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RELATÓRIO DE PESQUISA 1994

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Julho RP 31/94

INSTITUTO DE MATEMÁTICA
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ABSTRACT – Let G be a Lie group, $S \subset G$ a subsemigroup and G/H a homogeneous space of G. It is considered here the subsets of G/H, associated to the action of S, which corresponds to the chain control sets for control systems [4]. Their formal definition requires a family $\mathcal{F}$ of subsets of S, and when particularized to a control semigroup recovers its original form. It is shown, under broad conditions, that a chain control set is the intersection of control sets for semigroups generated by the neighborhoods of the subsets in $\mathcal{F}$. This fact permits to prove, for the chain control sets, results similar to those in [12] about control sets on the flag manifolds of semi-simple Lie groups.

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Julho – 1994

Chain Control Sets for Semigroup Actions on Homogeneous Spaces

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27/6/94

Abstract

Let G be a Lie group, $S \subset G$ a subsemigroup and G/H a homogeneous space of G. It is considered here the subsets of G/H, associated to the action of S, which corresponds to the chain control sets for control systems[4]. Their formal definition requires a family $\mathcal{F}$ of subsets of S, and when particularized to a control semigroup recovers its original form. It is shown, under broad conditions, that a chain control set is the intersection of control sets for semigroups generated by the neighborhoods of the subsets in $\mathcal{F}$. This fact permits to prove, for the chain control sets, results similar to those in [12] about control sets on the flag manifolds of semi-simple Lie groups.

AMS Subject classification: 22E15,22E46,93B03,93B05

Key words: Semigroups,control sets, chain control sets, semi-simple Lie groups, flag manifolds

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*This work was done while the author was a PhD student at the University of Campinas

†Research partially supported by CNPq grant 50.13.73/91-8

1 Introduction

One of the principal concepts appearing in the control theories of both continuous and discrete time systems is the semigroup of transformations defined by the flow of the control system, known as the system's semigroup. Many questions about the control system, specially those related to its controllability depends, in fact, only of the action of the semigroup of the system, so that it can be abstracted to arbitrary semigroup actions and solved in a more general setting. This procedure have provided a fruitful interrelation between the theories of Lie semigroups and control systems(as reference sources see [7, 8]).

The present article follows this line of questioning. It investigates the chain control sets for semigroup actions on homogeneous spaces. The chain control sets as well as the control sets for control systems were extensively studied by F.Colonius and W.Kliemann [2, 3, 4, 5, 6] in relation to some dynamical properties rounding the long time behavior of the trajectories of the control system. In the analysis of these sets, a particular attention is put on systems evolving on projective spaces (cf.[4]) because of their interest in the study of the linearized flow and thus in the study of the stability properties, in the sense of Lyapunov, of the system. This being so, one is interested in looking at the control sets and chain control sets for the action of linear semigroups in projective spaces. As to the control sets, this analysis was pursued in [10, 12] (see also [13]) in a situation which encompasses in a natural way the action of a linear semigroup in projective spaces, namely, the action of semigroups in non compact semi-simple Lie groups on the flag manifolds of the group, that is, its Furstenberg boundaries. The analysis of these control sets turned out to be an useful tool for the understanding of some properties of the semigroups in semi-simple Lie groups. For instance, in [12] a subgroup $W(S)$ of the Weyl group W of the Lie group G is built from a semigroup $S \subset G$ with non empty interior in G. The subgroup $W(S)$ reflects geometrical properties of the control sets on the boundaries of G, and the number of such sets is given by a double coset involving $W(S)$. Moreover, $W(S)$ describes exactly the type of diagonalizable elements that can be encountered inside the interior of the semigroup. In this paper we pursue the same kind of results for the chain control sets. Our main interest are in the setting adopted in [10, 12, 13]. There is however a basic question involving the formal definition of a chain control set. Certainly, a chain control set is a subset such that its points can be linked by chains in the semigroup (see Definition 2.1 below). However, it is not realistic to take arbitrary chains in the semigroup because this would, on one hand trivialize the concept, and on the other hand would not cover

the chain control sets for control systems as they require that the chains involve large time trajectories of the system. We define therefore the notion of $\mathcal{F}$-chain control set (Definition 2.2) where $\mathcal{F}$ is a family of subsets of the semigroup. The chain control sets are not, therefore, defined intrinsically from the action of the semigroup, depending on the family $\mathcal{F}$. We discuss these sets in Section 2 below for a general action of a semigroup in homogeneous spaces. After these generalities we go into the analysis of the $\mathcal{F}$-chain control sets on the flag manifolds. Our main technique is based on the fact that it is possible to get the chain control sets as intersections of control sets for semigroups generated by subsets of the original semigroup. This technique works in a very general situation which covers the action on the flag manifolds (see Theorem 3.7 below). After having means of describing the $\mathcal{F}$-chain control sets through control sets, we apply the results of [12] in order to define a semigroup $W_{\mathcal{F}}(S)$ of the Weyl group which furnishes the number of $\mathcal{F}$-chain control sets on the flag manifolds.

2 $\mathcal{F}$-Chain control sets

In this section we discuss the concept of chain control set for actions of subsemigroups of Lie groups in homogeneous spaces. The notion of chain control sets for control systems was extensively studied by F.Colonius and W. Kliemann (cf.[2, 3, 4, 5, 6]). Our purpose here is to extend this notion for semigroups which do not come from a control system. Thus, let G be a Lie group and $S \subset G$ a subsemigroup. It will be assumed throughout that $intS \neq \emptyset$. Although this condition is not needed everywhere, the main results require that S has interior points in G. Let H be a closed subgroup of G and form the homogeneous space G/H. Then S acts on G/H as a semigroup of diffeomorphisms. We fix a distance d in G/H and consider S-chains in G/H according to a family $\mathcal{F}$ of subsets of S.

Definition 2.1 *Let $\mathcal{F}$ be a family of subsets of S. Take $x, y \in G/H$, a real $\epsilon > 0$ and $A \in \mathcal{F}$. A (S, ϵ, A)-chain from x to y consists of $x_0 = x, x_1, \ldots, x_{n-1}, x_n = y$ in G/H and $g_0, \ldots, g_{n-1}$ in A such that $d(g_j x_j, x_{j+1}) < \epsilon$ for $j = 0, \ldots, n-1$.*

A $\mathcal{F}$-chain control set is a subset whose points can be linked by (S, ϵ, A)-chains. Precisely,

Definition 2.2 *A $\mathcal{F}$-chain control set for S on G/H is a subset $E \subset G/H$ that satisfies*

1. *int* $E \neq \emptyset$,

2. $\forall x, y \in E$, *there exists a* (S, ϵ, A)*-chain from* x *to* y, *for all* $\epsilon > 0$ *and* $A \in \mathcal{F}$ *and*

3. E *is maximal with these properties.*

The semigroup S is said to be $\mathcal{F}$-chain transitive if G/H is a $\mathcal{F}$-chain control set, that is, if any two of its points can be linked by (S, ϵ, A)-chains for any $\epsilon > 0$ and $A \in \mathcal{F}$.

We note that although the above definition mentions explicitly a distance in G/H, it is easily seen that the notion of $\mathcal{F}$-chain control set does not change if an equivalent distance is considered. Also, it follows quickly from condition 3 that two $\mathcal{F}$-chain control sets are either disjoint or coincident. On the other hand, a simple application of Zorn's Lemma shows that any subset satisfying the first two conditions is contained in a $\mathcal{F}$-chain control set.

The following families of subsets are the ones which will be highlighted in the sequel.

1. (*Control semigroups*)Let $X_0, X_1, \ldots, X_m$ be right invariant vector fields in G, and consider the control system

$$\dot{x}(t) = X_0(x(t)) + \sum_{i=1}^{m} u_i(t) X_i(x(t)) \tag{1}$$

where $u = (u_1, \ldots, u_m) \in \mathcal{U}$ for some class of admissible controls $\mathcal{U}$. Denote by $\pi(g, u, t)$ the solution of the system at time t given by the control u and starting at $g \in G$. It is given by $\pi(g, u, t) = \pi(1, u, t)g$ where 1 stands here for the identity in G. The attainable set from the identity at time t, $A(t)$ is given by $A(t) = \{\pi(1, u, t) : u \in \mathcal{U}\}$. Their union

$$S = \bigcup_{t \geq 0} A(t)$$

is a subsemigroup of G known as the system's semigroup.

Let $\mathcal{F}_{\text{control}}$ be the family of subsets of S defined by

$$\mathcal{F}_{\text{control}} = \{\bigcup_{t > T} A(t) : T \geq 0\}.$$

Then the $\mathcal{F}_{\text{control}}$-chain control sets on G/H are related to the chain control sets for control systems as defined by F.Colonius and W.Kliemann in [2]. In fact, the above right invariant control system on G induces on G/H a control system whose trajectories are given by $\pi(1, u, t)x, x \in G/H$ so that the set of attainability from x is given by its S orbit. In [2] F.Colonius and W. Kliemann defined a chain control set for the induced system to be a subset $E \subset G/H$ which satisfies

(a) For all $x, y \in E$ and all $\epsilon, T > 0$ there are $x_0, \ldots, x_n \in G/H$ with $x_0 = x, \ldots, x_n = y$, $t_0, \ldots, t_{n-1} > T$ and $u_0, \ldots, u_{n-1} \in \mathcal{U}$ such that

$$d(\pi(1, u_i, t_i)x_i, x_{i+1}) < \epsilon$$

for all $i = 0, \ldots, n-1$.

(b) For all $x \in E$ there exists $u \in \mathcal{U}$ with $\pi(1, u, t)x \in E$ for all possible times $t \geq 0$.

(c) E is maximal with these two properties.

Since, for $t_i > T, \pi(1, u, t_i) \in \bigcup_{t > T} A(t)$, it is readily seen that a chain control set for the induced system is an $\mathcal{F}_{\text{control}}$-chain control set if it has non void interior, a property which holds under the natural assumption that S has interior points in G (cf. [2]) Reciprocally, a $\mathcal{F}_{\text{control}}$-chain control set is a chain control set for the control system as in the above definition. For this statement, the only thing which needs to be checked is that an $\mathcal{F}_{\text{control}}$-chain control set satisfies the second condition, which will be verified below for chain control sets containing control sets.

2. Let $\mathcal{F}_{\infty}$ be the intersections with S of neighborhoods of ∞ in the one-point compactification of G, that is,

$$\mathcal{F}_{\infty} = \{S - K : K \subset S \text{ is compact in } G\}.$$

3. Let $\mathcal{F}_{\infty,S}$ be the neighborhoods of ∞ in the one point compactification of S, that is

$$\mathcal{F}_{\infty,S} = \{S - K : K \text{ is compact in } S\}$$

The $\mathcal{F}_{\text{control}}$-chain control sets are exclusive to the control semigroups while the $\mathcal{F}_{\infty}$ and the $\mathcal{F}_{\infty,S}$-chain controls sets are meaningful for general semigroups. Note that $\mathcal{F}_{\infty,S} \subset \mathcal{F}_{\infty}$, and that they coincide if S is closed in G.

Different chain control sets can be related as follows.

Proposition 2.3 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two families of subsets of S and suppose that for all $B \in \mathcal{F}_2$ there exists $A \in \mathcal{F}_1$ such that $A \subset B$. Then every $\mathcal{F}_1$-chain control set is contained in a $\mathcal{F}_2$-chain control set. In particular, this holds if $\mathcal{F}_2 \subset \mathcal{F}_1$.*

Proof. Let E be a $\mathcal{F}_1$-chain control set and take $B \in \mathcal{F}_2$, $\epsilon > 0$ and $x, y \in E$. By assumption, there exists $A \in \mathcal{F}_1$ with $A \subset B$. Since E is a $\mathcal{F}_1$-chain control set, there exists a (S, ϵ, A)-chain from x to y. As $A \subset B$, there exits also a (S, ϵ, B)-chain from x to y. Since B and ϵ were arbitrary, this shows that E satisfies 2 with $\mathcal{F} = \mathcal{F}_2$ so that it is contained in a $\mathcal{F}_2$-chain control set.□

Corollary 2.4 *The $\mathcal{F}_\infty$-chain control sets are contained in the $\mathcal{F}_{\infty,S}$-chain control set. The same happens with the $\mathcal{F}_{\mathrm{control}}$-chain control sets in place of $\mathcal{F}_{\infty,S}$ if S is a semigroup of control.*

Proof. The first statement follows from the fact that $\mathcal{F}_{\infty,S} \subset \mathcal{F}_\infty$. As to the $\mathcal{F}_{\mathrm{control}}$, suppose first that the control system (1) has restricted controls, that is, each control function u assume values in a compact subset of $I\!R^m$. In this case, the set $\bigcup_{0 \le t \le T} A(t)$ of the attainable points up to time T is relatively compact in G (see e.g. [9, Lemma 4.4]). Therefore, any $B \in \mathcal{F}_{\mathrm{control}}$ contains the complement in S of a compact subset of G, that is, a subset $A \in \mathcal{F}_\infty$. The general case follows by restricting the controls and observing that the sets of attainability for (1) with the controls restricted to a compact subset are contained in the sets of attainability without restriction of the controls.□

As to the converse inclusion, we have.

In order to continue our discussion of the chain control sets, it is convenient to recall some facts about control sets for semigroup actions (see [2, 3, 4, 5, 6] for results about control sets for control systems and [12, 13] for the control sets for the semigroup actions).

A control set for S on G/H is a subset $D \subset G/H$ which satisfies

1. $intD \neq \emptyset$,

2. $\forall x \in D, D \subset cl(Sx)$ and

3. D is maximal with these properties.

Note that contrary to the chain control sets, the control sets depend only of the action of the semigroup needing no reference to some other object. As stated

in 2 above, the control sets are the subsets where the semigroup are approximately transitive. This approximate transitivity can be improved to exact transitivity inside a dense subset of D as follows: let

$$D_0 = \{x \in D : x \in (intS)x\}.$$

be the set of those points of D which are self accessible by $intS$. In general, D_0 may be empty. However, in case it is not empty, it is open and dense in D, for all $x, y \in D_0$ there exists $g \in intS$ such that $gx = y$, and $D_0 = (intS)D \cap D$ (see [12, Proposition 2.2]). We shall refer to D_0 as the set of transitivity of S inside D, and D will be said to be an *effective* control set in case $D_0 \neq \emptyset$. The above definition of control set for semigoup actions differs slightly for the one considered in e.g. [2] for control systems. The difference comes from the assumption that a control set has non empty interior, which is not assumed beforehand in [2], but which turns out to hold in case the system satisfies the accessibility property(condition H in [2]). For right invariant control systems and the systems induced by them on the homogeneous spaces, this property holds in case the control semigroup has non empty interior, which is our basic assumption about the semigroup S. We note, furthermore, that for right invariant control systems, which satisfy the accessibility property, every control set is effective. In fact, in this case it is well known that the identity $1 \in cl(intS)$ so that $(intS)x \cap D \neq \emptyset$ if $x \in intD$.

In the sequel we are interested mainly in those chain control sets which contain the control sets. Note that in the full generality we have been working up to now, it is not clear that the control sets are contained in chain control sets. This is due to the fact that the elements of S in the chains linking points of a $\mathcal{F}$-chain control set are restricted to belong to the subsets $A \in \mathcal{F}$. Therefore, although two points of a control set can be linked, approximately, by the S-action, it is possible that this can not be done by $\mathcal{F}$-chains if the subsets $A \in \mathcal{F}$ are not large enough. Because of this, we consider the following conditions on $\mathcal{F}$.

Definition 2.5 *The family $\mathcal{F}$ is said to satisfy property P_l (respectively P_r) if for every $g \in intS, h \in S$ and $A \in \mathcal{F}$, there exists a positive integer n such that $g^n h \in A$ (respectively $hg^n \in A$)*

Note that by taking $h = g$, both P_l and P_r imply that $g^n \in A$ for some $n \geq 0$.

These properties are satisfied by the families mentioned above. In case of $\mathcal{F}_\infty$, they are satisfied except in the trivial case where $S = G$. In fact, $\{g^n : n \geq 0\}$ is relatively compact in case $g^n h$ or hg^n do not belong to some $A \in \mathcal{F}_\infty$, hence g is contained in a compact subgroup so that $intS$ intercepts a compact subgroup in case $g \in intS$, which shows that $S = G$. Since $\mathcal{F}_{\infty,S} \subset \mathcal{F}_{\infty}$ the same is true for $\mathcal{F}_{\infty,S}$. As to the family $\mathcal{F}_{\mathrm{control}}$ of a control semigroup, both P_l and P_r hold because if $g \in intS$ is attainable in time T by means of a control u, then g^n is attainable in time nT by means of the control u^n, which is the n times concatenation of u by itself (see e.g. [9, Lemma 4.5]).

For families satisfying the P-properties, it is possible to compare their chain control sets with the S-control sets. In order to do that, let us denote, for a subset $A \subset S$ by S_A the subsemigroup of S generated by A. In case A belongs to a family satisfying the P properties, the effective S-control sets are also control sets for S_A:

Proposition 2.6 *Suppose that the family $\mathcal{F}$ satisfies P_l and P_r. Take $A \in \mathcal{F}$ and let S_A denote the subsemigroup generated by A. Let D be an effective control set for S. Then $D \subset cl(S_A x)$ for any $x \in D$.*

Proof. Take $x, y \in D$ and let us show that $y \in cl(S_A x)$. Suppose first that $y \in D_0$. Then there exists $h \in S$ such that $y = hx$ because $D \subset (intS)^{-1}D_0$. Also, there exists $g \in intS$ such that $gy = y$. Since $g^n h \in A$ by P_l, and $g^n hx = y$, we get that $y \in S_A x$. Now, for arbitrary y, pick $z \in D_0$. Then $z \in S_A x$ so it is enough to show that $y \in cl(S_A z)$. For this, take $g \in intS$ such that $gz = z$ and a sequence $h_m \in S$ such that $h_m z \to y$. Then $h_m g^n \in S_A$ for large enough n, by P_r and since $h_m g^n z = h_m z$, this shows that there exists a sequence $g_k \in S_A$ with $g_k z \to y$, concluding the proof of the proposition.□

A consequence of this statement is that the effective control sets for S are contained in $\mathcal{F}$-chain control sets if $\mathcal{F}$ satisfies both P_l and P_r. In fact, the above proposition shows immediately that an effective control set satisfies 2 for every $A \in \mathcal{F}$ and since it has non empty interior, it is indeed contained in a $\mathcal{F}$-chain control set.

A $\mathcal{F}$-chain control set will be said to be *effective* if it contains an effective control set. In case $\mathcal{F}$ satisfies the properties P_l and P_r, a $\mathcal{F}$-chain control set is effective if and only if the subset $E_0 = \{x \in E : x \in (intS)x\}$ is not empty. In fact, $E_0 \neq \emptyset$ if E contains an effective control set by the very definition of the latter. Conversely, any $x \in E_0$ is fixed by some $g \in intS$ so that x belongs to some control set, say D, which is effective (cf. [14]). By the above proposition, D is contained in a $\mathcal{F}$-chain control set if $\mathcal{F}$ satisfies P_l and P_r. Since this chain control set meets E, it follows that $D \subset E$.

We note that under our basic assumption that the control system satisfies the accessibility property, the $\mathcal{F}_{\mathrm{control}}$-chain control sets which contain control sets are

effective because in this case the control sets are effective. This remark shows that the $\mathcal{F}_{\mathrm{control}}$-chain control sets which contain control sets are chain control sets for (1) in the sense of [2].

We conclude this section by showing that under equivariant fibrations, chain control sets are projected into chain control sets. This fact will be needed later in the analysis of the chain control sets on the flag manifolds.

Proposition 2.7 *Let $L_1 \subset L_2$ be closed subgroups of G and denote by $\pi : G/L_1 \to G/L_2$ the canonical equivariant fibration $\pi(gL_1) = gL_2$. Suppose that G/L_1 is compact, and let E be a $\mathcal{F}$-chain control set for S on G/L_1. Then $\pi(E)$ is contained in a $\mathcal{F}$-chain control set for S in G/L_2.*

Proof. Since $intE \neq \emptyset$ and π is an open map, we have that $\pi(E)$ has non empty interior.

Take $\epsilon > 0$, $A \in \mathcal{F}$ and $x', y' \in \pi(E)$. We shall show that there is a (S, ϵ, A)- chain from x' to y'. Pick $x, y \in E$ so that $\pi(x) = x'$ and $\pi(y) = y'$. Since G/L_1 is compact, π is uniformly continuous so that there exists $\delta > 0$ such that $d(\pi(z), \pi(z')) < \epsilon$ if $d(z, z') < \delta, z, z' \in G/L_1$. Let $x_0 = x, x_1, \ldots, x_{n-1}, x_n = y$ in G/L_1 together with $g_0, \ldots, g_{n-1}$in A form a (S, δ, A)-chain from x to y. As $d(g_i x_i, x_{i+1}) < \delta$, we have that $d(\pi(g_i x_i), \pi(x_{i+1})) < \epsilon$ which shows that $\pi(x_i), g_i$ form a (S, ϵ, A)-chain from x' to y'.

Since $\pi(E)$ is $\mathcal{F}$-chain transitive it is contained in a $\mathcal{F}$-chain control set. □

As a converse to this statement, we have.

Proposition 2.8 *Let $\mathcal{F}$ be a family satisfying P_l and P_r and $\pi : G/L_1 \to G/L_2$ as above with G/L_1 compact. Let also, F be an effective $\mathcal{F}$-chain control set for S on G/L_2. Then there exists an effective $\mathcal{F}$-chain control set E for S on G/L_1 such that $\pi(E) \subset F$.*

Proof. Take $x \in E_0$. Then there is $g \in intS$ such that $gx = x$. The fiber over x is compact and invariant under g. Therefore, there exists a minimal set for g, say M, in this fiber. Since each minimal set is contained in the interior of a control set (see [14, Proposition 2.5]), there exists a control set D on G/L_1 which contains M. This control set is effective because $gy \in (intS)D \cap D = D_0$ if $y \in M$ so that $D_0 \neq \emptyset$. Therefore, D is contained in a $\mathcal{F}$-chain control set E. By the last proposition, $\pi(E) \subset F$. □

3 Chain control sets and control sets

A chain in the semigroup S is given by interchanging the action of elements of S with small jumps of the elements of the homogeneous space G/H. Because of this, it is to be expected that the chain control sets for the S action could be obtained as intersections control sets for semigroups generated by neighborhoods of S. This turns out to be the case as will be shown in this section. Such a characterization of the chain control sets permits to apply the results on control sets in the study of the chain control sets.

We shall assume from now on that the homogeneous space G/H where S acts is compact. We assume also that the action of G on G/H is effective, that is, H does not contain normal subgroups. Fix, as before a metric d in G/H. The elements of G are viewed as homeomorphisms of G/H so it is possible to consider in G the metric of uniform convergence in G/H. It is given by

$$d'(g,h) = \sup_{x\in G/H} d(gx,hx),$$

which is a right invariant metric on G. We use the notation $B(A,\epsilon)$ for the ϵ-neighborhood with respect to d' of the subset A :

$$B(A,\epsilon) = \{g \in G : \exists h \in A, d'(h,g) < \epsilon\}.$$

Also, given a subset $A \subset S$, we shall denote by $S_{\epsilon,A}$ the subsemigroup of G generated by the ϵ-neighborhood of A in G :

$$S_{\epsilon,A} = < B(A,\epsilon) > .$$

We have,

Proposition 3.1 *Let $x,y \in G/H, A \subset S$. Then*

1. *there exists a (S,ϵ,A)-chain from x to y if $y \in S_{\epsilon,A}x$, and*

2. *There exists a (S,ϵ',A)-chain from x to y for every $\epsilon' > \epsilon$ if $y \in cl(S_{\epsilon,A}x)$.*

Proof.

1. Since $y \in S_{\epsilon,A}x$, there exists $g \in S_{\epsilon,A}$ such that $y = gx$. It follows from the definition of $S_{\epsilon,A}$ that $g = g_{k-1}\cdots g_0$ with $g_i \in B(A,\epsilon), i = 0,\dots k-1$. Choose $h_0,\dots,h_{k-1} \in A$ such that $d'(h_i,g_i) < \epsilon, i = 0,\dots,k-1$. Then the sequences $x_0 = x, x_1 = g_0x_0,\dots,x_k = g_{k-1}x_{k-1} = y$ and $h_0,\dots,h_{k-1} \in A$ form a (S,ϵ,A)-chain from x to y. In fact,
$$\begin{aligned} d(h_{i-1}x_{i-1},x_i) &= d(h_{i-1}x_{i-1},g_{i-1}x_{i-1}) \\ &\le d'(h_{i-1},g_{i-1}) \\ &< \epsilon \end{aligned}$$
for any $i = 1,\dots,k$, which shows the existence of a (S,ϵ,A)-chain from x to y.

2. Pick $y \in cl(S_{\epsilon,A}x)$. Then there exists a sequence $g_n \in S_{\epsilon,A}$ such that g_nx converges to y. Take $\epsilon' > \epsilon$, and let n_0 be such that $d(g_{n_0}x,y) < \epsilon' - \epsilon$. As in the proof of 1, there are $h_0,\dots,h_{n-1} \in A$, which together with $y_0 = x, y_1,\dots,y_n = g_{n_0}x$ becomes a (S,ϵ,A)- chain from x to $g_{n_0}x$. Changing in this sequence $g_{n_0}x$ by y, a (S,ϵ',A)-chain is obtained. In fact,
$$\begin{aligned} d(h_{n-1}y_{n-1},y) &\le d(h_{n-1}y_{n-1},g_{n_0}x) + d(g_{n_0}x,y) \\ &< \epsilon + (\epsilon'-\epsilon) = \epsilon' \end{aligned}$$
and $d(h_{i-1}y_{i-1},y_i) < \epsilon < \epsilon'$ for $i = 1,\dots,n-1$. □

The above statement shows that the points reachable through the semigroup generated by a perturbation of A are also reachable by (S,ϵ,A)-chains. We shall now get the converse of this statement by showing that the points reachable by chains can be reached by the action of the perturbed semigroup. For this it will be needed the following assumption about the action of G on G/H.

Hipothesis **H** : There are constants $c > 0$ and $\eta > 0$ such that the action of G on G/H satisfies $\forall x \in G/H, \forall y \in B_\eta(x)$ there exists $k \in G$ with $kx = y$ and $d(kx,x) \ge cd'(k,1)$.

For actions which satisfy **H**, it is possible link points which are close enough in G/H by elements of G whose distance to the identity are not too bigger than the distance between the points. Note that as $d'(k,1) = \sup_{y\in G/H} d(ky,y)$ and $d(kx,x) \ge cd'(k,1)$, we have that $c \le 1$, and for a given x it is possible to take $c = 1$ if and only if x is a point for which the sup in $d'(k,1)$ is attained.

Proposition 3.2 *Assume **H** holds and let ϵ be such that $0 < \epsilon < \eta$. Also, let $x_0,\dots,x_n \in G/H$ and $h_0,\dots,h_{n-1} \in G$ be sequences forming a (S,ϵ,A)-chain from x_0 to x_n. Then there exists $g \in S_{\epsilon',A}$ with $gx_0 = x_n$ where $\epsilon' = \epsilon/c$.*

Proof. Since $d(h_i x_i, x_{i+1}) < \epsilon < \eta$, we have, by **H**, that there are $k_i \in G$ such that

$$d(x_{i+1}, h_i x_i) = d(k_i h_i x_i, h_i x_i) \geq cd'(k_i, 1)$$

$i = 0, \ldots, n-1$, so that $d'(k_i, 1) < \epsilon/c = \epsilon'$. Let $g_i = k_i h_i$. Then $d'(g_i, h_i) = d'(k_i h_i, h_i) = d'(k_i, 1) < \epsilon'$ because of the right invariance of d'. Hence, $g_i \in B(A, \epsilon')$. However, $g_i x_i = k_i h_i x_i = x_{i+1}$, so that $x_n = g_{n-1} \ldots g_0 x_0$ which shows the desired result. □

This statement together with Proposition 3.1 show that the chain control sets in homogeneous spaces satisfying **H** can be studied through the control sets of the semigroups $S_{\epsilon,A}$. Before embarking on the discussion of this relation, we will ensure that hypothesis **H** holds for a large class of homogeneous spaces, namely those for which there is a compact subgroup $K \subset G$ which acts transitively in G/H. This class of homogeneous spaces includes the flag manifolds, which will be considered later.

Let K be a compact group and K/L a homogeneous space of K. It is well known that in K/L there is a K-invariant Riemannian metric $< \cdot, \cdot >$. This metric is built as follows. Let $\mathbf{k}$ be the Lie algebra of K and choose in it an inner product which is $Ad(K)$-invariant, that is, $< Ad(k)X, Ad(k)Y > = < X, Y >$ for all $X, Y \in \mathbf{k}$ and $k \in K$. Fixing $x \in G/H$ let $\mathbf{k}_x$ be the isotropy algebra at x, and denote by P_x the orthogonal projection onto $\mathbf{k}_x^\perp$ with respect to $< \cdot, \cdot >$, where $\mathbf{k}_x^\perp$ stands for the orthogonal complement to $\mathbf{k}_x$. Then the desired Riemannian metric is defined as

$$< \tilde{X}(x), \tilde{Y}(x) > = < P_x X, P_x Y >$$

where $\tilde{X}(x) = d/dt\,(\exp(tX)x)_{t=0}$ stands for the vector field induced by X on K/L. Of course, this expression defines an inner product in $T_x(K/L)$ as every $v \in T_x(K/L)$ is given as $v = \tilde{X}(x)$ for some $X \in \mathbf{k}$. Moreover, this Riemannian metric is K-invariant, that is, $< k_* u, k_* v > = < u, v >$ for all $k \in K$ and $u, v \in T_x(K/L)$, where k_* denotes the differential of the mapping $k : K/L \to K/L$.

We take in K/L the distance d given by this Riemannian metric. Since for $X \in \mathbf{k}$, $|\tilde{X}(x)| = < P_x X, P_x X >^{1/2}$, we have that $|\tilde{X}(x)| \leq |X|$. This inequality implies that the length of the curve $t \to \exp(tX), t \in [0,1]$, which is given by $\int_0^1 |\tilde{X}(\exp(tX)x)| dt$ is bounded above by $|X|$. Since the distance between two points is given by the infimum of the lengths of the curves joining them, we get, therefore, that $d(\exp(X)x, x) \leq |X|$. The inequality required in **H** is a kind of local equivalence between d and the norm in $\mathbf{k}$. We state

Proposition 3.3 *The homogeneous space K/L with K compact and the distance d given by a Riemannain metric satisfies* **H**.

Proof. Fix $x_0 \in K/L$. We check first that it is enough to show that **H** holds at x_0. So suppose that there are $c > 0$ and $\eta > 0$ such that for every $z \in B_\eta(x_0)$ there exists $k \in K$ with $kx_0 = z$ and $d(kx_0, x_0) \geq cd'(k, 1)$. Take $x \in K/L$ and $u \in K$ such that $x = ux_0$, and let $k' = uku^{-1}$. We have $k'x = y$ and since the Riemannian metric is K-invariant the same holds for the distance d, so that d' is both left and right invariant in K. Therefore, $d(k'x, x) = d(kx_0, x_0) \geq cd'(k, 1) = cd'(k', 1)$ and **H** holds at x as well with the same constants c and η.

In order to verify **H** at x_0, let $f : \mathbf{k} \to K/L$ be defined by $f(X) = \exp(X)$. Its differential at the origin is $(df)_0(X) = \tilde{X}(x_0)$ which shows that $\ker(df)_0 = \mathbf{k}_{x_0}$. Therefore, df_0 is one-to-one in $\mathbf{k}_{x_0}^\perp$ so that f is a diffeomorphism from a neighborhood of the origin of $\mathbf{k}_{x_0}^\perp$ onto an open ball $B_{\eta'}(x_0)$ for some $\eta' > 0$. Now, the distance d comes from a Riemannian metric so its square is differentiable. Since $X \to |X|^2$ is also differentiable, we have that

$$\liminf_{|X| \to 0} \frac{d(e^X x_0, x_0)}{|X|} > 0$$

for $X \in \mathbf{k}_{x_0}$. It follows that $d((\exp X)x_0, x_0)/\,|X|$ is bounded below in some neighborhood U of the origin so that there exists a constant $c > 0$ such that $d((\exp X)x_0, x_0) \geq c|X|$ for all $X \in U$. However, $d'(\exp X, 1) \leq |X|$, so that $d((\exp X)x_0, x_0) \geq cd'(\exp X, 1)$, and since f is onto a neighborhood of x_0, this shows that K/L satisfies **H** at x_0 and hence the proposition.□

This proposition implies immediately that a homogeneous space G/H satisfies **H** if G has a compact subgroup which acts transitively on G/H.

Corollary 3.4 *Let G/H be a homogeneous space and suppose that there exists a compact subgroup $K \subset G$ such that K acts transitively on G/H. Then G/H satisfies* **H** *with the distance d given by a Riemannian metric invariant by K.*

Recalling now Propositions 3.1 and 3.2 we get the following relationship between the (S, ϵ, A)-chains and the action of $S_{\epsilon,A}$.

Proposition 3.5 *Let G be a Lie group and G/H a homogeneous space which satisfies the condition of the above corollary. Let $S \subset G$ be a subsemigroup and take a subset $A \subset S$. Let also, $x, y \in G/H$. Then there exists a (S, ϵ, A)-chain from x to y if $y \in S_{\epsilon,A}x$. Conversely, $y \in S_{\epsilon/c,A}x$ if there exists a (S, ϵ, A)-chain from x to y.*

This last statement relating chain attainability with the action of a semigroup, permits to characterize the $\mathcal{F}$-chain control sets as intersections of control sets for the semigroups $S_{\epsilon,A}$. For this characterization, we shall need that the family $\mathcal{F}$ satisfies the properties P_l and P_r of Section 2. When these properties hold, we have at our disposal Proposition 2.6 so that the fact that $S_A \subset S_{\epsilon,A}$ for any $\epsilon > 0$, implies that the effective control sets for S are contained in the control sets for $S_{\epsilon,A}$. Note that under these circumstances, the $S_{\epsilon,A}$-control set containing the effective S-control set is also effective. Indeed, if $g \in intS$ fixes a point y then the same holds for $g^n, n \geq 0$, and clearly, $g^n \in intS_{\epsilon,A}$. For the perturbed semigroup $S_{\epsilon,A}$ Proposition 2.6 can be made more precise. In fact, we have

Proposition 3.6 *Assume that $\mathcal{F}$ satisfies P_l and P_r, and suppose furthermore that the homogeneous space under consideration satisfies* **H**. *Then $D \subset S_{\epsilon,A}x$ for any $x \in D$, $A \in \mathcal{F}$ and ϵ small enough.*

Proof. Let $x, y \in D$. Let us show that $y \in S_{\epsilon,A}x$. By Proposition 2.6, there exists, $g \in S_A$ such that gx is near y. Since S_A is generated by A, g is of the form $g = g_1 \dots g_n$ with $g_i \in A$. Using **H**, there is $k \in G$ with $d'(k, 1) < \epsilon$ and $kgx = y$ so that $kg \in S_{\epsilon,A}$ and $y \in S_{\epsilon,A}x$.□

We can now state the main result of this section which characterizes the $\mathcal{F}$-chain control sets as intersections of control sets for the semigroups $S_{\epsilon,A}$.

Theorem 3.7 *Suppose that G/H satisfies* **H** *and let $\mathcal{F}$ be a family of subsets of S satisfying P_l and P_r. Let D be an effective control set for S on G/H and for $\epsilon > 0$ and $A \in \mathcal{F}$, denote by $D_{\epsilon,A}$ the $S_{\epsilon,A}$-control set containing D. Then*

$$E = \bigcap_{\epsilon,A} D_{\epsilon,A}$$

is the only $\mathcal{F}$-chain control set containing D.

Proof. Clearly, $intE \neq \emptyset$ as $D \subset E$. Also, for any $x, y \in E, y \in cl(S_{\epsilon',A}x)$ for all $\epsilon' > 0$ and $A \in \mathcal{F}$. Therefore, there exists a (S, ϵ, A)-chain from x to y for all $\epsilon > 0$ and $A \in \mathcal{F}$, which shows that E is chain transitive. It lasts only to verify the maximality of E. For this, take $x \notin E$ and $y \in E$ and suppose that for every $\epsilon > 0$ and $A \in \mathcal{F}$ there are (S, ϵ, A)-chains from x to y and from y to x. Since G/H satisfies **H**, Proposition 3.5 shows that $x \in S_{\epsilon,A}y$ and $y \in S_{\epsilon,A}x$ for all $\epsilon > 0$ and $A \in \mathcal{F}$. However, $y \in D_{\epsilon,A}$, so that $x \in D_{\epsilon,A}$ for all ϵ, A contradicting the assumption that $x \notin E$. Hence, there is no chain either from x to y or from y to x, which shows that $E \cup \{x\}$ is not contained in any chain control set and so the maximality of E follows.□

As to the $\mathcal{F}$-chain control sets containing the invariant control sets, there is the following result which improves the above theorem when there is just one invariant control set.

Proposition 3.8 *Let the assumptions be as in the previous theorem and suppose moreover that there is just one invariant control set for S in G/H. Denote it by D and keep the notations as before. Then $D_{\epsilon,A}$ is the only $S_{\epsilon,A}$-invariant control set so that $D_{\epsilon,A}$ and E are closed subsets. Moreover,*

$$E = \bigcap_{\epsilon,A} (D_{\epsilon,A})_0 .$$

Proof. A well known fact about invariant control sets is that there is only one invariant control set for S if and only if

$$\bigcap_{x \in G/H} cl(Sx) \neq \emptyset$$

(c.f. [1, Lemma 3.1]). In this case, the invariant control set is given by this intersection. Of course, a similar statement holds also for $S_{\epsilon,A}$ so that in order to prove the proposition it is enough to show that

$$D_{\epsilon,A} \cap \bigcap_{x \in G/H} cl(S_{\epsilon,A}x) \neq \emptyset.$$

This being so, pick $x \in G/H$ and choose $g \in intS$ such that $gx \in D_0$ this is possible because $D \subset (intS)^{-1}x$ for every $x \in D_0$. Since $\mathcal{F}$ satisfies $P_l, g^n \in A$ for some integer n. We have that $g^n x \in D$ because D is invariant. As $D \subset D_{\epsilon,A}$, and $D_{\epsilon,A}$ is a control set for $S_{\epsilon,A}$ it follows that $D_{\epsilon,A} \subset cl(S_{\epsilon,A}x)$ showing that $D_{\epsilon,A}$ is indeed an invariant control set. As to the last statement, take $y \in E$ and $x \in D_0$. Then $x \in (D_{\epsilon,A})_0$ for every ϵ, A, and by Proposition 3.2, $y \in S_{\epsilon,A}x$ for all ϵ, A. The invariance of $D_{\epsilon,A}$ implies then that $y \in (D_{\epsilon,A})_0$ for arbitrary ϵ, A. □

4 Chain control sets on flag manifolds

We specialize now the preceding results on chain control sets to actions of subsemigroups of semi-simple Lie groups on their flag manifolds(Furstenberg boundaries). Thus, we let, in this section, G be a semi-simple Lie group and $S \subset G$ a semigroup with non empty interior in G. Also, we consider the homogeneous space G/H to be one of the (finite in number) flag manifolds of G, that is, H is a parabolic subgroup of G. We refer the reader to [15, 16] for the detailed theory of parabolic subgroups and flag manifolds. We refer also to [10, 12, 13] for an account according to the needs of this paper. In [12] the control sets on the flag manifolds were studied. There, their sets of transitivity were characterized as sets of fixed points of certain elements in $intS$. That characterization provided a means for counting and distinguish the control sets. In the sequel, we shall recall those results on control sets and then use them together with Theorem3.7 in order to have a procedure for counting the chain control sets on the flag manifolds.

Let $\mathbf{g}$ be the Lie algebra of G and select a Cartan decomposition $\mathbf{g} = \mathbf{k} + \mathbf{s}$ of $\mathbf{g}$, with Cartan involution θ, where $\mathbf{k}$ is a compactly embedded subalgebra and $\mathbf{s}$ is its orthogonal complement with respect the Cartan-Killing form. Let $\mathbf{a} \subset \mathbf{s}$ be a maximal abelian subalgebra, which in the sequel will be referred to as a split subalgebra. We have that $\mathbf{a}$ decomposes in Weyl chambers. Select one of them, say $\mathbf{a}^+$. Associated with this chamber there is a system of positive roots, denoted by Δ^+. The simple system of roots generating Δ^+ is denoted by Π, and the set of all roots is denoted by Δ, which is given by $\Delta = \Delta^+ \cup (-\Delta^+)$. For a root $\alpha \in \Delta$, we let $\mathbf{g}_\alpha = \{X \in \mathbf{g} : ad(H)X = \alpha(H)X\}$ be its root space. The subalgebra $\mathbf{n}^+ = \sum_{\alpha\in\Delta^+} \mathbf{g}_\alpha$ is nilpotent. It provides the Iwasawa decomposition of $\mathbf{g}$

$$\mathbf{g} = \mathbf{k} + \mathbf{a} + \mathbf{n}^+$$

with corresponding global decomposition $G = KAN^+$, where $K = \exp\mathbf{k}$, $A = \exp\mathbf{a}$ and $N^+ = \exp\mathbf{n}^+$. The subgroup A is a split subgroup of G. We use the notation $A^+ = \exp\mathbf{a}^+$ and refer to this subset as a Weyl chamber in G. For a subset $\Theta \subset \Pi$ we denote by P_Θ the parabolic subgroup defined by Θ. Its Lie algebra is the subalgebra $\mathbf{p}_\Theta$ generated by $\mathbf{n}_\Theta = \sum_{\alpha\in\pm\Theta} \mathbf{g}_\alpha$, and P_Θ is the normalizer of $\mathbf{p}_\Theta$. The associated flag manifold is $B_\Theta = G/P_\Theta$. When Θ is the empty set the subscript is omitted so that P is a minimal parabolic subgroup and $B = G/P$ the maximal flag manifold of G. We denote by b_0 the origin in G/P.

Of interest here is the action of the elements of A^+ in B. Given $h \in A^+$, it has a finite number of fixed points in B. They are $\{wb_0 : w \in W\}$ where W is the Weyl group $W = M^*/M$ where M^* is the normalizer and M is the centralizer of A in K. These fixed points are hyperbolic with unstable and stable manifolds given by N^+wb_0 and N^-wb_0 respectively where $N^- = \theta(N^+)$ is the nilpotent group opposed to N^+. The orbit N^-b_0 is open and dense so that b_0 is the only attractor for h, the other stable manifolds are lower dimensional. In the sequel we shall refer to wb_0 as the fixed point of type w for the elements of A^+. The choices of these objects

is no unique. In fact, given $g \in G$, a conjugation by g lead to another choice. In particular, the G-conjugates of A, gAg^{-1} are the split subgroups of G. The same way, the subsets gA^+g^{-1} are the Weyl chambers in G. As to the elements of A^+, the elements of gA^+g^{-1} also have a finite number of fixed points in their action on B. They are given by $gwb_0 = (gwg^{-1})gb_0, w \in W$ which are the translates by the Weyl group gWg^{-1} conjugate of W of the attractor gb_0. In the sequel we choose a basic chamber A^+ in the split subgroup A and refer to a fixed point gwb_0, $w \in W$ of an element of gA^+g^{-1} as the fixed point of type w. These fixed points play a central role in the description of the control sets of a semigroup in B.'

We recall now the results of [12] (see also [13]) about control sets on B. As before, we let S be a semigroup with non empty interior in G.

Proposition 4.1 *For each $w \in W$ there exists an effective control set D_w for S on B. Its set of transitivity $(D_w)_0$ consists of the fixed points of type w for the elements of the Weyl chambers in G meeting $intS$. There exists just one invariant control set D_1 whose set of transitivity is the set of attractors for the split elements in $intS$. Moreover, any effective control set on B is D_w for some $w \in W$.$\square$ (cf. [12, Theorems 3.2,3.5])*

Proposition 4.2 *The subset*

$$W(S) = \{w \in W : D_w = D_1\}$$

is a subgroup of W and $W(S)w_1 = W(S)w_2$ if and only if $D_{w_1} = D_{w_2}$ so that $w \in W \to D_w$ fibers through $W(S)\backslash W$ defining a bijection between the set of control sets and $W(S)\backslash W$. $\square$ (cf. [12, Section 4])

Proposition 4.3 *Suppose the basic chamber A^+ is chosen so that $A^+ \cap intS \neq \emptyset$. Then there exists a subset Θ of the simple system of roots Π such that $W(S) = W_\Theta$. In this case, the invariant control set is given as $D_1 = \pi^{-1}(C_\Theta)$ where C_Θ is the only invariant control set on B_Θ. Moreover, there exists h in closure of A^+ and $n \in N^+$ such that h is fixed by W_Θ and $hn \in intS$.$\square$(cf. [12, Corollary 4.4])*

These results are about the control sets on the maximal boundary. The control sets on the other boundaries are obtained from the control sets on the maximal one as follows.

Proposition 4.4 *Let $\pi : B \to B_\Theta$ be the canonical fibration onto the boundary B_Θ. Then $\pi((D_w)_0)$ is the set of transitivity of a control set on B_Θ. Reciprocally,* an effective control set E on B_Θ satisfies $\pi((D_w)_0) = E_0$ for any $w \in W$ such that $(D_w)_0 \cap \pi^{-1}(E_0) \neq \emptyset$, and the set of such control sets on B is not empty. On B_Θ there is just one invariant control set as well.□ (cf. [12, Proposition 5.1])

From now on, we shall denote by D_w^Θ the control set on B_Θ whose set of transitivity is the projection of the set of transitivity of the control set D_w on B. As a complement to this last statement we have the following fact which, although implicit were not proved in [12].

Proposition 4.5 *Let B_Θ be a boundary and denote by $\pi : B \to B_\Theta$ the canonical projection. Then $D_{w_1}^\Theta = D_{w_2}^\Theta$ if and only if $W(S)w_1W_\Theta = W(S)w_2W_\Theta$. Hence the number of control sets on B_Θ equals the order of $W(S)\backslash W/W_\Theta$.*

Proof. Suppose $D_{w_1}^\Theta = D_{w_2}^\Theta$. Then $(D_{w_1})_0$ and $(D_{w_2})_0$ project onto the set of transitivity of the same control set. Therefore, there is, on the same fiber as w_1b_0, a fixed point which belongs to D_{w_2}. Since w_1 maps fiber into fibers and fixed points into fixed points, that fixed point is of the form $w_1w'b_0$ with $w'b_0$ on the same fiber as b_0 so that $w' \in W_\Theta$. Now $D_{w_2} = D_{w_1w'}$ so that $W(S)w_2 = W(S)w_1w'$ showing that the condition is necessary.

Reciprocally, suppose that $w_2 = w''w_1w'$ with $w'' \in W(S), w' \in W_\Theta$. Then $D_{w_2} = D_{w_1w'}$, and since $D_{w_1}^\Theta = D_{w_1w'}^\Theta$ we have that $D_{w_1}^\Theta = D_{w_2}^\Theta$.□

These results apply, in particular to the semigroups $S_{\epsilon,A}$ whose control sets contain the $\mathcal{F}$-chain control sets of S. Because of this, the above statements on the number of control sets on the flag manifolds can be carried out to the chain control sets. In order to do this let, for $\epsilon > 0$ and a subset $A \in \mathcal{F}$, $W_{\epsilon,A}$ be the subgroup $W(S_{\epsilon,A})$ associated to the semigroup $S_{\epsilon,A}$. Consider also the subgroup

$$W_{\mathcal{F}}(S) = \bigcap_{\epsilon,A} W_{\epsilon,A}.$$

We assume from now on that the family $\mathcal{F}$ satisfies both P_l and P_r. Since the flag manifolds satisfy **H**, because a compact subgroup is transitive on it, we are under the situation covered by Theorem 3.7.

For the $\mathcal{F}$-chain control sets, the subgroup $W_{\mathcal{F}}(S)$ works the same way as $W(S)$ for the control sets. Indeed, for each $w \in W$, let E_w stand for the only effective $\mathcal{F}$-chain control set on B which contains D_w, and for a subset Θ of the simple system of roots, let E_w^Θ be the $\mathcal{F}$-chain control set which contains D_w^Θ. With these notations, the above statements for control sets extend to chain control sets as follows.

Proposition 4.6 *With the notations as above, we have*

1. $W_{\mathcal{F}}(S) = \{w \in W : E_w = E_1\}$.
2. $W_{\mathcal{F}}(S)w_1 = W_{\mathcal{F}}(S)w_2$ *if and only if* $E_{w_1} = E_{w_2}$.

Proof.

1. For $\epsilon > 0$, $A \in \mathcal{F}$,and $w \in W$, let $D_w^{\epsilon,A}$ be the corresponding control set for $S_{\epsilon,A}$ in B. By definition, $w \in W_{\mathcal{F}}(S)$ if and only if $w \in W_{\epsilon,A}$ for all ϵ, A, and this holds if and only if $D_w^{\epsilon,A} = D_1^{\epsilon,A}$. Therefore, $w \in W_{\mathcal{F}}(S)$ if and only if
$$\bigcap_{\epsilon,A} D_w^{\epsilon,A} = \bigcap_{\epsilon,A} D_1^{\epsilon,A}.$$
Which shows the first statement, because by Theorem3.7 the left hand side of this equality is E_w while the right one is E_1.
2. $W_{\mathcal{F}}(S)w_1 = W_{\mathcal{F}}(S)w_2$ if and only if $w_2w_1^{-1} \in W_{\mathcal{F}}(S) = \bigcap_{\epsilon,A} W_{\epsilon,A}$ for all ϵ, A, which holds if and only if $W_{\epsilon,A}w_1 = W_{\epsilon,A}w_2$, which in turn is equivalent to $D_{w_1}^{\epsilon,A} = D_{w_2}^{\epsilon,A}$ for all ϵ, A. Theorem3.7 shows then that $E_{w_1} = E_{w_2}$. □

Proposition 4.7 *Keeping the notations as above, we have that $E_{w_1}^\Theta = E_{w_2}^\Theta$ if and only if $W_{\mathcal{F}}(S)w_1W_\Theta = W_{\mathcal{F}}(S)w_2W_\Theta$.*

Proof. Is analogous to the above proposition.□

These statements have as consequences that the number of effective $\mathcal{F}$-chain control sets is given by $W_{\mathcal{F}}(S)$.

Corollary 4.8 *On the maximal boundary B the number of $\mathcal{F}$-chain control sets equals the number of elements in the coset space $W_{\mathcal{F}}(S)\backslash W$ while the number of $\mathcal{F}$-chain control sets on the boundary B_Θ equals the order of the double coset space $W_{\mathcal{F}}(S)\backslash W/W_\Theta$.*□

Apart from the information provided about the number of chain control sets, the subgroup $W_{\mathcal{F}}(S)$ has also something to say about the geometry of the chain control set which contains the invariant control set. We have from Proposition 4.3 that $W(S)$ is the subgroup generated by a subset Θ of the simple system of roots if the basic chamber A^+ is chosen so that it meets *intS*. Keeping fixed this chamber, the fact that $\mathcal{F}$ satisfies P_l or P_r implies that it meets also $intS_{\epsilon,A}$ for every ϵ, A. Therefore, $W_{\epsilon,A}$ is also generated by the reflexions of a subset, say $\Theta_{\epsilon,A}$, of the simple system of roots associated to A^+, that is, $W_{\epsilon,A} = W_{\Theta_{\epsilon,A}}$. Now, we have the following technical lemma about the intersections of the subgroups W_Θ.

Lemma 4.9 *Given any family $\{\Theta_i\}_{i\in I}$ of subsets of the simple system of roots, we have that*

$$\cap_i W_{\Theta_i} = W_{\cap_i\Theta_i}.$$

Proof. In fact, w belongs to a subgroup W_Θ if and only if $wH = H$ for every H in the subspace $\Theta^\perp$ of a annihilated by Θ (cf. [16, Thm.1.1.2.8]). Since the Θ_i's are subsets of a simple system of roots, we have that $(\cap_i\Theta_i)^\perp = \sum_i \Theta_i^\perp$. Therefore, w fixes the elements in $(\cap_i\Theta_i)^\perp$ if it belongs to $\cap_i W_{\Theta_i}$. This shows that $\cap_i W_{\Theta_i} \subset W_{\cap_i\Theta_i}$. Since the reverse inclusion is immediate, the equality follows. □

Applying this lemma to the subgroups $W_{\epsilon,A}$, we get that $W_{\mathcal{F}}(S)$ is generated by the reflections defined by some subset of the simple system of roots as well.

Proposition 4.10 *Suppose $\mathcal{F}$ satisfies P_l and P_r and take a basic chamber A^+ such that $A^+ \cap intS \neq \emptyset$. Let $\Theta_{\epsilon,A}$ be the subset of the simple system of roots associated to A^+ such that $W_{\epsilon,A} = W_{\Theta_{\epsilon,A}}$ and put $\Theta_{\mathcal{F}} = \cap_{\epsilon,A}\Theta_{\epsilon,A}$. Then*

$$W_{\mathcal{F}}(S) = W_{\Theta_{\mathcal{F}}}.$$

Moreover, let $B_{\Theta_{\mathcal{F}}}$ be the flag manifold corresponding to $\Theta_{\mathcal{F}}$, and denote by $E_{\Theta_{\mathcal{F}}}$ the $\mathcal{F}$-chain control set on $B_{\Theta_{\mathcal{F}}}$ which contains the invariant control set. Then

$$E = \pi^{-1}(E_{\Theta_{\mathcal{F}}})$$

where E is the $\mathcal{F}$-chain control set on B containing the invariant control set and $\pi : B \to B_{\Theta_{\mathcal{F}}}$ is the canonical fibration.

Proof. It is needed to prove only the last statement. We have from Proposition 4.3 that $D_{\epsilon,A} = \pi^{-1}(D_{\epsilon,A}^{\Theta_{\epsilon,A}})$ where $D_{\epsilon,A}$ is the $S_{\epsilon,A}$-invariant control set on B and $D_{\epsilon,A}^{\Theta_{\epsilon,A}}$ is the $S_{\epsilon,A}$-invariant control set on the boundary $B_{\Theta_{\epsilon,A}}$ corresponding to $\Theta_{\epsilon,A}$. Since $\Theta_{\mathcal{F}} \subset \Theta_{\epsilon,A}$, it follows that $B_{\Theta_{\mathcal{F}}}$ fibers over $B_{\Theta_{\epsilon,A}}$ so that $D_{\epsilon,A}$ is also the inverse image of the $S_{\epsilon,A}$-invariant control set on $B_{\Theta_{\mathcal{F}}}$ which is denoted by $D_{\epsilon,A}^{\Theta}$. Now, $D_{\epsilon,A}$ and $D_{\epsilon,A}^{\Theta}$ contain the S-invariant control sets on B and on $B_{\Theta_{\mathcal{F}}}$ respectively. Hence $E = \cap_{\epsilon,A}D_{\epsilon,A} = \pi^{-1}(\cap_{\epsilon,A}D_{\epsilon,A}^{\Theta}) = \pi^{-1}(E_{\Theta_{\mathcal{F}}})$. □

We finish this section with the following remarks about chain transitivity on the flag manifolds.

It was shown in [10, Thm.4.2] that the only semigroup with non empty interior of G which satisfies $W(S) = W$ is G itself in case G has finite center. Since, in any case, the action of G kills off the center, this fact implies that S is transitive on any boundary if $W(S) = W$. Since $W_{\mathcal{F}}(S) = \cap W_{\epsilon,A}$, it follows that the semigroups $S_{\epsilon,A}$ are transitive on the flag manifolds provided $W_{\mathcal{F}}(S) = W$ so that this equality implies that S is $\mathcal{F}$-chain transitive on any flag manifold as the chain control sets are intersections of $S_{\epsilon,A}$-control sets. Reciprocally, suppose that S is $\mathcal{F}$-chain transitive on some boundary. Then $S_{\epsilon,A}$ is transitive on this boundary for any $\epsilon > 0$ and $A \in \mathcal{F}$. Now, is was proved in [12, Thm.6.2] that the only semigroup which is transitive on some boundary is G itself if G is simple and has finite center. Again, the fact that the center of G is finite is not relevant as the action of the center on the flag manifolds is trivial . Therefore, under the condition that G is simple, $S_{\epsilon,A} = G$, $S_{\epsilon,A}$ is transitive on every boundary for all ϵ, A, and $W_{\mathcal{F}}(S) = W$ if S is $\mathcal{F}$-chain transitive on some flag manifold. Summarizing, we have

Proposition 4.11 *Assuming the above conditions on $\mathcal{F}$, S is $\mathcal{F}$-chain transitive on any flag manifold provided $W_{\mathcal{F}}(S) = W$. Reciprocally, $W_{\mathcal{F}}(S) = W$ if S is $\mathcal{F}$-chain transitive on some flag manifold and G is simple.*□

It is interesting to have conditions ensuring that a semigroup is chain transitive on the boundaries. Regarding this, it was shown in [10, Lemma 4.1] that a semigroup is transitive on the flag manifolds (and is in fact the whole group if it has finite center) in case it contains in its interior a nilpotent element. Since the chain control sets are intersections of control sets of a semigroup generated by neighborhoods of subsets, it can be seen that a semigroup is chain transitive provided it contains, not necessarily in its interior, nilpotent elements.

Proposition 4.12 *Let $\mathcal{F}$ be a family of subsets of S and suppose that for every $A \in \mathcal{F}$, there exists $n \in A$ such that $n = \exp X$ and $ad(X)$ is nilpotent in $\mathfrak{g}$. Then S is $\mathcal{F}$-chain transitive on any flag manifold.*

Proof. The semigroups $S_{\epsilon,A}$ are generated by a neighborhood of A. Thus $n \in intS_{\epsilon,A}$ if n is as in the statement. It follows then from [10, Lemma 4.1] that $S_{\epsilon,A}$ is transitive on the boundaries. Proposition 3.1 implies then that S is chain transitive.□

A case covered by this proposition is the case when $\mathcal{F} = \mathcal{F}_\infty$ and the semigroup contains an element n such that $n = \exp X$ and $ad(X)$ is nilpotent in g. Since the subset $\{n^k : k \geq 1\}$ is not compact, and n^k is also the exponential of a nilpotent element, the assumption holds for all $A \in \mathcal{F}_\infty$. As an example of a semigroup satisfying these conditions, let $Sl^+(n, \mathbb{R})$ be the semigroup of matrices in $Sl(n, \mathbb{R})$ which have positive entries. This semigroup has non empty interior and contains upper triangular matrices whose diagonal entries are all equals to 1. Since such a matrix is the exponential of a matrix X with $ad(X)$ nilpotent, it follows that $Sl^+(n, \mathbb{R})$ is $\mathcal{F}_\infty$-chain transitive on any flag manifold. Note that this holds despite the non transitivity of $Sl^+(n, \mathbb{R})$ itself.

In order to present another technique for checking chain transitivity, we consider the following example of semigroups of $Sl(n, \mathbb{R})$.

Example 4.13 *Let W be a pointed cone with non void interior in $\mathbb{R}^n$. This means that W is a closed convex cone which does not contain subspaces of positive dimension. Set*

$$S_W = \{g \in Sl(n, \mathbb{R}) : gW \subset W\}.$$

Take $v \in Wv \neq 0$ and complement it to a basis $\beta = \{v, e_2, \ldots, e_n\}$ such that the subspace spanned by $\{e_2, \ldots, e_n\}$ has zero intersection with W. Let H_v be the linear map which in basis β is $diag\{n-1, -1, \ldots, -1\}$. We claim that $\exp(tH_v) \in W$ for all $t \geq 0$. In fact, take $w \in W$ and write $w = a_1 v + \cdots + a_n e_n$. The choice of β implies that $a_1 > 0$. But

$$e^{tH_v} w = e^{t(n-1)} a_1 v + e^{-t} w$$

which shows that $\exp(tH_v)w \in W$. In case $v \in intW$ this equality shows moreover that $\exp(tH_v)w \in intW$. This fact together with a simple argument involving the continuity of the $Sl(n, \mathbb{R})$-action ensures that $\exp(tH_v) \in S_W$ if $v \in intW$, so that S_W has non empty interior. Now, $\exp(tH)$ belongs to a split subgroup of $Sl(n, \mathbb{R})$ and its attractor in the projective space is the line spanned by v. From this it follows that S_W has just two control sets in its action on the projective space. The invariant control set C is the set of lines which are contained in $W \cup -W$, while the other control set is the complement of C in $\mathbb{R}P^{n-1}$ (cf. [12, Thm. 6.11] for details).

These facts about S_W are enough to show that it is $\mathcal{F}_\infty$-chain transitive on $\mathbb{R}P^{n-1}$ and hence in any flag manifold. In fact, for any $A \in \mathcal{F}_\infty$ and positive ϵ the control sets of S_W are contained in control sets for $(S_W)_{\epsilon,A}$. Now, pick v in the boundary of W. Then for $t > 0$ large enough $\exp(tH_v) \in A$ because $\exp(tH_v)$ is not contained in any compact subset. Therefore, $\exp(tH_v) \in int(S_W)_{\epsilon,A}$ and since its attractor in

$\mathbb{R}P^{n-1}$ is the line spanned by v, it follows that this line is in the interior of $(S_W)_{\epsilon,A}$-invariant control set, which therefore meets the complement of C. This shows that $(S_W)_{\epsilon,A}$ has just one invariant control set and hence is transitive on the projective space. Since ϵ, A were arbitrary, it follows that S_W is $\mathcal{F}_\infty$-chain transitive on $\mathbb{R}P^{n-1}$ and thus on any flag manifold.

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