ-parallel vectors. Remember that for the initial geodesic flow g_t, the splitting is of the form $E^{ss} \oplus E^{ws} \oplus \mathbb{R}G(\theta) \oplus E^{wu} \oplus E^{uu}$. Call $E^s = E^{ss} \oplus E^{ws}$ and $E^u = E^{uu} \oplus E^{wu}$ and denote by $\mathcal{W}^s$ and $\mathcal{W}^u$ the stable and unstable foliations tangent to these bundles, respectively.

Proposition 2.5.1. *For every point $\theta \in T^1M$, there exist points $\theta^u \in \tilde{\mathcal{W}}^{uu}_1(\theta)$ and $\theta^s \in \tilde{\mathcal{W}}^{ss}_1(\theta)$ such that $\{\tilde{g}_t(\theta^u)\}_{t\geq 0}$ and $\{\tilde{g}_t(\theta^s)\}_{t\leq 0}$ never intersects $V_{\varepsilon,\theta}$.*

Proof. For each $\theta \in T^1M$ we have two possibilities: either $\theta \in V_\varepsilon$ or not. Suppose first that $\theta \in V_\varepsilon^c$ and denote by $\mathcal{W}^{uu}_\beta(\theta)$ its local strong unstable manifold with $\beta = \sup_\delta\{\delta \geq 0 : \mathcal{W}^{uu}_\delta(\theta) \cap V_\varepsilon = \emptyset\}$. The set V_ε is "ε-thin" in the direction transversal to the vertical fibers of $\pi : T^1M \to M$, and the strong foliations are never tangent to the vertical fibers. So, if β is bigger than 2ε and the orbit of θ enters the region V_ε sometime in the future, say $\tilde{g}_T(\theta)$, then we always have a disk D inside $\mathcal{W}^{uu}(\tilde{g}_T(\theta)) \cap \tilde{g}_T(\mathcal{W}^{uu}(\theta))$ of diameter bigger or equal to ε outside V_ε. The points in this disk can not enter the region for some short time, because they are growing apart from $\tilde{g}_T(\theta)$ exponentially fast, so they spend enough time outside to grow the diameter of $\tilde{g}_t(D)$ up to 2ε. So we have found a small disk D inside $\mathcal{W}^{uu}_\beta(\theta)$ and outside V_ε such that its orbit does not enter V_ε until it has a diameter bigger or equal to 2ε. By repeating the argument inductively we find a sequence of compact disks $\cdots \subset D_n \subset D_{n-1} \subset \cdots \subset D_0 = \mathcal{W}^{uu}_\beta(\theta)$ and an increasing sequence of real numbers converging to infinity $(t_n)_n$ such that $\varphi_t(D_n) \cap V_\varepsilon = \emptyset$ for all $t \leq t_n$. We conclude that there exists $\theta^u \in \cap_n D_n$ which never enters the region V_ε for the future iterates. If $\beta \leq 2\varepsilon$ and the orbit of θ spend enough time outside V_ε to have $\beta_T = \sup_\delta\{\delta \geq 0 : \mathcal{W}^{uu}_\delta(\varphi_T(\theta)) \cap V_\varepsilon = \emptyset\} \geq 2\varepsilon$ we apply the previous argument. Otherwise, the situation in which the orbit of θ enters the region in a shorter time happens when the underlying geodesic of $\tilde{g}_t(\theta)$ is a transversal one. In this case, it will stay in the region for a very short time and then it will spend a longer time outside, which is enough to make $\beta_T \geq 2\varepsilon$, so we repeat the argument again. The argument for the stable foliations is analogous. ■

Corollary 2.5.1. *$\tilde{g}_t$ satisfies the SH-property.*

Proof. Given $\theta \in T^1M$, then θ^u given by the previous Proposition has the desired property since it is hyperbolic. ■

Now, let us turn our attention towards some possible generalizations of our study. We have seen that geodesic flows are particular examples of a more general class called *Contact flows*, that is, the flow generated by the Reeb Vector field of some contact form. The results presented here raise several questions in this general setting:

Question 2.8. *What kind of dynamical properties partially hyperbolic contact flows can present?*

Question 2.9. *What kind of dynamical properties are general in this setting?*

Question 2.10. *What are other constructions of examples of partially hyperbolic contact flows?*

From the best of our knowledge, the only known examples are given by our results and by [CP14]. Indeed, in [CP14] the author prove the following result

Theorem 2.5.4. (Theorem 7.4 in [CP14]) *If the geodesic flow is partially hyperbolic, then the compact manifold M has even dimension.*

Remember that if $\dim M = n$, then $\dim T^1M = 2n-1$. Since for surfaces $\dim T^1S = 3$, then the previous Theorem implies that partially hyperbolic geodesic flows acting on T^1M may exist only for $\dim T^1M \geq 7$. Therefore, the following question is open in the general setting of contact flows:

Question 2.11. *Is there any example of a contact partially hyperbolic flow acting on a manifold of dimension 5?*

We can give a partial answer to this question in the non-compact case via the product of manifolds. To that end, let $M = M_1 \times M_2 \times \cdots \times M_k$ be a product manifold and denote by $\pi_i : M \to M_i$ the smooth projection. The following elementary result holds:

Lemma 2.5.1. *If M_i is an orientable manifold for every $i = 1, ..., k$, with orientation form ω_i, then M is an orientable manifold with orientation form $\omega = \pi_1^*(\omega_1) \wedge \cdots \wedge \pi_k^*(\omega_k)$, where π_i^* denotes the pullback by π_i.*

The construction goes as follow: let $\mathcal{S} := \mathbb{S}^2 \setminus \{N\}$ denote the 2-sphere minus the north pole in $\mathbb{R}^3$. It is known that $\mathcal{S}$ is orientable with orientation form in spherical coordinates (θ, ϕ) given by

$$\omega_{\mathcal{S}} = \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi.$$

Let (N, α) be a contact manifold with contact form α and suppose that its Reeb flow is an Anosov flow with splitting $TN = E^s \oplus \mathbb{R}R_\alpha \oplus E^u$ (e.g. let N be the unit tangent bundle of a negatively curved Riemannian manifold with canonical contact form).

Consider the product manifold $M = N \times \mathcal{S}$, with the product Riemannian metric, and define the 1-form in M by

$$\beta = \pi_1^*(\alpha) + \pi_2^*(-\cos\theta \mathrm{d}\phi).$$

Notice that $\mathrm{d}(-\cos\theta \mathrm{d}\phi) = \omega_{\mathcal{S}}$.

Lemma 2.5.2. *β is a contact form for M.*

Proof. Suppose $\dim N = 2n+1$, thus $\dim M = 2n+3$. Therefore we want to check that $\beta \wedge (\mathrm{d}\beta)^{n+1} \neq 0$. To see that, first notice that by the properties of pullback, we have

$$\mathrm{d}\beta = \pi_1^*(\mathrm{d}\alpha) + \pi_2^*(\sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi) = \pi_1^*(\mathrm{d}\alpha) + \pi_2^*(\omega_{\mathcal{S}}).$$

Since $\pi_1^*(\mathrm{d}\alpha)$ and $\pi_2^*(\omega_{\mathcal{S}})$ are 2-forms of M, then $\pi_1^*(\mathrm{d}\alpha) \wedge \pi_2^*(\omega_{\mathcal{S}}) = \pi_2^*(\omega_{\mathcal{S}}) \wedge \pi_1^*(\mathrm{d}\alpha)$. Moreover, $\dim N = 2n+1$ and $\dim \mathcal{S} = 2$ imply that $(\mathrm{d}\alpha)^{n+1} = 0$ and $\omega_{\mathcal{S}}^2 = 0$. We get

$$\begin{aligned}
(\mathrm{d}\beta)^{n+1} &= \sum_{k=0}^{n+1} \binom{n}{k} \pi_1^*(\mathrm{d}\alpha)^{n+1-k} \wedge \pi_2^*(\omega_{\mathcal{S}})^k \\
&= \pi_1^*(\mathrm{d}\alpha)^{n+1} + (n+1)\pi_1^*(\mathrm{d}\alpha)^n \wedge \pi_2^*(\omega_{\mathcal{S}}) \\
&= (n+1)\pi_1^*(\mathrm{d}\alpha)^n \wedge \pi_2^*(\omega_{\mathcal{S}})
\end{aligned}$$

Finally,

$$\beta \wedge (\mathrm{d}\beta)^{n+1} = (n+1)\pi_1^*(\alpha \wedge (\mathrm{d}\alpha)^n) \wedge \pi_2^*(\omega_{\mathcal{S}}).$$

By the previous Lemma, we get $\beta \wedge (\mathrm{d}\beta)^{n+1} \neq 0$. Therefore, β is a contact form for M. ∎

Lemma 2.5.3. $\ker\beta = (E^s \oplus E^u, 0) \oplus \mathbb{R}(0, \partial_\theta) \oplus \mathbb{R}\left(X_\alpha, \frac{1}{\cos\theta}\partial_\phi\right)$.

Proof. We know that $\dim\ker\beta = 2n+2$, because it is a contact form. It is clear that the space on the right-hand side also has a dimension equal to $2n+2$. We are left to prove that it is indeed contained in $\ker\beta$.

- $\beta(E^s \oplus E^u, 0) = \alpha(E^s \oplus E^u) = 0$.
- $\beta(0, \partial_\theta) = -\cos\theta \mathrm{d}\phi(\partial_\theta) = 0$.
- $\beta\left(X_\alpha, \frac{1}{\cos\theta}\partial_\phi\right) = \alpha(X_\alpha) - \cos\theta \mathrm{d}\phi\left(\frac{1}{\cos\theta}\partial_\phi\right) = 1 - 1 = 0$

∎

Lemma 2.5.4. *The Reeb vector field of* β *is given by* $R_\beta(p,q) = (R_\alpha(p), 0) \in T_pN \times T_q\mathcal{S}$*, where* R_α *is the Reeb vector field of* α*.*

Proof. We only need to check that $\beta(R_\beta) = 1$ and $\mathrm{d}\beta(R_\beta, \cdot) = 0$.

$$\begin{aligned}
\beta(R_\beta) =& \pi_1^*(\alpha)(R_\beta) + \pi_2^*(-\cos\theta\mathrm{d}\phi)(R_\beta) \\
&= \alpha(D\pi_1 R_\beta) - \cos\theta\mathrm{d}\phi(D\pi_2 R_\beta) \\
&= \alpha(R_\alpha) - \cos\theta\mathrm{d}\phi(0) \\
&= 1.
\end{aligned}$$

Moreover,

$$\begin{aligned} \mathrm{d}\beta(R_\beta,\cdot) &= \pi_1^*(\mathrm{d}\alpha)(R_\beta,\cdot) + \pi_2^*(\omega_{\mathcal{S}})(R_\beta,\cdot) \\ &= (\mathrm{d}\alpha)(D\pi_1 R_\beta,\cdot) + \omega_{\mathcal{S}}(D\pi_2 R_\beta,\cdot) \\ &= (\mathrm{d}\alpha)(R_\alpha,\cdot) + \omega_{\mathcal{S}}(0,\cdot) \\ &= 0. \end{aligned}$$

■

Lemma 2.5.5. *The Reeb flow is partially hyperbolic for the product Riemannian metric.*

Proof. Let φ_t^α be the contact Anosov flow generated by R_α and φ_t^β be the flow generated by R_β. It is not difficult to see that $\varphi_t^\beta(p,q) = (\varphi_t^\alpha(p), q)$, hence

$$D\varphi_t^\beta = \begin{bmatrix} D\varphi_t^\alpha & 0 \\ 0 & Id \end{bmatrix}$$

Therefore, there exists a $D\varphi_t$-invariant splitting $TM = (E^s \oplus E^u, 0) \oplus \mathbb{R}R_\beta \oplus E^c$, where E^s and E^u are the subbundles of TN given by the Anosov splitting for φ_t^α and $E^c = \mathbb{R}(0,\partial_\theta) \oplus \mathbb{R}\left(X_\alpha, \frac{1}{\cos\theta}\partial_\phi\right)$. Since $D\varphi_t^\alpha$ acts by isometry on the last space. We conclude that φ_t^β is a partially hyperbolic contact flow. ■

Consequently, we can construct examples of partially hyperbolic contact flows in dimension 5 by taking N to be the unit tangent bundle of a surface of negative curvature with a canonical contact structure given by the Riemannian metric. We can also produce several examples by iterating this construction. Starting with a partially hyperbolic contact flow, we can obtain:

Corollary 2.5.2. *For any $n, k \in \mathbb{N}$, with $k < n-1$, there exists a partially hyperbolic contact flow in dimension $2n+1$ with $2k$-dimensional center acting on a non-compact manifold.*

Remark 2.5.1. We must highlight once more that this construction does not work in the compact case because of Stokes' theorem. Also, this type of construction would work by switching $\mathbb{S}^2 \setminus \{N\}$ by $\mathbb{R}^2$ and $\omega_{\mathcal{S}}$ by $\mathrm{d}x \wedge \mathrm{d}y$.

From the best of our knowledge, the only work that deals with dynamical properties of partially hyperbolic contact flows is [FH22], where the authors prove that a partially hyperbolic contact flow can be perturbed inside this class to obtain *Accessibility*. By *Accessibility* we mean that any two points can be joined by a concatenation (called *su-path*) of a finite number of paths tangent to either E^s or E^u. On the other hand, for the above construction we get:

Lemma 2.5.6. *The Reeb flow is not accessible.*

Proof. The flow is not accessible since every path tangent to E^s or E^u remains inside N, therefore, there is no way to go from a point (p, q) to a point (p, q^*) for $q \neq q^*$. ■

Remark 2.5.2. Notice that we can explicitly verify that the center bundle is not integrable: $[(0, \partial_\theta), (X_\alpha, \frac{1}{\cos\theta}\partial_\phi)] = ([0, X_\alpha], [\partial_\theta, \frac{1}{\cos\theta}\partial_\phi]) = (0, \frac{\sin\theta}{\cos\theta}\partial_\phi)$ which does not belong to E^c.

The property verified in the previous remark is general in the compact case: let $\varphi_t : M \to M$ be a C^r partially hyperbolic flow contact flow on a compact manifold, then we obtain:

Proposition 2.5.2. *If $TM = E^s \oplus \mathbb{R}X_\alpha \oplus E^c \oplus E^u$ is the partially hyperbolic splitting, then the distributions $E^s \oplus E^u$ and E^c are not integrable at any point.*

Proof. First, notice that $\ker\alpha = E^s \oplus E^c \oplus E^u$ since $\dim\ker\alpha = 2n$ and $TM = E^s \oplus \mathbb{R}R_\alpha \oplus E^c \oplus E^u$, with R_α the Reeb vector field. Now, since R_α is the Reeb vector field, then its flow φ_t preserves α. Hence it also preserves $\mathrm{d}\alpha$ and $\alpha \wedge (\mathrm{d}\alpha)^n$.

Since E^s and E^u are integrable, then $\mathrm{d}\alpha(X^\tau, Y^\tau) = 0$, where $X^\tau, Y^\tau \in E^\tau$ and $\tau = s, u$:

$$\mathrm{d}\alpha(X^\tau, Y^\tau) = X^\tau(\alpha(Y^\tau)) - Y^\tau(\alpha(X^\tau)) - \alpha([X^\tau, Y^\tau]) = -\alpha([X^\tau, Y^\tau]) = 0$$

Since M is compact, then $\mathrm{d}\alpha$ is bounded, i.e. there exist a constant $C > 0$ such that $|d\alpha(X, Y)| \leq C \|X\| \|Y\|$ for any vectors $X, Y \in TM$. Then, we have for $X^s \in E^s$ and $X^c \in E^c$

$$|\mathrm{d}\alpha(X^s, X^c)| = |\mathrm{d}\alpha(D\varphi_t X^s, D\varphi_t X^c)| \leq C \|D\varphi_t X^s\| \|D\varphi_t X^c\| \xrightarrow{t\to\infty} 0,$$

because the action of $D\varphi_t$ on E^s dominates the action of $D\varphi_t$ on E^c. Analogously, $\mathrm{d}\alpha(X^u, X^c) = 0$.

Now, let $\{R_\alpha, X_1^c, ..., X_{2m}^c, X_1^s, ..., X_r^s, X_1^u, ..., X_r^u\}$ (check Lemma 2.5.7) be a local frame for TM defined on an open set U such that $X_i^\tau \in E^\tau$, for $\tau = c, s, u$. Since α is a contact form, then $\alpha \wedge (\mathrm{d}\alpha)^n$ is a volume form. On one side we have by the definition of wedge product and using that $\ker \alpha = E^s \oplus E^c \oplus E^u$

$$\alpha \wedge (\mathrm{d}\alpha)^n(X, X_1^c, ..., X_{2m}^c, X_1^s, ..., X_r^s, X_1^u, ..., X_r^u)$$
$$= \alpha(X)(\mathrm{d}\alpha)^n(X_1^c, ..., X_{2m}, X_1^s, ..., X_r^s, X_1^u, ..., X_r^u) \neq 0$$

On the other side, $(\mathrm{d}\alpha)^n(X_1^c, ..., X_{2m}, X_1^s, ..., X_r^s, X_1^u, ..., X_r^u)$ is composed by sums of n products of terms of the forms $\mathrm{d}\alpha(X_i^c, X_j^c)$ and $\mathrm{d}\alpha(X_i^s, X_j^u)$, once $\mathrm{d}\alpha(X_i^c, X_j^{s,u}) = 0$ for all i, j and E^s and E^u are integrable. Notice that we cannot have $\mathrm{d}\alpha(X_i^c, X_j^c) = 0$ for all i, j (for instance, when E^c is integrable) because this would contradict the inequality above. Then for at least one pair (i, j) we must have $\mathrm{d}\alpha(X_i^c, X_j^c) \neq 0$. Then, if $E^s \oplus E^u$ were jointly integrable we would get $\mathrm{d}\alpha(X_i^s, X_j^u) = -\alpha([X_i^s, X_j^u]) = 0$, thus

$$(\mathrm{d}\alpha)^n(X_1^c, ..., X_{2m}, X_1^s, ..., X_r^s, X_1^u, ..., X_r^u) = 0$$

which is a contradiction. We conclude that $E^s \oplus E^u$ is not jointly integrable, nor is E^c. ■

This property is interesting since the most standard obstruction to accessibility is joint integrability of $E^s \oplus E^u$. It is also known that if $\dim E^c = 1$, then if $E^s \oplus E^u$ is not jointly integrable at some point p, then its accessibility class (the set of points *su*-attainable from p) is open. Therefore, if $E^s \oplus E^u$ is not jointly integrable at any point, $\dim E^c = 1$ and the manifold is connected, then accessibility holds. On the other hand, in the case of partially hyperbolic contact flows with splitting $TM = E^s \oplus E^c \oplus \mathbb{R}R_\alpha \oplus E^u$ we have by the above computations

Lemma 2.5.7. $\dim E^c$ *is even and* $\dim E^s = \dim E^u$

Proof. Since α is a contact form, then $\mathrm{d}\alpha|_{\ker \alpha}$ is non-degenerate, that is given $X \in \ker \alpha$ there must exist $X' \in \ker \alpha$ such that

$$\mathrm{d}\alpha(X, X') \neq 0.$$

By the previous computation, it must happen that if $X \in E^c$, then $X' \in E^c$ and if $X \in E^s$, then $X' \in E^u$. ■

There are still several interesting open questions about partially hyperbolic contact flows. Since contact flows are naturally volume-preserving, accessibility would imply transitivity by the classical Brin's argument. What is known so far by [FH22] is that topological transitivity is dense among these systems.

Chapter 3

Ergodic Homoclinic Classes for Flows

The present chapter corresponds to the results obtained in the preprint [dJEP25], which is a work in collaboration with Marcielis Espitia (UCC - Colombia) and Gabriel Ponce (UNICAMP - Brazil), about the study of homoclinic classes for flows and its relationship to ergodicity and SRB measures. As mentioned in the introduction, this work is a flow version of the results obtained by Federico Rodriguez-Hertz, Jana Rodriguez-Hertz, Raul Ures and Ali Tahzibi in [RHRHTU11a] and [RHRHTU11b]. An important class of systems given by flows that we intend to obtain a better understanding with the techniques developed here is the class of geodesic flows. We have shown in the previous chapter that determining which conditions may imply or prevent the ergodicity of the Liouville measure for geodesic flows is far from been a completely solved problem. Therefore, we believe that our results can be a starting point of new directions to be explored.

As mentioned in the introduction, the strategy of the proofs of Theorems C, D and E follows the line of Hopf's Argument. This is the standard argument to obtain ergodicity (other than case-by-case studies) to obtain ergodicity of systems that present many forms of hyperbolicity. The argument was first developed by Eberhard Hopf in [Hop39] and [Hop40] to obtain ergodicity of the geodesic flow for metrics with constant negative curvature and for negatively curved surfaces; later this argument was extend to more general settings by Anosov, Sinai and Pesin (see [Ano67], [AS67], [Pes77a]). The main difference between our setting and the one considered by Hopf, Anosov and Sinai is that some important sets (which we will make precise) are not uniformly distributed along

Pesin's manifolds. Instead we will see that they are well distributed from the measure point of view, but, in general, they are not fully $su-$saturated.

To make precise our statements, let us recall some important notions such as Birkhoff average: for each function $f \in L^1_m(M;\mathbb{R})$ the Birkhoff Ergodic Theorem assures that $m-$almost everywhere the following limits are well-defined

$$f^{\pm}(x) = \lim_{T\to\pm\infty} \frac{1}{T}\int_0^T f(\varphi_t(x))dt.$$

We also have that $f^+(x) = f^-(x)$ for $m-$almost every $x \in M$ and $f^{\pm}$ are φ_t-invariant. The functions $f^{\pm}$ are called *The Birkhoff's average.* Remember that ergodicity is equivalent to f^+ being constant almost everywhere and the key point of the technique developed by Hopf is the use of transversality and absolute continuity of stable and unstable laminations to show that the image of two points by f^+ coincide (mod 0).

In the next section, we will see a description of the behavior of Birkhoff averages along Pesin's manifolds. Essentially, we will see that almost every point in the same (un)stable Pesin's manifold has the same Birkhoff average, in which case we say that the subset of Pesin's manifolds with constant Birkhoff average is well distributed along such manifolds from the measure theoretical point of view.

Remember that, given a hyperbolic closed orbit γ of a C^2-flow, we have defined its *stable and unstable homoclinic classes* as

$$\Lambda^s(\gamma) = \{x \in M : x \text{ is a regular point and } W^s(x) \pitchfork W^u(\gamma) \neq \emptyset\},$$

and

$$\Lambda^u(\gamma) = \{x \in M : x \text{ is a regular point and } W^u(x) \pitchfork W^s(\gamma) \neq \emptyset\}.$$

Therefore, we have defined the *Ergodic homoclinic classes* as

$$\Lambda(\gamma) := \Lambda^s(\gamma) \cap \Lambda^u(\gamma).$$

Remark 3.0.1. For the purpose of this chapter m will always denote a smooth measure. We will also denote by $m^{u,s}_x$ its conditional measures with respect to the partitions $W^{u,s}_{loc}(x)$. As stated in Chapter 1, these conditional measures are absolutely continuous with respect to the Riemannian measures on each $W^{u,s}_{loc}(x)$.

3.1 Typical points for smooth and SRB measures

The results of this section are completely analogous to those obtained in Section 4 of [RHRHTU11a] for diffeomorphisms; few changes are needed to adapt to the flow setting, but we are going to present the proofs for completeness. We remark that essentially no extra difficults are found here nor new techniques are needed.

Lemma 3.1.1 (Typical points for continuous functions)**.** *There exists a φ_t-invariant set M_0, with $m(M_0) = 1$, such that for any $f \in C^0(M)$ we have: if $x \in M_0$, then $f^+(x) = f^+(w)$, for any $w \in W^s(x)$ and m^u_x-almost every $w \in W^u(x)$.*

Proof. The conclusion for the stable manifold is a consequence of continuity. Let $w \in W^s(x)$, in particular $d(\varphi_t(x), \varphi_t(y)) \to 0$ as $t \to \infty$. Let us fix any $f \in C^0(M)$, then compactness of M implies that f is uniformly continuous. Therefore, let $\varepsilon > 0$ and chose $\delta > 0$ such that for all $x, y \in M$, with $d(x, y) < \delta$, it holds that $|f(x) - f(y)| < \frac{\varepsilon}{2}$. Now, let $T_0 > 0$ be such that $d(\varphi_t(x), \varphi_t(w)) < \delta$, for $t \geq T_0$. We get,

$$\begin{aligned}\left|\frac{1}{T}\int_0^T (f(\varphi_t(x)) - f(\varphi_t(w))\mathrm{d}t\right| &\leq \frac{1}{T}\int_0^{T_0} |f(\varphi_t(x)) - f(\varphi_t(w))|\mathrm{d}t \\ &+ \frac{1}{T}\int_{T_0}^{T} |f(\varphi_t(x)) - f(\varphi_t(w))|\mathrm{d}t \\ &\leq \frac{I}{T} + \frac{(T-T_0)}{T}\frac{\varepsilon}{2},\end{aligned}$$

where

$$I = \int_0^{T_0} |f(\varphi_t(x)) - f(\varphi_t(w))|\mathrm{d}t < \infty$$

It implies that for $T > 0$ sufficiently large, we get

$$\left|\frac{1}{T}\int_0^T (f(\varphi_t(x)) - f(\varphi_t(w))\mathrm{d}t\right| < \varepsilon.$$

Therefore, we conclude that $f^+(x) = f^+(w)$, for every $f \in C^0(M)$.

Now, the conclusion for the unstable manifold is a little bit more delicate. We proceed as follows: define the following full-measured set

$$S_0 = \{x \in M : f^+(x) = f^-(x) \text{ are well-defined, for all } f \in C^0(M)\}.$$

We will prove that $m-$almost every point x of S_0 is such that m^u_x-almost every $\xi \in W^u_{loc}(x)$ also lies inside S_0. In fact, suppose this is not the case. Thus there exists a subset $A \subset S_0$, with $m(A) > 0$, such that for any $x \in A$ there exists a set $B_x \subset W^u_{loc}(x) \setminus S_0$, with

$m^u_x(B_x) > 0$. Without losing generality, we can switch A by $A \cap \mathcal{R}^l$, where $\mathcal{R}^l$ is a Pesin block such that this intersection has positive measure. Let $y \in A$ be a Lesbegue density point (in fact we only need it to be in supp(m)), that is

$$\lim_{\varepsilon \to 0} \frac{m(A \cap B_\varepsilon(y))}{m(B_\varepsilon(y))} = 1$$

So, for $\varepsilon > 0$ small enough we can consider a small disk T inside $A \cap B_\varepsilon(y)$ that is transverse to its points unstable manifolds and the set

$$B = \bigcup_{x \in T} \tilde{B}_x,$$

where

$$\tilde{B}_x = \{w \in B_x : w \in B_\varepsilon(y)\}$$

satisfies

$$m(B) = \int_T m^u_x(\tilde{B}_x) dm_T(x) > 0.$$

This is a contradiction because B is a positive measure set outside S_0.
Define the full measure set

$$S_1 = \{x \in S_0 : m^u_x - \text{almost every } \xi \in W^u_{loc}(x) \text{ lies inside } S_0\}.$$

If $\xi \in W^u_{loc}(x)$, then for any $x \in M$, we have that $f^+(\xi) = f^+(x)$. In addition if $x \in S_0$ and $\xi \in W^u_{loc}(x) \cap S_0$, then by continuity $f^+(\xi) = f^-(\xi) = f^-(x) = f^+(x)$. We conclude that S_1 consist of points in S_0 such that m^u_x-almost every $\xi \in W^u_{loc}(x)$ satisfies $f^+(\xi) = f^+(x)$. Since f is continuous, then f^+ is constant on $W^s(x)$. Thus, by the invariance of f^+ we can find a full measure set M_0 with the desired property. ■

Remark 3.1.1. Notice that by continuity the set M_0 is $s-$saturated.

Lemma 3.1.2 (Typical set for L^1 functions)**.** *For any $f \in L^1(M;\mathbb{R})$ there exists a φ_t-invariant set $\mathcal{T}_f$, with $m(\mathcal{T}_f) = 1$, that satisfies: for all $x \in \mathcal{T}_f$ we have $f^+(z) = f^+(x)$ for m^s_x-almost every $z \in W^s(x)$ and m^u_x-almost every $z \in W^u(x)$.*

Proof. Let $f \in L^1(M;\mathbb{R})$ and consider a sequence $(f_n)_n \subset C^0(M)$ that converges to f in the L^1-norm. Then, we have that $(f^+_n)_n$ converges, in the L^1-norm, to f^+. It implies that there is a subsequence, say $(f^+_{n_k})_k$, that converges almost everywhere to f^+. Set

$$\mathcal{T}_0 = \{x \in M : f^+_{n_k}(x) \to f^+(x)\}$$

and define $\mathcal{T}_f := \mathcal{T}_0 \cap M_0$, where M_0 is the set of typical points given by the previous lemma. Of course $m(\mathcal{T}_f) = 1$. If $x \in \mathcal{T}_f$, then $f^+_{n_k}(x) \to f^+(x)$ and, by the previous Lemma, we also have $f^+_{n_k}(w) = f^+_{n_k}(x)$, for all $w \in W^s(x)$ and m^u_x-almost every $w \in W^u(x)$. Since m^s_x-almost every $w \in W^s(x)$ lies inside $\mathcal{T}_0$, we see that $f^+_{n_k}(w) = f^+_{n_k}(x)$ converges both to $f^+(w)$ and $f^+(x)$. The result follows. ■

If μ is an SRB measure, then its conditional measures are absolutely continuous along stable and unstable manifolds. Hence, the following two Lemmas can be proved analogously by using a point in the support of μ instead of using a Lesbegue density point.

Lemma 3.1.3. *Let μ be a SRB-measure, then there exists a φ_t-invariant set M_0, with $\mu(M_0) = 1$, such that for any $f \in C^0(M)$ we have: if $x \in M_0$, then $f^+(x) = f^+(w)$, for any $w \in W^s(x)$ and m^u_x-almost every $w \in W^u(x)$.*

Lemma 3.1.4. *For any $f \in L^1(M;\mathbb{R})$ there exists a φ_t-invariant set $\mathcal{T}_f$, with $\mu(\mathcal{T}_f) = 1$, that satisfies: for all $x \in \mathcal{T}_f$ we have $f^+(z) = f^+(x)$ for m^s_x-almost every $z \in W^s(x)$ and m^u_x-almost every $z \in W^u(x)$.*

3.2 Proof of Theorem C

We are going to split the proof of Theorem C into two parts: first we prove that $\varphi|_{\Lambda(\gamma)}$ is an ergodic flow. The proof of ergodicity is simpler and illustrates well the idea to prove the first statement $\Lambda^u(\gamma) = \Lambda^s(\gamma) \mod 0$.

The general idea is the following: for the second statement, we need to prove that Birkhoff averages of continuous functions are almost everywhere constant on the ergodic homoclinic classes. To this end, given two points x and y homoclinically related to γ, we are going to use this relation to transfer the set of typical points from $W^u(y)$ and $W^u(x)$ to $W^u(p)$, with $p \in \gamma$, via stable holonomy. The idea behind the first statement is similar, but slightly more delicate since we will need to deal with Birkhoff averages of measurable functions.

3.2.1 Proof of the second statement

For the sake of simplicity, let us fix some reference point $p \in \gamma$. We are going to consider the case where γ is not a singular orbit, otherwise the proof is exactly the same as for diffeomorphism. The difficulty for the flow setting is precisely considering the flow direction.

Define $\Delta = \Lambda(\gamma) \cap M_0$, where M_0 is given by Lemma 3.1.1 and $\mathcal{R}$ is the set of regular points. Let $f \in C^0(M)$ and let us prove that f^+ is constant on Δ. For each Pesin block $\mathcal{R}^{i,j}_{\varepsilon,l}$ intersecting Δ in a positive measure set, denote by $\Delta^{i,j}_{\varepsilon,l} = \mathcal{R}^{i,j}_{\varepsilon,l} \cap \Delta$. For points inside a fixed Pesin Block the local stable and unstable Pesin manifolds have diameter uniformly bounded away from 0, so let us by $\delta > 0$ this uniform lower bound. By the Poincaré Recurrence Theorem almost every points $x, y \in \Delta^{i,j}_{\varepsilon,l}$ return infinitely many times to $\Delta^{i,j}_{\varepsilon,l}$, i.e. there exist sequences $(t_k)_k$ and $(s_l)_l$ such that

(1) $t_k \to -\infty$, $x_k = \varphi_{t_k}(x) \in \Delta^{i,j}_{\varepsilon,l}$.

(2) $s_l \to \infty$, $y_l = \varphi_{s_l}(y) \in \Delta^{i,j}_{\varepsilon,l}$.

For all $q \in M_0$ we can define the m^u_q-full measure sets

$$A_q = \{\xi \in W^u(q) : f^+(\xi) = f^+(q)\} \subset M_0$$

Since $W^s(y_l) \pitchfork W^u(\gamma)$, we can find $l >> 1$ big enough such that $d(y_l, W^u(\gamma)) < \frac{\delta}{4}$. Because $y_l \in \Lambda(\gamma)$, then y_l is a hyperbolic point with dimensions of stable and unstable equal to those of $p \in \gamma$ and $W^{ws}(y_l) \pitchfork W^u(\varphi_T(p))$ (in a single point), where $\varphi_T(p)$ is the point in γ for which we have $W^s(y_l) \cap W^u(\varphi_T(p))$. Therefore, since y_l is in a Pesin Block, we can define the weak-stable holonomy in a positive $m^u_{y_l}-$measure set which intersects $A_{y_l} \cap W^u_{loc}(y_l)$ is a positive $m^u_{y_l}-$measure set. Since f^+ is constant along weak-stable manifolds and by absolute continuity, then the image of this intersection by the weak-stable holonomy is a set of $m_{\varphi_T(p)}-$positive measure with the same Birkhoff average as y_l. Call this set $B_{\varphi_T(p)}$ and notice that by $s-$saturation and invariance, we get $B_{\varphi_T(p)} \subset M_0$. Proceeding analogously with x_k we produce a set $B_{\varphi_{T'}(p)} \subset M_0 \cap W^u(\varphi_{T'}(p))$ with $m^u_{\varphi_{T'}(p)}-$positive measure and same Birkhoff average as x_k.
Now, since γ is periodic we take $\hat{T} > 0$ such that $\varphi_{T+\hat{T}}(p) = \varphi_{T'}(p)$, then $\varphi_{\hat{T}}(B_{\varphi_T(p)})$ is a set with same Birkhoff average as y_l of $m^u_{\varphi_{T'}(p)}-$positive measure. Finally, $B_{\varphi_{T'}(p)} \subset M_0$ and then there exist points in $\varphi_{\hat{T}}(B_{\varphi_T(p)})$ with the same Birkhoff average as x_k. We conclude that $f^+(x) = f^+(x_k) = f^+(y_l) = f^+(y)$. Thus the restriction of f^+ to Δ is constant.

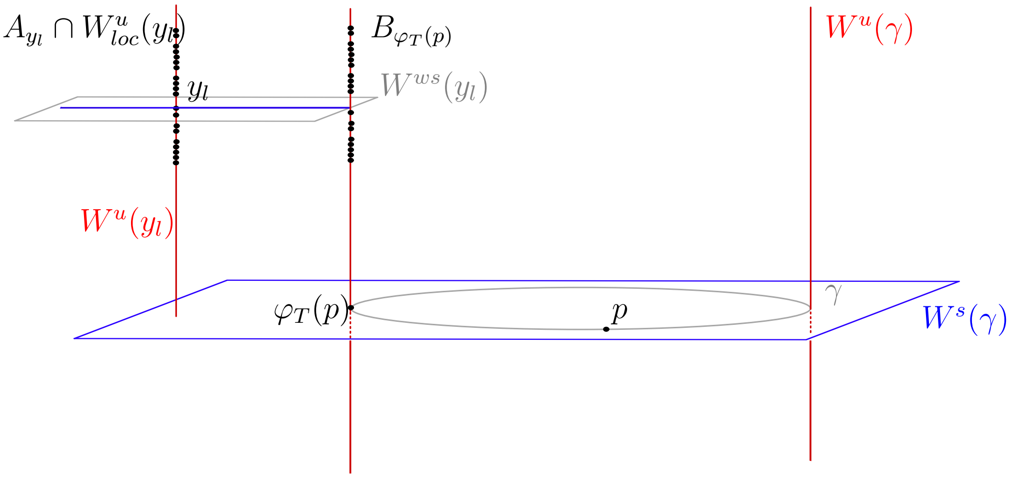


Figure 3.1: Construction of the set $B_{\varphi_T(p)}$

3.2.2 Proof of the first Statement

Consider the integrable function $f = 1_{\Lambda^s(\gamma)}$ and the full-measure set of typical points $\mathcal{T}_f$ given by Lemma 3.1.2. Define the φ_t-invariant set $\Delta^u = \Lambda^u(\gamma) \cap \mathcal{T}_f$. We will see that $\Delta^u \subset \Lambda^s(\gamma)$.

Let $x \in \Delta^u$, then $W^u(x) \pitchfork W^s(\gamma) \neq \emptyset$. The strategy is to use an auxiliary point to construct a subset of $W^{wu}(x)$ intersecting $\Lambda^s(\gamma)$ with m^{wu}_x-positive measure.

Let $\mathcal{R}^{i,j}_{\varepsilon,l}$ be a Pesin Block for which the set $\Delta^s := \Lambda^s(\gamma) \cap \mathcal{T}_f \cap R^{i,j}_{\varepsilon,l}$ has positive measure and consider a point $y \in \Delta^s$ such that there is a sequence of times $(s_l)_l$ converging to infinity satisfying $y_l := \varphi_{s_l}(y) \in \Delta^s$ and all y_l are Lesbegue density points (indeed we only need them to be in $\operatorname{supp}(m)$). Choose a point z in $W^s(y) \pitchfork W^u(\gamma)$ and consider $z_l := \varphi_{s_l}(z) \in W^s(y) \pitchfork W^u(\gamma)$. Inside $R^{i,j}_{\varepsilon,l}$ the Pesin's manifolds admit an uniform lower bound for the diameter, say $\operatorname{diam}(W^s_{loc}(q)) \geq \delta > 0$, for all $q \in R^{i,j}_{\varepsilon,l}$. By definition we have that $d(z_l, y_l) \to 0$, as $l \to \infty$ so we can choose $z_0 = \varphi_{s_{l_0}}(z)$ such that $d(z_0, y_0) < \frac{\delta}{2}$, where $y_0 = \varphi_{s_{l_0}}(y)$.

Since y_0 is a Lesbegue density point of Δ^s, we can find a small ball B centered in y_0 satisfying the condition $m(\Delta^s \cap B) > 0$ and we can assume its diameter is at least δ. Let $\mathcal{F}$ be a smooth foliation of B with dimension $n - \dim(W^s(y_0))$ such that each leaf $\mathcal{F}(\xi)$ is transverse to $W^s_{loc}(y_0)$. By the Fubini property

$$0 < m(\Delta^s \cap B) = \int_{W^s_{loc}(y_0)} m^{\mathcal{F}}_{\xi}(\mathcal{F}(\xi) \cap \Delta^s) dm^s_{y_0}(\xi),$$

we have that $m^{\mathcal{F}}_{\xi}(\mathcal{F}(\xi) \cap \Delta^s) > 0$ on a subset of $W^s_{loc}(y_0)$ of m^s_y-positive measure. Fix ξ_0 such that $m^{\mathcal{F}}_{\xi_0}(\mathcal{F}(\xi_0) \cap \Delta^s) > 0$.

We now prove that for some iterate of x we get a transversal intersection between $W^{wu}(\varphi_T(x))$ and $W^s_{loc}(y_0)$. To do that, we want to see that we can approximate $W^u(\gamma)$ by iterating $W^u(x)$. In fact, since $W^u(x) \pitchfork W^s(\gamma)$, we may divide the reasoning in two cases. First, if $\dim W^u(x) + \dim W^s(\gamma) = n$, we can call $\{w\} = W^u(x) \cap W^s(\gamma)$ and consider a small saturation along the flow direction

$$D := \bigcup_{t \in (-\varepsilon, \varepsilon)} W^u(\varphi_t(x))$$

Now, D is a small disk transverse to $W^s(q)$, for some $q \in \gamma$ and we can apply the $\lambda-$lemma for the diffeomorphism φ_T, where T is the period of γ. This implies that $\varphi_{nT}(D)$ converges in the C^1-topology to

$$D' = \bigcup_{t \in (-\varepsilon, \varepsilon)} W^u(q).$$

By taking big enough iterates we can make $\varphi_{nT}(D)$ close enough to D' such that after flowing by φ_t for some $0 \leq t \leq T$ it enters B and since $W^s_{loc}(y_0) \pitchfork W^u(\gamma)$ it must intersect $W^s_{loc}(y_0)$ transversally. The case where $\dim W^u(x) + \dim W^s(\gamma) > n$ is easier to deal with because we already have that $W^u(x) \pitchfork W^s(q)$ for some $q \in \gamma$, so we just apply the $\lambda-$lemma for φ_{nT} and $W^u(x)$ directly to get that $W^s_{loc}(y_0) \pitchfork W^u(\varphi_t(x))$. So, let us fix that $x_t := \varphi_t(x)$ satisfies $W^s_{loc}(y_0) \pitchfork W^{wu}(x_t)$ or $W^s_{loc}(y_0) \pitchfork W^u(x_t)$

To proceed we need to split each case into two more cases, but the analysis is very similar. First, if $\dim(W^s(y_0)) + \dim(W^{wu}(x_t)) = n$ (see Figure 3.2) we consider the holonomy map h between a subset of positive measure in $\mathcal{F}(\xi_0)$ and $W^{wu}(x_t)$, then h sends the positive measure set $\mathcal{F}(\xi_0) \cap \Delta^s$ into a set of $m^{wu}_{x_t}-$positive measure. Since $\Lambda^s(\gamma)$ is $s-$saturated, the last set lies inside $\Lambda^s(\gamma)$. The second case we need to consider occurs when $\dim(W^s(y_0)) + \dim(W^{wu}(x_t)) > n$. In this case, we can assume without losing any generality that we had choose the foliation $\mathcal{F}$ to be such that each leaf containing a point of $W^{wu}(x_t)$ is contained in $W^{wu}(x_t)$. Consider S to be an open submanifold inside $W^s(y_0) \cap W^{wu}(x_t)$. Integrating over S and using the holonomy maps from $\mathcal{F}(\xi_0)$ to $\mathcal{F}(q)$, for each $q \in S$, we construct a set A intersecting $\Lambda^s(\gamma)$ with $m^{wu}_{x_t}-$positive measure. In any of those cases, using that $\Lambda^s(\gamma)$ is φ_t-invariant, we found a set of $m^u_{x_t}-$positive measure inside $\Lambda^s(\gamma)$. It means that $W^u(x_t)$ contains a set of positive measure with Birkhoff average $f^+ \equiv 1$, but x_t is a typical point, so $f^+(x) = f^+(x_0) = 1$. We conclude that $x \in \Lambda^s(\gamma)$. The analysis for the case $W^s_{loc}(y_0) \pitchfork W^u(x_t)$ is exactly the same.

The proof of the reverse inclusion is analogous, so we conclude the proof of the theorem.

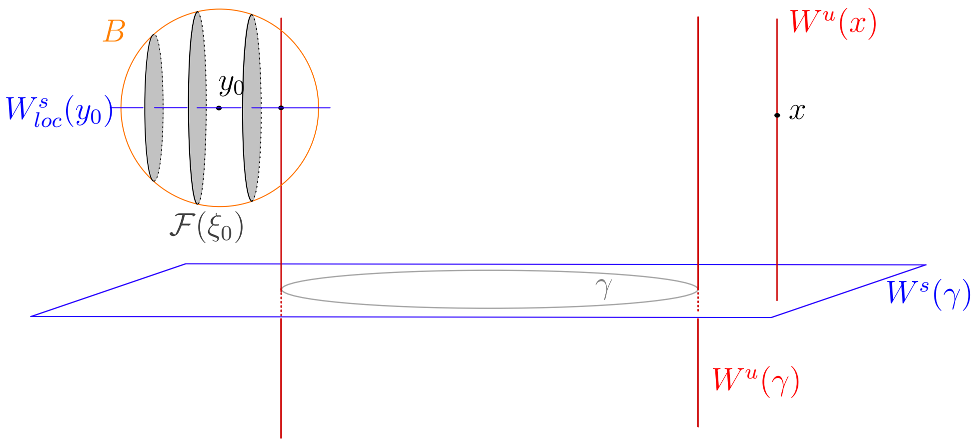


Figure 3.2: Case with $\dim W^s(y_0) + \dim W^u(\gamma) = n$ and $\dim W^u(x) + \dim W^s(\gamma) = n$.

3.3 Proof of Theorems D and E

The proof will be analogous to the proof of the first statement of Theorem C. Consider $f = 1_{\Lambda^s(\gamma)}$ and let $\mathcal{T}_f$ be as in Lemma 3.1.4 and define $\Delta^u := \Lambda^u(\gamma) \cap \mathcal{T}_f$. We are going to show that $\Delta^u \subset \Lambda^s(\gamma)$. To do that, let us prove that for every $x \in \Delta^u$ we get $f^+(x) = 1$. For now on fix $x \in \Delta^u$. The strategy is the same as before, we will consider an auxiliary point y to construct some local foliation of a small ball such that we find a leaf with intersection of positive measure with $\Lambda^s(\gamma)$, then we transfer this information to the unstable manifold of x also via holonomy. The only difference here is the choice of the point y. Insted of a Lesbegue density point we choose y to be a point in the support of the measure. The rest of the argument follows identically.

Let $R^{ij}_{\varepsilon,l}$ be a Pesin block such that $\Delta^s := R^{ij}_{\varepsilon,l} \cap \Lambda^s(\gamma)$ has positive measure and $y \in \Delta^s$ be such that there exists a sequence $(t_k)_k$ converging to infinity such that $y_k = \varphi_{t_k}(y) \in \Delta^s$ and y_k belongs to the support of μ restricted to $\Lambda^s(\gamma) \cap R^{ij}_{\varepsilon,l}$. Again, for points in $R^{ij}_{\varepsilon,l}$ their Pesin manifolds have a diameter bigger than a uniform constant $\delta > 0$. As before, let $y_0 = \varphi_{t_{k_0}}(y)$ be such that $d(y_0, W^u(\gamma)) < \frac{\delta}{2}$. Since y_0 is the support of μ, we can find a small ball B centered in y_0 satisfying the condition $\mu(\Delta^s \cap B) > 0$. Let $\mathcal{F}$ be a smooth foliation of B with dimension $n - \dim(W^s(y_0))$ such that each leaf $\mathcal{F}(\xi)$ is transverse to $W^s_{loc}(y_0)$. By the Fubini's property

$$0 < \mu(\Delta^s \cap B) = \int_{W^s_{loc}(y_0)} \mu^{\mathcal{F}}_{\xi}(\mathcal{F}(\xi) \cap \Delta^s) d\mu^s_{y_0}(\xi),$$

we have that $\mu^{\mathcal{F}}_{\xi}(\mathcal{F}(\xi) \cap \Delta^s) > 0$ on a subset of $W^s_{loc}(y_0)$ of μ^s_y−positive measure. Fix ξ_0 such that $\mu^{\mathcal{F}}_{\xi_0}(\mathcal{F}(\xi_0) \cap \Delta^s) > 0$. Since μ is a hyperbolic measure and has absolutely continuous conditional measures, the proof follows from the same arguments.

3.4 Proof of Theorem F

The proof will be a consequence of Theorem 1.4.15 and the next Lemma:

Lemma 3.4.1. *If μ is a regular, hyperbolic and SRB measure, then almost every one of its ergodic components is regular, hyperbolic and SRB.*

Proof. By the Ergodic Decomposition Theorem, there exist a partition $\mathcal{P}$ of M, a system of ergodic probability measures $\{\mu_P : P \in \mathcal{P}\}$, which form a decomposition for μ, and a measure $\hat{\mu}$ over $\mathcal{P}$. Since μ is regular, then

$$0 = \mu(Sing(X)) = \int \mu_P(Sing(X))d\hat{\mu}(P).$$

It follows that $\hat{\mu}-$almost every μ_P is regular. Now, we just need to check that $\hat{\mu}-$almost every μ_P satisfies Pesin's formula. Indeed, by the Margulis-Ruelle inequality and ergodicity we have

$$h_{\mu_P} \leq \int \sum_{\lambda(x)>0} \lambda(x)d\mu_P(x) = \sum_{\lambda(\mu_P)>0} \lambda(\mu_P).$$

On the other side, by Jacobs Theorem (check [VO16] Theorem 9.6.2) we get

$$h_\mu = \int h_{\mu_P} d\hat{\mu}(P),$$

and since μ is an SRB measure,

$$h_\mu = \int \sum_{\lambda(x)>0} \lambda(x)d\mu = \int \left(\int \sum_{\lambda(x)>0} \lambda(x)d\mu_P \right) d\hat{\mu}(P) = \int \sum_{\lambda(\mu_P)>0} \lambda(\mu_P)d\hat{\mu}(P).$$

It follows that

$$\int \underbrace{\left(h_{\mu_P} - \sum_{\lambda(\mu_P)>0} \lambda(\mu_P) \right)}_{\leq 0} d\hat{\mu}(P) = 0,$$

hence $\hat{\mu}-$almost every μ_P satisfies Pesin's formula. ∎

By the previous Lemma, we can assume that μ is an ergodic measure. Now Theorem 1.4.15 implies that there exists a periodic hyperbolic orbit γ such that $supp(\mu) \subset \overline{\Lambda(\gamma)}$ and, in particular, for $\mu-$almost every x satisfies $\varphi_t(W^u(x))$ acumulates on $W^u(\gamma)$ when t goes to infinity and $\varphi_t(W^s(x))$ acumulates on $W^s(\gamma)$ as t goes to minus infinity. The invariance of $\Lambda(\gamma)$ implies that $\mu(\Lambda(\gamma)) = 1$.

3.5 Proof of Theorem G

The strategy here is to prove that in this situation, we must have intersection of the basins of μ and ν, thus ergodicity will imply that they must coincide.

Let $B(\mu)$ and $B(\nu)$ be the basins of μ and ν, respectively. Since both measures are ergodic, it implies that $\mu(B(\mu)) = 1$ and $\nu(B(\nu)) = 1$. By Birkhoff Ergodic Theorem we can define the following $\mu-$full measure and $\nu-$full measure sets, respectively,

$$B_\mu = \left\{ x \in M : \lim_{T\to\pm\infty} \frac{1}{T}\int_0^T f(\varphi_t(x))dt = \int f d\mu, \forall f \in C^0(M) \right\}$$

$$B_\nu = \left\{ x \in M : \lim_{T\to\pm\infty} \frac{1}{T}\int_0^T f(\varphi_t(x))dt = \int f d\nu, \forall f \in C^0(M) \right\}.$$

By hypothesis we get $\mu(B_\mu \cap \Lambda(\gamma)) = 1$ and $\nu(B_\nu \cap \Lambda(\gamma)) = 1$. Apply Lemma 3.1.4 for μ and the function $f = 1_{B_\mu \cap \Lambda(\gamma)}$ to find $x \in \mathcal{T}_f \cap B_\mu \cap \Lambda(\gamma)$ such that $\mu_x^u(B_\mu \cap \Lambda(\gamma)) = 1$. Since μ is SRB, the conditionals μ_x^u are absolutly continuous with respect to the leaf volume measure thus $m_x^u(B_\mu \cap \Lambda(\gamma)) = 1$. Analogously, we find $y \in \mathcal{T}_{f'} \cap B_\nu \cap \Lambda(\gamma)$ such that $m_y^u(B_\nu \cap \Lambda(\gamma)) = 1$, where $f' = 1_{B_\nu \cap \Lambda(\gamma)}$. The strategy now is to transfer these sets to the unstable manifold of the orbit γ. More precisely, define

$$D_x = B_\mu \cap \Lambda(\gamma) \cap W^u(x),$$

$$D_y = B_\nu \cap \Lambda(\gamma) \cap W^u(y).$$

Fix the points $p, q \in \gamma$ such that p and q are the points satisfying $W^s(x) \cap W^u(p) \neq \emptyset$ and $W^s(x) \cap W^u(p) \neq \emptyset$. By the definition of $\Lambda(\gamma)$ and using the absolute continuity of weak-stable holonomies, for every point $z \in D_x$ we find a unique point $w \in W^u(p)$ such that $\{w\} = W^{ws}(z) \pitchfork W^u(p)$. Since $B_\mu \cap \Lambda(\gamma)$ is an invariant and $s-$saturated set, we get that $m_p^u(B_\mu \cap \Lambda(\gamma)) = 1$. Now, take $T \in \mathbb{R}$ such that $\varphi_T(q) = p$ and use the same argument to $D_{\varphi_T(y)}$ to get $m_p^u(B_\nu \cap \Lambda(\gamma)) = 1$. This implies that $B(\mu) \cap B(\nu) \neq \emptyset$, hence $\mu = \nu$.

3.5.1 Further questions

In this dissertation we have not discussed the applicability of the results presented in this chapter. This would be a very natural next step of our work.

Question 3.1. *Can we apply Theorem C to get ergodicity of some generic class of flows?*

The previous question is wide open, but it is worth mentioning that we look forward to understand homoclinic classes in the context of the geodesic flow, in special for manifolds with $rank(M) = 1$.

Another very natural question, now in the context of SRB measures is:

Question 3.2. *Can we apply Theorem G to get uniqueness of SRB measures in some general setting of flows?*

Bibliography

[ACW22] A. Avila, S. Crovisier, and A. Wilkinson, *Symplectomorphisms with positive metric entropy*, Proceedings of the London Mathematical Society **124** (2022), no. 5, 691–712. ↑23

[Ano67] D. Anosov, *Geodesic flows on closed riemannian manifolds with negative curvature*, Proc. Steklov Inst. Math. **90** (1967). ↑55, 115

[AS67] D. Anosov and Y. Sinai, *Some smooth ergodic systems*, Russian Mathematical Surveys **22** (1967), no. 5, 103. ↑55, 115

[Avi10] A. Avila, *On the regularization of conservative maps*, Acta Mathematica **205** (2010), no. 1, 5 –18. ↑23

[Bal95] W. Ballmann, *Lectures on spaces of nonpositive curvature*, Vol. 25, Springer Science & Business Media, 1995. ↑37, 56

[BB82] W. Ballmann and M. Brin, *On the ergodicity of geodesic flows*, Ergodic theory and Dynamical Systems **2** (1982), no. 3-4, 311–315. ↑15, 57, 105

[BBE85] W. Ballmann, M. Brin, and P. Eberlein, *Structure of manifolds of nonpositive curvature. i*, Annals of Mathematics **122** (1985), no. 1, 171–203. ↑57

[BCG95] G. Besson, G. Courtois, and S. Gallot, *Entropies et rigidités des espaces localement symétriques de courbure strictement négative*, Geometric & Functional Analysis GAFA **5** (1995), 731–799. ↑20

[BE21] T. Barthelmé and A. Erchenko, *Geometry and entropies in a fixed conformal class on surfaces*, Annales de l'Institut Fourier **71** (2021), 731–755. ↑17

[Bes07] A. L. Besse, *Einstein manifolds*, Springer, 2007. ↑43

[BG89] K. Burns and M. Gerber, *Real analytic bernoulli geodesic flows on s^2*, Ergodic theory and dynamical systems **9** (1989), no. 1, 27–45. ↑15, 57

[BP02] L. Barreira and Y. Pesin, *Lyapunov exponents and smooth ergodic theory*, Vol. 23, American Mathematical Soc., 2002. ↑63, 65, 67, 70

[BP07] ———, *Nonuniform hyperbolicity: Dynamics of systems with nonzero lyapunov exponents*, 2007. ↑63, 65

[BRHRH+08] K. Burns, F. Rodriguez-Hertz, M.A. Rodriguez-Hertz, A. Talitskaya, and R. Ures, *Density of accessibility for partially hyperbolic diffeomorphisms with one-dimensional center*, Discrete Contin. Dyn. Syst **22** (2008), no. 1-2, 75–88. ↑15

[Bur83] K. Burns, *Hyperbolic behaviour of geodesic flows on manifolds with no focal points*, Ergodic Theory and Dynamical Systems **3** (1983), no. 1, 1–12. ↑15, 57, 105

[BW10] K. Burns and A. Wilkinson, *On the ergodicity of partially hyperbolic systems*, Annals of Mathematics (2010), 451–489. ↑15

[Con02] G. Contreras, *Partially hyperbolic geodesic flows are anosov*, Comptes Rendus Mathematique **334** (2002), no. 7, 585–590. ↑60

[Cou16] Y. Coudène, *Ergodic theory and dynamical systems*, Springer, 2016. ↑69

[CP14] F. Carneiro and E. Pujals, *Partially hyperbolic geodesic flows*, Annales de l'IHP Analyse non linéaire **31(5)** (2014), 985–1014. ↑15, 17, 21, 24, 60, 62, 73, 76, 84, 85, 86, 88, 94, 97, 105, 106, 109

[DCF92] M. P. Do Carmo and J. Flaherty, *Riemannian geometry*, Vol. 2, Springer, 1992. ↑26, 29

[dJEP25] Y. de Jesus, M. Espitia, and G. Ponce, *Homoclinic classes for flows: ergodicity and srb measures*, arXiv preprint arXiv:2502.12260 (2025). ↑24, 115

[dJPR24] Y. de Jesus, L. P. Piñeyrúa, and S. Romaña, *Partially hyperbolic geodesic flow via conformal deformation*, arXiv preprint arXiv:2410.21519 (2024). ↑23, 73

[Don06] V. J. Donnay, *Geodesic flow on the two-sphere part ii: Ergodicity*, Dynamical systems: Proceedings of the special year held at the university of maryland, college park, 1986–87, 2006, pp. 112–153. ↑15, 57

[DP03] V. J. Donnay and C. Pugh, *Anosov geodesic flows for embedded surfaces*, Asterisque **287** (2003), 61–69. ↑55, 57

[Ebe72] P. Eberlein, *Geodesic flow in certain manifolds without conjugate points*, Transactions of the American Mathematical Society **167** (1972), 151–170. ↑106

[Ebe73] ———, *When is a geodesic flow of anosov type? i*, Journal of Differential Geometry **8** (1973), no. 3, 437–463. ↑15, 52, 55

[EO73] P. Eberlein and B. O'Neill, *Visibility manifolds*, Pacific Journal of Mathematics **46** (1973), no. 1, 45–109. ↑104

[FH22] T. Fisher and B. Hasselblatt, *Accessibility and centralizers for partially hyperbolic flows*, Ergodic Theory and Dynamical Systems **42** (2022), no. 3, 835–854. ↑15, 23, 111, 114

[Gol99] W. M. Goldman, *Complex hyperbolic geometry*, Oxford University Press, 1999. ↑43

[Gul75] R. Gulliver, *On the variety of manifolds without conjugate points*, Transactions of the American Mathematical Society **210** (1975), 185–201. ↑15, 36, 55, 57, 98

[Hop39] E. Hopf, *Statistik der geodätischen Linien in Mannigfaltigkeiten negativer Krümmung*, Ber. Verhandlungen der Sächsischen Akademie der Wissenschaften zu Leipzig **91** (1939), 261–304. ↑55, 115

[Hop40] ______, *Statistik der Lösungen geodätischer Probleme vom unstabilen Typus. ii*, Mathematische Annalen **117** (1940), no. 1, 590–608. ↑55, 115

[Jos08] J. Jost, *Riemannian geometry and geometric analysis*, Vol. 42005, Springer, 2008. ↑41, 44, 45

[Kat80] A. Katok, *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, Publications Mathématiques de l'IHÉS **51** (1980), 137–173. ↑72

[Kat82] ______, *Entropy and closed geodesies*, Ergodic theory and dynamical Systems **2** (1982), no. 3-4, 339–365. ↑17, 19, 20

[Kat88] ______, *Four applications of conformal equivalence to geometry and dynamics*, Ergodic Theory Dynam. Systems **8** (1988), no. Charles Conley Memorial Issue, 139–152. ↑17

[KKW91] A. Katok, G. Knieper, and H. Weiss, *Formulas for the derivative and critical points of topological entropy for Anosov and geodesic flows*, Comm. Math. Phys. **138** (1991), no. 1, 19–31. MR1108034 ↑20

[Kli74] W. Klingenberg, *Riemannian manifolds with geodesic flow of Anosov type*, Annals of Mathematics **99** (1974), no. 1, 1–13. ↑54

[Kni98] G. Knieper, *The uniqueness of the measure of maximal entropy for geodesic flows on rank 1 manifolds*, Annals of mathematics (1998), 291–314. ↑105

[Lee18] J. M. Lee, *Introduction to Riemannian manifolds*, Vol. 2, Springer, 2018. ↑26, 27, 37, 38

[LLL24] M. Li, C. Liang, and X. Liu, *A closing lemma for non-uniformly hyperbolic singular flows*, Communications in Mathematical Physics **405** (2024), no. 8, 195. ↑72

[LMM24] Y. Lima, C. Matheus, and I. Melbourne, *Polynomial decay of correlations for nonpositively curved surfaces*, Transactions of the American Mathematical Society **377** (2024), no. 09, 6043–6095. ↑16

[LY85] F. Ledrappier and L.-S. Young, *The metric entropy of diffeomorphisms: Part i: Characterization of measures satisfying Pesin's entropy formula*, Annals of Mathematics **122** (1985), no. 3, 509–539. ↑71

[MM63] F. K. Manasse and C. W. Misner, *Fermi normal coordinates and some basic concepts in differential geometry*, Journal of mathematical physics **4** (1963), no. 6, 735–745. ↑103

[Mor07] A. Moroianu, *Lectures on Kähler geometry*, Vol. 69, Cambridge University Press, 2007. ↑42

[Mos73] G. D. Mostow, *Strong rigidity of locally symmetric spaces*, Princeton University Press, 1973. ↑41

[Ose68] V. I. Oseledets, *A multiplicative ergodic theorem. Characteristic Ljapunov, exponents of dynamical systems*, Trudy Moskovskogo Matematicheskogo Obshchestva **19** (1968), 179–210. ↑64

[Pat93] M. Paternain, *Expansive geodesic flows on surfaces*, Ergodic Theory and Dynamical Systems **13** (1993), no. 1, 153–165. ↑16

[Pat99] G. Paternain, *Geodesic flows*, Vol. 180, Progress in Mathematics, 1999. ↑45, 50, 85

[Pes74] Y. Pesin, *Example of a nonergodic flow with nonzero characteristic numbers*, Functional Analysis and Its Applications **8** (1974), no. 3, 263–264. ↑62

[Pes76] ———, *Families of invariant manifolds corresponding to nonzero characteristic exponents*, Math. USSR-Izv **10** (1976), no. 6, 1261–1305. ↑62

[Pes77a] ———, *Characteristic lyapunov exponents and smooth ergodic theory*, Russian Mathematical Surveys **32** (1977), no. 4, 55. ↑62, 115

[Pes77b] ———, *Geodesic flows on closed riemannian manifolds without focal points*, Mathematics of the USSR-Izvestiya **11** (1977), no. 6, 1195. ↑14, 62

[Piñ23] L. P. Piñeyrúa, *Some hyperbolicity revisited and robust transitivity*, arXiv preprint arXiv:2302.01914 (2023). ↑107

[PS06] E. Pujals and M. Sambarino, *A sufficient condition for robustly minimal foliations*, Ergodic Theory and Dynamical Systems **26** (2006), no. 1, 281–289. ↑107

[RHRHTU11a] F. Rodriguez-Hertz, M. A. Rodriguez-Hertz, A. Tahzibi, and R. Ures, *New criteria for ergodicity and nonuniform hyperbolicity*, Duke Mathematical Journal **160** (2011), no. 3, 599 –629. ↑15, 17, 21, 115, 117

[RHRHTU11b] F. Rodriguez-Hertz, M.A. Rodriguez-Hertz, A. Tahzibi, and R. Ures, *Uniqueness of SRB measures for transitive diffeomorphisms on surfaces*, Communications in mathematical physics **306** (2011), 35–49. ↑18, 21, 115

[Roh52] V. A. Rohlin, *On the fundamental ideas of measure theory*, Amer. Math. Soc. Translation **1952** (1952), no. 71, 55. MR0047744 ↑69

[Rug91a] R. O. Ruggiero, *On the creation of conjugate points*, Mathematische Zeitschrift **208** (1991), 41–55. ↑16, 17, 21, 74

[Rug91b] ———, *Persistently expansive geodesic flows*, Communications in mathematical physics **140** (1991), 203–215. ↑60

[Rug97] ———, *Expansive geodesic flows in manifolds with no conjugate points*, Ergodic Theory and Dynamical Systems **17** (1997), no. 1, 211–225. ↑17

[VO16] M. Viana and K. Oliveira, *Foundations of ergodic theory*, Cambridge University Press, 2016. ↑69, 72, 124

[VP91] J. Von Plato, *Boltzmann's ergodic hypothesis*, Archive for History of Exact Sciences **42** (1991), 71–89. ↑14

[Wal04] G. Walschap, *Metric structures*, Springer, 2004. ↑26, 40

[You02] L.-S. Young, *What are SRB measures, and which dynamical systems have them?*, Journal of Statistical Physics **108** (2002), no. 5, 733–754. ↑70, 71