RELATÓRIO DE PESQUISA 1994

LINEAR SEMIGROUPS ACTING ON STIEFEL MANIFOLDS

Luiz A.B. San Martin

Julho RP 32/94

INSTITUTO DE MATEMÁTICA ESTATÍSTICA E CIÊNCIA DA COMPUTAÇÃO

UNICAMP

UNIVERSIDADE ESTADUAL DE CAMPINAS

ABSTRACT – This paper studies invariant control sets for d-dimensional bilinear control systems on the Stiefel manifolds $St_2(d)$ of two orthonormal frames in $\mathbb{R}^d$. It is shown that under an accessibility assumption the only non trivial case which remains to be analyzed is the case where the system's semigroup is a semigroup with non void interior in $Sl(d, \mathbb{R})$. For this case the quantity of invariant control sets on $St_2(d)$ is computed with the aid of the results of [7] about invariant control sets on the flag manifolds. Also, it is given an answer to the question of the existence or not, in an invariant control set on $St_2(d)$, of frames which span the same two dimensional subspace and are positively oriented inside this subspace. This question is suggested by the rotation numbers in higher dimensions as discussed in [2].

IMECC – UNICAMP
Universidade Estadual de Campinas
CP 6065
13081-970 Campinas SP
Brasil

Julho – 1994

Linear semigroups acting on Stiefel manifolds

Luiz A. B. San Martin*
Instituto de Matemática
Universidade Estadual de Campinas
Cx. Postal 6065, 13.084-100 Campinas-SP,Brasil

5/7/94

Abstract

This paper studies invariant control sets for d-dimensional bilinear control systems on the Stiefel manifolds $St_2(d)$ of two orthonormal frames in $\mathbb{R}^d$. It is shown that under an accessibility assumption the only non trivial case which remains to be analyzed is the case where the system's semigroup is a semigroup with non void interior in $Sl(d,\mathbb{R})$. For this case the quantity of invariant control sets on $St_2(d)$ is computed with the aid of the results of [7] about invariant control sets on the flag manifolds. Also, it is given an answer to the question of the existence or not, in an invariant control set on $St_2(d)$, of frames which span the same two dimensional subspace and are positively oriented inside this subspace. This question is suggested by the rotation numbers in higher dimensions as discussed in [2].

1 Introduction

Let

$$\Sigma : \qquad \dot{x} = Ax + \sum_{i=1}^{m} u_i B_i x$$

be a bilinear control system on $\mathbb{R}^d$, $d \geq 3$. Denote by S_Σ (or just by S) the system semigroup and by G_Σ (or G) the system group (see e.g. [6]).It is known that S_Σ is a

*Research partially supported by CNPq grant 50.13.73/91-8

semigroup with non void interior in G_Σ. The system induces a right invariant system on $Gl(d, \mathbb{R})$ and so it induces also control systems on the homogeneous spaces of $Gl(d, \mathbb{R})$. Their orbits, control sets, etc... are given by the action of G_Σ and S_Σ as a subgroup and subsemigroup of $Gl(d, \mathbb{R})$ respectively.

Of interest here are the invariant control sets for the system induced on the Stiefel manifolds $St_2(d)$ of two-orthonormal frames of $\mathbb{R}^d$, that is the invariant control sets for the action of S on $St_2(d)$. In particular, it will be discussed whether an invariant control set contain frames which are of the same orientation within a two-plane. This question is relevant for rotation numbers (c.f. [2, Thm. 4.1 ii]).

It will be assumed that $G_\Sigma \subset Sl(d, \mathbb{R})$. This assumption amounts to take A and B_i with zero trace, and can be done without loss of generality because $Gl(d, \mathbb{R})$ decomposes as $Sl(d, \mathbb{R})\, Z$ with Z the center which is the subgroup of the multiples of the identity and acts trivially on the Stiefel manifolds.

It will be assumed also that the system on $St_2(d)$ has the accessibility property, that is

$$\mathcal{A}: \qquad int\,(Sx) \neq \emptyset$$

This assumption, although natural from the points of view of e.g control theory; uniqueness of invariant probability inside the invariant control sets, etc.... (see[1]) it is highly restrictive from the point of view of group theory. In fact, $\mathcal{A}$ implies trivially that G is transitive on $St_2(d)$. On the other hand, there is the following

Fact: Except in dimensions 7 and 8 the only subgroups of $Sl(d, \mathbb{R})$ which are transitive on $St_2(d)$ (with the action induced by the $Sl(d, \mathbb{R})-$action) are $SO(d, \mathbb{R})$ and $Sl(d, \mathbb{R})$ itself. The other groups transitive in dimensions 7 and 8 are compact. This fact was stated without proof in [6, Example 3,page 58]. Its proof is based on Boothby[3] and Boothby and Wilson [4] classification of the groups which are transitive on $\mathbb{R}^d$.

This fact will be proved in section 2 below. From it, the accessibility assumption $\mathcal{A}$ reduces the possibilities only to those semigroups which are either with non void interior in $Sl(d, \mathbb{R})$ or with non void interior in a compact group. In the latter case, the semigroup is a group so there is no further analysis to be done. In the case of a semigroup with non void interior in $Sl(d, \mathbb{R})$, the analysis of its invariant control sets in the Stiefel manifold is provided in section 3. The main technique used comes from the results of [7] about the action of subsemigroups with non empty interior in $Sl(d, \mathbb{R})$ on the flag manifolds.

2 Groups Transitive on $St_2(d)$

The idea for classifying the linear groups (subgroups of $Sl(d, \mathbb{R})$), which are transitive on $St_2(d)$ is to look at those groups which are transitive on the sphere S^{d-1} and then use the fibration

$$\pi : St_2(d) \longrightarrow S^{d-1}$$

which maps the orthonormal two-frame $\{u, v\}$ into its first element $u \in S^{d-1}$. The action of $Sl(d, \mathbb{R})$ on $St_2(d)$ is given by orthonormalization of $\{gu, gv\}, g \in Sl(d, \mathbb{R}), \{u, v\} \in St_2(d)$. This implies, among other things, that the above fibration is equivariant *w.r.t* the $Sl(d, \mathbb{R})-$actions, which in turn implies that a necessary condition for a subgroup $G \in Sl(d, \mathbb{R})$ to be transitive on $St_2(d)$ is that it is transitive on S^{d-1}. On the other hand, fixing $u \in S^{d-1}$, the fiber $\pi^{-1}(u)$ over it is the set of frames $\{u, v\}$ with $v \in S^{d-1}$ orthogonal to u , so that it is homeomorphic to S^{d-2}. Also, in order that G is transitive on $St_2(d)$ it is necessary that it is transitive on the fiber over u all u, where transitivity on the fiber means that $\forall p, q \in \pi^{-1}(u)$ there exists $g \in G$ such that $gp = q$. Now, due to the equivariance of π, $gp = q$ implies that $g\,u = u$, so that G is transitive on the fiber over u if and only if the isotropy

$$G_u = \{g \in G : gu = u\}$$

is transitive in this fiber.

The job thus is to look at those groups which are transitive on S^{d-1} and check whether their isotropies are transitive on the fiber S^{d-2}. The linear groups transitive on S^{d-1} where classified by Boothby [3] and Boothby and Wilson [4].

The idea to check the transitivity of the isotropy on the fiber is to use Boothby's classification in dimension one less. There is however a problem about this which comes from the fact that this classification is exclusive for linear groups with action coming from the $Sl(d, \mathbb{R})-$action. So in order that that classification becomes available it is necessary to ensure first that the action of G_u on the fiber is linear, that is, comes from a linear representation of G_u on $\mathbb{R}^{d-1}$.This is assured by the

Proposition 1 *$St_2(d)$ is the sphere bundle $S(S^{d-1})$ of the spheres of the tangent bundle of S^{d-1}, and the action of $Sl(d, \mathbb{R})$ on it is just the differential action (lifting) of elements of $Sl(d, \mathbb{R})$ viewed as diffeomorphisms of S^{d-1}.*

Proof. In fact, the mapping $\{u,v\} \in St_2(d) \longrightarrow S(S^{d-1}), u \in S^{d-1}, v \in T_u S^{d-1}$ defines a bijection between $S(S^{d-1})$ and $St_2(d)$.

Denote for a moment by $g * u$ the action of $g \in Sl(d, \mathbb{R})$ in $u \in S^{d-1}$;

$$g * u = \frac{gu}{|gu|}$$

where gu is matrix multiplication(action on $\mathbb{R}^d$). Suppose $g * u = u$ and take $v \in T_u S^{d-1}$; $v \in \mathbb{R}^d$ and is orthogonal to u. Let u_t be a curve in S^{d-1} with $u_0 = u$ and whose derivative at 0 is v. Then

$$\begin{aligned} \frac{d}{dt}(g * u_t)_{t=0} &= \left(\frac{gu_t}{|gu_t|}\right)'_{t=0} \\ &= \frac{gv}{|gu_0|} - \frac{<gv, gu_0>}{|gu_0|^3} gu_0 \\ &= \frac{gv}{|gu|} - \frac{<gv, gu>}{|gu|} gu \end{aligned}$$

because $gu = |gu|u$ and $u_0 = u$. Therefore the differential of $y \longrightarrow g * y$ at u is

$$v \longrightarrow \frac{gv}{|gu|} - \frac{< gv, u >}{|gu|} u$$

which has the same direction as the second vector of the orthonormalization of $\{gu, gv\}$. Therefore the action of the isotropy is linear. □

Now it is possible to start.

Let G be a linear Lie group transitive on S^{d-1} and $\mathfrak{g}$ its Lie algebra. Transitivity of G implies its irreducibility and therefore $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{z}$$

with $\mathfrak{g}_0$ semi-simple and $\mathfrak{z}$ the center (compare with [3]). Let G_0 be the connected subgroup of G whose Lie algebra is $\mathfrak{g}_0$. It is known that G_0 is also transitive on S^{d-1} (see [3] and [4, page 216]).

The discussion to follow is to ensure that the analysis of the transitivity on the Stiefel manifold $St_2(d)$ can be reduced to the action of G_0, so that it is enough to look at semi-simple Lie groups. This facilitates the computations.

Let H be the isotropy (at some point of S^{d-1}) and denote by $\mathfrak{h}$ its Lie algebra. Assume that it is transitive on the fiber. Then, as to $\mathfrak{g}$, it acts irreducible on $\mathbb{R}^{d-1}$.

By the discussion above this action comes from a representation, say ρ, of H. Denoting also by ρ the representation of $\mathfrak{h}$, irreducibility implies that

$$\rho(\mathfrak{h}) = \mathfrak{l} \oplus \mathfrak{c}$$

with $\mathfrak{l}$ semi-simple and $\mathfrak{c}$ the center. The derived algebras are

$$\mathfrak{g}' = \mathfrak{g}_0 \quad \text{and} \quad \rho(\mathfrak{h})' = \mathfrak{l}$$

so that

$$\mathfrak{l} = \rho(\mathfrak{h})' = \rho(\mathfrak{h}') \subset \rho(\mathfrak{h} \cap \mathfrak{g}_0)$$

because $\mathfrak{h}' \subset \mathfrak{g}' = \mathfrak{g}_0$. Now, $\mathfrak{h} \cap \mathfrak{g}_0$ is the Lie algebra of $H \cap G_0$. Since the connected subgroup whose Lie algebra is $\mathfrak{l}$ is transitive (if H is transitive), this implies that $H \cap G_0$, which is the isotropy of the G_0−action is also transitive. Therefore, a linear group G is transitive on $St_2(d)$ if and only if its semi-simple component is also transitive.

The table below is extracted from [4] and shows the linear Lie algebras transitive on S^{d-1}. Transitivity of the Lie algebra $\mathfrak{g}$ means that the group G obtained by exponentiating a faithful representation of $\mathfrak{g}$ is transitive on the corresponding sphere. Due to the discussion above, it is reproduced here only the semi-simple components of the Lie algebras.

Notation d is the dimension of the representation an N is the dimension of the Lie algebra.

Type	d	N	*Algebra*
$I.1$	m	$m(m-1)/2$	$\mathfrak{so}(m)$
$I.2$	$2m$	m^2-1	$\mathfrak{su}(m)$
$I.3$	$4m$	$2m^2+m$	$\mathfrak{sp}(m)$
$I.4$	$4m$	$2m^2+m+3$	$\mathfrak{sp}(m) \oplus \mathbb{H}$
$I.5$	8	22	$\mathfrak{spin}(7)$
$I.6$	16	36	$\mathfrak{spin}(9)$
$I.7$	7	14	$\mathfrak{g}_{2(-14)}$
$II.1$	m	m^2-1	$\mathfrak{sl}(m, \mathbb{R})$
$II.2$	$2m$	$2(m^2-1)$	$\mathfrak{sl}(m, \mathbb{C})$
$II.3$	$4m$	$4m^2-1$	$\mathfrak{sl}(m, \mathbb{H})$
$II.4$	$2m$	$2m^2+m$	$\mathfrak{sp}(m, \mathbb{R})$
$II.5$	$4m$	$4m^2+2m$	$\mathfrak{sp}(m, \mathbb{C})$
III	$4m$	$4m^2+2$	$\mathfrak{sl}(m, \mathbb{H}) \oplus \mathfrak{su}(2)$

The algebras of class I are compact and those of classes II and III are non compact.

It is not worth to describe here all the representations. Except from $I.5$, 6 and 7 they are described by algebras of real, complex and quaternionic matrices. In particular, $I.1$ and $II.1$ are the canonical representations of $\mathfrak{so}$ and $\mathfrak{sl}$ respectively.

Clearly, $I.1$ and $II.1$ are transitive on $St_2(d)$.

The others non compact are not transitive. This can be easily checked by the dimension of the isotropy. This is a non compact Lie group acting transitively in a sphere in dimension one less, so that it is part of the above list. Now, the only non compact case occurring in *odd* dimension is $II.1$. Since the isotropy represents in dimension $d-1$, the inequality

$$di = N - d + 1 < ds$$

where di stands for the dimension of the isotropy and $ds = dim\, \mathrm{sl}(d-1, I\!R)$, shows that the isotropy is properly contained in $\mathrm{sl}(d-1, I\!R)$ so it cannot be transitive if d is even. The following list show this inequality for the non compact cases different from $II.1$. In it, rt are the roots of $ds - di$ which in all the cases is a quadratic polynomial in m with positive leader coefficient so that $ds - di > 0$ for m bigger than the highest root or all m if the roots are not real.

II.2

$$\begin{aligned} d &= 2m \\ di &= 2m^2 - 2m - 1 \\ ds &= 4m^2 - 4m \\ ds - di &= 2m^2 - 2m + 1 \\ rt & \quad complex \\ ds - di > 0 & \quad all \quad m \end{aligned}$$

II.3

$$\begin{aligned} d &= 4m \\ di &= 4m^2 - 4m \\ ds &= 16m^2 - 8m \\ ds - di &= 12m^2 - 4m \\ rt & \quad 0, 1/3 \\ ds - di > 0 & \quad m \geq 1 \end{aligned}$$

II.4

$$\begin{aligned} d &= 2m \\ di &= 2m^2 - m - 1 \\ ds &= 4m^2 - 4m \\ ds - di &= 2m^2 - 3m + 1 \\ rt & \quad 1/2, 1 \\ ds - di > 0 & \quad m \geq 2 \end{aligned}$$

when $m = 1$ there is isomorphism with $II.1$ with $m = 2$

II.5

$$\begin{aligned} d &= 4m \\ di &= 4m^2 - 2m + 1 \\ ds &= 16m^2 - 8m \\ ds - di &= 12m^2 - 6m - 1 \\ rt & \quad \tfrac{6 \pm \sqrt{84}}{24} \\ ds - di > 0 & \quad m \geq 1 \end{aligned}$$

The roots are smaller than 1

III

$$\begin{aligned} d &= 4m \\ di &= 4m^2 - 4m + 1 \\ ds &= 16m^2 - 8m \\ ds - di &= 12m^2 - 4m - 1 \\ rt & \quad \tfrac{1}{2}, -\tfrac{1}{6} \\ ds - di > 0 & \quad m \geq 1 \end{aligned}$$

3 The invariant control sets

By virtue of the above classification of the groups transitive on $St_2(d)$ the accessibility property $\mathcal{A}$ reduces everything to two possibilities: either the group is compact or it is $Sl(d, I\!R)$. Since for G compact $S = G$, the only case which needs further analysis is the case where

$$G_\Sigma = Sl(d, I\!R)$$

The counting and other descriptions of the invariant control sets on $St_2(d)$ to be done in the sequel is not restrictive for semigroups coming from a control system. So in what follows, S will stand for an arbitrary semigroup with non empty interior in $Sl(d, I\!R)$.

The $S - i.c.s$'s on $St_2(d)$ are looked through the fibration

$$\pi : St_2(d) \longrightarrow I\!F^d(1,2)$$

where $I\!F^d(1,2)$ is the manifold of flags $(V_1 \subset V_2)$ with $dim V_i = i$ and V_i are subspaces of $I\!R^d$. The above fibration is a four sheet covering and it can be seen in either one of the two ways

Geometricaly let (u,v) be an orthonormal two-frame in $\mathbb{R}^d$. Then $\pi(u,v) = (V_1 \subset V_2)$ where $V_1 = span\{u\}$ and $V_2 = span\{u,v\}$; let $\xi = (V_1 \subset V_2) \in \mathbb{F}^d(1,2)$ be a flag. Then its inverse image $\pi^{-1}\{\xi\}$ under π are the four possible orthonormal two-frames (u,v) with $u \in V_1$ and $v \in V_2$, that is the four orthonormal basis (u,v) of V_2 with $u \in V_1$.

As homogeneous spaces Let $\beta = \{e_1,\dots,e_d\}$ be a basis of $\mathbb{R}^d$. Then

$$St_2(d) = Sl(d,\mathbb{R})/H = SO(d,\mathbb{R})/L$$

and

$$\mathbb{F}^d(1,2) = Sl(d,\mathbb{R})/P(1,2) = SO(d,\mathbb{R})/M$$

where the isotropies are the groups of matrices which in basis β look like

$$\left(\begin{array}{cc|c} \lambda_1 & \epsilon_1 & \epsilon_2 \\ 0 & \lambda_2 & \\ \hline \multicolumn{2}{c|}{0} & A \end{array}\right)$$

with $A \in Gl(d-2,\mathbb{R})$, $\lambda_1\lambda_2 \det A = 1$, and

- $\lambda_1, \lambda_2 > 0$ for H,
- $\lambda_1 = \lambda_2 = 1$, $A \in SO(d-2)$ and $\epsilon_1 = \epsilon_2 = 0$ for L,
- λ_1 and λ_2 are ± 1, $A \in SO(d-2)$, and $\epsilon_1 = \epsilon_2 = 0$ for M.

There is the canonical fibration

$$SO(d,\mathbb{R})/L \longrightarrow SO(d,\mathbb{R})/M$$

which is the same as

$$Sl(d,\mathbb{R})/H \longrightarrow Sl(d,\mathbb{R})/P(1,2)$$

and maps cosets into cosets. The fiber is M/L which is isomorphic to the abelian group of four elements

$$\{(a_1,a_2,a_3) : a_i = \pm 1, a_1a_2a_3 = 1\}.$$

The fact that M/L is a group (that is, L is normal in M), implies that $\pi : St_2(d) \longrightarrow \mathbb{F}^d(1,2)$ is a principal bundle with structure group M/L and thus it is a four sheet covering and there is a right action of M/L identifying it with the fibers. Geometricaly this right action is given by changing signals of the elements of the two-frame (u,v) according to the signals of the first two coordinates of the element of M/L. This is because the right action of an element of M/L is given by the usual action of any of its representatives in M and a representative of (a_1,a_2,a_3) is any diagonal matrix such that the signals of the first two diagonal elements coincide with the signals of the first two coordinates of (a_1,a_2,a_3).

In $\mathbb{F}^d(1,2)$ there is just one invariant control set, say C, for S (c.f.[5]). Since $\pi : St_2(d) \longrightarrow \mathbb{F}^d(1,2)$ is equivariant with respect to the action of $Sl(d,\mathbb{R})$, the invariant control sets for S on $St_2(d)$ are contained in $\pi^{-1}(C)$ and project down onto C. Also, the fact that L is normal in M implies that for any $a \in M/L$, C_1a is an invariant control set if C_1 is an *i.c.s*. This fact implies that there exists a subgroup $\tilde{M}$ of M/L such that

$$\tilde{M} = \{g \in M/L : C_1\, g = C_1 \text{all i.c.s } C_1 \text{ on } St_2(d)\}$$

Therefore the number of *i.c.s*'s on $St_2(d)$ is 1, 2 or 4 according the order of $\tilde{M}$ is 4, 2 or 1 respectively.

For rotation numbers it is also relevant to know whether an *i.c.s* on $St_2(d)$ contains or not in the same fiber over $F^d(1,2)$ frames with different orientations inside the plane containing them (c.f [2, Th.4.1]). This question is answered by a quick glance at $\tilde{M}$: taking the realization of M/L as above, there are the possibilities

- $\tilde{M} = \{1\}$; four *i.c.s's* and the four frames over an element of C are in different invariant control sets.
- $\tilde{M} = \{1, (-1,-1,-1)\}$; two *i.c.s's*; along a fiber the frames in the same *i.c.s* have the same orientation.
- $\tilde{M} = \{1, (-1,1,-1)\}$ or $\{1, (1,-1,-1)\}$; two *i.c.s's*; along the fiber the frames in the same *i.c.s* have opposite orientation.
- $\tilde{M} = M/L$; just one i.c.s.

Remark Actually the question of having frames with the same or different orientation should be seen inside a two plane and not "inside" a flag as given by $\tilde{M}$

above. The discussion about the mutual orientations of frames of a given two-plane which are in the same *i.c.s* will be postponed till the end.

In what follows, $\tilde{M}$ will be given in its dependence to some structural properties of S which are given by its action on the flag manifolds. Because of this the following comments about the flag manifolds will be needed.

Denote by $\mathcal{P}$ a partition in intervals of $\{1, \ldots, d\}$, that is a decomposition of this set into non overlapping intervals defined with respect to its natural order. Such a partition will be written visually as

$$\mathcal{P} = [1, \ldots, r_1], [r_1 + 1, \ldots, r_2], \ldots, [r_k + 1, \ldots, d].$$

Associated to this $\mathcal{P}$ there is the flag manifold $\mathbb{F}^d(r_1, r_2, \ldots, r_k)$ of flags $(V_1 \subset \cdots \subset V_k)$ with $\dim V_i = r_i$, $i = 1, \ldots, k$. Henceforth this flag manifold will be denoted by $\mathbb{F}(\mathcal{P})$. It is a homogeneous space of $Sl(d, \mathbb{R})$ and of $SO(d, \mathbb{R})$ and is written as

$$\begin{aligned} \mathbb{F}(\mathcal{P}) &= Sl(d, \mathbb{R})/P(\mathcal{P}) \\ &= SO(d, \mathbb{R})/M(\mathcal{P}) \end{aligned}$$

where $P(\mathcal{P})$ is the (parabolic) subgroup of $Sl(d, \mathbb{R})$ of matrices of the form

$$\begin{array}{|c|c|c|} \hline * & * & * \\ \hline & \ddots & * \\ \hline & & * \\ \hline \end{array}$$

where the diagonal blocks decompose the diagonal exactly as $\mathcal{P}$ decomposes $\{1, \ldots, d\}$, and $M(\mathcal{P}) = Sl(d, \mathbb{R}) \cap SO(d, \mathbb{R})$. Reciprocally, given a flag manifold, its isotropy is a subgroup of matrices as above so it defines a partition of $\{1, \ldots, d\}$ in intervals and the flag is of the form $\mathbb{F}(\mathcal{P})$ some $\mathcal{P}$.

Let W be the permutation group of $\{1, \ldots, d\}$. It is the Weyl group of $Sl(d, \mathbb{R})$ and acts on the set of diagonal matrices by permuting its entries. Given a partition $\mathcal{P}$, let $W_{\mathcal{P}}$ be the subgroup of W generated by the permutation groups of the intervals defined by $\mathcal{P}$. That is

$$W_{\mathcal{P}} = \Pi(1, r_1)\Pi(r_1 + 1, r_2) \ldots \Pi(r_k + 1, d)$$

where $\Pi(i, j)$ is the permutation group of $\{i, \ldots, j\}, i < j$, and $\mathcal{P}$ is as above.These subgroups will play a role in the sequel. Note that the diagonal matrices which are keep fixed by every element of $W_{\mathcal{P}}$ are those which have repeated eigenvalues inside the diagonal blocks defined by $P(\mathcal{P})$.

EXAMPLES
$d = 4$

$$\mathcal{P} = [1][2][3][4]; \qquad W_{\mathcal{P}} = \{1\}; \qquad \mathbb{F}(\mathcal{P}) = \mathbb{F}^4(1, 2, 3)$$

$$\mathcal{P} = [1][2][34]; \qquad W_{\mathcal{P}} = \{1, (3, 4)\}; \qquad \mathbb{F}(\mathcal{P}) = \mathbb{F}^4(1, 2)$$

$$\mathcal{P} = [12][3][4]; \qquad W_{\mathcal{P}} = \{1, (1, 2)\}; \qquad \mathbb{F}(\mathcal{P}) = \mathbb{F}^4(2, 3)$$

$$\mathcal{P} = [1][23][4]; \qquad W_{\mathcal{P}} = \{1.(2, 3)\}; \qquad \mathbb{F}(\mathcal{P}) = \mathbb{F}^4(1, 3)$$

$$\mathcal{P} = [1][234]; \qquad W_{\mathcal{P}} = \Pi(2, 3, 4); \qquad \mathbb{F}(\mathcal{P}) = \mathbb{F}^4(1) = \mathbb{R}P^3$$

$$\mathcal{P} = [12][34]; \qquad W_{\mathcal{P}} = \{1, (1, 2), (3, 4), (1, 2)(3, 4)\}; \qquad \mathbb{F}(\mathcal{P}) = \mathbb{F}^4(2) = Gr_2(4)$$

$$\mathcal{P} = [123][4]; \qquad W_{\mathcal{P}} = \Pi(1, 2, 3); \qquad \mathbb{F}(\mathcal{P}) = \mathbb{F}^4(3) = Gr_3(4)$$

$$\mathcal{P} = [1234]; \qquad W_{\mathcal{P}} = W; \qquad \mathbb{F}(\mathcal{P}) \text{ reduces to one point}$$

In general,

$$\begin{aligned} \mathbb{F}(\mathcal{P}) &= \mathbb{R}P^{d-1}; & \mathcal{P} &= [1][2, \ldots, d] \\ \mathbb{F}(\mathcal{P}) &= Gr_2(d); & \mathcal{P} &= [12][3, \ldots, d] \\ \mathbb{F}(\mathcal{P}) &= \mathbb{F}^d(1, 2); & \mathcal{P} &= [1][2][3, \ldots, d] \end{aligned}$$

Returning now to the semigroups, in [7], it was associated to each semigroup $S \subset Sl(d, \mathbb{R})$, with $intS \neq \emptyset$, a subgroup $W(S)$ of W. This association is roughly as follows: In the full flag manifold there is, for each $w \in W$, a control set for S denoted by D_w. The invariant control set is D_1 and the D_w's are not necessarily distinct. $W(S)$ is a kind of kernel of the map $w \longrightarrow D_w$ and is given as $W(S) = \{w \in W : D_w = D_1\}$; it is a subgroup of W.

The properties of $W(S)$ which will be needed are given in the following statements:

Proposition 2 *A permutation w belongs to $W(S)$ if and only if there exists a basis $\{e_1, \dots, e_d\}$ of $\mathbb{R}^d$ and $h \in intS$ such that with respect to this basis*

$$h = diag\{\lambda_1, \dots, \lambda_d\} \qquad with \qquad \lambda_1 \geq \dots \geq \lambda_d > 0$$

(the ordering of the eigenvalues is essential) and $w\,h = h$ where w acts in h by permuting the eigenvalues. (c.f [7, Corollary4.4]).

Proposition 3 $W(S) = W_\mathcal{P}$ *some partition $\mathcal{P}$. For any $\mathcal{P}$ there exists S with $W(S) = W_\mathcal{P}$. [7, Theorem 4.3]*

Proposition 4 *If $W(S) = W_\mathcal{P}$ then the invariant control set on the full flag is given by $\pi^{-1}(C_\mathcal{P})$ where $C_\mathcal{P}$ is the invariant control set on $\mathbb{F}(\mathcal{P})$ and π is the canonical fibration $\pi : \mathbb{F}^d(1, \dots, d-1) \longrightarrow \mathbb{F}(\mathcal{P})$. [7, Theorem 4.3]*

In order to state the next property, let $\beta = \{e_1, \dots, e_d\}$ be a basis of $\mathbb{R}^d$ and denote by N_β the group of those matrices which in basis β are lower triangular with 1's in the diagonal. On any flag manifold $\mathbb{F}^d(r_1, \dots, r_k)$, N_β has just one open and dense orbit $N_\beta f_\beta$ where f_β is the flag built from $\{e_1, \dots, e_d\}$ as

$$f_\beta = (span\{e_1, \cdots e_{r_1}\} \subset span\{e_1, \cdots, e_{r_1+r_2}\} \subset \cdots \subset span\{e_1, \cdots, e_{r_1+\cdots+r_k}\})$$

These orbits are - as N_β - homeomorphic to Euclidian spaces.

Proposition 5 *Suppose $W(S) = W_\mathcal{P}$ and let $\{e_1, \dots, e_d\}$ be a basis realizing Proposition 2. Then $C_\mathcal{P}$ the invariant control set on $\mathbb{F}(\mathcal{P})$ is contained in $N_\beta f_\beta$. [7, Proposition 4.8]*

The fact stated in Proposition 2 shows one of the essential features of $W(S)$, namely it measures the possible repetitions of the eigenvalues of the diagonalizable matrices inside $intS$. For instance, $W(S) = \{1\}$ implies that there are in $intS$ only diagonal matrices with distinct eigenvalues. As another example, let $d = 4$ then $W(S) = W_\mathcal{P}$ with $\mathcal{P} = [12][34]$ if and only if there are, in $intS$, diagonalizable matrices (real eigenvalues) whose two biggest eigenvalues coincide, as well as the smallest ones.

The objective now is to determine $\tilde{M}$ from $W(S)$. That this is possible is suggested by Proposition 4 : $W(S)$ measures, in a certain sense how and how much the

invariant control set in the full flag turns around. Now, when lifting the invariant control sets from $\mathbb{F}^d(1,2)$ to $St_2(d)$ this turning around of the invariant control set will cause different ones to collapse and this will influence $\tilde{M}$. On the other hand Proposition 5 says that $W(S)$ measures exactly the turning around of the invariant control set because on the flag $\mathbb{F}_\mathcal{P}$ intrinsically associated to S there is no turning around of the invariant control set as it is contained in an N_β−orbit.

Examples 6 *1. $d = 2$ There are two possibilities: $W(S) = \{1\}$ or W and just one flag $\mathbb{R}P^1$. When $W(S) = W$ the semigroup is everything and the i.c.s is also everything, and $W(S)$ is proper if and only if S is proper if and only if the invariant control set is proper if and only if there is no turning around $\mathbb{R}P^1$ and of course the invariant control sets are contained in N_β−orbits which are the complementaries of one point sets.*

2. $d = 3$. The possibilities are:

$$(W(S), \mathbb{F}(\mathcal{P})) = \begin{matrix} \{1\} & \mathbb{F}^3(1,2) \\ \{1,(2,3)\} & \mathbb{R}P^2 \\ \{1,(1,2)\} & Gr_2(3) \end{matrix}$$

An example of the second type is the semigroup $Sl(d, \mathbb{R})^+$ of matrices which have positive entries, that is of matrices which map the first orthant into itself: if $\{v_1, v_2, v_3\}$ is a basis such that v_1 is in the interior of the first orthant and $span\{v_2, v_3\}$ meets it only in the origin then any linear map h which in this basis writes as $diag\{\lambda, \lambda^{-1/2}, \lambda^{-1/2}\}, \lambda > 1$ is in $intSl(d, \mathbb{R})^+$. The invariant control set on $\mathbb{R}P^2$ is the subset corresponding to the first orthant. Its lifting to the sphere splits out into two invariant control sets namely the positive and the negative orthants.

On the other hand, $S = Sl(d, \mathbb{R})^{+^{-1}}$ is of the third type because if h is as above, the two highest eigenvalues of h^{-1} coincide. Its invariant control set on $Gr_2(3)$ does not "turn around" but on projective space, it can be shown that the invariant control set is the closure of the complement of the first orthant. This invariant control set lifts to just one invariant control set on S^2.

Consider the following cases

a) $\mathcal{P} = [1, 2, 3, \dots] \dots [\dots]$

b) $\mathcal{P} = [1][2][3, \dots] \dots [\dots]$

c) $\mathcal{P} = [1][2,3,\ldots]\ldots[\ldots]$

d) $\mathcal{P} = [1,2][3,\ldots]\ldots[\ldots]$

Of course these four types of partitions cover all the possibilities.

In case a) holds, $\tilde{M} = M/L$ regardless any further assumption about S, so that there is just on invariant control set on $St_2(d)$. In fact, in this case it is possible to permute the first three eigenvalues with elements in $W(S)$ so that there exists a basis $\{e_1,\ldots,e_d\}$ and $h \in intS$ s.t with respect to this basis

$$h = \{\lambda_1, \lambda_1, \lambda_1, \ldots\} \qquad \lambda_1 > 0$$

Since $h \in intS$ it is possible to perturb it and get in $intS$ matrices whose upper left 3×3 corner is one of the following

$$\begin{pmatrix} \lambda_1 & -\epsilon & \\ \epsilon & \lambda_1 & \\ & & \lambda_1 \end{pmatrix} \quad \begin{pmatrix} \lambda_1 & & \\ & \lambda_1 & -\epsilon \\ & \epsilon & \lambda_1 \end{pmatrix} \quad \begin{pmatrix} \lambda_1 & & -\epsilon \\ & \lambda_1 & \\ \epsilon & & \lambda_1 \end{pmatrix}$$

These are exponentials of matrices of the form

$$\begin{pmatrix} a & -b & \\ b & a & \\ & & a \end{pmatrix} \quad \begin{pmatrix} a & & \\ & a & -b \\ & b & a \end{pmatrix} \quad \begin{pmatrix} a & & -b \\ & a & \\ b & & a \end{pmatrix}$$

Choosing b to be rational and taking powers of the perturbed matrices if needed, there are in $intS$ diagonal matrices whose first three eigenvalues are of the type

$$\{-\mu, -\mu, \mu\} \qquad \{\mu, -\mu, -\mu\} \qquad \{-\mu, \mu, -\mu\}$$

and these are representatives in M of the elements $(-1,-1,1), (1,-1,-1)$ and $(-1,1,-1)$ of $\tilde{M}$ respectively. Therefore $\tilde{M} = M/L$. (Note that this case cover the controllable case, that is, the case when $S = Sl(d, \mathbb{R})$ and $W(S) = W$ and $\mathcal{P} = [1\ldots d]$).

For the other cases it will be needed to assume that S is *connected* what happens to control semigroups.

Let $\mathcal{P} = [1][2][3\ldots]\ldots[\ldots]$ be a partition as in case b) and let $\mathcal{P}_0 = [1][2][3\ldots d]$ be the partition associated to $F^d(1,2)$. Since $\mathcal{P}$ is a sub partition of $\mathcal{P}_0$ it follows that $W_{\mathcal{P}} \subset W_{\mathcal{P}_0}$ and that $\mathbb{F}(\mathcal{P})$ fibers over $F^d(1,2)$, that is the flags in $\mathbb{F}(\mathcal{P})$

contain one and two dimensional subspaces. Let $\{e_1,\ldots,e_d\}$ be a basis as stated in Proposition 2 and denote by f_β the flag on $\mathbb{F}(\mathcal{P})$ as well as on $\mathbb{F}^d(1,2)$ built from it. Since the projection of $\mathbb{F}(\mathcal{P})$ onto $\mathbb{F}^d(1,2)$ is equivariant, the N_β−orbits and the invariant control set on $\mathbb{F}(\mathcal{P})$ project onto the N_β−orbits and invariant control set on $\mathbb{F}^d(1,2)$ respectively. This and Proposition 5 imply that the invariant control set on $\mathbb{F}^d(1,2)$, say $C_{1,2}$ is contained in $N_\beta f_\beta$. Now, there is the

Lemma 7 *Let $\pi : St_2(d) \longrightarrow \mathbb{F}^d(1,2)$ be the canonical projection. Then $\pi^{-1}(N_\beta f_\beta)$ is open dense and has four connected components. Moreover, M/L acts simply transitively on the connected components*

Proof. Choose an inner product $< \cdot, \cdot >$ on $\mathbb{R}^d$ such that β is orthonormal. $N_\beta f_\beta$ is the set of flags $(span\{v_1\} \subset span\{v_1, v_2\})$ such that

$$\begin{aligned} \pm v_1 &= f_1 + \alpha_2 f_2 + \cdots + \alpha_d f_d \\ \pm v_2 &= f_2 + \gamma_3 f_3 + \cdots + \gamma_d f_d \end{aligned}$$

that is

$$< v_1, f_1 > \neq 0 \qquad < v_2, f_2 > \neq 0$$

The set of orthonormal frames which satisfy these two conditions has four connected components. The fact that M/L acts simply transitively on the connected components comes from the transitive right action of M on the fiber. □

To conclude the analysis for case b) for connected S it is enough to observe now that the $i.c.s$'s are connected and contained in $\pi^{-1}(C_{1,2})$ so each one is contained in a connected component of $\pi^{-1}(N_\beta f_\beta)$, so there is at least four $i.c.s$'s on $St_2(d)$, and there is exactly four because of the transitivity of M/L on the fiber and the fact that elements of M/L maps $i.c.s$'s into $i.c.s$'s.

Turn now to case c) also with the assumption that S is connected. First of all, the form of $W(S) = W_{\mathcal{P}}$ and Proposition 2 ensures the existence of

$$diag\{\lambda_1, \lambda_2, \lambda_2, \ldots\} \in intS \qquad \lambda_1 > \lambda_2 > 0.$$

Perturbing this matrix as in case a), it is shown the existence of

$$diag\{\mu_1, -\mu_2, -\mu_2, \ldots\} \in intS \qquad \mu_1, \mu_2 > 0,$$

which is a representative in $P(1,2)$ of $(1,-1,-1)$ so this element of M/L belongs to $\tilde{M}$. Therefore $\tilde{M}$ has two or four elements. It actually has two elements. To show this it is needed the

Lemma 8 *Let $W(S)$ be as in c) and denote by C_1 the invariant control set on $\mathbb{R}P^{d-1}$. Then $C_{1,2} = \pi^{-1}(C_1)$ where π is the projection of f onto $\mathbb{R}P^{d-1}$*

Proof. $\mathcal{P}$ is a refinement of $\mathcal{P}_0 = [1][2,3,\ldots,d]$ which is associated to $\mathbb{R}P^{d-1}$ and is not a refinement of $[1][2][3,\ldots,d]$ which is associated to $F^d(1,2)$. Therefore $\mathbb{F}(\mathcal{P}) = \mathbb{F}^d(1,r_2,\ldots,r_s)$ with $r_2 \geq 3$ (r_2 might be d in which case $\mathbb{F}(\mathcal{P}) = \mathbb{R}P^{d-1}$).

Let $C_{\mathcal{P}}$ be the *i.c.s* on $\mathbb{F}(\mathcal{P})$. Then the invariant control set on the full flag manifold, say C is the inverse image of $C_{\mathcal{P}}$. Also, $C_{1,2}$ is the projection of C. Since $r_2 \geq 3$ this ensures that $C_{1,2} = \pi^{-1}(C_1)$. In fact, pick a flag

$$f^0 = (V_1^0 \subset V_{r_1}^0 \subset \cdots \subset V_{r_s}^0) \in \mathbb{F}(\mathcal{P}).$$

Its inverse image is the set of all flags in $\mathbb{F}^d(1,\ldots,d-1)$ which completes f^0. Since $dim\, V_{r_2} > 2$, every two-dimensional subspace appear in a full flag which projects into f^0. This means that the projection into $\mathbb{F}^d(1,2)$ of the inverse image of f^0 is the subset

$$(V_1^0 \subset V_2) \qquad dim V_2 = 2$$

The lemma follows now by varying f^0 along C_1. □

Now, let $\{e_1,\ldots,e_d\}$ be a basis like in Proposition 2. Since $\mathcal{P}$ is a refinement of the partition associated to the $\mathbb{R}P^{d-1}$, $\mathbb{F}(\mathcal{P})$ fibers over the projective space so that $C_1 \subset N_\beta f_\beta$ where C_1 is the *i.c.s* on $\mathbb{R}P^{d-1}$ and $f_\beta = span\{e_1\}$. But $N_\beta f_\beta$ in projective space is the set of lines spanned by $v \in \mathbb{R}^d$ such that $< e_1, v > \neq 0$ therefore its inverse image in $\mathbb{F}^d(1,2)$ is the set of flags $(span\{v_1\} \subset span\{v_1,v_2\})$ such that $< e_1, v_1 > \neq 0$. The set of orthonormal frames $\{v_1, v_2\}$ adapted to these flags is thus those which satisfy tie inequality. Hence it has two connected components, and an argument like in b) together with the above lemma ensures that S has at least two *i.c.s*'s on $St_2(d)$ if it is connected. Since it was shown before that $\tilde{M}$ has at least two elements, it follows that $\tilde{M} = \{1,(1,-1,-1)\}$ in case c).

Turn now to case d). An argument as in c) shows that $(-1,-1,1) \in \tilde{M}$. It must be checked that the order of $\tilde{M}$ is at most two, that is, that there is at least two *i.c.s*'s. The proof of this is similar to case c), only that instead of $\mathbb{R}P^{d-1}$ it is needed to take the partition $\mathcal{P}_0 = [12][3,\ldots,d]$ which is associated to $Gr_2(d)$. The flags in $\mathbb{F}(\mathcal{P})$ are of the form

$$(V_2 \subset \cdots) \qquad dim V_2 = 2$$

so similar to c) it follows that $C_{1,2}$ is the inverse image of C_2 where C_2 is the invariant control set on $Gr_2(d)$. The same way, $C_2 \subset N_\beta\, span\{e_1, e_2\}$ which is the subset of planes in $Gr_2(d)$ spanned by $\{v_1, v_2\}$ with

$$\begin{aligned} \pm v_1 &= f_1 + \alpha_2 f_2 + \ldots + \alpha_d f_d \\ \pm v_2 &= f_2 + \gamma_3 f_3 \ldots + \gamma_d f_d \end{aligned}$$

that is which

$$\det \begin{pmatrix} < v_1, f_1 > & < v_1, f_2 > \\ < v_2, f_1 > & < v_2, f_2 > \end{pmatrix} = < v_1 \wedge v_2, f_1 \wedge f_2 > \neq 0$$

Therefore the orthonormal frames adapted to this set has two connected components so $\tilde{M} = \{1,(-1,-1,1)\}$.

Summarizing, there is the

Theorem 9 *Suppose $W(S) = W_{\mathcal{P}}$ with $\mathcal{P}$ divided in the following cases*

a) $\mathcal{P} = [1,2,3,\ldots]\ldots[\ldots]$

b) $\mathcal{P} = [1][2][3,\ldots]\ldots[\ldots]$

c) $\mathcal{P} = [1][2,3,\ldots]\ldots[\ldots]$

d) $\mathcal{P} = [1,2][3,\ldots]\ldots[\ldots]$

Assume for cases b),c) and d) that S is connected.
Then

In case a) $\tilde{M} = M/L$ *and there is just one invariant control set for S on $St_2(d)$.*

In case b) $\tilde{M} = \{1\}$ *and there are four i.c.s's*

In case c) $\tilde{M} = \{1,(1,-1,-1)\}$ *so there are two i.c.s's on $St_2(d)$ and the frames belonging to an i.c.s and adapted to a given flag $(V_1 \subset V_2)$ have* **opposite orientation** *inside V_2.*

In case d) $\tilde{M} = \{1,(-1,-1,1)\}$ *so there are two i.c.s's on $St_2(d)$ and the frames belonging to a given i.c.s and adapted to a given flag $(V_1 \subset V_2)$ are* **positively oriented** *inside V_2.*

Remarks

1. In none of the cases the group $\{1,(-1,1,-1)\}$ appears. This is due to the way as M/L is realized and means that the frames $\{u,v\}$ and $\{-u,v\}$ belong to the same *i.c.s.* only if there is just one *i.c.s.*, that is, if $\{-u,v\}$ is attainable from $\{u,v\}$ then $\{-u,-v\}$ and $\{u,-v\}$ are also attainable.

2. The case $b),c)$ and $d)$ do not hold for non connected S. As an example, take d to be even and put $S' = \{-1\}S$ where S is a given semigroup and $-1 \in Sl(d,\mathbb{R})$ is the opposite of the identity. The characterization of $W(S)$ by diagonalizable elements in $intS$ shows that $W(S') = W(S)$. However, for S' in case $b)$, $\tilde{M} = \{1,(-1,-1,1)\}$ and in case $c)$, $\tilde{M} = M/L$.

3. In case $a)$ and $c)$ the *i.c.s* 's contain frames adapted to a flag which are in opposite orientation while positive orientation holds in $b)$ and $d)$. This is relevant for rotation numbers (c.f.[2, section 4]), causing them to be zero or not. The geometric difference from cases $a)$ and $c)$ to cases $b)$ and $d)$ is that in $a)$ and $c)$ the semigroup *turns around* the Grassmannian $Gr_2(d)$ while this turning does not happen in cases $b)$ and $d)$. Here turning around means, as above, that the invariant control set on $Gr_2(d)$ is not contained in a nilpotent orbit of the type $N_\beta f_\beta$.

Proposition 10 *Let C_2 be the i.c.s. on $Gr_2(d)$. Then in cases $b)$ and $d)$, C_2 is contained in a N_β-orbit for some β.*

Proof. In case b), $\mathcal{P}$ is a refinement of $[1][2][3,\ldots,d]$ which is associated to $F^d(1,2)$ and in case d) $\mathcal{P}$ is a refinement of $[1,2][3,\ldots]$ so in both cases $\mathbb{F}(\mathcal{P})$ fibers over $Gr_2(d)$ and the statement follows from Proposition 5. □

Proposition 11 *In cases $a)$ and $c)$, C_2 is not contained in N_β-orbits.*

Proof. In fact, realizing the oriented Grassmannian as a subset of the sphere in the two fold exterior product, an N_β−orbit becomes a set of the type $\{\xi \in \bigwedge^2 \mathbb{R}^d :< \xi,\eta >:> 0\}$. Since in $St_2(d)$ there are frames with opposite orientation in the same *i.c.s*, the same happens in the oriented Grassmannian, showing that in the Grassmannian itself C_2 is not contained in an N_β−orbit.□

Finally, in cases b) and d), the *i.c.s*'s can not contain frames of the same plane which are of opposite orientation (e.g. $\{u,v\}$ and $\{v,u\}$ which do not belong to

the same element of $F^d(1,2)$). This is because C_2 is contained in an N_β−orbit and these orbits on the oriented Grassmannian do not contain planes with different orientation.

References

[1] Arnold,L. W. Kliemann e E. Oeljeklaus : *Lyapunov exponents of linear stochastic systems.* In Lyapunov Exponents. Proceedings of a workshop. LNM-Springer n° **1186** (1986).

[2] Arnold,L. and L. San Martin: *A multiplicative ergodic theorem for rotation numbers.* J. of Dyn. and Diff. Eqs., vol.1, n°1 (1989), 95-119.

[3] Boothby,W. : *A transitivity problem from control theory.* J. Diff. Eqs. 17 (1975), 296-307.

[4] Boothby,W. and E. Wilson : *Determination of the transitivity of bilinear systems.* SIAM on Control, vol. 17, n°2 (1979), 212-221.

[5] San Martin, L. : *Invariant control sets on flag manifolds.* Math. of Control, Signals and Systems, 6 (1993), 41-61.

[6] San Martin,L. and L.Arnold: *A control problem related to the Lyapunov Exponents.* Matemática Aplicada e Computacional, vol.5 (1986), 31-61.

[7] San Martin,L. and P.A.Tonelli: *Semigroup actions on homogeneous spaces.* To appear in Semigroup Forum.

RELATÓRIOS DE PESQUISA — 1994