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Lagrangian solutions of Vlasov-Maxwell system and generalized SQG equation

Soluções lagrangianas do sistema de Vlasov-Maxwell e da equação SQG generalizada

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To my wife, Bruna.

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Resumo

Nessa tese, estudamos fluxos lagrangianos locais associado a campos vetoriais com estrutura do tipo onda ou convolução de uma função com núcleo muito singular, isto é, uma função cuja singularidade é pior do que a clássica da teoria de Calderón-Zygmund. Tais campos vetoriais são inspirados no sistema de Vlasov-Maxwell e da equação gSQG, onde ambos são equações não-lineares do transporte e da continuidade. Mais precisamente, provamos existência, unicidade e propriedade de semigrupo para o fluxo e boa colocação de soluções lagrangianas para as equações do transporte e do continuidade, isto é, soluções lagrangianas são soluções fracas ou renormalizadas. Como aplicação, provamos que caso soluções do sistema de Vlasov-Maxwell ou da gSQG sejam regulares o suficiente, então elas possuem estrutura lagrangiana.

Palavras-Chave: Equações diferenciais ordinárias para campos vetoriais não suaves. Equação do transporte. Fluxo regular lagraniano. Soluções renormalizadas. Integrais singulares. Sistema de Vlasov-Maxwell. Equação gSQG.

Classificação por assunto da AMS,2020: 34A12, 35F25, 35Q35, 35Q61, 35Q83, 37C10, 76B03.

Abstract

In this thesis, we study local Lagrangian flows associated to vector fields with wavelike structure or a convolution of a function and a very singular kernel, that is, a function whose singularity is worse than the classical one of Calderón-Zygmund theory. Such vector fields are inspired by the Vlasov-Maxwell system and the generalized SQG equations, which are nonlinear transport and continuity equations. More precisely, we prove existence, uniqueness and the semigroup property for the flow and the well posedness of the induced Lagrangian solutions for the transport or continuity equations, that is, Lagrangian solutions are weak or renormalized solutions. As an application, we prove that if solutions of the Vlasov-Maxwell system or the generalized SQG are regular enough, then they have a Lagrangian structure.

Keywords: Ordinary differential equations with nonsmooth vector fields. Transport equation. Regular Lagrangian flow. Renormalized solutions. Singular integral. Vlasov-Maxwell system. Generalized surface quasi-geostrophic equation.

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List of symbols

$\mathbb{N}$	set of natural numbers.
$\mathbb{R}$, $\mathbb{R}_+$	set of real and nonnegative real numbers, respectively.
$\mathbb{R}^d$	Euclidean space of dimension d.
x_j	$j-$th component of $x \in \mathbb{R}^d$.
$B_r(x)$, B_r	open balls in $\mathbb{R}^d$ of radius r and center x and 0, respectively.
$\partial B_1 = \mathbb{S}^{d-1}$	$d-1$ dimensional unit sphere.
Ω	open subset of $\mathbb{R}^d$.
$\bar{\Omega}$	closure of set Ω.
$\{u > \lambda\}$	set of $x \in \Omega$ such that $u(x) > \lambda$ for some $\lambda \in \mathbb{R}$.
κ_j	$j-$th component of the vector κ.
$\|\kappa\|$	size of the multi-index $\kappa = (\kappa_1, \ldots, \kappa_d) \in \mathbb{N}^d$, $\|\kappa\| = \sum_{i=1}^{d} \kappa_i$.
A^t	transpose of matrix A.
$v^\perp$	perpendicular vector to $v \in \mathbb{R}^2$.
$t \to a$	t tending towards a.
$t \nearrow a$	t increasingly tending towards a.
$t \searrow a$	t decreasingly tending towards a.
$\mathcal{L}^d$	$d-$dimensional Lebesgue measure.
$\|\Omega\|$	the measure of Ω with respect to Lebesgue measure.
$\phi_\# \mu$	pushforward of a measure μ by a function ϕ.
$\mu \llcorner A$	the restriction of a measure μ on a set A.
μ^a	the absolutely continuous part of μ with respect to Lebesgue measure.
μ^s	the singular part of μ with respect to Lebesgue measure.
$\|\mu\|$	total variation of a measure μ.

$\omega(z)$	direction of vector $z \in \mathbb{R}^d$.
δ_{ij}	Kronecker delta.
ϵ^{ijk}	Levi-Civita symbol.
δ_x	Dirac delta measure.
$\mathcal{M}(\Omega)$	space of finite measures in Ω.
$\mathcal{P}(X)$	space of probability measures in a Banach space.
$\mathrm{AC}(I)$	space of absolutely continuous functions in a interval I.
$L^p(\Omega, \mu)$	space of L^p integrable functions of Ω with μ; if μ not explicit, $\mu = \mathcal{L}^d$.
$L^p + L^q(\Omega)$	space of functions $f = g + h$, where $g \in L^p(\Omega)$ and $h \in L^q(\Omega)$.
$L^p_{\mathrm{loc}}(\Omega, \mu)$	space of locally L^p integrable functions of Ω with measure μ.
$L^p_c(\Omega, \mu)$	space of L^p integrable functions with compact support.
$\\| \cdot \\|_{L^p(\mu)}$	norm of L^p space with measure μ.
$L^p_w(\Omega, \mu)$	weak L^p space of functions of Ω with μ; if μ not explicit, $\mu = \mathcal{L}^d$.
$\\|\| \cdot \\|\|_{L^p_w(\mu)}$	“norm” of L^p_w space of measure μ.
$\mathbb{1}_A$	indicator function of the set A.
$\mathrm{dist}(x, \Omega)$	distance function of a point x and the set Ω.
$f\|_K$	restriction of a function f in $K \subset \Omega$.
$\mathcal{F}$	Fourier transform operator.
Id	Identity matrix.
$f * g$	convolution of a function f and a function or distribution g.
$f_t(x)$	function f evaluated at $(t, x) \in [0, \infty) \times \Omega$.
$\partial_i f$	partial derivative of f in the x_i direction, $i \in \{1, \dots, d\}$.
$\partial_t f$	partial derivative of f in the t (time) direction.
f'	derivative of f with interval domain.
$\dot{f}$	derivative of f with respect to t.
$\nabla^\perp f$	perpendicular gradient of f in $\mathbb{R}^2$.

$\operatorname{curl} \boldsymbol{b}$	curl of the vector field $\boldsymbol{b} = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_d)$.
$\operatorname{div} \boldsymbol{b}$	divergence of the vector field $\boldsymbol{b} = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_d)$.
$D\boldsymbol{b}$	Jacobian matrix of vector field $\boldsymbol{b}$.
$D^\kappa f$	partial derivative of f of order $\lvert\kappa\rvert$, where $D^\kappa f = \partial_1^{\kappa_1} \ldots \partial_d^{\kappa_d} f$.
$(-\Delta)^s$	fractional laplacian of order $2s$, where $s \in \mathbb{R}$.
$W^{k+\alpha,p}(\Omega)$	fractional Sobolev space in Ω with $k \in \mathbb{N}$ and $\alpha \in (0,1)$.
$B^{k+\alpha}_{p,q}(\mathbb{R}^d)$	Besov space in $\mathbb{R}^d$ with $k \in \mathbb{N}$ and $\alpha \in (0,1)$.
$\mathrm{BV}(\Omega)$	space of bounded variation functions in Ω.
X_c	space of compactly supported functions in a Banach space X.
$\mathrm{Lip}(\Omega)$	space of Lipschitz functions in Ω.
$C(\Omega)$	space of continuous functions in Ω.
$C^{k,\alpha}(\Omega)$	space of $k-$differentiable functions with kth derivative $\alpha-$Hölder continuous in Ω.
$C^\infty(\Omega)$	space of infinitely differentiable functions of Ω.
$C((0,T);X)$	space of time continuous function with values in Banach space X.

Contents

1 Introduction

One may frame the state of the art of Lagrangian solutions as the following: *if the vector field has structure* $\boldsymbol{b}_t = \Gamma * g$ *for a singular kernel* Γ *and a bounded variation function* g*, then there exists a unique solution of* (1.2) *satisfying the semigroup property. Moreover, it induces Lagrangian solutions* (1.3) *and* (1.5) *which are renormalized solutions of* (1.1) *and* (1.4)*, respectively.* The main motivation of the research of this thesis was to extend this result not by relaxing the regularity of functions g, but "changing the other two symbols" in $\Gamma * g$, that is, by considering different operators instead of the convolution "$*$" (Chapter 4) and more general singular kernels "Γ" not admissible for Calderón-Zygmund theory (Chapter 5). In what follows, we specify the considered changes and important examples which are not directly covered by the theory recalled in Chapter 2.

In the realm of differential equations, the transport equation is sometimes referred as the "simplest" partial differential equation (PDE) when the vector field $\boldsymbol{b}$ is constant; see Evans' book [49, Section 2.1]. Indeed, notice that the initial value problem

$$\begin{cases} \partial_t u + \boldsymbol{b} \cdot \nabla u = 0 & \text{in} \quad [0,\infty) \times \mathbb{R}^d; \\ u_{t=0} = u_0 & \text{on} \quad \mathbb{R}^d \end{cases} \tag{1.1}$$

only has first order derivatives with respect to all variables. Such partial differential equation is a simple model of the time evolution of some quantity u by a velocity $\boldsymbol{b}$, and it can be verified by its explicit solution in a smooth setting $u_t(x) = u_0(x - t\boldsymbol{b})$ if $\boldsymbol{b}$ does not depend on t; we remark that the subscript does not represent partial derivatives, as defined in list of symbols. Such representation of solutions holds for non-constant vector fields which can depend on the quantity u if $\boldsymbol{b}$ is smooth enough. The classical way to approach the aforementioned PDE is by solving the ordinary differential equation (ODE) with $x \in \mathbb{R}^d$ and $s \in [0,\infty)$ fixed

$$\begin{cases} \dot{\boldsymbol{X}}(t,s,x) = \boldsymbol{b}_t(\boldsymbol{X}(t,s,x)) & \text{in} \quad [0,\infty); \\ \boldsymbol{X}(s,s,x) = x. \end{cases} \tag{1.2}$$

The local well-posedness of the above ODE, i.e. existence and uniqueness of $\boldsymbol{X}(\cdot,x)$, is well-known by Cauchy-Lipschitz theory (also known as Picard-Lindelöf theorem) if one has $\boldsymbol{b} \in C([0,T];C_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d))$ with $\boldsymbol{b}_t \in \text{Lip}_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d))$ for all $t \in [0,T]$, or in the context of Sobolev spaces one may relax the hypothesis to $\boldsymbol{b} \in L^1([0,T];W^{1,\infty}_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d))$; for the global well-posedness is ensured if one drops the "loc" in the regularity hypothesis. Here, we considered a finite interval of existence $[0,T]$. Moreover, it satisfies the semigroup property, that is, for any $s, s', t \in [0,T]$, it holds

$$\boldsymbol{X}(t,s,\boldsymbol{X}(s,s',x)) = \boldsymbol{X}(t,s',x).$$

In particular, we have an inverse of $\boldsymbol{X}(t,0,x)$ (take $s=0$ and $s'=t$), with the identity

$$\boldsymbol{X}^{-1}(t,0,x) = \boldsymbol{X}(0,t,x).$$

The function $\boldsymbol{X}$ is often called the flow associated to the vector field $\boldsymbol{b}$, and with it one may construct a solution of the transport equation as

$$u(t,x) = u_0(\boldsymbol{X}(0,t,x)). \tag{1.3}$$

Such construction is called characteristic method, and it can be done for more general PDEs by solving an associated ODE; see Evans' book [49, Section 3.2] for a general overview of the technique.

We also remark the close relation of the transport equation with the continuity equation

$$\begin{cases} \partial_t u + \operatorname{div}(\boldsymbol{b}u) = 0 & \text{in} \quad [0,\infty) \times \mathbb{R}^d; \\ u_{t=0} = u_0 & \text{on} \quad \mathbb{R}^d. \end{cases} \tag{1.4}$$

Indeed, if the vector field $\boldsymbol{b}$ is divergence-free, then both equations coincide. Moreover, even if the latter does not hold, they still share the characteristic ODE (1.2). Therefore, by assuming $\operatorname{div} \boldsymbol{b} \in L^1([0,T]; L^\infty(\mathbb{R}^d))$, we may construct a solution as

$$u(t,x) = \exp\left(-\int_0^t \operatorname{div} \boldsymbol{b}_\tau(\boldsymbol{X}(\tau,t,x))\,\mathrm{d}\tau\right) u_0(\boldsymbol{X}(0,t,x)). \tag{1.5}$$

Concerning the regularity of u solution of (1.1), we notice that a simple computation gives that

$$\operatorname{Lip}(\boldsymbol{X}^{\pm 1}(t,0,\cdot)) \leqslant \exp\left(t \operatorname{Lip}(\boldsymbol{b})\right),$$

where $\operatorname{Lip}(f)$ stands for the uniform with respect to time Lipschitz constant of a function f. Hence, one may conclude by (1.3) that if $u_0 \in C^{0,\alpha}(\mathbb{R}^d)$ for $\alpha \in (0,1]$, then $u \in L^\infty([0,T]; C^{0,\alpha}(\mathbb{R}^d))$, that is, it preserves the regularity of the initial data. Since the continuity equation also has the same characteristic ODE, then the same regularity preservation holds for it.

The regularity assumption $\boldsymbol{b} \in \operatorname{Lip}(\mathbb{R}^d;\mathbb{R}^d)$ is however very restrictive, and many mathematicians tried to relax such hypothesis. An positive result in this direction is due to Osgood, by assuming that $\boldsymbol{b}$ has a uniform in time modulus of continuity ω such that

$$\int_0^1 \omega^{-1}(s)\,\mathrm{d}s = \infty.$$

The classical example of a non-Lipschitz function satisfying the above (known as Osgood condition) is $|x|\log|x|$, and more generally log −Lipschitz functions. However, notice that Hölder functions do not satisfy Osgood condition, and in fact fail uniqueness of solutions of (1.2) by considering the vector field $\boldsymbol{b}_t(x) = |x|^{1/3}$ (for simplicity, we assume $d = 1$). Indeed, we have that $\boldsymbol{X}(t,0,0) \equiv 0$ and $\boldsymbol{X}(t,0,0) = (2t/3)^{3/2}$ both solve the ODE

$$\begin{aligned}\dot{\boldsymbol{X}}(t,0,0) &= \boldsymbol{b}_t(\boldsymbol{X}(t,0,0));\\ \boldsymbol{X}(0,0,0) &= 0.\end{aligned}$$

As for the regularity for vector fields satisfying Osgood condition, as well as general loss of regularity for Besov vector fields, we refer to Bahouri-Chemin-Danchin's book [13, Chapter 3]. Moreover, we also refer to Crippa-Mazzucato and collaborators works [3, 40] and references therein for very recent results concerning loss or regularity for Sobolev vector fields for $d \geqslant 2$, and a classical result by Colombini-Luo-Rauch [34] for the case $d = 2$. Nevertheless, as mentioned in [3], the one dimensional case preserves BV regularity, and also there exists some very weak notion of regularity preserved; see Crippa's thesis [38, Section 7.5] and a very recent result of Bruè and Nguyen of "log-Sobolev" type of preserved regularity [26]. Finally, we also refer to [38, Chapter 1] for a result concerning one-sided Lipschitz condition and a brief summary of these classical results.

Concerning existence and uniqueness of solutions of (1.2), the first major breakthrough was by an indirect proof by DiPerna-Lions in the seminal paper [48], whose proof lies on the renormalization property for Sobolev vector fields. The motivation follows from the computation for smooth solutions u of (1.1)

$$\begin{aligned}\partial_t(\beta\circ u) + \operatorname{div}(\boldsymbol{b}\,\beta\circ u) - \beta\circ u\operatorname{div}\boldsymbol{b} &= \partial_t(\beta\circ u) + \boldsymbol{b}\cdot\nabla(\beta\circ u)\\ &= \beta'(u)\left(\partial_t u + \boldsymbol{b}\cdot\nabla u\right) = 0,\end{aligned}$$

where β is any smooth function. Analogously, we may compute assuming (1.4) that

$$\begin{aligned}\partial_t(\beta\circ u) + \operatorname{div}(\boldsymbol{b}\,\beta\circ u) &= \beta'(u)\left(\partial_t u + \boldsymbol{b}\cdot\nabla u\right) + (\beta\circ u)\operatorname{div}\boldsymbol{b}\\ &= \operatorname{div}\boldsymbol{b}\left(\beta\circ u - u\beta'(u)\right).\end{aligned}$$

We remark that in the transport equation case, we have that a smooth solution u implies that $\beta\circ u$ is also a solution, provided that β is regular enough, e.g. $\beta\in C^1(\mathbb{R})$. However, the divergence structure of the continuity equation is more intricate, with a nonlinear right-hand term. We summarize the idea above as a property for weak solutions of (1.1) by simply integrating by parts the divergence and time derivative.

Condition 1.1 (Renormalization property)**.** Let $\boldsymbol{b}\in L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^d;\mathbb{R}^d)$ such that $\operatorname{div}\boldsymbol{b}\in L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^d)$. We say that $\boldsymbol{b}$ has the renormalization property if for any weak solution $u\in L^\infty_{\mathrm{loc}}([0,T]\times\mathbb{R}^d)$ of the transport equation, $\beta\circ u$ is also a weak solution of (1.1) for all $\beta\in C^1(\mathbb{R})$.

Such condition, albeit present implicitly in [48], is not posed in it, and so we follow a more modern presentation, as in De Lellis' seminar [44]. We remark that

Condition 1.1 implies uniqueness and stability of solutions of (1.1) and (1.4) if the vector field $\boldsymbol{b}$ also satisfies $\operatorname{div} \boldsymbol{b} \in L^1([0,T]; L^\infty(\mathbb{R}^d))$ and the growth assumption

$$\int_0^T \left\| \frac{\boldsymbol{b}_t}{1+|\cdot|} \right\|_{L^1+L^\infty(\mathbb{R}^d)} \mathrm{d}t < \infty. \tag{1.6}$$

We sketch the proof of uniqueness and stability for the transport equation (the continuity equation case follows from the subsequent referred results of [48]), as in [44]: notice that for u_1, u_2 solution of transport equation with vector field $\boldsymbol{b}$ with initial data u_0 and holding the renormalization property, the difference $v = u_1 - u_2$ satisfies

$$\begin{cases} \partial_t v^2 + \operatorname{div}(\boldsymbol{b}\, v^2) = v^2 \operatorname{div} \boldsymbol{b} & \text{in} \quad [0,\infty) \times \mathbb{R}^d; \\ v_{t=0} = 0 & \text{on} \quad \mathbb{R}^d, \end{cases}$$

A formal computation gives that $(\|v_t\|^2_{L^2(\mathbb{R}^d)})' \leqslant \|\operatorname{div} \boldsymbol{b}_t\|_{L^\infty(\mathbb{R}^d)} \|v_t\|^2_{L^2(\mathbb{R}^d)}$, and so the Gronwall Lemma gives that $v \equiv 0$. If one assumes that $\boldsymbol{b} \in L^1([0,\infty); L^\infty(\mathbb{R}^d; \mathbb{R}^d))$, a precise proof is done in [44, Proposition 1.6] with a weak Gronwall version and suitable test functions, and with the growth assumption (1.6) in [48, Theorem II.2].

As for stability result, that is, assuming smooth sequences $\boldsymbol{b}_k$ and $(u_0)_k$ with limits $\boldsymbol{b}_k \to \boldsymbol{b}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^d)$ and $(u_0)_k \to u_0$ $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ with a uniform bound $(u_0)_k \in L^\infty(\mathbb{R}^d)$, and u_k solutions of

$$\begin{cases} \partial_t u_k + \operatorname{div}(\boldsymbol{b}_k\, u_k) = u_k \operatorname{div} \boldsymbol{b}_k & \text{in} \quad [0,\infty) \times \mathbb{R}^d; \\ (u_{t=0})_k = (u_0)_k & \text{on} \quad \mathbb{R}^d, \end{cases}$$

we prove that there exists u solution of (1.1) with vector field $\boldsymbol{b}$ and initial data u_0. Indeed, by Cauchy-Lipschitz theory, u_k is smooth and satisfies

$$\|u_k\|_{L^\infty([0,\infty)\times\mathbb{R}^d)} \leqslant \sup_k \|(u_0)_k\|_{L^\infty(\mathbb{R}^d)} \leqslant C$$

and the above uniform bound combined with the uniqueness results gives that u_k converges weakly* to some function $u \in L^\infty([0,\infty) \times \mathbb{R}^d)$. Moreover, since $\boldsymbol{b}$ has the renormalization property, then u_k^2 satisfies (1.1) with initial data $(u_k)_0^2$. By the same argument as before and by the renormalization property, u_k^2 converges weakly* to u^2. Therefore, we have that $u_k \to u$ in $L^1_{\mathrm{loc}}([0,\infty) \times \mathbb{R}^d)$ as $k \to \infty$. By the strong limits of $\boldsymbol{b}_k$ and $(u_0)_k$, we may pass the limit in the weak setting of (1.1) and so the result follows. We refer to the original proof in [48, Theorem II.4] and [38, Theorem 2.3.3] for a modern presentation.

The above uniqueness and stability result is called the "soft" part of DiPerna-Lions result [44], which juxtapose the regularity assumption for $\boldsymbol{b}$ to satisfy Condition 1.1—or the "hard" part—and it is proven in [48, Section II] that it suffices that $\boldsymbol{b}$ is in the Sobolev vector field $\boldsymbol{b} \in L^1((0,T); W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^d))$. The key idea is to mollify the transport equation in space variable with $\rho_\epsilon = \epsilon^{-d}\rho(\epsilon^{-1}\,\cdot\,)$, where ρ is a smooth and even kernel, and then write the transport equation as

$$\partial_t u_\epsilon + \boldsymbol{b}\cdot\nabla u_\epsilon = \boldsymbol{b}\cdot\nabla u_\epsilon - (\boldsymbol{b}\cdot\nabla u) * \rho_\epsilon =: r_\epsilon,$$

where $u_\epsilon = u * \rho_\epsilon$. The right-hand term r_ϵ is known as "commutator", since it can be written as the difference of changing order of derivatives and convolutions. Since u_ϵ is smooth in space and $\partial_t u^\epsilon = -(\boldsymbol{b}\cdot\nabla u) * \rho_\epsilon$, we may use the chain rule for Sobolev maps so that

$$\partial_t \beta(u_\epsilon) + \boldsymbol{b}\cdot\nabla\beta(u_\epsilon) = \beta'(u_\epsilon) r_\epsilon \tag{1.7}$$

for every $\beta \in C^1(\mathbb{R})$. We wish to pass the limit in the above equation in the weak sense. Notice that each function on the right-hand side converges weakly, but this is not enough for the convergence of the product. Since $\beta'(u_\epsilon)$ is locally uniformly bounded, it suffices to show a strong convergence $r_\epsilon \to 0$ in $L^1_{\text{loc}}(\mathbb{R}^d)$ as $\epsilon \to 0$. This is precisely [48, Lemma II.1], but we refer to [38, Lemma 2.5.2] for a more direct presentation.

In the subsequent years, the interest of extending DiPerna-Lions result for vector fields was latent. The first main improvement was by relaxing the summability condition of the Jacobian matrix of $\boldsymbol{b}$ by only assuming the summability of symmetric part $D\boldsymbol{b} + (D\boldsymbol{b})^t \in L^1((0,T)L^1_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^{d\times d}))$; see Capuzzo-Dolcetta-Perthame paper [27]. Later, the major breakthrough was due to Bouchut [23] for Vlasov equations, that is, transport equations in $\mathbb{R}^{2d}$ with vector field $\boldsymbol{b}_t(x,v) = (\xi(v), F_t(x,v))$, where ξ is the "velocity" and E is the "acceleration" with regularity $\xi_t \in W^{1,1}_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d)$, $D_v F \in L^1((0,T);L^1_{\text{loc}}(\mathbb{R}^{2d};\mathbb{R}^{d\times d}))$, and the assumption $D_x F \in L^1((0,T);\text{BV}_{\text{loc}}(\mathbb{R}^{2d};\mathbb{R}^{d\times d}))$. Notice that it was not assumed that $F_t \in W^{1,1}_{\text{loc}}(\mathbb{R}^{2d};\mathbb{R}^{2d})$, for its derivative with respect to x variable is only a Radon measure. Finally, the full generalization was due to Ambrosio [4] for vector fields $\boldsymbol{b} \in L^1((0,T);\text{BV}_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d))$ with $\operatorname{div}\boldsymbol{b} \in L^1((0,T);L^1_{\text{loc}}(\mathbb{R}^d))$, with the aid of rank one theorem for bounded variations functions of Alberti [1]. Informally, the rank one theorem states that if $f \in \text{BV}_{\text{loc}}(\mathbb{R}^d)$, then one can explicitly write the matrix function M which satisfies $D^s f = M|D^s f|$, where $D^s f$ is the singular part of Df with respect to the Lebesgue measure. They key idea is to consider once again (1.7), but now they split the derivative (which is a Radon measure) into absolutely continuous and singular parts with respect to the Lebesgue measure. Therefore, the absolutely continuous part is computed analogously as in [48] using the fact that $\operatorname{div}\boldsymbol{b}$ is absolutely continuous, but if one carries the same computations for the singular part one only gets a locally finite measure σ (known as "defect measure") such that

$$\sigma = \partial_t\beta(u) + \boldsymbol{b}\cdot\nabla\beta(u).$$

The computations *à la* DiPerna-Lions imply that σ is singular with respect to the Lebesgue measure in spacetime, and it has being coined as "isotropic estimate", for it does not assume any further conditions on the convolution kernel. In order to conclude the renormalization property, one needs to show that in fact $\sigma = 0$. This was done using the so called "anisotropic estimate" (see [38, Subsection 2.6.3]), which in turn implies that the measure σ is estimated by an infimum over the convolution kernels of a linear functional, and by [1], it can be shown that such infimum equals zero. More recently, it has been shown that the aforementioned infimum can be explicitly computed, simplifying the original proof [38, Subsection 2.6.6]. Furthermore, there exists a more modern proof for Alberti's rank one theorem by De Lellis [43], and also a more general result concerning the explicit computation of the singular parts of measures with respect to their total variation in De Philippis-Rindler's paper [72].

We summarize the above as the following: if $\boldsymbol{b} \in L^1([0,T]; \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d))$ has integrability $\operatorname{div} \boldsymbol{b} \in L^1((0,T); L^1_{\mathrm{loc}}(\mathbb{R}^d))$, then Condition 1.1 holds. In particular, if one further assumes that $\operatorname{div} \boldsymbol{b} \in L^1([0,T]; L^\infty(\mathbb{R}^d))$ and the growth assumption (1.6), then there exists a unique renormalized solution of (1.1) and (1.4). We remark that existence of weak solutions of transport and continuity equations is a trivial result, and so the existence of renormalized ones and their uniqueness follow from Condition 1.1. Indeed, by considering the convolution of the vector field and initial data with a mollifier and denoting them as $\boldsymbol{b}_\epsilon$ and $(u_0)_\epsilon$, respectively, we have by Cauchy-Lipschitz theory a unique solution u_ϵ of

$$\begin{cases} \partial_t u_\epsilon + \operatorname{div}(\boldsymbol{b}_\epsilon u_\epsilon) = 0 & \text{in} \quad [0,\infty) \times \mathbb{R}^d; \\ (u_{t=0})_\epsilon = (u_0)_\epsilon & \text{on} \quad \mathbb{R}^d. \end{cases}$$

In particular, we have that u_ϵ is uniformly bounded in $L^\infty((0,\infty) \times \mathbb{R}^d)$, and so we have a subsequence u_{ϵ_k} weakly* converging to some u in $L^\infty((0,\infty) \times \mathbb{R}^d)$ as $\epsilon_k \to 0$. Since $\boldsymbol{b}_\epsilon$ and $(u_0)_\epsilon$ converges strongly in $L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$, the result follows. The transport equation case is analogous with the aditional assumption that $\operatorname{div} \boldsymbol{b} \in L^1_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^d)$.

There are also results further that extend the admissible vector fields satisfying Condition 1.1, and we highlight two results; first due to Ambrosio-Crippa-Maniglia [8] for renormalization property for special bounded deformation vector fields, that is, functions with symmetric derivative being a Radon measure with zero Cantor part; the second due to Miot and Sharples [67] for the renormalization property for BV vector fields off a "small" set S in spacetime, provided that the vector field has appropriate integrability with respect to the normal component of S. We also highlight the Alberti-Bianchini-Crippa's paper [2] for uniqueness of weak solutions of the continuity equation in 2D for autonomous vector fields with Hamiltonian structure and satisfying a weak Sard property, where the proof does not rely on the renormalization property and Le Bris-Lions work [25] for existence and uniqueness of renormalized solutions of the transport equation for vector fields $\boldsymbol{b}(x,y) = (\boldsymbol{b}_1(x), \boldsymbol{b}_2(x,y))$, where $(x,y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and $\boldsymbol{b}_2$ has only regularity $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^{d_2};\mathbb{R}^{d_2})$ on the variable y and only the summability in t, x variables. We also refer to [38, Sections 2.7–2.8, Chapter 3] and references therein for a general discussion of the renormalization property—in particular for nearly incompressible vector fields. Finally, we also highlight some very recent results for vector fields in infinitely many coordinates found in [9, 12, 65] and references therein.

We emphasize that the renormalization approach is purely from a PDE's perspective, and as a byproduct it is possible to prove existence and uniqueness of solutions of the ODE (1.2). Indeed, the results in [48, Section III] and [4, Section 6] for existence, uniqueness and semigroup property for (1.2) follow analogous approximation procedures as before and using the "soft" part of DiPerna-Lions theorem. Moreover, the solution has a Lagrangian structure, that is, if it is a solution of the transport or continuity equation, then it satisfies almost everywhere (1.3) or (1.5), respectively. Furthermore, the uniqueness for (1.2) from the renormalization property Condition 1.1 is quite rigid: it has been known since DiPerna-Lions' work [48, Section IV] that vector fields with unbounded divergence or without integrability assumptions on their derivatives (e.g., vector fields in any fractional Sobolev space $W^{\alpha,p}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ for any $\alpha < 1$) do not have unique flows. Quite remarkably, there are finer results for nonuniqueness of flows: the example by Depauw in [45] shows that the integrability in time for Ambrosio's theory needs to be global, that is, if $\boldsymbol{b} \in L^1_{\mathrm{loc}}((0,T);\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2))$, then uniqueness of flows are not guaranteed; the example by Colombini-Luo-Rauch in [33] shows nonuniqueness for bounded autonomous vector fields $\boldsymbol{b}$ in $\mathbb{R}^3$ with $x_3 D\boldsymbol{b}$ being a finite Borel matrix measure. Therefore, the $\mathrm{BV}(\mathbb{R}^d)$ space is very almost the best space for the renormalization property to hold.

We organize the thesis as follows: in Chapter 2, we recall the Lagrangian approach for (1.1) and (1.4) introduced in [39] by Crippa-De Lellis, that is, a direct study of existence, uniqueness and semigroup property of (1.2), and so it is possible to construct a renormalized solution for transport and continuity equations for $\boldsymbol{b} \in L^1((0,T);W^{1,p}(\mathbb{R}^d;\mathbb{R}^d))$ for $p > 1$. This result was further extended independently by Jabin [59] and Bouchut-Crippa [22] for vector fields in $L^1((0,T);W^{1,1}(\mathbb{R}^d;\mathbb{R}^d))$; the former in fact also extended the proof for vector fields in the space of special bounded variation functions (SBV) and the latter for convolutions of singular kernels *à la* Calderón-Zygmund theory with a $W^{1,1}(\mathbb{R}^d)$ function among other cases. Moreover, the results was extended for vector fields with anisotropic regularity in [16] by Crippa-Bohun-Bouchut in a similar spirit as the aforementioned [25], where the vector field $\boldsymbol{b}(x,y) = (\boldsymbol{b}_1(x,y), \boldsymbol{b}_2(x,y))$ has derivative $D_y\boldsymbol{b}_2$ being singular kernel convolution with a finite measure, while the others are singular kernel convolution with an summable function. Finally, Nguyen [70] extended the result for vector fields which are singular kernels convoluted with a function of bounded variation. In particular, it extends Ambrosio's results [4] on the ODE level. All of the aforementioned works assume the bounded divergence in space of its vector fields and the growth assumption (1.6).

In Chapter 3 we recall the work of Ambrosio-Colombo-Figalli [5] and [6, Sections

4 and 5] on local flows without assuming the growth assumption (1.6), that is, given a $x \in \mathbb{R}^d$, there exists $T = T(x) > 0$ and a unique solution of (1.2) in $[0, T(x))$. They prove existence, uniqueness and semigroup property of flows in this setting, as well existence of renormalized solutions of (1.1) for divergence-free vector fields. Furthermore, they show a sort of "weak" renormalization property if one assumes uniqueness of nonnegative solutions of (1.4). The latter result has not been reproduced in the previous setting, i.e., where vector fields satisfy (1.6). Finally, we also recall the extension done by the author with Marcon [21] for quasistatic Vlasov-Maxwell systems, extending the result of [6] for Vlasov-Poisson equation.

We now explain somewhat informally the contents of Chapters 4 and 5, which are the author's original contributions: in Chapter 4, we study solutions of (1.2) with vector fields which are wavelike convolutions of a singular kernel and some $L^1((0,T); L^p(\mathbb{R}^d))$ function, where $p \geqslant 1$. More precisely, we consider vector fields

$$\boldsymbol{b}_t^i(x) = \int_{B_t(x)} K^i(x-y) g_{t-|x-y|}(y)\, \mathrm{d}y =: K \star g_t(x),$$

where K is a singular vector kernel near the origin of order at most $|x|^{1-d}$. Such vector fields arise from inhomogeneous solutions of the wave equation [49, Section 2.4], where in the three dimensional case the kernel is precisely computed as $K^i(x) = (4\pi|x|)^{-1} e_i$, e_i being the canonical basis in $\mathbb{R}^3$. We remark that this "hyperbolic convolution" operator $\star$ is not symmetric nor associative in both variables, and so many of the techniques employed in [22] are not available. In order to prove existence, uniqueness and semigroup property for solutions of (1.2), it was necessary to extend an estimate for a composition of grand maximal functions and singular kernels [22, Theorem 3.3], where in our case the kernel are not necessarily smooth, i.e. $C^1(\mathbb{R}^d)$ functions outside the origin. An important application is in Vlasov-Maxwell system, where we consider vector fields (as aforementioned in the discussion of [23]) with further structure of the acceleration F, that is

$$\boldsymbol{b}_t(x, v) = (\xi(v), E_t(x) + \xi(v) \times H_t(x)).$$

The physical velocities $\xi(v)$ are either nonrelativistic, where it coincides with phase velocity v and so $\xi(v) = v$; or relativistic, i.e. $\xi(v) = (1 + |v|^2)^{-1/2} v$, where the first term is the Lorentz correction factor (we consider the speed of light $c = 1$). Moreover, the electric and magnetic fields E and H, respectively satisfy the wave equation

$$\partial_{tt} E - \Delta E = -\nabla \rho - \partial_t j \quad \text{and} \quad \partial_{tt} H - \Delta H = \operatorname{curl} j$$

for particle and current densities ρ, j, respectively; we remark that these physical quantities depend on the solution of the transport equation, hence so does the vector field $\boldsymbol{b}$. In particular, the Vlasov-Maxwell system is a nonlinear transport/continuity equation. We shall make an explicit derivation of the above wave equations with physical units from

Maxwell's equations in Chapter 4, as well as write the electromagnetic field depending only on the densities ρ, j, and $\partial_t j$. We prove that if densities and some time derivatives are summable, then weak, renormalized, and Lagrangian solutions are all equivalent. All of the aforementioned new results can be found in [20].

In Chapter 5, we consider vector fields $\boldsymbol{b}_t = K^\alpha * g_t$ and $\boldsymbol{b}_t = \Gamma^\alpha * g_t$, where $0 < \alpha < 1$ and the kernels K^α and Γ^α satisfy the decay

$$|K^\alpha(x)| \leqslant \frac{C}{|x|^{d-1+\alpha}} \quad \text{and} \quad |\Gamma^\alpha(x)| \leqslant \frac{C}{|x|^{d+\alpha}}.$$

Since we shall need to compute the jacobian of $\boldsymbol{b}_t$, we distinguish the kernels for which term in the convolution we differentiate: in the former, the kernel is less singular, and so we have enough room to compute it as $\partial_j \boldsymbol{b}^i = (\partial_j K_i^\alpha) * g$, and so we assume that either

$$K_i^\alpha(y) = \frac{\Omega_i(y)}{|y|^{d-1+\alpha}},$$

where Ω_i are smooth zeroth order homogeneous functions with average zero on the sphere $\mathbb{S}^{d-1}$; in the latter, we differentiate the functions g, i.e.

$$\partial_j \boldsymbol{b}_t^i(x) = \Gamma_i^\alpha * \partial_j g_t(x) \quad \text{for kernels} \quad \Gamma_i^\alpha(y) = \frac{\Omega_i(y)}{|y|^{d+\alpha}}.$$

We shall consider ever more singular kernels in Chapter 5, where we shall assume more cancellation properties for Ω_i, e.g. zero average on the sphere in all directions. Concerning the regularity needed for the functions g, the results of [22, 70] suggests that one should consider an "intermediate" space between $L^1(\mathbb{R}^d)$ and $\mathrm{BV}(\mathbb{R}^d)$ in cases $\boldsymbol{b}_t = K^\alpha * g_t$, and a more regular space than $\mathrm{BV}(\mathbb{R}^d)$ in cases $\boldsymbol{b}_t = \Gamma^\alpha * g_t$. Moreover, such spaces should take into account the parameter α, and we were able to prove that the Besov spaces $B^\alpha_{p,1}(\mathbb{R}^d)$ are the appropriate ones when considering the less singular kernels; the other cases one takes $B^{1+\alpha}_{p,1}(\mathbb{R}^d)$. The even more singular cases needed not only better cancellation properties for the kernels, but also stricter Besov spaces for the functions g. The proof of such results is in fact an simplification of the Nguyen's work [70] for the aforementioned special case.

The quintessential example of vector fields with such structure are from the generalized surface quasi-geostrophic equations (gSQG) with parameter $\alpha \in (0,2)\backslash\{1\}$:

$$\begin{cases} \partial_t \theta + [\nabla^\perp (-\Delta)^{\frac{\alpha}{2}-1}\theta] \cdot \nabla \theta = 0 & \text{in} \quad (0,T) \times \mathbb{R}^2; \\ \theta_{t=0} = \theta_0 & \text{on} \quad \mathbb{R}^2. \end{cases} \tag{1.8}$$

Notice that cases $\alpha = 0$ and $\alpha = 1$ are the 2D vorticity Euler and surface quasi-geostrophic equations, respectively. Indeed, writing the vector field explicitly (see Silvestre's thesis [74, Chapter 2] for an extensive study of fractional Laplace operator), we have that

$$\boldsymbol{b}_t(x) = \nabla^\perp (-\Delta)^{\frac{\alpha}{2}-1}\theta_t(x) = C_\alpha \nabla^\perp \int_{\mathbb{R}^2} \frac{\theta(y)}{|x-y|^\alpha}\,\mathrm{d}y = C_\alpha \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{2+\alpha}}\theta(y)\,\mathrm{d}y.$$

In this context, we prove that if solutions are nonnegative and in the appropriate Besov space, then we have equivalence of weak, renormalized, and Lagrangian solutions; if we do not know a priori the sign of solutions, we still have existence of renormalized/weak solutions. The latter can also be extended for bounded divergence vector fields.

Finally, in Chapter 6, we summarize the results for wavelike and very singular vector fields, and discuss the possible extension of these type of results for less regular vector fields, such as bounded deformation or with anisotropic regularity, and Euler equations with measure vorticity.

2 Global results with Lagrangian approach

In this chapter, we revisit the results without relying on the renormalization property Condition 1.1 introduced by Crippa-De Lellis in [39]. This is now called Lagrangian approach and we shall expose it among several generalizations, namely [16, 22, 70]. We begin by recalling classical results regarding maximal functions and singular integrals.

2.1 Maximal functions and singular integrals

In this section, we recall the estimates of maximal functions (also known as Hardy-Littlewood maximal function) and singular integrals. We first define maximal,local maximal functions, and grand maximal functions.

Definition 2.1. For $u \in L^1_{\text{loc}}(\mathbb{R}^d)$, we define the maximal function Mu as

$$Mu(x) := \sup_{\epsilon>0} \frac{1}{|B_\epsilon(x)|} \int_{B_\epsilon(x)} |u(y)| \,\mathrm{d}y,$$

for each $x \in \mathbb{R}^d$. We define the local maximal function $M_R u$ as

$$M_R u(x) := \sup_{0<\epsilon<R} \frac{1}{|B_\epsilon(x)|} \int_{B_\epsilon(x)} |u(y)| \,\mathrm{d}y.$$

Finally, given a family of functions $\{\rho^\nu\}_\nu \subset L^\infty_c(\mathbb{R}^d)$, we define the grand maximal function $M_{\rho^\nu} u$ associated to $\{\rho^\nu\}$ as

$$M_{\rho^\nu} u(x) := \sup_\nu \sup_{\epsilon>0} |(\rho^\nu_\epsilon * u)(x)|,$$

where $\rho^\nu_\epsilon(x) = \epsilon^{-d}\rho^\nu(\epsilon^{-1}x)$ for all $x \in \mathbb{R}^d$. If $\{\rho^\nu\}_\nu \subset C^\infty_c(\mathbb{R}^d)$, then one may consider finite measures u.

We remark that $M_R u(x) \leqslant Mu(x)$ for all $R > 0$. Moreover, by assuming uniform boundedness of the functions ρ^ν and uniform support $\operatorname{supp} \rho^\nu \subset B_1$, we have that

$$M_{\rho^\nu} u(x) = \sup_\nu \sup_{\epsilon>0} \left| \frac{1}{\epsilon^d} \int_{\mathbb{R}^d} \rho^\nu \left(\frac{x-y}{\epsilon} \right) u(y) \,\mathrm{d}y \right| \leqslant |B_1| \sup_\nu \|\rho^\nu\|_{L^\infty(\mathbb{R}^d)} Mu(x).$$

In particular, if we take $\rho^\nu(x) = |B_1|^{-1}\mathbb{1}_{B_1}(x)$ (so that it is independent of ν), and so

$$M_{\rho^\nu} u(x) = \sup_{\epsilon>0} \left| \frac{1}{\epsilon^d |B_1|} \int_{\mathbb{R}^d} \mathbb{1}_{B_1} \left(\frac{x-y}{\epsilon} \right) u(y) \,\mathrm{d}y \right| = \sup_{\epsilon>0} \left| \frac{1}{|B_\epsilon(x)|} \int_{B_\epsilon(x)} u(y) \,\mathrm{d}y \right|,$$

that is, the maximal function with the absolute value outside of the integral.

We begin by recalling the classical result concerning estimates of the operator M in $L^p(\mathbb{R}^d)$ for all the range $p \in [1,\infty]$. For a concise presentation, we refer to the seminal book of Stein [76, Chapter I]. By the above considerations, the following also holds for the grand maximal operators.

Lemma 2.1 (Hardy-Littlewood estimates)**.** *Let $u \in L^p(\mathbb{R}^d)$. If $1 < p \leqslant \infty$, then there exists a constant $C_p > 0$ depending only on p such that*

$$\|Mu\|_{L^p(\mathbb{R}^d)} \leqslant C_p \|u\|_{L^p(\mathbb{R}^d)}.$$

If $p = 1$, there exists a constant $C_d > 0$ depending on the dimension d such that it holds the weak estimate

$$|||Mu|||_{L^1_w(\mathbb{R}^d)} \leqslant C_d \|u\|_{L^1(\mathbb{R}^d)}.$$

Moreover, let $\lambda > 0$ and $r > 0$. If $p > 1$, there exists a constant $C_{d,p} > 0$ depending only on d, p such that

$$\|M_\lambda u\|_{L^p(B_r)} \leqslant C_{d,p} \|u\|_{L^p(B_{\lambda+r})}.$$

If $p = 1$, there exists a constant $C_d > 0$ depending only on d such that

$$|||M_\lambda u|||_{L^1_w(B_\lambda)} \leqslant C_d \|u\|_{L^1(B_{\lambda+r})}.$$

We also recall the pointwise estimates for the difference quotient of a function; they are finer estimates than those typically presented in graduate studies, e.g. [49, Section 5.8.2]. They are sometimes referred as Lusin-Lipschitz inequalities.

Lemma 2.2 (Difference quotient estimates)**.** *Let $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$. Then there exists a measure zero set (with respect to the Lebesgue measure) E such that for any $x, y \in \mathbb{R}^d \setminus E$, it holds*

$$|u(x) - u(y)| \leqslant |x-y| (M_{|x-y|} \nabla u(x) + M_{|x-y|} \nabla u(y)).$$

More generally, if $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ (but not necessarily in $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$), consider the family of functions

$$\left\{ \Upsilon^{j,\xi}(x) = h\left(\frac{\xi}{2} - x\right) x_j : h \in C^\infty_c(\mathbb{R}^d),\ \int_{\mathbb{R}^d} h(y)\,\mathrm{d}y = 1,\ \operatorname{supp} h \subset B_{1/2} \right\}_{j=1,\dots,d;\ \xi \in \mathbb{S}^{d-1}}$$

and the associated grand maximal operator $M_{\Upsilon^{j,\xi}}$. Moreover, let $\eta \in C^\infty_c(\mathbb{R}^d)$ be a function satisfying

$$\int_{\mathbb{R}^d} \eta(y)\,\mathrm{d}y = 1 \quad \textit{and} \quad \operatorname{supp} \eta \subset B_\alpha \quad \textit{for some } \alpha > 0 \textit{ such that } \operatorname{supp} \eta_n \subset B_{1/2-\alpha},$$

where $\eta_n(x) := n^d \eta(nx)$ for $n \in \mathbb{N}$. Finally, denote E the measure zero set such that $\eta_n * u(x) \to u(x)$ for all $x \in \mathbb{R}^d \setminus E$. If we further assume that there exists a measure zero set F such that the weak derivative ∇u satisfies $M_{\Upsilon^{j,\xi}}(\nabla u)(x) < \infty$ for all $x \in \mathbb{R}^d \setminus F$, then for all x, $y \in \mathbb{R}^d \setminus (E \cup F)$, it holds

$$|u(x) - u(y)| \leqslant |x-y| \left(M_{\Upsilon^{j,\xi}} \nabla u(x) + M_{\Upsilon^{j,\xi}} \nabla u(y) \right).$$

We refer to Bojarski-Hajłasz's work [19, Theorem 1] for the first claim of Lemma 2.2. For the second, we remark that the aforementioned result is due to Bouchut-Crippa work [22, Proposition 4.2]–albeit not stated as above, but rather in the specific application for their subject of study. Nevertheless, their thesis can be extended as stated in Lemma 2.2 with the same proof.

Concerning the theory of singular kernels, we refer once again to Stein's classical book [76, Chapter II], and also the Muscalu and Schlag's chapter [68]. By assuming a regular enough kernel, the singular operator defined by it enjoys estimates similar to Lemma 2.1, and are usually refer as Calderón-Zygmund theory.

Lemma 2.3 (Calderón-Zygmund estimates)**.** *Let $\Gamma \in L^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\})$ be a function for which there exists a constant $C > 0$ such that*

$$\int_{B_R} |y||\Gamma(y)|\,\mathrm{d}y \leqslant CR \quad \text{for all } R > 0;$$

$$\int_{\mathbb{R}^d \setminus B_{2|x|}} |\Gamma(y-x) - \Gamma(y)|\,\mathrm{d}y \leqslant C \quad \text{for all } x \in \mathbb{R}^d;$$

$$\left| \int_{B_R \setminus B_r} \Gamma(y)\,\mathrm{d}y \right| \leqslant C \quad \text{for all } 0 < r < R < \infty.$$

If $p \in (1,\infty)$, there exists a constant $C' > 0$ such that it holds

$$\|\Gamma * u\|_{L^p(\mathbb{R}^d)} \leqslant C' \|u\|_{L^p(\mathbb{R}^d)}$$

for all $u \in L^p(\mathbb{R}^d)$. If $p = 1$, it holds the weak estimate

$$|||\Gamma * u|||_{L^1_w(\mathbb{R}^d)} \leqslant C' \|u\|_{L^1(\mathbb{R}^d)}.$$

In the above, the convolution is understood in the sense

$$\Gamma * u(x) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^d \setminus B_\epsilon(x)} \Gamma(x-y) u(y)\,\mathrm{d}y.$$

We remark that the case $p = \infty$ is intentionally omitted for it is beyond the scope of this work. Nevertheless, there are results concerning estimates of $\Gamma * u$ in bounded mean oscillation (BMO) spaces; see [68].

The kernel conditions in Lemma 2.3 which are known as *singular kernels* usually are too relaxed for some applications, e.g. [22, Theorem 3.3] and [68, Section 7.3]. The first one is usually restricted to kernels with pointwise estimate

$$|\Gamma(x)| \leqslant \frac{C}{|x|^d} \quad \text{for all } x \in \mathbb{R}^d \setminus \{0\};$$

the second one[1] follows if we assume pointwise estimate on the kernel's derivative:

$$|\nabla\Gamma(x)| \leqslant \frac{C}{|x|^{d+1}} \quad \text{for all } x \in \mathbb{R}^d \setminus \{0\};$$

[1] Such estimate is known as Hörmander's condition.

the third estimate, sometimes called "cancellation property" is a finer condition, for it allows us to prove that a composition of singular operators are also singular operators; see the comment in the beginning of [22, Section 3]. Therefore, it is usual to assume a structure for the kernels rather than an estimate, as in

$$\Gamma(x) = \frac{\Omega(x)}{|x|^d}, \quad \text{with} \quad \int_{\mathbb{S}^{d-1}} \Omega(y)\,\mathrm{d}S_y = 0. \tag{2.1}$$

Here, Ω is a zeroth order homogeneous function, that is, $\Omega(tx) = \Omega(x)$ for all $x \in \mathbb{R}^d$ and $t \in \mathbb{R}$, and so it is completely determined by its values in $\mathbb{S}^{d-1}$.

We also recall the results in [68, Section 7.3] concerning a truncated version of the singular operator associated to kernels of form (2.1):

$$T_* u(x) := \sup_{\epsilon>0} \left| \int_{\mathbb{R}^d \setminus B_\epsilon(x)} \Gamma(x-y)u(y)\,\mathrm{d}y \right|.$$

Lemma 2.4 (Calderón-Zygmund estimates for truncated kernel)**.** *Let $u \in L^p(\mathbb{R}^d)$ and Γ as in (2.1). Then there exists a constant $C_d > 0$ depending only on the dimension such that*

$$T_* u(x) \leqslant C_d(M(\Gamma * u)(x) + Mu(x)).$$

In particular, if $p \in (1,\infty)$, there exists a constant $C_d' > 0$ such that

$$\|T_* u\|_{L^p(\mathbb{R}^d)} \leqslant C_d' \|u\|_{L^p(\mathbb{R}^d)}.$$

Moreover, if $p = 1$, it satisfies the weak estimate

$$|||T_* u|||_{L^1_w(\mathbb{R}^d)} \leqslant C_d' \|u\|_{L^1(\mathbb{R}^d)}.$$

Remark 1. In Section 2.5, we shall need specialized versions of Lemma 2.2 and Lemma 2.3. By the niche nature of these results, we shall state them afterwards in Proposition 2.5, Theorem 2.5, and Lemma 2.12.

2.2 Dawn of Lagrangian theory: Crippa-De Lellis' result

Albeit pioneering the study of (1.2) by Crippa and De Lellis' work [39], their conclusions can be further improved following the presentation of Bouchut-Crippa [22]. Moreover, inspired by [6, Proposition 4.11], we further improved Crippa-De Lellis' result, and so we present the latter. We begin by defining the key mathematical object throughout the thesis, which in a sense is the "correct" solution of (1.2).

Definition 2.2 (Renormalized regular Lagrangian flow)**.** Let $\boldsymbol{b} : (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$ be a vector field satisfying the growth assumption

$$\int_0^T \left\| \frac{\boldsymbol{b}_t}{(1+|\cdot|)\log(2+|\cdot|)} \right\|_{L^1+L^\infty(\mathbb{R}^d)} \mathrm{d}t < \infty. \tag{2.2}$$

For a fixed $s \in [0,T)$, consider the map $\boldsymbol{X}$ continuous in time and locally measurable in space such that for almost any $t \in [s,T]$ it holds

$$\int_{B_R} \log\log(e + |\boldsymbol{X}(t,s,x)|)\,\mathrm{d}x < \infty.$$

We say that $\boldsymbol{X}$ is a renormalized regular Lagrangian flow[2] of $\boldsymbol{b}$ starting at s if

(i) it satisfies in the weak sense the ODE

$$\begin{cases} \partial_t(\beta(\boldsymbol{X}(t,s,x))) = \nabla\beta(\boldsymbol{X}(t,s,x)) \cdot \boldsymbol{b}_t(\boldsymbol{X}(t,s,x)) & \text{in } (s,T) \times \mathbb{R}^d; \\ \boldsymbol{X}(s,s,x) = x & \text{on } \mathbb{R}^d; \end{cases}$$

for all $\beta \in C^1(\mathbb{R}^d;\mathbb{R})$ such that

$$|\beta(x)| \leqslant C\,(1 + \log\log(e + |x|)) \quad \text{and} \quad |\nabla\beta(x)| \leqslant \frac{C}{(1+|x|)\log(2+|x|)}$$

for all $x \in \mathbb{R}^d$ and some constant $C > 0$.

(ii) there exists a compressibility constant $L > 0$ such that for every $t \in [s,T)$, it holds

$$\boldsymbol{X}(t,s,\cdot)_{\#}\mathcal{L}^d \leqslant L\mathcal{L}^d.$$

We remark that the above definition is not contained in [39] nor [22]: the former defines a Lagrangian flow via almost everywhere well-posedness of (1.2) instead of Definition 2.2 (i), while the latter assumes a stricter class of test functions β, replacing their estimates by

$$|\beta(x)| \leqslant C\,(1 + \log(1 + |x|)) \quad \text{and} \quad |\nabla\beta(x)| \leqslant \frac{C}{1+|x|};$$

see also [42] for a similar approach of renormalized Lagrangian flows. Definition 2.2 (ii) and on β and $\boldsymbol{b}$ shall imply that $t \mapsto \beta(\boldsymbol{X}(t,s,x))$ is an absolutely continuous function for almost every $t \in [s,T]$ and $x \in \mathbb{R}^d$. Assuming the integrability of $\boldsymbol{b}$ (2.2) and the estimates of β in Definition 2.2 (i), we may take $\beta(x) = \log\log(e + |x|)$, and so we heuristically conclude for any $R > 0$ that

$$\int_{B_R} \log\log(e + |\boldsymbol{X}(t,s,x)|)\,\mathrm{d}x \leqslant C_{R,L} + L\int_0^T \left\| \frac{\boldsymbol{b}_t}{\log(2+|\cdot|)\,(1+|\cdot|)} \right\|_{L^1+L^\infty(\mathbb{R}^d)} \mathrm{d}t < \infty$$

for some $C_{R,L} > 0$ depending on the subscripts, where we have used Definition 2.2 (ii). The same computation is done in [22] when assuming (1.6), and so by taking $\beta(x) = \log(1+|x|)$, their flow is logarithmically more integrable then what we consider in this section. Before we make the above computation precise, we define the sublevels of a flow.

[2] An important question is whether Lagrangian flows are well-posed in the sense of Lebesgue classes, namely if $\nabla\beta(\boldsymbol{X})\cdot\boldsymbol{b}(\boldsymbol{X})$ depends on the representation of its Lebesgue representative. For this discussion, we refer to Remark 22.

Definition 2.3 (Sublevels)**.** For a measurable in spacetime flow $\boldsymbol{X}(\cdot, s, \cdot)$ starting at s and a fixed $\lambda > 0$, we call the sets

$$G_\lambda := \{x \in \mathbb{R}^d : |\boldsymbol{X}(t, s, x)| \leqslant \lambda \text{ for a.e. } t \in [s, T]\}$$

the sublevels of the flow.

Finally, we shall write vector fields satisfying (2.2) as

$$\frac{\boldsymbol{b}_t(x)}{\log(2 + |x|)\,(1 + |x|)} = \tilde{\boldsymbol{b}}_t^1(x) + \tilde{\boldsymbol{b}}_t^2(x), \tag{2.3}$$

where $\tilde{\boldsymbol{b}}^1 \in L^1((0,T) \times \mathbb{R}^d; \mathbb{R}^d)$ and $\tilde{\boldsymbol{b}}^2 \in L^1((0,T); L^\infty(\mathbb{R}^d; \mathbb{R}^d))$. We are now ready to prove the "$\log\log L$" integrability of the flow.

Lemma 2.5 (Integrability of flow)**.** *Let $\boldsymbol{b}$ be a vector field satisfying (2.2) and an associated renormalized regular Lagrangian flow $\boldsymbol{X}$ starting at time s with compressibility constant L. Then for any $R > 0$ and $t \in [s, T]$, it holds*

$$\int_{B_R} \log\log(e + |\boldsymbol{X}(t, s, x)|)\,\mathrm{d}x \leqslant \int_{B_R} \log\log(e + |x|)\,\mathrm{d}x + L\|\tilde{\boldsymbol{b}}^1\|_{L^1((0,T)\times\mathbb{R}^d)} + |B_R|\|\tilde{\boldsymbol{b}}^2\|_{L^1((0,T);L^\infty(\mathbb{R}^d))}$$

and also for any $R > 0$

$$\int_{B_R} \sup_{t\in[s,T]} \log\log(e + |\boldsymbol{X}(t, s, x)|)\,\mathrm{d}x \leqslant \int_{B_R} \log\log(e + |x|)\,\mathrm{d}x + L\|\tilde{\boldsymbol{b}}^1\|_{L^1((0,T)\times\mathbb{R}^d)} + |B_R|\|\tilde{\boldsymbol{b}}^2\|_{L^1((0,T);L^\infty(\mathbb{R}^d))}.$$

In particular, it follows that there exists function $f(r, \lambda)$, $g(r, \lambda)$ depending only on $\|\tilde{\boldsymbol{b}}^1\|_{L^1((0,T)\times\mathbb{R}^d)}$, $\|\tilde{\boldsymbol{b}}^2\|_{L^1((0,T);L^\infty(\mathbb{R}^d))}$, and the constant L such that

$$|B_r \setminus G_\lambda| \leqslant f(r, \lambda) \quad \text{and} \quad |G_\lambda \setminus B_r| \leqslant g(r, \lambda),$$

where r, $\lambda > 0$. Moreover, for a fixed $r > 0$, the function $f(r, \cdot)$ satisfies the limit

$$\lim_{\lambda\nearrow\infty} f(r, \lambda) \searrow 0;$$

and for a fixed $\lambda > 0$, the function $g(\cdot, \lambda)$ satisfies

$$\lim_{r\nearrow\infty} g(r, \lambda) \searrow 0.$$

Proof. We follow very closely the proof of [22, Lemma 5.5]. We consider for $\epsilon > 0$ the function

$$\beta_\epsilon(x) = \log\log(e + \sqrt{|x|^2 + \epsilon}).$$

By Definition 2.2 (i), Definition 2.2 (ii), and (2.2) we have that

$$\partial_t(\beta_\epsilon(\boldsymbol{X}(\cdot,s,\cdot))) \in L^1((s,T);L^1_{\mathrm{loc}}(\mathbb{R}^d)).$$

In particular, for almost every $x \in \mathbb{R}^d$, $\partial_t(\beta_\epsilon(\boldsymbol{X}(\cdot,s,x)))$ is summable in (s,T), and so for almost every $x \in \mathbb{R}^d$ and for all $t \in (s,T)$, we claim that

$$\beta_\epsilon(\boldsymbol{X}(t,s,x)) = \beta_\epsilon(\boldsymbol{X}(s,s,x)) + \int_s^t \nabla\beta_\epsilon(\boldsymbol{X}(\tau,s,x)) \cdot \boldsymbol{b}_\tau(\boldsymbol{X}(\tau,s,x))\,\mathrm{d}\tau. \tag{2.4}$$

Indeed, since $\beta_\epsilon(\boldsymbol{X}(\cdot,s,x))$ coincides with an absolutely continuous function $\xi_\epsilon(\cdot,x)$ in (s,T), and so we have that

$$\beta_\epsilon(\boldsymbol{X}(t,s,x)) = \xi_\epsilon(s,x) + \int_s^t \nabla\beta_\epsilon(\boldsymbol{X}(\tau,s,x)) \cdot \boldsymbol{b}_\tau(\boldsymbol{X}(\tau,s,x))\,\mathrm{d}\tau.$$

Since the integral on the right-hand-side and $t \mapsto \beta_\epsilon \circ (\boldsymbol{X}(t,s,x))$ are an continuous function in time for almost all $s \in [0,T]$ and $x \in \mathbb{R}^d$, we conclude that $\xi_\epsilon(s,x) = \beta_\epsilon(\boldsymbol{X}(s,s,x))$.

By the above claim, we may take $\epsilon \to 0$ to obtain that

$$\log\log(e + |\boldsymbol{X}(t,s,x)|) \leqslant \log\log(e+|x|) + \int_0^T |\tilde{\boldsymbol{b}}^1_\tau(\boldsymbol{X}(\tau,s,x))| + |\tilde{\boldsymbol{b}}^2_\tau(\boldsymbol{X}(\tau,s,x))|\,\mathrm{d}\tau \tag{2.5}$$

for all $t \in (s,T)$. Integrating with respect to x in B_R for any $R > 0$ implies the first inequality in the thesis of Lemma 2.5; taking the supremum with respect to t in $[s,T]$ and then integrating in B_R implies the second one.

Integrating (2.5) in $B_r \setminus G_\lambda$, we have that

$$|B_r \setminus G_\lambda| \leqslant \frac{|B_r|\log\log(e+r) + L\left(\|\tilde{\boldsymbol{b}}^1\|_{L^1((0,T)\times\mathbb{R}^d)} + |B_r|\|\tilde{\boldsymbol{b}}^2\|_{L^1((0,T);L^\infty(\mathbb{R}^d))}\right)}{\log\log(e+\lambda)} =: f(r,\lambda).$$

Notice that we may estimate (2.4) from the other side, and so by taking $\epsilon \to 0$, we have that

$$\log\log(e+|x|) \leqslant \log\log(e + |\boldsymbol{X}(t,s,x)|) + \int_0^T |\tilde{\boldsymbol{b}}^1_\tau(\boldsymbol{X}(\tau,s,x))| + |\tilde{\boldsymbol{b}}^2_\tau(\boldsymbol{X}(\tau,s,x))|\,\mathrm{d}\tau. \tag{2.6}$$

Integrating in $G_\lambda \setminus B_r$ and an analogous estimate as before, we have that

$$|G_\lambda \setminus B_r| \leqslant \frac{L\Big(|B_\lambda|\log\log(e+\lambda) + \|\tilde{\boldsymbol{b}}^1\|_{L^1((0,T)\times\mathbb{R}^d)} + |B_\lambda|\|\tilde{\boldsymbol{b}}^2\|_{L^1((0,T);L^\infty(\mathbb{R}^d))}\Big)}{\log\log(e+r)} =: g(r,\lambda).$$

The limits of $f(r,\cdot)$ and $g(\cdot,\lambda)$ follow, and so does the lemma. □

Remark 2. An analogous thesis of Lemma 2.5 holds if one assumes (1.6) instead of (2.2). Indeed, by dropping the logarithm in the denominator in (2.3), we have that

$$\int_{B_R} \sup_{t\in[s,T]} \log(1 + |\boldsymbol{X}(t,s,x)|)\,\mathrm{d}x \leqslant \int_{B_R} \log(1+|x|)\,\mathrm{d}x + L\|\tilde{\boldsymbol{b}}^1\|_{L^1((0,T)\times\mathbb{R}^d)} + |B_R|\|\tilde{\boldsymbol{b}}^2\|_{L^1((0,T);L^\infty(\mathbb{R}^d))}.$$

In particular, the control of $|B_r \setminus G_\lambda|$ and $|G_\lambda \setminus B_r|$ holds.

We are now ready to state the now called "fundamental estimate for flows", coined by Bouchut-Crippa in [22, Proposition 5.9]. We shall follow the former approach for a more modern presentation comparatively to [39].

Proposition 2.1. *Let* $\boldsymbol{b}$ *and* $\bar{\boldsymbol{b}}$ *be vector fields satisfying* (2.2)*, and* $\boldsymbol{X}$*,* $\bar{\boldsymbol{X}}$ *their renormalized regular Lagrangian flows starting at time* s *with compressibility constants* L *and* $\bar{L}$*, respectively. Moreover, assume that* $D\boldsymbol{b} \in L^1((0,T); L^p_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^{d\times d}))$ *for some* $p \in (1,\infty)$*. Then for every* $\gamma > 0$*,* $\eta > 0$*, and* $r > 0$*, there exists* $\lambda > 0$ *and a constant* $C_{\gamma,\eta,r} > 0$ *such that*

$$|B_r \cap \{|\boldsymbol{X}(t,s,\cdot) - \bar{\boldsymbol{X}}(t,s,\cdot)| > \gamma\}| \leqslant C_{\gamma,\eta,r}\|\boldsymbol{b} - \bar{\boldsymbol{b}}\|_{L^1((0,T)\times B_\lambda)} + \eta$$

uniformly in $s \in [0,T]$ *and* $t \in [s,T]$*. The constant* $C_{\gamma,\eta,r}$ *depends only on its subscripts, the compressibility constants* L *and* $\bar{L}$*, the norms* (2.2) *of* $\boldsymbol{b}$ *and* $\bar{\boldsymbol{b}}$ *and* $\|D\boldsymbol{b}\|_{L^1((0,T);L^p(B_{3\lambda}))}$.

Proof. We begin by defining for any $\delta > 0$, $\lambda > 0$, and $t \in [s,T]$ the function

$$\Phi_\delta(t) = \int_{B_r\cap G_\lambda\cap\bar{G}_\lambda} \log\log\left(e + \frac{|\boldsymbol{X}(t,s,x) - \bar{\boldsymbol{X}}(t,s,x)|}{\delta}\right) \mathrm{d}x,$$

where $\bar{G}_\lambda$ is the sublevel associated to $\bar{\boldsymbol{X}}(\cdot,s,\cdot)$. By the assumed regularity of $\boldsymbol{X}$ and $\bar{\boldsymbol{X}}$, we have that Φ_δ is absolutely continuous, and it satisfies

$$\begin{aligned}
\Phi_\delta'(t) &\leqslant \int_{B_r\cap G_\lambda\cap\bar{G}_\lambda} \frac{|\boldsymbol{b}_t(\boldsymbol{X}(t,s,x)) - \bar{\boldsymbol{b}}_t(\bar{\boldsymbol{X}}(t,s,x))|}{|\boldsymbol{X}(t,s,x) - \bar{\boldsymbol{X}}(t,s,x)| + \delta}\,\mathrm{d}x \\
&\leqslant \int_{B_r\cap G_\lambda\cap\bar{G}_\lambda} \frac{|\boldsymbol{b}_t(\bar{\boldsymbol{X}}(t,s,x)) - \bar{\boldsymbol{b}}_t(\bar{\boldsymbol{X}}(t,s,x))|}{|\boldsymbol{X}(t,s,x) - \bar{\boldsymbol{X}}(t,s,x)| + \delta}\,\mathrm{d}x \\
&\quad + \int_{B_r\cap G_\lambda\cap\bar{G}_\lambda} \frac{|\boldsymbol{b}_t(\boldsymbol{X}(t,s,x)) - \boldsymbol{b}_t(\bar{\boldsymbol{X}}(t,s,x))|}{|\boldsymbol{X}(t,s,x) - \bar{\boldsymbol{X}}(t,s,x)| + \delta}\,\mathrm{d}x.
\end{aligned} \tag{2.7}$$

The first integral on the right-hand side is bounded by using Definition 2.2 (ii) for $\bar{\boldsymbol{X}}$, and so

$$\int_{B_r\cap G_\lambda\cap\bar{G}_\lambda} \frac{|\boldsymbol{b}_t(\bar{\boldsymbol{X}}(t,s,x)) - \bar{\boldsymbol{b}}_t(\bar{\boldsymbol{X}}(t,s,x))|}{|\boldsymbol{X}(t,s,x) - \bar{\boldsymbol{X}}(t,s,x)| + \delta}\,\mathrm{d}x \leqslant \frac{\bar{L}}{\delta}\|\boldsymbol{b}_t - \bar{\boldsymbol{b}}_t\|_{L^1(B_\lambda)}.$$

For the second integral, we use Lemma 2.2, and so

$$\begin{aligned}
\int_{B_r\cap G_\lambda\cap\bar{G}_\lambda} \frac{|\boldsymbol{b}_t(\boldsymbol{X}(t,s,x)) - \boldsymbol{b}_t(\bar{\boldsymbol{X}}(t,s,x))|}{|\boldsymbol{X}(t,s,x) - \bar{\boldsymbol{X}}(t,s,x)| + \delta}\,\mathrm{d}x \leqslant \int_{B_r\cap G_\lambda\cap\bar{G}_\lambda} &|M_{2\lambda}D\boldsymbol{b}_t(\boldsymbol{X}(t,s,x))| \\
&+ |M_{2\lambda}D\boldsymbol{b}_t(\bar{\boldsymbol{X}}(t,s,x))|\,\mathrm{d}x.
\end{aligned}$$

Therefore, we have by (2.7), Hölder inequality, Lemma 2.1, and Definition 2.2 (ii) that

$$\Phi_\delta'(t) \leqslant \frac{\bar{L}}{\delta}\|\boldsymbol{b}_t - \bar{\boldsymbol{b}}_t\|_{L^1(B_\lambda)} + (L + \bar{L})|B_r|^{1-1/p}\|D\boldsymbol{b}_t\|_{L^p(B_{3\lambda})}.$$

Integrating with respect to t in $[s, s+\tau]$ and noticing that $\Phi_\delta(s) = 0$, we have that

$$\Phi_\delta(\tau) \leqslant \frac{\bar{L}}{\delta}\|\boldsymbol{b} - \bar{\boldsymbol{b}}\|_{L^1((0,T)\times B_\lambda)} + (L + \bar{L})|B_r|^{1-1/p}\|D\boldsymbol{b}\|_{L^1((0,T);L^p(B_{3\lambda}))}. \tag{2.8}$$

On the other hand, we have a lower bound for $\Phi_\delta(\tau)$:

$$\Phi_\delta(\tau) \geqslant \log\log\left(e+\frac{\gamma}{\delta}\right)|B_r \cap \{|\boldsymbol{X}(\tau,s,\cdot)-\bar{\boldsymbol{X}}(\tau,s,\cdot)|>\gamma\}\cap G_\lambda \cap \bar{G}_\lambda|.$$

In particular, we have that

$$|B_r \cap \{|\boldsymbol{X}(\tau,s,\cdot)-\bar{\boldsymbol{X}}(\tau,s,\cdot)|>\gamma\}| \leqslant \frac{\Phi_\delta(\tau)}{\log\log\left(e+\frac{\gamma}{\delta}\right)} + |B_r \setminus G_\lambda| + |B_r \setminus \bar{G}_\lambda|.$$

By the above, Lemma 2.5, and (2.8), we conclude

$$\begin{aligned}|B_r \cap \{|\boldsymbol{X}(\tau,s,\cdot)-\bar{\boldsymbol{X}}(\tau,s,\cdot)|>\gamma\}| \leqslant & \frac{\bar{L}\|\boldsymbol{b}-\bar{\boldsymbol{b}}\|_{L^1((0,T)\times B_\lambda)}}{\delta\log\log\left(e+\frac{\gamma}{\delta}\right)} \\ &+ \frac{(L+\bar{L})|B_r|^{1-1/p}\|D\boldsymbol{b}\|_{L^1((0,T);L^p(B_{3\lambda}))}}{\log\log\left(e+\frac{\gamma}{\delta}\right)} \\ &+ f(r,\lambda)+\bar{f}(r,\lambda),\end{aligned}$$

where $\bar{f}(r,\lambda)$ is the function which comes from Lemma 2.5 associated to $\bar{\boldsymbol{X}}$. Now, by Lemma 2.5, we may choose λ large enough so that $f(r,\lambda)<\eta/3$ and $\bar{f}(r,\lambda)<\eta/3$. Moreover, since λ is now fixed, we choose δ small enough so that

$$\frac{(L+\bar{L})|B_r|^{1-1/p}\|D\boldsymbol{b}\|_{L^1((0,T);L^p(B_{3\lambda}))}}{\log\log\left(e+\frac{\gamma}{\delta}\right)} < \frac{\eta}{3}.$$

We recall that since the functions f and $\bar{f}$ depend on the norm (2.2) of $\boldsymbol{b}$ and $\bar{\boldsymbol{b}}$, then so does η, and therefore so does δ. Defining the constant

$$C_{\gamma,\eta,r} := \frac{\bar{L}}{\delta\log\log\left(e+\frac{\gamma}{\delta}\right)},$$

we have that the proposition follows. □

From Proposition 2.1, we may derive uniqueness, stability, and existence of renormalized regular Lagrangian flows. We remark that such results depend only on the estimate contained in the aforementioned proposition, and so in the forthcoming results, in order to obtain uniqueness, stability, and existence of renormalized regular Lagrangian flows, it suffices to prove an analogous result of Proposition 2.1 for more general vector fields. Before we prove the compactness result for renormalized regular Lagrangian flows, we recall a criterion for $L^1(\mathbb{R}^d)$ sequence convergence. For a proof the result below, see Bogachev's book [15, Section 4.5].

Lemma 2.6 (Vitali convergence theorem)**.** *Let (X,μ) be a measure, $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions and f a function in $L^1(\mu)$. Then f_n converges to f in $L^1(\mu)$ if and only if it converges locally in measure, it is uniformly integrable, and has uniformly absolutely continuous integrals, that is,*

$$\lim_{n\to\infty}\int_X |f_n(x)-f(x)|\,\mathrm{d}\mu = 0$$

if and only if

- *(Local convergence in measure) for any $\gamma > 0$ and K any compact set of X, it holds*

$$\lim_{n\to\infty} \mu(K \cap \{|f_n - f| > \gamma\}) = 0.$$

- *(Uniform integrability) for every $\epsilon > 0$, there exists a finite measure set $\Omega_\epsilon \subset \mathbb{R}^d$ such that for any $n \in \mathbb{N}$*

$$\int_{\mathbb{R}^d \setminus \Omega_\epsilon} |f_n(x)| \,\mathrm{d}\mu < \epsilon.$$

- *(Uniform absolutely continuity of integrals) it holds for any $n \in \mathbb{N}$ that*

$$\lim_{|\Omega|\to 0} \int_\Omega |f_n(x)| \,\mathrm{d}\mu = 0.$$

Notice that if μ is a finite measure, e.g. $\mu_R = \mathcal{L}^d \llcorner B_R$ for $R > 0$, then uniform integrability of $\{f_n\}$ is equivalent to uniformly absolutely continuity of integrals and uniform bound of $\|f^n\|_{L^1(\mu)}$; see [15, Proposition 4.5.3]. For a clean presentation of the compactness result, we first prove a technical lemma concerning composition of sequences converging locally in measure.

Lemma 2.7. *Let $\{\boldsymbol{X}^n\}_{n\in\mathbb{N}}$ be a sequence of renormalized regular Lagrangian flows with compressibility constants L_n satisfying $\sup_{n\in\mathbb{N}} L_n < \infty$ such that $\boldsymbol{X}^n(t,s,\cdot)$ converges locally in measure to a renormalized regular Lagrangian flow $\boldsymbol{X}(t,s,\cdot)$ with compressibility constant L for any $s \in [0,T]$ and $t \in [s,T]$. Assume also that its associated sublevels G^n_λ of $\boldsymbol{X}^n$ and G_λ of $\boldsymbol{X}$ have the control of $|B_r \setminus G^n_\lambda|$, $|B_r \setminus G_\lambda|$ as in Lemma 2.5 (e.g. if the sequence of vector fields $\boldsymbol{b}^n$ which $\boldsymbol{X}^n$ is associated can be written as in (2.3)). Moreover, let $\{\phi^n\}_{n\in\mathbb{N}}$ be a sequence of Lebesgue functions converging locally in measure to a Lebesgue measurable function ϕ. Then for any $r > 0$, $\gamma > 0$, $s \in [0,T]$, and $t \in [s,T]$, it holds*

$$\lim_{n\to\infty} |B_r \cap \{|\phi^n(\boldsymbol{X}^n(t,s,\cdot)) - \phi(\boldsymbol{X}(t,s,\cdot))| > \gamma\}| = 0,$$

that is, the composition $\phi^n \circ \boldsymbol{X}^n(t,s,\cdot)$ also converges locally in measure to $\phi \circ \boldsymbol{X}(t,s,\cdot)$.

Proof. We proceed by adapting the proof of [22, Theorem 6.4]: for a fixed $r > 0$,$\gamma > 0$, $s \in [0,T]$, and $t \in [s,T]$, we have

$$\begin{aligned}
&|B_r \cap \{|\phi^n(\boldsymbol{X}^n(t,s,\cdot)) - \phi(\boldsymbol{X}(t,s,\cdot))| > \gamma\}| \\
&\leqslant |B_r \cap G^n_\lambda \cap \{|\phi^n(\boldsymbol{X}^n(t,s,\cdot)) - \phi(\boldsymbol{X}^n(t,s,\cdot))| > \gamma/4\}| \\
&\quad + |B_r \cap G^n_\lambda \cap G_\lambda \cap \{|\phi(\boldsymbol{X}^n(t,s,\cdot)) - \phi(\boldsymbol{X}(t,s,\cdot))| > 3\gamma/4\}| \\
&\quad + |B_r \setminus G_\lambda| + |B_r \setminus G^n_\lambda|,
\end{aligned} \tag{2.9}$$

where G^n_λ is the sublevel of flow $\boldsymbol{X}^n$ and λ is chosen such that for a given $\eta > 0$,

$$|B_r \setminus G_\lambda| + |B_r \setminus G^n_\lambda| < \eta;$$

such choice is possible by Lemma 2.5. Now, by the local convergence in measure of ϕ^n, then we may take n large enough so it holds

$$|B_\lambda \setminus (B_\lambda \cap \{|\phi^n - \phi| < \gamma/4\})| < \eta.$$

Therefore, we have that the first term in (2.9) is bounded by $L_n\eta$. Since the third and fourth terms are bounded by η by the choice of λ, it remains to estimate the second one. By Lusin's theorem, there exists a function $\hat{\phi}$ such that $\hat{\phi} \in C(\bar{B}_\lambda)$ and

$$|\bar{B}_\lambda \cap \{\hat{\phi} \neq \phi\}| \leqslant \eta. \tag{2.10}$$

Moreover, by the continuity of $\hat{\phi}$, then there exists $\alpha > 0$ such that

$$|y - x| \leqslant \alpha \implies |\hat{\phi}(y) - \hat{\phi}(x)| \leqslant \frac{3}{4}\gamma, \tag{2.11}$$

Therefore, we have that

$$\begin{aligned} &|B_r \cap G_\lambda^n \cap G_\lambda \cap \{|\phi(\boldsymbol{X}^n(t,s,\cdot)) - \phi(\boldsymbol{X}(t,s,\cdot))| > 3\gamma/4\}| \\ &\leqslant |B_r \cap G_\lambda^n \cap G_\lambda \cap \{|\hat{\phi}(\boldsymbol{X}^n(t,s,\cdot)) - \hat{\phi}(\boldsymbol{X}(t,s,\cdot))| > 3\gamma/4\}| \\ &\quad + |B_r \cap G_\lambda^n \cap \{\hat{\phi}(\boldsymbol{X}^n(t,s,\cdot)) \neq \phi(\boldsymbol{X}^n(t,s,\cdot))\}| \\ &\quad + |B_r \cap G_\lambda \cap \{\hat{\phi}(\boldsymbol{X}(t,s,\cdot)) \neq \phi(\boldsymbol{X}(t,s,\cdot))\}|. \end{aligned}$$

The second and third terms above can be bounded by $L_n\eta$ and $L\eta$, respectively by (2.10) and the compressibility of $\boldsymbol{X}$ and $\boldsymbol{X}^n$. For the first term, notice that by (2.11) that

$$|\boldsymbol{X}^n(t,s,x) - \boldsymbol{X}(t,s,x)| > \alpha \quad \text{for every } x \in B_r.$$

Choosing n even larger, by the local convergence in measure of $\boldsymbol{X}^n$ we have that

$$|B_r \setminus (B_r \cap \{|\boldsymbol{X}^n(t,s,x) - \boldsymbol{X}(t,s,x)| > \alpha\})| < \eta.$$

Using all of the above, we conclude by (2.9) that

$$|B_r \cap \{|\phi^n(\boldsymbol{X}^n(t,s,\cdot)) - \phi(\boldsymbol{X}(t,s,\cdot))| > \gamma\}| \leqslant C\eta,$$

to some $C > 0$, and so the lemma follows. □

Lemma 2.8 (Stability and compactness of flows)**.** *Let $\{\boldsymbol{b}^n\}_{n\in\mathbb{N}}$ a sequence such that for every $R > 0$*

$$\sup_{n\in\mathbb{N}} \left(\|\tilde{\boldsymbol{b}}^{n,1}\|_{L^1((0,T)\times\mathbb{R}^d)} + \|\tilde{\boldsymbol{b}}^{n,2}\|_{L^1((0,T);L^\infty(\mathbb{R}^d))} + \|\boldsymbol{b}^n\|_{L^1((0,T);L^q(B_R))} \right) < \infty, \tag{2.12}$$

where the decomposition of $\boldsymbol{b}^n$ as in (2.3) and $q > 1$. Moreover, let $\boldsymbol{X}^n$ be the renormalized regular Lagrangian flow associated to $\boldsymbol{b}^n$ starting at $s \in [t,T]$ with compressibility constant L_n, where $\sup_{n\in\mathbb{N}} L_n < \infty$. Assume that $\boldsymbol{b}^n$ converges in $L^1((0,T);L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d))$ to $\boldsymbol{b} \in L^1((0,T);L^q_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d))$ with regularity $D\boldsymbol{b} \in L^1((0,T);L^p_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^{d\times d}))$ for some $p > 1$. Then $\{\boldsymbol{X}^n\}_{n\in\mathbb{N}}$ converges to $\boldsymbol{X}$ locally in measure in $\mathbb{R}^d$ uniformly in s and t, where $\boldsymbol{X}$ is a renormalized regular Lagrangian flow associated to $\boldsymbol{b}$ starting at s.

Proof. **Step 1.** We begin by proving the stability result, that is, if the existence of $\boldsymbol{X}$ is *known*, then the convergence of $\boldsymbol{X}^n$ claimed in the thesis follows. But this is a corollary from Proposition 2.1 by taking $\bar{\boldsymbol{b}} = \boldsymbol{b}^n$, as by our assumptions $C_{\gamma,\eta,r}$ is uniformly bounded with respect to n, and so we may take n large enough in order to obtain for any $r > 0$, $\gamma > 0$ that

$$|B_r \cap \{|\boldsymbol{X}(t,s,\cdot) - \boldsymbol{X}^n(t,s,\cdot)| > \gamma\}| \leqslant C_{\gamma,\eta,r}\|\boldsymbol{b} - \boldsymbol{b}^n\|_{L^1((0,T)\times B_\lambda)} + \eta \leqslant 2\eta$$

uniformly in $s \in [0,T]$ and $t \in [s,T]$. This is precisely the desired convergence.

Step 2. We now prove the compactness result, i.e., we *do not* assume the existence of $\boldsymbol{X}$. For this purpose, we may proceed analogously as before, but now with a Cauchy sequence

$$|B_r \cap \{|\boldsymbol{X}^m(t,s,\cdot) - \boldsymbol{X}^n(t,s,\cdot)| > \gamma\}| \leqslant 2\eta.$$

This implies that there exists a locally measurable in space and continuous in time function $\boldsymbol{X}$ such that

$$|B_r \cap \{|\boldsymbol{X}(t,s,\cdot) - \boldsymbol{X}^n(t,s,\cdot)| > \gamma\}| \to 0 \quad \text{as } n \to \infty, \quad \text{uniformly in } s,\, t.$$

Moreover, by the uniform global bound (2.12), we have by Lemma 2.5 that $\boldsymbol{X}$ satisfies the integrability assumptions of Definition 2.2 (i). Moreover, by Fatou's Lemma, we have that

$$\boldsymbol{X}(t,s,\cdot)_{\#}\mathcal{L}^d \leqslant \liminf_{n\to\infty} \boldsymbol{X}^n(t,s,\cdot)_{\#}\mathcal{L}^d \leqslant \liminf_{n\to\infty} L_n\mathcal{L}^d,$$

and so Definition 2.2 (ii) follows with $L := \liminf_{n\to\infty} L_n$. Hence, it suffices to show that $\boldsymbol{X}$ satisfies the ODE in Definition 2.2 (i). We claim that it suffices to consider $\beta \in C^1_c(\mathbb{R}^d)$; indeed consider for any β satisfying the hypothesis in Definition 2.2 (i) the approximation

$$\beta_\epsilon(x) := \beta(x)\chi\left(\epsilon \log\log(e + \sqrt{|x|^2 + \epsilon})\right),$$

where $\chi \in C^\infty_c([0,\infty))$ is a nonnegative function such that $\chi(z) = 1$ for $z \leqslant 1$ and $\epsilon \geqslant 0$.

Notice that since $\boldsymbol{X}^n(t,s,\cdot)$ converges locally in measure uniformly in s,t, then taking $\varphi \in C^1_c([0,T) \times \mathbb{R}^d)$ with $\operatorname{supp}\varphi_t \subset B_R$ for some $R > 0$, we have

$$\begin{aligned}
&\int_s^T \int_{\mathbb{R}^d} \partial_t\varphi_t(x)[\beta(\boldsymbol{X}^n(t,s,x)) - \beta(\boldsymbol{X}(t,s,x))]\,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant 2T\|\partial_t\varphi\|_{L^\infty((0,T)\times\mathbb{R}^d)} \sup_{s\in[0,T]} \sup_{t\in[s,T]} \int_{B_R} |\beta(\boldsymbol{X}^n(t,s,x)) - \beta(\boldsymbol{X}(t,s,x))|\,\mathrm{d}x.
\end{aligned}$$

By Lemma 2.7, $\beta(\boldsymbol{X}^n(t,s,\cdot)) \to \beta(\boldsymbol{X}(t,s,\cdot))$ locally in measure uniformly in s,t. Moreover, $\beta(\boldsymbol{X}^n(t,s,\cdot))$ has uniform absolutely continuous integrals, for

$$\int_{\Omega\cap B_R} |\beta(\boldsymbol{X}^n(t,s,x))|\,\mathrm{d}x \leqslant \|\beta\|_{L^\infty(\mathbb{R}^d)}|\Omega|.$$

Therefore, we conclude by Lemma 2.6 that

$$\lim_{n\to\infty}\int_0^T\int_{\mathbb{R}^d}\partial_t\varphi_t(x)\beta(\boldsymbol{X}^n(t,s,x))\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_{\mathbb{R}^d}\partial_t\varphi_t(x)\beta(\boldsymbol{X}(t,s,x))\,\mathrm{d}x\,\mathrm{d}t.$$

It remains to show that

$$\begin{aligned}\lim_{n\to\infty}&\int_0^T\int_{\mathbb{R}^d}\varphi_t(x)\nabla\beta(\boldsymbol{X}^n(t,s,x))\cdot\boldsymbol{b}_t^n(\boldsymbol{X}^n(t,s,x))\,\mathrm{d}x\,\mathrm{d}t\\&=\int_0^T\int_{\mathbb{R}^d}\varphi_t(x)\nabla\beta(\boldsymbol{X}(t,s,x))\cdot\boldsymbol{b}_t(\boldsymbol{X}(t,s,x))\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Notice that since $\beta\in C_c^1(\mathbb{R}^d)$, we may consider $\operatorname{supp}\beta\subset B_R$ for $R>0$ large enough. By the same argument as above, it suffices to show that

$$\nabla\beta(\boldsymbol{X}^n(t,s,\cdot))\cdot\boldsymbol{b}_t^n(\boldsymbol{X}^n(t,s,\cdot))\to\nabla\beta(\boldsymbol{X}(t,s,\cdot))\cdot\boldsymbol{b}_t(\boldsymbol{X}(t,s,\cdot))$$

locally in measure uniformly in s,t, for the uniform compressibility constant and uniform bound of $\boldsymbol{b}^n\in L^1((0,T);L^q(B_R;\mathbb{R}^d))$ implies that

$$\begin{aligned}&\int_s^T\int_{\Omega\cap B_R}|\nabla\beta(\boldsymbol{X}^n(t,s,x))\cdot\boldsymbol{b}_t^n(\boldsymbol{X}^n(t,s,x))|\,\mathrm{d}x\,\mathrm{d}t\\&\leqslant|\Omega|^{1-\frac{1}{q}}\|\nabla\beta\|_{L^\infty(\mathbb{R}^d)}\sup_{n\in\mathbb{N}}\left(L_n\|\boldsymbol{b}^n\|_{L^1((0,T);L^q(B_R))}\right)\end{aligned}$$

and so uniform absolutely continuity of integrals holds. But local convergence in measure follows from Lemma 2.7 with $\phi^n=\nabla\beta\cdot\boldsymbol{b}^n$, and so does the lemma. □

We are now ready to prove existence and uniqueness of renormalized regular Lagrangian flows.

Theorem 2.1 (Existence and uniqueness of flows). *Let $\boldsymbol{b}\in L^1((0,T);L^q_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d))$ be a vector field satisfying (2.2) for some $q>1$, $D\boldsymbol{b}\in L^1((0,T);L^p_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^{d\times d}))$ for some $p>1$, and $\operatorname{div}\boldsymbol{b}\geqslant m$ in $(0,T)\times\mathbb{R}^d$ for some $m\in L^1((0,T))$. Then there exists a unique regular renormalized Lagrangian flow, as in Definition 2.2. Moreover, it holds the forward semigroup property: for every $0\leqslant s\leqslant\tau\leqslant t\leqslant T$, it follows that*

$$\boldsymbol{X}(t,\tau,\boldsymbol{X}(\tau,s,x))=\boldsymbol{X}(t,s,x)\quad\textit{for almost every } x\in\mathbb{R}^d.$$

Finally, if one further assumes that $\operatorname{div}\boldsymbol{b}\in L^1((0,T);L^\infty(\mathbb{R}^d))$, then the above also holds backward in time, that is, for any s, τ, $t\in[0,T]$. In particular, taking $t=s$, we have the well posedness of the inverse of $\boldsymbol{X}(\tau,s,\cdot)$, with $\boldsymbol{X}^{-1}(\tau,s,x)=\boldsymbol{X}(s,\tau,x)$ for almost every $x\in\mathbb{R}^d$, and the following regularity holds: $\boldsymbol{X}$ is continuous in $t\in[0,T]$, $s\in[0,T]$ and locally measurable in space, and

$$\int_{B_R}\log\log(e+|\boldsymbol{X}(t,s,x)|)\,\mathrm{d}x<\infty$$

for all $R>0$ and $s,t\in[0,T]$.

Proof. Notice that uniqueness follows from Proposition 2.1, for if $\boldsymbol{X}$ and $\bar{\boldsymbol{X}}$ are flows associated to $\boldsymbol{b}$, then for all $r > 0$ and $\gamma > 0$ it holds

$$|B_r \cap \{|\boldsymbol{X}(t,s,\cdot) - \bar{\boldsymbol{X}}(t,s,\cdot)| > \gamma\}| \leqslant \eta \quad \text{for all } \eta > 0 \text{ and } t \in [s,T],$$

and so uniqueness holds. For existence, we shall use Lemma 2.8 with $\boldsymbol{b}^n = \zeta^n * \boldsymbol{b}$, where $\zeta^n(x) = n\zeta(nx)$ and ζ is a nonnegative radial convolution kernel with support in B_1. By Young convolution inequality, $\boldsymbol{b}^n$ satisfies (2.12) and converges in $L^1((0,T); L^1_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d))$ to $\boldsymbol{b}$. Moreover, since $\operatorname{div} \boldsymbol{b}^n_t \geqslant m(t)$ for ζ is nonnegative, we have that $\boldsymbol{X}^n$ has compressibility constant $L_n = \exp(\|m\|_{L^1((0,T))})$. Therefore, the existence follows.

For the forward semigroup property, notice that for the approximate flows it holds for every $0 \leqslant s \leqslant \tau \leqslant t \leqslant T$ that

$$\boldsymbol{X}^n(t,\tau,\boldsymbol{X}^n(\tau,s,x)) = \boldsymbol{X}^n(t,s,x) \quad \text{for every } x \in \mathbb{R}^d.$$

By considering $\varphi^n := \boldsymbol{X}^n(t,s,\cdot)$ and $\varphi := \boldsymbol{X}(t,s,\cdot)$, we have by Lemma 2.7 that uniformly in $s \in [0,T]$, $\tau \in [s,T]$, and $t \in [\tau,T]$ it holds

$$\boldsymbol{X}^n(t,\tau,\boldsymbol{X}^n(\tau,s,\cdot)) \to \boldsymbol{X}(t,\tau,\boldsymbol{X}(\tau,s,\cdot)) \quad \text{locally in measure in } \mathbb{R}^d.$$

Therefore, the forward semigroup property follows. Finally, if $\operatorname{div} \boldsymbol{b} \in L^1((0,T); L^\infty(\mathbb{R}^d))$, one has existence and uniqueness for the backward flow, that is, a renormalized regular Lagrangian flow for the vector field $-\boldsymbol{b}_{T-t}(x)$, with the compressibility constant $L = \exp(\|\operatorname{div} \boldsymbol{b}\|_{L^1((0,T);L^\infty(\mathbb{R}^d))})$, and so by the same argument as before, now for any time $s,\tau,t \in [0,T]$, the result follows. □

In the more regular case of $\operatorname{div} \boldsymbol{b} \in L^1((0,T); L^\infty(\mathbb{R}^d))$, we are also able to prove the change of variables associated to $x \mapsto \boldsymbol{X}(t,s,x)$.

Proposition 2.2 (Change of variables)**.** *Let $\boldsymbol{b} \in L^1((0,T); L^1_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d))$ be a vector field satisfying (2.2), $D\boldsymbol{b} \in L^1((0,T); L^p_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^{d\times d}))$ for some $p > 1$, and $\operatorname{div} \boldsymbol{b} \in L^1((0,T); L^\infty(\mathbb{R}^d))$. Then it holds*

$$\mathcal{L}^d = \boldsymbol{X}(t,s,\cdot)_\# \exp\left(\int_s^t \operatorname{div} \boldsymbol{b}_\tau(\boldsymbol{X}(\tau,s,\cdot))\,\mathrm{d}\tau\right)\mathcal{L}^d. \tag{2.13}$$

Moreover, if there exists some function $F(t,s,x)$ such that

$$\mathcal{L}^d = \boldsymbol{X}(t,s,\cdot)_\# F(t,s,\cdot)\mathcal{L}^d, \tag{2.14}$$

then we have for almost every $x \in \mathbb{R}^d$ and for all $t,s \in [0,T]$ that

$$F(t,s,x) = \exp\left(\int_s^t \operatorname{div} \boldsymbol{b}_\tau(\boldsymbol{X}(\tau,s,\cdot))\,\mathrm{d}\tau\right).$$

In particular, we have also, by the semigroup property, the other change of variables

$$\exp\left(-\int_s^t \operatorname{div} \boldsymbol{b}_\tau(\boldsymbol{X}(\tau,t,\cdot))\,\mathrm{d}\tau\right)\mathcal{L}^d = \boldsymbol{X}(t,s,\cdot)_\#\mathcal{L}^d. \tag{2.15}$$

Proof. Notice that if there exists F and F' satisfying (2.14), then

$$\boldsymbol{X}(t,s,\cdot)_{\#}F(t,s,\cdot)\mathcal{L}^d = \boldsymbol{X}(t,s,\cdot)_{\#}F'(t,s,\cdot)\mathcal{L}^d.$$

On the other hand, given $g \in L^1(\mathbb{R}^d)$, we have that $f(x) = g(\boldsymbol{X}(s,t,x))$ defines a function in $f \in L^1(\mathbb{R}^d)$, by the compressibility both forward and backward in time, and so by testing the above with f, we have that

$$\int_{\mathbb{R}^d} g(x)F(t,s,x)\,\mathrm{d}x = \int_{\mathbb{R}^d} g(x)F'(t,s,x)\,\mathrm{d}x,$$

where we have used that $\boldsymbol{X}^{-1}(t,s,x) = \boldsymbol{X}(s,t,x)$. Therefore, we conclude that $F(t,s,\cdot) = F'(t,s,\cdot)$ for almost every $x \in \mathbb{R}^d$. Hence, it suffices to prove (2.13)

For this purpose, notice that for approximation flows $\boldsymbol{X}^n$ associated to $\boldsymbol{b}^n = \zeta^n * \boldsymbol{b}$, where ζ^n as in the previous proof, it holds

$$\mathcal{L}^d = \boldsymbol{X}^n(t,s,\cdot)_{\#}\exp\left(\int_s^t \operatorname{div}\boldsymbol{b}^n_\tau(\boldsymbol{X}^n(\tau,s,\cdot))\,\mathrm{d}\tau\right)\mathcal{L}^d.$$

Notice that $\operatorname{div}\boldsymbol{b}^n \to \operatorname{div}\boldsymbol{b}$ in $L^1((0,T);L^1_{\mathrm{loc}}(\mathbb{R}^d))$ and $\operatorname{div}\boldsymbol{b}^n \in L^1((0,T);L^\infty(\mathbb{R}^d))$ uniformly in $n \in \mathbb{N}$, and so by the local measure convergence of $\boldsymbol{X}^n(\tau,s,\cdot)$ uniformly in $\tau, s \in [0,T]$, we have that

$$\operatorname{div}\boldsymbol{b}^n(\boldsymbol{X}^n(\cdot,s,\cdot)) \to \operatorname{div}\boldsymbol{b}(\boldsymbol{X}(\cdot,s,\cdot)) \quad \text{in } L^1((0,T);L^1_{\mathrm{loc}}(\mathbb{R}^d))$$

uniformly in $s \in [0,T]$. Indeed, notice that since

$$\int_0^T\int_\Omega |\operatorname{div}\boldsymbol{b}_\tau(\boldsymbol{X}^n(\tau,s,x))|\,\mathrm{d}x\,\mathrm{d}\tau \leqslant |\Omega|\sup_{n\in\mathbb{N}}\|\operatorname{div}\boldsymbol{b}^n\|_{L^1((0,T);L^\infty(\mathbb{R}^d))},$$

which combined with Lemma 2.7 with $\phi^n = \operatorname{div}\boldsymbol{b}^n_t$ and Lemma 2.6 implies the L^1 convergence. In particular, it holds uniformly in $s,t \in [0,T]$ that

$$\int_s^t \operatorname{div}\boldsymbol{b}^n_\tau(\boldsymbol{X}^n(\tau,s,\cdot))\,\mathrm{d}\tau \to \int_s^t \operatorname{div}\boldsymbol{b}_\tau(\boldsymbol{X}(\tau,s,\cdot))\,\mathrm{d}\tau \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^d).$$

Now, in order to conclude (2.13), it suffices to show that it holds uniformly in $s,t \in [0,T]$

$$\varphi(\boldsymbol{X}^n(t,s,\cdot)) \to \varphi(\boldsymbol{X}(t,s,\cdot)) \quad \text{in } L^1(\mathbb{R}^d)$$

for all $\varphi \in C_c(\mathbb{R}^d)$. This holds once again by Lemma 2.6: the local convergence in measure follows from Lemma 2.7, since that, by the compressibility of flows, we have

$$\|\varphi \circ \boldsymbol{X}^n(t,s,\cdot)\|_{L^1(\mathbb{R}^d)} \leqslant \sup_{n\in\mathbb{N}} L_n\|\varphi\|_{L^1(\mathbb{R}^d)};$$

the uniform absolutely continuity of integrals follows by

$$\int_\Omega |\varphi(\boldsymbol{X}^n(t,s,x))|\,\mathrm{d}x \leqslant |\Omega|\|\varphi\|_{L^\infty(\mathbb{R}^d)};$$

and the uniform integrability follows from Lemma 2.5, for $\operatorname{supp}\varphi \subset B_\lambda$ for some $\lambda > 0$, and so

$$\int_{\mathbb{R}^d \setminus B_r} |\varphi(\boldsymbol{X}^n(t,s,x))| \,\mathrm{d}x \leqslant \|\varphi\|_{L^\infty(\mathbb{R}^d)} |(\mathbb{R}^d \setminus B_r) \cap G_\lambda^n| \leqslant \|\varphi\|_{L^\infty(\mathbb{R}^d)} g(r,\lambda).$$

Thus, (2.15) follows, which is equivalent to (2.13). □

Remark 3. The existence of a nonnegative function

$$F \in C([0,T] \times [0,T]; L^\infty(\mathbb{R}^d) - w*)$$

satisfying (2.14) is guaranteed by only assuming the weaker condition $\operatorname{div} \boldsymbol{b} \geqslant m$ in $(0,T) \times \mathbb{R}^d$; see [22, Proposition 6.7, proof of step 2].

Finally, we show that solutions defined by the transport of the initial data by the regular flow, in the sense (1.3) or (1.5) for the transport and continuity equations, respectively, are also renormalized solutions, in a sense similar as in Condition 1.1.

Theorem 2.2 (Existence of renormalized solutions)**.** *Let $\boldsymbol{b} \in L^1((0,T); L^q_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d))$ be a vector field satisfying (2.2), with $D\boldsymbol{b} \in L^1((0,T); L^p_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^{d\times d}))$ for some $p,q > 1$, and $\operatorname{div} \boldsymbol{b} \in L^1((0,T); L^\infty(\mathbb{R}^d))$. Moreover, let u as in (1.3). Then for any $\beta \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$, the function $\beta \circ u$ is a weak solution of the initial value problem of transport equation (1.1), that is, it solves*

$$\int_0^T \int_{\mathbb{R}^d} [\partial_t \varphi_t(x) + \varphi_t(x) \operatorname{div} \boldsymbol{b}_t(x) + \nabla\varphi_t(x) \cdot \boldsymbol{b}_t(x)] \beta(u_t(x)) \,\mathrm{d}x \,\mathrm{d}t = \int_{\mathbb{R}^d} \varphi_0(x) \beta(u_0(x)) \,\mathrm{d}x$$

for all $\varphi \in C^1_c([0,T) \times \mathbb{R}^d)$ and with initial data $u_0 \in L^1(\mathbb{R}^d)$. Analogously, let v as in (1.5). Then for any $\beta \in L^\infty(\mathbb{R}) \cap C^1(\mathbb{R})$ and $\log(2+|\cdot|)(1+|\cdot|)|\beta'(\cdot)|$ also bounded, the function $\beta \circ v$ is a weak solution of the initial value problem of continuity equation (1.4), that is, it solves

$$\int_0^T \int_{\mathbb{R}^d} [\partial_t \varphi_t(x) + \nabla\varphi_t(x) \cdot \boldsymbol{b}_t(x) + \varphi_t(x) \operatorname{div} \boldsymbol{b}_t(x)] \beta(v_t(x)) - \varphi_t(x) \operatorname{div} \boldsymbol{b}_t(x) \beta'(v_t(x)) v_t(x) \,\mathrm{d}x \,\mathrm{d}t = \int_{\mathbb{R}^d} \varphi_0(x) \beta(v_0(x)) \,\mathrm{d}x$$

for all $\varphi \in C^1_c([0,T) \times \mathbb{R}^d)$ and with initial data $u_0 \in L^1(\mathbb{R}^d)$.

Proof. Notice that for the transport equation case, the function $\beta \circ u$ is given by $\beta \circ u_0(\boldsymbol{X}(0,t,x))$. Therefore, by a density argument, it suffices to show that $w(\boldsymbol{X}(0,t,x))$ is a solution (in the weak sense) to the transport equation starting at $w \in C^\infty_c(\mathbb{R}^d)$. But this can be proven by considering approximations flows $\boldsymbol{X}^n$ associated to $\boldsymbol{b}^n_t = \zeta^n * \boldsymbol{b}_t$ in a similar fashion as in Proposition 2.2. Indeed, $\boldsymbol{b}^n$ and $\operatorname{div} \boldsymbol{b}^n$ converge in $L^1((0,T); L^1_{\mathrm{loc}}(\mathbb{R}^d))$

to $\boldsymbol{b}$ and $\operatorname{div} \boldsymbol{b}$, respectively, and we have also the uniform in time convergence in measure of $\boldsymbol{X}^n$, and the uniform bounds

$$\int_0^T \int_{B_R} |w(\boldsymbol{X}^n(0,t,x))| \, \mathrm{d}x \, \mathrm{d}t \leqslant T|B_R| \|w\|_{L^\infty(\mathbb{R}^d)};$$
$$\int_0^T \int_{\Omega \cap B_R} |w(\boldsymbol{X}^n(0,t,x))| \, \mathrm{d}x \, \mathrm{d}t \leqslant T|\Omega| \|w\|_{L^\infty(\mathbb{R}^d)},$$

where $\operatorname{supp} \varphi \subset B_R$, and so $w(\boldsymbol{X}^n(0,t,\cdot))$ converges to $w(\boldsymbol{X}(0,t,\cdot))$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ uniformly in $t \in [0,T]$.

For the continuity equation case, it suffices to consider $\beta \in C^1_c(\mathbb{R}^d)$ using the approximations β_ϵ in the proof of Lemma 2.8. Moreover, by a density argument it suffices to consider $v_0 \in C^\infty_c(\mathbb{R}^d)$. By considering $\boldsymbol{b}^n = \zeta^n * \boldsymbol{b}$ with associated $\boldsymbol{X}^n$, we have as in the transport case that $\boldsymbol{b}^n$ and $\operatorname{div} \boldsymbol{b}^n$ converge in $L^1((0,T); L^1_{\mathrm{loc}}(\mathbb{R}^d))$ to $\boldsymbol{b}$ and $\operatorname{div} \boldsymbol{b}$, respectively, as well as $v_0(\boldsymbol{X}^n(0,t,\cdot))$ converges to $v_0(\boldsymbol{X}(0,t,\cdot))$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ uniformly for $t \in [0,T]$. Moreover, we have the following uniform convergence with respect to $t \in [0,T]$ proven in Proposition 2.2:

$$\exp\left(- \int_0^t \operatorname{div} \boldsymbol{b}^n_\tau(\boldsymbol{X}^n(\tau,t,\cdot)) \, \mathrm{d}\tau \right) \to \exp\left(- \int_0^t \operatorname{div} \boldsymbol{b}_\tau(\boldsymbol{X}(\tau,t,\cdot)) \, \mathrm{d}\tau \right) \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^d).$$

In particular, this implies that

$$v^n_t := v_0(\boldsymbol{X}^n(0,t,\cdot)) \exp\left(- \int_0^t \operatorname{div} \boldsymbol{b}^n_\tau(\boldsymbol{X}^n(\tau,t,\cdot)) \, \mathrm{d}\tau \right) \to v_t \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^d)$$

uniformly for $t \in (0,T)$. Of course, this implies that $\beta(v^n_t) \to \beta(v_t)$ and $\beta'(v^n_t) \to \beta'(v_t)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ uniformly for $t \in (0,T)$. Therefore, the theorem follows. □

Remark 4. If one replaces the assumption $\operatorname{div} \boldsymbol{b} \in L^1((0,T); L^\infty(\mathbb{R}^d))$ with

$$\operatorname{div} \boldsymbol{b} \in L^1_{\mathrm{loc}}((0,T) \times \mathbb{R}^d) \quad \text{and} \quad \operatorname{div} \boldsymbol{b} \geqslant m \text{ in } (0,T) \times \mathbb{R}^d,$$

then it is possible to show that

$$u_t(x) = u_T(\boldsymbol{X}(T,t,x))$$

is a solution of the terminal value problem

$$\begin{cases} \partial_t u + \boldsymbol{b} \cdot \nabla u = 0 & \text{in} \quad [0,T) \times \mathbb{R}^d; \\ u_{t=T} = u_T & \text{on} \quad \mathbb{R}^d; \end{cases}$$

see [22, Proposition 7.2]. Moreover, in [22, Propositions 7.2 and 7.5], it was proven that if $u^T \in L^q(\mathbb{R}^d)$ for some $q \in [1,\infty]$, then the Lagrangian solution for the transport and continuity equation $u \in C((0,T); L^q(\mathbb{R}^d))$ if $q \in [1,\infty)$ or

$$u \in C((0,T); L^\infty(\mathbb{R}^d) - w*) \cap C((0,T); L^1_{\mathrm{loc}}(\mathbb{R}^d))$$

if $q = \infty$. If one only has that u^T is a measurable function, then the Lagrangian solution is continuous in time and locally measurable in space, and it is a renormalized solutions of the transport or continuity equation.

Remark 5. If one assumes that $\boldsymbol{b}^n$ has the relevant uniform boundedness with respect to n, namely the ones of $\boldsymbol{b}$ from Theorem 2.2, then we have stability of Lagrangian solutions associated to transport and continuity equations [22, Propositions 7.3 and 7.6]. With this result at hand, it is possible to prove that if one has initial data u_0^n weakly converging to u_0 in $L^q(\mathbb{R}^d)$ for some $q \in [1, \infty]$, then one has weak convergence if $q \in [1, \infty)$ (and weak* convergence if $q = \infty$) of Lagrangian solutions u^n starting at u_0^n to u Lagrangian solutions with initial data u; see [22, Proposition 7.7] for these results for transport and continuity equations.

2.3 Singular kernels and $W^{1,1}$: the Bouchut-Crippa's result

In the previous section, the key estimate in Proposition 2.1 and the control Lemma 2.5 are sufficient to imply uniqueness, existence and forward semigroup property for the flow of a vector field $\boldsymbol{b} \in L^1((0,T); L^q_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d))$ (for some $q > 1$) satisfying also $\operatorname{div} \boldsymbol{b} \geqslant m$ and the growth assumption (2.2). Indeed, notice that Lemma 2.8, Theorem 2.1, and even the results assuming $\operatorname{div} \boldsymbol{b} \in L^1((0,T); L^\infty(\mathbb{R}^d))$, such as Proposition 2.2 and Theorem 2.2 do not depend explicitly on the assumption on the Jacobian $D\boldsymbol{b} \in L^1((0,T); L^p_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^{d\times d}))$ for some $p > 1$ nor on (2.2), but rather on the fundamental estimate

$$|B_r \cap \{|\boldsymbol{X}(t,s,\cdot) - \bar{\boldsymbol{X}}(t,s,\cdot)| > \gamma\}| \leqslant C_{\gamma,\eta,r} \|\boldsymbol{b} - \bar{\boldsymbol{b}}\|_{L^1((0,T)\times B_\lambda)} + \eta$$

(see Proposition 2.1) and on the control of sub and superlevels

$$|B_r \setminus G_\lambda| \leqslant f(r,\lambda) \quad \text{and} \quad |G_\lambda \setminus B_r| \leqslant g(r,\lambda)$$

(see Lemma 2.5).

Accordingly, we shall reproduce Bouchut-Crippa's proof of the analogous result Proposition 2.1 for vector fields satisfying (1.6) (instead of the milder (2.2)) and

$$\begin{aligned} &\boldsymbol{b} \in L^p_{\text{loc}}((0,T) \times \mathbb{R}^d; \mathbb{R}^d) \quad \text{for some } p > 1; \\ &\partial_j \boldsymbol{b}^i_t = \sum_{k=1}^m \Gamma^{ijk} * g^{ijk}_t \quad \text{in the weak sense,} \end{aligned} \tag{2.16}$$

where Γ^{ijk} is a singular kernel as in Section 2.1 and $g^{ijk} \in L^1((0,T) \times \mathbb{R}^d)$ for each $i,j,k \in \{1,\dots,d\}$.

Remark 6. Notice that the space of vector fields satisfying (2.16) are not contained nor contains the space $L^1((0,T); \mathrm{BV}(\mathbb{R}^d))$. On one hand, this implies that the class of vector fields studied by Bouchut-Crippa did not provide a direct Lagrangian approach for the vector fields studied by Ambrosio [4]; on the other hand, notice that it provides a suitable flow for vector fields beyond the scope of renormalization technique.

For this purpose, we begin by recalling two slightly obscure results concerning the space $L^1(\mathbb{R}^d)$. The first one is an interpolation result between integrable function spaces and weak Lebesgue spaces. For a proof, we refer to [22, Lemma 2.2].

Lemma 2.9 (Interpolation for L^1)**.** *Let Ω be a finite measurable subset of $\mathbb{R}^d$ and $u \in L^q_w(\Omega) \cap L^1_w(\Omega)$ for some $q > 1$. Then $u \in L^1(\Omega)$, and it holds the estimate*

$$\|u\|_{L^1(\Omega)} \leqslant \frac{q}{q-1} |||u|||_{L^1_w(\Omega)} \left[1 + \log \left(\frac{|\Omega|^{1-\frac{1}{q}} |||u|||_{L^q_w(\Omega)}}{|||u|||_{L^1_w(\Omega)}} \right) \right] \quad \text{if } q < \infty;$$

$$\|u\|_{L^1(\Omega)} \leqslant |||u|||_{L^1_w(\Omega)} \left[1 + \log \left(\frac{|\Omega| \|u\|_{L^\infty(\Omega)}}{|||u|||_{L^1_w(\Omega)}} \right) \right] \quad \text{if } q = \infty.$$

We also recall a uniform decomposition for a sequence of functions uniformly integrable and with uniform absolutely continuous integrals; as mentioned in [22, Proposition 5.8], the proof is quite straightforward.

Lemma 2.10. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $L^1(\mathbb{R}^d)$ and let $p \in [1, \infty)$. Then the sequence is uniformly integrable and with uniform absolutely continuous integrals if and only if given $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ and a finite measure Borel set $\Omega_\epsilon \subset \mathbb{R}^d$ such that for any $n \in \mathbb{N}$, we have*

$$f_n = f_n^1 + f_n^2,$$

where $\|f_n^1\|_{L^1(\mathbb{R}^d)} \leqslant \epsilon$ and $\operatorname{supp} f_n^2 \subset \Omega_\epsilon$, $\|f_n^2\|_{L^p(\mathbb{R}^d)} \leqslant C_\epsilon$.

Finally, as stated in Section 2.1, the cancellation property associated to singular kernels implies that a "composition" of them is also a singular kernel, i.e. for Γ^1 and Γ^2 singular kernels, $\Gamma^1 * \Gamma^2$ is also one. We shall state an analogous result for singular kernels and grand maximal operator which were defined in Section 2.1. For a proof, we refer to [22, Theorem 3.3]; we shall revisit this type of result in Chapter 4 in an analogous case, and since the proof is quite similar, we skip the one for the following result.

Definition 2.4 (Singular kernel of fundamental type)**.** We say Γ is a singular kernel of fundamental type if it satisfies the following: there exists a constant $C_0 > 0$ such that

$$|\Gamma(x)| \leqslant C_0|x|^{-d}, \quad |\nabla\Gamma(x)| \leqslant C_0|x|^{-d-1}\, \forall x \in \mathbb{R}^d \setminus \{0\}, \text{ and } \quad \left| \int_{B_R \setminus B_r} \Gamma(x)\, \mathrm{d}x \right| \leqslant C_0$$

for all $0 < r < R < \infty$.

Theorem 2.3 (Weak Lebesgue space estimate)**.** *Let Γ be a singular kernel of fundamental type and $\{\rho^\nu\}_\nu$ be a family of functions in $L^\infty(\mathbb{R}^d)$ such that $\operatorname{supp} \rho^\nu \subset B_1$, $\sup_\nu \|\rho^\nu\|_{L^1(\mathbb{R}^d)} \leqslant$*

C_0, and

$$\sup_{\epsilon>0}\sup_{\nu}\|(\epsilon^d\Gamma(\epsilon\,\cdot\,)) * \rho^\nu\|_{L^\infty(\mathbb{R}^d)} \leqslant C_0.$$[3]

Therefore, it holds the following:

(i) there exists a constant $C > 0$ depending on C_0 and on the dimension d such that

$$|||M_{\rho^\nu}(\Gamma * u)|||_{L^1_w(\mathbb{R}^d)} \leqslant C\|u\|_{L^1(\mathbb{R}^d)} \quad \text{for all } u \in L^1(\mathbb{R}^d);$$

(ii) if $\rho^\nu \in C_c^\infty(\mathbb{R}^d)$, then it holds with the same constant $C > 0$ as before that

$$|||M_{\rho^\nu}(\Gamma * \mu)|||_{L^1_w(\mathbb{R}^d)} \leqslant C\mu(\mathbb{R}^d) \quad \text{for all } \mu \in \mathcal{M}(\mathbb{R}^d);$$

(iii) if $\sup_\nu \|\rho^\nu\|_{L^\infty(\mathbb{R}^d)} \leqslant C_0$, then there exists a constant $C' > 0$ depending on $C_0 > 0$, on dimension d, and on $p \in (1,\infty)$ such that

$$\|M_{\rho^\nu}(\Gamma * u)\|_{L^p(\mathbb{R}^d)} \leqslant C'\|u\|_{L^p(\mathbb{R}^d)} \quad \text{for all } u \in L^p(\mathbb{R}^d).$$

We are now ready to state the Proposition below, which states fundamental estimate (the same as Proposition 2.1) associated to vector fields satisfying (2.16) found in [22, Proposition 5.9]. We remark that (2.2) is no longer sufficient for our computations, and so we assume the classical (1.6).

Proposition 2.3. *Let $\boldsymbol{b}$ and $\bar{\boldsymbol{b}}$ be vector fields satisfying (1.6), with $\boldsymbol{b}$ as in (2.16), and $\boldsymbol{X}$, $\bar{\boldsymbol{X}}$ their renormalized regular Lagrangian flows starting at time s with compressibility constants L and $\bar{L}$, respectively. Then for every $\gamma > 0$, $\eta > 0$, and $r > 0$, there exists $\lambda > 0$ and a constant $C_{\gamma,\eta,r} > 0$ such that*

$$|B_r \cap \{|\boldsymbol{X}(t,s,\cdot) - \bar{\boldsymbol{X}}(t,s,\cdot)| > \gamma\}| \leqslant C_{\gamma,\eta,r}\|\boldsymbol{b} - \bar{\boldsymbol{b}}\|_{L^1((0,T)\times B_\lambda)} + \eta$$

uniformly in $s \in [0,T]$ and $t \in [s,T]$. The constant $C_{\gamma,\eta,r}$ depends only on its subscripts, the compressibility constants L and $\bar{L}$, the norms (1.6) of $\boldsymbol{b}$ and $\bar{\boldsymbol{b}}$, $\|\boldsymbol{b}\|_{L^p((s,t)\times B_\lambda)}$ for any $t \in (s,T)$, $\|g^{ijk}\|_{L^1((0,T)\times\mathbb{R}^d)}$, and the constants at Theorem 2.3.

Proof. Analogously to the proof of Proposition 2.1, we consider now for fixed $\delta > 0$, $\lambda > 0$, and $t \in [s,T]$ the function

$$\Phi_\delta(t) = \int_{B_r\cap G_\lambda\cap\bar{G}_\lambda} \log\left(1 + \frac{|\boldsymbol{X}(t,s,x) - \bar{\boldsymbol{X}}(t,s,x)|}{\delta}\right)\mathrm{d}x,$$

[3] Such condition is fulfilled if the family ρ^ν is regular enough, e.g. $\rho^\nu \in H^s(\mathbb{R}^d)$ for $s > d/2$.

and it satisfies for all $\tau \in [s,T]$

$$\begin{aligned}\Phi_\delta(\tau) = \int_s^\tau \Phi'_\delta(t)\,\mathrm{d}t &\leqslant \int_s^\tau \int_{B_r\cap G_\lambda\cap\bar G_\lambda} \frac{|\boldsymbol{b}_t(\boldsymbol{X}(t,s,x)) - \bar{\boldsymbol{b}}_t(\bar{\boldsymbol{X}}(t,s,x))|}{|\boldsymbol{X}(t,s,x) - \bar{\boldsymbol{X}}(t,s,x)| + \delta}\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant \int_s^\tau \int_{B_r\cap G_\lambda\cap\bar G_\lambda} \frac{|\boldsymbol{b}_t(\bar{\boldsymbol{X}}(t,s,x)) - \bar{\boldsymbol{b}}_t(\bar{\boldsymbol{X}}(t,s,x))|}{|\boldsymbol{X}(t,s,x) - \bar{\boldsymbol{X}}(t,s,x)| + \delta}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_s^\tau \int_{B_r\cap G_\lambda\cap\bar G_\lambda} \frac{|\boldsymbol{b}_t(\boldsymbol{X}(t,s,x)) - \boldsymbol{b}_t(\bar{\boldsymbol{X}}(t,s,x))|}{|\boldsymbol{X}(t,s,x) - \bar{\boldsymbol{X}}(t,s,x)| + \delta}\,\mathrm{d}x\,\mathrm{d}t \\ &=: \mathrm{I}(\tau) + \mathrm{II}(\tau)\end{aligned}$$

The first integral on the right-hand side is bounded as in Proposition 2.1:

$$\mathrm{I}(\tau) \leqslant \frac{\bar L}{\delta}\|\boldsymbol{b} - \bar{\boldsymbol{b}}\|_{L^1((0,T)\times B_\lambda)}.$$

For the second integral, we use the second part of Lemma 2.2, for Theorem 2.3 gives that $M_{\Upsilon^{j,\xi}}(\partial_j \boldsymbol{b}_t^i)(x) < \infty$[4] for all $i,j \in \{1,\dots,d\}$ and almost every $x \in \mathbb{R}^d$, $t \in [s,T]$, and so

$$\begin{aligned}\mathrm{II}(\tau) \leqslant \sum_{i,j=1}^d \int_s^\tau \int_{B_r\cap G_\lambda\cap\bar G_\lambda} \min\Bigg\{ & M_{\Upsilon^{j,\xi}}\partial_j \boldsymbol{b}_t^i(\boldsymbol{X}(t,s,x)) + M_{\Upsilon^{j,\xi}}\partial_j \boldsymbol{b}_t^i(\bar{\boldsymbol{X}}(t,s,x)), \\ & \frac{|\boldsymbol{b}_t^i(\boldsymbol{X}(t,s,x))|}{\delta} + \frac{|\boldsymbol{b}_t^i(\bar{\boldsymbol{X}}(t,s,x))|}{\delta}\Bigg\}\,\mathrm{d}x.\end{aligned}$$

By Lemma 2.10 with $n = (i,j,k)$ and (2.16), we may write

$$\partial_j \boldsymbol{b}_t^i(x) = \sum_{k=1}^m \Gamma^{ijk} * g_t^{ijk}(x) = \sum_{k=1}^m \Gamma^{ijk} * \bar g_t^{ijk}(x) + \Gamma^{ijk} * \tilde g_t^{ijk,2}(x),$$

where $\|\bar g^{ijk}\|_{L^1((0,T)\times\mathbb{R}^d)} \leqslant \epsilon$ and $\|\tilde g^{ijk,2}\|_{L^2((0,T)\times\mathbb{R}^d)} \leqslant C_\epsilon$ uniformly in i,j,k. Therefore, we have that

$$\begin{aligned}\mathrm{II}(\tau) \leqslant \sum_{k=1}^m \sum_{i,j=1}^d \int_s^\tau \int_{B_r\cap G_\lambda\cap\bar G_\lambda} \min\Bigg\{ & M_{\Upsilon^{j,\xi}}\Gamma^{ijk} * \bar g_t^{ijk}(\boldsymbol{X}(t,s,x)) \\ & + M_{\Upsilon^{j,\xi}}\Gamma^{ijk} * \bar g_t^{ijk}(\bar{\boldsymbol{X}}(t,s,x)), \\ & \frac{|\boldsymbol{b}_t^i(\boldsymbol{X}(t,s,x))|}{\delta} + \frac{|\boldsymbol{b}_t^i(\bar{\boldsymbol{X}}(t,s,x))|}{\delta}\Bigg\} \\ & + M_{\Upsilon^{j,\xi}}\Gamma^{ijk} * \tilde g_t^{ijk}(\boldsymbol{X}(t,s,x)) \\ & + M_{\Upsilon^{j,\xi}}\Gamma^{ijk} * \tilde g_t^{ijk}(\bar{\boldsymbol{X}}(t,s,x))\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

The second term is simply bounded by the compressibility assumption Definition 2.2 (ii) and Theorem 2.3, so that

$$\begin{aligned}&\int_s^\tau \int_{B_r\cap G_\lambda\cap\bar G_\lambda} M_{\Upsilon^{j,\xi}}\Gamma^{ijk} * \tilde g_t^{ijk}(\boldsymbol{X}(t,s,x))\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant [(\tau - s)|B_r|]^{1/2} L \|M_{\Upsilon^{j,\xi}}\Gamma^{ijk} * \tilde g^{ijk}\|_{L^2((s,\tau)\times B_\lambda)} \\ &\leqslant C[T|B_r|]^{1/2} L \|\tilde g^{ijk}\|_{L^2((0,T)\times\mathbb{R}^d)} \leqslant C_0 C_\epsilon,\end{aligned}$$

[4] Recall that $\partial_i \boldsymbol{b}_j = \Gamma^{ij} * g$, and an weakly summable function is finite almost everywhere.

and analogously the third term is bounded. Therefore, it suffices to estimate the first integral. For this purpose, we shall exploit Lemma 2.9: we have a L^1_w-bound by Definition 2.2 (ii) and Theorem 2.3, for

$$\left\| \left| M_{\Upsilon^{j,\xi}} \Gamma^{ijk} * \bar{g}^{ijk}(\boldsymbol{X}(\cdot, s, \cdot)) \right| \right\|_{L^1_w((s,\tau)\times G_\lambda)} \mathrm{d}t \leqslant CL \|\bar{g}^{ijk}\|_{L^1((0,T)\times\mathbb{R}^d)} \leqslant C_1\epsilon,$$

and also the L^p_w-bound follows by (2.16) and Definition 2.2 (ii):

$$\| \boldsymbol{b}(\boldsymbol{X}(\cdot, s, \cdot)) \|_{L^p_w((s,\tau)\times G_\lambda)} \leqslant CL \|\boldsymbol{b}\|_{L^p((s,\tau)\times B_\lambda)} \leqslant C_2.$$

Therefore, we have that (recall that $t \mapsto t\log(C/t))$ is and increasing function)

$$\begin{aligned} \int_{B_r\cap G_\lambda \cap \bar{G}_\lambda} \min \Bigg\{ \int_s^\tau & M_{\Upsilon^{j,\xi}}\Gamma^{ijk} * \bar{g}^{ijk}_t(\boldsymbol{X}(t,s,x)) + M_{\Upsilon^{j,\xi}}\Gamma^{ijk} * \bar{g}^{ijk}_t(\bar{\boldsymbol{X}}(t,s,x))\,\mathrm{d}t, \\ & \int_s^\tau \frac{|\boldsymbol{b}^i_t(\boldsymbol{X}(t,s,x))|}{\delta} + \frac{|\boldsymbol{b}^i_t(\bar{\boldsymbol{X}}(t,s,x))|}{\delta}\,\mathrm{d}t \Bigg\}\,\mathrm{d}x \\ & \leqslant 2C_1\epsilon\left[1 + \log\left(\frac{[T|B_r|]^{1-\frac{1}{p}}C_2}{C_1\delta\epsilon}\right)\right], \end{aligned}$$

and so it holds

$$\Phi_\delta(\tau) \leqslant \frac{C}{\delta}\|\boldsymbol{b} - \bar{\boldsymbol{b}}\|_{L^1((0,T)\times B_\lambda)} + C_0C_\epsilon + C_1\epsilon\left[1 + \log\left(\frac{[T|B_r|]^{1-\frac{1}{p}}C_2}{C_1\epsilon\delta}\right)\right].$$

On the other hand, we have a lower bound for $\Phi_\delta(\tau)$ computed in Proposition 2.1, but now with $\log(1 + |x|)$ instead of $\log\log(e + |x|)$:

$$\begin{aligned} |B_r \cap \{|\boldsymbol{X}(\tau,s,\cdot) - \bar{\boldsymbol{X}}(\tau,s,\cdot)| > \gamma\}| \leqslant & \frac{\Phi_\delta(\tau)}{\log\left(1+\frac{\gamma}{\delta}\right)} + f(r,\lambda) + \bar{f}(r,\lambda) \\ \leqslant & \frac{C}{\log\left(1+\frac{\gamma}{\delta}\right)\delta}\|\boldsymbol{b} - \bar{\boldsymbol{b}}\|_{L^1((0,T)\times B_\lambda)} \\ & + \frac{C_0C_\epsilon}{\log\left(1+\frac{\gamma}{\delta}\right)} \\ & + \frac{C_1\epsilon}{\log\left(1+\frac{\gamma}{\delta}\right)}\left[1 + \log\left(\frac{[T|B_r|]^{1-\frac{1}{p}}C_2}{C_1\epsilon\delta}\right)\right] \\ & + f(r,\lambda) + \bar{f}(r,\lambda) \end{aligned}$$

We now proceed analogously to the conclusion of Proposition 2.1: by Lemma 2.5, we may choose λ large enough so that $f(r,\lambda) < \eta/4$ and $\bar{f}(r,\lambda) < \eta/4$. Moreover, we may take ϵ small enough so that

$$\frac{C_1\epsilon}{\log\left(1+\frac{\gamma}{\delta}\right)}\left[1 + \log\left(\frac{[T|B_r|]^{1-\frac{1}{p}}C_2}{C_1\epsilon\delta}\right)\right] < \frac{\eta}{4},$$

for the above is uniformly bounded with respect to δ[5]. Moreover, since λ and ϵ are now fixed, we choose δ small enough so that

$$\frac{C_0 C_\epsilon}{\log\left(1+\frac{\gamma}{\delta}\right)} < \frac{\eta}{4},$$

Defining the constant

$$C_{\gamma,\eta,r} := \frac{C}{\delta \log\left(1+\frac{\gamma}{\delta}\right)},$$

we have that the proposition follows. □

Remark 7. We conclude this section by proving that Proposition 2.3 can be applied to vector fields $\boldsymbol{b} \in L^p((0,T); W^{1,1}_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d))$ if $p > 1$. For this purpose, we proceed as Nguyen [70]: first, recall that we have the identity

$$\boldsymbol{b}_t = \sum_{k=1}^{d} \mathcal{R}_k^2(\chi_R \boldsymbol{b}_t) \quad \text{in } B_R, \tag{2.17}$$

where $\chi_R \in C_c^\infty(\mathbb{R}^d)$ satisfies $\chi_R \equiv 1$ in B_{2R} and $\chi_R \equiv 0$ in $\mathbb{R}^d \setminus B_{4R}$ and $\mathcal{R}_k$ is the Riesz transforms in $\mathbb{R}^d$ for $k \in \{1,\dots,d\}$. Now, notice that the proof of Proposition 2.3 also holds for vector fields satisfying a similar structure as in (2.17) for every $R > 0$, that is, for any given $R > 0$, there exists singular kernels Γ_R^j and functions $g^{j,R} \in L^p((0,T); W^{1,1}(\mathbb{R}^d))$ for some $p > 1$ such that

$$\boldsymbol{b}_t = \sum_{k=1}^{m} \Gamma^{k,R} * g_t^{k,R} \quad \text{in } B_R \quad \text{for almost all } t \in (0,T).$$

Indeed, the Sobolev embedding implies that $g^{j,R} \in L^q((0,T)\times\mathbb{R}^d)$ for $q = \min\{p, 3/2\}$. Moreover, we have that

$$\partial_j \boldsymbol{b}_t^i(x) = \sum_{k=1}^{m} \Gamma^{ik,R} * \partial_j g_t^{k,R} \quad \text{in } B_R \quad \text{for almost all } t \in (0,T),$$

and $\partial_j g^{k,R} \in L^1((0,T)\times\mathbb{R}^d)$ uniformly in j, k, and $R > 0$. Since the integral defining Φ_δ is local in B_r, then the thesis of Proposition 2.3 follows. In particular, by (2.17), we have that results of Section 2.1 also hold for $\boldsymbol{b} \in L^p((0,T); W^{1,1}_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d))$ if $p > 1$.

It is interesting to observe that in the result Proposition 2.3 we assumed a mild higher integrability in time, which is undesirable not only from a mathematical perspective, but also from a compatibility one: we only assumed so far that $\boldsymbol{b} \in L^1((0,T); L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d))$, so that (1.6) holds, and $D\boldsymbol{b} \in L^1((0,T); W^{1,p}(\mathbb{R}^d;\mathbb{R}^{d\times d}))$ if $p > 1$. Since (2.17) is just a technical consideration, we only needed to assume higher integrability in time to "match" the hypothesis on Proposition 2.3. This shall be addressed in Section 2.5 in a more general context of BV vector fields.

[5] Here, the proof requires the growth assumption (1.6) instead of the more mild (2.2).

Remark 8. It is possible to extend the aforementioned results for vector fields with Vlasov-type structure, as in Remark 9. For this purpose, the authors in [18] proved that with the additional structure of the vector field, one may relax (1.6) for only assuming the control of sublevels, namely $|B_r \setminus G_\lambda|$ being arbitrary small for a fixed r; see Lemma 2.5. Moreover, the same authors proved in [17] that one has compatibility of solutions to the $2D$ Euler equation and $2D$ Euler vorticity equation

$$\begin{cases} \partial_t u + \omega u^\perp + \nabla \left(p + \dfrac{|u|^2}{2} \right) = 0, \\ \operatorname{div} u = 0, \\ u_{t=0} = u_0; \end{cases} \qquad \begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \\ \omega_{t=0} = \omega_0, \end{cases}$$

where $u = K * \omega$ and ∇K is a singular kernel of fundamental type. More precisely, they show that solutions transported by the flow, weak and symmetrized solutions are all equivalent notions.

2.4 Anisotropic vector fields: Bohun-Bouchut-Crippa's result

The duo Bouchut-Crippa further explored the result of Section 2.3–joined with Bohun–for vector fields satisfying analogous assumptions of (2.16):

$$\begin{aligned} \boldsymbol{b}_t(x_1, x_2) &= (\boldsymbol{b}_t^1(x_1, x_2), \boldsymbol{b}_t^2(x_1, x_2)) && \text{with the splitting } \mathbb{R}^d = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}; \\ \boldsymbol{b} &\in L^p_{\mathrm{loc}}((0,T) \times \mathbb{R}^d; \mathbb{R}^d) && \text{for some } p > 1; \\ D\boldsymbol{b}_t &= \begin{pmatrix} D_1 \boldsymbol{b}^1 & D_2 \boldsymbol{b}^1 \\ D_1 \boldsymbol{b}^2 & D_2 \boldsymbol{b}^2 \end{pmatrix} && \text{in the weak sense,} \end{aligned} \tag{2.18}$$

as well as the ever present growth assumption (1.6). The structure of the derivative is assumed to be the following:

$$\begin{aligned} (D_1 \boldsymbol{b}_t^1)^{ij}(x_1, x_2) &= \sum_{k=1}^m \alpha_t^{ijk}(x_2) \Gamma_1^{ijk} * g_t^{ijk}(x_1) && \text{for } i, j \in \{1, \dots, n_1\}; \\ (D_2 \boldsymbol{b}_t^1)^{ij}(x_1, x_2) &= \sum_{k=1}^m \beta_t^{ijk}(x_2) \Gamma_2^{ijk} * \bar{g}_t^{ijk}(x_1) && \text{for } i \in \{1, \dots, n_1\},\ j \in \{n_1 + 1, \dots, d\}; \\ (D_1 \boldsymbol{b}_t^2)^{ij}(x_1, x_2) &= \sum_{k=1}^m \gamma_t^{ijk}(x_2) \Gamma_3^{ijk} * \mu_t^{ijk}(x_1) && \text{for } i \in \{n_1 + 1, \dots, d\},\ j \in \{1, \dots, n_1\}; \\ (D_2 \boldsymbol{b}_t^2)^{ij}(x_1, x_2) &= \sum_{k=1}^m \psi_t^{ijk}(x_2) \Gamma_4^{ijk} * \tilde{g}_t^{ijk}(x_1) && \text{for } i, j \in \{n_1 + 1, \dots, d\}, \end{aligned} \tag{2.19}$$

where for all suitable i, j, k, it holds $\alpha^{ijk}, \beta^{ijk}, \gamma^{ijk}, \psi^{ijk} \in L^\infty((0,T); L^q(\mathbb{R}^{n_2}))$ for some $q > 1$, $\Gamma_1^{ijk}, \dots, \Gamma_4^{ijk}$ are singular kernels of fundamental type in $\mathbb{R}^{n_1}$ as in Theorem 2.3, $g^{ijk}, \bar{g}^{ijk}, \tilde{g}^{ijk} \in L^1((0,T) \times \mathbb{R}^{n_1})$, and $\mu \in L^1((0,T); \mathcal{M}(\mathbb{R}^{n_1}))$.

Remark 9. The natural application of vector fields with structure (2.18) in Vlasov type systems, where $\boldsymbol{b}_t(x,v) = (\xi(v), E_t(x))$ and $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$, being ξ usually a Lipschitz function and $E_t(x) = \nabla(-\Delta)^{-1}\mu_t$ for μ a integrable in time finite measure, and so

$$D\boldsymbol{b}_t(x,v) = \begin{pmatrix} 0 & \nabla_v \xi(v) \\ D^2(-\Delta)^{-1}\mu_t(x) & 0 \end{pmatrix} \quad \text{in the weak sense.}$$

Notice that for every $i,j \in \{1,\dots,d\}$

$$\partial_{ij}(-\Delta)^{-1}\mu_t(x) = \int_{\mathbb{R}^d} \frac{\delta_{ij} - d\omega_i(x-y)\omega_j(x-y)}{|x-y|^d}\,\mathrm{d}\mu_t(y) =: \Gamma^{ij} * \mu_t(x),$$

where $\omega(z) = z/|z|$ is the direction vector field. It is straightforward to show that Γ^{ij} is a singular kernel of fundamental type. The authors of [18] remarked that the strategy they used for the case $\mu_t = \rho_t \mathcal{L}^d$ for some bounded in time summable function ρ was not suitable for a precise definition of a Lagrangian solution without energy hypothesis. This was later remedied by Ambrosio-Colombo-Figalli in [6] with a notion of generalized solution of such equation; we shall explore this in Section 3.2.

In order to prove the analogous result of Proposition 2.1 and Proposition 2.3, we shall state the "anisotropic" version of Lemma 2.2 proved in [16, Lemma 5.5]:

Lemma 2.11. *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and A the matrix*

$$A = \operatorname{diag}\{\underbrace{\delta_1,\dots,\delta_1}_{n_1 \text{ times}}, \underbrace{\delta_2,\dots,\delta_2}_{n_2 \text{ times}}\}.$$

Assume further that it holds in the weak sense

$$\partial_j u(x_1,x_2) = \sum_{k=1}^m \alpha^{jk}(x_2)\Gamma^{jk} * \mu^{jk}(x_1)$$

for some $\alpha^{jk} \in L^1((0,T);L^q(\mathbb{R}^{n_2}))$ for some $q > 1$, being μ^{jk} a finite measure in $\mathbb{R}^{n_1}$, and Γ^{jk} a singular kernel of fundamental type for all $j \in \{1,\dots,d\}$ and $k \in \{1,\dots,m\}$. Then it holds for almost every $x,y \in \mathbb{R}^d$ that

$$|u(x) - u(y)| \leqslant |A^{-1}[x-y]| \left(U(A^{-1}x) + U(A^{-1}y)\right)$$

where

$$U(x) := \sum_{j=1}^d \sum_{k=1}^m A_{jj} M_{\Upsilon_1^{j,\xi} \otimes \Upsilon_2^{j,\xi}} \left[\alpha^{jk}(\delta_2\,\cdot\,)(\delta_1^{n_1}\Gamma^{jk}(\delta_1\,\cdot\,) * \mu^{jk}(\delta_1\,\cdot\,))\right](x).$$

Moreover, the above satisfies the following estimates: denoting B^1_R and B^2_R the balls in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively, there exists a constant $C_{q,R,m,d} > 0$ depending only on the subscripts and on Γ^{jk} such that

$$|||U(A^{-1}\,\cdot\,)|||_{L^1_w(B^1_R \times B^2_R)} \leqslant C_{q,R,m,d}\Big(\delta_1 \sum_{j=1}^{n_1}\sum_{k=1}^m \|\alpha^{jk}\|_{L^q(\mathbb{R}^{n_2})}\mu^{jk}(\mathbb{R}^{n_1}) + \delta_2 \sum_{j=n_1+1}^{d}\sum_{k=1}^m \|\alpha^{jk}\|_{L^q(\mathbb{R}^{n_2})}\mu^{jk}(\mathbb{R}^{n_1})\Big).$$

Finally, if $\mu^{jk} = g^{jk}\mathcal{L}^d$ *for some* $g^{jk} \in L^q(\mathbb{R}^{n_1})$*, then there exists a constant* $C_{q,d} > 0$ *depending only on the subscripts such that*

$$\|U(A^{-1}\cdot)\|_{L^q(\mathbb{R}^d)} \leqslant C_{q,d}\Big(\delta_1 \sum_{j=1}^{n_1}\sum_{k=1}^{m} \|\alpha^{jk}\|_{L^q(\mathbb{R}^{n_2})}\|g^{jk}\|_{L^q(\mathbb{R}^{n_1})} + \delta_2 \sum_{j=n_1+1}^{d}\sum_{k=1}^{m} \|\alpha^{jk}\|_{L^q(\mathbb{R}^{n_2})}\|g^{jk}\|_{L^q(\mathbb{R}^{n_1})}\Big).$$

We are now ready to prove the fundamental estimate for the anisotropic case (2.18) found in [16, Theorem 6.1]. The proof follows the same idea as the proof of Proposition 2.3.

Proposition 2.4. *Let* $\boldsymbol{b}$ *and* $\bar{\boldsymbol{b}}$ *be vector fields satisfying* (1.6)*, with* $\boldsymbol{b}$ *as in* (2.18) *and* (2.19)*, and* $\boldsymbol{X}$*,* $\bar{\boldsymbol{X}}$ *their renormalized regular Lagrangian flows starting at time* s *with compressibility constants* L *and* $\bar{L}$*, respectively. Then for every* $\gamma > 0$*,* $\eta > 0$*, and* $r > 0$*, there exists* $\lambda > 0$ *and a constant* $C_{\gamma,\eta,r} > 0$ *such that*

$$|B_r \cap \{|\boldsymbol{X}(t,s,\cdot) - \bar{\boldsymbol{X}}(t,s,\cdot)| > \gamma\}| \leqslant C_{\gamma,\eta,r}\|\boldsymbol{b} - \bar{\boldsymbol{b}}\|_{L^1((0,T)\times B_\lambda)} + \eta$$

uniformly in $s \in [0,T]$ *and* $t \in [s,T]$*. The constant* $C_{\gamma,\eta,r}$ *depends on its subscripts, as well as the compressibility constants* L *and* $\bar{L}$*, on the norms* (1.6) *of* $\boldsymbol{b}$ *and* $\bar{\boldsymbol{b}}$*, on* $\|\boldsymbol{b}\|_{L^p((s,t)\times B_\lambda)}$ *for any* $t \in (s,T)$*, on the norms* $\alpha^{ijk}, \beta^{ijk}, \gamma^{ijk}, \psi^{ijk} \in L^\infty((0,T);L^q(\mathbb{R}^{n_2}))$ *for some* $q > 1$*,* $g^{ijk}, \bar{g}^{ijk}, \tilde{g}^{ijk} \in L^1((0,T)\times\mathbb{R}^{n_1})$*, and* $\mu \in L^1((0,T);\mathcal{M}(\mathbb{R}^{n_1}))$*, and on the constants at Theorem 2.3 for* $\Gamma_1^{ijk}, \dots, \Gamma_4^{ijk}$*.*

Proof. For a fixed matrix A as in Lemma 2.11 with $\delta_1 \leqslant \delta_2$ and $s \in [0,T]$, we define for $t \in [s,T]$

$$\Phi_{\delta_1,\delta_2}(t) = \int_{B_r\cap G_\lambda\cap\bar{G}_\lambda} \log\big(1 + |A^{-1}[\boldsymbol{X}(t,s,x) - \bar{\boldsymbol{X}}(t,s,x)]|\big)\,\mathrm{d}x.$$

Using the same argument as in Proposition 2.3, we have that

$$\begin{aligned}\Phi_{\delta_1,\delta_2}(\tau) = \int_s^\tau \Phi'_{\delta_1,\delta_2}(t)\,\mathrm{d}t &\leqslant \int_s^\tau\int_{B_r\cap G_\lambda\cap\bar{G}_\lambda} \frac{|A^{-1}[\boldsymbol{b}_t(\boldsymbol{X}(t,s,x)) - \bar{\boldsymbol{b}}_t(\bar{\boldsymbol{X}}(t,s,x))]|}{|A^{-1}[\boldsymbol{X}(t,s,x) - \bar{\boldsymbol{X}}(t,s,x)]| + 1}\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant \int_s^\tau\int_{B_r\cap G_\lambda\cap\bar{G}_\lambda} |A^{-1}[\boldsymbol{b}_t(\bar{\boldsymbol{X}}(t,s,x)) - \bar{\boldsymbol{b}}_t(\bar{\boldsymbol{X}}(t,s,x))]|\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_s^\tau\int_{B_r\cap G_\lambda\cap\bar{G}_\lambda} \frac{|A^{-1}[\boldsymbol{b}_t(\boldsymbol{X}(t,s,x)) - \boldsymbol{b}_t(\bar{\boldsymbol{X}}(t,s,x))]|}{|A^{-1}[\boldsymbol{X}(t,s,x) - \bar{\boldsymbol{X}}(t,s,x)]| + 1}\,\mathrm{d}x\,\mathrm{d}t \\ &=: \mathrm{I}(\tau) + \mathrm{II}(\tau).\end{aligned}$$

The first term is estimated using the compressibility and $\delta_1 \leqslant \delta_2$:

$$\mathrm{I}(\tau) \leqslant \frac{\|\boldsymbol{b} - \bar{\boldsymbol{b}}\|_{L^1((s,\tau)\times G_\lambda)}}{\delta_1}. \tag{2.20}$$

The second term can be estimated as in Proposition 2.3:

$$\begin{aligned}\mathrm{II}(\tau) \leqslant \int_s^\tau \int_{B_r\cap G_\lambda\cap\bar{G}_\lambda} \min\Bigg\{ &|A^{-1}[\boldsymbol{b}_t(\boldsymbol{X}(t,s,x)) - \boldsymbol{b}_t(\bar{\boldsymbol{X}}(t,s,x))]|, \\ &\frac{1}{\delta_1}\frac{|[\boldsymbol{b}_t^1(\boldsymbol{X}(t,s,x)) - \boldsymbol{b}_t^1(\bar{\boldsymbol{X}}(t,s,x))]|}{|A^{-1}[\boldsymbol{X}(t,s,x) - \bar{\boldsymbol{X}}(t,s,x)]|} \\ &+ \frac{1}{\delta_2}\frac{|[\boldsymbol{b}_t^2(\boldsymbol{X}(t,s,x)) - \boldsymbol{b}_t^2(\bar{\boldsymbol{X}}(t,s,x))]|}{|A^{-1}[\boldsymbol{X}(t,s,x) - \bar{\boldsymbol{X}}(t,s,x)]|}\Bigg\}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.21}$$

By Lemma 2.11, we have that

$$\begin{aligned}\frac{|[\boldsymbol{b}_t^1(\boldsymbol{X}(t,s,x)) - \boldsymbol{b}_t^1(\bar{\boldsymbol{X}}(t,s,x))]|}{|A^{-1}[\boldsymbol{X}(t,s,x) - \bar{\boldsymbol{X}}(t,s,x)]|} &\leqslant U^1(A^{-1}\boldsymbol{X}(t,s,x)) + U^1(A^{-1}\bar{\boldsymbol{X}}(t,s,x)); \\ \frac{|[\boldsymbol{b}_t^2(\boldsymbol{X}(t,s,x)) - \boldsymbol{b}_t^2(\bar{\boldsymbol{X}}(t,s,x))]|}{|A^{-1}[\boldsymbol{X}(t,s,x) - \bar{\boldsymbol{X}}(t,s,x)]|} &\leqslant U^2(A^{-1}\boldsymbol{X}(t,s,x)) + U^2(A^{-1}\bar{\boldsymbol{X}}(t,s,x)),\end{aligned}$$

where U^1 and U^2 are defined analogously as in Lemma 2.11:

$$\begin{aligned}U_t^1(x) := \sum_{i,j=1}^d \sum_{k=1}^m A_{jj} M_{\Upsilon_1^{j,\xi}\otimes\Upsilon_2^{j,\xi}}\Big[&\alpha_t^{ijk}(\delta_2\,\cdot)(\delta_1^{n_1}\Gamma_1^{ijk}(\delta_1\,\cdot) * g_t^{ijk}(\delta_1\,\cdot)) \\ &+ \beta_t^{ijk}(\delta_2\,\cdot)(\delta_1^{n_1}\Gamma_2^{ijk}(\delta_1\,\cdot) * \bar{g}_t^{ijk}(\delta_1\,\cdot))\Big](x),\end{aligned}$$

$$\begin{aligned}U_t^2(x) := \sum_{i,j=1}^d \sum_{k=1}^m A_{jj} M_{\Upsilon_1^{j,\xi}\otimes\Upsilon_2^{j,\xi}}\Big[&\gamma_t^{ijk}(\delta_2\,\cdot)(\delta_1^{n_1}\Gamma_3^{ijk}(\delta_1\,\cdot) * \mu_t^{ijk}(\delta_1\,\cdot)) \\ &+ \psi_t^{ijk}(\delta_2\,\cdot)(\delta_1^{n_1}\Gamma_2^{ijk}(\delta_1\,\cdot) * \tilde{g}_t^{ijk}(\delta_1\,\cdot))\Big](x).\end{aligned}$$

By Lemma 2.10, we have the decomposition for each $i,j \in \{1,\dots,d\}$ and $k \in \{1,\dots,m\}$:

$$g^{ijk} = g^{ijk,1} + g^{ijk,2}, \quad \bar{g}^{ijk} = \bar{g}^{ijk,1} + \bar{g}^{ijk,2}, \quad \tilde{g}^{ijk} = \tilde{g}^{ijk,1} + \tilde{g}^{ijk,2},$$

where $\operatorname{supp} g^{ijk,2}$, $\operatorname{supp} \bar{g}^{ijk,2}$, $\operatorname{supp} \tilde{g}^{ijk,2} \subset \Omega_\epsilon$ and

$$\begin{aligned}&\|g^{ijk,1}\|_{L^1((0,T)\times\mathbb{R}^{n_1})} < \epsilon, \quad \|\bar{g}^{ijk,1}\|_{L^1((0,T)\times\mathbb{R}^{n_1})} < \epsilon, \quad \|\tilde{g}^{ijk,1}\|_{L^1((0,T)\times\mathbb{R}^{n_1})} < \epsilon, \\ &\|g^{ijk,2}\|_{L^q((0,T)\times\mathbb{R}^{n_1})} < C_\epsilon, \quad \|\bar{g}^{ijk,2}\|_{L^q((0,T)\times\mathbb{R}^{n_1})} < C_\epsilon, \quad \|\tilde{g}^{ijk,2}\|_{L^q((0,T)\times\mathbb{R}^{n_1})} < C_\epsilon.\end{aligned}$$

Now, we may further split the functions U^1 and U^2 as

$$U_t^1(x) = U_t^{11}(x) + U_t^{12}(x) \quad \text{and} \quad U_t^2(x) = U_t^{20}(x) + U_t^{21}(x) + U_t^{22}(x)$$

where for $l \in \{1,2\}$ we define

$$\begin{aligned}U_t^{1l}(x) := \sum_{i,j=1}^d \sum_{k=1}^m A_{jj} M_{\Upsilon_1^{j,\xi}\otimes\Upsilon_2^{j,\xi}}\Big[&\alpha_t^{ijk}(\delta_2\,\cdot)(\delta_1^{n_1}\Gamma_1^{ijk}(\delta_1\,\cdot) * g_t^{ijk,l}(\delta_1\,\cdot)) \\ &+ \beta_t^{ijk}(\delta_2\,\cdot)(\delta_1^{n_1}\Gamma_2^{ijk}(\delta_1\,\cdot) * \bar{g}_t^{ijk,l}(\delta_1\,\cdot))\Big](x),\end{aligned}$$

and for the U^2 splitting we define

$$U_t^{20}(x) := \sum_{i,j=1}^{d}\sum_{k=1}^{m} A_{jj} M_{\Upsilon_1^{j,\xi}\otimes\Upsilon_2^{j,\xi}}\left[\gamma_t^{ijk}(\delta_2\,\cdot)(\delta_1^{n_1}\Gamma_3^{ijk}(\delta_1\,\cdot) * \mu_t^{ijk}(\delta_1\,\cdot))\right](x),$$

$$U_t^{2l}(x) := \sum_{i,j=1}^{d}\sum_{k=1}^{m} A_{jj} M_{\Upsilon_1^{j,\xi}\otimes\Upsilon_2^{j,\xi}}\left[\psi_t^{ijk}(\delta_2\,\cdot)(\delta_1^{n_1}\Gamma_4^{ijk}(\delta_1\,\cdot) * \tilde{g}_t^{ijk,l}(\delta_1\,\cdot))\right](x).$$

Notice that by Lemma 2.11 and $\delta_1 \leqslant \delta_2$ we have

$$|||U^{11}(A^{-1}\,\cdot)|||_{L^1_w((0,T)\times B_\lambda)} \leqslant C\epsilon\,(\delta_1+\delta_2) \leqslant C\epsilon\delta_2, \quad |||U^{21}(A^{-1}\,\cdot)|||_{L^1_w((0,T)\times B_\lambda)} \leqslant C\epsilon\delta_2,$$
$$\|U^{12}(A^{-1}\,\cdot)\|_{L^q((0,T)\times B_\lambda)} \leqslant C_\epsilon\delta_2, \qquad \|U^{22}(A^{-1}\,\cdot)\|_{L^q((0,T)\times B_\lambda)} \leqslant C_\epsilon\delta_2.$$

Moreover, we have

$$|||U^{20}(A^{-1}\,\cdot)|||_{L^1_w((0,T)\times B_\lambda)} \leqslant C\delta_1.$$

Now, by (2.21) and the above definitions, we may estimate

$$\begin{aligned}\mathrm{II}(\tau) \leqslant \int_s^\tau\int_{\mathcal{D}} &\min\left\{|A^{-1}[\boldsymbol{b}_t(\boldsymbol{X})-\boldsymbol{b}_t(\bar{\boldsymbol{X}})]|, \frac{U_t^{20}(\boldsymbol{Y})+U_t^{20}(\bar{\boldsymbol{Y}})}{\delta_2}\right\}\\ &+\min\left\{|A^{-1}[\boldsymbol{b}_t(\boldsymbol{X})-\boldsymbol{b}_t(\bar{\boldsymbol{X}})]|, \frac{U_t^{21}(\boldsymbol{Y})+U_t^{21}(\bar{\boldsymbol{Y}})}{\delta_2}\right\}\\ &+\frac{U_t^{22}(\boldsymbol{Y})+U_t^{22}(\bar{\boldsymbol{Y}})}{\delta_2}+\frac{U_t^{12}(\boldsymbol{Y})+U_t^{12}(\bar{\boldsymbol{Y}})}{\delta_1}\\ &+\min\left\{|A^{-1}[\boldsymbol{b}_t(\boldsymbol{X})-\boldsymbol{b}_t(\bar{\boldsymbol{X}})]|, \frac{U_t^{11}(\boldsymbol{Y})+U_t^{11}(\bar{\boldsymbol{Y}})}{\delta_1}\right\}\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where we simplified the notation; $\mathcal{D} = B_r \cap G_\lambda \cap \bar{G}_\lambda$, $\boldsymbol{Y} = A^{-1}\boldsymbol{X}(t,s,x)$, and $\bar{\boldsymbol{Y}} = A^{-1}\bar{\boldsymbol{X}}(t,s,x)$. The third and fourth integrals are trivially estimated using Hölder inequality and $\delta_1 \leqslant \delta_2$:

$$\int_s^\tau\int_{\mathcal{D}}\frac{U_t^{12}(\boldsymbol{Y})+U_t^{12}(\bar{\boldsymbol{Y}})}{\delta_1}\,\mathrm{d}x\,\mathrm{d}t \leqslant |B_r|^{1-\frac{1}{q}}C_\epsilon\frac{\delta_2}{\delta_1}$$
$$\int_s^\tau\int_{\mathcal{D}}\frac{U_t^{22}(\boldsymbol{Y})+U_t^{22}(\bar{\boldsymbol{Y}})}{\delta_2}\,\mathrm{d}x\,\mathrm{d}t \leqslant |B_r|^{1-\frac{1}{q}}C_\epsilon.$$

Now, notice that by the compressibility of the flows, $\delta_1 \leqslant \delta_2$, and (2.18) that

$$\|A^{-1}[\boldsymbol{b}_t(\boldsymbol{X})-\boldsymbol{b}_t(\bar{\boldsymbol{X}})]\|_{L^q((s,\tau)\times\mathcal{D})} \leqslant \frac{L+\bar{L}}{\delta_1}|B_r|^{1-\frac{1}{p}}\|\boldsymbol{b}\|_{L^p((s,\tau)\times B_\lambda)} \leqslant \frac{C'}{\delta_1}.$$

Therefore, the first integral is estimated by using Lemma 2.9:

$$\begin{aligned}&\int_s^\tau\int_{\mathcal{D}}\min\left\{|A^{-1}[\boldsymbol{b}_t(\boldsymbol{X})-\boldsymbol{b}_t(\bar{\boldsymbol{X}})]|, \frac{U_t^{20}(\boldsymbol{X})+U_t^{20}(\bar{\boldsymbol{X}})}{\delta_2}\right\}\mathrm{d}x\,\mathrm{d}t\\ &\qquad\leqslant \frac{C\delta_1 q}{\delta_2(q-1)}\left[1+\log\left(\frac{\delta_2 C'}{\delta_1^2 C}\right)\right] \leqslant \frac{C\delta_1}{\delta_2}\left[1+\log\left(\frac{\delta_2}{\delta_1^2}\right)\right],\end{aligned}$$

and so does the second and fifth integrals:

$$\int_s^\tau \int_{\mathcal{D}} \min\left\{ |A^{-1}[\boldsymbol{b}_t(\boldsymbol{X}) - \boldsymbol{b}_t(\bar{\boldsymbol{X}})]|, \frac{U_t^{21}(\boldsymbol{X}) + U_t^{21}(\bar{\boldsymbol{X}})}{\delta_2} \right\} \mathrm{d}x\,\mathrm{d}t$$
$$\leqslant \frac{C\epsilon q}{(q-1)} \left[1 + \log\left(1 + \frac{C'}{C\epsilon\delta_1}\right)\right] \leqslant C\epsilon \log\left(\frac{1}{\epsilon\delta_1}\right),$$

$$\int_s^\tau \int_{\mathcal{D}} \min\left\{ |A^{-1}[\boldsymbol{b}_t(\boldsymbol{X}) - \boldsymbol{b}_t(\bar{\boldsymbol{X}})]|, \frac{U_t^{11}(\boldsymbol{X}) + U_t^{11}(\bar{\boldsymbol{X}})}{\delta_2} \right\} \mathrm{d}x\,\mathrm{d}t$$
$$\leqslant \frac{C\epsilon q\delta_2}{(q-1)\delta_1} \left[1 + \log\left(1 + \frac{C'}{C\epsilon\delta_2}\right)\right] \leqslant C\epsilon \frac{\delta_2}{\delta_1} \log\left(\frac{1}{\epsilon\delta_2}\right).$$

Therefore by denoting $\delta_1/\delta_2 = \alpha$ we have the following estimate

$$\mathrm{II}(\tau) \leqslant C_\epsilon \left[\frac{1+\alpha}{\alpha} + \alpha \log\left(\frac{1}{\delta_1\alpha}\right)\right] + C\epsilon \left[\frac{1}{\alpha}\left(\log\left(\frac{1}{\epsilon\delta_2}\right) + 1\right) + 1 + \log\left(\frac{1}{\epsilon\delta_1}\right)\right]. \tag{2.22}$$

Using the same idea as in Proposition 2.3 and $\delta_1 \leqslant \delta_2$, then

$$|B_r \cap \{|\boldsymbol{X}(\tau,s,\cdot) - \bar{\boldsymbol{X}}(\tau,s,\cdot)| > \gamma\}| \leqslant \frac{\Phi_{\delta_1,\delta_2}(\tau)}{\log\left(1 + \frac{\gamma}{\delta_2}\right)} + f(r,\lambda) + \bar{f}(r,\lambda).$$

By the same idea as in Lemma 2.5, then $f(r,\lambda) + \bar{f}(r,\lambda) \leqslant \eta/2$ for λ large enough. Now, choose α small enough so that

$$\frac{C_\epsilon}{\log\left(1 + \frac{\gamma}{\delta_2}\right)} \alpha \log\left(\frac{1}{\delta_1\alpha}\right) \leqslant \frac{\eta}{6}.$$

and $\epsilon \leqslant \alpha^2$ so that

$$\frac{C\epsilon}{\log\left(1 + \frac{\gamma}{\delta_2}\right)} \left[\frac{1}{\alpha}\left(\log\left(\frac{1}{\epsilon\delta_2}\right) + 1\right) + 1 + \log\left(\frac{1}{\epsilon\delta_1}\right)\right] \leqslant \frac{\eta}{6}.$$

As long as the ratio $\alpha = \delta_1/\delta_2$ is constant, we have δ_1 and δ_2 are free–only α, ϵ, and λ are fixed. Therefore, we may choose δ_2 small enough so that

$$\frac{C_\epsilon(1+\alpha)}{\alpha \log\left(1 + \frac{\gamma}{\delta_2}\right)} \leqslant \frac{\eta}{6}.$$

Hence, we have by (2.20), (2.22), and the above estimates that

$$|B_r \cap \{|\boldsymbol{X}(\tau,s,\cdot) - \bar{\boldsymbol{X}}(\tau,s,\cdot)| > \gamma\}| \leqslant \frac{\|\boldsymbol{b} - \bar{\boldsymbol{b}}\|_{L^1((s,\tau)\times G_\lambda)}}{\delta_1 \log\left(1 + \frac{\gamma}{\delta_2}\right)} + \eta.$$

Since δ_2 and α are fixed, then so is δ_1, and so the proposition follows. □

Remark 10. There exists a closely related result of anisotropic vector fields by Crippa and Ligabue [41]. They assume (1.6) and a similar structure of (2.18), that is

$$\begin{aligned}
&\boldsymbol{b}_t(x_1,x_2) = (\boldsymbol{b}_t^1(x_1), \boldsymbol{b}_t^2(x_1,x_2)), \text{ where } \boldsymbol{b}^1 \text{ and } \boldsymbol{b}^2 \text{ satisfies} \\
&\boldsymbol{b}^1 \in L^1((0,T); W^{1,p}(\mathbb{R}^{n_1};\mathbb{R}^{n_1})); \\
&\boldsymbol{b}^2 \in L^1((0,T)\times\mathbb{R}^{n_2}; W^{\alpha,1}(\mathbb{R}^{n_1};\mathbb{R}^{n_2})) \cap L^1((0,T)\times\mathbb{R}^{n_1}; W^{1,p}(\mathbb{R}^{n_2};\mathbb{R}^{n_2})),
\end{aligned}$$

where $\alpha \in (0,1)$ and $p > 1$. Notice that in this setting $D_1\boldsymbol{b}^1$ is only a distribution, and the lack of derivatives of $\boldsymbol{b}^1$ in x_1 is "compensated" by the fact that $D_2\boldsymbol{b}^1 = 0$. The strategy of proving the fundamental estimate for such vector fields follows very closely Proposition 2.4, but adding and subtracting the mollified vector fields $\boldsymbol{b} = (\boldsymbol{b}^1, \boldsymbol{b}^{2,\epsilon})$ and using the fact that

$$\|u - u^\epsilon\|_{W^{s,p}(\mathbb{R}^d)} \leqslant C\epsilon^s \|u\|_{W^{s,p}(\mathbb{R}^d)} \quad \text{and} \quad \|\nabla u^\epsilon\|_{W^{s,p}(\mathbb{R}^d)} \leqslant C\epsilon^{s-1}\|u\|_{W^{s,p}(\mathbb{R}^d)};$$

see [41, Lemma 2.4] for a detailed proof of the above. Then the proof follows by choosing ϵ depending on the quotient $\alpha = \delta_1/\delta_2$.

2.5 Singular kernels and BV: the Nguyen's result

Very recently, the Lagrangian approach was fully extended by Nguyen [70] to cover Ambrosio's result for vector fields of bounded variation. The difficulty lies in the singular part (with respect to the Lebesgue measure) of ∇g, where $g \in \mathrm{BV}(\mathbb{R}^d)$. The Nguyen's approach also uses Alberti's rank one theorem–which we briefly discussed in Chapter 1, but now we give a precise statement of the result.

Theorem 2.4 (Alberti's rank one theorem)**.** *Let $\boldsymbol{b} \in \mathrm{BV}(\mathbb{R}^d;\mathbb{R}^m)$. Then there exists unit vectors $a(x) \in \mathbb{R}^d$ and $b(x) \in \mathbb{R}^m$ such that*

$$\mathrm{d}(D\boldsymbol{b})^s(x) = a(x) \otimes b(x)\mathrm{d}|(D\boldsymbol{b})^s|(x),$$

where $\mathrm{d}\mu$ means the differential of the measure μ.

We begin this section by recalling the generalization of Theorem 2.3 for rough kernels and more general family of functions considered in the grand maximal operator. Moreover, we also recall the crucial result associated to Kakeya singular integral operator. The name is an allusion to the Kakeya maximal function M^δ, $\delta > 0$ of a measure μ in $\mathbb{R}^d$:

$$M^\delta\mu(x) = \sup_{\xi\in\mathbb{S}^{d-1}} \sup_{\epsilon>0} \frac{1}{|B_\epsilon|}\int_{B_\epsilon(x)} \frac{\mathbb{1}_{|\omega(y-x)-\xi|\leqslant\delta}}{\delta^{d-1}}\,\mathrm{d}|\mu|(y),$$

where $\omega(z) = z/|z|$ as in Remark 9. Loosely speaking, it considers a family of functions favoring a direction $\xi \in \mathbb{S}^{d-1}$ and then takes a supremum over all ξ. This is analogous to the anisotropic estimate Ambrosio used to prove the renormalization property. Before we state the results, we begin by establishing the considered singular kernels.

Definition 2.5 (Singular rough kernels)**.** Let $\Gamma \in C^1(\mathbb{R}^d \setminus \{0\}; \mathbb{R}^d)$ be a singular kernel with decay as in Theorem 2.3, that is, there exists a constant $C > 0$ such that

$$|\Gamma(x)| \leqslant C|x|^{-d} \quad \text{and} \quad |\nabla\Gamma(x)| \leqslant C|x|^{-d-1}.$$

Moreover, let Ω be a zeroth order homogeneous function as in Section 2.1 satisfying

$$\|\Omega\|_{W^{\alpha,1}(B_2\setminus B_1)} := \int_{B_2\setminus B_1} |\Omega(y)|\,\mathrm{d}y + \int_{B_2\setminus B_1} \frac{|\Omega(x)-\Omega(y)|}{|x-y|^{d+\alpha}}\,\mathrm{d}y \leqslant C_1$$

for some $\alpha \in (0,1)$ and $C_1 > 0$. We say that

$$\tilde{\Gamma}(x) := \Omega(x)\Gamma(x) \quad \text{for all} \quad x \in \mathbb{R}^d \setminus \{0\}$$

is a singular rough kernel if it satisfies the following cancellation property: there exists a constant $C_2 > 0$ such that

$$\left|\int_{B_R\setminus B_r} \tilde{\Gamma}(x)\,\mathrm{d}x\right| \leqslant C_2 \quad \text{for all} \quad 0 < r < R < \infty.$$

In particular, if $\Omega \in C^1(\mathbb{S}^{d-1})$, then $\tilde{\Gamma}$ is a singular kernel of fundamental type.

We are now ready to state one of the main theorems in [70]; for a proof of the results below, we refer to the original ones [70, Proposition 2.13, Theorem 3.3]. They are very technical and are only used in the proof of Theorem 2.6.

Proposition 2.5. *Let $\{\rho^\nu\}_\nu$ be a family of functions satisfying $\rho^\nu \in C^1(\mathbb{R}^d \setminus \{0\})$ such that*

$$\operatorname{supp}\rho^\nu \subset B_1 \quad \textit{and} \quad \sup_{x\in\mathbb{R}^d}\sup_{\nu} |\rho^\nu(x)| + |x||\nabla\rho^\nu(x)| \leqslant C_0.$$

Therefore for any $\gamma \in (0,\infty)$ and $\tilde{\Gamma}$ singular rough kernel, the operator

$$T\mu(x) := \sup_{\nu}\sup_{\epsilon>0}\left|\left(\frac{\epsilon^{-\gamma}}{|\cdot|^{d-\gamma}}\rho^\nu(\epsilon^{-1}\,\cdot)\right) * \tilde{\Gamma} * \mu(x)\right| \quad \textit{for all } x \in \mathbb{R}^d$$

is a bounded from L^p to L^p for any $p \in (1,\infty)$ and from bounded signed measure space to L^1_w, with estimates

$$\begin{aligned} \|Tu\|_{L^p(\mathbb{R}^d)} &\leqslant C_0(C_1+C_2)\|u\|_{L^p(\mathbb{R}^d)} \quad &&\textit{for any } u \in L^p(\mathbb{R}^d) \\ |||T\mu|||_{L^1_w(\mathbb{R}^d)} &\leqslant C_0(C_1+C_2)|\mu|(\mathbb{R}^d) \quad &&\textit{for any bounded signed measure } \mu. \end{aligned}$$

Moreover, it holds

$$\limsup_{\lambda\to\infty} \lambda|\{T\mu > \lambda\}| \leqslant C_0(C_1+C_2)|\mu|^s(\mathbb{R}^d).$$

Finally, for $\epsilon_0 > 0$, $\gamma_0 \in (0,\gamma]$ and the operator

$$T^{\gamma_0}\mu(x) := \sup_{\nu}\sup_{\epsilon\in(0,\epsilon_0)}\left|\left(\frac{\epsilon^{\gamma_0-\gamma}}{|\cdot|^{d-\gamma}}\rho^\nu(\epsilon^{-1}\,\cdot)\right) * \tilde{\Gamma} * \mu(x)\right| \quad \textit{for all } x \in \mathbb{R}^d,$$

there exists a constant $C > 0$ such that

$$\|T^{\gamma_0}\mu\|_{L^q(\mathbb{R}^d)} \leqslant C(C_1 + C_2)|\mu|(\mathbb{R}^d)$$

where

$$q := \frac{d}{d - 4^{-1}\min\{\alpha, \gamma, \gamma_0\}}.$$

Finally, if $\psi \in L^\infty((0,T); W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d))$, then for all $R > 0$ there exists a constant $C_R > 0$ such that

$$\int_0^T \|T^{\gamma_0}[(\psi_t - \psi_t(\cdot))\mu_t]\|_{L^q(B_R)}\,\mathrm{d}t \leqslant C_R\|\psi\|_{L^\infty((0,T);W^{1,\infty}(\mathbb{R}^d))}\int_0^T |\mu_t|(\mathbb{R}^d)\,\mathrm{d}t$$

for any $\mu \in L^1([0,T];\mathcal{M}(\mathbb{R}^d))$.

Notice that we may write T as a grand maximal operator, with the family of functions $\{|\cdot|^{\gamma-d}\rho^\nu\}_\nu$. In particular, if $\gamma = d$, then

$$T\mu(x) = M_{\rho^\nu}\left(\tilde\Gamma * \mu\right)(x) \quad \text{for all } x \in \mathbb{R}^d,$$

and so the above result generalizes Theorem 2.3 for singular rough kernels $\tilde\Gamma$.

Theorem 2.5 (Kakeya singular operator estimates)**.** *Let $\{\rho^{\delta,\xi}\}_{\epsilon,\xi}$ be a family of functions satisfying $\rho^{\delta,\xi} \in C^1(\mathbb{R}^d \setminus \{0\};\mathbb{R}^d)$ such that for some $\delta_0 > 0$ it holds*

$$\begin{aligned}&\operatorname{supp}\rho^{\delta,\xi} \subset B_1 \cap \{x \in \mathbb{R}^d : |\omega(x) - \xi| < \delta\},\\ &\sup_{x\in\mathbb{R}^d}\sup_{(\delta,\xi)\in(0,\delta_0)\times\mathbb{S}^{d-1}} |\rho^{\delta,\xi}(x)| + \delta|x||\nabla\rho^{\delta,\xi}(x)| \leqslant C_0.\end{aligned}$$

Let $\tilde\Gamma(x) = \Omega(x)\Gamma(x)$ for $\mathbb{R}^d\setminus\{0\}$ be a singular rough kernel. Moreover, assume that there exists a bounded zeroth degree homogeneous function $\Omega_n \in C^2(\mathbb{S}^{d-1})$ such that

$$\|\Omega_n\|_{W^{\alpha,1}(B_2\setminus B_1)} \leqslant 2C_1 \quad \textit{and} \quad \lim_{n\to\infty}\|\Omega_n - \Omega\|_{W^{\alpha,1}(B_2\setminus B_1)} = 0 \tag{2.23}$$

for some $\alpha \in (0,1)$, and $\tilde\Gamma^n(x) := \Omega_n(x)\Gamma(x)$ satisfies

$$\begin{aligned}&\left|\int_{B_R\setminus B_r}\tilde\Gamma^n(x)\,\mathrm{d}x\right| \leqslant C_3 \quad \textit{for all} \quad 0 < r < R < \infty,\\ &\lim_{n\to\infty}\sup_{0<r<R<\infty}\left|\int_{B_R\setminus B_r}\tilde\Gamma^n(x) - \Gamma(x)\,\mathrm{d}x\right| = 0.\end{aligned}$$

Then, for the operator–which we shall name Kakeya singular operator defined as

$$K^\gamma_{\rho^{\delta,\xi}}\mu(x) := \sup_{\xi\in\mathbb{S}^{d-1}}\sup_{\epsilon>0}\left|\left(\frac{\delta^{1-d}\epsilon^{-\gamma}}{|\cdot|^{d-\gamma}}\rho^{\delta,\xi}(\epsilon^{-1}\cdot)\right) * \tilde\Gamma * \mu(x)\right| \quad \textit{for all } x \in \mathbb{R}^d$$

for $\gamma \in (0,\infty)$ satisfies for all $u \in \mathrm{BV}(\mathbb{R}^d)$ the estimate

$$\limsup_{\lambda\to\infty}\lambda|\{K^\gamma_{\rho^{\delta,\xi}}(\nabla u) > \lambda\}| \leqslant C|\log(\delta)||\nabla u|^s(\mathbb{R}^d).$$

In particular, the operator has the above bound uniformly with respect to $\gamma \in (0,\infty)$.

The aforementioned result does not hold for any $\mu \in \mathcal{M}(\mathbb{R}^d)$; see the fairly simple counterexample by Nguyen in [70, Remark 3.4] in Remark 13.

Remark 11. It is not trivial task to show examples of $\tilde{\Gamma}$ with the approximations properties assumed in Theorem 2.5. Nevertheless, Nguyen proved in [70, Remark 2.6] that if the singular rough kernel satisfies (2.1), i.e. $\Gamma(x) = |x|^{-d}$, then

$$\Omega_n(x) := \frac{1}{\log 2}\int_0^\infty \left(\frac{\Omega(\cdot)}{|\cdot|^d}\mathbb{1}_{B_2\setminus B_1}\right) * \zeta^n(\tau\omega(x))\tau^{d-1}\,\mathrm{d}\tau$$

for ζ the standard mollifier and $\zeta^n(x) = n^d\zeta(nx)$ satisfies the desired properties with (2.23) holding for $\alpha/2$ instead of α. In particular, it holds for $\Omega \in \mathrm{BV}(\mathbb{S}^{d-1})$ satisfying (2.1).

We shall also need a version of Lemma 2.2 for BV functions and with T^δ as above replacing the maximal operators. For a proof, see [70, Lemmas 4.5 and 4.6]. The key idea is that by [70, Lemma 2.3], we may write

$$|\boldsymbol{u}_j(x) - \boldsymbol{u}_j(x)| \leqslant |x-y|(f^\delta(D\boldsymbol{u}_j)(x) + f^\delta(D\boldsymbol{u}_j)(y) + \delta g^\delta(D\boldsymbol{u}_j)(x) + \delta g^\delta(D\boldsymbol{u}_j)(y))$$

for some subadditive functions f^δ and g^δ. Using the Lebesgue decomposition, the Alberti's theorem gives $\mathrm{d}(\partial_i \boldsymbol{u}^j) = \mathrm{d}(\partial_i \boldsymbol{u}^j)^a + a^i b^j \mathrm{d}|(D\boldsymbol{u})^s|$. Applying the above on $f(D\boldsymbol{u}_j)$ and using its subadditivity, we have the following:

Lemma 2.12 (Difference quotient for BV functions)**.** *Let Ω as in Theorem 2.5 and $\tilde{\Gamma}(x) = \Omega(x)\Gamma(x)$ be a singular rough kernel as in Definition 2.5. Moreover, for every $\boldsymbol{u} \in \mathrm{BV}(\mathbb{R}^d;\mathbb{R}^m)$, let $a(x) \in \mathbb{R}^d$ and $b(x) \in \mathbb{R}^d$ be the unit vector fields given by Theorem 2.4 such that*

$$\mathrm{d}(D\boldsymbol{u})^s(x) = a(x) \otimes b(x)\mathrm{d}|(D\boldsymbol{u})^s|(x),$$

and consider $a^\epsilon(x) \in C^\infty(\mathbb{R}^d;\mathbb{R}^d)$ be unit vector field approximating $a(x)$ in the sense

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^d} |a^\epsilon(x) - a(x)|\,\mathrm{d}|(D\boldsymbol{u})^s|(x) = 0.$$

Then for every $x, y \in \mathbb{R}^d$, $x \neq y$, it holds

$$\begin{aligned}\frac{|\tilde{\Gamma} * \boldsymbol{u}_j(x) - \tilde{\Gamma} * \boldsymbol{u}_j(y)|}{|x-y|} \leqslant & S_{\Theta^{\delta,\xi}}(\omega\cdot(D\boldsymbol{u}_j)^a)(x) + S_{\Theta^{\delta,\xi}}(\omega\cdot(D\boldsymbol{u}_j)^a)(y)\\ &+ S_{\Theta^{\delta,\xi}}(\omega\cdot(a-a^\epsilon)b_j|(D\boldsymbol{u})^s|)(x)\\ &+ S_{\Theta^{\delta,\xi}}(\omega\cdot(a-a^\epsilon)b_j|(D\boldsymbol{u})^s|)(y)\\ &+ S_{\Theta^{\delta,\xi}}(\omega\cdot(a^\epsilon-a^\epsilon(x))b_j|(D\boldsymbol{u})^s|)(x)\\ &+ S_{\Theta^{\delta,\xi}}(\omega\cdot(a^\epsilon-a^\epsilon(y))b_j|(D\boldsymbol{u})^s|)(y)\\ &+ \|\nabla a^\epsilon\|_{L^\infty(\mathbb{R}^d)} S_{\Theta^{\delta,\xi}}(b_j|(D\boldsymbol{u})^s|)(y)\\ &+ \delta[S_{\Xi^{\delta,\xi}}(D\boldsymbol{u}_j)(x) + S_{\Xi^{\delta,\xi}}(D\boldsymbol{u}_j)(y)]\\ &+ \omega\cdot a^\epsilon(x)[S_{\Theta^{\delta,\xi}}(b_j|(D\boldsymbol{u})^s|)(x) + S_{\Theta^{\delta,\xi}}(b_j|(D\boldsymbol{u})^s|)(y)],\end{aligned}$$

where for simplicity we denote $\omega = \omega(x-y)$. In the above, there exist a family of functions $\{\Theta^{\delta,\xi}\}_{(\delta,\xi)\in((0,\delta_0),\mathbb{S}^{d-1})}$, $\{\Xi^{\delta,\xi}\}_{(\delta,\xi)\in((0,\delta_0),\mathbb{S}^{d-1})}$ for some $\delta_0 > 0$ such that $\Theta^{\delta,\xi}$ is nonnegative, $\Theta^{\delta,\xi} \in C^\infty(\mathbb{R}^d \setminus \{0\})$ and $\Xi^{\delta,\xi} \in C^\infty(\mathbb{R}^d \setminus \{0\};\mathbb{R}^d)$, and there exists a constant $C > 0$ so that

$$\begin{aligned}
&\operatorname{supp}\Theta^{\delta,\xi}, \operatorname{supp}\Xi^{\delta,\xi} \subset B_{3/4} \cap \{x \in \mathbb{R}^d : |\omega(x) - \xi| < \delta\};\\
&\sup_{x\in\mathbb{R}^d}\ \sup_{\delta\in(0,\delta_0)}\ \sup_{\xi\in\mathbb{S}^{d-1}} \left(|\Theta^{\delta,\xi}(x)| + \delta|x||\nabla\Theta^{\delta,\xi}(x)|\right) \leqslant C;\\
&\sup_{x\in\mathbb{R}^d}\ \sup_{\delta\in(0,\delta_0)}\ \sup_{\xi\in\mathbb{S}^{d-1}} \left(|\Xi^{\delta,\xi}(x)| + \delta|x||\nabla\Xi^{\delta,\xi}(x)|\right) \leqslant C;\\
&\sup_{\xi\in\mathbb{S}^{d-1}} \delta^{1-d}\left(\|\Theta^{\delta,\xi}\|_{L^1(\mathbb{R}^d)} + \|\Xi^{\delta,\xi}\|_{L^1(\mathbb{R}^d)}\right) \leqslant C.
\end{aligned}$$

We also have used the following operator defined on x, y for $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $\nu \in \mathcal{M}(\mathbb{R}^d;\mathbb{R}^d)$:

$$\begin{aligned}
S_{\Theta^{\delta,\xi}}\mu(x) &:= \sup_{\xi\in\mathbb{S}^{d-1}} \left(\frac{\delta^{1-d}|x-y|^{-1}}{|\cdot|^{d-1}}\Theta^{\delta,\xi}(|x-y|^{-1}\,\cdot\,)\right) * \tilde{\Gamma} * \mu(x);\\
S_{\Theta^{\delta,\xi}}\mu(y) &:= \sup_{\xi\in\mathbb{S}^{d-1}} \left(\frac{\delta^{1-d}|x-y|^{-1}}{|\cdot|^{d-1}}\Theta^{\delta,\xi}(|x-y|^{-1}\,\cdot\,)\right) * \tilde{\Gamma} * \mu(y);\\
S_{\Xi^{\delta,\xi}}\nu(x) &:= \sup_{\xi\in\mathbb{S}^{d-1}} \sum_{i=1}^d \left(\frac{\delta^{1-d}|x-y|^{-1}}{|\cdot|^{d-1}}\Xi_i^{\delta,\xi}(|x-y|^{-1}\,\cdot\,)\right) * \tilde{\Gamma} * \nu_i(x);\\
S_{\Xi^{\delta,\xi}}\nu(y) &:= \sup_{\xi\in\mathbb{S}^{d-1}} \sum_{i=1}^d \left(\frac{\delta^{1-d}|x-y|^{-1}}{|\cdot|^{d-1}}\Xi_i^{\delta,\xi}(|x-y|^{-1}\,\cdot\,)\right) * \tilde{\Gamma} * \nu_i(y);\\
\bar{S}_{\Theta^{\delta,\xi}}\mu(x) &:= \sup_{\xi\in\mathbb{S}^{d-1}} \left(\frac{\delta^{1-d}|x-y|^{-1}}{|\cdot|^{d-1}}\Theta^{\delta,\xi}(|x-y|^{-1}\,\cdot\,)\right) * \mu(x);\\
\bar{S}_{\Theta^{\delta,\xi}}\mu(y) &:= \sup_{\xi\in\mathbb{S}^{d-1}} \left(\frac{\delta^{1-d}|x-y|^{-1}}{|\cdot|^{d-1}}\Theta^{\delta,\xi}(|x-y|^{-1}\,\cdot\,)\right) * \mu(y);
\end{aligned}$$

Moreover, it holds

$$\begin{aligned}
S_{\Theta^{\delta,\xi}}(a^\epsilon(x)\cdot b|(D\boldsymbol{u})^s|)(x) + S_{\Theta^{\delta,\xi}}(a^\epsilon(x)\cdot b|(D\boldsymbol{u})^s|)(y) \leqslant{}& S_{\Theta^{\delta,\xi}}(D\boldsymbol{u})^a(x) + S_{\Theta^{\delta,\xi}}(D\boldsymbol{u})^a(y)\\
&+ S_{\Theta^{\delta,\xi}}((a-a^\epsilon)b|(D\boldsymbol{u})^s|)(x)\\
&+ S_{\Theta^{\delta,\xi}}((a-a^\epsilon)b|(D\boldsymbol{u})^s|)(y)\\
&+ S_{\Theta^{\delta,\xi}}((a^\epsilon-a^\epsilon(x))b|(D\boldsymbol{u})^s|)(x)\\
&+ S_{\Theta^{\delta,\xi}}((a^\epsilon-a^\epsilon(y))b|(D\boldsymbol{u})^s|)(y)\\
&+ \|\nabla a^\epsilon\|_{L^\infty(\mathbb{R}^d)}\tilde{S}_{\Theta^{\delta,\xi}}(b|(D\boldsymbol{u})^s|)(y)\\
&+ \bar{S}_{\Theta^{\delta,\xi}}\operatorname{div}\boldsymbol{u}(x) + \bar{S}_{\Theta^{\delta,\xi}}\operatorname{div}\boldsymbol{u}(y),
\end{aligned}$$

where $\tilde{S}_{\Theta^{\delta,\xi}} = |x-y|S_{\Theta^{\delta,\xi}}$ and for vectors u, v, we denote for simplicity the tensor product $(u\otimes v)_{ij} = (uv)_{ij} = u_i v_j$ and the operator S acting on vector valued functions as the sum of its action on each component.

Remark 12. Notice that by taking the supremum over $\epsilon = |x-y|$ and recalling the definitions in Proposition 2.5 and Theorem 2.5, we have for $\rho = \Theta, \Xi$ that

$$\begin{aligned}
S_{\rho^{\delta,\xi}}\mu(x) \leqslant K^1_{\rho^{\delta,\xi}}\mu(x), &\quad |x-y|S_{\rho^{\delta,\xi}}\mu(x) \leqslant T^1\mu(x),\\
\bar{S}_{\rho^{\delta,\xi}}\mu(x) \leqslant \delta^{1-d}CM\mu(x), &\quad |x-y|\bar{S}_{\rho^{\delta,\xi}}\mu(x) \leqslant \delta^{1-d}C\mathcal{I}_1\mu(x),
\end{aligned}$$

where M is the classical maximal function and $\mathcal{I}_1$ is the Riesz potential (without renormalization constant; see [76, Chapter III])

$$\mathcal{I}_1 u(x) := \left(\frac{1}{|\cdot|^{d-1}}\right) * u(x).$$

We are now ready to state the main result of this section originally proved in [70, Theorem 4.3]. It extends Proposition 2.3 and Proposition 2.4 in four ways: it relaxes the conditions on singular kernels $\tilde{\Gamma}$ and on densities g, assuming that $\tilde{\Gamma}$ is singular rough kernel and $g \in \mathrm{BV}(\mathbb{R}^d)$; it does not assume the convolution structure $\tilde{\Gamma} * g$ (or more generally a sum of $\tilde{\Gamma}^k * g^k$) in all space, but rather a localized version on balls, with kernels and densities depending on the radius; it splits the growth assumption (1.6) and of the structure and integrability of its divergence; finally, it does not require the extra integrability on time, since it swaps Lemma 2.9 for the following (for a proof, see [70, Lemma 2.4, Remark 2.16]):

Lemma 2.13. [6] *Let $q > 1$ and $R > 0$. Then if one has an operator T bounded from $\mathcal{M}(\mathbb{R}^d)$ to $L^1_w(\mathbb{R}^d)$, with local estimate*

$$\int_0^T \limsup_{\lambda\to\infty} \lambda |B_R \cap \{T\mu_t > \lambda\}| \, \mathrm{d}t \leqslant C \int_0^T |\mu_t^s|(\mathbb{R}^d) \, \mathrm{d}t,$$

$\mu \in L^1((0,T); \mathcal{M}(\mathbb{R}^d))$, and $f \in L^1((0,T); L^q(B_R))$, we have that

$$\limsup_{\delta\to 0} \frac{1}{|\log \delta|} \int_0^T \int_{B_R} \min\left\{\frac{f_t(x)}{\delta}, T\mu_t(x)\right\} \mathrm{d}x \, \mathrm{d}t \leqslant \frac{q}{q-1} C \int_0^T |\mu_t^s|(\bar{B}_R) \, \mathrm{d}t.$$

Proof. We first give the prove for the time independent case. By layer cake representation, we have that for $\lambda_1 < \lambda_2$ that

$$\begin{aligned}
\int_{B_R} \min\left\{\frac{f(x)}{\delta}, T\mu(x)\right\} \mathrm{d}x &= \int_0^\infty \left|\left\{\min\left\{\frac{f}{\delta}, T\mu\right\} > \lambda\right\} \cap B_R\right| \mathrm{d}\lambda \\
&\leqslant \int_0^{\lambda_1} |B_R| \, \mathrm{d}\lambda + \int_{\lambda_1}^{\lambda_2} |\{T\mu > \lambda\} \cap B_R| \, \mathrm{d}\lambda \\
&\quad + \int_{\lambda_2}^\infty \left|\left\{\frac{f}{\delta} > \lambda\right\} \cap B_R\right| \mathrm{d}\lambda \\
&\leqslant \lambda_1 |B_R| + \log\left(\frac{\lambda_2}{\lambda_1}\right) \sup_{\lambda>\lambda_1} \lambda \, |\{T\mu > \lambda\} \cap B_R| \\
&\quad + \frac{\lambda_2^{1-q}}{q\delta^q} \|f\|^q_{L^q(B_R)}.
\end{aligned}$$

Choosing $\lambda_1 = |\log \delta|^{1/2}$ and $\lambda_2 = \delta^{\frac{q}{1-q}}$ and considering δ small enough so that $\lambda_1 < \lambda_2$, we obtain that

$$\limsup_{\delta\to 0} \frac{1}{|\log \delta|} \int_{B_R} \min\left\{\frac{f(x)}{\delta}, T\mu(x)\right\} \mathrm{d}x \leqslant \frac{q}{q-1} C |\mu^s|(\bar{B}_R).$$

The time dependent case is a simple application of dominated convergence theorem. □

[6] This result could be applied in Section 2.3 and the main estimate would be valid without requiring $\boldsymbol{b} \in L^p_{\mathrm{loc}}((0,T) \times \mathbb{R}^d; \mathbb{R}^d)$ for $p > 1$, but rather $\boldsymbol{b} \in L^1((0,T) L^p_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^d))$.

Theorem 2.6. *Let $R > 0$ and $\boldsymbol{c} \in L^1((0,T); L^1_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d))$ be a vector field such that for any $R > 0$, there exists an integer m_R and (vector valued) singular rough kernels $\tilde{\Gamma}^{k,R}$ with constants C_{1R}, C_{2R}, $g^{k,R} \in L^1((0,T); \mathrm{BV}(\mathbb{R}^d))$, and Ω as in Theorem 2.5 such that*

$$\boldsymbol{c}_t(x) = \sum_{k=1}^{m_R} \tilde{\Gamma}^{k,R} * g_t^{k,R}(x) \quad \text{in } B_R,$$

$$\operatorname{div} \boldsymbol{c} \in L^1((0,T); \mathcal{M}_{\text{loc}}(\mathbb{R}^d)) \quad \text{and} \quad (\operatorname{div} \boldsymbol{c})^+ \in L^1((0,T); L^1_{\text{loc}}(\mathbb{R}^d)).$$

Moreover, let $\boldsymbol{b}$, $\bar{\boldsymbol{b}} \in L^1((0,T); L^1_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d))$ be vector fields satisfying (1.6) and $\boldsymbol{X}$, $\bar{\boldsymbol{X}}$ their renormalized regular Lagrangian flows starting at time s with compressibility constants L and $\bar{L}$, respectively. Then for any $\gamma' > 0$, $\eta > 0$, and $r > 0$, there exists $\lambda > 0$ and a constant $C_{\gamma',\eta,r} > 0$ such that it holds

$$|B_r \cap \{|\boldsymbol{X}(t,s,\cdot) - \bar{\boldsymbol{X}}(t,s,\cdot)| > \gamma'\}| \leqslant C_{\gamma',\eta,r}\big(\|\boldsymbol{b} - \boldsymbol{c}\|_{L^1((0,T)\times B_\lambda)} + \|\boldsymbol{c} - \bar{\boldsymbol{b}}\|_{L^1((0,T)\times B_\lambda)}\big) + \eta$$

uniformly in $s \in [0,T]$ and $t \in [s,T]$. The constant $C_{\gamma',\eta,r}$ depends on its subscripts, as well as the compressibility constants L and $\bar{L}$, on the norms (1.6) of $\boldsymbol{b}$ and $\bar{\boldsymbol{b}}$, on the norm $\sup_{k\in\{1,\dots,m_{r_0}\}} \|g^{k,r_0}\|_{L^1((0,T);\mathrm{BV}(\mathbb{R}^d))}$, and on the constants associated to the singular rough kernel C_{1r_0} and C_{2r_0}, with radius r_0 depending on r, on the norms (1.6) of $\boldsymbol{b}$ and $\bar{\boldsymbol{b}}$, on compressibility constants L and $\bar{L}$, and on η.

Proof. Let $a_t(x), b_t(x)$ be the unit vector fields given by Theorem 2.4 such that

$$\mathrm{d}(D\boldsymbol{b}_t)^s(x) = a_t(x) \otimes b_t(x) \mathrm{d}|(D\boldsymbol{b}_t)^s|(x).$$

Moreover, let $a_t^\epsilon(x)$, $b_t^\epsilon(x) \in C^\infty((0,T) \times \mathbb{R}^d;\mathbb{R}^d)$ be unit vector fields approximating $a_t(x)$, $b_t(x)$ in the sense

$$\lim_{\epsilon\to 0} \int_0^T \int_{\mathbb{R}^d} |a_t^\epsilon(x) - a_t(x)| + |b_t^\epsilon(x) - b_t(x)| \,\mathrm{d}|(D\boldsymbol{b}_t)^s|(x)\,\mathrm{d}t = 0.$$

Now, for $\delta > 0$, $\gamma \in (0, |\log\delta|)$, $\epsilon > 0$, and $t \in [s,T]$, we consider

$$\Phi_\delta^{\gamma,\epsilon}(t) = \frac{1}{2} \int_{\mathcal{D}} \log\left(1 + \frac{|\boldsymbol{X} - \bar{\boldsymbol{X}}|^2 + \gamma[a_t^\epsilon(\boldsymbol{X}) \cdot (\boldsymbol{X} - \bar{\boldsymbol{X}})]^2}{\delta^2}\right) \mathrm{d}x, \tag{2.24}$$

where $\mathcal{D} = B_r \cap G_\lambda \cap \bar{G}_\lambda$, $\boldsymbol{X} = \boldsymbol{X}(t,s,x)$, and $\bar{\boldsymbol{X}} = \bar{\boldsymbol{X}}(t,s,x)$ for simplicity. Then

$$\begin{aligned}
\Phi_\delta^{\gamma,\epsilon}(\tau) = \int_s^\tau (\Phi_\delta^{\gamma,\epsilon})'(t)\,\mathrm{d}t = & \int_s^\tau \int_{\mathcal{D}} \frac{(\boldsymbol{X}-\bar{\boldsymbol{X}})\cdot[\boldsymbol{b}_t(\boldsymbol{X})-\bar{\boldsymbol{b}}_t(\bar{\boldsymbol{X}})]}{\delta^2+|\boldsymbol{X}-\bar{\boldsymbol{X}}|^2+\gamma[a_t^\epsilon(\boldsymbol{X})\cdot(\boldsymbol{X}-\bar{\boldsymbol{X}})]^2}\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_s^\tau \int_{\mathcal{D}} \frac{\gamma[a_t^\epsilon(\boldsymbol{X})\cdot(\boldsymbol{X}-\bar{\boldsymbol{X}})]a_t^\epsilon(\boldsymbol{X})\cdot[\boldsymbol{b}_t(\boldsymbol{X})-\bar{\boldsymbol{b}}_t(\bar{\boldsymbol{X}})]}{\delta^2+|\boldsymbol{X}-\bar{\boldsymbol{X}}|^2+\gamma[a_t^\epsilon(\boldsymbol{X})\cdot(\boldsymbol{X}-\bar{\boldsymbol{X}})]^2}\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_s^\tau \int_{\mathcal{D}} \frac{\gamma[a_t^\epsilon(\boldsymbol{X})\cdot(\boldsymbol{X}-\bar{\boldsymbol{X}})][Da_t^\epsilon(\boldsymbol{X})\boldsymbol{b}_t(\boldsymbol{X})\cdot(\boldsymbol{X}-\bar{\boldsymbol{X}})]}{\delta^2+|\boldsymbol{X}-\bar{\boldsymbol{X}}|^2+\gamma[a_t^\epsilon(\boldsymbol{X})\cdot(\boldsymbol{X}-\bar{\boldsymbol{X}})]^2}\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_s^\tau \int_{\mathcal{D}} \frac{\gamma[a_t^\epsilon(\boldsymbol{X})\cdot(\boldsymbol{X}-\bar{\boldsymbol{X}})][\partial_t a_t^\epsilon(\boldsymbol{X})\cdot(\boldsymbol{X}-\bar{\boldsymbol{X}})]}{\delta^2+|\boldsymbol{X}-\bar{\boldsymbol{X}}|^2+\gamma[a_t^\epsilon(\boldsymbol{X})\cdot(\boldsymbol{X}-\bar{\boldsymbol{X}})]^2}\,\mathrm{d}x\,\mathrm{d}t \\
=: & \mathrm{I}_\delta^{\gamma,\epsilon}(\tau) + \mathrm{II}_\delta^{\gamma,\epsilon}(\tau) + \mathrm{III}_\delta^{\gamma,\epsilon}(\tau) + \mathrm{IV}_\delta^{\gamma,\epsilon}(\tau).
\end{aligned}$$

By the trivial estimate $2|uv| \leqslant |u|^2 + |v|^2$, we use the compressibility condition on the flows to estimate the third and fourth terms by

$$\begin{aligned}
\mathrm{III}_\delta^{\gamma,\epsilon}(\tau) &\leqslant L\gamma^{1/2}\|Da^\epsilon\|_{L^\infty((0,T)\times\mathbb{R}^d)}\|\boldsymbol{b}\|_{L^1((0,T)\times B_\lambda)}; \\
\mathrm{IV}_\delta^{\gamma,\epsilon}(\tau) &\leqslant T|B_r|\gamma^{1/2}\|\partial_t a^\epsilon\|_{L^\infty((0,T)\times\mathbb{R}^d)}.
\end{aligned} \tag{2.25}$$

For the first and second terms, we add and subtract the respective terms with $\boldsymbol{c}$, and so we have the estimate

$$\begin{aligned}
\mathrm{I}_\delta^{\gamma,\epsilon}(\tau) \leqslant & \int_s^\tau \int_{\mathcal{D}} \frac{(\boldsymbol{X}-\bar{\boldsymbol{X}})\cdot[\boldsymbol{b}_t(\boldsymbol{X})-\boldsymbol{c}_t(\boldsymbol{X})]}{\delta^2+|\boldsymbol{X}-\bar{\boldsymbol{X}}|^2+\gamma[a_t^\epsilon(\boldsymbol{X})\cdot(\boldsymbol{X}-\bar{\boldsymbol{X}})]^2}\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_s^\tau \int_{\mathcal{D}} \frac{(\boldsymbol{X}-\bar{\boldsymbol{X}})\cdot[\boldsymbol{c}_t(\boldsymbol{X})-\boldsymbol{c}_t(\bar{\boldsymbol{X}})]}{\delta^2+|\boldsymbol{X}-\bar{\boldsymbol{X}}|^2+\gamma[a_t^\epsilon(\boldsymbol{X})\cdot(\boldsymbol{X}-\bar{\boldsymbol{X}})]^2}\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_s^\tau \int_{\mathcal{D}} \frac{(\boldsymbol{X}-\bar{\boldsymbol{X}})\cdot[\boldsymbol{c}_t(\bar{\boldsymbol{X}})-\bar{\boldsymbol{b}}_t(\bar{\boldsymbol{X}})]}{\delta^2+|\boldsymbol{X}-\bar{\boldsymbol{X}}|^2+\gamma[a_t^\epsilon(\boldsymbol{X})\cdot(\boldsymbol{X}-\bar{\boldsymbol{X}})]^2}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

The first and third integrals are bounded analogously as above:

$$\begin{aligned}
\int_s^\tau \int_{\mathcal{D}} \frac{(\boldsymbol{X}-\bar{\boldsymbol{X}})\cdot[\boldsymbol{b}_t(\boldsymbol{X})-\boldsymbol{c}_t(\boldsymbol{X})]}{\delta^2+|\boldsymbol{X}-\bar{\boldsymbol{X}}|^2+\gamma[a_t^\epsilon(\boldsymbol{X})\cdot(\boldsymbol{X}-\bar{\boldsymbol{X}})]^2}\,\mathrm{d}x\,\mathrm{d}t &\leqslant \frac{L}{\delta}\|\boldsymbol{b}-\boldsymbol{c}\|_{L^1((0,T)\times B_\lambda)}; \\
\int_s^\tau \int_{\mathcal{D}} \frac{(\boldsymbol{X}-\bar{\boldsymbol{X}})\cdot[\boldsymbol{c}_t(\bar{\boldsymbol{X}})-\bar{\boldsymbol{b}}_t(\bar{\boldsymbol{X}})]}{\delta^2+|\boldsymbol{X}-\bar{\boldsymbol{X}}|^2+\gamma[a_t^\epsilon(\boldsymbol{X})\cdot(\boldsymbol{X}-\bar{\boldsymbol{X}})]^2}\,\mathrm{d}x\,\mathrm{d}t &\leqslant \frac{\bar{L}}{\delta}\|\boldsymbol{c}-\bar{\boldsymbol{b}}\|_{L^1((0,T)\times B_\lambda)}.
\end{aligned}$$

By performing the same computation for $\mathrm{II}_\delta^{\gamma,\epsilon}(\tau)$, we conclude that

$$\begin{aligned}
\mathrm{I}_\delta^{\gamma,\epsilon}(\tau) \leqslant & \frac{L+\bar{L}}{\delta}\left(\|\boldsymbol{b}-\boldsymbol{c}\|_{L^1((0,T)\times B_\lambda)} + \|\boldsymbol{c}-\bar{\boldsymbol{b}}\|_{L^1((0,T)\times B_\lambda)}\right) \\
&+ \int_s^\tau \int_{\mathcal{D}} \frac{(\boldsymbol{X}-\bar{\boldsymbol{X}})\cdot[\boldsymbol{c}_t(\boldsymbol{X})-\boldsymbol{c}_t(\bar{\boldsymbol{X}})]}{\delta^2+|\boldsymbol{X}-\bar{\boldsymbol{X}}|^2+\gamma[a_t^\epsilon(\boldsymbol{X})\cdot(\boldsymbol{X}-\bar{\boldsymbol{X}})]^2}\,\mathrm{d}x\,\mathrm{d}t; \\
\mathrm{II}_\delta^{\gamma,\epsilon}(\tau) \leqslant & \frac{(L+\bar{L})\gamma^{1/2}}{\delta}\left(\|\boldsymbol{b}-\boldsymbol{c}\|_{L^1((0,T)\times B_\lambda)} + \|\boldsymbol{c}-\bar{\boldsymbol{b}}\|_{L^1((0,T)\times B_\lambda)}\right) \\
&+ \int_s^\tau \int_{\mathcal{D}} \frac{\gamma[a_t^\epsilon(\boldsymbol{X})\cdot(\boldsymbol{X}-\bar{\boldsymbol{X}})]a_t^\epsilon(\boldsymbol{X})\cdot[\boldsymbol{c}_t(\boldsymbol{X})-\bar{\boldsymbol{c}}_t(\bar{\boldsymbol{X}})]}{\delta^2+|\boldsymbol{X}-\bar{\boldsymbol{X}}|^2+\gamma[a_t^\epsilon(\boldsymbol{X})\cdot(\boldsymbol{X}-\bar{\boldsymbol{X}})]^2}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{2.26}$$

The estimate of $\Phi_\delta^{\gamma,\epsilon}(\tau)$ from below follows the same idea as Proposition 2.3 and Proposition 2.4, using the fact that we chose $\gamma < |\log\delta|$:

$$|B_r \cap \{|\boldsymbol{X}(\tau,s,\cdot) - \bar{\boldsymbol{X}}(\tau,s,\cdot)| > \delta^{1/2}\}| \leqslant |B_r \setminus G_\lambda| + |B_r \setminus \bar{G}_\lambda| + \frac{\Phi_\delta^{\gamma,\epsilon}(\tau)}{|\log\delta|}.$$

Combining the above with Lemma 2.5, (2.25), and (2.26), we have that

$$\begin{aligned}|B_r \cap \{|\boldsymbol{X}(\tau,s,\cdot) - \bar{\boldsymbol{X}}(\tau,s,\cdot)| > \delta^{1/2}\}| \leqslant & f(r,\lambda) + \bar{f}(r,\lambda) + \frac{C_r}{|\log\delta|^{1/2}}\left(1 + \|\boldsymbol{b}\|_{L^1((0,T)\times B_\lambda)}\right) \\ &+ \frac{(L+\bar{L})}{\delta}\|\boldsymbol{b} - \boldsymbol{c}\|_{L^1((0,T)\times B_\lambda)} \\ &+ \frac{(L+\bar{L})}{\delta}\|\boldsymbol{c} - \bar{\boldsymbol{b}}\|_{L^1((0,T)\times B_\lambda)} \\ &+ \frac{1}{|\log\delta|}\left(\mathrm{V}_\delta^{\gamma,\epsilon}(\tau) + \mathrm{VI}_\delta^{\gamma,\epsilon}(\tau)\right),\end{aligned} \tag{2.27}$$

where we have used that $\gamma^{1/2} < \log|\delta|$ for δ small enough and

$$\mathrm{V}_\delta^{\gamma,\epsilon}(\tau) = \int_s^\tau \int_{\mathcal{D}} \frac{(\boldsymbol{X} - \bar{\boldsymbol{X}})\cdot[\boldsymbol{c}_t(\boldsymbol{X}) - \boldsymbol{c}_t(\bar{\boldsymbol{X}})]}{\delta^2 + |\boldsymbol{X} - \bar{\boldsymbol{X}}|^2 + \gamma[a_t^\epsilon(\boldsymbol{X})\cdot(\boldsymbol{X} - \bar{\boldsymbol{X}})]^2}\,\mathrm{d}x\,\mathrm{d}t;$$
$$\mathrm{VI}_\delta^{\gamma,\epsilon}(\tau) = \int_s^\tau \int_{\mathcal{D}} \frac{\gamma[a_t^\epsilon(\boldsymbol{X})\cdot(\boldsymbol{X} - \bar{\boldsymbol{X}})]a_t^\epsilon(\boldsymbol{X})\cdot[\boldsymbol{c}_t(\boldsymbol{X}) - \bar{\boldsymbol{c}}_t(\bar{\boldsymbol{X}})]}{\delta^2 + |\boldsymbol{X} - \bar{\boldsymbol{X}}|^2 + \gamma[a_t^\epsilon(\boldsymbol{X})\cdot(\boldsymbol{X} - \bar{\boldsymbol{X}})]^2}\,\mathrm{d}x\,\mathrm{d}t.$$

We shall use Lemma 2.12 with $\delta = \epsilon_1$, $x = \boldsymbol{X}$, $y = \bar{\boldsymbol{X}}$, and estimate each term separately. For this purpose, notice that by the inequality $2|uv| \leqslant |u|^2 + |v|^2$, Proposition 2.5, and Remark 12, we have that

$$\frac{|\boldsymbol{X} - \bar{\boldsymbol{X}}|^2 S_{\Theta^{\delta,\xi}}(D\boldsymbol{c}_t)^a(\boldsymbol{X})}{\delta^2 + |\boldsymbol{X} - \bar{\boldsymbol{X}}|^2 + \gamma[a_t^\epsilon(\boldsymbol{X})\cdot(\boldsymbol{X} - \bar{\boldsymbol{X}})]^2} \leqslant \min\left\{\frac{1}{\delta}T^1(D\boldsymbol{c}_t)^a(\boldsymbol{X}), K^1_{\Theta^{\delta,\xi}}(D\boldsymbol{c}_t)^a(\boldsymbol{X})\right\},$$

and analogously for $(D\boldsymbol{c}_t)^a(\bar{\boldsymbol{X}})$. Moreover, we have

$$\begin{aligned}\frac{|\boldsymbol{X} - \bar{\boldsymbol{X}}|^2 S_{\Theta^{\epsilon_1,\xi}}((a_t - a_t^\epsilon)b_t|(D\boldsymbol{c}_t)^s|)(\boldsymbol{X})}{\delta^2 + |\boldsymbol{X} - \bar{\boldsymbol{X}}|^2 + \gamma[a_t^\epsilon(\boldsymbol{X})\cdot(\boldsymbol{X} - \bar{\boldsymbol{X}})]^2} \leqslant \min\Bigg\{&\frac{1}{\delta}T^1((a_t - a_t^\epsilon)b_t|(D\boldsymbol{c}_t)^s|)(\boldsymbol{X}), \\ &K^1_{\Theta^{\epsilon_1,\xi}}((a_t - a_t^\epsilon)b_t|(D\boldsymbol{c}_t)^s|)(\boldsymbol{X})\Bigg\};\end{aligned}$$

$$\frac{|\boldsymbol{X} - \bar{\boldsymbol{X}}|^2 S_{\Theta^{\epsilon_1,\xi}}((a_t^\epsilon - a_t^\epsilon(\boldsymbol{X}))b_t|(D\boldsymbol{c}_t)^s|)(\boldsymbol{X})}{\delta^2 + |\boldsymbol{X} - \bar{\boldsymbol{X}}|^2 + \gamma[a_t^\epsilon(\boldsymbol{X})\cdot(\boldsymbol{X} - \bar{\boldsymbol{X}})]^2} \leqslant K^1_{\Theta^{\epsilon_1,\xi}}((a_t - a_t^\epsilon(\boldsymbol{X}))b_t|(D\boldsymbol{c}_t)^s|)(\boldsymbol{X});$$
$$\frac{|\boldsymbol{X} - \bar{\boldsymbol{X}}|^2\|\nabla a_t^\epsilon\|_{L^\infty(\mathbb{R}^d)}\tilde{S}_{\Theta^{\epsilon_1,\xi}}(b_t|(D\boldsymbol{c}_t)^s|)(\bar{\boldsymbol{X}})}{\delta^2 + |\boldsymbol{X} - \bar{\boldsymbol{X}}|^2 + \gamma[a_t^\epsilon(\boldsymbol{X})\cdot(\boldsymbol{X} - \bar{\boldsymbol{X}})]^2} \leqslant \|\nabla a_t^\epsilon\|_{L^\infty(\mathbb{R}^d)}T^1(b_t|(D\boldsymbol{c}_t)^s|)(\bar{\boldsymbol{X}});$$
$$\frac{|\boldsymbol{X} - \bar{\boldsymbol{X}}|^2\epsilon_1 S_{\Xi^{\epsilon_1,\xi}}(D\boldsymbol{c}_t)(\boldsymbol{X})}{\delta^2 + |\boldsymbol{X} - \bar{\boldsymbol{X}}|^2 + \gamma[a_t^\epsilon(\boldsymbol{X})\cdot(\boldsymbol{X} - \bar{\boldsymbol{X}})]^2} \leqslant \epsilon_1\min\left\{\frac{1}{\delta}T^1(D\boldsymbol{c}_t)(\boldsymbol{X}), K^1_{\Xi^{\epsilon_1,\xi}}(D\boldsymbol{c}_t)(\boldsymbol{X})\right\};$$

$$\frac{|\boldsymbol{X}-\bar{\boldsymbol{X}}|(\boldsymbol{X}-\bar{\boldsymbol{X}})\cdot a^{\epsilon}(\boldsymbol{X})S_{\Theta^{\epsilon_1,\xi}}(b_t|(D\boldsymbol{c}_t)^s|)(\boldsymbol{X})}{\delta^2+|\boldsymbol{X}-\bar{\boldsymbol{X}}|^2+\gamma[a^{\epsilon}_t(\boldsymbol{X})\cdot(\boldsymbol{X}-\bar{\boldsymbol{X}})]^2}\leqslant\gamma^{-1/2}\min\left\{\frac{1}{\delta}T^1(b_t|(D\boldsymbol{c}_t)^s|)(\boldsymbol{X}),\right.$$
$$\left.K^1_{\Theta^{\epsilon_1,\xi}}(b_t|(D\boldsymbol{c}_t)^s|)(\boldsymbol{X})\right\}.$$

All of the above holds changing the argument $\boldsymbol{X}$ for $\bar{\boldsymbol{X}}$. Now, by the compressibility of flows, Proposition 2.5, Theorem 2.5, and Lemma 2.13, we have that (recall $b_t(x)$ is a unit vector)

$$\begin{aligned}\limsup_{\delta\to0}\frac{1}{|\log\delta|}\mathrm{V}^{\gamma,\epsilon}_{\delta}(\tau)\leqslant&C_{\epsilon_1}\int_0^T\int_{\mathbb{R}^d}|a_t(x)-a^{\epsilon}_t(x)|\,\mathrm{d}|(D\boldsymbol{c}_t)^s|(x)\,\mathrm{d}t\\&+\epsilon_1C|\log(\epsilon_1)|\int_0^T|(D\boldsymbol{c}_t)^s|(\mathbb{R}^d)\,\mathrm{d}t\\&+\gamma^{-1/2}C_{\epsilon_1}\int_0^T|(D\boldsymbol{c}_t)^s|(\mathbb{R}^d)\,\mathrm{d}t.\end{aligned}\tag{2.28}$$

The analogous estimate for $\mathrm{VI}^{\gamma,\epsilon}_{\delta}$ follows very similarly, e.g.

$$\frac{\gamma|\boldsymbol{X}-\bar{\boldsymbol{X}}|[a^{\epsilon}_t(\boldsymbol{X})\cdot(\boldsymbol{X}-\bar{\boldsymbol{X}})]S_{\Theta^{\epsilon_2,\xi}}(D\boldsymbol{c}_t)^a(\boldsymbol{X})}{\delta^2+|\boldsymbol{X}-\bar{\boldsymbol{X}}|^2+\gamma[a^{\epsilon}_t(\boldsymbol{X})\cdot(\boldsymbol{X}-\bar{\boldsymbol{X}})]^2}\leqslant\gamma^{1/2}\min\left\{\frac{1}{\delta}T^1(D\boldsymbol{c}_t)^a(\boldsymbol{X}),\right.$$
$$\left.K^1_{\Theta^{\epsilon_2,\xi}}(D\boldsymbol{c}_t)^a(\boldsymbol{X})\right\},$$

and so we have

$$\begin{aligned}\limsup_{\delta\to0}\frac{1}{|\log\delta|}\mathrm{VI}^{\gamma,\epsilon}_{\delta}(\tau)\leqslant&\gamma^{1/2}C_{\epsilon_2}\int_0^T\int_{\mathbb{R}^d}|a_t(x)-a^{\epsilon}_t(x)|\,\mathrm{d}|(D\boldsymbol{c}_t)^s|(x)\,\mathrm{d}t\\&+\gamma^{1/2}\epsilon_2C|\log(\epsilon_2)|\int_0^T|(D\boldsymbol{c}_t)^s|(\mathbb{R}^d)\,\mathrm{d}t\\&+\limsup_{\delta\to0}\frac{1}{|\log\delta|}\mathrm{VII}^{\gamma,\epsilon}_{\delta}(\tau).\end{aligned}\tag{2.29}$$

In the above, we have defined

$$\begin{aligned}\mathrm{VII}^{\gamma,\epsilon}_{\delta}(\tau):=&\int_s^\tau\int_{\mathcal{D}}\frac{\gamma[a^{\epsilon}_t(\boldsymbol{X})\cdot(\boldsymbol{X}-\bar{\boldsymbol{X}})]^2S_{\Theta^{\epsilon_2,\xi}}(a^{\epsilon}_t(\boldsymbol{X})b_t|(D\boldsymbol{c}_t)^s|)(\boldsymbol{X})}{\delta^2+|\boldsymbol{X}-\bar{\boldsymbol{X}}|^2+\gamma[a^{\epsilon}_t(\boldsymbol{X})\cdot(\boldsymbol{X}-\bar{\boldsymbol{X}})]^2}\,\mathrm{d}x\,\mathrm{d}t\\&+\int_s^\tau\int_{\mathcal{D}}\frac{\gamma[a^{\epsilon}_t(\boldsymbol{X})\cdot(\boldsymbol{X}-\bar{\boldsymbol{X}})]^2S_{\Theta^{\epsilon_2,\xi}}(a^{\epsilon}_t(\boldsymbol{X})b_t|(D\boldsymbol{c}_t)^s|)(\bar{\boldsymbol{X}})}{\delta^2+|\boldsymbol{X}-\bar{\boldsymbol{X}}|^2+\gamma[a^{\epsilon}_t(\boldsymbol{X})\cdot(\boldsymbol{X}-\bar{\boldsymbol{X}})]^2}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Now, by the nonnegativity of $\Theta^{\epsilon_2,\xi}$, $(\operatorname{div}\boldsymbol{c})^s\leqslant0$–for $(\operatorname{div}\boldsymbol{c})^+\in L^1((0,T);L^1_{\mathrm{loc}}(\mathbb{R}^d))$–as well as $a^{\epsilon}_t(x)$ being a unit vector and the uniform boundedness of $\Theta^{\epsilon_2,\xi}$, we have that

$$\begin{aligned}&\frac{\gamma[a^{\epsilon}_t(\boldsymbol{X})\cdot(\boldsymbol{X}-\bar{\boldsymbol{X}})]^2\bar{S}_{\Theta^{\epsilon_2,\xi}}\mathbb{1}_{\mathcal{D}}\operatorname{div}\boldsymbol{c}_t(\boldsymbol{X})}{\delta^2+|\boldsymbol{X}-\bar{\boldsymbol{X}}|^2+\gamma[a^{\epsilon}_t(\boldsymbol{X})\cdot(\boldsymbol{X}-\bar{\boldsymbol{X}})]^2}\\&\leqslant C\epsilon_2^{1-d}\gamma^{1/2}\min\left\{\frac{1}{\delta}\mathcal{I}_1[\mathbb{1}_{\mathcal{D}}(\operatorname{div}\boldsymbol{c}_t)^+](\boldsymbol{X}),M[\mathbb{1}_{\mathcal{D}}(\operatorname{div}\boldsymbol{c}_t)^+](\boldsymbol{X})\right\},\end{aligned}$$

By a simple application of Young convolution inequality, we have for every $R > 0$ that

$$\|\mathcal{I}_1 u\|_{L^p(B_R)} \leqslant C_R \|u\|_{L^1(\mathbb{R}^d)} \quad \text{for all } 1 \leqslant p < \frac{d}{d-1}.$$

Using the second part of Lemma 2.12 with $\delta = \epsilon_2$ and repeating the argument as above with Lemma 2.13, we have that

$$\begin{aligned} \limsup_{\delta \to 0} \frac{1}{|\log \delta|} \mathrm{VII}_\delta^{\gamma,\epsilon}(\tau) \leqslant & \gamma^{1/2} C_{\epsilon_2} \int_0^T \int_{\mathbb{R}^d} |a_t(x) - a_t^\epsilon(x)| \, \mathrm{d}|(D\boldsymbol{c}_t)^s|(x) \, \mathrm{d}t \\ & + \gamma^{1/2} \epsilon_2 C |\log(\epsilon_2)| \int_0^T |(D\boldsymbol{c}_t)^s|(\mathbb{R}^d) \, \mathrm{d}t. \end{aligned} \tag{2.30}$$

Combining (2.27), (2.28), (2.29), and (2.30), there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$, it holds

$$\begin{aligned} |B_r \cap \{|\boldsymbol{X} - \bar{\boldsymbol{X}}| > \delta^{1/2}\}| \leqslant & \big(f(r,\lambda) + \bar{f}(r,\lambda)\big) + \frac{C_r}{|\log \delta|^{1/2}} \big(1 + \|\boldsymbol{b}\|_{L^1((0,T)\times B_\lambda)}\big) \\ & + \frac{(L + \bar{L})}{\delta} (\|\boldsymbol{b} - \boldsymbol{c}\|_{L^1((0,T)\times B_\lambda)} + \|\boldsymbol{c} - \bar{\boldsymbol{b}}\|_{L^1((0,T)\times B_\lambda)}) \\ & + C_{\epsilon_1,\epsilon_2,\gamma} \int_0^T \int_{\mathbb{R}^d} |a_t(x) - a_t^\epsilon(x)| \, \mathrm{d}|(D\boldsymbol{c}_t)^s|(x) \, \mathrm{d}t \\ & + \gamma^{1/2} \epsilon_2 C |\log(\epsilon_2)| \int_0^T |(D\boldsymbol{c}_t)^s|(\mathbb{R}^d) \, \mathrm{d}t \\ & + (\epsilon_1 C |\log(\epsilon_1)| + \gamma^{-1/2} C_{\epsilon_1}) \int_0^T |(D\boldsymbol{c}_t)^s|(\mathbb{R}^d) \, \mathrm{d}t. \end{aligned}$$

In order to conclude, we take the limits in the following order: firstly $\epsilon \to 0$ so that the fourth term vanishes; secondly $\epsilon_2 \to 0$ so that the fifth term vanishes; thirdly, $\gamma \to \infty$ (recall that the range of γ after the limit superior is $(0, \infty)$) and then $\epsilon_1 \to 0$ for the sixth term to vanish. Now, for any $\eta > 0$, we choose δ small enough so that

$$\frac{C_{\gamma,r}}{|\log \delta|^{1/2}} \big(1 + \|\boldsymbol{b}\|_{L^1((0,T)\times B_\lambda)}\big) \leqslant \frac{\eta}{2}.$$

Recalling that Lemma 2.5 implies that the first term is less than $\eta/2$, provided λ is large enough, and so for all $\gamma' > 0$, it holds

$$|B_r \cap \{|\boldsymbol{X} - \bar{\boldsymbol{X}}| > \gamma'\}| \leqslant \eta + C_{\gamma',\eta,r}(\|\boldsymbol{b} - \boldsymbol{c}\|_{L^1((0,T)\times B_\lambda)} + \|\boldsymbol{c} - \bar{\boldsymbol{b}}\|_{L^1((0,T)\times B_\lambda)}),$$

provided we also restrict δ so that $\delta^{1/2} < \gamma'$. □

Remark 13. Before we finish the chapter, we recall the striking counterexample of generalizing the results of Section 2.3 for bounded measure. More precisely, it is a natural question whether one can weaken the assumption on vector field

$$\boldsymbol{b}_t^i(x) = \sum_{k=1}^{m_R} \tilde{\Gamma}^{ik,R} * g_t^{ik,R}(x) \quad \text{in } B_R$$

where $\tilde{\Gamma}^{ik,R}$ is a singular rough kernels as in Theorem 2.5 and $g^{ik,R} \in L^1((0,T);\mathrm{BV}(\mathbb{R}^d))$ for all $i \in \{1,\dots,d\}$ and $k \in \{1,\dots,m_R\}$ for an analogous version (2.16), namely

$$\partial_j \boldsymbol{b}_t^i = \sum_{k=1}^{m_R} \tilde{\Gamma}^{ijk,R} * \mu_t^{ijk,R} \qquad \text{in the weak sense in } B_R, \tag{2.31}$$

where $\tilde{\Gamma}^{ijk,R}$ is a singular rough kernels as in Theorem 2.5 and $\mu^{ijk,R} \in L^1((0,T);\mathcal{M}(\mathbb{R}^d))$ for all $i,j \in \{1,\dots,d\}$ and $k \in \{1,\dots,m_R\}$. The result of Nguyen [70, Proposition 1.2] states that (2.31) is not enough to ensure uniqueness of flows, and so in particular the fundamental estimate Theorem 2.6 does not hold. The proof is a clever repurposed example of DiPerna-Lions [48, Section IV.2], where the vector field has no integrable first derivative, and so Nguyen was able to show that such example can be written as (2.31) with additional structure explained in Remark 11. We refer to Chapter 6 for a discussion of related open problems.

3 The Ambrosio-Colombo-Figalli theory: local results with Lagrangian approach

In this chapter, we shall present the local version of the results of Chapter 2 developed by Ambrosio-Colombo-Figalli [5] in the abstract form, i.e. not assuming explicitly any structure for vector fields, but rather a uniqueness hypothesis on a class of solutions of continuity equation (1.4); see Condition 3.1. The motivation follows from the classical Cauchy-Lipschitz theory: local Lipschitz regularity in space variable of vector fields implies local well-posedness on (1.2), and the global regularity implies global well-posedness. Notice that the latter is addressed in Chapter 2, since it was assumed growth assumptions (2.2) or (1.6), but the former is not, for the aforementioned technique heavily relies on the control of the sublevels obtained in Lemma 2.5, which is a consequence of (2.2) and (1.6).

We also present an important application of the local theory for Vlasov systems in Section 3.2, namely the (nonrelativistic) Vlasov-Poisson equation by Ambrosio-Colombo-Figalli[1] [6] and later the more general quasistatic Vlasov-Maxwell approximations by the author and Marcon [21].

3.1 Local flows: Ambrosio-Colombo-Figalli's result

In order to present the results of [5], we first recall the fundamental theorems proven in Chapter 2: it was necessary the control of the sublevels, namely

$$|B_r \setminus G_\lambda| \to 0 \quad \text{for a fixed } r \text{ as } \lambda \to \infty \tag{3.1}$$

proven in Lemma 2.5–which followed by assuming either (2.2) or (1.6). Moreover, it was crucial the fundamental estimate associated to flows: for every $s \in [0,T]$, $t \in [s,T]$, $\gamma > 0$, $r > 0$, and $\eta > 0$, there exists a constant $C_{\gamma,\eta,r} > 0$ and $\lambda > 0$ such that

$$|B_r \cap \{|\boldsymbol{X}(t,s,\cdot) - \bar{\boldsymbol{X}}(t,s,\cdot)| > \gamma\}| < C_{\gamma,\eta,r}\|\boldsymbol{b} - \bar{\boldsymbol{b}}\|_{L^1((0,T)\times B_\lambda)} + \eta \tag{3.2}$$

proven in Propositions 2.1, 2.3 and 2.4 and Theorem 2.6. In order to prove this, besides the aforementioned control of sublevels, it was necessary to assume some structure to the vector fields, e.g. (2.16) and (2.18)-(2.19). Finally, in order to establish well-posedness for the flows, it was necessary to assume (at least) a lower bound on the divergence of the vector field, as in Theorem 2.1. Notice that all the developed theory in Section 2.2 can be reproduced if one assumes that the vector field satisfies (3.1), (3.2), and a lower bound on the divergence as in (H3) in Condition 3.1 below.

[1] Throughout this chapter, we shall refer to them as "trio".

The major change in the work [5] is not assume (3.1) and (3.2), but rather local properties (H1) and (H2) in Condition 3.1 below.

Condition 3.1. We say that a Borel vector field $\boldsymbol{b} : (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$ is admissible if the following are satisfied:

(H1) it holds $\boldsymbol{b} \in L^1((0,T); L^1_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d))$;

(H2) for any nonnegative $\bar{u} \in L^\infty_c(\mathbb{R}^d)$ and $[a,b] \subset [0,T]$, there exists at most one nonnegative weak solution $u \in L^\infty_c([a,b] \times \mathbb{R}^d)$ of the continuity equation

$$\begin{cases} \partial_t u + \operatorname{div}(\boldsymbol{b}u) = 0 & \text{in} \quad [a,b] \times \mathbb{R}^d; \\ u_{t=a} = \bar{u} & \text{on} \quad \mathbb{R}^d. \end{cases}$$

such that $u \in C([a,b]; L^\infty(\mathbb{R}^d) - w*)$;

(H3) for all $t \in (0,T)$, it holds in the sense of distributions $\operatorname{div} \boldsymbol{b}_t \geqslant m_\Omega(t)$ in Ω for all compact $\Omega \subset \mathbb{R}^d$ and some $m_\Omega \in L^1((0,T))$.

Remark 14. Although (H1) and (H3) are clearly local hypothesis in space, it is not straightforward to see that (H2) also is. In fact, the trio showed in [5, Lemma 8.1] that (H1)-(H2) are equivalent to (H1)-(H2'), where

(H2') for any $t_0 \geqslant 0$ and $x_0 \in \mathbb{R}^d$, there exists $\epsilon = \epsilon_{t_0,x_0} > 0$ such that for any nonnegative $\bar{u} \in L^\infty(\mathbb{R}^d)$ with compact support contained in $B_\epsilon(x_0)$ and $[a,b] \subset [t_0 - \epsilon, t_0 + \epsilon] \cap [0,T]$, there exists at most one nonnegative solution $u_t \in L^\infty_c(B_\epsilon(x_0))$ for all $t \in [a,b]$ of the continuity equation

$$\begin{cases} \partial_t u + \operatorname{div}(\boldsymbol{b}u) = 0 & \text{in} \quad [a,b] \times \mathbb{R}^d; \\ u_{t=a} = \bar{u} & \text{on} \quad \mathbb{R}^d. \end{cases}$$

such that $u \in C([a,b]; L^\infty(\mathbb{R}^d) - w*)$,

and so (H2') can be seen as a local version of (H2). Since it combined with (H1) is equivalent to (H2), we can assume (H2') in place of (H2) in Condition 3.1.

Remark 15. As mentioned in Chapter 1, by the striking example of Depauw [45], we know that in order to establish uniqueness of the flow–even if only a local one–it is not sufficient to assume a local in time (H1), that is, to assume that $\boldsymbol{b} \in L^1_{\text{loc}}((0,T) \times \mathbb{R}^d; \mathbb{R}^d)$; it is however enough to assume $\boldsymbol{b} \in L^1((0,T_0) \times \mathbb{R}^d; \mathbb{R}^d)$ for all $T_0 < T$. Moreover, as commented in [5, Remark 5.1], the hypothesis (H3) can be weakened for (H3'), where

(H3') for $T_0 < T$ and all $t \in (0,T_0)$, it holds in the sense of distributions $\operatorname{div} \boldsymbol{b}_t \geqslant m_\Omega(t)$ in Ω for all compact $\Omega \subset \mathbb{R}^d$ and some $m_\Omega \in L^1((0,T_0))$.

Notice that in both cases the hypothesis in not truly local in time, for we cannot relax the assumption from local in $[0, T)$ to local in $(0, T)$. Hence, (H3) are chosen for the sake of simplicity and they are not essential, for it could be instead assumed (H3'), and likewise the assumption $\boldsymbol{b} \in L^1((0, T_0) \times \mathbb{R}^d; \mathbb{R}^d)$ in place of (H1).

Finally, we remark that $t = 0$ being the initial time is for the sake of simplicity, and all of the above considerations can be made *mutatis mutandis* for any initial time $s \in [0, T]$.

A natural question is what class of vector fields satisfy (H2); condition (H1) is very mild and (H3) is usually an assumption. We shall see that in fact proofs of (3.2) can be adapted to prove (H2). In particular, (H2) holds for vector fields considered in Section 2.5, namely $\boldsymbol{b}$ being locally written as convolution of a zero average singular kernel with a BV function in space an summable in time. Before we prove it, we first state some useful functional analysis results. For a detailed proof of them, we refer to [15, Theorems 4.7.18 and 8.6.2] and [11, Theorem 5.3.1]; they may be stated more generally, but for the sake of simplicity we restrict their thesis for a clearer presentation.

Theorem 3.1 (Dunford-Pettis property)**.** *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$. Then a family in $L^1(\mu)$ has compact closure in the weak topology of $L^1(\mu)$ if and only if it is uniformly integrable (recall its definition in Lemma 2.6).*

A natural comparison can be made between Lemma 2.6 and Theorem 3.1: if the sequence is uniformly bounded in $L^1(\mu)$ but lacks local convergence in measure, then it does not converge strongly in $L^1(\mu)$, but has a subsequence converging weakly in $L^1(\mu)$.

Theorem 3.2 (Prokhorov compactness criterion)**.** *Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of Radon signed measures in X. Then such sequence has compact closure in the weak topology of $\mathcal{M}(X)$ if and only if $|\mu_n(X)|$ is uniformly bounded and uniformly tight, that is, if for every $\epsilon > 0$, there exists a compact K_ϵ such that $|\mu_n(X \setminus K_\epsilon)| < \epsilon$ for all n.*

Remark 16. We shall use an equivalent definition of uniform tightness [7, Remark 5.1.5], which says that there exists a coercive functional $F : X \to [0, \infty]$ such that

$$\sup_{n\in\mathbb{N}} \int_X F(x)\, \mathrm{d}\mu_n(x) < \infty$$

and has compact sublevels, that is, for each $\lambda > 0$, $\{x \in X : F(x) < \lambda\}$ is a compact subset of X.

Before we state the disintegration theorem, we recall the definition of Radon spaces: a separable metric space X is a Radon space if every probability measure $\mu \in \mathcal{P}(X)$ is inner regular for every Borel set.

Theorem 3.3 (Disintegration)**.** *Let X and Y be Radon spaces, $\mu \in \mathcal{P}(Y)$ and $\pi : Y \to X$ a Borel map. Then by denoting $\nu := \pi_{\#}\mu \in \mathcal{P}(X)$, there exists an $\nu-$almost everywhere uniquely determined Borel family $\{\mu_x\}_{x\in X} \subset \mathcal{P}(Y)$ such that $\mu_x(Y \setminus \pi^{-1}(x)) = 0$ for $\nu-$almost every $x \in X$ and for every Borel map $f : X \to \mathbb{R}$ it holds*

$$\int_Y f(y)\,\mathrm{d}\mu(y) = \int_X \int_{\pi^{-1}(x)} f(y)\,\mathrm{d}\mu_x(y)\,\mathrm{d}\nu(x).$$ [2]

One of the simplest motivations for Theorem 3.3 is the following (see [32, Example 2]): consider a unit square $Q = [0,1] \times [0,1]$ in $\mathbb{R}^2$ and a probability measure $\mu \in \mathcal{P}(Q)$ defined as $\mu := \mathcal{L}^2 \llcorner Q$. Notice that for any line segment $L_x = \{x\} \times [0,1]$, the measure μ "does not see" any probability in it, that is $\mu(L_x) = 0$. However, it seems plausible that by "restricting" the measure μ in L_x–not by simply considering $\mu \llcorner L_x$, for it is the zero measure, hence the quotation marks–we would have the one dimensional Lebesgue measure restricted to L_x. This is precisely what the disintegration does, for considering the projection $\pi : Q \to [0,1]$, $\pi(x,y) = x$, we obtained a one dimensional measure μ_x such that

$$\mu(E) = \int_{[0,1]} \mu_x(E)\,\mathrm{d}\nu(x) = \int_{[0,1]} \mu_x(E \cap L_x)\,\mathrm{d}\nu(x) \quad \text{for any Borel set} \quad E \subset Q.$$

Finally, we shall need the so called "superposition principle" stated in [5, Theorem 2.1] with a detailed proof in [7, Theorem 4.4]; it heavily uses Theorem 3.1 and Theorem 3.2. We shall use throughout this chapter the evaluation function at time $t \in I$

$$e_t : C(I;\mathbb{R}^d) \to \mathbb{R}^d, \quad e_t(\gamma) = \gamma(t),$$

where I is an interval.

Theorem 3.4 (Superposition principle)**.** *Let $\boldsymbol{b} : (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$ be a Borel vector field and $\{\mu_t\}_{t\in(0,T)}$ be a family of measures in $\mathcal{M}(\mathbb{R}^d)$ such that $t \mapsto \mu_t$ is weakly continuous, that is, for any bounded $\varphi \in C(\mathbb{R}^d)$, the map $t \mapsto \int_{\mathbb{R}^d} \varphi(x)\,\mathrm{d}\mu_t(x)$ is continuous. Moreover, assume that $\mu : (0,T) \mapsto \mathbb{R}^d$, $\mu(t) = \mu_t$ solves in the weak sense $\partial_t\mu + \mathrm{div}(\boldsymbol{b}\mu) = 0$ in $(0,T) \times \mathbb{R}^d$ with the following integrability:*

$$\int_0^T \int_{\mathbb{R}^d} \frac{|\boldsymbol{b}_t(x)|}{1+|x|}\,\mathrm{d}\mu_t(x)\,\mathrm{d}t < \infty. \tag{3.3}$$

Then there exists a measure $\boldsymbol{\eta} \in \mathcal{P}(C([0,T];\mathbb{R}^d))$ concentrated on absolutely continuous in times curves $\gamma(t)$ solving $\dot\gamma(t) = \boldsymbol{b}_t(\gamma(t))$ for almost every $t \in (0,T)$ such that $\mu_t = (e_t)_{\#}\boldsymbol{\eta}$ for all $t \in [0,T]$.

[2] When there is no confusion, we shall write it for simplicity in a more compact form $\mu = \int \mu_x \mathrm{d}\nu(x)$.

In the above, we did not use the renormalized formulation of (1.2), namely Definition 2.2 (i), but rather the almost everywhere notion, and so in order to make it precise the curves need to be absolutely continuous.

By [7, Remark 4.3], it is equivalent to state Theorem 3.4 for probability measures $\boldsymbol{\sigma}$ on $C([0,T];\mathbb{R}^d)$ in place of $\boldsymbol{\eta}$, and Theorem 3.4 would state that μ_t satisfy the identity

$$\int_{\mathbb{R}^d} \varphi(x)\mathrm{d}\mu_t(x) = \int_{\mathbb{R}^d\times\Gamma} \varphi(e_t(\gamma))\mathrm{d}\boldsymbol{\sigma}(x,\gamma),$$

where $\Gamma = \{\gamma \in \mathrm{AC}([0,T];\mathbb{R}^d) : \dot{\gamma}(t) = \boldsymbol{b}_t(\gamma(t)) \text{ and } \gamma(0) = x\}$. This justifies the name "superposition principle", for solutions of continuity equation can be written as a sum of integral curves associated to $\boldsymbol{b}$.

Remark 17. The Theorem 3.4 can be stated for a family of measures in other sets besides the Euclidean spaces. For the sake of simplicity, we remark that the result holds for $\{\mu_t\}_{t\in(0,T)}$ being a family of measures in $\mathcal{M}(\mathbb{S}^d)$ solving the continuity equation in $(0,T)\times\mathbb{S}^d$ and satisfying the assumptions of Theorem 3.4. More generally, the result would hold for any subset $\Omega\subset\mathbb{R}^d$, for the result only uses the structure of Euclidean spaces. Indeed, this can be done by extending $\mu_t\in\mathcal{M}(\Omega)$ as zero in $\mathbb{R}^d\setminus\Omega$, and so the proof of [7, Theorem 4.4] can be used for such case. We emphasize that it is not used any intrinsic property of Ω, e.g. topology, metric, etc. For these type of results, we refer to [12].

We now show an argument of the trio [5, Remark 3.2] regarding the idea of adapting the global results of Chapter 2 without the growth assumptions.

Remark 18. Notice that if one considers as in (H2) a nonnegative solution $u\in L^\infty_c([a,b]\times\mathbb{R}^d)$ of the continuity equation on $[a,b]\times\mathbb{R}^d\subset[0,T]\times\mathbb{R}^d$, then considering $\mathrm{d}\mu_t = u_t\mathrm{d}\mathcal{L}^d$ and a vector field $\boldsymbol{b}$ satisfying (H1), we have that (3.3) holds. Moreover, if one further assumes the structure for $\boldsymbol{b}$ studied in Section 2.5, namely the one for $\boldsymbol{c}$ in Theorem 2.6, we have that (H2) holds. Indeed, consider two solutions u^1 and u^2 of the continuity equation with same vector field and initial data, with support in space inside B_R for some $R>0$. By Theorem 3.4, there exists two measures $\boldsymbol{\eta}^1,\boldsymbol{\eta}^2\in\mathcal{P}(C([0,T];B_R))$ concentrated on absolutely continuous in times curves $\gamma(t)$ solving $\dot{\gamma}(t) = \boldsymbol{b}_t(\gamma(t))$ for almost every $t\in(0,T)$ such that $\mu_t = (e_t)_\#\boldsymbol{\eta}$. If we are only interested in compressible curves in the sense that they satisfy Definition 2.2 (ii), we consider the analogous for measures in trajectories[3], that is, for all $t\in(a,b)$ it holds

$$(e_t)_\#\boldsymbol{\eta}^1 \leqslant L^1\mathcal{L}^d \quad \text{and} \quad (e_t)_\#\boldsymbol{\eta}^2 \leqslant L^2\mathcal{L}^d$$

for some constants $L_1, L_2 > 0$. Moreover, notice that $(e_a)_\#\boldsymbol{\eta}^1 = \mu_a = (e_a)_\#\boldsymbol{\eta}^2$. Denoting the probability measure $\boldsymbol{\eta} = (\boldsymbol{\eta}^1+\boldsymbol{\eta}^2)/2$, if one proves that its disintegration with respect

[3] Notice that if one has uniqueness for the flow equation (1.2), then $\boldsymbol{\eta}$ is concentrated on $\boldsymbol{X}$, and so we have $\mu_t = (e_t)_\#\boldsymbol{\eta} = (e_t)_\#\int_{\mathbb{R}^d}\delta_{\boldsymbol{X}(\cdot,x)}\,\mathrm{d}\mu_0(x)$, and so $\mu_t = \boldsymbol{X}(t,\cdot)_\#\mu_0$. Since we may take $\mu_0 = f\mathcal{L}^d$ for nonnegative $f\in L^1(\mathbb{R}^d)$, it is analogous to Definition 2.2 (ii).

to $(e_a)_\#\boldsymbol{\eta}$ denoted $\boldsymbol{\eta}_x$ is a Dirac measure for μ_a−almost every x, then

$$\boldsymbol{\eta}_x = \boldsymbol{\eta}_x^1 = \boldsymbol{\eta}_x^2 \quad \text{for } \mu_a - \text{almost every } x.$$

Therefore, we have that $\boldsymbol{\eta}^1 = \boldsymbol{\eta}^2$, and so (H2) would follow. In order to prove that $\boldsymbol{\eta}_x$ is a Dirac delta measure, we adapt (2.24) for the context of probability measures: let

$$\Psi_\delta^{\alpha,\epsilon}(t) = \frac{1}{2}\int_{\mathbb{R}^d}\int\int \log\left(1 + \frac{|\gamma_t - \eta_t|^2 + \alpha[a_t^\epsilon(\gamma_t)\cdot(\gamma_t - \eta_t)]^2}{\delta^2}\right) \mathrm{d}\boldsymbol{\eta}_x(\gamma)\, \mathrm{d}\boldsymbol{\eta}_x(\eta)\mathrm{d}\mu_a(x);$$

we changed the γ parameter to α for we now use the former to denote curves and for simplicity we write $\gamma_t = \gamma(t)$, $\eta_t = \eta(t)$. Since it holds $(e_t)_\#\boldsymbol{\eta} \leqslant L\mathcal{L}^d$ for $L = (L_1 + L_2)/2$, the same upper estimates done for $\Phi_\delta^{\alpha,\epsilon}$ can be done for the above $\Psi_\delta^{\alpha,\epsilon}$, namely it holds for all $t \in [a, b]$ that for any $\epsilon' > 0$, there exists δ_0 small enough such that

$$\lim_{\alpha\to\infty}\lim_{\epsilon\to 0}\Psi_\delta^{\alpha,\epsilon}(t) \leqslant |\log\delta|\epsilon' \quad \text{for all } \delta < \delta_0. \tag{3.4}$$

In order to conclude it, we proceed as in [6, Theorem 4.4]: assume by contradiction that $\boldsymbol{\eta}_x$ is not a Dirac delta. Then there exists a constant $c > 0$ such that

$$\iiint\int_a^b \min\{1, |\gamma_t - \eta_t|\}\, \mathrm{d}t\, \mathrm{d}\boldsymbol{\eta}_x(\gamma)\, \mathrm{d}\boldsymbol{\eta}_x(\eta)\mathrm{d}\mu_a(x) = c.$$

Indeed, let us prove the contraposition, and so by assuming that $\boldsymbol{\eta}_x$ are Dirac deltas, notice that for almost every $t \in [a, b]$ and μ_a−almost every $x \in \mathbb{R}^d$, we would have that

$$\iint \min\{1, |\gamma_t - \eta_t|\}\, \mathrm{d}\boldsymbol{\eta}_x(\gamma)\, \mathrm{d}\boldsymbol{\eta}_x(\eta) = 0.$$

Now, it follows by using the proof found in [77, Theorem 3.1] that if

$$\iint |\gamma_t - \eta_t|\, \mathrm{d}\boldsymbol{\eta}_x(\gamma)\, \mathrm{d}\boldsymbol{\eta}_x(\eta) = 0,$$

then $\boldsymbol{\eta}_x$ is not a Dirac delta.

By Fubini theorem there exists $t_0 \in [a, b]$ such that

$$\iiint \min\{1, |\gamma_{t_0} - \eta_{t_0}|\}\, \mathrm{d}\boldsymbol{\eta}_x(\gamma)\, \mathrm{d}\boldsymbol{\eta}_x(\eta)\mathrm{d}\mu_a(x) > \frac{c}{T}.$$

Now, consider the set

$$\Omega := \left\{(\gamma, \eta, x) : \min\{1, |\gamma_{t_0} - \eta_{t_0}|\} \geqslant \frac{c}{2T}\right\}$$

and notice that since $\boldsymbol{\eta}_x$ are probability measures, we have

$$\iiint \mathbb{1}_\Omega \min\{1, |\gamma_{t_0} - \eta_{t_0}|\}\, \mathrm{d}\boldsymbol{\eta}_x(\gamma)\, \mathrm{d}\boldsymbol{\eta}_x(\eta)\mathrm{d}\mu_a(x) > \frac{c}{2T}.$$

Therefore, without loss of generality assuming $c < 2T$, we have that

$$|\gamma_{t_0} - \eta_{t_0}| \geqslant \frac{c}{2T} \quad \text{for all } (\gamma, \eta, x) \in \Omega.$$

Hence, we have a lower bound

$$\Psi_\delta^{\alpha,\epsilon}(t_0) \geqslant \frac{c}{4T}\log\left(1+\frac{c^2}{(2T\delta)^2}\right) \geqslant \frac{c}{2T}\left[\log\left(\frac{c}{2T}\right)+|\log\delta|\right]. \tag{3.5}$$

Combining (3.4) and (3.5), we conclude

$$\left(\frac{c}{2T}-\epsilon'\right)|\log\delta| < \frac{c}{2T}\log\left(\frac{2T}{c}\right).$$

Taking $\epsilon' < c/2T$ and letting $\delta \to 0$, we have a contradiction to the assumption and so $\boldsymbol{\eta}_x$ is a Dirac delta measure.

We may state a abstract form of the aforementioned result concerning the proof of the disintegration $\boldsymbol{\eta}_x$ being a Dirac delta measure [5, Theorem 3.4]: It states that if one assumes (H1) and (H2) for $\boldsymbol{b}$, and given $\boldsymbol{\eta}$ concentrated on absolutely continuous curves associated to $\boldsymbol{b}$ and $(e_t)_\#\boldsymbol{\eta} \leqslant L\mathcal{L}^d$ for some $L > 0$, then its disintegration with respect to e_0 is an Dirac delta.

Theorem 3.5. *Let Ω be a subset of $\mathbb{R}^d$ and $\boldsymbol{b}$ satisfying (H1) and (H2) in Ω*[4] *and $\boldsymbol{\eta} \in \mathcal{P}(C([0,T];\mathbb{R}^d))$ concentrated on*

$$\{\gamma \in \mathrm{AC}([0,T];\Omega) : \dot\gamma(t) = \boldsymbol{b}_t(\gamma(t)) \quad \textit{for almost every } t \in (0,T)\} \tag{3.6}$$

and such that $(e_t)_\#\boldsymbol{\eta} \leqslant L\mathcal{L}^d$ for some $L > 0$ for all $t \in [0,T]$. Then its disintegration with respect to e_0 is a Dirac delta measure; equivalently, there exists curves γ_x in (3.6) such that $\gamma_x(0) = x$ such that $\boldsymbol{\eta} = \int_\Omega \delta_{\gamma_x}\,\mathrm{d}(e_0)_\#\boldsymbol{\eta}(x)$.

Remark 19. As discussed in [7, Theorem 4.1], notice that Theorem 3.5 implies local uniqueness of compressible Lagrangian flows, for it suffices to consider

$$\boldsymbol{\eta} = \frac{1}{2|B_R|}\int_{B_R}\delta_{\boldsymbol{X}(\cdot,x)} + \delta_{\bar{\boldsymbol{X}}(\cdot,x)}\,\mathrm{d}x$$

for $R > 0$ and $\boldsymbol{X}, \bar{\boldsymbol{X}}$ regular Lagrangian flows in $[a,b] \times B_R$. On the other hand, local uniqueness of flows implies (H2) by Theorem 3.4, for one may consider weak solutions of continuity equation $u_t\mathcal{L}^d = \boldsymbol{X}(t,\cdot)_\#(\chi_{B_R}\mathcal{L}^d)$ and $\bar u_t\mathcal{L}^d = \boldsymbol{X}(t,\cdot)_\#(\chi_{B_R}\mathcal{L}^d)$. Moreover, (3.6) implies the consistency of regular Lagrangian flows [5, Lemma 4.2]: for $\boldsymbol{X} : [0,\tau]\times\Omega \to \mathbb{R}^d$ and $\bar{\boldsymbol{X}} : [0,\bar\tau] \times \bar\Omega \to \mathbb{R}^d$ regular Lagrangian flows ($\Omega, \bar\Omega$ Borelian sets), it follows by considering the probability measure

$$\boldsymbol{\eta} = \frac{1}{2|\Omega\cap\bar\Omega|}\int_{\Omega\cap\bar\Omega}\delta_{\boldsymbol{X}(\cdot,x)} + \delta_{\bar{\boldsymbol{X}}(\cdot,x)}\,\mathrm{d}x$$

on time interval $[0,\min\{\tau,\bar\tau\}]$ that $\boldsymbol{X}(\cdot,x) = \bar{\boldsymbol{X}}(\cdot,x)$ in $[0,\min\{\tau,\bar\tau\}]$ for almost every $x \in \Omega\cap\bar\Omega$.

[4] By that we mean $\boldsymbol{b} \in L^1((0,T);L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^d))$ and the uniqueness condition of (H2) holds replacing $\mathbb{R}^d$ with Ω.

Notice that the lower bound estimate (3.5) is completely independent of the growth assumption (1.6) and even of (3.3), where the latter was only used for the application of Theorem 3.4. This key observation led the trio to prove the so called "extended superposition principle" [6, Theorem 5.1] which is completely independent of (3.3). In particular, we do not have to restrict solutions u of continuity equation being bounded and of compact support as in (H2), but rather a much milder assumption $u \in L^\infty((0,T); L^1(\mathbb{R}^d))$. Before we prove it, we recall the result [6, Lemma 5.3] which states that we may construct a "damped stereographic projection", that is, given a nonincreasing function g, there exists a diffeomorphism between the $d-$sphere without the north pole and $(d+1)-$Euclidean space such that its gradient is less than g. We denote N the north pole of the $d-$sphere.

Lemma 3.1. *Let $g : [0,\infty) \to (0,1]$ be a monotone nonincreasing function. Then there exists $r_0 > 0$ and a smooth diffeomorphism $\psi : \mathbb{R}^d \to \mathbb{S}^d \setminus \{N\} \subset \mathbb{R}^{d+1}$ such that*

$$\begin{aligned} \psi(x) &\to N \quad \text{as } |x| \to \infty; \\ |\nabla\psi(x)| &\leqslant g(0) \quad \text{for all } x \in \mathbb{R}^d; \\ |\nabla\psi(x)| &\leqslant g(|x|) \quad \text{for all } x \in \mathbb{R}^d \setminus B_{r_0}. \end{aligned} \tag{3.7}$$

We shall denote $\bar{\mathbb{R}}^d = \mathbb{R}^d \cup \{\infty\}$ the one-point compactification (sometimes referred as Alexandroff compactification) of the $d-$Euclidean space. Moreover, we recall an equivalent formulation of weak solution of the continuity equation, which typically of form

$$\int_a^b \int_\Omega \partial_t\varphi_t(x) + \boldsymbol{b}_t(x) \cdot \nabla\varphi_t(x) \,\mathrm{d}\mu_t(x) \,\mathrm{d}t = 0$$

for all $\varphi \in C_c^\infty((a,b) \times \Omega)$, where μ is said to be a weak solution of continuity equation with vector field $\boldsymbol{b}$ in $(a,b) \times \Omega$. As proven in [11, Section 8.1], the above is equivalent of proving that for almost every $t \in (a,b)$, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \varphi(x) \,\mathrm{d}\mu_t(x) = \int_\Omega \boldsymbol{b}_t(x) \cdot \nabla\varphi(x) \,\mathrm{d}\mu_t(x) \tag{3.8}$$

for all $\varphi \in \mathrm{Lip}_c(\Omega)$, where the left-hand side should be understood as $\int_\Omega \varphi(x) \,\mathrm{d}\mu_t(x)$ being equal almost everywhere to an absolutely continuous function. If one consider an initial value problem, we further assume that such function coincides at $t = 0$ to $\int_\Omega \varphi(x) \,\mathrm{d}\mu_0(x)$.

Theorem 3.6 (Extended superposition principle)**.** *Let $u \in L^\infty((0,T); L^1(\mathbb{R}^d))$ be a non-negative solution of the continuity equation with vector field $\boldsymbol{b}$ satisfying (H1) such that the map $t \mapsto u_t(x)$ is weakly continuous in duality with $C_c(\mathbb{R}^d)$. Moreover, assume that $|\boldsymbol{b}|u \in L^1((0,T); L^1_{\mathrm{loc}}(\mathbb{R}^d))$. Then there exists $\boldsymbol{\eta} \in \mathcal{M}(C([0,T]; \bar{\mathbb{R}}^d))$ concentrated on the set*

$$\begin{aligned} &\{\gamma \in C([0,T]; \bar{\mathbb{R}}^d) : \gamma \in \mathrm{AC}_{\mathrm{loc}}(\{\gamma \neq \infty\}; \mathbb{R}^d) \\ &\quad \text{and} \quad \dot\gamma(t) = \boldsymbol{b}_t(\gamma(t)) \text{ for almost every } t \in \{\gamma \neq \infty\}.\} \end{aligned} \tag{3.9}$$

such that $u_t\mathcal{L}^d = (e_t)_\#\boldsymbol{\eta} \llcorner \mathbb{R}^d$ for all $t \in [0,T]$ and $\boldsymbol{\eta}(C([0,T]; \bar{\mathbb{R}}^d)) \leqslant \|u\|_{L^\infty((0,T);L^1(\mathbb{R}^d))}$.

Proof. We begin by considering in Lemma 3.1 the function

$$g(r) = \begin{cases} 1 & \text{if} \quad r \in [0,1); \\ 2^{-n}\left(1 + \displaystyle\int_0^T \int_{B_{2^n}} |\boldsymbol{b}_t(x)| u_t(x) \,\mathrm{d}x \,\mathrm{d}t\right)^{-1} & \text{if} \quad r \in [2^{n-1}, 2^n), \end{cases}$$

and so we obtain the existence of a smooth diffeomorphism $\psi : \mathbb{R}^d \to \mathbb{S}^d \setminus \{N\}$ that it can be extended as $\psi(\infty) = N$ and such that $|\nabla\psi(x)| \leqslant g(0) = 1$ for all $x \in \mathbb{R}^d$, and there exists some n_0 such that $r_0 \leqslant 2^{n_0}$ and for all $n \geqslant n_0$

$$|\nabla\psi(x)| \leqslant 2^{-n}\left(1 + \int_0^T \int_{B_{2^n}} |\boldsymbol{b}_t(x)| u_t(x) \,\mathrm{d}x \,\mathrm{d}t\right)^{-1} \quad \text{for all } x \in B_{2^n} \setminus B_{2^{n-1}}.$$

Therefore, we conclude that

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d} |\nabla\psi(x)||\boldsymbol{b}_t(x)| u_t(x) \,\mathrm{d}x \,\mathrm{d}t \leqslant & \int_0^T \int_{B_{2^{n_0}}} |\boldsymbol{b}_t(x)| u_t(x) \,\mathrm{d}x \,\mathrm{d}t \\ & + \sum_{i=n_0+1}^{\infty} \int_0^T \int_{B_{2^i} \setminus B_{2^{i-1}}} |\nabla\psi(x)||\boldsymbol{b}_t(x)| u_t(x) \,\mathrm{d}x \,\mathrm{d}t \\ \leqslant & \int_0^T \int_{B_{2^{n_0}}} |\boldsymbol{b}_t(x)| u_t(x) \,\mathrm{d}x \,\mathrm{d}t + \sum_{i=n_0+1}^{\infty} 2^{-i} < \infty. \end{aligned} \tag{3.10}$$

We now construct $\boldsymbol{\eta}$ with desired properties. Without loss of generality, assume that $\|u\|_{L^\infty((0,T);L^1(\mathbb{R}^d))} = 1$ and write ϕ as the inverse of the diffeomorphism ψ constructed above. Let $m_t = \|u_t\|_{L^1(\mathbb{R}^d)}$,

$$\boldsymbol{c}_t(y) = \begin{cases} \nabla\psi(\phi(y))\boldsymbol{b}_t(\phi(y)) & \text{if} \quad y \in \mathbb{S}^d \setminus \{N\}; \\ 0 & \text{if} \quad y = N, \end{cases}$$

and consider the measure

$$\mu_t = \psi_\#(u_t \mathcal{L}^d) + (1 - m_t)\delta_N \quad \text{for} \quad t \in [0,T].$$

Notice that $\phi_\#(\mu_t \llcorner (\mathbb{S}^d \setminus \{N\})) = (\psi \circ \phi)_\#(u_t \mathcal{L}^d) = u_t \mathcal{L}^d$, and since $\boldsymbol{c}_t(N) = 0$, it holds

$$\begin{aligned} \int_0^T \int_{\mathbb{S}^d} |\boldsymbol{c}_t(y)| \,\mathrm{d}\mu_t(y) \,\mathrm{d}t &= \int_0^T \int_{\mathbb{R}^d} |\nabla\psi(x)||\boldsymbol{b}_t(x)| \,\mathrm{d}\phi_\#(\mu_t \llcorner (\mathbb{S}^d \setminus \{N\}))(x) \,\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{R}^d} |\nabla\psi(x)||\boldsymbol{b}_t(x)| u_t(x) \,\mathrm{d}x \,\mathrm{d}t < \infty, \end{aligned} \tag{3.11}$$

where the last inequality follows from (3.10), and so $\boldsymbol{c}$ satisfies (3.3). Therefore, in order to apply Theorem 3.4 (recall Remark 17), it suffices to show that μ is a solution of continuity equation with vector field $\boldsymbol{c}$ on $(0,T) \times \mathbb{S}^d$ and that it is weakly continuous in time in duality with bounded $C(\mathbb{R}^{d+1})$[5]. The latter follows by the weak continuity of u_t and the

[5] Recall that $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, and so the weak continuity in duality with bounded $C(\mathbb{R}^{d+1})$ means that the map $t \to \int_{\mathbb{S}^d} \varphi(y) \,\mathrm{d}\mu_t(y)$ is continuous for all bounded $\varphi \in C(\mathbb{R}^{d+1})$.

fact that μ_t is a probability measure. Indeed, recall that the continuity of μ_t in duality with $C_c(\mathbb{R}^{d+1})$ is equivalent to bounded $C(\mathbb{R}^{d+1})$ if μ_t is a probability measure (see [11, Remark 5.1.6]), which is easily verified by its definition. Notice that by the weak continuity in time of u, then assuming that μ_t has two limit points λ, ν as $t \to s$, then for all $\varphi \in C_c(\mathbb{R}^{d+1})$, it holds (recall that $m_t = \mu_t(\mathbb{S}^d \setminus \{N\})$)

$$\begin{aligned} \int_{\mathbb{S}^d} \varphi(y)\,\mathrm{d}\lambda(y) &= \int_{\mathbb{R}^d} \varphi(\phi(x))\,\mathrm{d}\mu_s(x) + (1 - \lambda(\mathbb{S}^d \setminus \{N\}))\varphi(N); \\ \int_{\mathbb{S}^d} \varphi(y)\,\mathrm{d}\nu(y) &= \int_{\mathbb{R}^d} \varphi(\phi(x))\,\mathrm{d}\mu_s(x) + (1 - \nu(\mathbb{S}^d \setminus \{N\})))\varphi(N), \end{aligned} \tag{3.12}$$

and so both limit points are uniquely determined in $\mathbb{S}^d \setminus \{N\}$. Subtracting both equations in (3.12) and considering $\varphi \in C_c(\mathbb{R}^{d+1} \setminus \{N\})$, we have that

$$\int_{\mathbb{S}^d} \varphi(y)\,\mathrm{d}(\lambda - \nu)(y) = (\nu - \lambda)(\mathbb{S}^d \setminus \{N\})\varphi(N) = 0,$$

and so $\lambda = \nu$ in $\mathbb{S}^d \setminus \{N\}$. Since the right-hand of (3.12) coincide, we conclude $\lambda = \nu$ in $\mathbb{S}^d$. Now, since the limit point is unique, the same reasoning implies that such limit point is μ_s, and so the claim follows.

We now prove that μ solves the continuity equation in the equivalent sense (3.8), i.e. for almost every $t \in (0, T)$ it holds

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{S}^d} \varphi(y)\,\mathrm{d}\mu_t(y) = \int_{\mathbb{S}^d} \boldsymbol{c}_t(y) \cdot \nabla\varphi(y)\,\mathrm{d}\mu_t(y)$$

for all $\varphi \in C^\infty(\mathbb{R}^{d+1})$. Of course, if one has $\varphi \in C_c^\infty(\mathbb{R}^{d+1} \setminus \{N\})$, then since u is a weak solution of continuity equation with vector field $\boldsymbol{b}$ in $(0, T) \times \mathbb{R}^d$ by assumption, then the result follows easily by the definition of $\boldsymbol{c}$ and μ, since

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{S}^d} \varphi(y)\,\mathrm{d}\mu_t(y) = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \varphi(\psi(x)) u_t(x)\,\mathrm{d}x &= \int_{\mathbb{R}^d} \boldsymbol{b}_t(x) \cdot \nabla(\varphi \circ \psi)(x) u_t(x)\,\mathrm{d}x \\ &= \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} \boldsymbol{b}_t^i(x) \partial_i \psi^j(x) \partial_j \varphi(\psi(x)) u_t(x)\,\mathrm{d}x \\ &= \int_{\mathbb{S}^d \setminus \{N\}} (\nabla\psi \boldsymbol{b}_t)(\phi(y)) \cdot \nabla\varphi(y) u_t(\phi(y))\,\mathrm{d}y \\ &= \int_{\mathbb{S}^d} \boldsymbol{c}_t(y) \cdot \nabla\varphi(y)\,\mathrm{d}\mu_t(y). \end{aligned}$$

Nevertheless, we still need to verify when φ is not necessarily zero at N. For this purpose, since $\mu_t(\mathbb{S}^d) = 1$ for all $t \in [0, T]$, we have

$$\int_{\mathbb{S}^d} \varphi(y)\,\mathrm{d}\mu_t(y) = \varphi(N) + \int_{\mathbb{S}^d} \varphi(y) - \varphi(N)\,\mathrm{d}\mu_t(y). \tag{3.13}$$

Now, consider a cutoff function χ_ϵ such that it vanishes in $B_\epsilon(N)$, equals one in $\mathbb{R}^{d+1} \setminus B_{2\epsilon}(N)$, and whose gradient is bounded by $2\epsilon^{-1}$. Since $\chi_\epsilon(y)[\varphi(y) - \varphi(N)] \in C_c^\infty(\mathbb{R}^{d+1} \setminus \{N\})$, we may apply the previous result and so

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^d}\chi_\epsilon(y)[\varphi(y)-\varphi(N)]\,\mathrm{d}\mu_t(y) &= \int_{\mathbb{S}^d}[\varphi(y)-\varphi(N)]\boldsymbol{c}_t(y)\cdot\nabla\chi_\epsilon(y)\,\mathrm{d}\mu_t(y)\\ &\quad+\int_{\mathbb{S}^d}\chi_\epsilon(y)\boldsymbol{c}_t(y)\cdot\nabla\varphi(y)\,\mathrm{d}\mu_t(y).\end{aligned}$$

Since $|\varphi(y)-\varphi(N)| \leqslant C|y-N| < C\epsilon$ for $y \in B_{2\epsilon}(N)$, it follows that $[\varphi(y)-\varphi(N)]\nabla\chi_\epsilon(y)$ is uniformly bounded with respect to ϵ, and so the first integral is bounded by

$$\int_{\mathbb{S}^d}[\varphi(y)-\varphi(N)]\boldsymbol{c}_t(y)\cdot\nabla\chi_\epsilon(y)\,\mathrm{d}\mu_t(y) \leqslant C\int_{B_{2\epsilon}(N)\setminus B_\epsilon(N)}|\boldsymbol{c}_t(y)|\,\mathrm{d}\mu_t(y).$$

By (3.11) and dominated convergence theorem, such term vanishes as $\epsilon \to 0$. Since the second term converges in $L^1((0,T))$ to $\int_{\mathbb{S}^d}\boldsymbol{c}_t(y)\cdot\nabla\varphi(y)\,\mathrm{d}\mu_t(y)$ as $\epsilon \to 0$. Therefore, it follows that $t \mapsto \int_{\mathbb{S}^d}\varphi(y)-\varphi(N)\,\mathrm{d}\mu_t(y)$ is absolutely continuous in $[0,T]$, with for almost every $t \in (0,T)$ it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^d}\varphi(y)-\varphi(N)\,\mathrm{d}\mu_t(y) = \int_{\mathbb{S}^d}\boldsymbol{c}_t(y)\cdot\nabla\varphi(y)\,\mathrm{d}\mu_t(y).$$

By the above and (3.13), it follows that μ is a weak solution of continuity equation with vector field $\boldsymbol{c}$ in $(0,T)\times\mathbb{S}^d$.

Hence, by Theorem 3.4 and Remark 17, it follows that there exists probability measure $\boldsymbol{\sigma} \in \mathcal{P}(C([0,T];\mathbb{S}^d))$ concentrated on absolutely continuous in time curves solving $\dot\gamma(t) = \boldsymbol{c}_t(\gamma(t))$ for almost every $t \in (0,T)$ and $\mu_t = (e_t)_\#\boldsymbol{\sigma}$ for all $t \in [0,T]$. In order to transport it back to the desired space $\bar{\mathbb{R}}^d$, we consider

$$\Xi : C([0,T];\mathbb{S}^d) \to C([0,T];\bar{\mathbb{R}}^d), \quad \Xi(\gamma) := \phi\circ\gamma.$$

Setting $\boldsymbol{\eta} := \Xi_\#\boldsymbol{\sigma} \in \mathcal{P}(C([0,T];\bar{\mathbb{R}}^d))$, we notice that it is concentrated on (3.9). Indeed, denoting $\Lambda := \{\gamma \in \mathrm{AC}([0,T];\mathbb{S}^d) : \dot\gamma(t) = \boldsymbol{c}_t(\gamma(t))\}$, we have for any set $\Gamma \subset C([0,T];\bar{\mathbb{R}}^d)$ that

$$\boldsymbol{\eta}(\Gamma) = \boldsymbol{\sigma}((\Xi)^{-1}\Gamma) = \boldsymbol{\sigma}((\Xi)^{-1}\Gamma\cap\Lambda) = \boldsymbol{\sigma}((\Xi)^{-1}(\Gamma\cap\Xi\Lambda)) = \boldsymbol{\eta}(\Gamma\cap\Xi\Lambda).$$

By a simple computation and recalling the definition of $\boldsymbol{c}$, we obtain that $\Xi\Lambda$ is a subset of (3.9), as desired. Moreover, we have that

$$(e_t)_\#\boldsymbol{\eta}\llcorner\mathbb{R}^d = [\Xi_\#((e_t)_\#\boldsymbol{\sigma})]\llcorner\mathbb{R}^d = \phi_\#\mu_t\llcorner\mathbb{R}^d = u_t\mathcal{L}^d,$$

and so the theorem follows. □

Remark 20. In the proof of Theorem 3.6, we did not verify that the set (3.9) is Borel in $C([0,T];\bar{\mathbb{R}}^d)$. We refer to [6, Footnote 3] for a proof of such result.

Notice that we have been using the almost everywhere notion of solution of the flow equation (1.2). We now give a precise notion of such regular Lagrangian flows in a local sense, that is, solutions in a Borel set of $\mathbb{R}^d$ and in time $[0,\tau]$. By Remark 19, notice that one may assume the largest τ possible, which shall be called maximal time. Before we give such definition, we define the concept of hitting time of a curve in a set Ω.

Definition 3.1 (Hitting time in Ω)**.** Let $\tau > 0$ and Ω an open subset of $\mathbb{R}^d$ and $\gamma \in C([0,\tau);\mathbb{R}^d)$. We say that $h_\Omega(\gamma)$ is the hitting time of γ in Ω as

$$h_\Omega(\gamma) := \sup\left\{ t \in [0,\tau) : \max_{s\in[0,t]} V_\Omega(\gamma(s)) < \infty \right\},$$

if $\gamma(0) \in \Omega$, and $h_\Omega(\gamma) = 0$ otherwise, where $V_\Omega(x) = \max\{[\mathrm{dist}(x,\mathbb{R}^d \setminus \Omega)]^{-1}, |x|\}$.

The above definition of hitting time was chosen for convenience, for one could have consider a more general function V_Ω. More precisely, one could replace the above function V_Ω by any continuous function $V_\Omega : \Omega \to [0,\infty)$ such that $\lim_{x\to\partial\Omega} V_\Omega(x) = \infty$ in the sense that for any $M > 0$, there exists a compact set $K \subset \Omega$ such that $V_\Omega \geqslant M$ in $\Omega \setminus K$.

Remark 21. Notice that h_Ω is a lower-semicontinuous function. Indeed, let $\gamma^n \to \gamma$ in $C([0,\tau);\mathbb{R}^d)$. For any $\delta > 0$, consider $\Omega^\delta = \{x \in \Omega : \mathrm{dist}(x,\mathbb{R}^d \setminus \Omega) \geqslant \delta\}$. Denoting $a = h_\Omega(\gamma)$, for any $\epsilon > 0$, there exists $\delta > 0$ such that $\gamma(t) \in \Omega^{2\delta}$ for all $t \in [0, a-\epsilon]$, and by the convergence of γ^n, we have that $\gamma^n(t) \in \Omega^\delta$ for all $t \in [0, a-\epsilon]$. Then we have that

$$V_\Omega(\gamma^n(t)) = [V_\Omega(\gamma^n(t)) - V_\Omega(\gamma(t))] + V_\Omega(\gamma(t))$$

is finite for all $t \in [0, a-\epsilon]$. Therefore we have for all $\epsilon > 0$ that

$$h_\Omega(\gamma^n) \geqslant a - \epsilon = h_\Omega(\gamma) - \epsilon.$$

Definition 3.2 (Maximal regular flow)**.** Let Ω a Borel set of $\mathbb{R}^d$ and $\boldsymbol{b} : (0,T) \times \Omega \to \mathbb{R}^d$ be a Borel map. We say that $\boldsymbol{X}$ is the maximal regular flow associated to $\boldsymbol{b}$ if there exists a Borel map–which shall be called maximal time–$T^+_{\boldsymbol{X}} : \Omega \to (0,T]$[6] such that $\boldsymbol{X}$ is defined on $\{(t,x) : t < T^+_{\boldsymbol{X}}(x)\}$ and

(Flow) for almost every $x \in \Omega$, $\boldsymbol{X}(\cdot,x) \in \mathrm{AC}_{\mathrm{loc}}([0,T^+_{\boldsymbol{X}}(x);\mathbb{R}^d)$ and it solves for almost every $t \in (0,T^+_{\boldsymbol{X}}(x))$

$$\begin{aligned} \partial_t \boldsymbol{X}(t,x) &= \boldsymbol{b}_t(\boldsymbol{X}(t,x)); \\ \boldsymbol{X}(0,x) &= x; \end{aligned} \tag{3.14}$$

[6] The notation may seem unnecessarily cumbersome, but in [6] they study the case when the initial time is $s > 0$, and they also need a minimal time $T^-_{\boldsymbol{X}} : \Omega \to [0,s)$, and the maximal time should read $T^+_{\boldsymbol{X}} : \Omega \to (s,T]$.

(Regular) for any compact $K \subset \Omega$, there exists a constant $C_{K,\boldsymbol{X}} > 0$ such that for all $t \in [0,T]$

$$\boldsymbol{X}(t,\cdot)_{\#}(\mathcal{L}^d \llcorner \{T_K > t\}) \leqslant C_{K,\boldsymbol{X}} \mathcal{L}^d, \tag{3.15}$$

where $T_K(x)$ is the hitting time of the curve $\boldsymbol{X}(\cdot,x)$ at the boundary of K. More precisely,

$$T_K(x) = \begin{cases} h_K(\boldsymbol{X}(\cdot,x)) & \text{if } x \in K; \\ 0 & \text{if } x \in \Omega \setminus K; \end{cases}$$

(Maximal) for almost every $x \in \Omega$ such that $T^+_{\boldsymbol{X}}(x) < T$, it holds

$$\limsup_{t \nearrow T^+_{\boldsymbol{X}}(x)} V_\Omega(\boldsymbol{X}(t,x)) = \infty. \tag{3.16}$$

Remark 22. Notice that for any negligible set $E \subset (0,T) \times \mathbb{R}^d$, by the compressibility of flows it follows for any Borelian set Ω that

$$\left\{x \in \Omega : |\{t \in (0, T^+_{\boldsymbol{X}}(x)) : (t, \boldsymbol{X}(t,x)) \in E\}| > 0\right\}$$

is negligible. In particular, (3.14) does not depend on the representative (in Lebesgue equivalence class) of $\boldsymbol{b}$.

Notice that (3.15) is a weaker notion than Definition 2.2 (ii). Although many results follow from the former, which is intrinsically a local property, for proper blowup–which means that lim sup in (3.16) is in fact a limit–of flows as the time approaches the maximal time we shall need a global version of it akin to the latter.

In order to establish existence of maximal regular flows, we define a probability measure version of Definition 3.2; it is precisely the one we have used in Theorem 3.5.

Definition 3.3 (Regular generalized flow in $\bar{\Omega}$)**.** Let Ω an open set of $\mathbb{R}^d$ and let a Borel vector field $\boldsymbol{b} : (0,T) \times \bar{\Omega} \to \mathbb{R}^d$. A measure $\boldsymbol{\eta} \in \mathcal{P}(C([0,T];\mathbb{R}^d))$ is said to be a regular generalized flow[7] in $\bar{\Omega}$ if it is concentrated in

$$\{\gamma \in \mathrm{AC}([0,T];\bar{\Omega}) : \dot{\gamma}(t) = \boldsymbol{b}_t(\gamma(t)) \quad \text{for almost every } t \in (0,T)\}$$

and there exists a constant $L_{\boldsymbol{\eta}} > 0$ such that for all $t \in [0,T]$ it holds

$$[(e_t)_{\#}\boldsymbol{\eta}] \llcorner \Omega \leqslant L_{\boldsymbol{\eta}} \mathcal{L}^d.$$

We now prove the tightness and stability associated to regular generalized flows by the trio in [5, Theorem 4.4]. It parallels the already proven stability and compactness result for global regular flows Lemma 2.8. The proof does not rely on Lemma 2.6, but rather on Theorems 3.1 and 3.2.

[7] Notice that when there exists a regular flow $\boldsymbol{X}$, then $(e_t)_{\#}\boldsymbol{\eta} = \delta_{\boldsymbol{X}(t,\cdot)}$.

Theorem 3.7 (Tightness and stability of regular generalized flows)**.** *Let Ω an open bounded set of $\mathbb{R}^d$ and let $\boldsymbol{c}, \boldsymbol{c}^n : (0,T)\times\bar{\Omega}\to\mathbb{R}^d$ Borel vector fields satisfying $\boldsymbol{c}=\boldsymbol{c}^n=0$ on $(0,T)\times\partial\Omega$*[8] *and*

$$\lim_{n\to\infty}\boldsymbol{c}^n=\boldsymbol{c}\quad in\quad L^1((0,T)\times\Omega).$$

Moreover, let $\boldsymbol{\eta}^n\in\mathcal{P}(C([0,T];\bar{\Omega}))$ be regular generalized flows of $\boldsymbol{c}^n$ with compressibility constants L_n uniformly bounded. Then $\{\boldsymbol{\eta}^n\}_{n\in\mathbb{N}}$ is tight, with limit point $\boldsymbol{\eta}$ being a regular generalized flow of $\boldsymbol{c}$, and for any $\Gamma\subset C([0,T];\bar{\Omega})$ and $\Omega'\subset\Omega$, it holds

$$\begin{aligned}&[(e_t)_\#(\boldsymbol{\eta}^n \llcorner \Gamma)]\llcorner\Omega'\leqslant C_n\mathcal{L}^d\quad \textit{for some } C_n>0\\ \Longrightarrow\ &[(e_t)_\#(\boldsymbol{\eta}\llcorner\Gamma)]\llcorner\Omega'\leqslant(\liminf_{n\to\infty}C_n)\mathcal{L}^d.\end{aligned}\tag{3.17}$$

Proof. By the convergence of $\{\boldsymbol{c}^n\}_{n\in\mathbb{N}}$ in $L^1((0,T)\times\bar{\Omega})$, for $\boldsymbol{c}^n$ and $\boldsymbol{c}$ vanish at the boundary of Ω, then by Theorem 3.1 and [62, Theorem 6.19] there exists an increasing, convex, and superlinear function $F:[0,\infty)\to[0,\infty)$ such that[9] $F(0)=0$ and

$$\sup_{n\in\mathbb{N}}\int_0^T\int_{\bar{\Omega}}F(|\boldsymbol{c}_t^n(x)|)\,\mathrm{d}x\,\mathrm{d}t<\infty.$$

Define the functional $G:C([0,T];\mathbb{R}^d)\to[0,\infty]$ as

$$G(\gamma)=\begin{cases}\displaystyle\int_0^T F(|\dot{\gamma}(t)|)\,\mathrm{d}t & \text{if } \mathrm{AC}([0,T];\bar{\Omega});\\ \infty & \text{if } C([0,T];\mathbb{R}^d)\setminus\mathrm{AC}([0,T];\bar{\Omega}).\end{cases}$$

Since $\boldsymbol{\eta}^n$ is concentrated on $\mathrm{AC}([0,T];\bar{\Omega})$, we have by Theorem 3.3 and the regularity (in the compressibility sense) of $\boldsymbol{\eta}^n$ that

$$\begin{aligned}\int G(\gamma)\,\mathrm{d}\boldsymbol{\eta}^n(\gamma)=\int\int_0^T F(|\dot{\gamma}(t)|)\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\eta}^n(\gamma)&=\int_0^T\int_{\bar{\Omega}}F(|\boldsymbol{c}_t^n(x)|)\,\mathrm{d}[(e_t)_\#\boldsymbol{\eta}^n](x)\,\mathrm{d}t\\ &\leqslant L_n\int_0^T\int_{\bar{\Omega}}F(|\boldsymbol{c}_t^n(x)|)\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant\sup_{n\in\mathbb{N}}L_n\int_0^T\int_{\bar{\Omega}}F(|\boldsymbol{c}_t^n(x)|)\,\mathrm{d}x\,\mathrm{d}t<\infty.\end{aligned}\tag{3.18}$$

If one proves that G has compact sublevels, by Theorem 3.2 and Remark 16 we have that $\boldsymbol{\eta}^n$ is uniformly tight and there exists a limit point $\boldsymbol{\eta}$ (since $\{\boldsymbol{\eta}^n\}_{n\in\mathbb{N}}$ are probability measures, the uniform boundedness of follows trivially). Moreover, notice that it suffices to prove that its sublevels are sequentially compact, for $C([0,T];\mathbb{R}^d)$ is a metric space. For this purpose, fix $M>0$ and consider a sequence $\gamma^k\in\{\gamma:G(\gamma)<M\}$ for $k\in\mathbb{N}$. Since F is an increasing and convex function, we have by Jensen's inequality that

$$|\gamma^k(t)-\gamma^k(s)|\leqslant|t-s|F^{-1}\left(|t-s|^{-1}\int_s^t F(|\dot{\gamma}(\tau)|)\,\mathrm{d}\tau\right)\leqslant M\frac{F^{-1}(M|t-s|^{-1})}{M|t-s|^{-1}}.$$

[8] Although the $\boldsymbol{c}$, $\boldsymbol{c}^n$ are invariant on Lebesgue measure zero sets, the vanishing hypothesis on the boundary of Ω is understood in a pointwise sense.

[9] It is known as modulus of integrability, and by superlinear we mean $\lim_{z\to\infty}\frac{F(z)}{z}=\infty$.

Since F is superlinear, then it follows that $\lim_{z\to\infty} \frac{F^{-1}(z)}{z} = 0$, and so for every $\epsilon > 0$, there exists $N_\epsilon > 0$ such that $F^{-1}(z) < \epsilon M^{-1} z$ whenever $z > N$. Choosing $z = M|t-s|^{-1}$, we conclude that $|\gamma^k(t) - \gamma^k(s)| < \epsilon$ whenever $|t-s| < MN_\epsilon^{-1} =: \delta$. Since M does not depend on k, we have that $\{\gamma^k\}_{k\in\mathbb{N}}$ is equicontinuous, which combined with its uniform boundedness–for $\bar{\Omega}$ is a compact subset–gives that such sequence has a convergent subsequence by Arzelà-Ascoli theorem.

Moreover, notice that G is lower semicontinuous by the classical result due to Ioffe originally proved in [58]; see a modern presentation in [10, Theorem 5.8] and [63, Theorem 1.1]. Indeed, consider $\{\gamma^k\}_{k\in\mathbb{N}}$ a sequence converging to $\gamma \in C([0,T];\mathbb{R}^d)$ with respect to the uniform norm. Notice that if $\gamma \in C([0,T];\mathbb{R}^d) \setminus \mathrm{AC}([0,T];\bar{\Omega})$, then $G(\gamma) = \infty$, so it amounts to show that $\liminf_{k\to\infty} G(\gamma^k) = \infty$. Since $\mathrm{AC}([0,T];\bar{\Omega})$ is a closed subset of $C([0,T];\mathbb{R}^d)$ with respect to the topology induced by the uniform norm, we have that there exists N such that for all $k > N$, $\gamma^k \in C([0,T];\mathbb{R}^d) \setminus \mathrm{AC}([0,T];\bar{\Omega})$, and so we are done.

If $\gamma \in \mathrm{AC}([0,T];\bar{\Omega})$, then it suffices to prove when $L := \liminf_{k\to\infty} G(\gamma^k) < \infty$. Notice that there exists a subsequence $\{\gamma^{k_j}\}_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} G(\gamma^{k_j}) = L$. By the definition of G we have that $\gamma^j \in \mathrm{AC}([0,T];\bar{\Omega})$ for $j > N$ for some $N \in \mathbb{N}$ (we have relabeled k_j as j). Since $\gamma^j \to \gamma$ in $C([0,T];\bar{\Omega})$, we have for every $t, s \in [0,T]$ that

$$\int_s^t \dot{\gamma}^j(\tau)\,\mathrm{d}\tau \to \int_s^t \dot{\gamma}(\tau)\,\mathrm{d}\tau \quad \text{as } j \to \infty,$$

and so it follows that $\dot{\gamma}^j \rightharpoonup \dot{\gamma}$ weakly in $L^1([0,T])$. Since F is a convex function, it follows from Ioffe's result that

$$G(\gamma) = \int_0^T F(|\dot{\gamma}(t)|)\,\mathrm{d}t \leqslant \liminf_{k\to\infty} \int_0^T F(|\dot{\gamma}^k(t)|)\,\mathrm{d}t = \liminf_{k\to\infty} G(\gamma^k),$$

and so the lower semicontinuity of G follows. We now claim that $\int G(\gamma)\,\mathrm{d}\boldsymbol{\eta}(\gamma) < \infty$, so that $\boldsymbol{\eta}$ is concentrated on $\mathrm{AC}([0,T];\bar{\Omega})$. Indeed, following the proof of [78, Theorem 4.3], we have that $G = \lim_{k\to\infty} G_k$ in a pointwise sense, where $\{G_k\}_{k\in\mathbb{N}}$ is a nondecreasing sequence of continuous functions[10]. Therefore by monotone convergence theorem, the weak convergence $\boldsymbol{\eta}^n \rightharpoonup \boldsymbol{\eta}$ in $\mathcal{M}(C([0,T];\mathbb{R}^d))$ (given by Theorem 3.2), and (3.18), we have that

$$\begin{aligned}\int G(\gamma)\,\mathrm{d}\boldsymbol{\eta}(\gamma) &= \lim_{k\to\infty} \int G_k(\gamma)\,\mathrm{d}\boldsymbol{\eta}(\gamma) = \lim_{k\to\infty}\lim_{n\to\infty} \int G_k(\gamma)\,\mathrm{d}\boldsymbol{\eta}^n(\gamma) \\ &\leqslant \liminf_{n\to\infty} \int G(\gamma)\,\mathrm{d}\boldsymbol{\eta}^n(\gamma) < \infty.\end{aligned}$$

[10] For instance, consider $G_k(\gamma) = \inf_{\eta\in C([0,T];\mathbb{R}^d)} \{G(\eta) + k\|\gamma - \eta\|_{L^\infty([0,T])}\}$.

Now, since $(e_t)_{\#}\boldsymbol{\eta}^n$ converges to $(e_t)_{\#}\boldsymbol{\eta}$ in duality with bounded continuous functions and the compressibility of $\boldsymbol{\eta}^n$, we have for any $\Omega' \subset \Omega$ and $t \in [0,T]$ that

$$(e_t)_{\#}\boldsymbol{\eta}(\Omega') \leqslant \liminf_{n\to\infty}(e_t)_{\#}\boldsymbol{\eta}^n(\Omega') \leqslant (\liminf_{n\to\infty} L_n)|\Omega'|,$$

and so $\boldsymbol{\eta}$ is regular in the compressibility sense by the arbitrariness of Ω'. The same argument can be done to prove (3.17).

In order to conclude the theorem, we must show that $\boldsymbol{\eta}$ is concentrated on integral curves of $\boldsymbol{c}$, and it suffices to prove that for any $t \in [0,T]$

$$\int \left|\gamma(t) - \gamma(0) - \int_0^t \boldsymbol{c}_\tau(\gamma(\tau))\,\mathrm{d}\tau\right| \mathrm{d}\boldsymbol{\eta}(\gamma) = 0.$$

For this purpose, consider $\boldsymbol{c}' \in C([0,T]\times\bar{\Omega};\mathbb{R}^d)$ satisfying $\boldsymbol{c}' = 0$ on $[0,T]\times\partial\Omega$. We begin by proving that

$$\int \left|\gamma(t) - \gamma(0) - \int_0^t \boldsymbol{c}'_\tau(\gamma(\tau))\,\mathrm{d}\tau\right| \mathrm{d}\boldsymbol{\eta}(\gamma) \leqslant C\int_0^T\int_\Omega |\boldsymbol{c}_t(x) - \boldsymbol{c}'_t(x)|\,\mathrm{d}x\,\mathrm{d}t.$$

The above follows by using that the approximation $\{\boldsymbol{\eta}^n\}_{n\in\mathbb{N}}$ is a regular (in the compressibility sense) measure concentrated on integral curves of vector fields $\boldsymbol{c}^n$:

$$\begin{aligned}\int \left|\gamma(t) - \gamma(0) - \int_0^t \boldsymbol{c}'_\tau(\gamma(\tau))\,\mathrm{d}\tau\right| \mathrm{d}\boldsymbol{\eta}^n(\gamma) &= \int \left|\int_0^t \boldsymbol{c}^n_\tau(\gamma(\tau)) - \boldsymbol{c}'_\tau(\gamma(\tau))\,\mathrm{d}\tau\right| \mathrm{d}\boldsymbol{\eta}^n(\gamma)\\ &\leqslant \int_0^T\int_\Omega |\boldsymbol{c}^n_\tau(x) - \boldsymbol{c}'_\tau(x)|\,\mathrm{d}[(e_\tau)_{\#}\boldsymbol{\eta}^n](x)\,\mathrm{d}\tau\\ &\leqslant \sup_{n\in\mathbb{N}} L_n \int_0^T\int_\Omega |\boldsymbol{c}^n_\tau(x) - \boldsymbol{c}'_\tau(x)|\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned}$$

Taking the limit in $n\to\infty$ in the above and using the convergences $\boldsymbol{c}^n \to \boldsymbol{c}$ in $L^1((0,T)\times\Omega)$ and $\boldsymbol{\eta}^n \to \boldsymbol{\eta}$ in duality with bounded continuous functions, we conclude that

$$\int \left|\gamma(t) - \gamma(0) - \int_0^t \boldsymbol{c}'_\tau(\gamma(\tau))\,\mathrm{d}\tau\right| \mathrm{d}\boldsymbol{\eta}(\gamma) \leqslant \sup_{n\in\mathbb{N}} L_n \int_0^T\int_\Omega |\boldsymbol{c}_\tau(x) - \boldsymbol{c}'_\tau(x)|\,\mathrm{d}x\,\mathrm{d}\tau.$$

Therefore, we conclude by the above that that

$$\int \left|\gamma(t) - \gamma(0) - \int_0^t \boldsymbol{c}_\tau(\gamma(\tau))\,\mathrm{d}\tau\right| \mathrm{d}\boldsymbol{\eta}(\gamma) = \left(1 + \sup_{n\in\mathbb{N}} L_n\right)\int_0^T\int_\Omega |\boldsymbol{c}_\tau(x) - \boldsymbol{c}'_\tau(x)|\,\mathrm{d}x\,\mathrm{d}\tau.$$

Choosing $\boldsymbol{c}'$ as a sequence converging in $L^1((0,T)\times\bar{\Omega})$ to $\boldsymbol{c}$, the result follows. □

We now prove the local existence of local regular flows for vector fields satisfying Condition 3.1 (the first "local" refers to the space variable). This is a combination of Theorems 5.2 and 5.5 of trio [5].

Theorem 3.8 (Local existence)**.** *Let $\boldsymbol{b} : (0,T)\times\mathbb{R}^d \to \mathbb{R}^d$ be a Borel vector field satisfying Condition 3.1 and let Ω be an open subset of $\mathbb{R}^d$ with compact closure. Then there exists Borel maps $T_\Omega : \Omega \to (0,T]$ (as in Definition 3.2) and a Borel map $\boldsymbol{X}(t,x)$ for $x\in\Omega$ and $t \in [0, T_\Omega(x)]$ such that* *(i) for almost every* $x \in \Omega$*,* $\boldsymbol{X}(\cdot,x) \in \mathrm{AC}([0,T_\Omega(x)];\mathbb{R}^d)$*,* $\boldsymbol{X}(0,x) = x$*,* $\boldsymbol{X}(t,x) \in \Omega$ *for all* $t \in [0,T_\Omega(x))$*, and* $\boldsymbol{X}(T_\Omega(x),x) \in \partial\Omega$ *for* $T_\Omega(x) < T$*;*

(ii) for almost every $x \in \Omega$*,* $\boldsymbol{X}(\cdot,x)$ *is an integral curve associated to the vector field* $\boldsymbol{b}$ *almost everywhere in* $(0,T_\Omega(x))$*;*

(iii) $\boldsymbol{X}(t,\cdot)_\#(\mathcal{L}^d \llcorner \{T_\Omega > t\}) \leqslant \exp\left(\int_0^T m_\Omega(\tau)\,\mathrm{d}\tau\right)\mathcal{L}^d$ *for all* $t \in [0,T]$*, where* m_Ω *is the function in (H3).*

Proof. We shall split the proof in three steps: the first is concerned with existence of flows for the truncated vector field $\mathbb{1}_\Omega\boldsymbol{b}$ given a regular generalized measure $\boldsymbol{\eta}$ in $\bar{\Omega}$; the second ensures existence of such $\boldsymbol{\eta}$ if $\boldsymbol{b}$ is a smooth function; finally, the third concludes the theorem for the whole $\boldsymbol{b}$. We remark that steps 1 and 2 do not rely on hypothesis (H3).

Step 1: Let us denote $\Gamma^t = \{\gamma : h_\Omega(\gamma) > t\}$ for $t \in (0,T)$ (recall the definition of hitting time in Definition 3.1) and $\Sigma^t : \Gamma^t \to C([0,t];\Omega)$ the restriction of curves on $[0,t]$, i.e. $\Sigma^t(\gamma) = \gamma|_{[0,t]}$. We begin by proving that given $\delta \in (0,1)$ and a regular generalized flow $\boldsymbol{\eta}$ in $\bar{\Omega}$ with respect to vector field $\mathbb{1}_\Omega\boldsymbol{b}$ with compressibility constant $L > 0$ and such that $(e_0)_\#\boldsymbol{\eta} = \rho_0\mathcal{L}^d$ with $\rho_0 > 0$ almost everywhere in Ω, it holds that

$$\boldsymbol{\sigma}_x := \frac{1}{\boldsymbol{\eta}_x(\Gamma^t)}\Sigma^t_\#(\boldsymbol{\eta}_x \llcorner \Gamma^t) \in \mathcal{P}(C([0,t];\Omega))$$

is a Dirac measure for $\rho_0\mathcal{L}^d-$almost every x such that $\boldsymbol{\eta}_x(\Gamma^t) \geqslant \delta$, where $\boldsymbol{\eta}_x$ is the disintegration of $\boldsymbol{\eta}$ with respect to e_0. Since $\boldsymbol{\eta}_x$ is concentrated on integral curves in $[0,T]$ of $\mathbb{1}_\Omega\boldsymbol{b}$ with initial data x, the probability measure

$$\boldsymbol{\sigma} := \int_{\{x\in\Omega:\boldsymbol{\eta}_x(\Gamma^t)\geqslant\delta\}} \boldsymbol{\sigma}_x\rho_0(x)\,\mathrm{d}x$$

satisfies the hypothesis of Theorem 3.5 with $T = t$, for the definition of $\boldsymbol{\sigma}$ gives that

$$\int_{\mathbb{R}^d}\varphi(x)\,(e_s)_\#\boldsymbol{\sigma}(x) \leqslant \frac{1}{\delta}\int_{\mathbb{R}^d}\varphi(x)\,(e_s)_\#\boldsymbol{\eta}(x) \leqslant \frac{L}{\delta}\int_{\mathbb{R}^d}\varphi(x)\,\mathrm{d}x$$

for all $\varphi \in C_c(\Omega)$, and so $\boldsymbol{\sigma}$ has compressibility constant $\delta^{-1}L$. Therefore Theorem 3.5 implies that $\boldsymbol{\sigma}_x$ is a Dirac measure for $\rho_0\mathcal{L}^d-$almost everywhere in $\{x \in \Omega : \boldsymbol{\eta}_x(\Gamma^t) \geqslant \delta\}$. In particular, by the definition of ρ_0, we have for almost every $x \in \Omega$ and all $t \in (0,T)$ that $\Sigma^t_\#(\boldsymbol{\eta}_x \llcorner \Gamma^t)$ is either a multiple of Dirac delta or null measure.

The above gives that, for $\rho_0\mathcal{L}^d-$almost every $x \in \Omega$, $h_\Omega(\gamma)$ equals a positive constant for $\boldsymbol{\eta}_x-$almost every γ. Indeed, let

$$\begin{aligned}\mathbb{Q}_1 &:= \{q \in \mathbb{Q} : \Sigma^q_\#(\boldsymbol{\eta}_x \llcorner \Gamma^q) \text{ is a null measure}\};\\ \mathbb{Q}_2 &:= \{q \in \mathbb{Q} : \Sigma^q_\#(\boldsymbol{\eta}_x \llcorner \Gamma^q) \text{ is a multiple of Dirac measure}\},\end{aligned}$$

and notice by the above result that $\mathbb{Q} = \mathbb{Q}_1 \cup \mathbb{Q}_2$. By their definition, $\boldsymbol{\eta}_x(\Gamma^q) = 0$ for $q \in \mathbb{Q}_1$ and there exists $\gamma^q \in C([0,q];\Omega)$ such that $\boldsymbol{\eta}_x((\Sigma^q)^{-1}C([0,q];\Omega) \setminus \{\gamma^q\}) = 0$ for

$q \in \mathbb{Q}_2$. Moreover, notice that if $\mathbb{Q}_1 = \varnothing$, then $h_\Omega(\gamma) = T$ for all $\gamma \in C([0,T];\bar{\Omega})$. Let us consider the $\boldsymbol{\eta}_x$−null set

$$E := (\cup_{q\in\mathbb{Q}_1}\Gamma^q) \cup \left(\cup_{q\in\mathbb{Q}_2}(\Sigma^q)^{-1}C([0,q];\Omega) \setminus \{\gamma^q\}\right) = Q_1 \cup Q_2.$$

Then we have that

$$\begin{aligned} C([0,T];\bar{\Omega}) \setminus E &= \left(C([0,T];\bar{\Omega}) \setminus Q_1\right) \cup \left(C([0,T];\bar{\Omega}) \setminus Q_2\right) \\ &= \{\gamma \in C([0,T];\bar{\Omega}) : h_\Omega(\gamma) \leqslant q \text{ for all } q \in \mathbb{Q}_1 \\ &\qquad \text{and } \Sigma^q(\gamma) = \gamma^q \text{ for all } q \in \mathbb{Q}_2\}. \end{aligned}$$

Therefore by defining $q_1 = \inf \mathbb{Q}_1$ and $q_2 = \sup \mathbb{Q}_2$, we have that $q_2 \leqslant q_1$. In order to conclude the claim, we prove that $q_2 = q_1$. Assuming otherwise, that is $q_2 < q_1$, there exists $q_0 \in \mathbb{Q} \cap (0,T)$ such that $q_2 < q_0 < q_1$. Since q_0 is either in $\mathbb{Q}_1$ or in $\mathbb{Q}_2$, we get a contradiction to the definition of q_1 or q_2, and so the claim follows. In particular, taking $T_\Omega(x)$ as the constant $h_\Omega(\gamma)$ for $\gamma \notin E$, then $(e_t)_\#\boldsymbol{\eta}_x$ is a Dirac measure for all $t \in [0, T_\Omega(x)]$. Indeed, notice that for any Borel $A \subset \mathbb{R}^d$, we have for almost every $x \in \mathbb{R}^d$ and all $t \in [0, T_\Omega(x)]$ that

$$\boldsymbol{\eta}_x((e_t)^{-1}A) = \boldsymbol{\eta}_x\left(\{\gamma \in C([0,T];\bar{\Omega}) : \Sigma^q(\gamma) = \gamma^q \text{ for any } q \in \mathbb{Q}_2 \text{ and } \gamma(t) \in A\}\right).$$

Now, if $\gamma^{q_2}(t) := \lim\limits_{q \nearrow q_2} \gamma^q(t) \in A$, then $\gamma|_{[0,q_2]}(t) \in A$; if $\gamma^{q_2}(t) \notin A$, then $\gamma|_{[0,q_2]}(t) \notin A$. Summarizing, we have that

$$\boldsymbol{\eta}_x((e_t)^{-1}A) = \begin{cases} 1 & \text{if } \gamma^{q_2}(t) \in A \\ 0 & \text{if } \gamma^{q_2}(t) \notin A, \end{cases}$$

and so $(e_t)_\#\boldsymbol{\eta}$ is a Dirac delta measure for all $t \in [0, T_\Omega(x)]$.

Hence, we now define the curve

$$\boldsymbol{X}(t,x) := \int e_t(\gamma)\, \mathrm{d}\boldsymbol{\eta}_x(\gamma),$$

for $t < T_\Omega(x)$ and $x \in \Omega$, we conclude by Theorem 3.3 that $\boldsymbol{X}$ is a Borel map and (i) and (ii) of Theorem 3.8 holds for such vector fields. Moreover, Remark 21 gives that T_Ω is a Borel map, and by the definition of $\boldsymbol{X}$, we conclude that

$$\boldsymbol{X}(t,\cdot)_\#[\rho_0\mathcal{L}^d \llcorner \{T_\Omega > t\}] \leqslant L\mathcal{L}^d \quad \text{for all } t \in [0,T].$$

Step 2: Assuming that $\boldsymbol{b} \in C^\infty([0,T] \times \bar{\Omega}; \mathbb{R}^d)$, we shall prove that there exists a regular generalized flow $\boldsymbol{\eta}$ associated to $\mathbb{1}_\Omega \boldsymbol{b}$ such that

$$(e_0)_\#\boldsymbol{\eta} = \frac{1}{|\Omega|}\mathcal{L}^d \llcorner \Omega \quad \text{and} \quad [(e_t)_\#\boldsymbol{\eta} \llcorner \{h_K > t\}] \llcorner K \leqslant \frac{1}{|\Omega|} \exp\left(\int_0^T m_K(\tau)\,\mathrm{d}\tau\right)\mathcal{L}^d \tag{3.19}$$

for all $t \in [0,T]$ and any compact subset $K \subset \Omega$, Ω am open subset of $\mathbb{R}^d$. For this purpose, notice that we may apply the classical Cauchy-Lipschitz theory and denote $\boldsymbol{X}(\cdot,x)$ the unique integral curve of $\boldsymbol{b}$ with initial data $x \in \Omega$ in the interval $[0, T_\Omega(x)]$, where $T_\Omega(x)$ denotes the first time where the flow hits the boundary of Ω. Therefore, we may extend $\boldsymbol{X}(\cdot,x)$ to $[0,T]$ as $\boldsymbol{X}(T_\Omega(x),x)$ for $t \in [T_\Omega(x),T]$. We thus define $\boldsymbol{\eta}$ as the law under $|\Omega|^{-1}\mathcal{L}^d \llcorner \Omega$ of $\boldsymbol{X}$[11] and we prove that it is a generalized regular flow associated to $\mathbb{1}_\Omega \boldsymbol{b}$ and satisfies (3.19). By its definition, $\boldsymbol{\eta}$ is concentrated on integral curves of $\mathbb{1}_\Omega \boldsymbol{b}$–namely, the flow $\boldsymbol{X}(t,\cdot)$–and the first property in (3.19). Hence, it suffices to prove that it satisfies the compressibility property in the sense of the second property in (3.19). For this purpose, we begin by recalling the change of variables formula

$$\int_{\mathbb{R}^d} \varphi(\boldsymbol{X}(t,x)) \exp\left(\int_0^t \operatorname{div} \boldsymbol{b}_\tau(\boldsymbol{X}(\tau,x))\,\mathrm{d}\tau\right)\mathrm{d}x = \int_{\mathbb{R}^d} \varphi(x)\,\mathrm{d}x$$

for all $\varphi \in C_c(\mathbb{R}^d)$, which holds since $\boldsymbol{b}$ is a smooth vector field. Now, for any Ω open subset of $\mathbb{R}^d$ and any compact $K \subset \Omega$, we consider a nonnegative function $\varphi \in C_c(K)$. Notice that $\varphi(\boldsymbol{X}(t,x)) = 0$ if $t \geqslant h_K(\boldsymbol{X}(\cdot,x))$, hence $\operatorname{supp}\varphi(\boldsymbol{X}(t,\cdot))$ is a compact subset of $\{x \in K : t < h_K(\boldsymbol{X}(\cdot,x))\}$. Therefore, we have for all $t \in [0,T]$ that

$$\int_{K \cap \{x : t < h_K(\boldsymbol{X}(\cdot,x))\}} \varphi(\boldsymbol{X}(t,x))\,\mathrm{d}x \leqslant \exp\left(\int_0^T m_K(\tau)\,\mathrm{d}\tau\right)\int_{\mathbb{R}^d} \varphi(x)\,\mathrm{d}x,$$

where m_K is the lower bound on the divergence of $\boldsymbol{b}$, as in (H3). By the definition of $\boldsymbol{\eta}$, we conclude that the above is equivalent to

$$[(e_t)_\# \boldsymbol{\eta} \llcorner \{h_K(\cdot) > t\}] \llcorner K \leqslant \frac{1}{|\Omega|} \exp\left(\int_0^T m_K(\tau)\,\mathrm{d}\tau\right) \mathcal{L}^d \quad \text{for all } t \in [0,T].$$

Step 3: We now conclude Theorem 3.8. By step 1, it suffices to construct a regular generalized flow $\boldsymbol{\eta}$ in $\bar{\Omega}$ associated to vector field $\mathbb{1}_\Omega \boldsymbol{b}$ such that $(e_0)_\# \boldsymbol{\eta} = \rho_0 \mathcal{L}^d$ for some $\rho_0 > 0$ almost everywhere in Ω. For this purpose, we shall use step 2 and construct via approximation such $\boldsymbol{\eta}$ by considering the mollification of $\boldsymbol{b}$ in spacetime (possibly extending $\boldsymbol{b}$ to be the null vector field for $\mathbb{R}^{1+d}$), which we denote as $\boldsymbol{b}^\epsilon$. Indeed, step 2 gives the existence of a regular generalized flow $\boldsymbol{\eta}^\epsilon$ in $\bar{\Omega}$ associated to $\mathbb{1}_\Omega \boldsymbol{b}^\epsilon$ with

$$[(e_t)_\# \boldsymbol{\eta}^\epsilon \llcorner \{h_K(\cdot) > t\}] \llcorner K \leqslant \frac{1}{|\Omega|} \exp\left(\int_0^T m_K^\epsilon(\tau)\,\mathrm{d}\tau\right) \mathcal{L}^d \quad \text{for all } t \in [0,T],$$

where $\operatorname{div} \boldsymbol{b}^\epsilon \geqslant m_K^\epsilon$ in $(0,T) \times K$. By the definition of $\boldsymbol{b}^\epsilon$ and m_K^ϵ, we have for any compact $K \subset \Omega$ that

$$\limsup_{\epsilon \searrow 0} \frac{1}{|\Omega|} \exp\left(\int_0^T m_K^\epsilon(\tau)\,\mathrm{d}\tau\right) \leqslant \frac{1}{|\Omega|} \exp\left(\int_0^T m_\Omega(\tau)\,\mathrm{d}\tau\right),$$

[11] Recall that given $(\Omega, \mathcal{F}, P)$ a probability space, (S, Σ) a measure space, and $\boldsymbol{Y} : I \times \Omega \to S$ a map such that $\boldsymbol{Y}(t,\cdot)$ is a measurable function for each index $t \in I$, the law of $\boldsymbol{Y}$ is defined as $(\Phi_{\boldsymbol{Y}})_\# P$, where $[\Phi_{\boldsymbol{Y}}(x)](t) = \boldsymbol{Y}(t,x)$.

and so by Theorem 3.7, it follows that

$$[(e_t)_{\#}\boldsymbol{\eta} \llcorner \{h_K(\cdot) > t\}] \llcorner K \leqslant \frac{1}{|\Omega|} \exp\left(\int_0^T m_\Omega(\tau)\,\mathrm{d}\tau\right) \mathcal{L}^d \quad \text{for all } t \in [0,T].$$

By taking the limit $K \nearrow \Omega$[12], we may drop the restriction in K on the left-hand side. Moreover, by step 1 one takes T_Ω as the hitting time and $\rho_0 = |\Omega|^{-1}$, we conclude

$$\boldsymbol{X}(t,\cdot)_{\#}(\mathcal{L}^d \llcorner \{T_\Omega > t\}) \leqslant \exp\left(\int_0^T m_\Omega(\tau)\,\mathrm{d}\tau\right) \mathcal{L}^d \quad \text{for all } t \in [0,T],$$

which gives (iii), and so the theorem follows. □

We now state and prove an uniqueness and existence theorem, which is one of the main results of the trio [5, Theorem 5.7]. It heavily relies on Theorem 3.8 by taking the limit of local regular flows in Ω_n as $n \to \infty$. Moreover, we also prove that if (H3) is strengthened to

$$\operatorname{div} \boldsymbol{b} \geqslant m \text{ in } (0,T) \times \mathbb{R}^d \quad \text{for some } m \in L^1((0,T)), \tag{3.20}$$

which implies that (3.15) holds with $\{T^+_{\boldsymbol{X}} > t\}$ instead of $\{T_K > t\}$ and with compressibility constant independent of the compact set K, then limit superior in (3.16) is in fact a limit; this is precisely the trio's result [5, Theorem 7.1].

Theorem 3.9 (Existence and uniqueness of maximal regular flow)**.** *Let $\boldsymbol{b} : (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$ be a Borel vector field satisfying Condition 3.1 and let Ω be a compact subset of $\mathbb{R}^d$. Then there exists an unique maximal regular flow associated to $\boldsymbol{b}$, with compressibility constant $\exp\left(\int_0^T m_\Omega(\tau)\,\mathrm{d}\tau\right)$. Moreover, let $\boldsymbol{Y} : [0,\tau) \times \Omega' \to \mathbb{R}^d$ for $\tau \in (0,T]$ and a compact set $\Omega' \subset \mathbb{R}^d$ such that $\boldsymbol{Y}(\cdot,x) \in \Omega$ is an integral curve of $\boldsymbol{b}$ in $[0,\tau)$ for $x \in \Omega'$ satisfying*

$$\boldsymbol{Y}(t,\cdot)_{\#}\mathcal{L}^d \llcorner \Omega' \leqslant L\mathcal{L}^d$$

for some $L > 0$. Then $\tau < T^+_{\boldsymbol{X}}(x)$ and $\boldsymbol{X}(\cdot,x) = \boldsymbol{Y}(\cdot,x)$ in $[0,\tau)$ for almost every $x \in \Omega'$. Finally, if (3.20) holds, then for all $t \in [0,T]$, it follows that

$$\boldsymbol{X}(t,\cdot)_{\#}\mathcal{L}^d \llcorner \{T^+_{\boldsymbol{X}} > t\} \leqslant \exp\left(\int_0^T m(\tau)\,\mathrm{d}\tau\right) \mathcal{L}^d, \tag{3.21}$$

and the limit superior in Definition 3.2 is in fact an limit, that is,

$$\lim_{t \nearrow T^+_{\boldsymbol{X}}(x)} |\boldsymbol{X}(t,x)| = \infty.$$

[12] Such limit is the usual definition: for $K_1 \subset K_2 \subset \dots K_n \subset \dots \subset \Omega$, we define $K_n \nearrow \Omega$ for $n \to \infty$ as the union $\cup_{n \geqslant 1} K_n = \Omega$.

Proof. By Theorem 3.5 and Remark 19, for maximal regular flows $\boldsymbol{X}$ and $\bar{\boldsymbol{X}}$ with maximal times $T^+_{\boldsymbol{X}}$ and $T^+_{\bar{\boldsymbol{X}}}$, respectively, then $\boldsymbol{X}(\cdot,x) = \bar{\boldsymbol{X}}(\cdot,x)$ in $[0,\min\{T^+_{\boldsymbol{X}},T^+_{\bar{\boldsymbol{X}}}\})$ for almost every $x \in \Omega$. Now, notice that for almost every $x \in \{T^+_{\boldsymbol{X}} > T^+_{\bar{\boldsymbol{X}}}\}$, then for any V_Ω as in Definition 3.1, we have that the image of $[0,T^+_{\bar{\boldsymbol{X}}}(x)]$ through $V_\Omega(\boldsymbol{X}(\cdot,x))$ is bounded, and of $[0,T^+_{\bar{\boldsymbol{X}}}(x)$ through $V_\Omega(\bar{\boldsymbol{X}}(\cdot,x))$ is not (by the maximality of $T^+_{\bar{\boldsymbol{X}}}$). Therefore, we have that $\{T^+_{\boldsymbol{X}} > T^+_{\bar{\boldsymbol{X}}}\}$ has measure zero. Applying the same argument swapping the flows, we conclude that $T^+_{\boldsymbol{X}}(x) = T^+_{\bar{\boldsymbol{X}}}(x)$ for almost every $x \in \Omega$, and so uniqueness of maximal regular flow follows. The same argument holds when considering flows $\boldsymbol{Y}$ as in Theorem 3.9.

In order to ensure existence of maximal regular flows, we begin by constructing auxiliary local flows $\boldsymbol{X}^n$ in Ω_n, where $\Omega_n \nearrow \mathbb{R}^d$ such that Ω_n is compactly contained in Ω_{n+1}, with Borel map $T_n := T_{\Omega_n}$–whose existence follows from Theorem 3.8–such that

(i) for almost every $x \in \Omega_n$, $\boldsymbol{X}^n(\cdot,x) \in \mathrm{AC}([0,T_n(x));\mathbb{R}^d)$, $\boldsymbol{X}(0,x) = x$, $\boldsymbol{X}(t,x) \in \Omega_n$ for all $t \in [0,T_n(x))$, and $\boldsymbol{X}(T_n(x),x) \in \partial\Omega_n$ for $T_n(x) < T$, so $T_n(x) = h_{\Omega_n}(\boldsymbol{X}^n(\cdot,x))$;

(ii) for almost every $x \in \Omega_n$, $\boldsymbol{X}^n(\cdot,x)$ is an integral curve associated to the vector field $\boldsymbol{b}$ almost everywhere in $(0,T_n(x))$;

(iii) $\boldsymbol{X}^n(t,\cdot)_\#(\mathcal{L}^d \llcorner \{T_n > t\}) \leqslant \exp\left(\int_0^T m_{\Omega_n}(\tau)\,\mathrm{d}\tau\right)\mathcal{L}^d$ for all $t \in [0,T]$.

Now, we define for almost every $x \in \mathbb{R}^d$

$$T^+_{\boldsymbol{X}}(x) := \lim_{n\to\infty} T_n(x), \quad \boldsymbol{X}(t,x) := \boldsymbol{X}^n(t,x) \quad \text{for } t \in [0,T^+_{\boldsymbol{X}}(x)).$$

The first limit is well-posed by the monotonicity of the sequence $\{T_n(x)\}_{n\in\mathbb{N}}$ for almost every $x \in \Omega_n$, which follows by using the same argument for the proof of uniqueness of maximal regular flows. As a consequence, we have for $n \leqslant k$ that $\boldsymbol{X}^n(\cdot,x) = \boldsymbol{X}^m(\cdot,x)$ in $[0,T_n(x)]$ for almost every $x \in \Omega_n$. We now prove that $\boldsymbol{X}$ is a maximal regular flow with maximal time $T^+_{\boldsymbol{X}}$. By the definition of $\boldsymbol{X}$ and the fact that $\boldsymbol{X}^n$ is an integral curve of vector field $\boldsymbol{b}$ in $(0,T_n(x))$ implies that $\boldsymbol{X}$ satisfies the flow property in Definition 3.2. For the regular property, notice that the compressibility of the auxiliar flows $\boldsymbol{X}^n$ and the definition of $\boldsymbol{X}$ gives that for all $t \in [0,T]$,

$$\boldsymbol{X}(t,\cdot)_\#(\mathcal{L}^d \llcorner \{T_n > t\}) \leqslant \exp\left(\int_0^T m_{\Omega_n}(\tau)\,\mathrm{d}\tau\right)\mathcal{L}^d.$$

Now, for any compact set $\Omega \subset \mathbb{R}^d$, take $\Omega_n \nearrow \Omega$ and since in this case holds

$$\boldsymbol{X}(t,\cdot)_\#(\mathcal{L}^d \llcorner \{T_n > t\}) \leqslant \exp\left(\int_0^T m_{\Omega}(\tau)\,\mathrm{d}\tau\right)\mathcal{L}^d,$$

we may pass the limit using the fact that $T_n = h_{\Omega_n}(\boldsymbol{X}(\cdot,x))$. Finally, for the maximal property in Definition 3.2, notice that for any compact $\Omega \in \mathbb{R}^d$, we may take Ω_n containing it for n sufficiently large, and since $\boldsymbol{X}(T_n(x),x) \in \partial\Omega_n$, and so the property follows trivially.

If the vector field also satisfies (3.20), notice that by the previous construction, (3.21) follows. Now, since we consider $\Omega_n \nearrow \mathbb{R}^d$ such that Ω_n is compactly contained in Ω_{n+1}, we may consider $\psi_n \in C_c^\infty(\Omega_{n+1})$ cutoff functions such that $0 \leqslant \psi_n \leqslant 1$ and $\psi_n \equiv 1$ in $\bar{\Omega}_n$. Therefore, denoting $L := \exp\left(\int_0^T m_\Omega(\tau)\,\mathrm{d}\tau\right)$, we have that

$$\int_{\mathbb{R}^d}\int_0^{T^+_{\boldsymbol{X}}(x)}\left|\frac{\mathrm{d}}{\mathrm{d}t}\psi_n(\boldsymbol{X}(t,x))\right|\mathrm{d}t\,\mathrm{d}x \leqslant \int_0^T\int_{\{T^+_{\boldsymbol{X}}>t\}}|\nabla\psi_n(\boldsymbol{X}(t,x))||\boldsymbol{b}_t(\boldsymbol{X}(t,x))|\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant L\|\nabla\psi_n\|_{L^\infty(\mathbb{R}^d)}\int_0^T\int_{\Omega_{n+1}}|\boldsymbol{b}_t(x)|\,\mathrm{d}x\,\mathrm{d}t < \infty.$$

Hence for almost every $x \in \mathbb{R}^d$ $\psi_n(\boldsymbol{X}(\cdot,x))$ is the restriction absolutely continuous function in $[0, T^+_{\boldsymbol{X}}(x)]$. In order to conclude that the limit superior

$$\limsup_{t\nearrow T^+_{\boldsymbol{X}}(x)}|\boldsymbol{X}(t,x)| = \infty$$

is a limit, it suffices to show that limit inferior also diverges to infinity. For this purpose, fix $x \in \mathbb{R}^d$ such that the maximal property in Definition 3.2 holds and notice by the above that $\psi_n(\boldsymbol{X}(\cdot,x))$ is uniformly continuous in $[0, T^+_{\boldsymbol{X}}(x))$. This combined with the compact support of ψ_n implies that

$$\lim_{t\nearrow T^+_{\boldsymbol{X}}(x)}\psi_n(\boldsymbol{X}(\cdot,x)) = 0 \quad \text{for any } n \in \mathbb{N}.$$

Assuming by contradiction that limit inferior of $|\boldsymbol{X}(t,x)|$ as $t \nearrow T^+_{\boldsymbol{X}}(x)$ is finite, there would exist an integer N and a sequence of times $t_k \nearrow T^+_{\boldsymbol{X}}(x)$ such that $\boldsymbol{X}(t_k,x) \in \Omega_n$; this is not possible, for $\psi_{N+1}(\boldsymbol{X}(t_k,x)) = 1$, and so the theorem follows. □

Remark 23. The proof of Theorem 3.9 exemplifies why the growth assumptions (1.6) and (3.3) for vector fields $\boldsymbol{b}$ are crucial in order to ensure global well-posedness in time of integral curves associated to $\boldsymbol{b}$, for even in the classical case for Cauchy-Lipschitz theory. Indeed, the very simple one dimensional example $\boldsymbol{b}(x) = x^2$ illustrate this phenomenon. For also a simple example in higher dimension, consider the divergence-free time independent two-dimensional vector field

$$\boldsymbol{b}(x,y) = (x^2, -2xy).$$

Then the associated flow can be explicitly computed, and so denoting $\boldsymbol{X} = (\boldsymbol{X}_1, \boldsymbol{X}_2)$, we have that

$$\begin{pmatrix}\boldsymbol{X}_1(t,x,y)\\ \boldsymbol{X}_2(t,x,y)\end{pmatrix} = \begin{pmatrix}x(1-xt)^{-1}\\ y\exp\left(2\log|1-xt|\right).\end{pmatrix}$$

Notice that although $\boldsymbol{X}$ is smooth, the lack of global control does not guarantee that $\boldsymbol{X}$ is well-posed for all times. In this particular example, we have $T^+_{\boldsymbol{X}}(x) = |x|^{-1}$.

So far, we have recovered in Theorem 3.9 the existence and uniqueness analogous to Theorem 2.1 without any growth assumptions. The complete analogy shall be done once we obtain a forward semigroup property first proven in [5, Theorem 6.1], in the sense that

$$\boldsymbol{X}(\cdot, s, \boldsymbol{X}(s,0,x)) = \boldsymbol{X}(\cdot, 0, x) \quad \text{in } [s, T^+_{\boldsymbol{X}}(x)) \tag{3.22}$$

for almost every $x \in \{T^+_{\boldsymbol{X}} > s\}$, where we once again denote $\boldsymbol{X}(t,s,x)$ as a flow with initial time s and $T^+_{\boldsymbol{X},s}(x)$ the maximal time of $\boldsymbol{X}(\cdot,s,x)$. In this notation, $\boldsymbol{X}(t,0,x) = \boldsymbol{X}(t,x)$ and $T^+_{\boldsymbol{X},0}(x) = T^+_{\boldsymbol{X}}(x)$.

Theorem 3.10 (Semigroup property)**.** *Let $\boldsymbol{b}$ satisfying Condition 3.1 and $s \in [0,T]$. Then the maximal regular flow associated to $\boldsymbol{b}$ satisfies (3.22) for almost every $x \in \{T^+_{\boldsymbol{X}} > s\}$ and $T^+_{\boldsymbol{X},s}(\boldsymbol{X}(s,x)) = T^+_{\boldsymbol{X}}(x)$ for almost every $x \in \{T^+_{\boldsymbol{X}} > s\}$.*

Proof. Since both thesis assume that $x \in \{T^+_{\boldsymbol{X}} > s\}$, without loss generality assume $|\{T^+_{\boldsymbol{X}} > s\}| > 0$. Let $\Omega_s \subset \{T^+_{\boldsymbol{X}} > s\}$ such that $|\Omega_s| > 0$, and by the compressibility of $\boldsymbol{X}$, there exists a bounded function ρ_s such that

$$\frac{1}{|\Omega_s|}\boldsymbol{X}(s,\cdot)_{\#}\mathcal{L}^d \llcorner \Omega_s = \rho_s \mathcal{L}^d.$$

Consider $\pi := |\Omega_s|^{-1}(\mathrm{Id} \times \boldsymbol{X}(s,\cdot))_{\#}\mathcal{L}^d \llcorner \Omega_s$ a probability measure, and by disintegrating π with respect to $\rho_s\mathcal{L}^d$ we obtain a family of probability measures $\{\pi_y\}_{y\in\mathbb{R}^d}$ such that

$$\pi := \int [\pi_y \otimes \delta_y]\rho_s(y)\,\mathrm{d}y.$$

Now, for $\epsilon > 0$, we construct the probability measure

$$\pi_\epsilon := \int_{\{\rho_s \geqslant \epsilon\}} \pi_y \otimes \delta_y \,\mathrm{d}y,$$

and since $\epsilon\pi_\epsilon \leqslant \pi$, the first marginal of π_ϵ[13] is bounded by the first marginal of π over ϵ, i.e. $(\epsilon|\Omega_s|)^{-1}\mathcal{L}^d \llcorner \Omega_s$, by the definition of π. Therefore, we have that the first marginal of π_ϵ can be written as $\tilde{\rho}_\epsilon\mathcal{L}^d$ for some bounded function $\tilde{\rho}_\epsilon$. Moreover, notice that

$$\pi \leqslant \|\rho_s\|_{L^\infty(\mathbb{R}^d)}\pi_\epsilon + \epsilon\int_{\{\rho_s < \epsilon\}} \pi_y \otimes \delta_y \,\mathrm{d}y,$$

which gives that

$$\frac{1}{|\Omega_s|}\mathcal{L}^d \llcorner \Omega_s \leqslant \|\rho_s\|_{L^\infty(\mathbb{R}^d)}\tilde{\rho}_\epsilon\mathcal{L}^d + \epsilon\int_{\{\rho_s<\epsilon\}} \pi_y \,\mathrm{d}y,$$

and so we conclude that $\tilde{\rho}_\epsilon > 0$ almost everywhere in Ω_s for ϵ small enough.

[13] Recall that the first marginal of a measure $\mu \in \mathcal{P}(X \times Y)$ is a probability measure in $\mathcal{P}(Y)$ defined as $\tilde{\pi}_{\#}\mu$, where $\tilde{\pi}(x,y) = x$ is the canonical projection.

We now construct a generalized regular flow $\boldsymbol{\eta}_{\tau,\epsilon} \in \mathcal{P}(C([s,\tau];\mathbb{R}^d))$ for fixed $\tau > s$ and $\epsilon > 0$ as

$$\boldsymbol{\eta}_{\tau,\epsilon} := \int_{\{\rho_s \geqslant \epsilon\}} \int_{\{T^+_{\boldsymbol{X},s} > \tau\}} \delta_{\boldsymbol{X}(\cdot,x)} \,\mathrm{d}\pi_y(x)\,\mathrm{d}y = \int_{\{T^+_{\boldsymbol{X},s} > \tau\}} \delta_{\boldsymbol{X}(\cdot,x)} \tilde{\rho}_\epsilon(x)\,\mathrm{d}x. \tag{3.23}$$

Notice that by its definition, $\boldsymbol{\eta}_{\tau,\epsilon}$ is concentrated on integral curves of $\boldsymbol{b}$, and the compressibility follows by the computation

$$\int_{\mathbb{R}^d} \phi(x)\,\mathrm{d}[(e_{\tau'})_\#\boldsymbol{\eta}_{\tau,\epsilon}](x) = \int_{\{T^+_{\boldsymbol{X},s} > \tau\}} \phi(\boldsymbol{X}(\tau',x))\tilde{\rho}_\epsilon(x)\,\mathrm{d}x \leqslant L\|\tilde{\rho}_\epsilon\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} \phi(x)\,\mathrm{d}x$$

for any bounded nonnegative $\phi \in C(\mathbb{R}^d)$ and $\tau' \in [s,\tau]$, where L is the compressibility constant for the maximal regular flow $\boldsymbol{X}$. By Theorem 3.5 with interval $[s,\tau]$, we have

$$\boldsymbol{\eta}_{\tau,\epsilon} = \int \delta_{\gamma_x}\,\mathrm{d}[(e_s)_\#\boldsymbol{\eta}_{\tau,\epsilon}](x),$$

and by the uniqueness proven in Theorem 3.9, we have that $\gamma_x(t) = \boldsymbol{X}(t,s,x)$ for $[(e_s)_\#\boldsymbol{\eta}_{\tau,\epsilon}]$−almost every $x \in \mathbb{R}^d$ and all $t \in [s,\tau]$. Using the fact that by definition $[(e_s)_\#\boldsymbol{\eta}_{\tau,\epsilon}] = \boldsymbol{X}(s,\cdot)_\#[\mathbb{1}_{\{T^+_{\boldsymbol{X},s} > \tau\}}\tilde{\rho}_\epsilon]$, we conclude

$$\boldsymbol{\eta}_{\tau,\epsilon} = \int_{\{T^+_{\boldsymbol{X},s} > \tau\}} \delta_{\boldsymbol{X}(\cdot,s,\boldsymbol{X}(s,x))}\tilde{\rho}_\epsilon(y)\,\mathrm{d}y. \tag{3.24}$$

By (3.23) and (3.24), the positivity of $\tilde{\rho}_\epsilon$ on Ω_s, and the arbitrariness of τ, we conclude (3.22) and $T^+_{\boldsymbol{X},s}(\boldsymbol{X}(s,x)) > T^+_{\boldsymbol{X}}(x)$ for almost every $x \in \Omega_s$. Notice that by the maximal property for maximal regular flows and the identity (3.22), we conclude that

$$T^+_{\boldsymbol{X},s}(\boldsymbol{X}(s,x)) = T^+_{\boldsymbol{X}}(x) \quad \text{for almost every } x \in \Omega_s.$$

By the arbitrariness of Ω_s, the theorem follows. □

To conclude the trio's main results concerning the general theory, we now present a criterion for maximal regular flows to be global in time, with first iteration in [5, Theorem 7.6], but we shall present a more general version found in [6, Proposition 4.11]. For this purpose, we shall prove a result essentially contained in [6, Theorem 4.7].

Proposition 3.1. *Let $\boldsymbol{b}$ a vector field satisfying Condition 3.1 and $\boldsymbol{X}$ its associated maximal regular flow (existence, uniqueness and semigroup property of it follows from Theorems 3.9 and 3.10). Moreover, let $\boldsymbol{\eta} \in \mathcal{M}(C([0,T];\bar{\mathbb{R}}^d))$ concentrated on (3.9), and such that there exists a constant $L > 0$ satisfying $(e_t)_\#\boldsymbol{\eta} \llcorner \mathbb{R}^d \leqslant L\mathcal{L}^d$ for all $t \in [0,T]$. Then for $(e_0)_\#\boldsymbol{\eta}$-almost every $x \in \mathbb{R}^d$, the disintegration $\boldsymbol{\eta}_x$ of $\boldsymbol{\eta}$ with respect to e_0 is concentrated on the set*

$$\{\gamma \in C([0,T];\bar{\mathbb{R}}^d) : \gamma(0) = x \text{ and } \gamma(t) = \boldsymbol{X}(t,x) \text{ for all } t \in [0,T^+_{\boldsymbol{X}}(x))\}.$$

In particular, $\boldsymbol{\eta}$ is concentrated on

$$\{\gamma \in C([0,T];\bar{\mathbb{R}}^d) : \gamma(0) = \infty \text{ or } \gamma(t) = \boldsymbol{X}(t,\gamma(0)) \text{ for all } t \in [0,T^+_{\boldsymbol{X}}(x))\}.$$

Proof. Since $T^+_{\boldsymbol{X}}(x)$ and $\gamma \mapsto \boldsymbol{X}(t,x) = \boldsymbol{X}(t,\gamma(0))$ are a Borel maps, then so is the above set by first considering $t \in \mathbb{Q} \cap (0,T)$. By the definition of $\boldsymbol{\eta}_x$, it suffices to show that it is concentrated on

$$\{\gamma \in C([0,T];\bar{\mathbb{R}}^d) : \gamma(t) = \boldsymbol{X}(t,x) \text{ for all } t \in [0, T^+_{\boldsymbol{X}}(x))\}. \tag{3.25}$$

For this purpose, we consider for every $R > 0$ and $t \in (0,T]$ the measure

$$\boldsymbol{\eta}^{R,t} := \Sigma^t_\# \left(\boldsymbol{\eta} \llcorner \{\gamma : |\gamma(s)| < R \text{ for all } s \in [0,t]\}\right),$$

where Σ^t the restriction of a curve as in Theorem 3.8. By its definition, $\boldsymbol{\eta}^{R,t}$ satisfies the hypothesis of Theorem 3.4, and so there exists $\boldsymbol{Y}(\cdot,x)$ an integral curve of $\boldsymbol{b}$ in $[0,t]$ such that $\boldsymbol{Y}(0,x) = x$ for $(e_0)_\#\boldsymbol{\eta}^{R,t}$−almost every $x \in \mathbb{R}^d$ and

$$\boldsymbol{\eta}^{R,t} = \int \delta_{\boldsymbol{Y}(\cdot,x)}\, \mathrm{d}(e_0)_\#\boldsymbol{\eta}^{R,t}(x). \tag{3.26}$$

By the compressibility of $\boldsymbol{\eta}$, there exists a function $\rho^{R,t}$ bounded by L such that $(e_0)_\#\boldsymbol{\eta}^{R,t} = \rho^{R,t}\mathcal{L}^d$. Moreover, we have that

$$\boldsymbol{Y}(t,\cdot)_\#(\mathcal{L}^d \llcorner \{\rho^{R,t} > \delta\}) \leqslant \delta^{-1}(e_t)_\# \int_{\{\rho^{R,t}>\delta\}} \delta_{\boldsymbol{Y}(\cdot,x)} \mathrm{d}(e_0)_\#\boldsymbol{\eta}^{R,t}(x) \leqslant \delta^{-1}L\mathcal{L}^d,$$

and so $\boldsymbol{Y}(\cdot,x)$ is a regular flow in $[0,t]$ for almost every $x \in \{\rho^{R,t} > \delta\}$ with compressibility constant $\delta^{-1}L$. Hence by Theorem 3.9, we conclude that $\boldsymbol{Y}(\cdot,x) = \boldsymbol{X}(\cdot,x)$ in $[0,t]$ for almost every $x \in \{\rho^{R,t} > \delta\}$, and so by letting $\delta \to 0$ and then $R \to \infty$, we have that $\boldsymbol{Y}(\cdot,x) = \boldsymbol{X}(\cdot,x)$ in $[0,t]$ for $(e_0)_\#\boldsymbol{\sigma}^t$−almost every $x \in \mathbb{R}^d$, where

$$\boldsymbol{\sigma}^t := \Sigma^t_\# \left(\boldsymbol{\eta} \llcorner \{\gamma : \gamma(s) \neq \infty \text{ for all } s \in [0,t]\}\right)$$

satisfies by (3.26) for all $t \in (0,T]$ that

$$\boldsymbol{\sigma}^t = \int \delta_{\boldsymbol{X}(\cdot,x)}\, \mathrm{d}(e_0)_\#\boldsymbol{\sigma}^t(x).$$

By way of contradiction, assume that there exists a Borel set $\Omega \subset \mathbb{R}^d$ such that it is not $(e_0)_\#\boldsymbol{\eta}$−negligible and for every $x \in \Omega$, and so the set

$$\cup_{q \in \mathbb{Q}\cap(0,T^+_{\boldsymbol{X}}(x))} \{\gamma \in C([0,T];\bar{\mathbb{R}}^d) : \gamma(t) \neq \boldsymbol{X}(t,x) \text{ for all } t \in [0,q], \gamma([0,q]) \subset \mathbb{R}^d\}$$

is not $\boldsymbol{\eta}_x$−negligible; notice that this is equivalent to proving (3.25). Of course, this implies that for every $x \in \Omega$, there exists $r_x \in \mathbb{Q} \cap (0, T^+_{\boldsymbol{X}}(x))$ such that

$$\Sigma^{r_x}_\#(\boldsymbol{\eta}_x \llcorner \{\gamma : \gamma(t) \neq \infty \text{ for every } t \in [0,r_x]\})$$

is not a null measure nor a multiple of $\delta_{\boldsymbol{X}(\cdot,x)}$. Therefore, there exists a Borel set with positive measure $\Omega' \subset \Omega$ with respect to $(e_0)_\#\boldsymbol{\eta}$ and $r \in \mathbb{Q} \cap (0,T]$ such that for every $x \in \Omega'$,

$$\Sigma^r_\#(\boldsymbol{\eta}_x \llcorner \{\gamma : \gamma(t) \neq \infty \text{ for every } t \in [0,r]\})$$

is not a null measure nor a multiple of $\delta_{\boldsymbol{X}(\cdot,x)}$. By the definition of $\boldsymbol{\sigma}^t$, $(e_0)_\#\boldsymbol{\sigma}^t \leqslant (e_0)_\#\boldsymbol{\eta}$, and so $\delta_{\boldsymbol{X}(\cdot,x)} \geqslant \Sigma^r_\#(\boldsymbol{\eta}_x \llcorner \{\gamma : \gamma(t) \neq \infty \text{ for every } t \in [0,r]\})$ for $(e_0)_\#\boldsymbol{\eta}$−almost every x. Notice that this is a contradiction of the existence of Ω', and so $\boldsymbol{\eta}_x$ is concentrated on (3.25). For the concentration of $\boldsymbol{\eta}$, it follows from the concentration of $\boldsymbol{\eta}_x$ and Theorem 3.3. □

We are now ready to prove a criterion for global well-posedness in time of maximal regular flow.

Proposition 3.2 (No-blow-up criterion)**.** *Let $\boldsymbol{b}$ be a Borel vector field satisfying (H1), $\boldsymbol{\eta} \in \mathcal{M}(C([0,T];\bar{\mathbb{R}}^d))$ concentrated on the set* (3.9) *such that $\boldsymbol{\eta}(\{\gamma : \gamma(0) = \infty\}) = 0$, and for $\mu_t = (e_t)_\#\boldsymbol{\eta} \llcorner \mathbb{R}^d$ assume that*

$$\int_0^T \int_{\mathbb{R}^d} \frac{|\boldsymbol{b}_t(x)|}{(1+|x|)\log(2+|x|)} \, \mathrm{d}\mu_t(x)\mathrm{d}t < \infty.$$

Then $\boldsymbol{\eta}(\{\gamma \in C([0,T];\bar{\mathbb{R}}^d) : \gamma(t) = \infty \text{ for some } t \in (0,T)\}) = 0$. In particular, if μ_t is absolutely continuous with respect to the Lebesgue measure for all $t \in [0,T]$ and $\boldsymbol{\eta}$ is concentrated on the maximal regular flow $\boldsymbol{X}$ associated to $\boldsymbol{b}$, then $\boldsymbol{X}(\cdot,x) \in \mathrm{AC}([0,T];\mathbb{R}^d)$ for μ_0−almost every $x \in \mathbb{R}^d$, and $\mu_t = \boldsymbol{X}(t,\cdot)_\#\mu_0$ for all $t \in [0,T]$.

Proof. Since $\boldsymbol{\eta}(\{\gamma : \gamma(0) = \infty\}) = 0$, for $\boldsymbol{\eta}$−almost every curve there exists a time which it is finite, and so $\boldsymbol{\eta}$ is concentrated on

$$\Gamma := \left\{\gamma \in C([0,T];\bar{\mathbb{R}}^d) : \gamma(0) \in \mathbb{R}^d\right\},$$

and so it suffices to show that $\boldsymbol{\eta}\llcorner\Gamma$ is concentrated on non-blow-up curves. For this purpose, notice that by Proposition 3.1, $\boldsymbol{\eta}\llcorner\Gamma$ is concentrated on integral curves in $[0, T^+_{\boldsymbol{X}}(\gamma(0)))$. Therefore, we have by Fubini theorem and the definition of μ_t that

$$\begin{aligned}
&\int\int_0^{T^+_{\boldsymbol{X}}(\gamma(0))} \left|\frac{\mathrm{d}}{\mathrm{d}t}[\log\log(2+|\gamma(t)|)]\right| \mathrm{d}t\, \mathrm{d}[\boldsymbol{\eta}\llcorner\Gamma](\gamma) \\
&\leqslant \int_0^T \int \frac{|\boldsymbol{b}_t(\gamma(t))|}{(1+|\gamma(t)|)\log(2+|\gamma(t)|)} \, \mathrm{d}(e_t)_\#[\boldsymbol{\eta}\llcorner\Gamma](\gamma)\, \mathrm{d}t \\
&\leqslant \int_0^T \int_{\mathbb{R}^d} \frac{|\boldsymbol{b}_t(x)|}{(1+|x|)\log(2+|x|)} \, \mathrm{d}\mu_t(x)\mathrm{d}t < \infty,
\end{aligned}$$

and so for $\boldsymbol{\eta}\llcorner\Gamma$−almost every γ, it holds

$$\sup_{0\leqslant s<t\leqslant T^+_{\boldsymbol{X}}(\gamma(0))} |\log\log(2+|\gamma(t)|) - \log\log(2+|\gamma(s)|)| < \infty.$$

Therefore, we have that $T^+_{\boldsymbol{X}}(\gamma(0)) = T$ and γ does not blow up on $[0,T]$. In particular, the disintegration $\boldsymbol{\eta}_x$ of $\boldsymbol{\eta}$ with respect to e_0 is concentrated on

$$\{\gamma \in C([0,T];\bar{\mathbb{R}}^d) : \gamma(0) = x,\ \gamma(t) \neq \infty \text{ and } \gamma = \boldsymbol{X}(\cdot,x) \text{ for all } t \in [0,T]\}$$

for μ_0−every $x \in \mathbb{R}^d$. Since $\boldsymbol{\eta}_x$ is a probability measure, we conclude that $\boldsymbol{\eta}_x = \delta_{\boldsymbol{X}(\cdot,x)}$, and so for all $t \in [0,T]$ it holds

$$\mu_t = (e_t)_\# \boldsymbol{\eta} \llcorner \mathbb{R}^d = (e_t)_\# \int_{\mathbb{R}^d} \delta_{\boldsymbol{X}(\cdot,x)} \,\mathrm{d}\mu_0(x) = \boldsymbol{X}(t,\cdot)_\# \mu_0. \qquad \square$$

Before we proceed to the next section, we emphasize that all of the results were streamlined for a cleaner presentation. For instance, the theorems extracted from [5] can be proven for local flows, that is, for integral curves of $\boldsymbol{b} \in L^1((0,T); L^1_{\text{loc}}(\Omega;\mathbb{R}^d))$ for some Borel set $\Omega \subset \mathbb{R}^d$. Moreover, the results from [6] can be stated for flows starting from any $s \in [0,T]$ if one assumes divergence-free vector fields; we shall revisit this case in Chapter 5 without the divergence-free assumption. We also remark that

- we skipped the analogous results of Theorem 2.2, for the available ones are either only concerned with weak solutions of continuity equation [5, Remark 7.2] or with renormalized and weak solutions of transport/continuity equations for divergence-free vector fields [6, Proposition 4.10]. We shall provide a complete analogous of Theorem 2.2 in Chapter 5 by using a slight variation of Theorem 3.9 for vector fields with bounded divergence;

- by [5, Theorem 6.2 and Proposition 6.5], it is possible to extend Theorem 3.7 without the strong convergence of $\boldsymbol{c}^n$, replacing it by the weak convergence of $\boldsymbol{c}^n$ and uniform convergence (with respect to n) of $\mathbb{1}_{\Omega^{|h|}}(x+h)\boldsymbol{c}^n_t(x+h)$ as $h \to 0$ in $L^1((0,T)\times\Omega)$, where $\Omega^{|h|} := \{x \in \Omega : \operatorname{dist}(x, \mathbb{R}^d \setminus \Omega) > |h|\}$; such condition is akin to the classical DiPerna-Lions' one [48, Theorem II.7];

- the global assumption (3.20) is optimal (at least in dimensions $d \geqslant 3$) in order to obtain proper blow in Theorem 3.9, namely the maximal property in Definition 3.2 is in fact a limit. Indeed, the trio provided a very intricate counterexample in [5, Proposition 7.3] of a time independent vector field $\boldsymbol{b} \in W^{1,p}_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ for $p > 1$ with $\operatorname{div} \boldsymbol{b} \in L^\infty_{\text{loc}}(\mathbb{R}^d)$, and a positive Borel measure $\Omega \subset \mathbb{R}^d$ such that for every $x \in \Omega$, it holds $T^+_{\boldsymbol{X}}(x) \leqslant 2$, $\liminf_{t \nearrow T^+_{\boldsymbol{X}}(x)} |\boldsymbol{X}(t,x)| = 0$, and $\limsup_{t \nearrow T^+_{\boldsymbol{X}}(x)} |\boldsymbol{X}(t,x)| = \infty$ for $d \geqslant 3$. Nevertheless, in the case $d = 2$, if $\boldsymbol{b} \in \mathrm{BV}_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ with $\operatorname{div} \boldsymbol{b} \in L^\infty_{\text{loc}}(\mathbb{R}^d)$, then for almost every $x \in \mathbb{R}^2$, the proper blow-up occurs [5, Proposition 7.4]; the authors conjectured whether if one may construct a time dependent two dimensional vector field on the same lines as in $d \geqslant 3$ case for an counterexample;

- if one assumes $\boldsymbol{b} \in L^1((0,T)\times\Omega;\mathbb{R}^d)$, (H2), and (H3), then $\boldsymbol{X}(\cdot,x) \in \mathrm{AC}([0,T^+_{\boldsymbol{X}}(x)])$ for almost every $x \in \Omega$, and $\lim_{t \nearrow T^+_{\boldsymbol{X}}(x)} \boldsymbol{X}(t,x) \in \partial\Omega$ if $T^+_{\boldsymbol{X}}(x) < T$.

3.2 Vlasov-Poisson and quasistatic Vlasov-Maxwell

The trio rapidly applied their theory of local flows for the Vlasov-Poisson system [6, Sections 2 and 3]. The main advantage over the global theory shown in Chapter 2 is that we may define a suitable notion of a solution of Vlasov systems without energy hypothesis; as proven in [18] and briefly exposed in Remark 9, such assumptions were necessary for a precise notion of Lagrangian solution.

Before we make a rigorous comparison between the results of [6] and [18], we motivate the general Vlasov-system (see [28, Chapter II] for a extensive derivation of the more general Boltzmann equation): for a distribution of particles $f : (0, \infty) \times \mathbb{R}^{2d}$ in phase space $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$–$x$ and v being the physical space and the phase velocity coordinates, respectively–over time $t \in (0, \infty)$, we may compute its time derivative (assuming that x and v are functions depending of t rather than independent variables) and conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t} f = \partial_t f + \dot{x} \cdot \nabla_x f + \dot{v} \cdot \nabla_v f,$$

where the subscript on symbol ∇ signifies with respect to which variable is being differentiated. In classical mechanics, physical and phase velocity coincide, so $\dot{x} = v$ and $\dot{v} = m^{-1}F(f)$, where $\xi(v)$ is the physical velocity, i.e. the velocities of each particle, m is the mass of a particle (we assume that all particles are identical), and F is the force experienced by each particle, which may depend on the distribution of particles, that is, on f. In most applications, F may depend on f only in the physical space $x \in \mathbb{R}^d$, and so we rather consider F depending on x, v and an integral over $v \in \mathbb{R}^d$ of f multiplied by some function $g(x, v)$, that is, for $\bar{\rho}_t(x) = \int_{\mathbb{R}^d} g(x, v) f_t(x, v)\,\mathrm{d}v$ we have $F = F(x, v, \bar{\rho}_t(x))$. In special relativity case, physical and phase velocities $\xi(v)$ and v, respectively no longer coincide, but have a correction factor $\xi(v) = [1 + (c^{-1}|v|)^2]^{-1/2} v$, where c is the speed the light; this can be derived by the Lorentz factor $\gamma(\xi(v)) := [1 - (c^{-1}|\xi(v)|)^2]^{-1/2}$ and the relation $v = \gamma(\xi(v))\xi(v)$; see [60, Section 11.4]. Hence we have $\dot{x} = \xi(v)$ and $\dot{v} = m^{-1}F(x, v, \bar{\rho}(x))$, where F is again the force experienced by the particles, but now in the relativistic case, and so in fact F may only depend of $\xi(v)$ rather than v. Hence, we write

$$\frac{\mathrm{d}}{\mathrm{d}t} f = \partial_t f + \xi(v) \cdot \nabla_x f + m^{-1} F(x, v, \bar{\rho}(x)) \cdot \nabla_v f,$$

where from now on $\xi(v)$ is either v or $[1 + |v|^2]^{-1/2} v$ depending on which framework we are working on; we shall always assume that $c = 1$ for simplicity.

We have not considered so far the collision between particles: the most general form is to consider a function G depending on f (known as collision operator) such that

$$\partial_t f + \xi(v) \cdot \nabla_x f + m^{-1} F(x, v, \bar{\rho}(x)) \cdot \nabla_v f = G(f). \tag{3.27}$$

The simplest–although very complicated–collision operator is to consider only collision pairwise, which gives a bilinear operator Q and the above evolution equation now reads

$$\partial_t f + \xi(v) \cdot \nabla_x f + m^{-1} F(x, v, \bar{\rho}(x)) \cdot \nabla_v f = Q(f, f); \tag{3.28}$$

see [28, Chapter II] for a complete derivation of such operator for hard sphere models and its generalizations, as well as a probability-based justification on why it suffices to consider pairwise collisions.

Remark 24. It is common to refer to both (3.27) and (3.28), even in the particular case $F \equiv 0$ as Boltzmann equation, although they have different levels of generality.

We shall consider only the collisionless case–known as Vlasov equation

$$\partial_t f + \xi(v) \cdot \nabla_x f + m^{-1} F(x, v, \bar{\rho}(x)) \cdot \nabla_v f = 0,$$

which implies a transport equation structure (1.1) with vector field

$$\boldsymbol{b}_t(x, v) = (\xi(v), m^{-1} F_t(x, \xi(v), \bar{\rho}_t(x))).$$

It is expected, given some structure for the force function F, that the theory of Chapter 2 and Chapter 3 can be applied to this transport equation. For this purpose, let us consider a force given by the sum of Lorentz and Newton gravitational laws (we assume $d = 3$)

$$F_t(x, v) = q\,(E_t(x) + \xi(v) \times H_t(x)) + m g_t(x),$$

where q is the charge of particles, E, H, and g are the electric, magnetic, and gravitational acceleration fields satisfying

$$\begin{aligned} \operatorname{div} E &= (\epsilon_0)^{-1} q\rho, & \operatorname{curl} E &= -\partial_t H; \\ \operatorname{div} H &= 0, & \operatorname{curl} H &= \mu_0 q j + \partial_t E; \\ \operatorname{div} g &= -4\pi m G \rho, & \operatorname{curl} g &= 0. \end{aligned} \tag{3.29}$$

Here, the first four equations are Maxwell equations, G is Gravitational constant, ϵ_0 and μ_0 are vacuum permittivity and permeability, respectively, and ρ, j are number and momentum densities, respectively, defined as

$$\rho_t(x) = \int_{\mathbb{R}^3} f_t(x, v)\,\mathrm{d}v \quad \text{and} \quad j_t(x) = \int_{\mathbb{R}^3} \xi(v) f_t(x, v)\,\mathrm{d}v. \tag{3.30}$$

Following the same computations in [21, Appendix] based on quasi-static limits proven in [64], we have that either

$$\begin{aligned} \operatorname{div} E &= (\epsilon_0)^{-1} q\rho, & \operatorname{curl} E &= -\partial_t H; \\ \operatorname{div} H &= 0, & \operatorname{curl} H &= \mu_0 q j \end{aligned}$$

which is the quasi-magnetostatic limit, or

$$\begin{aligned} \operatorname{div} E &= (\epsilon_0)^{-1} q\rho, & \operatorname{curl} E &= 0; \\ \operatorname{div} H &= 0, & \operatorname{curl} H &= \mu_0 q j + \partial_t E, \end{aligned}$$

which is the quasi-electrostatic limit. In either case, Maxwell equations become decoupled, and so

$$(-\Delta)E = -(\epsilon_0)^{-1}q\nabla\rho + \mu_0 q\partial_t j \quad \text{and} \quad (-\Delta)H = \mu_0 q\,\mathrm{curl}\,H$$

or

$$(-\Delta)E = -(\epsilon_0)^{-1}q\nabla\rho \quad \text{and} \quad (-\Delta)H = \mu_0 q\,\mathrm{curl}\,j.$$

Hence it motivates us to consider the electric magnetic fields as

$$E_t(x) = -(\epsilon_0)^{-1}q\nabla[\Gamma * \rho_t(x)] \quad \text{and} \quad H = \mu_0 q\,\mathrm{curl}[\Gamma * j_t(x)],$$

where $\Gamma(x) = (4\pi|x|)^{-1}$. Notice that since $(-\Delta)g = 4\pi mG\nabla\rho$, we may write the force term as

$$m^{-1}F_t(x,v) = \left(\frac{q^2}{4\pi\epsilon_0 m} - Gm\right)\nabla[\Gamma * \rho_t(x)] + \frac{\mu_0 q^2}{4\pi m}\xi(v) \times \mathrm{curl}[\Gamma * j_t(x)].$$

Notice that the first constant has no definite sign, whereas the second is nonnegative. Hence, upon redefining densities ρ and j, then we may further write $E_t(x) = -\nabla[\Gamma * \rho_t(x)]$, $H_t(x) = \mathrm{curl}[\Gamma * j_t(x)]$, and

$$\boldsymbol{b}_t(x,v) = \left(\xi(v), \sigma_E E_t(x) + \sigma_H \xi(v) \times H_t(x)\right), \tag{3.31}$$

where $\sigma_E \in \{0, \pm 1\}$ and $\sigma_H \in \{0, 1\}$. This class of vector fields has two important cases: $\sigma_E \neq 0$ and $\sigma_H = 0$ is the Vlasov-Poisson equation; $\sigma_E = 0$ na $\sigma_H = 1$ is the Vlasov-Biot-Savart equation[14]. Notice that $\mathrm{div}_{x,v}\,\boldsymbol{b}_t(x,v) = \mathrm{div}_v(\xi(v) \times H_t(x)) = 0$. Moreover, a simple application of Young convolution inequality implies that $\boldsymbol{b} \in L^\infty((0,T); L^p_{\mathrm{loc}}(\mathbb{R}^6; \mathbb{R}^6))$ for any $p \in [1, 3/2)$.

We are now ready to state the main results of [21] by Borrin-Marcon; since it contains the Vlasov-Poisson case, we shall also cover the results of [6]. We begin with the consistency result [21, Theorem 1.1] which extends [6, Theorem 2.2].

Theorem 3.11 (Consistency of solutions)**.** *Let $f \in L^\infty([0,T]; L^1(\mathbb{R}^6))$ be a nonnegative function weakly continuous in time in duality with $C_c(\mathbb{R}^6)$ and either $f \in L^\infty((0,T) \times \mathbb{R}^6)$ is a weak solution of (1.1) with vector field (3.31); or f is a renormalized solution of (1.1) with vector field (3.31). Then f is a Lagrangian solution with respect to maximal regular flow, that is, $f_t\mathcal{L}^d = \boldsymbol{X}(t,\cdot)_\# \left[f_0\mathcal{L}^d \llcorner \{T^+_{\boldsymbol{X}} > t\}\right]$. In particular, f is a renormalized solution.*

The result follows from Theorem 3.6 and an application of [6, Proposition 4.10]; we shall prove an analogous result in Chapter 5 for non-divergence-free vector fields.

In order to establish existence of renormalied solutions of nonrelativistic Vlasov-Poisson equations, i.e. the transport/continuity equation with vector field (3.31) with

[14] Their names follow from the fact that E satisfies the Poisson equation $-\Delta E = \sigma_E \nabla\rho$ and H the Biot-Savart law.

$\xi(v) = v$, $\sigma_E = 1$, and $\sigma_H = 0$, the authors of [18, Theorem 8.4] assumed finite initial energy

$$\int_{\mathbb{R}^6} |v|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v + \int_{\mathbb{R}^3} |E_0(x)|^2\,\mathrm{d}x < \infty \tag{3.32}$$

in order to ensure the control of sublevels, i.e. (3.1). With minor modifications, one can extend to the relativistic case replacing $|v|^2$ with $(1+|v|^2)^{1/2}$. The key innovation of the trio [6, Theorem 2.7] was to obtain a "generalized" solution of the Vlasov-Poisson equation. The main advantage is that it is only assumed that $f_0 \in L^1(\mathbb{R}^6)$ is a nonnegative function, and if (3.32) holds, then generalized and renormalized solutions are equivalent [6, Theorem 2.8], as well as continuity in time in L^1_{loc} for ρ, E and an energy inequality. Moreover, in [6, Remark 2.9], the trio gave a sketch for a proof when $\sigma_E = -1$, and the full generality was achieved in [21, Theorems 1.2 and 1.3].

Theorem 3.12 (Existence of generalized solutions)**.** *Let $f_0 \in L^1(\mathbb{R}^6)$ be a nonnegative function. Then there exists a renormalized solution of transport/continuity equation with vector field*

$$\boldsymbol{b}^{\mathrm{eff}} = (\xi(v), \sigma_E E^{\mathrm{eff}} + \sigma_H \xi(v) \times H^{\mathrm{eff}}),$$

where $\xi(v) = (1+|v|^2)^{-1/2}v$, $E^{\mathrm{eff}}_t(x) = -\nabla[\Gamma * \rho^{\mathrm{eff}}_t(x)]$, $H^{\mathrm{eff}}_t(x) = \mathrm{curl}[\Gamma * j^{\mathrm{eff}}_t(x)]$ for some measures $\rho^{\mathrm{eff}}, j^{\mathrm{eff}}$ such that $\rho_t\mathcal{L}^3 \leqslant \rho^{\mathrm{eff}}_t$ and $|j^{\mathrm{eff}}_t| < \rho^{\mathrm{eff}}_t$ as measures, $\rho^{\mathrm{eff}}_t(\mathbb{R}^3) \leqslant \|f_0\|_{L^1(\mathbb{R}^6)}$, and $\partial_t\rho^{\mathrm{eff}} + \mathrm{div}\, j^{\mathrm{eff}} = 0$ with initial data ρ_0. Moreover, f is continuous in time in L^1_{loc}, i.e.

$$\lim_{t\to s} \int_{B_R} |f_t(x,v) - f_s(x,v)|\,\mathrm{d}v\,\mathrm{d}x = 0 \quad \text{for any } R < \infty,$$

and by Theorem 3.11, $f_t\mathcal{L}^d = \boldsymbol{X}^{\mathrm{eff}}(t,\cdot)_\# \left[f_0\mathcal{L}^d \llcorner \{T^+_{\boldsymbol{X}^{\mathrm{eff}}} > t\}\right]$, where $\boldsymbol{X}^{\mathrm{eff}}$ is the maximal regular flow associated to $\boldsymbol{b}^{\mathrm{eff}}$.

The proof is via approximation by smoothing the singular kernels in E, H, and when passing to the limit for approximation densities ρ^n, j^n, we cannot ensure that they converge to ρ, j in $L^\infty((0,\infty); L^1(\mathbb{R}^3))$, respectively; we can only ensure that $\rho^n\mathcal{L}^d$, $j^n\mathcal{L}^d$ converges weakly* in $L^\infty((0,\infty); \mathcal{M}(\mathbb{R}^3))$ to some measures ρ^{eff}, j^{eff} with the properties listed in Theorem 3.12.

Finally, we have the existence of renormalized solutions of transport/continuity equation with vector field (3.31) if one assumes that f_0 has finite energy, that is

$$\int_{\mathbb{R}^6} (1+|v|^2)^{1/2} f_0\,\mathrm{d}v\,\mathrm{d}x + \int_{\mathbb{R}^3} \frac{\sigma_E}{2}\Gamma * \rho_0(x)\rho_0(x) + \frac{\sigma_H}{2}\Gamma * j_0(x)j_0(x)\,\mathrm{d}x \tag{3.33}$$

and has integrability

$$f_0 \in \begin{cases} L^1(\mathbb{R}^6) & \text{if } \sigma_E = 1; \\ L^1(\mathbb{R}^6) \cap L^{3/2}(\mathbb{R}^6) & \text{if } \sigma_E = 0; \\ L^1(\mathbb{R}^6) \cap L^{3/2}(\mathbb{R}^6) \text{ and } \|f_0\|_{L^{3/2}(\mathbb{R}^6)} < \epsilon_0 & \text{if } \sigma_E = -1, \end{cases} \tag{3.34}$$

where ϵ_0 is any constant such that $\epsilon_0 < C_{\mathrm{HLS}}^{-1}$ and $C_{\mathrm{HLS}} > 0$ is a constant in the Hardy-Littlewood-Sobolev inequality $\|\Gamma * g\|_{L^6(\mathbb{R}^3)} \leqslant C_{\mathrm{HLS}}\|g\|_{L^{6/5}(\mathbb{R}^3)}$.

Theorem 3.13 (Existence of renormalized solutions)**.** *Let f_0 be a nonnegative function satisfying (3.33) and (3.34). Then there exists a maximal regular flow associated to vector field $\boldsymbol{b}$ as in (3.31) with $\xi(v) = (1+|v|^2)^{-1/2}v$ such that $f_t\mathcal{L}^6 = \boldsymbol{X}(t,\cdot)_\# \left[f_0\mathcal{L}^6 \llcorner \{T_{\boldsymbol{X}}^+ > t\}\right]$ is a renormalized solution of transport/continuity equation with vector field $\boldsymbol{b}$ and initial data f_0, the flow is globally defined in $[0,\infty)$ for $f_0\mathcal{L}^6-$almost every $(x,v) \in \mathbb{R}^6$, and ρ, j, E, and H are continuous in time in L^1_{loc}. Moreover, it holds for all $t \geqslant 0$ that*

$$\begin{aligned}
&\int_{\mathbb{R}^6} (1+|v|^2)^{1/2} f_t \,\mathrm{d}v\,\mathrm{d}x + \frac{\sigma_E}{2}\int_{\mathbb{R}^3} \Gamma * \rho_t(x)\rho_t(x)\,\mathrm{d}x \\
&\leqslant \int_{\mathbb{R}^6} (1+|v|^2)^{1/2} f_0 \,\mathrm{d}v\,\mathrm{d}x + \frac{\sigma_E}{2}\int_{\mathbb{R}^3} \Gamma * \rho_0(x)\rho_0(x)\,\mathrm{d}x.
\end{aligned}$$

The main strategy to prove this theorem is to show that the measures in Theorem 3.12 satisfy $\rho^{\mathrm{eff}} = \rho\mathcal{L}^3$ and the inequalities

$$\begin{aligned}
&\int_{\mathbb{R}^3} |E_t(x)|^2\,\mathrm{d}x \leqslant \int_{\mathbb{R}^3} \Gamma * \rho_t(x)\rho_t(x)\,\mathrm{d}x; \\
&\int_{\mathbb{R}^3} |H_t(x)|^2\,\mathrm{d}x \leqslant \int_{\mathbb{R}^3} \Gamma * j_t(x) j_t(x)\,\mathrm{d}x - \int_{\mathbb{R}^3} [\operatorname{div}\Gamma * j_t(x)]^2\,\mathrm{d}x.
\end{aligned}$$

We shall not provide the proof for Theorems 3.11 to 3.13 as the techniques are very specific for vector fields with structure (3.31) and do not seem to be applicable for the cases presented in Chapters 4 and 5.

Notice that Theorems 3.12 and 3.13 do not contemplate the classical case $\xi(v) = v$. The main difficulty is that in this case the electromagnetic force $E_t(x) + v \times H_t(x)$ is not bounded by a function independent of v. Moreover, it does not follow that $|j| < \rho$, and so the properties of ρ does not translate to j via dominated convergence theorem.

4 Extension for wavelike vector fields

In this chapter, we shall study the transport/continuity equation

$$\partial_t f + \operatorname{div}_{x,v}(\boldsymbol{b} f) = 0,$$

where the vector field $\boldsymbol{b}$ has structure (3.31) (hence, divergence-free) with $\sigma_E = \sigma_H = 1$ and the electromagnetic field E and H satisfies the rescaled first four equations (3.29)–recall that they are the so called Maxwell's equations–with densities ρ and j as in (3.30). More precisely, we are interested in the Vlasov-Maxwell system (in Gaussian units and with speed of light $c = 1$):

$$\begin{cases} \partial_t f + \xi(v) \cdot \nabla_x f + (E + \xi(v) \times H) \cdot \nabla_v f = 0 & \text{in} \quad (0,\infty) \times \mathbb{R}^3 \times \mathbb{R}^3; \\ \operatorname{div} E = 4\pi\rho, \quad \operatorname{div} H = 0 & \text{in} \quad (0,\infty) \times \mathbb{R}^3; \\ \partial_t H + \operatorname{curl} E = 0, \quad \partial_t E - \operatorname{curl} H = -4\pi j & \text{in} \quad (0,\infty) \times \mathbb{R}^3, \end{cases} \tag{4.1}$$

where the densities ρ, j satisfy (3.30). Once again we shall write it as a transport/continuity equation with vector field

$$\boldsymbol{b}(x,v) := (\xi(v), E + \xi(v) \times H). \tag{4.2}$$

We now follow the same idea as in Section 3.2 and we decouple Maxwell equations with respect to electric and magnetic fields: by the quintessential vector calculus identity

$$\operatorname{curl}\operatorname{curl} u = \nabla(\operatorname{div} u) - \Delta u,$$

one may formally compute that

$$\begin{aligned} -\Delta E &= -\nabla(\operatorname{div} E) - \operatorname{curl} \partial_t H = -4\pi \nabla\rho - 4\pi \partial_t j - \partial_{tt} E; \\ -\Delta H &= -\nabla(\operatorname{div} H) + \operatorname{curl}(\partial_t E + 4\pi j) = 4\pi \operatorname{curl} j - \partial_{tt} H. \end{aligned}$$

Therefore one has that E and H solves the non-homogeneous wave equation

$$\begin{aligned} (\partial_{tt} - \Delta) E &= -4\pi\nabla\rho - 4\pi\partial_t j; \\ (\partial_{tt} - \Delta) H &= 4\pi \operatorname{curl} j. \end{aligned}$$

As argued in [52], the above is equivalent to Maxwell's equations provided that we assume initial data E_0 and H_0, as well as f_0, with compatibility equations

$$\begin{aligned} \operatorname{div} E_0 = 4\pi\rho_0, \quad \partial_t E|_{t=0} &= -4\pi j_0 + \operatorname{curl} H_0; \\ \operatorname{div} H_0 = 0, \quad \partial_t H|_{t=0} &= -\operatorname{curl} E_0; \end{aligned} \tag{4.3}$$

Following the explicit formula for solutions of wave equation; see [49, Section 2.4], we have the so called Jefimenko's equations, as computed in [60, 66, 75]: the eletric field reads

$$\begin{aligned} E_t(x) =& E_t^0(x) + \int_{B_t(x)} \frac{\omega(x-y)}{|x-y|^2} \left[\rho_t(y)\right]_{ret}(x)\mathrm{d}y \\ &+ \int_{B_t(x)} \frac{\omega(x-y)}{|x-y|^2} \omega(x-y) \cdot \left[j_t(y)\right]_{ret}(x)\mathrm{d}y \\ &+ \int_{B_t(x)} \frac{\omega(x-y)}{|x-y|^2} \times \left(\omega(x-y) \times \left[j_t(y)\right]_{ret}(x)\right)\mathrm{d}y \\ &+ \int_{B_t(x)} \frac{\omega(x-y)}{|x-y|} \times \left(\omega(x-y) \times \left[\partial_t j_t(y)\right]_{ret}(x)\right)\mathrm{d}y, \end{aligned} \tag{4.4}$$

while the magnetic field reads

$$\begin{aligned} H_t(x) =& H_t^0(x) - \int_{B_t(x)} \frac{\omega(x-y)}{|x-y|^2} \times \left[j_t(y)\right]_{ret}(x)\mathrm{d}y \\ &- \int_{B_t(x)} \frac{\omega(x-y)}{|x-y|} \times \left[\partial_t j_t(y)\right]_{ret}(x)\mathrm{d}y. \end{aligned} \tag{4.5}$$

where we recall the notation of Section 2.5 $\omega(z) := z/|z|$, and we denote $\left[f_t(y)\right]_{ret}(x) := f_{t-|y-x|}(y)$[1], and (E^0, H^0) are functionals depending only on initial data $(f_0,\, E_0,\, H_0)$:

$$\begin{aligned} E_t^0(x) :=& E_t^H(x) - \frac{1}{t}\int_{\partial B_t(x)} \rho_0(y)\omega(x-y)\,\mathrm{d}S_y \\ &- \frac{1}{t}\int_{\partial B_t(x)} j_0(y)\cdot\omega(x-y)\omega(x-y)\,\mathrm{d}S_y; \\ H_t^0(x) :=& H_t^H(x) + \frac{1}{t}\int_{\partial B_t(x)} j_0(y)\times\omega(x-y)\,\mathrm{d}S_y, \end{aligned} \tag{4.6}$$

where E^H, H^H are homogeneous solutions of Maxwell's equations (see "Kirchhoff's formula" in [49, Section 2.4]):

$$\begin{aligned} E_t^H(x) =& \frac{1}{4\pi t^2}\int_{\partial B_t(x)} t(\operatorname{curl} H_0(y) - 4\pi j_0(y)) + E_0(y) + DE_0(y)(y-x)\,\mathrm{d}S_y; \\ H_t^H(x) =& \frac{1}{4\pi t^2}\int_{\partial B_t(x)} -t\operatorname{curl} E_0(y) + H_0(y) + DH_0(y)(y-x)\,\mathrm{d}S_y. \end{aligned} \tag{4.7}$$

There are two main results concerning existence of solutions of (4.1): the first one is due to Glassey and Strauss [52] for distribution functions with a cutoff at high velocities, that is, they prove existence of C^1 solutions with an a priori assumption that $f(t,x,v) = 0$ for $v > \alpha(t)$ for some continuous function α, provided that the initial data f_0 is a C^1 function with compact support and the initial electromagnetic field E_0 and H_0 are C^2 functions satisfying the compatibility conditions (4.3); see also in [52] a result with a modification

[1] Notice that the "retarded bracket" function $[f_t(x)]_{ret}(y)$ is a measurable function as a function of t, x, y, if f is measurable as a function of t, y. Indeed, it is the composition of f and the Lipschitz function $t, x, y \mapsto (t - |y-x|, y)$.

of (4.1) with acceleration term now reading $\zeta(v)[E_t(x) + \xi(v) \times H_t(x)]$ for some cutoff function ζ and a modern presentation of the original result in [24, 61]. Moreover, we also refer to [53] for a substitution of the aforementioned a priori assumption for an a priori bound on first $v-$moments. The second one is due to DiPerna and Lions [47] via the now called "average lemma" result. Loosely speaking, weak solutions of (4.1) enjoys a $H^{1/4}-$regularity when averaged with respect to $v-$variable and with any smooth weight. We refer to [73, 75] and references therein for a very recent survey of results concerning Vlasov systems.

We now present some results we obtained in this thesis and are already published in [20].

4.1 Singular kernels and "hyperbolic convolution"

Notice that any of the terms in (4.4)-(4.5) cannot be written as a convolution of a singular kernel and a density function, and so the previous theory presented in Chapter 2 is not applicable. Hence, here we study vector fields in the form

$$\boldsymbol{b}_t(x) = \sum_{i=1}^{m} \int_{B_t(x)} K^i(x-y) g^i_{t-|y-x|}(y) \mathrm{d}y =: \sum_{i=1}^{m} K^i \star g^i_t(x), \tag{4.8}$$

where K^i is a kernel with suitable singularity at the origin and g^i are summable functions in spacetime. In (4.8) we introduce the "$\star$" operator and we refer to it as hyperbolic convolution. If one considers a non-unit speed of propagation v, (4.8) should now read

$$\boldsymbol{b}_t(x) = \sum_{i=1}^{m} \int_{B_{vt}(x)} K^i(x-y) g^i_{t-v^{-1}|y-x|}(y) \mathrm{d}y =: \sum_{i=1}^{m} K^i \star g^i_t(x);$$

if one formally takes $v \to \infty$, then the classical convolution is recovered. More generally, vector fields with such hyperbolic convolution come from solutions of wavelike equations; for instance, if $\boldsymbol{b}$ satisfies the wave equation for $d = 3$ with zero initial data

$$(v^{-2}\partial_{tt} - \Delta)\boldsymbol{b}_t = (g^1, g^2, g^3),$$

then $\boldsymbol{b}$ has structure (4.8) with $K^i(x) = (4\pi|x|)^{-1}e_i$, where $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^3$.

Let us make more precise the hypothesis on the kernels $K \equiv K^i$:

Condition 4.1. We shall consider kernels $K \in C^1(\mathbb{R}^d \setminus \{0\}; \mathbb{R}^d)$ such that there exists a bounded set $A \ni 0$ and a constant $C > 0$ in which $|K(x)| \leqslant C|x|^{1-d}$ for $x \in A \setminus \{0\}$ and $|K(x)| \leqslant C$ for $x \in \mathbb{R}^d \setminus A$.

Moreover, let us make some comments on the hyperbolic convolution: the first one is that an analogous result related to the classical convolution, namely the associative property $(f * g) * h = f * (g * h)$ does not hold for the hyperbolic variant, that is, generally,

$$K \star (f * g_t)(x) \neq (K * f) \star g_t(x),$$

although a similar result (which is a simple exercise on changing the order of integrals) does hold:

$$K \star (f * g_t)(x) = f * (K \star g_t)(x).$$

While the latter allow us to use the result Lemma 2.2, the former implies that the proof of Theorem 2.3 found in [22, Theorem 3.3] is no longer adaptable. Moreover, many of classical results about convolution–namely Young's convolution inequality–do not readily extend for the hyperbolic convolution. Nevertheless, by using that the integrability is local, we prove that $K \star g$ have similar inequalities to Young's convolution case if one assumes that K satisfies Condition 4.1 and g has enough integrability in spacetime. In particular, we conclude that it is locally summable in space and globally in time, i.e. it satisfies property (H1).

Lemma 4.1 (Young's convolution inequality and property (H1))**.** *Let $\boldsymbol{b}$ be a vector field with structure*

$$\boldsymbol{b}_t(x) = K \star g_t(x) = \int_{B_t(x)} K(x-y)[g(y)]_{ret}(x)\,\mathrm{d}y,$$

where $g \in L^1((0,T);L^p(\mathbb{R}^d))$ for $p \geqslant 1$ and K satisfies Condition 4.1 with set A and constant $C_A > 0$. Then there exists a constant $C_{A,T} > 0$ such that

$$\|K \star g\|_{L^1((0,T);L^p(\mathbb{R}^d))} \leqslant C_{A,T}\|g\|_{L^1((0,T);L^p(\mathbb{R}^d))},$$

and for $s \in [1, d/(d-1))$, $g \in L^1((0,T)\times\mathbb{R}^d)$, $\partial_t g \in L^1((0,T);L^q(\mathbb{R}^d))$ for $q \geqslant 1$, it holds

$$\|K \star g\|_{L^1((0,T);L^s(\mathbb{R}^d))} \leqslant C_{A,T,s}\left(\|g\|_{L^1((0,T)\times\mathbb{R}^d)} + \|\partial_t g\|_{L^1((0,T);L^q(\mathbb{R}^d))}\right)$$

for a constant $C_{A,T,s} > 0$. In particular, $\boldsymbol{b}$ satisfies (H1).

Proof. By changing variables, we have that

$$\begin{aligned}\left[\int_{\mathbb{R}^d}\left|\int_{B_t(x)} K(x-y)[g_t(y)]_{ret}(x)\,\mathrm{d}y\right|^p \mathrm{d}x\right]^{\frac{1}{p}} \leqslant & C\left[\int_{\mathbb{R}^d}\left|\int_{A\cap B_t} |y|^{-d+1} g_{t-|y|}(x-y)\,\mathrm{d}y\right|^p \mathrm{d}x\right]^{\frac{1}{p}} \\ & + \left[\int_{\mathbb{R}^d}\left|\int_{B_t\setminus A} K(y) g_{t-|y|}(x-y)\,\mathrm{d}y\right|^p \mathrm{d}x\right]^{\frac{1}{p}} \\ \leqslant & C\int_{B_t}\left(|y|^{-d+1}+1\right)\|g_{t-|y|}\|_{L^p(\mathbb{R}^d)}\,\mathrm{d}y \\ = & C\int_0^t\int_{\partial B_1}\left(1+\tau^{d-1}\right)\|g_\tau\|_{L^p(\mathbb{R}^d)}\,\mathrm{d}S_\omega\,\mathrm{d}\tau \\ \leqslant & C_{A,T}\|g\|_{L^1((0,T);L^p(\mathbb{R}^d))}.\end{aligned}$$

Integrating with respect to $t \in [0,T]$, the first inequality follows. The second one follows from the fact that

$$\begin{aligned}|K \star g_t(x)| &= \left| \int_{\mathbb{R}^d} K(x-y)\mathbb{1}_{B_t}(x-y) g_{t-|x-y|}(y)\,\mathrm{d}y \right| \\ &= \left| \int_{\mathbb{R}^d} K(y)\mathbb{1}_{B_t}(y) \left[g_t(x-y) + \int_t^{t-|y|} \partial_t g_\tau(x-y)\,\mathrm{d}\tau \right] \mathrm{d}y \right| \\ &\leqslant (|K|\mathbb{1}_{B_t}) * |g_t|(x) + (|K|\mathbb{1}_{B_t}) * \int_0^T |\partial_t g_\tau|\,\mathrm{d}\tau(x).\end{aligned}$$

Taking the L^s norm, using Young's convolution inequality, we have that

$$\|K \star g_t\|_{L^s(\mathbb{R}^d)} \leqslant \|K\|_{L^s(B_T)}\|g_t\|_{L^1(\mathbb{R}^d)} + \|K\|_{L^{s'}(B_T)}\|\partial_t g_t\|_{L^q(\mathbb{R}^d)},$$

where $s' \leqslant s$ satisfies

$$1 + \frac{1}{s} = \frac{1}{s'} + \frac{1}{q}.$$

Integrating with respect to time, the result follows. □

As mentioned in Chapter 1, it is well-known since [48, Section IV] some regularity on the derivatives of vector fields, and so we shall compute it for (4.8). For this purpose, we need to assume some further regularity on kernesl K^i, namely on its derivative.

Condition 4.2. We shall also assume that K^i as in Condition 4.1 has derivative

$$\partial_j K^i(x) = \frac{\Omega_{ij}(x)}{|x|^d},$$

where Ω_{ij} is a zero order homogeneous function with average zero over the $\mathbb{S}^{d-1}$ sphere, that is, $\Omega_{ij}(tx) = \Omega_{ij}(x)$ for all $t \in \mathbb{R}$ and

$$\int_{\mathbb{S}^{d-1}} \Omega_{ij}(y)\,\mathrm{d}S_y = 0.$$

Lemma 4.2. *Let $\boldsymbol{b}$ as in (4.8), where for each $i = 1, \dots, m$, the kernels K satisfying Condition 4.1 and Condition 4.2, the functions $g^i \in L^1((0,T); L^p(\mathbb{R}^d))$ with further regularity $\partial_t g^i \in L^1((0,T); L^q(\mathbb{R}^d))$ and $g_0^i \in L^r(\mathbb{R}^d)$ for $p,q,r \geqslant 1$. Then for each $j = 1, \dots, d$, it holds in the weak sense*

$$\partial_j \boldsymbol{b}_t(x) = \sum_{i=1}^m - \int_{\partial B_t} \omega_j(y) K^i(y) g_0^i(x-y)\,\mathrm{d}S_y + \partial_j K^i \star g_t^i(x) + (\omega_j K^i) \star \partial_t g_t^i(x). \quad (4.9)$$

Proof. Notice that the first and third terms are L^r and L^q functions, as for the third term follows from Lemma 4.1 and the first one is trivial. For the second, consider $\phi \in C_c^\infty(\mathbb{R}^d)$.

Then

$$\begin{aligned}
&\int_{\mathbb{R}^d} \phi(x) \int_{B_t(x)} \frac{\Omega_{ij}(x-y)}{|x-y|^d} [g_t(y)]_{ret}(x)\, \mathrm{d}y\, \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \phi(x) \int_0^t \int_{\partial B_1} \frac{\Omega_{ij}(\omega)}{t-\tau} g_\tau(x+\omega(t-\tau))\, \mathrm{d}S_\omega \mathrm{d}\tau \mathrm{d}x \\
&= \int_{\mathbb{R}^d} \int_0^t \int_{\partial B_1} [\phi(x-\omega(t-\tau)) - \phi(x)] \frac{\Omega_{ij}(\omega)}{t-\tau} g_\tau(x)\, \mathrm{d}S_\omega \mathrm{d}\tau \mathrm{d}x,
\end{aligned}$$

and so we have that

$$\left| \int_{\mathbb{R}^d} \phi(x) \int_{B_t(x)} \frac{\Omega_{ij}(x-y)}{|x-y|^d} [g_t(y)]_{ret}(x)\, \mathrm{d}y\, \mathrm{d}x \right| \leqslant C \|\nabla \phi\|_{L^\infty(\mathbb{R}^d)} \|g\|_{L^1((0,T);L^p(\mathbb{R}^d))}.$$

Therefore, the derivative is well-defined in the weak sense.

We shall prove (4.9) when $m = 1$, for the case $m > 1$ the argument is done for each term $K^i \star g^i$, where K and g satisfy the conditions for any K^i and g^i, respectively. Moreover, without loss of generality, we shall assume that g is a smooth function, for otherwise we may use a standard density argument. By change of variables, we have

$$\partial_j \boldsymbol{b}_t(x) = \int_{B_t} K(y) [\partial_j g_t(x-y)]_{ret}(0)\, \mathrm{d}y.$$

We now want to integrate by parts the gradient, and so we recall the commutation between the retarded brackets and derivatives with respect to y, t:

$$\begin{aligned}
\partial_{y_j} [g_t(y)]_{ret}(x) &= [\partial_{y_j} g_t(y)]_{ret}(x) + \omega_j(x-y) \partial_t [g_t(y)]_{ret}(x), \\
\partial_t [g_t(y)]_{ret}(x) &= [\partial_t g_t(y)]_{ret}(x).
\end{aligned} \tag{4.10}$$

Therefore, we have

$$\begin{aligned}
\partial_j \boldsymbol{b}_t(x) = -\int_{\partial B_t} \omega_j(y) K(y) g_0(x-y)\, \mathrm{d}S_y &+ \int_{B_t} \partial_j K(y) [g_t(x-y)]_{ret}(0)\, \mathrm{d}y \\
&+ \int_{B_t} \omega_j(y) K(y) [\partial_t g_t(x-y)]_{ret}(0)\, \mathrm{d}y.
\end{aligned}$$

Changing variables once again, we obtain the desired result. □

Now, we recall the grand maximal function studied in [22] and introduced in Section 2.1. For the convenience of the reader, we also recall all the relevant properties and desired estimates.

Definition 4.1. Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Given a family of functions $\{\rho^\nu\}_\nu$ such that $\operatorname{supp} \rho^\nu \subset B_1$, and $\|\rho^\nu\|_{L^1(\mathbb{R}^d)} + \|\rho^\nu\|_{L^\infty(\mathbb{R}^d)} \leqslant C_0$ for all ν, we define the grand maximal function of u associated to ρ^ν as

$$M_{\rho^\nu} u(x) := \sup_\nu \sup_{\epsilon > 0} |\rho^\nu_\epsilon * u(x)|,$$

where $\rho^\nu_\epsilon(x) = \epsilon^{-d} \rho^\nu(x/\epsilon)$.

Recall that the grand maximal function behaves like the classical maximal function

$$Mu(x) = \sup_{\epsilon} \frac{1}{|B_\epsilon|} \int_{B_\epsilon} |u(x-y)| \,\mathrm{d}y,$$

in the sense that it is a bounded operator from L^p to L^p for $p > 1$ and from L^1 to L^1_w, where L^1_w stands for the weak L^1 space. Indeed, as argued in Section 2.1, we have that

$$M_{\rho^\nu} u(x) \leqslant \sup_{\nu} \|\rho^\nu\|_{L^\infty(\mathbb{R}^d)} Mu(x) \leqslant C_0 Mu(x), \tag{4.11}$$

and so the aforementioned bounds hold. We would like to take $u_j = \partial_j(K \star g)$ for each $j = 1, \ldots, d$, but it is not clear that such function is in any L^p, and so we shall compute it explicitly. For this purpose, we recall the definition in Section 2.3 of singular kernels of fundamental type, which now will further assume the structure analogous to Condition 4.2:

Definition 4.2. A function Γ is said to be a singular kernel of fundamental type if

1. $\Gamma|_{\mathbb{R}^d \setminus \{0\}} \in C^1(\mathbb{R}^d \setminus \{0\})$;
2. There exists constants $C \geqslant 0$ such that $|\Gamma(x)| \leqslant C|x|^{-d}$ and $|D\Gamma(x)| \leqslant C|x|^{-d-1}$ for every $x \neq 0$;
3. The kernel can be written as $\Gamma(x) = \Omega(x/|x|)|x|^{-d}$, where

$$\int_{\partial B_1} \Omega(\omega) \,\mathrm{d}S_\omega = 0.$$

In the following, we shall use the fact that if ρ^ν is sufficiently regular (e.g. $\rho^\nu \in H^s(\mathbb{R}^d)$ for $s > d/2$ with uniform bounds), then $\mathcal{F}\rho^\nu \in L^1(\mathbb{R}^d)$, where $\mathcal{F}$ is the Fourier transform. Taking Γ a singular kernel of fundamental type, since $\mathcal{F}(\Gamma \mathbb{1}_{B_\delta^c}) \in L^\infty(\mathbb{R}^d)$ by Calderón-Zygmund theory (see [68, Estimate 7.3]), it holds $\mathcal{F}\rho^\nu \mathcal{F}(\Gamma \mathbb{1}_{B_\delta^c})(\epsilon \cdot) \in L^1(\mathbb{R}^d)$ uniformly for $\epsilon > 0$ and $\delta > 0$. Therefore, we have that there exists a constant $C_1 > 0$ such that for all $\epsilon > 0$, ν and $\delta > 0$,

$$\|\epsilon^d \rho^\nu_\epsilon * (\Gamma \mathbb{1}_{B_\delta^c})\|_{C_b(\mathbb{R}^d)} = \|\rho^\nu * (\epsilon^d \Gamma \mathbb{1}_{B_\delta^c})(\epsilon \cdot)\|_{C_b(\mathbb{R}^d)} \leqslant C_1. \tag{4.12}$$

We begin by proving that $M_{\rho^\nu}(\Gamma \mathbb{1}_{B_\delta^c} * g)$ is in $L^1_w(\mathbb{R}^d)$.

Proposition 4.1. *Let $\delta > 0$, Γ a kernel satisfying Definition 4.2, and a function $g \in L^1((0,T); L^p(\mathbb{R}^d))$ for $1 \leqslant p < \infty$. Then there exists a constant $C_d > 0$ independent of δ such that*

$$\left|\left|\left| M_{\rho^\nu}(\Gamma \mathbb{1}_{B_\delta^c} * g_t) \right|\right|\right|_{L^1_w(\mathbb{R}^d)} \leqslant C_d (C_0 + C_1) \|g_t\|_{L^1(\mathbb{R}^d)}$$

if $p = 1$, and if $p > 1$, it follows that

$$\|M_{\rho^\nu}(\Gamma \mathbb{1}_{B_\delta^c} * g_t)\|_{L^p(\mathbb{R}^d)} \leqslant C_d C_0 \|g_t\|_{L^p(\mathbb{R}^d)}.$$

Proof. If $p > 1$, the result follows easily from (4.11), the boundedness of the maximal operator from L^p to L^p, the boundedness of the operator $u \mapsto \Gamma * u$ from L^p to L^p (see [68, Theorem 7.5]), and the estimate [68, Proposition 7.10]

$$\sup_{\delta>0} |\Gamma \mathbb{1}_{B_\delta^c} * g_t(x)| \leqslant C \left(M(\Gamma * g_t(x)) + M g_t(x) \right).$$

Hence, it remains to prove the harder case $p = 1$. Notice that by the definition of grand maximal function, we have that

$$\begin{aligned} M_{\rho^\nu}(\Gamma \mathbb{1}_{B_\delta^c} * g_t)(x) \leqslant & \sup_{\nu} \sup_{\epsilon>0} \left| \rho_\epsilon^\nu * \Gamma \mathbb{1}_{B_\delta^c} * g_t(x) - (\Gamma \mathbb{1}_{B_\eta^c}) * g_t(x) \int_{\mathbb{R}^d} \rho^\nu(y)\, \mathrm{d}y \right| \\ & + \sup_{\nu} \|\rho^\nu\|_{L^1(\mathbb{R}^d)} \sup_{\epsilon>0} \left| \Gamma \mathbb{1}_{B_\eta^c} * g_t(x) \right| \end{aligned}$$

where $\eta := \max\{2\epsilon, \delta\}$. Now, the second term can be estimated as

$$\sup_{\nu} \|\rho^\nu\|_{L^1(\mathbb{R}^d)} \sup_{\epsilon>0} \left| \Gamma \mathbb{1}_{B_\eta^c} * g_t(x) \right| \leqslant C_0 \sup_{\eta>0} \left| \Gamma \mathbb{1}_{B_\eta^c} * g_t(x) \right|.$$

By the L^1 to L^1_w boundedness of maximal truncated operator (see [68, Proposition 7.10]), it suffices to estimate the first term. For this purpose, we shall split it as the supremum for $\epsilon \geqslant \delta/2$ and $0 < \epsilon < \delta/2$.

Notice that

$$\begin{aligned} \sup_{\nu} \sup_{\epsilon \geqslant \delta/2} & \left| \rho_\epsilon^\nu * \Gamma \mathbb{1}_{B_\delta^c} * g_t(x) - (\Gamma \mathbb{1}_{B_\eta^c}) * g_t(x) \int_{\mathbb{R}^d} \rho^\nu(y)\, \mathrm{d}y \right| = \\ & \sup_{\nu} \sup_{\epsilon \geqslant \delta/2} \left| \left[\rho_\epsilon^\nu * \Gamma \mathbb{1}_{B_\delta^c} - \Gamma \mathbb{1}_{B_{2\epsilon}^c} \int_{\mathbb{R}^d} \rho^\nu(y)\, \mathrm{d}y \right] * g_t(x) \right|. \end{aligned}$$

Now, by (4.12), we have that for all $x \in \mathbb{R}^d$ and $\epsilon \geqslant \delta/2$ that

$$\left| \rho_\epsilon^\nu * \Gamma \mathbb{1}_{B_\delta^c}(x) - (\Gamma \mathbb{1}_{B_{2\epsilon}^c})(x) \int_{\mathbb{R}^d} \rho^\nu(y)\, \mathrm{d}y \right| \leqslant \frac{C_1 + 2^{-d} C_0}{\epsilon^d}.$$

Moreover, for $|x| > 3\epsilon$, we have that

$$\begin{aligned} \left| \rho_\epsilon^\nu * \Gamma \mathbb{1}_{B_\delta^c}(x) - (\Gamma \mathbb{1}_{B_{2\epsilon}^c})(x) \int_{\mathbb{R}^d} \rho^\nu(y)\, \mathrm{d}y \right| \leqslant & \left| \int_{B_\delta^c} [\Gamma(y) - \Gamma(x)] \rho_\epsilon^\nu(x-y)\, \mathrm{d}y \right| \\ & + |\Gamma(x)| \int_{B_\delta} |\rho_\epsilon^\nu(x-y)|\, \mathrm{d}y \\ \leqslant & C \int_0^1 \int_{\mathbb{R}^d} \frac{\epsilon |\rho_\epsilon^\nu(x-y)|}{|x - s(x-y)|^{d+1}}\, \mathrm{d}y\, \mathrm{d}s, \end{aligned}$$

since $|x-y| > |x| - |y| > 3\epsilon - \delta \geqslant 3\epsilon - 2\epsilon = \epsilon$, and so the second integral vanishes. Since

$$|x - s(x-y)| \geqslant |x| - |x-y| \geqslant |x| - \epsilon > \frac{2}{3}|x|,$$

we have

$$\left| \rho_\epsilon^\nu * \Gamma \mathbb{1}_{B_\delta^c}(x) - (\Gamma \mathbb{1}_{B_{2\epsilon}^c})(x) \int_{\mathbb{R}^d} \rho^\nu(y)\, \mathrm{d}y \right| \leqslant \frac{3^{d+1} C_0 \epsilon}{2^{d+1} |x|^{d+1}}$$

and so
$$\left|\rho_\epsilon^\nu * \Gamma \mathbb{1}_{B_\delta^c}(x) - (\Gamma \mathbb{1}_{B_{2\epsilon}^c})(x) \int_{\mathbb{R}^d} \rho^\nu(y)\,\mathrm{d}y\right| \leqslant \frac{C_d(C_0+C_1)}{\epsilon^d\left(1+\left(\frac{|x|}{\epsilon}\right)^{d+1}\right)}.$$

By [76, Chapter III, Section 2.2, Theorem 2], we conclude that

$$\sup_\nu \sup_{\epsilon\geqslant\delta/2}\left|\left[\rho_\epsilon^\nu * \Gamma \mathbb{1}_{B_\delta^c} - \Gamma \mathbb{1}_{B_{2\epsilon}^c}\int_{\mathbb{R}^d}\rho^\nu(y)\,\mathrm{d}y\right] * g_t(x)\right| \leqslant C_d(C_0+C_1)Mg_t(x),$$

and so the above boundedness implies the desired L^1_w to L^1 boundedness.

It remains to study

$$\sup_\nu \sup_{0<\epsilon<\delta/2}\left|\left[\rho_\epsilon^\nu * \Gamma \mathbb{1}_{B_\delta^c} - \Gamma \mathbb{1}_{B_\delta^c}\int_{\mathbb{R}^d}\rho^\nu(y)\,\mathrm{d}y\right] * g_t(x)\right|. \tag{4.13}$$

We begin by writing it as

$$\begin{aligned}
&\sup_\nu \sup_{0<\epsilon<\delta/2}\left|\int_{\mathbb{R}^d}\left[\rho_\epsilon^\nu * \Gamma \mathbb{1}_{B_\delta^c}(z) - (\Gamma \mathbb{1}_{B_\delta^c})(z)\int_{\mathbb{R}^d}\rho^\nu(y)\,\mathrm{d}y\right] g_t(x-z)\,\mathrm{d}z\right| \\
&\quad\leqslant \sup_\nu \sup_{0<\epsilon<\delta/2}\left|\int_{B_{2\epsilon}}\left[\rho_\epsilon^\nu * \Gamma \mathbb{1}_{B_\delta^c}(z) - (\Gamma \mathbb{1}_{B_\delta^c})(z)\int_{\mathbb{R}^d}\rho^\nu(y)\,\mathrm{d}y\right] g_t(x-z)\,\mathrm{d}z\right| \\
&\quad\quad + \sup_\nu \sup_{0<\epsilon<\delta/2}\left|\int_{B_{2\epsilon}^c}\rho_\epsilon^\nu * \Gamma \mathbb{1}_{B_\delta^c}(z) - (\Gamma \mathbb{1}_{B_\delta^c})(z)\int_{\mathbb{R}^d}\rho^\nu(y)\,\mathrm{d}y\, g_t(x-z)\,\mathrm{d}z\right|
\end{aligned}$$

By (4.12), we have for all $z \in \mathbb{R}^d$ and $0<\epsilon<\delta/2$ that

$$\left|\rho_\epsilon^\nu * \Gamma \mathbb{1}_{B_\delta^c}(z) - \left(\Gamma \mathbb{1}_{B_\delta^c}\right)(z)\int_{\mathbb{R}^d}\rho^\nu(y)\,\mathrm{d}y\right| \leqslant \frac{C_1}{\epsilon^d} + \frac{C_0}{\delta^d} \leqslant \frac{C_1 + 2^d C_0}{\epsilon^d},$$

and so

$$\begin{aligned}
&\sup_\nu \sup_{0<\epsilon<\delta/2}\left|\int_{B_{2\epsilon}}\left[\rho_\epsilon^\nu * \Gamma \mathbb{1}_{B_\delta^c}(z) - (\Gamma \mathbb{1}_{B_\delta^c})(z)\int_{\mathbb{R}^d}\rho^\nu(y)\,\mathrm{d}y\right] g_t(x-z)\,\mathrm{d}z\right| \\
&\quad\leqslant C_d(C_1+C_0)\sup_{\epsilon>0}\epsilon^{-d}\int_{B_{2\epsilon}}|g_t(x-z)|\,\mathrm{d}z \leqslant C_d(C_1+C_0)Mg_t(x).
\end{aligned} \tag{4.14}$$

The second term can be written as

$$\begin{aligned}
&\sup_\nu \sup_{0<\epsilon<\delta/2}\left|\int_{B_{2\epsilon}^c}\rho_\epsilon^\nu * \Gamma \mathbb{1}_{B_\delta^c}(z) - (\Gamma \mathbb{1}_{B_\delta^c})(z)\int_{\mathbb{R}^d}\rho^\nu(y)\,\mathrm{d}y g_t(x-z)\,\mathrm{d}z\right| \\
&\quad= \sup_\nu \sup_{0<\epsilon<\delta/2}\left|\int_{\mathbb{R}^d}\int_{B_{2\epsilon}^c}\left[\Gamma \mathbb{1}_{B_\delta^c}(z-y) - (\Gamma \mathbb{1}_{B_\delta^c})(z)\right] g_t(x-z)\,\mathrm{d}z\,\rho_\epsilon^\nu(y)\,\mathrm{d}y\right|.
\end{aligned}$$

We may write

$$\begin{aligned}
B_{2\epsilon}^c = (\{|z|>2\epsilon\} \cap \{|z-y|\geqslant\delta\} \cap \{|z|\geqslant\delta\}) &\cup (\{|z|>2\epsilon\} \cap \{|z-y|<\delta\} \cap \{|z|\geqslant\delta\}) \\
&\cup (\{|z|>2\epsilon\} \cap \{|z-y|\geqslant\delta\} \cap \{|z|<\delta\}) \\
&=: A \cup B \cup C.
\end{aligned}$$

At set A, we have

$$\begin{aligned}\left[\Gamma\mathbb{1}_{B_\delta^c}(z-y)-(\Gamma\mathbb{1}_{B_\delta^c})(z)\right]=\Gamma(z-y)-\Gamma(z)&\leqslant|y|\int_0^1|\nabla\Gamma(z-sy)|\,\mathrm{d}s\\&\leqslant\frac{2^{d+1}|y|}{|z|^{d+1}}\\&\leqslant\frac{C_d}{\epsilon^d\left(1+\left(\frac{|z|}{\epsilon}\right)^{d+1}\right)}\end{aligned}$$

since $|y|<\epsilon$ and $|z-sy|\geqslant|z|-|y|\geqslant|z|-\epsilon>|z|/2$. Therefore, we have by [76, Chapter III, Section 2.2, Theorem 2] that

$$\sup_{\nu}\sup_{0<\epsilon<\delta/2}\left|\int_{\mathbb{R}^d}\int_A\left[\Gamma\mathbb{1}_{B_\delta^c}(z-y)-(\Gamma\mathbb{1}_{B_\delta^c})(z)\right]g_t(x-z)\,\mathrm{d}z\,\rho_\epsilon^\nu(y)\,\mathrm{d}y\right|\leqslant C_dC_0Mg_t(x). \tag{4.15}$$

At set B, we have

$$\begin{aligned}&\sup_{\nu}\sup_{0<\epsilon<\delta/2}\left|\int_{\mathbb{R}^d}\int_B\left[\Gamma\mathbb{1}_{B_\delta^c}(z-y)-(\Gamma\mathbb{1}_{B_\delta^c})(z)\right]g_t(x-z)\,\mathrm{d}z\,\rho_\epsilon^\nu(y)\,\mathrm{d}y\right|\\&\qquad=\sup_{\nu}\sup_{0<\epsilon<\delta/2}\left|\int_{\mathbb{R}^d}\int_B\Gamma(z)g_t(x-z)\,\mathrm{d}z\,\rho_\epsilon^\nu(y)\,\mathrm{d}y\right|\leqslant CC_0\int_B\frac{|g_t(x-z)|}{|z|^d}\,\mathrm{d}z.\end{aligned}$$

Now, notice that

$$B\subset\{\delta\leqslant|z|<2\delta\}.$$

Indeed, the lower bound is by the definition of the set and for the upper bound, we have

$$|z|\leqslant|z-y|+|y|<\delta+|y|\leqslant\delta+\epsilon\leqslant\delta+\frac{|z|}{2}.$$

Therefore, we have that

$$\begin{aligned}&\sup_{\nu}\sup_{0<\epsilon<\delta/2}\left|\int_{\mathbb{R}^d}\int_B\left[\Gamma\mathbb{1}_{B_\delta^c}(z-y)-(\Gamma\mathbb{1}_{B_\delta^c})(z)\right]g_t(x-z)\,\mathrm{d}z\rho_\epsilon^\nu(y)\,\mathrm{d}y\right|\\&\leqslant CC_0\int_{B_{2\delta}\setminus B_\delta}\frac{|g_t(x-z)|}{|z|^d}\,\mathrm{d}z.\end{aligned} \tag{4.16}$$

At set C, we proceed analogously, noticing that

$$C\subset\left\{\delta\leqslant|y-z|<\frac{3}{2}\delta\right\},$$

as the lower bound is trivial and the upper bound follows easily:

$$|y-z|\leqslant|y|+|z|<\epsilon+\delta<\delta/2+\delta.$$

Hence, we have by (4.11) that

$$
\begin{aligned}
&\sup_{\nu}\sup_{0<\epsilon<\delta/2}\left|\int_{\mathbb{R}^d}\int_{C}\left[\Gamma\mathbb{1}_{B_\delta^c}(z-y)-(\Gamma\mathbb{1}_{B_\delta^c})(z)\right]g_t(x-z)\,\mathrm{d}z\,\rho_\epsilon^\nu(y)\,\mathrm{d}y\right|\\
&\leqslant C\sup_{\nu}\sup_{0<\epsilon<\delta/2}\int_{\mathbb{R}^d}\int_{B_{3\delta/2}(y)\backslash B_\delta(y)}\frac{|g_t(x-z)|}{|y-z|^d}\,\mathrm{d}z\,|\rho_\epsilon^\nu(y)|\,\mathrm{d}y\\
&= C\sup_{\nu}\sup_{0<\epsilon<\delta/2}\int_{\mathbb{R}^d}\frac{\mathbb{1}_{B_{3\delta/2}\backslash B_\delta}}{|\cdot|^d}*|g_t|(x-y)|\rho_\epsilon^\nu(y)|\,\mathrm{d}y\\
&\leqslant C_dC_0M\left(\frac{\mathbb{1}_{B_{3\delta/2}\backslash B_\delta}}{|\cdot|^d}*|g_t|\right)(x).
\end{aligned}
\tag{4.17}
$$

Combining the estimates (4.14), (4.15), (4.16), and (4.17), we have the boundedness for (4.13):

$$
\begin{aligned}
\sup_{\nu}\sup_{0<\epsilon<\delta/2}\left|\left[\rho_\epsilon^\nu*\Gamma\mathbb{1}_{B_\delta^c}-\Gamma\mathbb{1}_{B_\delta^c}\int_{\mathbb{R}^d}\rho^\nu(y)\,\mathrm{d}y\right]*g_t(x)\right| \leqslant\, & C_d(C_1+C_0)Mg_t(x)\\
&+CC_0\frac{\mathbb{1}_{B_{2\delta}\backslash B_\delta}}{|\cdot|^d}*|g_t|(x)\\
&+C_dC_0M\left(\frac{\mathbb{1}_{B_{3\delta/2}\backslash B_\delta}}{|\cdot|^d}*|g_t|\right)(x).
\end{aligned}
$$

Therefore, it suffices to prove that the last two terms are in L^1_w. Notice that

$$
\left\|\frac{\mathbb{1}_{B_{2\delta}\backslash B_\delta}}{|\cdot|^d}*|g_t|+M\left(\frac{\mathbb{1}_{B_{3\delta/2}\backslash B_\delta}}{|\cdot|^d}*|g_t|\right)\right\|_{L^1_w(\mathbb{R}^d)}\leqslant\left\|\frac{\mathbb{1}_{B_{2\delta}\backslash B_\delta}}{|\cdot|^d}\right\|_{L^1(\mathbb{R}^d)}\|g_t\|_{L^1(\mathbb{R}^d)}\leqslant C_d\|g_t\|_{L^1(\mathbb{R}^d)},
$$

and so the proof is complete. □

Notice that the above is a direct extension of Theorem 2.3 for truncated kernels. Moreover, the result of Nguyen Proposition 2.5 does not cover the case treated in Proposition 4.1, and so we have a new fundamental estimate for vector fields $\boldsymbol{b}$ whose derivative can be written as in Definition 4.2, following the same lines as in Section 2.2 and Section 2.3. More precisely, we have the following:

Proposition 4.2. *Let $\boldsymbol{b}$ and $\bar{\boldsymbol{b}}$ be vector fields satisfying (1.6), with $\boldsymbol{b}$ with regularity*

$$
\begin{aligned}
&\boldsymbol{b}\in L^p_{\mathrm{loc}}((0,T)\times\mathbb{R}^d;\mathbb{R}^d)\quad\textit{for some } p>1;\\
&\partial_j\boldsymbol{b}_t^i=\sum_{k=1}^m\Gamma^{ijk}\mathbb{1}_{\mathbb{R}^d\backslash B_{\delta_{ijk}}}*g_t^{ijk}\qquad\textit{in the weak sense,}
\end{aligned}
$$

where δ_{ijk} are constants, Γ^{ijk} as in Definition 4.2, and $g^{ijk}\in L^1((0,T)\times\mathbb{R}^d)$. Moreover, let $\boldsymbol{X}$, $\bar{\boldsymbol{X}}$ renormalized regular Lagrangian flows with respect to $\boldsymbol{b}$ and $\bar{\boldsymbol{b}}$ starting at time s with compressibility constants L and $\bar{L}$, respectively. Then for every $\gamma>0$, $\eta>0$, and $r>0$, there exists $\lambda>0$ and a constant $C_{\gamma,\eta,r}>0$ such that

$$
|B_r\cap\{|\boldsymbol{X}(t,s,\cdot)-\bar{\boldsymbol{X}}(t,s,\cdot)|>\gamma\}|\leqslant C_{\gamma,\eta,r}\|\boldsymbol{b}-\bar{\boldsymbol{b}}\|_{L^1((0,T)\times B_\lambda)}+\eta
$$

uniformly in $s \in [0, T]$ and $t \in [s, T]$. The constant $C_{\gamma,\eta,r}$ depends on its subscripts, as well as the compressibility constants L and $\bar{L}$, on the norms (1.6) of $\boldsymbol{b}$ and $\bar{\boldsymbol{b}}$, on $\|\boldsymbol{b}\|_{L^p((s,t)\times B_\lambda)}$ for any $t \in (s, T)$, on $\|g^{ijk}\|_{L^1((0,T)\times\mathbb{R}^d)}$, and on the constants at Definition 4.2.

We now estimate each term in the derivative of the vector field computed in Lemma 4.2.

Proposition 4.3. *Let, for each $i = 1, \dots m$, $g^i \in L^1((0,T)\ L^p(\mathbb{R}^d))$ such that $\partial_t g^i \in L^1((0,T); L^q(\mathbb{R}^d))$ and $g^i_0 \in L^r(\mathbb{R}^d)$, where $1 \leqslant p < \infty$ and $1 \leqslant q, r \leqslant \infty$, and K^i as in Condition 4.1 and Condition 4.2. Then, for any $R > 0$, there exist constants $C_{d,R,r}$ and $C_{d,T,R,q}$ depending only on the dimension quantities specified as subscripts, such that the following holds for a.e. $t > 0$:*

$$\begin{aligned}\left\|\int_{\partial B_t} \omega_j(y) K^i_l(y) g^i_0(x-y)\,\mathrm{d}S_y\right\|_{L^1(B_R)} &\leqslant C_{d,R,r}\|g^i_0\|_{L^r(\mathbb{R}^d)};\\ \left\|(\omega_j K_l) \star \partial_t g^i\right\|_{L^1((0,T)\times B_R)} &\leqslant C_{d,T,R,q}\|\partial_t g^i\|_{L^1((0,T);L^q(\mathbb{R}^d))};\\ \int_0^T \left\|[(\partial_j K^i_l) \star g^i_t - (\partial_j K^i_l \mathbb{1}_{B_t}) * g^i_t]\right\|_{L^1(B_R)}\,\mathrm{d}t &\leqslant C_{d,T,R,q}\|\partial_t g^i\|_{L^1((0,T);L^q(\mathbb{R}^d))}\end{aligned}$$

Moreover, by further assuming that ρ^ν satisfies Definition 4.1, there exists a constant $C_{d,p,R}$ depending only on d, R, p such that

$$\begin{cases}\|M_{\rho^\nu}(\partial_j K_l \mathbb{1}_{B_t}) * g_t\|_{L^1(B_R)} \leqslant C_{d,R,p} C_0 \|g^i_t\|_{L^p(\mathbb{R}^d)} & \text{if } p > 1;\\ \left\|M_{\rho^\nu}(\partial_j K^i_l \mathbb{1}_{B_t}) * g^i_t\right\|_{L^1_w(\mathbb{R}^d)} \leqslant C_{d,R}\,(C_0 + C_1)\,\|g^i_t\|_{L^1(\mathbb{R}^d)} & \text{if } p = 1.\end{cases}$$

Proof. We shall omit the index i, for the sake of clarity. Notice that

$$\begin{aligned}\left\|\int_{\partial B_t} \omega_l(y) K^i_j(y) g^i_0(x-y)\,\mathrm{d}S_y\right\|_{L^1(B_R)} &\leqslant C_d \int_{\partial B_1} \|g_0(x - t\omega)\|_{L^r(B_R)}\,\mathrm{d}S_\omega\\ &\leqslant C_{d,R,r}\|g_0\|_{L^r(\mathbb{R}^d)},\end{aligned}$$

and so integrating in time gives the first estimate. The second estimate follows from Lemma 4.1. For the third inequality, denoting $\Gamma_{lj} = \partial_l K_j = |x|^{-d}\Omega_{lj}(x/|x|)$, we may write

$$\Gamma_{lj} \star g_t(x) - (\Gamma_{lj}\mathbb{1}_{B_t}) * g_t(x) = \int_{B_t} \Gamma_{lj}(y)(g_{t-|y|}(x-y) - g_t(x-y))\,\mathrm{d}y.$$

Now, the integral can be written as

$$-\int_{B_t}\int_0^1 \Gamma_{lj}(y)\partial_t g_{t-s|y|}(x-y)|y|\,\mathrm{d}s\,\mathrm{d}y = -\int_{B_t}\int_0^1 |y|^{-d+1}\Omega_{lj}(y/|y|)\partial_t g_{t-s|y|}(x-y)\,\mathrm{d}s\,\mathrm{d}y.$$

Extending $g_t(x) = 0$ for $t < 0$ for all $x \in \mathbb{R}^d$, we may reproduce the proof of Lemma 4.1, and so the third estimate follows. The fourth and fifth estimate follow from Proposition 4.1 and Theorem 2.3, since we may write

$$(\partial_l K^i_j \mathbb{1}_{B_t}) * g^i_t = \partial_l K^i_j * g^i_t - (\partial_l K^i_j \mathbb{1}_{B^c_t}) * g^i_t. \qquad \square$$

We now adapt the notion of Lagrangian solutions of (1.1) found in (1.3) in the context of maximal regular flows, as in Chapter 3:

Definition 4.3 (Lagrangian solution)**.** For vector fields $\boldsymbol{b}$ with a well defined maximal regular flow $\boldsymbol{X}$, we say that $f_t \in L^1(\mathbb{R}^d)$ is a Lagrangian solution of the transport equation with vector field $\boldsymbol{b}$ and initial data f_0 if

$$f_t\mathcal{L}^d = \boldsymbol{X}(t,0,\cdot)_{\#}f_0 \exp\left(\int_0^t \operatorname{div} \boldsymbol{b}_\tau(\boldsymbol{X}(\tau,0,\cdot))\,\mathrm{d}\tau\right)\mathcal{L}^d \llcorner \{x \in \mathbb{R}^d : T^+_{0,\boldsymbol{X}}(x) > t\}.$$

We are now ready to state the main result of this section and the first original theorem of this thesis.

Theorem 4.1 (Lagrangian flow and renormalized solution)**.** *Let $T > 0$ and $\boldsymbol{b}$ be a vector field satisfying $\operatorname{div}\boldsymbol{b}_t \geqslant \alpha(t)$, for $\alpha \in L^1((0,T))$ and (4.8) with kernels K^i satisfying Condition 4.1 and Condition 4.2 and $g^i \in L^1((0,T);L^p(\mathbb{R}^d))$ with further regularity $\partial_t g^i \in L^1((0,T);L^q(\mathbb{R}^d))$ and $g^i_0 \in L^r(\mathbb{R}^d)$ for $1 \leqslant q,r \leqslant \infty$, and $1 \leqslant p < \infty$. Then*

1. *there exists a maximal regular flow $\boldsymbol{X}$ associated to $\boldsymbol{b}$ (see Definition 3.2) starting from 0;*
2. *if $\boldsymbol{b}f \in L^1((0,T);L^\infty(\mathbb{R}^d))$, the Lagrangian solution, i.e., the transport of f_0 by $\boldsymbol{X}$ (see Definition 4.3) is a renormalized solution of (1.1) with initial data f_0; if we further assume that $\boldsymbol{b}f \in L^1((0,T);L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d))$, then the Lagrangian solution is a distributional solution of (1.1);*
3. *assuming a divergence-free vector field $\boldsymbol{b}$, if a nonnegative function f is weakly continuous in $[0,T]$ in duality with $C_c(\mathbb{R}^d)$ and it is either a renormalized or distributional solution of (1.1), then f is a Lagrangian solution.*

Proof. Notice that by our assumptions and Lemma 4.1, we only need to prove that property (H2) holds for such vector fields, and so existence, uniqueness and semigroup property for the flow will follow.

We split the proof in three steps: we begin by proving that property (H2) holds for such vector fields, and so existence, uniqueness and semigroup property for the flow will follow; in step two, we prove that Lagrangian solutions are renormalized ones; finally, in step three we show that renormalized/distributional solutions are Lagrangian.

Step 1. By Theorems 3.9 and 3.10, existence, uniqueness, and semigroup property of maximal regular flow follow once we prove that (H2) holds, and so (1) is ensured. For this purpose, we follow the same strategy as [6, 21] presented in Chapter 3. We begin by defining $\mathcal{P}(X)$ as the set of probability measures on $X = C([0,T];B_R)$, being $R > 0$ arbitrary and the evaluation map e_t, $t \in [0,T]$ of curves $\gamma \in X$, i.e. $e_t(\gamma) := \gamma(t)$.

Using the same argument as in [6], by extended superposition principle [6, Theorem 5.1]), it is sufficient to show that a measure $\boldsymbol{\eta} \in \mathcal{P}(X)$ satisfying $(e_t)_\# \boldsymbol{\eta} \leqslant \tilde{C}\mathcal{L}^d$ for all $t \in [0,T]$ concentrated on integral curves $\boldsymbol{X}(\cdot, 0, x)$ of $\boldsymbol{b}$ has Dirac delta disintegration $\boldsymbol{\eta}_x$ with respect to e_0. More precisely, $\boldsymbol{\eta}_x = \delta_x$ for $(e_0)_\#\boldsymbol{\eta}$−a.e. $x \in B_R$.

For this purpose, we consider an "local" adaptation of the function Φ_δ in Proposition 2.1 presented in Chapter 3:

$$\Phi_\delta(t) := \iiint \log\left(1 + \frac{|\gamma(t) - \eta(t)|}{\delta}\right) \mathrm{d}\mu(\eta, \gamma, z),$$

where $\mathrm{d}\mu(\eta,\gamma,z) := \mathrm{d}\boldsymbol{\eta}_z(\gamma)\mathrm{d}\boldsymbol{\eta}_z(\eta)\mathrm{d}(e_0)_\#\boldsymbol{\eta}(z)$, $\delta \in (0,1)$ is arbitrary and $t \in [0,T]$; note that $\mu \in \mathcal{P}(C([0,T];B_R)^2 \times B_R)$ and $\Phi_\delta(0) = 0$, as curves begin at the same point. Now, assuming by contradiction that $\boldsymbol{\eta}_x$ is not a Dirac delta for $(e_0)_\#\boldsymbol{\eta}$−a.e. x, we may computed a lower bound to Φ_δ as in (3.5), and so

$$\Phi_\delta(t_0) \geqslant \frac{a}{2T}\log\left(1 + \frac{a}{2\delta T}\right). \tag{4.18}$$

Computing the time derivative of Φ_δ, we have that

$$\frac{\mathrm{d}\Phi_\delta}{\mathrm{d}t}(t) \leqslant \iiint \frac{|\boldsymbol{b}_t(\gamma(t)) - \boldsymbol{b}_t(\eta(t))|}{\delta + |\gamma(t) - \eta(t)|}\,\mathrm{d}\mu(\eta,\gamma,z) \leqslant \iiint h_t(\eta,\gamma)\,\mathrm{d}\mu(\eta,\gamma,z), \tag{4.19}$$

where by Lemma 2.2 and Proposition 4.3, we have

$$h_t(\eta,\gamma) := \min\left\{\sum_{j=1}^d \left(M_{\rho_0^{\nu,j}}\partial_j \boldsymbol{b}_t(\gamma(t)) + M_{\rho_0^{\nu,j}}\partial_j \boldsymbol{b}_t(\eta(t))\right), \frac{|\boldsymbol{b}_t(\gamma(t))| + |\boldsymbol{b}_t(\eta(t))|}{\delta}\right\}.$$

Now, notice that

$$h_t(\eta,\gamma) \leqslant h_t^1(\eta,\gamma) + h_t^2(\eta,\gamma) + h_t^3(\eta,\gamma),$$

where

$$\begin{aligned} h_t^1(\eta,\gamma) := \min\Bigg\{ & \sum_{j,l=1}^d \sum_{i=1}^m \Bigg(M_{\rho_0^{\nu,j}} \int_{\partial B_t} \omega_j(y) K_l^i(y) g_0^i(\gamma(t) - y)\,\mathrm{d}S_y \\ & + M_{\rho_0^{\nu,j}} \int_{\partial B_t} \omega_j(y) K_l^i(y) g_0^i(\eta(t) - y)\,\mathrm{d}S_y\Bigg), \frac{|\boldsymbol{b}_t(\gamma(t))| + |\boldsymbol{b}_t(\eta(t))|}{\delta}\Bigg\}; \end{aligned}$$

$$\begin{aligned} h_t^2(\eta,\gamma) := \min\Bigg\{ & \sum_{j=1}^d \sum_{i=1}^m \Bigg(M_{\rho_0^{\nu,j}}(\omega_j K_l) \star \partial_t g_t^i(\gamma(t)) \\ & + M_{\rho_0^{\nu,j}}[(\partial_j K_l^i) \star g_t^i - (\partial_j K_l^i \mathbb{1}_{B_t}) * g_t^i](\gamma(t)) \\ & + M_{\rho_0^{\nu,j}}[(\partial_j K_l^i) \star g_t^i - (\partial_j K_l^i \mathbb{1}_{B_t}) * g_t^i](\eta(t)) \\ & + M_{\rho_0^{\nu,j}}(\omega_j K_l) \star \partial_t g_t^i(\eta(t))\Bigg), \frac{|\boldsymbol{b}_t(\gamma(t))| + |\boldsymbol{b}_t(\eta(t))|}{\delta}\Bigg\}; \end{aligned}$$

$$h_t^3(\eta,\gamma) := \min\Bigg\{ \sum_{j=1}^{d}\sum_{i=1}^{m} \Big(M_{\rho_0^{\nu,j}}(\partial_j K_l^i \mathbb{1}_{B_t}) * g_t^i(\gamma(t)) + M_{\rho_0^{\nu,j}}(\partial_j K_l^i \mathbb{1}_{B_t}) * g_t^i(\eta(t))\Big), \frac{|\boldsymbol{b}_t(\gamma(t))| + |\boldsymbol{b}_t(\eta(t))|}{\delta} \Bigg\}.$$

With the above definitions, we have that

$$\Phi_\delta(t_0) \leqslant \int_0^{t_0} \iiint h_t^1(\eta,\gamma) + h_t^2(\eta,\gamma) + h_t^3(\eta,\gamma)\, \mathrm{d}\mu(\eta,\gamma,z)\,\mathrm{d}t.$$

Now, firstly notice that if $r > 1$, the condition $(e_t)_{\#}\boldsymbol{\eta} \leqslant \tilde{C}\mathcal{L}^d\ \boldsymbol{\eta}$ and (4.11) imply that

$$\begin{aligned}\|h^1\|_{L^1((0,T)\times\mu)} &\leqslant 2\sum_{j,l=1}^{d}\sum_{i=1}^{m}\int_0^T \left\| M_{\rho_0^{\nu,j}} \int_{\partial B_t} \omega_j(y) K_l^i(y) g_0^i(\cdot - y)\mathrm{d}S_y \right\|_{L^1(B_R)} \mathrm{d}t \\ &\leqslant \tilde{C} C_{d,T,R,r} \sum_{i=1}^{m} \|g_0^i\|_{L^r(\mathbb{R}^d)},\end{aligned}$$

and analogously if $q > 1$,

$$\begin{aligned}\|h^2\|_{L^1((0,T)\times\mu)} \leqslant& 2\sum_{j,l=1}^{d}\sum_{i=1}^{m} + \|M_{\rho_0^{\nu,j}}(\omega_j K_l) \star \partial_t g^i\|_{L^1((0,T)\times B_R)} \\ &+ \int_0^T \|M_{\rho_0^{\nu,j}}[(\partial_j K_l^i)\star g_t^i - (\partial_j K_l^i \mathbb{1}_{B_t}) * g_t^i]\|_{L^1(B_R)}\,\mathrm{d}t \\ \leqslant& \tilde{C} C_{d,T,R,q} \sum_{i=1}^{m} \|\partial_t g^i\|_{L^1((0,T);L^q(\mathbb{R}^d))},\end{aligned}$$

and if $p > 1$,

$$\begin{aligned}\|h^3\|_{L^1((0,T)\times\mu)} &\leqslant 2\sum_{j,l=1}^{d}\sum_{i=1}^{m}\int_0^T \|(\partial_j K_l^i \mathbb{1}_{B_t}) * g_t^i\|_{L^1(B_R)}\,\mathrm{d}t \\ &\leqslant \tilde{C} C_{d,T,R,p} \sum_{i=1}^{m} \|g^i\|_{L^1((0,T);L^p(\mathbb{R}^d))},\end{aligned}$$

Therefore, we shall study separately when each exponent p, q, r equals 1.

Case p=1: Notice that by Proposition 4.3, we only need to consider the integral of h^3. Recalling Lemma 2.10, for fixed $\epsilon > 0$, we may write

$$g^i = g^{i,1} + g^{i,2},$$

where

$$\|g^{i,1}\|_{L^1((0,T)\times\mathbb{R}^d)} \leqslant \epsilon, \quad \|g^{i,2}\|_{L^2((0,T)\times\mathbb{R}^d)} \leqslant C_\epsilon,$$

we may estimate h^3 as $h^3 \leqslant h^{3,1} + h^{3,2}$, where for $k = \{1,2\}$, we define

$$h_t^{3,k}(\eta,\gamma) := \min\Bigg\{ \sum_{j=1}^{d}\sum_{i=1}^{m} \Big(M_{\rho_0^{\nu,j}}(\partial_j K_l^i \mathbb{1}_{B_t}) * g_t^{i,k}(\gamma(t)) + M_{\rho_0^{\nu,j}}(\partial_j K_l^i \mathbb{1}_{B_t}) * g_t^{i,k}(\eta(t))\Big), \frac{|\boldsymbol{b}_t(\gamma(t))| + |\boldsymbol{b}_t(\eta(t))|}{\delta} \Bigg\}.$$

By Proposition 4.3 and the condition $(e_t)_{\#}\boldsymbol{\eta} \leqslant \tilde{C}\mathcal{L}^d$, we have that

$$\|h^{3,2}\|_{L^1((0,T)\times\mu)} \leqslant \tilde{C}C_{d,T}\sum_{i=1}^{m}\|g^i\|_{L^2((0,T)\times\mathbb{R}^d)} \leqslant C_\epsilon.$$

On the other hand, by combining (4.8), the interpolation estimate in Lemma 2.9, the condition $(e_t)_{\#}\boldsymbol{\eta} \leqslant \tilde{C}\mathcal{L}^d$, and Proposition 4.3, we have

$$\|h^{3,1}\|_{L^1((0,T)\times\mu)} \leqslant C\sum_{i=1}^{m}\|g^{i,1}\|_{L^1((0,T)\times\mathbb{R}^d)}\log\left(\frac{C}{\delta}\right) \leqslant \epsilon\log(\delta^{-1});$$

details on the above estimate can be found in the proof of [6, Theorem 4.4]. Hence, we have that

$$\|h^3\|_{L^1((0,T)\times\mu)} \leqslant C_\epsilon + \epsilon\log(\delta^{-1}).$$

Cases q=1 and r=1: Proceeding analogously as the previous case, and so for a fixed $\epsilon > 0$, we write $\partial_t g^i$ as

$$\partial_t g^i = f^{i,1} + f^{i,2},$$

where

$$\|f^{i,1}\|_{L^1((0,T)\times\mathbb{R}^d)} \leqslant \epsilon, \quad \|f^{i,2}\|_{L^2((0,T)\times\mathbb{R}^d)} \leqslant C_\epsilon,$$

and also

$$g_0^i = g_0^{i,1} + g_0^{i,2},$$

where

$$\|g_0^{i,1}\|_{L^1(\mathbb{R}^d)} \leqslant \epsilon, \quad \|g_0^{i,2}\|_{L^2((\mathbb{R}^d)} \leqslant C_\epsilon,$$

and so *mutatis mutandis*, we have

$$\|h^1\|_{L^1((0,T)\times\mu)} + \|h^2\|_{L^1((0,T)\times\mu)} \leqslant C_\epsilon + \epsilon\log(\delta^{-1}).$$

Hence, we have that

$$\|h\|_{L^1((0,T)\times\mu)} \leqslant C_\epsilon + \epsilon\log(\delta^{-1}).$$

Therefore, we have by (4.18) and the above estimate of h that

$$\frac{a}{2T}\log\left(1 + \frac{a}{2\delta T}\right) \leqslant C_{\epsilon,T,R} + \epsilon\log\left(CC_R(\epsilon\delta)^{-1}\right) \leqslant C_{\epsilon,T,R} + \epsilon\log(\delta^{-1}).$$

Taking $\epsilon < a/2T$, we have a contradiction by letting $\delta \to 0^+$, and so (H2) holds.

Step 2. For the statement (2) in Theorem 4.1, we extend [6, Theorem 4.10] for vector fields (4.8) with bounded divergence in space and integrable in time. We recall Definition 4.3, which stated that

$$f_t\mathcal{L}^d = \boldsymbol{X}(t,0,\cdot)_{\#}f_0\exp\left(\int_0^t \operatorname{div}\boldsymbol{b}_\tau(\boldsymbol{X}(\tau,0,\cdot))\,\mathrm{d}\tau\right)\mathcal{L}^d \llcorner \{x\in\mathbb{R}^d : T^+_{0,\boldsymbol{X}}(x) > t\},$$

and notice that for all $\varphi \in C_c^1(\mathbb{R}^d)$ and $t \in [0,T)$, we have the following equalities in the set $A_t := \{x \in \mathbb{R}^d : T_{0,\boldsymbol{X}}^+(x) > t\}$:

$$\begin{aligned}\int_{A_t} \varphi(\boldsymbol{X}(t,0,x)) f_t(\boldsymbol{X}(t,0,x))\,\mathrm{d}x &= \int_{\boldsymbol{X}(t,0,\cdot)A_t} \varphi(x) f_t(x) \\ &\quad \times \exp\left(-\int_0^t \operatorname{div} \boldsymbol{b}_\tau(\boldsymbol{X}(\tau,t,\cdot))\,\mathrm{d}\tau\right)\mathrm{d}x \\ &= \int_{A_t} \varphi(\boldsymbol{X}(t,0,x)) f_0(x)\,\mathrm{d}x.\end{aligned}$$

In the first equality above, we performed the change of variables $x \mapsto \boldsymbol{X}(t,0,x)$ and noticed that the jacobian determinant, which is given by the exponential in the second integral above is well-defined by the hypothesis on the divergence of $\boldsymbol{b}$ stated in (2) and the fact that by Lemma 4.1, we have that $\boldsymbol{b} \in L^1((0,T);L^p(\mathbb{R}^d)) \subset L^1((0,T);L^1(\mathbb{R}^d)+L^\infty(\mathbb{R}^d))$; see [51, Proposition 6.9], and so one can apply Proposition 2.2. In the second equality we used the definition of Lagrangian solution. Therefore

$$f_t(\boldsymbol{X}(t,0,x)) = f_0(x) \quad \text{for a.e. } x \in A_t.$$

Then we have that for $\beta \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and so we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi(x)\beta(f_t(x))\,\mathrm{d}x &= \int_{\boldsymbol{X}(t,0,\cdot)A_t} \varphi(x)\beta(f_t(x))\,\mathrm{d}x \\ &= \int_{A_t} \varphi(\boldsymbol{X}(t,0,x))\beta(f_0(x)) \exp\left(\int_0^t \operatorname{div} \boldsymbol{b}_\tau(\boldsymbol{X}(\tau,0,x))\,\mathrm{d}\tau\right)\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \varphi(\boldsymbol{X}(t,0,x))\beta(f_0(x)) \exp\left(\int_0^t \operatorname{div} \boldsymbol{b}_\tau(\boldsymbol{X}(\tau,0,x))\,\mathrm{d}\tau\right)\mathrm{d}x,\end{aligned}$$

since by *(iii)* of Definition 3.2, we have that $\varphi(\boldsymbol{X}(t,0,x))$ continuously vanishes at $\{x \in \mathbb{R}^d : T_{0,\boldsymbol{X}}^+(x) \leqslant t\} = A_t^c$ if the maximal time does not reach T. Furthermore, we have that (see [6, Equation 4.31]) for a.e. $t \in [0,T]$

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(\boldsymbol{X}(t,0,x)) = \mathbb{1}_{[0,T_{0,\boldsymbol{X}}^+(x))}\boldsymbol{b}_t(\boldsymbol{X}(t,0,x)) \cdot \nabla\varphi(\boldsymbol{X}(t,0,x)),$$

and so by denoting the above jacobian determinant as $J_t(x)$, we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \varphi(x)\beta(f_t(x))\,\mathrm{d}x &= \int_{A_t} \boldsymbol{b}_t(\boldsymbol{X}(t,0,x)) \cdot \nabla\varphi(\boldsymbol{X}(t,0,x))\beta(f_0(x))J_t(x)\,\mathrm{d}x \\ &\quad + \int_{A_t} \varphi(\boldsymbol{X}(t,0,x)) \operatorname{div} \boldsymbol{b}_t(\boldsymbol{X}(t,0,x))\beta(f_0(x))J_t(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} [\boldsymbol{b}_t(x) \cdot \nabla\varphi(x) + \varphi(x) \operatorname{div} \boldsymbol{b}_t(x)]\beta(f_t(x))\,\mathrm{d}x.\end{aligned}$$

Since this is equivalent to stating that $\beta \circ f$ is a distributional solution of the transport equation with vector field $\boldsymbol{b}$ (see [11, Section 8.1]), we are done. For the case $\boldsymbol{b}f \in L^1((0,T);L^1_{\mathrm{loc}})$, we proceed analogously, without taking the composition of f with β.

Step 3. Finally, the statement (3) in Theorem 4.1 follow from [6, Theorem 5.1]. More precisely, it follows by Theorem 3.6. Moreover, if f is a bounded distributional solution, then it is a Lagrangian solution, and so by the previous step, also a renormalized solution.

This concludes the proof of Theorem 4.1. □

The global in time well-posedness of the maximal regular flow is ensured by Proposition 3.2 for vector fields $\boldsymbol{b}$ with extra integrability assumption, namely, for all nonnegative $\beta \in L^\infty(\mathbb{R}) \cap C^1(\mathbb{R})$, it holds

$$\int_0^T \int_{\mathbb{R}^d} \frac{|\boldsymbol{b}_t(x)|\beta(f_t(x))}{(1+|x|)\log(2+|x|)} \,\mathrm{d}x\,\mathrm{d}t < \infty. \tag{4.20}$$

Notice that if $p \in [1, d/(d-1)]$ and $d \geqslant 2$, then by Lemma 4.1 we have that the vector field $\boldsymbol{b} \in L^1((0,T); L^p(\mathbb{R}^d; \mathbb{R}^d))$, and so (4.20) follows for Lagrangian solutions. Indeed, notice that by taking $\beta \equiv 1$, we have that

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} \frac{|\boldsymbol{b}_t(x)|\beta(f_t(x))}{(1+|x|)\log(2+|x|)} \,\mathrm{d}x\,\mathrm{d}t \leqslant &C_d \|\boldsymbol{b}\|_{L^1((0,T);L^p(\mathbb{R}^d))} \\ &\times \left(1 + \int_1^\infty \frac{\tau^{d-1}}{[(1+\tau)\log(2+\tau)]^q}\,\mathrm{d}\tau\right)^{\frac{1}{q}},\end{aligned}$$

where $q = p/(p-1) \geqslant d$, and so the integral on the right hand side is finite, and so by Proposition 3.2, we have that $\boldsymbol{X}(\cdot, 0, x) \in \mathrm{AC}((0,T); \mathbb{R}^d)$ for a.e. $x \in \mathbb{R}^d$.

We now prove an analogous result for $p \in (d/(d-1), \infty)$, provided that $f \in L^\infty((0,T); L^1(\mathbb{R}^d))$ and $d \geqslant 2$; such condition follows if one has that f is a Lagrangian (hence, renormalized by Theorem 4.1) solution of transport equation, provided that $\operatorname{div} \boldsymbol{b} \in L^1((0,T); L^\infty(\mathbb{R}^d))$. The proof is straightforward: let $\beta_n := \frac{2}{\pi} \arctan \circ \zeta_n \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, where $(\zeta_n)_{n\in\mathbb{N}}$ is a nonnegative monotonous sequence such that $\zeta_n \nearrow |\cdot|$. Then

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} \frac{|\boldsymbol{b}_t(x)|\beta_n(f_t(x))}{(1+|x|)\log(2+|x|)} \,\mathrm{d}x\,\mathrm{d}t &\leqslant \|\boldsymbol{b}\|_{L^1((0,T);L^p(\mathbb{R}^d))} \|\beta_n(f)\|_{L^\infty((0,T);L^q(\mathbb{R}^d))} \\ &\leqslant \|\boldsymbol{b}\|_{L^1((0,T);L^p(\mathbb{R}^d))} \|\zeta_n(f)\|^{1/q}_{L^\infty((0,T);L^1(\mathbb{R}^d))},\end{aligned}$$

where $q = p/(p-1)$ and we have used that $\left|\frac{2}{\pi}\arctan(x)\right|^q \leqslant \left|\frac{2}{\pi}\arctan(x)\right| \leqslant |x|$. By monotone convergence theorem, we have that

$$\int_0^T \int_{\mathbb{R}^d} \frac{|\boldsymbol{b}_t(x)|\beta_n(f_t(x))}{(1+|x|)\log(2+|x|)} \,\mathrm{d}x\,\mathrm{d}t \leqslant \|\boldsymbol{b}\|_{L^1((0,T);L^p(\mathbb{R}^d))} \|f\|^{1/q}_{L^\infty((0,T);L^1(\mathbb{R}^d))} < \infty.$$

Therefore, by Proposition 3.2, $\boldsymbol{X}(\cdot, 0, x) \in \mathrm{AC}((0,T); \mathbb{R}^d)$ for $\beta_n(f_0)\mathcal{L}^d$–a.e. $x \in \mathbb{R}^d$ and all $n \in \mathbb{N}$. Since the tangent function is a diffeomorphism, we have that it holds for $\zeta_n(f_0)\mathcal{L}^d$–a.e., and by monotone convergence theorem, we finally conclude that it holds for $|f_0|\mathcal{L}^d$–a.e. $x \in \mathbb{R}^d$. We remark that the result holds for the full range $p \in [1, \infty)$.

4.2 An application for Vlasov-Maxwell system

We now consider the case (4.1) with electromagnetic fields with structure (4.4) and (4.5). Recall that $\xi(v)$ is either equals v or $(1+|v|^2)^{-1/2}v$. We begin by verifying (H1):

Lemma 4.3. *[Property (H1) for Vlasov-Maxwell system] Let $\rho \in L^1((0,T)\times\mathbb{R}^3)$, $j \in W^{1,1}((0,T);L^1(\mathbb{R}^3))$, with $\rho_0 \in L^1(\mathbb{R}^3)$ and $E_0, H_0 \in W^{1,1}(\mathbb{R}^3;\mathbb{R}^3)$. Then the vector field (4.2) satisfies property (H1).*

Proof. Notice that since $\xi(v) \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^3_v)$, it is trivial to verify that the vector field (4.2) satisfies (H1). Moreover, since the electromagnetic field E, H does not depend on v, it suffices to show that they are in $L^1((0,T)\times\mathbb{R}^3)$. By the definition of the functionals E^0, H^0 and the homogeneous solutions of 3D wave equation (4.7), we have that

$$\begin{aligned}\|E^0\|_{L^1((0,T)\times\mathbb{R}^3)} + \|H^0\|_{L^1((0,T)\times\mathbb{R}^3)} \leqslant & C_T\big(\|E_0\|_{W^{1,1}(\mathbb{R}^3)} + \|H_0\|_{W^{1,1}(\mathbb{R}^3)} \\ & + \|\rho_0\|_{L^1(\mathbb{R}^3)} + \|j_0\|_{L^1(\mathbb{R}^3)}\big).\end{aligned}$$

The estimate for the non-homogeneous terms follow from Lemma 4.1. □

As we shall apply the same idea as in the proof of Theorem 4.1, we need to compute the spatial derivative of the electromagnetic fields given in (4.4) and (4.5). We begin by computing the derivative of the homogeneous terms:

Lemma 4.4. *Let $\rho_0 \in W^{1,1}(\mathbb{R}^3)$, $j_0 \in W^{1,1}(\mathbb{R}^3;\mathbb{R}^3)$, $E_0, H_0 \in W^{2,1}(\mathbb{R}^3;\mathbb{R}^3)$. Then there exists a constant $C_T > 0$ depending only on T such that*

$$\begin{aligned}\|DE^0\|_{L^1((0,T)\times\mathbb{R}^3)} + \|DH^0\|_{L^1((0,T)\times\mathbb{R}^3)} \leqslant C_T\big(&\|E_0\|_{W^{2,1}(\mathbb{R}^3)} + \|H_0\|_{W^{2,1}(\mathbb{R}^3)} \\ & + \|\rho_0\|_{W^{1,1}(\mathbb{R}^3)} + \|j_0\|_{W^{1,1}(\mathbb{R}^3)}\big).\end{aligned}$$

Proof. We begin by assuming $E_0, H_0 \in C^\infty_0(\mathbb{R}^3;\mathbb{R}^3) \cap W^{2,1}(\mathbb{R}^3;\mathbb{R}^3)$. Recall that by classical results of wave equation (see [71]), we have that E^H, H^H is smooth, and

$$\|DE^H\|_{L^1((0,T)\times\mathbb{R}^3)} + \|DH^H\|_{L^1((0,T)\times\mathbb{R}^3)} \leqslant C_T\left(\|E_0\|_{W^{2,1}(\mathbb{R}^3)} + \|H_0\|_{W^{2,1}(\mathbb{R}^3)}\right).$$

Moreover, assuming that $\rho_0 \in W^{1,1}(\mathbb{R}^3) \cap C^\infty_0(\mathbb{R}^3)$ and $j_0 \in W^{1,1}(\mathbb{R}^3;\mathbb{R}^3) \cap C^\infty_0(\mathbb{R}^3;\mathbb{R}^3)$, we have that

$$\begin{aligned}&\int_0^T \left\|\frac{1}{t}\int_{\partial B_t} D\rho_0(\cdot - y)\omega(y)\,\mathrm{d}S_y\right\|_{L^1(\mathbb{R}^3)} + \left\|\frac{1}{t}\int_{\partial B_t} Dj_0(\cdot - y)\times\omega(y)\,\mathrm{d}S_y\right\|_{L^1(\mathbb{R}^3)} \\ &+ \int_0^T \left\|\frac{1}{t}\int_{\partial B_t(x)} j_0(y)\cdot\omega(x-y)\omega(x-y)\,\mathrm{d}S_y\right\|_{L^1(\mathbb{R}^3)} \mathrm{d}t \\ &\leqslant C_T\left(\|D\rho_0\|_{L^1(\mathbb{R}^3)} + \|Dj_0\|_{L^1(\mathbb{R}^3)}\right).\end{aligned}$$

By a density argument, we have the result for the general case. □

We now compute the derivative of the first term of the electric and magnetic fields, namely

$$\int_{B_t(x)} \frac{\omega(x-y)}{|x-y|^2} \left[\rho_t(y)\right]_{ret}(x)\mathrm{d}y, \quad \int_{B_t(x)} \frac{\omega(x-y)}{|x-y|^2} \times \left[j_t(y)\right]_{ret}(x)\mathrm{d}y;$$

we will split the derivative computation of each term of the electromagnetic field for the sake of clarity.

Lemma 4.5. *Let K_i and Γ_{ij} denote the kernels*

$$K_i(x) := \frac{\omega_i(x)}{|x|^2}, \quad \Gamma_{ik}(x) := \partial_k K_i(x) = \frac{\delta_{ik} - 3\omega_i(x)\omega_k(x)}{|x|^3}.$$

Assume that $\rho_0 \in L^1(\mathbb{R}^3)$, $j_0 \in L^1(\mathbb{R}^3;\mathbb{R}^3)$, $\rho \in L^1((0,T)\times\mathbb{R}^3)$, $j \in W^{1,1}((0,T);L^1(\mathbb{R}^3))$, and that $\partial_t\rho + \operatorname{div} j = 0$ holds in the weak sense. Then, it holds in the weak sense

$$\begin{aligned}\partial_k(K_i \star \rho_t)(x) = &-\int_{\partial B_t} \omega_k(y)K_i(y)\rho_0(x-y) - \omega_k(y)K_i(y)\omega(y)\cdot j_0(x-y)\,\mathrm{d}S_y \\ &+ \Gamma_{ik}\star\rho_t(x) - \sum_{l=1}^{3}(\omega_k\Gamma_{il})\star j_t^l(x) + (K_i\partial_l\omega_k)\star j_t^l(x) + (\omega_k K_i\omega_l)\star\partial_t j_t^l(x),\end{aligned}$$

$$\partial_k(K_i \star j_t^l)(x) = -\int_{\partial B_t} \omega_k(y)K_i(y)j_0^l(x-y)\,\mathrm{d}S_y + \Gamma_{ik}\star j_t^l(x) + (\omega_k K_i)\star\partial_t j_t^l(x).$$

Proof. The second equality is a simple application of Lemma 4.2, since Γ_{ij} has average zero. Therefore, it suffices to prove the first equality. Furthermore, by density argument, we only need to consider smooth densities ρ and j. By Lemma 4.2, we have

$$\partial_k(K_i \star \rho_t)(x) = -\int_{\partial B_t} \omega_k(y)K_i(y)\rho_0(x-y)\,\mathrm{d}S_y + \Gamma_{ik}\star\rho_t(x) + (\omega_k K_i)\star\partial_t\rho_t(x).$$

Now, since it holds the continuity equation $\partial_t\rho + \operatorname{div} j = 0$, then

$$(\omega_k K_i)\star\partial_t\rho_t(x) = -(\omega_k K_i)\star\operatorname{div} j_t(x) = -\int_{B_t}\omega_k(y)K_i(y)[\operatorname{div} j_t(y)]_{ret}(x-y)\,\mathrm{d}y.$$

Using the commuting relation of the retarded brackets and derivatives (4.10), we have

$$\begin{aligned}(\omega_k K_i)\star\partial_t\rho_t(x) &= \int_{B_t}\omega_k(y)K_i(y)\operatorname{div}[j_t(y)]_{ret}(x-y)\,\mathrm{d}y - \sum_{l=1}^{3}(\omega_k K_i\omega_l)\star\partial_t j_t^l(x) \\ &= \int_{\partial B_t}\omega_k(y)K_i(y)\omega(y)\cdot j_0(x-y)\,\mathrm{d}S_y + \sum_{l=1}^{3} -(\omega_k\Gamma_{il})\star j_t^l(x) \\ &\quad - (K_i\partial_l\omega_k)\star j_t^l(x) - (\omega_k K_i\omega_l)\star\partial_t j_t^l(x).\end{aligned}$$

In order to conclude the lemma, one needs to verify that $\omega_k\Gamma_{il}$ and $K_i\partial_l\omega_k$ satisfy Definition 4.2. Since the first two properties are trivial, it suffices to compute the averages of $\omega_k(\delta_{ik} - 3\omega_i\omega_l)$ and $\omega_i\partial_l\omega_k = \omega_i(\delta_{lk} - \omega_l\omega_k)$ on the sphere. This is verified by the oddness of the kernel. □

We now compute the derivative the second and third terms of the electric field, namely

$$\int_{B_t(x)} \frac{\omega(x-y)}{|x-y|^2}\omega(x-y)\cdot [j_t(y)]_{ret}(x)\mathrm{d}y + \int_{B_t(x)} \frac{\omega(x-y)}{|x-y|^2} \times (\omega(x-y)\times [j_t(y)]_{ret}(x))\,\mathrm{d}y;$$

the main difference of them from the previous terms is the kernel, and so the proof is similar and we shall skip it. More precisely we have the following lemma and we will skip its proof since it is similar to previous one.

Lemma 4.6. *Let $\tilde{K}_{il}$ and $\tilde{\Gamma}_{ilk}$ denote the kernels*

$$\tilde{K}_{il}(x) := \frac{\omega_i(x)\omega_l(x)}{|x|^2}, \quad \tilde{\Gamma}_{ilk}(x) := \partial_k \tilde{K}_{il}(x) = \frac{\delta_{ik}\omega_l(x) + \delta_{lk}\omega_i(x) - 4\omega_i(x)\omega_l(x)\omega_k(x)}{|x|^3}.$$

Assume that $j_0 \in L^1(\mathbb{R}^3;\mathbb{R}^3)$ and $j \in W^{1,1}((0,T);L^1(\mathbb{R}^3))$. Then, it holds in the weak sense

$$\partial_k(K_{il} \star j_t^m)(x) = -\int_{\partial B_t} \omega_k(y)\tilde{K}_{il}(y) j_0^m(x-y)\,\mathrm{d}S_y + \tilde{\Gamma}_{ilk} \star j_t^m(x) + (\omega_k \tilde{K}_{il}) \star \partial_t j_t^m(x).$$

Finally, we compute the derivative of the radiation terms

$$\int_{B_t(x)} \frac{\omega(x-y)}{|x-y|} \times (\omega(x-y)\times [\partial_t j_t(y)]_{ret}(x))\,\mathrm{d}y, \quad \int_{B_t(x)} \frac{\omega(x-y)}{|x-y|} \times [\partial_t j_t(y)]_{ret}(x)\mathrm{d}y.$$

Of course, if $\nabla \partial_t j \in L^1((0,T)\times\mathbb{R}^3)$, there is nothing to prove, as the hyperbolic convolution and the derivative commutes in the second variable, that is, $\nabla(K \star g_t) = K \star \nabla g_t$, and so by assuming K as in Condition 4.1, we have that

$$\|K \star \nabla g\|_{L^1((0,T)\times\mathbb{R}^3)} \leqslant C_T \|\nabla g\|_{L^1((0,T)\times\mathbb{R}^3)}.$$

Notice that in this case the only important feature of the kernel is its decay, that is, it need not to be smooth and with zero average. Therefore, we only need to prove the case where we assume $\partial_t j \in W^{1,1}((0,T);L^1(\mathbb{R}^3;\mathbb{R}^3))$ and $\xi(v)(\xi(v)\cdot\nabla_x f_0) \in W^{1,1}(\mathbb{R}^6;\mathbb{R}^3))$. Notice that the latter assumption follows (as well as $j_0 \in W^{1,1}(\mathbb{R}^3;\mathbb{R}^3)$ and $\rho_0 \in W^{1,1}(\mathbb{R}^3))$) if one assumes that the initial distribution satisfies $\nabla_x f_0 \in L^1(\mathbb{R}^6)$ for the relativistic case, as $|\xi(v)| < 1$.

Lemma 4.7. *Let $\bar{K}_i|_{\mathbb{R}^d\backslash\{0\}} \in C^1(\mathbb{R}^3\backslash\{0\};\mathbb{R}^3)$ with $\bar{\Gamma}_{ik} := \partial_k \bar{K}_i$ be kernels satisfying Condition 4.1. Moreover, let $E_0, H_0 \in L^3(\mathbb{R}^3;\mathbb{R}^3)$, $\rho_0 \in L^{3/2}(\mathbb{R}^3)$, $j_0 \in L^{3/2}(\mathbb{R}^3;\mathbb{R}^3)$, $\xi(v)(\xi(v)\cdot\nabla_x f_0) \in L^1(\mathbb{R}^6;\mathbb{R}^3)$, and $j \in W^{2,1}((0,T);L^1(\mathbb{R}^3))$ and the compatibility relation at Theorem 4.2 be satisfied. Then it holds*

$$\partial_k(\bar{K}_i * \partial_t j_t^l)(x) = -\int_{\partial B_t} \omega_k(y)\bar{K}_i(y)\partial_t j_0^l(x-y)\,\mathrm{d}S_y + \bar{\Gamma}_{ik} \star \partial_t j_t^l(x) + (\bar{K}_i \omega_l) \star \partial_{tt} j_t^l(x). \quad (4.21)$$

Proof. We begin by claiming that $\partial_t j|_{t=0} \in L^1(\mathbb{R}^3;\mathbb{R}^3)$, where

$$\partial_t j_0^l = -\int_{\mathbb{R}^3} \xi(v)_l\,(\xi(v)\cdot\nabla_x f_0(\cdot,v))\,\mathrm{d}v + \int_{\mathbb{R}^3} (\partial_{v_k}\xi(v)_l)\,(E_0 + \xi(v)\times H_0)^k f_0(\cdot,v)\,\mathrm{d}v = 0.$$

The first term on the right hand side is integrable by our integrability assumption of f_0, hence it remains to show that the second one is also in L^1. In the relativistic case, notice that $\partial_{v_k}\xi(v)_l = (1+|v|^2)^{-1/2}\left(\delta_{kl} - \xi(v)_k\xi(v)_l\right)$, and so by Hölder inequality we have

$$\begin{aligned}\int_{\mathbb{R}^6} |\partial_{v_k}\xi(v)_l| \left|(E_0 + \xi(v)\times H_0)^k\right| f_0(\cdot,v)\,\mathrm{d}v\,\mathrm{d}x &\leqslant 2\int_{\mathbb{R}^6}(|E_0| + |H_0|) f_0(\cdot,v)\,\mathrm{d}v\,\mathrm{d}x \\ &\leqslant 2(\|E_0\|_{L^3(\mathbb{R}^3)} + \|H_0\|_{L^3(\mathbb{R}^3)})\|\rho_0\|_{L^{3/2}(\mathbb{R}^3)}.\end{aligned}$$

In the nonrelativistic case, we have that $\partial_{v_k}\xi(v)_l = \delta_{kl}$, and so

$$\int_{\mathbb{R}^3} (\partial_{v_k}\xi(v)_l)\left(E_0 + \xi(v)\times H_0\right)^k f_0(\cdot,v)\,\mathrm{d}v = (\rho_0 E_0 + j_0\times H_0)^l.$$

By Hölder inequality, the claim follows.

Therefore, we may repeat the proof of Lemma 4.2, but now since $\bar{\Gamma}_{ik}$ is less singular than a singular kernel of fundamental type at the origin, we can use Lemma 4.1 to conclude that the second term on the right hand side of (4.21) is an integrable function. By the above claim, the first term is clearly integrable and the integrability of the third follows by an analogous proof of Lemma 4.1. □

We remark that the integrability assumption of $\rho_0 \in L^{3/2}(\mathbb{R}^3)$, $j_0 \in L^{3/2}(\mathbb{R}^3;\mathbb{R}^3)$, $E_0, H_0 \in L^3(\mathbb{R}^3;\mathbb{R}^3)$ follows from the hypothesis $\rho_0 \in W^{1,1}(\mathbb{R}^3)$, $j_0 \in W^{1,1}(\mathbb{R}^3;\mathbb{R}^3)$, $E_0, H_0 \in W^{2,1}(\mathbb{R}^3;\mathbb{R}^3)$ at Theorem 4.2 by the Sobolev embedding.

We now prove that $\boldsymbol{b} \in L^1((0,T);L^p_{\mathrm{loc}}(\mathbb{R}^6;\mathbb{R}^6))$ for some $p>1$, where $\boldsymbol{b}$ as in (4.2). Since $\xi(v) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^3)$, it suffices to show that $E,\ H \in L^1((0,T);L^p_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^3))$.

Lemma 4.8 ($L^p_{\mathrm{loc}}-$estimate for the electromagnetic field)**.** *Let the densities $\rho \in L^1((0,T)\times\mathbb{R}^3)$, $j \in W^{1,1}((0,T);L^1(\mathbb{R}^3;\mathbb{R}^3)) \cap L^1((0,T);L^p_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^3))$ for some $p>1$, the continuity equation $\partial_t\rho + \operatorname{div} j = 0$ be satisfied in the distributional sense, $\rho_0 \in W^{1,1}(\mathbb{R}^3)$, $j_0 \in W^{1,1}(\mathbb{R}^3;\mathbb{R}^3)$, $E_0, H_0 \in W^{2,1}(\mathbb{R}^3;\mathbb{R}^3)$, and either $\partial_t j \in L^1((0,T);W^{1,1}(\mathbb{R}^3;\mathbb{R}^3))$ or $\partial_{tt} j \in L^1((0,T)\times\mathbb{R}^3;\mathbb{R}^3)$. Then it holds that*

$$E,\ H \in L^1((0,T);L^q_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^3)), \quad \textit{where } 1 < q < \min\left\{\frac{3}{2}, p\right\}.$$

Proof. By the proof of Lemma 4.3 and Lemma 4.4, we have that

$$E^0, H^0 \in L^1((0,T);L^{3/2}(\mathbb{R}^3;\mathbb{R}^3)).$$

Moreover, by Lemma 4.1 we have that for any $s \in (1,3/2)$, it holds

$$\begin{aligned}\int_0^T \Bigg\| \int_{B_t} \frac{\omega(y)}{|y|^2} \times \left[j_t(\cdot - y)\right]_{ret}(0)\mathrm{d}y\Bigg\|_{L^s(\mathbb{R}^3)} + \Bigg\|\int_{B_t} \frac{\omega(y)}{|y|^2}\omega(y)\cdot\left[j_t(\cdot-y)\right]_{ret}(0)\mathrm{d}y\Bigg\|_{L^s(\mathbb{R}^3)} \\ + \Bigg\|\int_{B_t}\frac{\omega(y)}{|y|^2}\times\left(\omega(y)\times\left[j_t(\cdot-y)\right]_{ret}(0)\right)\mathrm{d}y\Bigg\|_{L^s(\mathbb{R}^3)}\mathrm{d}t \leqslant C_T\|j\|_{W^{1,1}((0,T);L^1(\mathbb{R}^3))}.\end{aligned}$$

Furthermore, if $\partial_t j \in L^1((0,T); W^{1,1}(\mathbb{R}^3;\mathbb{R}^3))$ then Sobolev embedding gives that $\partial_t j \in L^1((0,T); L^{3/2}(\mathbb{R}^3;\mathbb{R}^3))$, and so Lemma 4.1 gives that

$$\begin{aligned} \int_0^T \Big\| \int_{B_t} \frac{\omega(y)}{|y|} \times (\omega(y) \times [\partial_t j_t(\cdot - y)]_{ret}(0))\, \mathrm{d}y \Big\|_{L^{3/2}(\mathbb{R}^3)} \\ + \Big\| \int_{B_t} \frac{\omega(y)}{|y|} \times [\partial_t j_t(\cdot - y)]_{ret}(0) \mathrm{d}y \Big\|_{L^{3/2}(\mathbb{R}^3)} \mathrm{d}t \leqslant C_T \|\partial_t j\|_{L^1((0,T);L^{3/2}(\mathbb{R}^3))}. \end{aligned} \tag{4.22}$$

If $\partial_{tt} j \in L^1((0,T)\times\mathbb{R}^3;\mathbb{R}^3)$, then it follows from Lemma 4.1 that for any $s \in (1, 3/2)$, it holds

$$\begin{aligned} \int_0^T \Big\| \int_{B_t} \frac{\omega(y)}{|y|} \times (\omega(y) \times [\partial_t j_t(\cdot - y)]_{ret}(0))\, \mathrm{d}y \Big\|_{L^{s}(\mathbb{R}^3)} \\ + \Big\| \int_{B_t} \frac{\omega(y)}{|y|} \times [\partial_t j_t(\cdot - y)]_{ret}(0) \mathrm{d}y \Big\|_{L^{s}(\mathbb{R}^3)} \mathrm{d}t \leqslant C_T \|j\|_{W^{2,1}((0,T);L^1(\mathbb{R}^3))}. \end{aligned} \tag{4.23}$$

Hence, it remains to estimate the term

$$\int_{B_t} \frac{\omega(y)}{|y|^2} [\rho_t(x-y)]_{ret}(0)\mathrm{d}y.$$

For this purpose, notice that

$$\begin{aligned} \int_{B_t} \frac{\omega_i(y)}{|y|^2} [\rho_t(x-y)]_{ret}(0)\mathrm{d}y &= \left(\frac{\omega_i}{|\cdot|^2}\mathbb{1}_{B_t}\right) * \rho_t(x) - \int_{B_t}\int_0^1 \frac{\omega_i(y)}{|y|} \partial_t \rho_{t-\tau|y|}(x-y)\,\mathrm{d}\tau\,\mathrm{d}y \\ &= \left(\frac{\omega_i}{|\cdot|^2}\mathbb{1}_{B_t}\right) * \rho_t(x) + \int_{B_t}\int_0^1 \frac{\omega_i(y)}{|y|} \operatorname{div} j_{t-\tau|y|}(x-y)\,\mathrm{d}\tau\,\mathrm{d}y \\ &= \left(\frac{\omega_i}{|\cdot|^2}\mathbb{1}_{B_t}\right) * \rho_t(x) \\ &\quad - \int_0^1 \int_{\partial B_t} \frac{\omega_i(y)}{|y|} [\omega(y)\cdot j_{t(1-\tau)}(x-y)]\,\mathrm{d}S_y\,\mathrm{d}\tau \\ &\quad + \int_0^1 \int_{B_t} \sum_{k=1}^3 \frac{\delta_{ik} - 2\omega_i(y)\omega_k(y)}{|y|^2} j^k_{t-\tau|y|}(x-y)\,\mathrm{d}y\,\mathrm{d}\tau \\ &\quad + \int_0^1 \int_{B_t} \tau \frac{\omega_i(y)}{|y|} \omega(y)\cdot \partial_t j_{t-\tau|y|}(x-y)\,\mathrm{d}y\,\mathrm{d}\tau. \end{aligned}$$

The first term is an $L^1((0,T); L^s(\mathbb{R}^3;\mathbb{R}^3))$ for any $s \in (1,3/2)$; the third is bounded as

$$\int_0^T \int_0^1 \Big\| \int_{B_t} \sum_{k=1}^3 \frac{\delta_{ik} - 2\omega_i(y)\omega_k(y)}{|y|^2} j^k_{t-\tau|y|}(\cdot - y)\,\mathrm{d}y \Big\|_{L^s(\mathbb{R}^3)} \mathrm{d}\tau\,\mathrm{d}t \leqslant C_T \|j\|_{W^{1,1}((0,T);L^1(\mathbb{R}^3))}.$$

The fourth term is bounded analogously as in (4.22) or (4.23). For the second one, notice that

$$\int_0^1 \int_{\partial B_t} \frac{\omega_i(y)}{|y|} [\omega(y)\cdot j_{t(1-\tau)}(x-y)]\,\mathrm{d}S_y\,\mathrm{d}\tau = \int_0^t \int_{\partial B_1} \omega_i(y)[\omega(y)\cdot j_\tau(x-ty)]\,\mathrm{d}S_y\,\mathrm{d}\tau,$$

and so for any $R > 0$, we have

$$\int_0^T \left\| \int_0^1 \int_{\partial B_t} \frac{\omega_i(y)}{|y|} [\omega(y) \cdot j_{t(1-\tau)}(\cdot - y)] \, \mathrm{d}S_y \, \mathrm{d}\tau \right\|_{L^p(B_R)} \mathrm{d}t \leqslant C_T \|j\|_{L^1((0,T);L^p(B_{R+T}))}.$$

Therefore, the lemma follows. □

We now state the main result of this section, and the second original theorem of this thesis.

Theorem 4.2 (Existence of flow and consistency)**.** *Let $T > 0$ and f_0 a nonnegative function. Moreover, assume that $j_0 \in W^{1,1}(\mathbb{R}^3;\mathbb{R}^3)$, $\rho_0 \in W^{1,1}(\mathbb{R}^3)$, E_0, $H_0 \in W^{2,1}(\mathbb{R}^3;\mathbb{R}^3)$ satisfying (4.3), and $\rho \in L^1((0,T)\times\mathbb{R}^3)$, $j \in W^{1,1}((0,T);L^1(\mathbb{R}^3;\mathbb{R}^3))$, $j \in L^1((0,T);L^p_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^3))$ for some $p > 1$, f is weakly continuous in $[0,T]$ in duality with $C_c(\mathbb{R}^6)$, and either*

- *$\partial_{tt} j \in L^1((0,T)\times\mathbb{R}^3;\mathbb{R}^3)$, $\xi(v)(\xi(v)\cdot\nabla_x f_0) \in L^1(\mathbb{R}^6;\mathbb{R}^3)$, and the compatibility condition*

$$\partial_t j^l\big|_{t=0} + \int_{\mathbb{R}^3} \xi(v)_l \left(\xi(v)\cdot\nabla_x f_0(\cdot,v)\right) \mathrm{d}v - \sum_{k=1}^{3} \int_{\mathbb{R}^3} (\partial_{v_k}\xi(v)_l)\left(E_0 + \xi(v)\times H_0\right)^k f_0(\cdot,v)\,\mathrm{d}v = 0; \quad \textit{or}$$

- *$\partial_t j \in L^1((0,T);W^{1,1}(\mathbb{R}^3;\mathbb{R}^3))$.*

Now assume that either $f \in L^\infty((0,T)\times\mathbb{R}^6)$ is a distributional solution or f is a renormalized solution of (4.1) and $\partial_t\rho + \operatorname{div} j = 0$ in the distributional sense.

Then, there exists a unique maximal regular flow $\boldsymbol{X}$ associated to the vector field (4.2) (see Definition 3.2) starting at 0 such that f is the transport of f_0 by $\boldsymbol{X}$ (see Definition 4.3). In particular, all the above notions of solutions are consistent for bounded (in phase space) distribution function f.

Before we prove Theorem 4.2, we shall make a few comments on its hypothesis.

Remark 25. The integrability hypothesis $\rho_0 \in W^{1,1}(\mathbb{R}^3)$, $j_0 \in W^{1,1}(\mathbb{R}^3;\mathbb{R}^3)$ and $E_0, H_0 \in W^{2,1}(\mathbb{R}^3;\mathbb{R}^3)$ is so that the functionals E^0, H^0 have the regularity $E^0_t, H^0_t \in \mathrm{BV}(\mathbb{R}^3;\mathbb{R}^3)$, which follows from classical wave equation results; see [71]. Indeed, such hypothesis implies that the homogeneous solution $E^H_t, H^H_t \in W^{1,1}(\mathbb{R}^3;\mathbb{R}^3)$, and so we conclude the result by the embedding $W^{1,1} \subset \mathrm{BV}$. The author did not find explicit results (possibly relaxing the integrability) for BV spaces, so it is worth pointing that our result holds for the space of BV homogeneous solutions.

Remark 26. Notice that the vector field $\boldsymbol{b}$ in Theorem 4.2 cannot be written as in Theorem 4.1, only the electromagnetic fields E and H. This is similar to the Vlasov-Poisson result as in Section 2.4, where they consider the vector field $\boldsymbol{b}_t(x,v) = (v, K * \rho_t(x))$. In order to obtain the existence and consistency of notions of solutions, the authors could not directly apply the results of Section 2.3, for only the second part of the vector field has derivative which can be written as convolution of singular kernel and a density. Moreover, in the case of Theorem 4.2, one has the second part $E + \xi(v) \times H$, which its cannot be written as "hyperbolic convolutions", only E and H; we circumvent this problem by adapting the proof found in [21]. Therefore Theorem 4.1 should be read as a standalone abstract result parallel to the one in Section 2.3.

Let us make some comments on the (myriad of) hypothesis on Theorem 4.2. Notice that if we only used the regularity results for the wave equation (see [71]), that is, not relying on Jefimenko's equations (4.4) and (4.5), then one would need to assume that $\rho_t \in W^{2,1}(\mathbb{R}^3)$, $j_t \in W^{2,1}(\mathbb{R}^3;\mathbb{R}^3)$, and $\partial_t j_t \in W^{1,1}(\mathbb{R}^3;\mathbb{R}^3)$. Therefore Theorem 4.2 is a relaxation of hypothesis on the densities, as it does not assume any integrability of Hessian matrix of ρ, j. However, the hypothesis on the second derivative of j_t is far from optimal, for (4.1) only has first order ones. Moreover, by [47, Theorem 2], if one assumes finite initial total energy (here, we assume that $c = 1$), i.e.

$$\int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_0(x,v)\, \mathrm{d}x\, \mathrm{d}v + \int_{\mathbb{R}^3} \tfrac{1}{2}|E_0(x)|^2 + \tfrac{1}{2}|H_0(x)|^2\, \mathrm{d}x < \infty$$

in the case $\xi(v) = (1+|v|^2)^{-1/2} v$, or electromagnetic and kinetic initial energies are finite, i.e.,

$$\int_{\mathbb{R}^6} |v|^2 f_0(x,v)\, \mathrm{d}x\, \mathrm{d}v + \int_{\mathbb{R}^3} |E_0(x)|^2 + |H_0(x)|^2\, \mathrm{d}x < \infty$$

in the case $\xi(v) = v$, as well as $f_0 \in L^2(\mathbb{R}^6)$ one has finite energy at later times (and also $f_t \in L^2(\mathbb{R}^6)$). Moreover, the continuity equation $\partial_t \rho + \operatorname{div} j = 0$ holds in the distributional sense. In particular, by [35, Lemma 8.15] and [21, Lemma 4.1], we conclude that ρ_t and j_t are in $L^p(\mathbb{R}^3)$ and $L^q(\mathbb{R}^3;\mathbb{R}^3)$ for some $p, q > 1$, respectively, and so the local L^p integrability of j and the continuity equation assumptions at Theorem 4.2 follow if one assumes finite initial energy.

Now, notice that by multiplying the Vlasov equation (4.1) by $\xi(v)$ and integrating in the $v-$marginal, we have

$$\partial_t j^l + \sum_{k=1}^{3} \int_{\mathbb{R}^3} \xi(v)_l \left(\xi(v) \cdot \nabla_x f(\cdot,v)\right) \mathrm{d}v - \int_{\mathbb{R}^3} \left(\partial_{v_k} \xi(v)_l\right) \left(E + \xi(v) \times H\right)^k f(\cdot,v)\, \mathrm{d}v = 0^2.$$

Therefore, the compatibility relation in the first alternative of Theorem 4.2 is physically justified, and we only need to assume for the initial data; see Lemma 4.7 for a proof that such term is in fact a summable function at $t = 0$.

Since the results of Theorem 4.2 are purely local (see Definition 3.2), one may ask for conditions of global existence of the Maximal Regular Flow, so that Proposition 3.2

[2] By $(E + \xi(v) \times H)^k$ we mean the $k-$th component of the vector field $E + \xi(v) \times H$.

is applicable. We have a positive answer for this question by assuming integrable finite total energy in the relativistic case

$$\int_0^T \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_t(x,v)\,\mathrm{d}x\,\mathrm{d}v + \int_{\mathbb{R}^3} \tfrac{1}{2}|E_t(x)|^2 + \tfrac{1}{2}|H_t(x)|^2\,\mathrm{d}x\,\mathrm{d}t < \infty,$$

which follows if the initial energy is finite. Moreover, by the incompressibity of $\boldsymbol{b}_t$, integrals of $\psi \circ f_t$ in $\mathbb{R}^6$ are conserved through time for all Borel function ψ. Such result is a direct adaptation of results found in [21, Corollary 2.1], since they do not use the explicit formulation of electromagnetic fields. We remark that such integrability in the relativistic case follows from a priori energy estimates from Vlasov-Maxwell system; see [47].

Proof of Theorem 4.2. As remarked in the beginning of this section, it suffices to verify assumption (H2).

Once again following the idea and notations of Theorem 4.1, we now study the adapted function

$$\Phi_{\delta,\epsilon}(t) := \iiint \log\left(1 + \frac{|\gamma^1(t)-\eta^1(t)|}{\delta\epsilon} + \frac{|\gamma^2(t)-\eta^2(t)|}{\delta}\right) \mathrm{d}\mu(\eta,\gamma,z),$$

where now the trajectories are denoted as $\gamma = (\gamma^1,\gamma^2) \in \mathbb{R}^3 \times \mathbb{R}^3$, and $\delta, \epsilon \in (0,1)$ to be chosen later, and assume by contradiction that $\boldsymbol{\eta}_x$ is not a Dirac delta for $(e_0)_\#\boldsymbol{\eta}$−a.e. x. Taking the derivative of $\Phi_{\delta,\epsilon}$ and then integrating at $[0,t_0]$, we have that

$$\begin{aligned}
\Phi_{\delta,\epsilon}(t_0) \leqslant & \int_0^{t_0} \iiint \frac{|\xi(\gamma^2(t)) - \xi(\eta^2(t))|}{\epsilon\,(\delta + |\gamma^2(t)-\eta^2(t)|)} \mathrm{d}\mu(\eta,\gamma,z)\,\mathrm{d}t \\
& + \epsilon \int_0^{t_0} \iiint \frac{|E(\gamma^1(t)) - E(\eta^1(t))|}{\delta\epsilon + |\gamma^1(t)-\eta^1(t)| + \epsilon|\gamma^2(t)-\eta^2(t)|} \mathrm{d}\mu(\eta,\gamma,z)\,\mathrm{d}t, \\
& + \epsilon \int_0^{t_0} \iiint \frac{|\xi(\gamma^2(t)) \times H(\gamma^1(t)) - \xi(\eta^2(t)) \times H(\eta^1(t))|}{\delta\epsilon + |\gamma^1(t)-\eta^1(t)| + \epsilon|\gamma^2(t)-\eta^2(t)|} \mathrm{d}\mu(\eta,\gamma,z)\,\mathrm{d}t,
\end{aligned}$$

By mean value inequality, recalling that the derivative of the velocity is bounded (see the proof of Lemma 4.7) and that the μ has measure 1, we have that

$$\int_0^{t_0} \iiint \frac{|\xi(\gamma^2(t)) - \xi(\eta^2(t))|}{\epsilon\,(\delta + |\gamma^2(t)-\eta^2(t)|)} \mathrm{d}\mu(\eta,\gamma,z)\,\mathrm{d}t \leqslant \frac{2t_0}{\epsilon}. \tag{4.24}$$

The third integral is bounded by

$$\begin{aligned}
& \int_0^{t_0} \iiint \frac{|E(\gamma^1(t)) - E(\eta^1(t))|}{\delta\epsilon + |\gamma^1(t)-\eta^1(t)|} \mathrm{d}\mu(\eta,\gamma,z)\,\mathrm{d}t \\
& + \int_0^{t_0} \iiint \frac{|(\xi(\gamma^2(t)) - \xi(\eta^2(t))) \times H(\gamma^1(t))|}{\epsilon|\gamma^2(t)-\eta^2(t)|} \mathrm{d}\mu(\eta,\gamma,z)\,\mathrm{d}t \\
& + \int_0^{t_0} \iiint \frac{|\xi(\gamma^2(t)) \times (H(\gamma^1(t)) - H(\eta^1(t)))|}{\delta\epsilon + |\gamma^1(t)-\eta^1(t)|} \mathrm{d}\mu(\eta,\gamma,z)\,\mathrm{d}t.
\end{aligned} \tag{4.25}$$

By the condition $(e_t)_\# \boldsymbol{\eta} \leqslant \tilde{C}\mathcal{L}^d$, the boundedness of derivative of the velocity, and the integrability of the magnetic field (Lemma 4.3) the second term in (4.25) is bounded as

$$\int_0^{t_0} \iiint \frac{|(\xi(\gamma^2(t)) - \xi(\eta^2(t))) \times H(\gamma^1(t))|}{\epsilon|\gamma^2(t) - \eta^2(t)|} \mathrm{d}\mu(\eta,\gamma,z)\,\mathrm{d}t \leqslant \frac{2|B_R|}{\epsilon} \|H\|_{L^1((0,T)\times\mathbb{R}^3)}.$$

Now, by Hölder inequality, the condition $(e_t)_\# \boldsymbol{\eta} \leqslant \tilde{C}\mathcal{L}^d$, and the local boundedness of the velocity, we have that

$$\begin{aligned}
&\int_0^{t_0} \iiint \frac{|\xi(\gamma^2(t)) \times (H(\gamma^1(t)) - H(\eta^1(t)))|}{\delta\epsilon + |\gamma^1(t) - \eta^1(t)|} \mathrm{d}\mu(\eta,\gamma,z)\,\mathrm{d}t \\
&\leqslant \|\xi(\gamma^2)\|_{L^1((0,T)\times\mu)} \int_0^{t_0} \iiint \frac{|H(\gamma^1(t)) - H(\eta^1(t))|}{\delta\epsilon + |\gamma^1(t) - \eta^1(t)|} \mathrm{d}\mu(\eta,\gamma,z)\,\mathrm{d}t \\
&\leqslant |B_R|^2 T \|\xi\|_{L^\infty(B_R)} \int_0^{t_0} \iiint \frac{|H(\gamma^1(t)) - H(\eta^1(t))|}{\delta\epsilon + |\gamma^1(t) - \eta^1(t)|} \mathrm{d}\mu(\eta,\gamma,z)\,\mathrm{d}t.
\end{aligned}$$

Therefore, it remains to estimate the terms

$$\begin{aligned}
&\int_0^{t_0} \iiint \frac{|E(\gamma^1(t)) - E(\eta^1(t))|}{\delta\epsilon + |\gamma^1(t) - \eta^1(t)|} \mathrm{d}\mu(\eta,\gamma,z)\,\mathrm{d}t \quad \text{and} \\
&\int_0^{t_0} \iiint \frac{|H(\gamma^1(t)) - H(\eta^1(t))|}{\delta\epsilon + |\gamma^1(t) - \eta^1(t)|} \mathrm{d}\mu(\eta,\gamma,z)\,\mathrm{d}t.
\end{aligned}$$

We split the proof for each of the above integrals.

Step 1: Once again, we estimate

$$\int_0^{t_0} \iiint \frac{|E(\gamma^1(t)) - E(\eta^1(t))|}{\delta\epsilon + |\gamma^1(t) - \eta^1(t)|} \mathrm{d}\mu(\eta,\gamma,z)\,\mathrm{d}t \leqslant \int_0^{t_0} \iiint h_t^E(\gamma^1,\eta^1)\mathrm{d}\mu(\gamma,\eta,z)\,\mathrm{d}t,$$

where by Lemma 2.2, we have

$$h_t^E(\gamma^1,\eta^1) := \min\left\{ \sum_{k=1}^3 \left(M_{\rho_0^{\nu,k}} \partial_k E_t(\gamma(t)) + M_{\rho_0^{\nu,k}} \partial_k E_t(\eta(t)) \right), \frac{|E_t(\gamma(t))| + |E_t(\eta(t))|}{\delta\epsilon} \right\}.$$

Now, by Lemmas 4.4 to 4.7, we have that

$$\begin{aligned}
\partial_k E_t^i(x) = &\Bigg[\left(\partial_k E_t^0\right)^i(x) - \int_{\partial B_t} \omega_k(y) K_i(y) \rho_0(x-y) - \omega_k(y) K_i(y) \omega(y) \cdot j_0(x-y)\,\mathrm{d}S_y \\
&- \sum_{l=1}^3 \int_{\partial B_t} \omega_k(y) \tilde{K}_{il}(y) j_0^l(x-y)\,\mathrm{d}S_y \\
&- \sum_{l,m,a,b=1}^3 \varepsilon^{bai}\varepsilon^{alm} \int_{\partial B_t} \omega_k(y) \tilde{K}_{bl}(y) j_0^m(x-y)\,\mathrm{d}S_y \Bigg] + \left[\Gamma_{ik} \star \rho_t(x)\right] \\
&- \Bigg[\sum_{l=1}^3 \left((\omega_k \Gamma_{il}) \star j_t^l(x) + (K_i \partial_l \omega_k) \star j_t^l(x) - \tilde{\Gamma}_{ilk} \star j_t^l(x) \right) \\
&- \sum_{l,m,a,b=1}^3 \varepsilon^{bai}\varepsilon^{alm} \tilde{\Gamma}_{blk} \star j_t^m(x) \Bigg] + \Bigg[\sum_{l=1}^3 \left((\omega_k K_i \omega_l) \star \partial_t j_t^l(x) - (\omega_k \tilde{K}_{il}) \star \partial_t j_t^l(x) \right) \\
&+ \sum_{l,m,a,b=1}^3 \varepsilon^{bai}\varepsilon^{alm} \left((\omega_k \tilde{K}_{bl}) \star \partial_t j_t^m(x) - \partial_k (\omega_b \omega_l |\cdot|^{-1} \star \partial_t j_t^m)(x) \right) \Bigg],
\end{aligned}$$

where ε stands for the Levi-Civita symbol. By Lemmas 4.1, 4.4 and 4.7, we have that the first and fourth brackets are integrable in spacetime, and so the grand maximal operator applied to them lie in $L^1_w(\mathbb{R}^3)$. Moreover, by adding and subtracting in the third bracket the analogous convolution term, that is,

$$\Bigg[\sum_{l=1}^{3}\Big((\omega_k\Gamma_{il})\star j_t^l(x)+(K_i\partial_l\omega_k)\star j_t^l(x)-\tilde{\Gamma}_{ilk}\star j_t^l(x)\Big)-\sum_{l,m,a,b=1}^{3}\varepsilon^{bai}\varepsilon^{alm}\tilde{\Gamma}_{blk}\star j_t^m(x)$$
$$-\sum_{l=1}^{3}\Big((\omega_k\Gamma_{il})*j_t^l(x)+(K_i\partial_l\omega_k)*j_t^l(x)-\tilde{\Gamma}_{ilk}*j_t^l(x)\Big)-\sum_{l,m,a,b=1}^{3}\varepsilon^{bai}\varepsilon^{alm}\tilde{\Gamma}_{blk}*j_t^m(x)\Bigg]$$
$$+\sum_{l=1}^{3}\Big((\omega_k\Gamma_{il})*j_t^l(x)+(K_i\partial_l\omega_k)*j_t^l(x)-\tilde{\Gamma}_{ilk}*j_t^l(x)\Big)-\sum_{l,m,a,b=1}^{3}\varepsilon^{bai}\varepsilon^{alm}\tilde{\Gamma}_{blk}*j_t^m(x),$$

we have by Propositions 4.1 and 4.3 that the grand maximal operator applied to the above is in $L^1_w(\mathbb{R}^3)$. Therefore, it suffices to estimate the grand maximal function applied to the second bracket. Once again, by adding and subtracting a convolution term, we have

$$M_{\rho_0^{\nu,k}}\Gamma_{ik}\star\rho_t(x)\leqslant\Big[M_{\rho_0^{\nu,k}}(\Gamma_{ik}\star\rho_t(x)-\Gamma_{ik}*\rho_t(x))\Big]+M_{\rho_0^{\nu,k}}\Gamma_{ik}*\rho_t(x).$$

The second term is an $L^1_w(\mathbb{R}^3)$ function by Proposition 4.1, and the first term can be estimated as

$$M_{\rho_0^{\nu,k}}(\Gamma_{ik}\star\rho_t(x)-\Gamma_{ik}*\rho_t(x))=M_{\rho_0^{\nu,k}}\int_{B_t}\int_0^1\Gamma_{ik}(y)|y|\partial_t\rho_{t-s|y|}(x-y)\,\mathrm{d}s\,\mathrm{d}y.$$

Once again, we may use the continuity equation $\partial_t\rho=-\operatorname{div} j$ and the commuting relation (4.10) to obtain

$$\begin{aligned}\int_{B_t}\int_0^1\Gamma_{ik}(y)|y|\partial_t\rho_{t-s|y|}(x-y)\,\mathrm{d}s\,\mathrm{d}y=\sum_{l=1}^{3}\Bigg[&\int_{B_t}\int_0^1 s\Gamma_{ik}(y)\omega_l(y)|y|\partial_t j^l_{t-s|y|}(x-y)\,\mathrm{d}s\,\mathrm{d}y\\&-\int_{\partial B_t}\int_0^1\omega_l(y)\Gamma_{ik}(y)|y|j^l_{t(1-s)}(x-y)\,\mathrm{d}s\,\mathrm{d}S_y\\&+\int_{B_t}\int_0^1\partial_l(\Gamma_{ik}(y)|y|)j^l_{t-s|y|}(x-y)\,\mathrm{d}s\,\mathrm{d}y\Bigg].\end{aligned}$$

Notice that the first term is an $L^1((0,T)\times\mathbb{R}^3;\mathbb{R}^3)$ function, by the proof of Proposition 4.3. Since $j\in W^{1,1}((0,T);L^1(\mathbb{R}^3;\mathbb{R}^3))\subset L^\infty((0,T);L^1(\mathbb{R}^3;\mathbb{R}^3))$, we have that

$$\left\|\int_{\partial B_t}\int_0^1\omega_l(y)\Gamma_{ik}(y)|y|j^l_{t(1-s)}(\cdot-y)\,\mathrm{d}s\,\mathrm{d}S_y\right\|_{L^1(\mathbb{R}^3)}\leqslant C\|j\|_{W^{1,1}((0,T);L^1(\mathbb{R}^3))},$$

and so the second term is also a $L^1((0,T)\times\mathbb{R}^3;\mathbb{R}^3)$. For the third term, recalling the definition of Γ_{ik} at Lemma 4.5, we have that

$$\begin{aligned}\partial_l(\Gamma_{ik}(y)|y|)=&\partial_l\left(\frac{\delta_{ik}-3\omega_i(y)\omega_k(y)}{|y|^2}\right)\\=&\frac{-2\delta_{ik}\omega_l(y)+12\omega_l(y)\omega_i(y)\omega_k(y)-3\delta_{il}\omega_k(y)-3\delta_{kl}\omega_i(y)}{|y|^3}\\=:&\Gamma'_{ikl}(y).\end{aligned}$$

Notice that Γ'_{ikl} satisfies Definition 4.2. Therefore, we estimate the third term as

$$\begin{aligned}\int_{B_t}\int_0^1 \Gamma'_{ikl}(y) j^l_{t-s|y|}(x-y)\,\mathrm{d}s\,\mathrm{d}y &= \int_{B_t}\int_0^1 \Gamma'_{ikl}(y)[j^l_{t-s|y|} - j^l_t](x-y)\,\mathrm{d}s\,\mathrm{d}y \\ &\quad + \int_{B_t}\int_0^1 \Gamma'_{ikl}(y) j^l_t(x-y)\,\mathrm{d}s\,\mathrm{d}y \\ &= -\int_{B_t}\int_0^1\int_0^1 \Gamma'_{ikl}(y) s|y|\partial_t j^l_{t-ss'|y|}(x-y)\,\mathrm{d}s'\,\mathrm{d}s\,\mathrm{d}y \\ &\quad + \int_{B_t}\int_0^1 \Gamma'_{ikl}(y) j^l_t(x-y)\,\mathrm{d}s\,\mathrm{d}y.\end{aligned}$$

Therefore, we have that

$$\begin{aligned}M_{\rho_0^{\nu},k}(\Gamma_{ik}\star\rho_t(x) - \Gamma_{ik}*\rho_t(x)) \leqslant \sum_{l=1}^{3}\Bigg[& M_{\rho_0^{\nu},k}\int_{B_t}\int_0^1 \Gamma_{ik}(y)\omega_l(y)|y|\partial_t j^l_{t-s|y|}(x-y)\,\mathrm{d}s\,\mathrm{d}y \\ & + M_{\rho_0^{\nu},k}\int_{\partial B_t}\int_0^1 \omega_l(y)\Gamma_{ik}(y)|y| j^l_{t(1-s)}(x-y)\,\mathrm{d}s\,\mathrm{d}S_y \\ & + M_{\rho_0^{\nu},k}\int_{B_t}\int_0^1 \partial_l(\Gamma_{ik}(y)|y|) j^l_{t-s|y|}(x-y)\,\mathrm{d}s\,\mathrm{d}y \\ & + M_{\rho_0^{\nu},k}\int_{B_t}\int_0^1\int_0^1 \Gamma'_{ikl}(y) s|y|\partial_t j^l_{t-ss'|y|}(x-y)\,\mathrm{d}s'\,\mathrm{d}s\,\mathrm{d}y \\ & + M_{\rho_0^{\nu},k}\int_{B_t} \Gamma'_{ikl}(y) j^l_t(x-y)\,\mathrm{d}y\Bigg].\end{aligned}$$

By the previous estimates and Propositions 4.1 and 4.3, there exists a constant $C_{d,T} > 0$ depending only on d and T such that

$$\left\| \left\| M_{\rho_0^{\nu},k}(\Gamma_{ik}\star\rho - \Gamma_{ik}*\rho)\right\|_{L^1_w(\mathbb{R}^3)}\right\|_{L^1(0,T)} \leqslant C_{d,T}\|j\|_{W^{1,1}((0,T);L^1(\mathbb{R}^3))}.$$

Therefore, we have that

$$\begin{aligned}\left\|\left\| M_{\rho_0^{\nu},k}\partial_k E\right\|_{L^1_w(\mathbb{R}^3)}\right\|_{L^1(0,T)} \leqslant C_{d,T}\Big(& \|E_0\|_{W^{2,1}(\mathbb{R}^3)} + \|H_0\|_{W^{2,1}(\mathbb{R}^3)} + \|\rho_0\|_{W^{1,1}(\mathbb{R}^3)} + \|j_0\|_{W^{1,1}(\mathbb{R}^3)} \\ & + \|j\|_{W^{1,1}((0,T);L^1(\mathbb{R}^3))} + \|\rho\|_{L^1((0,T)\times\mathbb{R}^3)} \\ & + \|D(\omega\otimes\omega|\cdot|^{-1}\star\partial_t j)\|_{L^1((0,T)\times\mathbb{R}^3)}\Big),\end{aligned}$$

where the last term is bounded by Lemma 4.7. Hence, by the condition $(e_t)_\#\boldsymbol{\eta} \leqslant \tilde{C}\mathcal{L}^d$, we have that

$$\begin{aligned}\left\|\left\| h^E_t\right\|_{L^1_w(\mu)}\right\|_{L^1(0,T)} \leqslant & 2\left\|\left\| M_{\rho_0^{\nu},k}\partial_k E\right\|_{L^1_w(\mu)}\right\|_{L^1(0,T)} \leqslant 2\tilde{C}|B_R|\left\|\left\| M_{\rho_0^{\nu},k}\partial_k E\right\|_{L^1_w(\mathbb{R}^3)}\right\|_{L^1(0,T)} \\ \leqslant & C_{d,T,R}\tilde{C}\Big(\|E_0\|_{W^{2,1}(\mathbb{R}^3)} + \|H_0\|_{W^{2,1}(\mathbb{R}^3)} + \|\rho_0\|_{W^{1,1}(\mathbb{R}^3)} + \|j_0\|_{W^{1,1}(\mathbb{R}^3)} \\ & + \|j\|_{W^{1,1}((0,T);L^1(\mathbb{R}^3))} + \|\rho\|_{L^1((0,T)\times\mathbb{R}^3)} \\ & + \|D(\omega\otimes\omega|\cdot|^{-1}\star\partial_t j)\|_{L^1((0,T)\times\mathbb{R}^3)}\Big).\end{aligned}$$

Hence, by (2.9) and Lemma 4.8, we have that

$$\int_0^{t_0} \iiint h_t^E(\gamma^1, \eta^1) \mathrm{d}\mu(\gamma, \eta, z)\,\mathrm{d}t \leqslant C \log\left(\frac{C}{\delta\epsilon}\right) \leqslant C\left(1 + |\log\epsilon| + |\log\delta|\right). \tag{4.26}$$

Step 2: We analogously estimate the magnetic field term as

$$\int_0^{t_0} \iiint \frac{|H(\gamma^1(t)) - H(\eta^1(t))|}{\delta\epsilon + |\gamma^1(t) - \eta^1(t)|} \mathrm{d}\mu(\eta, \gamma, z)\,\mathrm{d}t \leqslant \int_0^{t_0} \iiint h_t^H(\gamma^1, \eta^1) \mathrm{d}\mu(\gamma, \eta, z)\,\mathrm{d}t,$$

where by Lemma 2.2, we have

$$h_t^H(\gamma^1, \eta^1) := \min\left\{\sum_{k=1}^{3}\left(M_{\rho_0^{\nu,k}}\partial_k H_t(\gamma(t)) + M_{\rho_0^{\nu,k}}\partial_k H_t(\eta(t))\right), \frac{|H_t(\gamma(t))| + |H_t(\eta(t))|}{\delta\epsilon}\right\}.$$

Computing the derivative of the magnetic field, we have by Lemmas 4.5 to 4.7 that

$$\begin{aligned}\partial_k H_t^i(x) = (\partial_k H_t^0)^i(x) + \sum_{m,l=1}^{3} \varepsilon^{lmi}\Big(& -\omega_k K_i(y) j_0(x-y)\,\mathrm{d}S_y + \Gamma_{il} \star j_t^m(x) \\ & + (\omega_k K_i) \star j_t^m(x) - \partial_k(\omega_l |\cdot|^{-1} \star \partial_t j_t^m)(x)\Big).\end{aligned}$$

Moreover, by Propositions 4.1 and 4.3 and lemma 4.4, we conclude that

$$\begin{aligned}\Big\|\Big\|M_{\rho_0^{\nu,k}}\partial_k H\Big\|_{L^1_w(\mathbb{R}^3)}\Big\|_{L^1(0,T)} \leqslant C_{d,T}\Big(& \|E_0\|_{W^{2,1}(\mathbb{R}^3)} + \|H_0\|_{W^{2,1}(\mathbb{R}^3)} + \|j_0\|_{W^{1,1}(\mathbb{R}^3)} \\ & + \|j\|_{W^{1,1}((0,T);L^1(\mathbb{R}^3))} + \|D(\omega|\cdot|^{-1} \star \partial_t j)\|_{L^1((0,T)\times\mathbb{R}^3)}\Big).\end{aligned}$$

Therefore, by Lemma 4.8, we may conclude analogously as in (4.26) that

$$\int_0^{t_0} \iiint h_t^H(\gamma^1, \eta^1) \mathrm{d}\mu(\gamma, \eta, z)\,\mathrm{d}t \leqslant C\left(1 + |\log\epsilon| + |\log\delta|\right). \tag{4.27}$$

Finally, we have by (4.24), (4.25), (4.26), and (4.27) that

$$\Phi_{\delta,\epsilon}(t_0) \leqslant \frac{C}{\epsilon} + C\epsilon\left(1 + |\log\epsilon| + |\log\delta|\right).$$

Taking $\epsilon < a/(2CT)$ and recalling that the assumption that "$\boldsymbol{\eta}_x$ is not a Dirac delta for $(e_0)_\#\boldsymbol{\eta}$−a.e. x" implies (4.18), we have the desired contradiction.

Hence, property (H2) holds, and so by Theorems 3.9 and 3.10, existence, uniqueness, and semigroup property of maximal regular flow follow. Moreover, as the vector field is divergence free, Lagrangian solutions are renormalized; finally, since f is nonnegative and weakly continuous in $[0,T]$ in duality with $C_c(\mathbb{R}^3)$, the consistency of solutions follows from [6, Theorem 5.1]. □

5 A study for gSQG-type vector fields

A natural question when comparing the results in Section 2.3 and Section 2.5 is if one can consider the intermediate cases. More precisely, recall that by Proposition 2.3, the fundamental estimate holds for vector fields $\boldsymbol{b} = K * g$, where $g \in L^1((0,T) \times \mathbb{R}^d)$ and K has estimate $|K(x)| \leqslant C|x|^{-(d-1)}$ has derivative $\Gamma = DK$ which is a singular kernel of fundamental type. Moreover, in Theorem 2.6, the fundamental estimate also holds for vector fields $\boldsymbol{b} = \Gamma * g$, where $g \in L^1((0,T); \mathrm{BV}(\mathbb{R}^d))$ and $\Gamma = |\cdot|^{-d}\Omega$ is a singular kernel of fundamental type with average zero, that is, $\int_{\mathbb{S}^{d-1}} \Omega(y)\, \mathrm{d}S_y = 0$. Hence, we ask if one can consider $\boldsymbol{b} = K^\alpha * g$, with a kernel K^α having appropriate cancellation properties and an estimate $|K^\alpha(x)| \leqslant C|x|^{-(d-1+\alpha)}$ for some $\alpha \in (0,1)$, and g in a space between $L^1((0,T); \mathrm{BV}(\mathbb{R}^d))$ and $L^1((0,T) \times \mathbb{R}^d)$. A natural candidate for this space would be the fractional Sobolev space $L^1((0,T); W^{\alpha,1}(\mathbb{R}^d))$.

Analogously, by Proposition 2.1 in Section 2.2 the fundamental estimate holds for vector fields $\boldsymbol{b} = K * g$ for $g \in L^1((0,T); L^p(\mathbb{R}^d))$ and $\boldsymbol{b} = \Gamma * g$ for $g \in L^1((0,T); W^{1,p}(\mathbb{R}^d))$, where K and Γ as before and $p \in (1,\infty)$. In this case, the candidate of an intermediate space for g is not that clear. For instance, one might expect the fractional Sobolev space $L^1((0,T); W^{\alpha,p}(\mathbb{R}^d))$ or the Besov space $L^1((0,T); B^\alpha_{p,q}(\mathbb{R}^d))$ or even the Triebel-Lizorkin space $L^1((0,T); F^\alpha_{p,q}(\mathbb{R}^d))$ for some $q \in [1,\infty]$; notice that in the $p = q = 1$ case, all of the above spaces are equal. We shall consider g in the space $L^1((0,T); B^\alpha_{p,1}(\mathbb{R}^d))$. We did not investigate whether the other aforementioned spaces are also suitable.

Accordingly, in this chapter we shall study the transport and continuity equations

$$\partial_t u + \boldsymbol{b} \cdot \nabla u = 0;$$
$$\partial_t u + \operatorname{div}(\boldsymbol{b} u) = 0;$$

where the vector field $\boldsymbol{b}$ has one of the following structures: the more regular form

$$\boldsymbol{b}_t^i(x) = K_i^\alpha * g_t(x), \quad \text{where} \quad K_i^\alpha(y) = \frac{\Omega_i(y)}{|y|^{d-1+\alpha}} \quad \text{and} \quad |\partial_j K_i^\alpha(y)| \leqslant \frac{C}{|y|^{d+\alpha}} \tag{5.1}$$

or the more singular form

$$\boldsymbol{b}_t^i(x) = \Gamma_i^{k,\alpha} * g_t(x), \quad \text{where} \quad \Gamma_i^{k,\alpha}(y) = \frac{\Omega_i(y)}{|y|^{d+k+\alpha}}. \tag{5.2}$$

In the above, we assume that g is in an appropriate fractional Sobolev/Besov space, the indexes k, α are $k \in \{0,1,2\}$, $\alpha \in (0,1)$ (we shall refer to $K^\alpha, \Gamma^{k,\alpha}$ as kernels and g as density of the vector field), and the functions Ω are zero order homogeneous function and satisfy the appropriate cancellation condition:

Definition 5.1 (Cancellation property)**.** For vector fields (5.1), we say that Ω has cancellation property if for every $i, j = 1, \dots, d$, we have that

$$\int_{\partial B_1} \Omega_i(y)\,\mathrm{d}S_y = 0.$$

We additionally assume in (5.1) that $|\Omega_i(\cdot)| \in W^{1,\infty}(\mathbb{S}^{d-1})$. Moreover, for vector fields (5.2), we always assume that $|\Omega_i(\cdot)| \in L^{\infty}(\mathbb{S}^{d-1})$; in the case $k = 0$, we assume that for every $i = 1, \dots, d$ that

$$\int_{\partial B_1} \Omega_i(y)\,\mathrm{d}S_y = 0.$$

In the case (5.2) with $k = 1$, we further assume that for every $i, j = 1, \dots, d$, we have that

$$\int_{\partial B_1} \Omega_i(y)\,\mathrm{d}S_y = 0 \quad \text{and} \quad \int_{\partial B_1} \omega_j(y)\Omega_i(y)\,\mathrm{d}S_y = 0,$$

where $\omega_j(y) = y_j|y|^{-1}$. Finally, in the case (5.2) with $k = 2$, we assume that for every index, $i, j, l = 1, \dots, d$, it holds

$$\int_{\partial B_1} \Omega_i(y)\,\mathrm{d}S_y = 0, \quad \int_{\partial B_1} \omega_j(y)\Omega_i(y)\,\mathrm{d}S_y = 0, \quad \text{and} \quad \int_{\partial B_1} \omega_l(y)\omega_j(y)\Omega_i(y)\,\mathrm{d}S_y = 0.$$

Remark 27. Notice that in the case of gSQG equation (1.8) (changing the parameter α to γ in order to avoid any confusion), the associated kernels can be written as $K^{\gamma}(y) = |y|^{-(2+\gamma)}y^{\perp}$ if $\gamma \in (0,1)$ and $\Gamma^{0,\gamma-1}(y) = |y|^{-(2+\gamma)}y^{\perp}$ if $\gamma \in (1,2)$.

Remark 28. This chapter is in fact a simplified proof of Nguyen's work [70] for vector fields with structure (5.1) and (5.2) and Definition 5.1. It is an open problem if the well-posedness of regular flows holds if one does not assume the latter.

The quintessential examples of Ω satisfying Definition 5.1 are $\Omega_i(y) = \omega_i(y)$ and $\Omega_{ij}(y) = \delta_{ij} - d\,\omega_i(y)\omega_j(y)$. Moreover, notice that $\Omega_i(y) = \delta_{i1} - d\,\omega_i(y)\omega_1(y)$ satisfies Definition 5.1 in the case $k = 1$. Finally, as an example for Definition 5.1 in the case $k = 2$ in dimension $d = 2$, we have

$$\Omega_i(y) = \omega_1(y)\left(\delta_{i1} - \frac{4}{3}\omega_i(y)\omega_1(y)\right) + \omega_2(y)\left(\delta_{i2} - \frac{4}{3}\omega_i(y)\omega_2(y)\right).$$

We first recall the definition of Besov spaces: for $k \in \mathbb{N}$, $\alpha \in (0,1)$, $p \in [1,\infty]$, $q \in [1,\infty)$, we define such space as

$$B^{k+\alpha}_{p,q}(\mathbb{R}^d) = \left\{ g \in W^{k,p}(\mathbb{R}^d) : \left(\int_{\mathbb{R}^d} \frac{\|D^{\sigma}g(\cdot + y) - D^{\sigma}g\|^q_{L^p(\mathbb{R}^d)}}{|y|^{d+q\alpha}}\,\mathrm{d}y \right)^{1/q} < \infty, \right.$$
$$\left. \text{where } |\sigma| = k \right\},$$

and so we may define the associated norm $\|g\|_{B^{k+\alpha}_{p,q}(\mathbb{R}^d)} := \|g\|_{W^{k,p}(\mathbb{R}^d)} + [D^\sigma g]_{B^{\alpha}_{p,q}(\mathbb{R}^d)}$ with $|\sigma| = k$, where

$$[f]_{B^{\alpha}_{p,q}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} \frac{\|f(\cdot + y) - f\|^q_{L^p(\mathbb{R}^d)}}{|y|^{d+q\alpha}} \, \mathrm{d}y \right)^{1/q} .$$

We now make a precise assumption on the regularity of densities g:

Condition 5.1 (Regularity of densities)**.** We shall assume in the case (5.1) that $g \in L^1((0,T); B^{\alpha}_{p,1}(\mathbb{R}^d))$ for some $p \in [1, \infty]$. In the case (5.2) with $k = 0$, we assume that $g \in L^1((0,T); B^{1+\alpha}_{p,1}(\mathbb{R}^d))$ for some $p \in [1, \infty]$.

Notice that by Definition 5.1 and assuming that Ω is a zeroth homogeneous function, the vector fields $\boldsymbol{b} = \Gamma^{k,\alpha} * g$ and $\boldsymbol{b} = \left(\Omega |\cdot|^{-(d+k)}\right) * g$ in the case $k \in \{1, 2\}$ can be rewritten as

$$\begin{aligned} \Gamma^{1,\alpha} * g_t(x) &= \int_{\mathbb{R}^d} \frac{\Omega(y)}{|y|^{d+1+\alpha}} [g_t(x-y) - g_t(x)] \, \mathrm{d}y \\ &= - \int_0^1 \int_{\mathbb{R}^d} \frac{\Omega(y)}{|y|^{d+\alpha}} \omega(y) \cdot \nabla g_t(x - \tau y) \, \mathrm{d}y \, \mathrm{d}\tau \\ &= - \int_0^1 \tau^\alpha \, \mathrm{d}\tau \int_{\mathbb{R}^d} \frac{\Omega(y)}{|y|^{d+\alpha}} \omega(y) \cdot \nabla g_t(x-y) \, \mathrm{d}y \\ &= \frac{-1}{1+\alpha} \int_{\mathbb{R}^d} \frac{\Omega(y)}{|y|^{d+\alpha}} \omega(y) \cdot \nabla g_t(x-y) \, \mathrm{d}y \end{aligned} \tag{5.3}$$

where $\omega(y) := |y|^{-1} y$, and so we may write $\boldsymbol{b} = \Gamma^{1,\alpha} * g$ as finite sum of $\Gamma^{0,\alpha}_i * h^i_t$, where

$$\Gamma^{0,\alpha}_i(y) = \frac{-\Omega(y)\omega_i(y)}{(1+\alpha)|y|^{d+\alpha}} \quad \text{and} \quad h^i_t(y) = \partial_i g_t(y).$$

By Definition 5.1, the kernels satisfy the $k = 0$ of Definition 5.1 (and so the notation is consistent) and we shall assume that densities h^i have the integrability

$$\int_0^T \|h^i_t\|_{B^{1+\alpha}_{p,1}(\mathbb{R}^d)} \, \mathrm{d}t < \infty \iff \int_0^T \|D^\sigma g_t\|_{B^{1+\alpha}_{p,1}(\mathbb{R}^d)} \, \mathrm{d}t < \infty, \quad \text{where } |\sigma| = 1.$$

Applying the same idea twice, we may write

$$\Gamma^{2,\alpha} * g_t(x) = \frac{1}{(1+\alpha)^2} \int_{\mathbb{R}^d} \frac{\Omega(y)}{|y|^{d+\alpha}} [\omega(y) \otimes \omega(y)] : D^2 g_t(x-y) \, \mathrm{d}y, \tag{5.4}$$

and so we may write it as a sum of kernels $\Gamma^{0,\alpha}_{ij}$ satisfying Definition 5.1 for $k = 0$ and assume that densities $h^{ij}_t = \partial_{ij} g_t$ with integrability

$$\int_0^T \|h^{ij}_t\|_{B^{1+\alpha}_{p,1}(\mathbb{R}^d)} \, \mathrm{d}t < \infty \iff \int_0^T \|D^\sigma g_t\|_{B^{1+\alpha}_{p,1}(\mathbb{R}^d)} \, \mathrm{d}t < \infty \quad \text{where } |\sigma| = 2.$$

Hence, any property to be obtained for a finite sum $\Gamma^{0,\alpha} * g$ shall be translated for vector fields (5.2) with $k \in \{1, 2\}$ by assuming that the density function g has the aforementioned suitable regularity. Therefore we restrict ourselves to prove the desired estimates for the case $k = 0$ in (5.2).

Remark 29. One could iterate the above process for $k \geqslant 3$, and the above computation would follow for even more singular kernels, provided that we increase the required regularity of the density g and the cancellation property for Ω. We do not include such cases for we do not find any explicit examples of functions satisfying Definition 5.1 in the case $k = 2$, as well as for all $i, j, l, m = 1, \dots, d$

$$\int_{\partial B_1} \omega_m \omega_l(y)\omega_j(y)\Omega_i(y)\,\mathrm{d}S_y = 0.$$

Remark 30. The same computation (5.3) holds for $\boldsymbol{b} = \left(\Omega|\cdot|^{-(d+1)}\right) * g$, and the only difference is now the regularity $h_t^i \in \mathrm{BV}(\mathbb{R}^d)$ or $h_t^i \in W^{1,p}(\mathbb{R}^d)$, since it now becomes a particular case of Theorem 2.6 and Proposition 2.1. Similarly, the computation (5.4) holds for $\boldsymbol{b} = \left(\Omega|\cdot|^{-(d+2)}\right) * g$, where now one assumes $h_t^{ij} \in \mathrm{BV}(\mathbb{R}^d)$ or $h_t^{ij} \in W^{1,p}(\mathbb{R}^d)$.

We begin by proving that (H1) and (1.6) hold, that is, the local integrability and growth assumption of $\boldsymbol{b}$. We recall that the Besov space in this case gives $B^{\alpha}_{1,1}(\mathbb{R}^d) = W^{\alpha,1}(\mathbb{R}^d)$, where the latter is the fractional Sobolev space.

Lemma 5.1 (Property (H1) and (1.6))**.** *Let* $\boldsymbol{b}$ *be a vector field satisfying* (5.1) *and* (5.2) *with Definition 5.1 and Condition* 5.1 *with* $p \in [1, \infty]$*. Then it holds that* $\boldsymbol{b} \in L^1((0,T); L^p(\mathbb{R}^d))$*. In particular,* $\boldsymbol{b} \in L^1((0,T) \times \mathbb{R}^d) + L^1((0,T); L^\infty(\mathbb{R}^d))$*.*

Proof. Let us first consider $K_i^\alpha = |\cdot|^{-(d-1+\alpha)}\Omega_i$. Then by the computation and zero average of Ω_i, we have that

$$\begin{aligned}\|\boldsymbol{b}_t\|_{L^p(\mathbb{R}^d)} \leqslant& \|\Omega\|_{L^\infty(\mathbb{S}^{d-1})} \int_{B_1} \frac{1}{|y|^{d-1+\alpha}} \|g_t\|_{L^p(\mathbb{R}^d)}\,\mathrm{d}y \\ &+ \|\Omega\|_{L^\infty(\mathbb{S}^{d-1})} \int_{\mathbb{R}^d\setminus B_1} \frac{1}{|y|^{d-1+\alpha}} \|g_t(\cdot+y) - g_t\|_{L^p(\mathbb{R}^d)}\,\mathrm{d}y \\ \leqslant& C_{d,\alpha}\|g_t\|_{B^{\alpha}_{p,1}(\mathbb{R}^d)},\end{aligned}$$

we conclude that $\boldsymbol{b} \in L^1((0,T); L^p(\mathbb{R}^d))$. For the case $\Gamma_i^\alpha = |\cdot|^{-(d+\alpha)}\Omega_i$, it follows from the zero average of Ω_i that

$$\|\boldsymbol{b}_t\|_{L^p(\mathbb{R}^d)} \leqslant \|\Omega\|_{L^\infty(\mathbb{S}^{d-1})}[g_t]_{B^{\alpha}_{p,1}(\mathbb{R}^d)} \leqslant C_\alpha \|g_t\|_{B^{1+\alpha}_{p,1}(\mathbb{R}^d)}.$$

Integrating with respect to time, the lemma follows. □

We are now ready to prove the fundamental estimate associated to vector fields (5.1) and (5.2) and the third main result of this thesis.

Proposition 5.1. *Let* $\bar{\boldsymbol{b}}$ *be vector field satisfying* (1.6) *and* $\boldsymbol{b}$ *as in* (5.1) *with* $g \in L^1((0,T); B^{\alpha}_{p,1}(\mathbb{R}^d))$ *or* (5.2) *with* $k = 0$*, with Definition 5.1 and* $g \in L^1((0,T); B^{1+\alpha}_{p,1}(\mathbb{R}^d))$ *for* $p \in [1, \infty]$*, and* $\boldsymbol{X}$*,* $\bar{\boldsymbol{X}}$ *their renormalized regular Lagrangian flows starting at time s* *with compressibility constants* L *and* $\bar{L}$*, respectively. Then for every* $\gamma > 0$*,* $\eta > 0$*, and* $r > 0$*, there exists* $\lambda > 0$ *and a constant* $C_{\gamma,\eta,r} > 0$ *such that*

$$|B_r \cap \{|\boldsymbol{X}(t,s,\cdot) - \bar{\boldsymbol{X}}(t,s,\cdot)| > \gamma\}| \leqslant C_{\gamma,\eta,r}\|\boldsymbol{b} - \bar{\boldsymbol{b}}\|_{L^1((0,T)\times B_\lambda)} + \eta$$

uniformly in $s \in [0,T]$ *and* $t \in [s,T]$*. The constant* $C_{\gamma,\eta,r}$ *depends on its subscripts, as well as the compressibility constants* L *and* $\bar{L}$*, on the norms* (1.6) *of* $\boldsymbol{b}$ *and* $\bar{\boldsymbol{b}}$*, and on the* $W^{1,\infty}(\mathbb{S}^{d-1})$ *(or on* $L^\infty(\mathbb{S}^{d-1})$*) norm of* Ω*.*

Proof. As in Proposition 2.3, we consider for fixed $\delta > 0$, $\lambda > 0$, and $t \in [s,T]$ the function

$$\Phi_\delta(t) = \int_{B_r\cap G_\lambda\cap\bar{G}_\lambda} \log\left(1 + \frac{|\boldsymbol{X}(t,s,x) - \bar{\boldsymbol{X}}(t,s,x)|}{\delta}\right) \mathrm{d}x,$$

and it satisfies for all $\tau \in [s,T]$

$$\begin{aligned}
\Phi_\delta(\tau) = \int_s^\tau \Phi_\delta'(t)\,\mathrm{d}t &\leqslant \int_s^\tau \int_{B_r\cap G_\lambda\cap\bar{G}_\lambda} \frac{|\boldsymbol{b}_t(\boldsymbol{X}(t,s,x)) - \bar{\boldsymbol{b}}_t(\bar{\boldsymbol{X}}(t,s,x))|}{|\boldsymbol{X}(t,s,x) - \bar{\boldsymbol{X}}(t,s,x)| + \delta}\,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant \int_s^\tau \int_{B_r\cap G_\lambda\cap\bar{G}_\lambda} \frac{|\boldsymbol{b}_t(\bar{\boldsymbol{X}}(t,s,x)) - \bar{\boldsymbol{b}}_t(\bar{\boldsymbol{X}}(t,s,x))|}{|\boldsymbol{X}(t,s,x) - \bar{\boldsymbol{X}}(t,s,x)| + \delta}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_s^\tau \int_{B_r\cap G_\lambda\cap\bar{G}_\lambda} \frac{|\boldsymbol{b}_t(\boldsymbol{X}(t,s,x)) - \boldsymbol{b}_t(\bar{\boldsymbol{X}}(t,s,x))|}{|\boldsymbol{X}(t,s,x) - \bar{\boldsymbol{X}}(t,s,x)| + \delta}\,\mathrm{d}x\,\mathrm{d}t \\
&=: \mathrm{I}(\tau) + \mathrm{II}(\tau).
\end{aligned}$$

The first integral on the right-hand side is bounded as in Proposition 2.1:

$$\mathrm{I}(\tau) \leqslant \frac{\bar{L}}{\delta}\|\boldsymbol{b} - \bar{\boldsymbol{b}}\|_{L^1((0,T)\times B_\lambda)}.$$

For the second integral, we use the second part of Lemma 2.2, for Theorem 2.3 gives that $M_{\Upsilon^{j,\xi}}(\partial_j \boldsymbol{b}_t^i)(x) < \infty$ for all $i,j \in \{1,\dots,d\}$ and almost every $x \in \mathbb{R}^d$, $t \in [s,T]$, and so

$$\begin{aligned}
\mathrm{II}(\tau) \leqslant \sum_{i,j=1}^d \int_s^\tau \int_{B_r\cap G_\lambda\cap\bar{G}_\lambda} \min\Bigg\{ & M_{\Upsilon^{j,\xi}}\partial_j \boldsymbol{b}_t^i(\boldsymbol{X}(t,s,x)) + M_{\Upsilon^{j,\xi}}\partial_j \boldsymbol{b}_t^i(\bar{\boldsymbol{X}}(t,s,x)), \\
& \frac{|\boldsymbol{b}_t^i(\boldsymbol{X}(t,s,x))|}{\delta} + \frac{|\boldsymbol{b}_t^i(\bar{\boldsymbol{X}}(t,s,x))|}{\delta}\Bigg\}\,\mathrm{d}x.
\end{aligned}$$

We shall split the proof in the simpler $p > 1$ case and in the more challenging $p = 1$ case.

The $p > 1$ case: Notice that if $\boldsymbol{b} = K^\alpha * g$, then

$$\partial_j \boldsymbol{b}_t^i(x) = \partial_j \int_{\mathbb{R}^d} \frac{\Omega_i(x-y)}{|x-y|^{d-1+\alpha}}[g_t(y) - g_t(x)]\,\mathrm{d}y = \int_{\mathbb{R}^d} \partial_j\left(\frac{\Omega_i(x-y)}{|x-y|^{d-1+\alpha}}\right)[g_t(y) - g_t(x)]\,\mathrm{d}y,$$

where the above computation can be justified by the regularity of g. By the compressibility of flows $\boldsymbol{X}$, $\bar{\boldsymbol{X}}$, we have that

$$\|D\boldsymbol{b}\|_{L^1((0,T);L^p(\mathbb{R}^d))} \leqslant C_{\alpha,d}\|\Omega\|_{W^{1,\infty}(\mathbb{S}^{d-1})}\|g\|_{L^1((0,T);B^\alpha_{p,1}(\mathbb{R}^d))},$$

and so we are in fact in the same setting as in Proposition 2.1. Analogously, if $\boldsymbol{b} = \Gamma^{0,\alpha} * g$, then

$$\|D\boldsymbol{b}\|_{L^1((0,T);L^p(\mathbb{R}^d))} \leqslant C_{\alpha,d}\|\Omega\|_{L^\infty(\mathbb{S}^{d-1})}\|\nabla g\|_{L^1((0,T);B^\alpha_{p,1}(\mathbb{R}^d))},$$

and once again the desired estimate follows from Proposition 2.1.

The $\boldsymbol{p = 1}$ case: We shall apply Lemma 2.13 with operator $T_j = M_{\Upsilon^{j,\xi}}$ and $f^i_t(x) = |\boldsymbol{b}^i_t(\boldsymbol{X}(t,s,x))| + |\boldsymbol{b}^i_t(\bar{\boldsymbol{X}}(t,s,x))|$. Indeed, since the grand maximal operator is a bounded operator from $L^1(\mathbb{R}^d)$ to $L^1_w(\mathbb{R}^d)$. In the $p = 1$ case, we have either $B^\alpha_{1,1} = W^{\alpha,1}$ in the (5.1) case or $B^{1+\alpha}_{1,1} = W^{1+\alpha,1}$ (5.2) with $k = 0$ case. By fractional Sobolev embedding (see [46, Chapter 6]), we have that $g \in L^1((0,T);L^q(\mathbb{R}^d))$ if $g \in L^1((0,T);W^{\alpha,1}(\mathbb{R}^d))$ and $\nabla g \in L^1((0,T);L^q(\mathbb{R}^d))$ for any $q \in [1, d/(d-\alpha)]$ if $g \in L^1((0,T);W^{1+\alpha,1}(\mathbb{R}^d))$, and so for any $R > 0$ that

$$\begin{aligned}\|K^\alpha * g\|_{L^1((0,T);L^q(B_R))} \leqslant& \|(\mathbb{1}_{B_1}K^\alpha) * g\|_{L^1((0,T);L^q(B_R))} + \|(\mathbb{1}_{\mathbb{R}^d\setminus B_1}K^\alpha) * g\|_{L^1((0,T);L^q(B_R))}\\ \leqslant& \|K^\alpha\|_{L^1(B_1)}\|g\|_{L^1((0,T);L^q(\mathbb{R}^d))}\\ &+ |B_R|^{1-1/q}\|K^\alpha\|_{L^\infty(\mathbb{R}^d\setminus B_1)}\|g\|_{L^1((0,T)\times\mathbb{R}^d)}\\ \leqslant& C_{\alpha,R,d}\|g\|_{L^1((0,T);W^{\alpha,1}(\mathbb{R}^d))},\end{aligned}$$

and analogously for the case $\boldsymbol{b} = \Gamma^{0,\alpha} * g$ by the identity (the computation is *mutatis mutandis* the same as in (5.3))

$$\boldsymbol{b} = \Gamma^{0,\alpha} * g_t(x) = -\frac{1}{\alpha}\int_{\mathbb{R}^d}\frac{\Omega(y)}{|y|^{d-1+\alpha}}\omega(y)\cdot\nabla g_t(x-y)\,\mathrm{d}y,$$

we have for any $R > 0$ that

$$\|\Gamma^{0,\alpha} * g\|_{L^1((0,T);L^q(B_R))} \leqslant C_{\alpha,R,d}\|\nabla g\|_{L^1((0,T);W^{\alpha,1}(\mathbb{R}^d))} \leqslant C_{\alpha,R,d}\|g\|_{L^1((0,T);W^{1+\alpha,1}(\mathbb{R}^d))}.$$

By the compressibility of flows $\boldsymbol{X}$, $\bar{\boldsymbol{X}}$, we conclude that there exists $q > 1$ such that $f \in L^1((0,T);L^q(B_R))$. Finally, we shall consider as measures $\mu^{ij}_t(x) = \partial_j\boldsymbol{b}^i_t(\boldsymbol{X}(t,s,x))\mathcal{L}^d$ or $\bar{\mu}^{ij}_t(x) = \partial_j\boldsymbol{b}^i_t(\bar{\boldsymbol{X}}(t,s,x))\mathcal{L}^d$, which by the above considerations and the compressibility of flows $\boldsymbol{X}$, $\bar{\boldsymbol{X}}$, we have that μ, $\bar{\mu}$ are finite measures in space and integrable in time. By Lemma 2.13, we conclude for each $i, j = 1, \dots, d$ that

$$\limsup_{\delta\to 0}\frac{1}{|\log\delta|}\int_0^T\int_{B_r}\min\left\{\frac{f^i_t(x)}{\delta}, T_j\mu^{ij}_t(x)\right\}\mathrm{d}x\,\mathrm{d}t = 0. \tag{5.5}$$

Using the same lower bound for Φ_δ as in Proposition 2.3, we have that

$$\begin{aligned}|B_r \cap \{|\boldsymbol{X}(\tau,s,\cdot) - \bar{\boldsymbol{X}}(\tau,s,\cdot)| > \gamma\}| \leqslant& \frac{\Phi_\delta(\tau)}{\log\left(1+\frac{\gamma}{\delta}\right)} + f(r,\lambda) + \bar{f}(r,\lambda)\\ \leqslant& \frac{C}{\log\left(1+\frac{\gamma}{\delta}\right)\delta}\|\boldsymbol{b} - \bar{\boldsymbol{b}}\|_{L^1((0,T)\times B_\lambda)}\\ &+ \frac{1}{|\log\delta|}\int_0^T\int_{B_r}\min\left\{\frac{f^i_t(x)}{\delta}, T_j\mu^{ij}_t(x)\right\}\mathrm{d}x\,\mathrm{d}t\\ &+ f(r,\lambda) + \bar{f}(r,\lambda),\end{aligned}$$

where f^i, T_j, μ^{ij} as before and f, $\bar{f}$ the sublevel's control as in Lemma 2.5. By choosing λ large enough so that $f(r,\lambda)+\bar{f}(r,\lambda)<\eta/2$ and using (5.5) to obtain for δ small enough so that

$$\frac{1}{|\log\delta|}\int_0^T\int_{B_r}\min\left\{\frac{f_t^i(x)}{\delta},T_j\mu_t^{ij}(x)\right\}\,\mathrm{d}x\,\mathrm{d}t\leqslant\frac{\eta}{2},$$

and so the proposition follows by defining

$$C_{\gamma,\eta,r}:=\frac{C}{\log\left(1+\frac{\gamma}{\delta}\right)\delta},$$

since the choice of δ depends on η and r. □

Using the machinery majorly developed in [22] presented in Section 2.2, we may state existence and uniqueness of Lagrangian solutions of transport/continuity equations associated to vector fields (5.1) and (5.2). More precisely, following the proofs of Theorem 2.1 and Theorem 2.2, we conclude the result below:

Theorem 5.1. *Let* $\boldsymbol{b}$ *as in* (5.1) *or* (5.2)*, where* Ω_i *satisfies Definition 5.1,* $\alpha\in(0,1)$*,* $p\in[1,\infty]$*, and the density* g *has one of the integrabilities below:*

$$\begin{aligned}\int_0^T\|g_t\|_{B^{\alpha}_{p,1}(\mathbb{R}^d)}\,\mathrm{d}t<\infty\quad&\text{if } \boldsymbol{b}=K^{\alpha}*g;\\ \int_0^T\|g_t\|_{B^{1+\alpha}_{p,1}(\mathbb{R}^d)}\,\mathrm{d}t<\infty\quad&\text{if } \boldsymbol{b}=\Gamma^{0,\alpha}*g;\end{aligned}\tag{5.6}$$

$$\int_0^T\|D^{\sigma}g_t\|_{B^{1+\alpha}_{p,1}(\mathbb{R}^d)}\,\mathrm{d}t<\infty\quad\text{if } \boldsymbol{b}=\Gamma^{1,\alpha}*g \text{ with } |\sigma|=1;$$

$$\int_0^T\|D^{\sigma}g_t\|_{B^{1+\alpha}_{p,1}(\mathbb{R}^d)}\,\mathrm{d}t<\infty\quad\text{if } \boldsymbol{b}=\Gamma^{2,\alpha}*g \text{ with } |\sigma|=2.$$

Then there exists an unique renormalized regular flow in the sense of Definition 2.2 (i) and Definition 2.2 (ii). Moreover, there exists a Lagrangian solution (1.3)/(1.5) *of transport equation* (1.1)*/continuity equation* (1.4)*.*

We now turn our attention to the gSQG equation (1.8). We recall that general interest in the literature for range of α is $[0,2)$, where $\alpha=0$ and $\alpha=1$ corresponds to the 2D-Euler vorticity equation and the SQG equation, respectively; see [30, 31] and references therein. We remark that θ is taken as the vorticity if $\alpha=0$ and as potential temperature if $\alpha=1$ (see the comment in Chapter 1). Of course, taking $\alpha=2$ implies that $\theta_t=\theta_0$, and so (5.2) with $k=1$ no longer has a direct application. Nevertheless, we refer to [30, Theorem 1.3] where the authors study the dissipative version of (1.1) with the vector field $\boldsymbol{b}_t=\nabla^{\perp}(\log(1-\Delta))^{\mu}\theta_t$ for any $\mu>0$, which is logarithmically more singular than the case $\alpha=2$.

Concerning existence of solutions, there are results for global weak solutions of (1.8) in bounded domains for $1\leqslant\alpha\leqslant2$ (see [37, 69]). Also, existence and uniqueness of local in time classical solutions for $0 \leqslant \alpha \leqslant 2$ has been obtained in [29, 30, 56, 57, 79]. More precisely, if $\theta_0 \in H^{1+\alpha+\epsilon}(\mathbb{R}^2)$ for any $\epsilon > 0$, then there exists a finite time $T > 0$ depending on the initial data and a unique solution $\theta \in C([0,T]; H^{1+\alpha+\epsilon}(\mathbb{R}^2))$ of (1.8). Notice that by the relation (for $p \in [2, \infty]$)

$$H^{1+\alpha+\epsilon}(\mathbb{R}^2) = B^{1+\alpha+\epsilon}_{2,2}(\mathbb{R}^2) \subset B^{1+\alpha}_{2,1}(\mathbb{R}^2) \subset B^{\alpha}_{p,1}(\mathbb{R}^2),$$

our regularity assumptions (5.6) in Proposition 5.1 much milder than the ones for classical solutions. Moreover, since θ represent the potential temperature, we have that $\theta \geqslant 0$. Finally, notice that by (1.8), the associated vector field $\boldsymbol{b}_t(x) = \nabla^{\perp}(-\Delta)^{\frac{\alpha}{2}-1}\theta_t(x)$ is divergence-free. Combining all the above, we may apply Theorem 3.6 and Proposition 5.1 and obtain a criteria for Lagrangian solutions and the fourth and final main result of this thesis, as described below.

Theorem 5.2 (Lagrangian solutions of gSQG)**.** *If θ is a nonnegative distributional or renormalized solution of (1.8) with regularity $\theta \in L^1([0,T]; B^{\alpha}_{p,1}(\mathbb{R}^2))$, where $\alpha \in (0,2) \setminus \{1\}$, $p \in [1, \infty]$, and $t \mapsto u_t(x)$ is weakly continuous in duality with $C_c(\mathbb{R}^d)$, then there exists a renormalized regular Lagrangian flow $\boldsymbol{X}$ in the sense Definition 2.2 (i) and Definition 2.2 (ii) (and also in the sense Definition 3.2 with maximal time T) and θ can be written as a Lagrangian solution $\theta_t(x) = \theta_0(\boldsymbol{X}(0,t,x))$.*

Proof. If θ is a distributional solution, then one may apply Theorem 3.6 and Proposition 3.1, with properties (H1) and (H2) ensured by Lemma 5.1 and Proposition 5.1 to obtain that θ is Lagrangian solution. If θ is a renormalized solution, then $\beta \circ \theta$ is distributional solution of the continuity equation with vector field $\boldsymbol{b}_t(x) = \nabla^{\perp}(-\Delta)^{\frac{\alpha}{2}-1}\theta_t(x)$ for any $\beta \in L^{\infty}(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$.[1] We now follow the proof found in [6, Step 3 of Theorem 5.1]: we can consider a family of functions

$$\beta_k(z) := \begin{cases} 0 & \text{if } z \leqslant k; \\ z-k & \text{if } k < z < k+1; \\ 1 & \text{if } z \geqslant k+1, \end{cases}$$

and so by Theorem 3.6, we conclude that there exists $\boldsymbol{\eta}_k \in \mathcal{M}(C([0,T]; \bar{\mathbb{R}}^d))$ $\beta_k \circ \theta_t \mathcal{L}^d = (e_t)_{\#}\boldsymbol{\eta}_k \llcorner \mathbb{R}^d$ for all $t \in [0,T]$ and $\boldsymbol{\eta}_k(C([0,T]; \bar{\mathbb{R}}^d)) \leqslant \|\beta_k \circ \theta\|_{L^{\infty}((0,T);L^1(\mathbb{R}^d))}$. Therefore, since $\sum_{k=0}^{\infty} \beta_k(z) = z$, we have that $\boldsymbol{\eta} := \sum_{k=0}^{\infty} \boldsymbol{\eta}_k$ satisfies the same properties as in Theorem 3.6. Moreover, we may apply Proposition 3.1 for each k and then summing over all possible k, we conclude that θ is also a Lagrangian solution. $\square$

[1] We have assumed so far that the admissible class of β is $L^{\infty}(\mathbb{R}) \cap C^1(\mathbb{R})$, but a simple approximation argument can be done to allow a more general class $L^{\infty}(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$.

6 Conclusion and Future problems

In Chapters 4 and 5, we were able to extend the Lagrangian approach technique of obtaining a solution via the transport of the initial data by its associated flow for vector fields with the following structures:

- it can be written as hyperbolic convolution in the sense of (4.8), as proven in Theorem 4.1;
- if it can be written as a convolution of singular kernel with non-fundamental decay and a density in a Besov space as in (5.1) or (5.2), as proven in a simplified version of Nguyen's result [70] in Theorem 5.1.

As an application, we were able to present a criteria of Lagrangian structure for the Vlasov-Maxwell system (4.1) and the gSQG equation (1.8), provided we assume enough regularity of solutions; see Theorem 4.2 and Theorem 5.1. We have extensively used the ideas presented in Chapters 2 and 3 in order to conclude our four aforementioned main theorems.

Concerning future problems we are interested in investigating, we list them below without any order of preference:

(i) obtain a extension of Theorem 2.5 in two main branches: using the ideas of Proposition 4.1 to allow truncated singular rough kernels; obtaining a non average-zero singular rough kernel with desired approximating sequence of kernels (recall that the only example given by Nguyen for such vector fields is the one in Remark 11);

(ii) generalize Proposition 4.1 for more complicated truncated kernels, e.g. with truncation $\mathbb{1}_{A_\delta}$ for some general family of sets $\{A_\delta\}_{\delta>0}$ with suitable hypothesis;

(iii) drop the second order derivative integrability of the density j in the Vlasov-Maxwell system result Theorem 4.2, possibly exploring the ideas of [52, 53];

(iv) obtain Theorem 5.1 for vector fields with structure (5.1) and (5.2) without assuming Definition 5.1;

(v) obtain a Besov type continuation of regularity result *a la* [31, Theorem 1.5] for the gSQG equation. More precisely, we would like to ensure that if one assumes that the initial data is in some appropriate Besov space B and the solution is a less regular Besov space $B' \supset B$, then solutions are also in B for later times. The main motivation is that in the proof of [31, Theorem 1.5], they use the Lagrangian structure via the Cauchy-Lipschitz approach;

(vi) obtain a Lagrangian structure for the Landau-Vlasov equation [55], that is, the Vlasov equation as in Section 3.2 with $F_t(x) = -\nabla(V(x) + \rho_t(x))$, where V is a given potential and ρ the $v-$marginal of f;

(vii) obtain a Lagrangian structure for the quantum Vlasov equation [54], that is, the Vlasov equation as in Section 3.2 replacing $F \cdot \nabla_v f$ with

$$\int_{\mathbb{R}^6} \frac{i}{\epsilon} \left(V\left(x + \frac{\epsilon\tilde{v}}{2}\right) - V\left(x - \frac{\epsilon\tilde{v}}{2}\right)\right) f_t(x, y) e^{i(v-v')\cdot\tilde{v}} \,\mathrm{d}v' \,\mathrm{d}\tilde{v}.$$

This is the quantum version of the classical conservative force $F_t(x) = -\nabla V(x)$ with parameter ϵ (in a unit system, ϵ should read $\hbar$). An interesting question is whether we have stability of solutions and/or of flows as $\epsilon \to 0$;

(viii) in the same spirit as before, we would like to generalize the idea heavily utilized in Chapter 2 of obtaining a fundamental estimate for vector fields which we only have its the Fourier symbol in the sense $\mathcal{F}(\boldsymbol{b}_t)(\xi) = m(\xi)\mathcal{F}g_t(\xi)$. This is a natural extension of assuming that the vector field can be written as convolution of singular kernel and a density, which gives that $m \in L^\infty(\mathbb{R}^d)$ among other properties;

(ix) a natural extension of a Lagrangian structure for the Vlasov-Poisson equation studied in [6] is to consider bounded physical domains, in the sense that $(x, v) \in \Omega \times \mathbb{R}^3$ for some open and bounded $\Omega \subset \mathbb{R}^3$. The first major step in this direction was due to Fernández-Real [50], where they consider $\partial\Omega$ as a perfect conductor with specular reflection, that is, the electric field has no tangential component on the boundary and the particles' collision to the boundary is perfectly elastic, so that the angle of incidence matches the reflection's one. It is worth investigating whether the same type of result holds for magnetic analogous case, i.e. for $(x, v) \in (\mathbb{R}^3 \setminus \Omega) \times \mathbb{R}^3$, we only consider particles under magnetic field (the Vlasov-Biot-Savart equation, see Section 3.2), and still assuming that $\partial\Omega$ is a perfect conductor, we should assume now that the magnetic field has no normal component at the boundary. It is also of interest whether one can drop the perfect elastic collision hypothesis;

(x) following the work of [22], more precisely Theorem 2.3, it should be possible to extend their result for kernels without pointwise estimates, but rather integrals ones *a la* Lemma 2.3;

(xi) it has been proven by Nguyen [70, Proposition 1.2], the extension of Theorem 2.3 for finite measure densities is not possible. Notice, however, that it does not cover the case $K * \mu$, where K is a singular kernel with singularity of order $d - 1$ at the origin and μ a finite density. This is particularly interesting for the 2D-Euler equation with finite measure vorticity;

(xii) it has been proven by Colombo-Crippa-Spirito [36] that if the vector field has the renormalization property Condition 1.1, as well as satisfies the growth assumption (1.6) and integrable in time, bounded in space divergence, then there exists a unique solution of the so called damped continuity equation

$$\begin{cases} \partial_t u + \operatorname{div}(\boldsymbol{b}u) = cu & \text{in} \quad [0,\infty) \times \mathbb{R}^d; \\ u_{t=0} = u_0 & \text{on} \quad \mathbb{R}^d \end{cases}$$

and has a Lagrangian structure, that is, it can be written as

$$u(t,x) = \exp\left(\int_0^t c_\tau(\boldsymbol{X}(\tau,t,x)) - \operatorname{div} \boldsymbol{b}_\tau(\boldsymbol{X}(\tau,t,x))\, \mathrm{d}\tau\right) u_0(\boldsymbol{X}(0,t,x)),$$

even if one considers $c \in L^1((0,T) \times \mathbb{R}^d)$; the DiPerna-Lions result [48] needed to assume that $c \in L^1((0,T); L^\infty(\mathbb{R}^d))$, and so $\operatorname{div} \boldsymbol{b}$, c are treated the same. Moreover, it has been hinted by Ambrosio-Colombo-Figalli in a comment before [6, Theorem 4.10] that in the $c \equiv 0$ case, i.e. (1.4), that the Lagrangian solution (1.5) is a renormalized or weak solution of continuity equation (and analogously (1.3) should be a Lagrangian solution of transport equation (1.1)) without the growth assumption, but they only prove it for divergence-free vector fields. Therefore, it should be possible to extend at least the existence result in [36] without the growth assumption;

(xiii) finally, the generalization of Theorem 2.6 for densities in the space of bounded deformation, that is

$$E\boldsymbol{b} = \frac{1}{2}\left(D\boldsymbol{b} + (D\boldsymbol{b})^t\right) \in L^1((0,T); \mathcal{M}(\mathbb{R}^d; \mathbb{R}^d)).$$

Of course, the Theorem 2.4 is no longer available in this case, but an analogous result proven by De Philippis-Rindler [72] gives that

$$\mathrm{d}(E\boldsymbol{b})^s(x) = \left(\frac{a(x) \otimes b(x) + b(x) \otimes a(x)}{2}\right) \mathrm{d}|(E\boldsymbol{b})^s|(x).$$

This is by far the most difficult open problem listed in this chapter, for no analogous result is available apart from [8], which still heavily restricts the type of vector fields considered in order to obtain the renormalization property.

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