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Mathematical Analysis of Chemotaxis Models with Fractional Diffusion

Análise Matemática de Modelos de Quimiotaxia com Difusão Fracionária

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Orientadora: Profa. Dra. Anne Caroline Bronzi

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In memory of my dad,
who longed for this moment more than anyone else.

Em memória do meu pai,
que aguardava por este momento mais do que
qualquer outra pessoa.

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Resumo

O modelo de Keller-Segel é um sistema de equações diferenciais parciais que descreve a quimiotaxia, movimento de organismos em resposta a gradientes químicos, crucial em muitos processos biológicos. Este trabalho estende o modelo clássico incorporando difusão anômala, especificamente superdifusão, através do uso do Laplaciano fracionário. Essa modificação captura efeitos difusivos não locais observados em experimentos, especialmente em ambientes com alvos esparsos.

A primeira parte desta tese analisa um sistema de Keller-Segel fracionário duplamente parabólico, com Laplacianos fracionários modelando tanto a difusão celular quanto a química, usando expoentes diferentes. A boa colocação local de soluções brandas é estabelecida, e, sob certas condições, a boa colocação global é demonstrada. O estudo também explora o comportamento assintótico de das soluções.

A segunda parte estende o modelo de Keller-Segel, incluindo processos adicionais, como escoamento oceânico e reações biológicas, enfatizando o papel da quimiotaxia na sustentação de reações no contexto da difusão anômala da densidade celular. O modelo é particularmente motivado pela desova por dispersão em organismos aquáticos. Este estudo foi realizado durante o estágio de pesquisa na Duke University, sob a supervisão do Prof. Dr. Kiselev.

Palavras-chave: Modelo de Keller-Segel; Quimiotaxia; Laplaciano Fracionário; Difusão anômala; Soluções brandas; Reações biológicas.

Abstract

The Keller-Segel model is a system of partial differential equations that describes chemotaxis, the movement of organisms in response to chemical gradients, which is crucial in many biological processes. This work extends the classical model by incorporating anomalous diffusion, specifically superdiffusion, through the use of the fractional Laplacian. This modification captures nonlocal diffusive effects observed in experimental settings, particularly in environments with sparse targets.

The first part of this thesis analyzes a doubly parabolic fractional Keller-Segel system, with fractional Laplacians modeling both cellular and chemical diffusion using different exponents. Local well-posedness is established, and global well-posedness is shown under certain conditions. The study also explores the asymptotic behavior of solutions.

The second part extends the Keller-Segel model by including additional processes such as fluid flow and biological reactions, emphasizing the role of chemotaxis in sustaining reactions within the context of anomalous diffusion of cell density. The model is particularly motivated by broadcast spawning in aquatic organisms. This study was carried out during the research internship conducted at Duke University, under the supervision of Prof. Dr. Kiselev.

Keywords: Keller-Segel Model; Chemotaxis; Fractional Laplacian; Anomalous diffusion; Mild solutions; Biological reactions.

Index of Notation

Notation used only in the section in which it is introduced is largely omitted from this list.

A^T	Transpose of a matrix A.
$C_0(\mathbb{R}^d)$	Space of continuous functions that tend to zero at infinity.
$\det A$	Determinant of a matrix A.
Γ	Gamma function.
$(-\Delta)^{-1}$	Inverse of Laplace operator in $\mathbb{R}^d$.
$\lceil x \rceil$	The smallest integer greater than or equal to x.
$\mathcal{S}(\mathbb{R}^d)$	Schwartz space.
∂^η	$\partial^\eta = \left(\frac{\partial}{\partial x_1}\right)^{\eta_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\eta_d}$, for $\eta = (\eta_1, \ldots, \eta_d)$ a multi-index.[1]
$\lvert\alpha\rvert$	$\lvert\alpha\rvert = \sum_1^n \alpha_j$, for $\eta = (\eta_1, \ldots, \eta_d)$ a multi-index.
$\alpha!$	$\alpha! = \prod_1^n \alpha_j!$, for $\eta = (\eta_1, \ldots, \eta_d)$ a multi-index.
ξ^η	$\xi^\eta = \prod_{j=1}^d \xi_j^{\eta_j}$, for $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$.

[1] A multi-index is an ordered d-tuple of nonnegative integers.

Contents

Introduction **12**

Research Problem and Relevance of the Study 12

Structure of the Thesis 14

1 Literature Review **18**

1.1 Chemotaxis 18

1.2 Mathematical Models of Chemotaxis 19

1.2.1 The Minimal Model 20

1.2.2 Literature Review of Key Mathematical Results for the Classical Keller-Segel Model 21

1.2.3 Chemotactic Collapse in Keller-Segel Systems 24

1.3 Anomalous Diffusion Process 25

1.3.1 Superdiffusion Process in Biology 27

1.4 Partial Differential Equations and Fourier Analysis 29

1.4.1 Applications 30

1.5 Self-Similar Solutions and Critical spaces 33

1.5.1 Scaling Heuristics 34

1.5.2 Analyses of Critical spaces 36

1.6 Literature Review of Key Mathematical Results for the Fractional Keller-Segel Model 38

1.6.1 The Parabolic-elliptic Fractional Keller-Segel Model 39

1.6.2 Generalized Parabolic-elliptic fractional Keller-Segel model 43

1.6.3 The Parabolic-parabolic Fractional Keller-Segel Model 45

1.7 Further PDE models for chemotaxis 46

2 Preparations **47**

2.1 Introduction 47

2.2 Mild solution to Keller-Segel system 49

2.2.1 Contraction Mapping Principle 50

2.3 Preliminary . . . 52
2.4 Properties of Solutions . . . 58

3 Generalized Keller-Segel model **64**
3.1 Introduction . . . 64
3.2 Local existence of solutions . . . 65
3.3 Global in time solutions . . . 75
3.4 Nonnegative Solution . . . 81
3.5 Well-posedness in Weighted spaces . . . 81

4 Chemotaxis and Reactions **85**
4.1 Introduction . . . 85
4.2 Global Existence of smooth solutions . . . 87
4.2.1 Nonnegative Solution . . . 92
4.3 Reaction efficiency . . . 93
4.3.1 Reaction in a Chemotaxis-Free Environment . . . 94
4.3.2 Reaction in a Chemotactic Environment . . . 98
4.3.2.1 Lower bound for the total fraction of unfertilized eggs . . . 98
4.3.2.2 Upper bound for the total fraction of unfertilized eggs . . . 101
4.3.3 Chemotaxis Impact on Reaction Efficiency . . . 110

Bibliography **112**

Appendix A Fractional Laplacian **119**

Appendix B Parameters for Lebesgue Spaces **123**

Introduction

Research Problem and Relevance of the Study

This thesis is centered around the analysis of nonlinear nonlocal parabolic equations modeling the evolution of the density of mutually interacting particles. These equations incorporate inertial-type nonlinearities and anomalous diffusive terms that describe nonlocal interactions. A prototype of this model can be given by

$$\begin{cases} \partial_t \rho = -(-\Delta)^{\alpha/2}\rho - \chi\nabla\cdot(\rho\mathcal{B}(c)) + F(\rho) & x \in \mathbb{R}^d, \quad t > 0, \\ \tau\partial_t c = -\kappa(-\Delta)^{\beta/2}c + G(\rho, c) & x \in \mathbb{R}^d, \quad t > 0, \end{cases} \tag{1}$$

where $\rho = \rho(x,t)$ and $c = c(x,t)$ are the unknown, $\mathcal{B}$ is an integral operator, $F = F(\rho)$ and $G = G(\rho, c)$ are functions of the unknown, and the diffusion in both equations is given by a fractional power of the Laplacian operator.

For appropriate choices of the functions $\mathcal{B}$, F, and G, these nonlinear partial differential equations become the ones representing models in physics, mathematical biology, fluid mechanics, chemistry, engineering, and social science. The shared mathematical structures in different models, stemming from the similarity in their dynamics, enable the use of the same mathematical techniques across a wide range of distinct physical contexts.

A notable example of such a system can be found in the Keller-Segel models, widely studied in mathematical biology to describe chemotaxis. In these models, cells with density ρ move in response to chemical gradients, represented as $\mathcal{B}(c) = \nabla c$. When $G(\rho, c) = \zeta\rho - \gamma c$, the model captures the dynamics of a chemoattractant with density c, which is secreted by the cells themselves at a rate $\zeta > 0$ while degrades at a rate $\gamma \geq 0$ due to chemical reactions. The inclusion of a damping term, such as $F(\rho) = \rho(a - b\rho)$, $a,\ b > 0$, enables the capture of birth and death processes, where the coefficient a represents the intrinsic growth rate of cells and b measures its intraspecific competition.

Chemotaxis systems describe two contrasting processes: the diffusion of cells driven by their random motion, and their directed movement toward regions with higher concentrations of a chemical. The latter may lead to the aggregation of cells (chemotactic collapse), which is viewed as a finite-time blow-up of solutions [15, 23]. For instance, in the case of the classical parabolic-elliptic Keller-Segel model given by (1) with $\alpha = \beta = 2$, $F = 0$, $G(\rho, c) = \rho$, and $\tau = 0$, the integral operator $\mathcal{B}$ becomes a function only of ρ and x, i. e., $\mathcal{B}(\rho) = CK * \rho$, where $K(x) = x|x|^{-2-d+\beta}$ and $C > 0$ is a constant. Since $|K(x)|$ is a radially symmetric, nonincreasing function of $r = |x|$, equation (1) has a nonlinear drift term that concentrates the distribution of particles, acting in the opposite direction of the diffusive term. Therefore, a competition arises between the nonlinear transport and the linear dissipative terms [12]. For this system, in the one-dimensional case, standard diffusion is strong enough to ensure the global existence of solution; however, for $d \geq 2$, there is a balance between diffusion and aggregation, and the existence of global solution depends on the size of the initial condition in a suitable norm. Hence, it is mathematically intriguing to determine the minimal diffusion strength required to outweigh chemotactic forces so that the global existence of regular solutions is ensured [23].

For $\alpha < 2$, the anomalous diffusion processes introduce challenges, such as nonlocality and weaker dissipation effects (compared to the classical one, $\alpha = 2$), making the system more susceptible to blow-ups in finite time [25, 33]. These issues demand a deeper investigation into the interplay between nonlocality, diffusion strength, and chemotactic forces. This thesis seeks to address these challenges by analyzing generalized Keller-Segel models with fractional diffusion, aiming to establish conditions for local and global well-posedness of solutions.

Moreover, in classical models, $\alpha = \beta = 2$, local diffusion dynamics are assumed, which correspond to cellular motion governed by Brownian diffusion. However, in real-world phenomena, cellular populations often exhibit nonlocal interactions, particularly in situations when chemoattractants, food, or other targets are sparse or rare in the environment. In such cases, Lévy walk – a form of anomalous diffusion – provides a more precise description of cellular interactions [33, 34]. Furthermore, the nonlocal nature of fractional Laplacians introduces rich mathematical structures that mirror the long-range interactions observed in these systems. Therefore, the study of fractional diffusion equations is not only mathematically interesting but also practically relevant due to their wide applicability in modeling complex systems. In biological contexts, for example, the bacterium *Escherichia coli* has an exploratory behavior like anomalous diffusion in an environment with fluctuating levels of a type of protein, while the myxamoebae *Dictyostelium discoideum* displays this behavior in nutrient-scarce environments. Beyond biology, these equations also find applications in modeling turbulence in fluid dynamics, energy dissipation in fractional viscoelastic materials, and decision-making processes in social systems. Superdiffusion type of behavior can also be observed in chaotic transport in turbulent flows [68, 74, 79], which can justify the use of fractional Laplacian in both equations of model (1).

This research bridges gaps in the understanding of how nonlocality influences solution behaviors, particularly in terms of regularity and asymptotic properties. By addressing the challenges posed by nonlocal fractional diffusion, this thesis contributes to the theoretical understanding of nonlinear partial differential equations and their applications. To the best of our knowledge, despite significant advancements, the fundamental question of global and local well-posedness for the fully parabolic-parabolic fractional Keller-Segel system (model (1) with $F = 0$ and $G(\rho, c) = \zeta\rho - \gamma c$) in $\mathbb{R}^d$, involving distinct exponents α and β, remained unresolved before this thesis. In Chapter 3, we thoroughly investigate this specific system.

Structure of the Thesis

Here we outline the structure of the thesis and provide a roadmap for the topics that are addressed in subsequent chapters. The thesis is organized into four chapters, supplemented by two appendices. Its structure aims to provide a solid theoretical foundation and insights, contributing to the understanding of the mathematical model for chemotaxis in scenarios of fractional diffusion. Each chapter addresses a distinct aspect of the research, as outlined below.

Chapter 1 presents the context and motivates the research problems addressed in this thesis. It includes an overview of various chemotaxis models and fractional diffusion processes, highlighting key mathematical results from the literature and some tools to analyze them. As the chapter provides a background to equip the readers with an understanding of foundational topics, its key aspects are referenced throughout the outline that follows.

Chapter 2 offers a detailed review of the mathematical tools and concepts required to approach a fractional Keller-Segel system. In this chapter, we reformulate the system under analysis into an equivalent integral equation, the mild solution, constructed using Duhamel's and the contraction mapping principles. The connection to Fixed-Point Theorems arises as this integral form is treated as a fixed-point problem in a suitable Banach space, leading to the establishment of the existence and uniqueness of the mild solution. Next, key properties and estimates involving the kernel functions K_t^α and K^α, defined in (1.17) and (1.19) respectively, are established, setting the stage for the application of fixed-point theorems. Finally, some properties of the solution are discussed, with further exploration in later chapters. Overall, this chapter lays the groundwork for the rigorous treatment of the fractional Keller-Segel model and its extensions, explored in detail in Chapter 3 and Chapter 4.

Chapter 3 focuses on the generalized fractional Keller-Segel model (1.5), providing results on the existence and uniqueness of the mild solution to this system. The discussion on the local well-posedness of solutions in Lebesgue space is displayed in Section 3.2, while their global-in-time existence and uniqueness are covered in Section 3.3. Additionally, in Section 3.5, results regarding local existence in weighted spaces are presented. Each of these sections begins by establishing estimates that demonstrate the fulfillment of the contraction mapping principles' premises. Specifically, in **Theorem 3.4 (Local existence of solutions)**, local well-posedness for initial conditions $\rho_0 \in L^p(\mathbb{R}^d)$ and $\nabla c_0 \in L^\wp(\mathbb{R}^d)$ is proven for specific values of p and $\wp$, with $p, \wp \geq 1$, ensuring that the mild solution $(\rho, \nabla c)$ exists in $[0, T]$, for $T = T(\|\rho_0\|_{L^p}, \|\nabla c_0\|_{L^r})$, in which $\rho \in C\big([0, T], L^p(\mathbb{R}^d)\big)$, $\nabla c \in C\big([0, T], L^r(\mathbb{R}^d)\big)$, for certain values of r. Likewise, in **Theorem 3.12 (Global in time solutions)**, global well-posedness is proven under smallness assumptions in Lebesgue norms for initial data, $\rho_0 \in L^{p_1}(\mathbb{R}^d)$ and $\nabla c_0 \in L^{p_2}(\mathbb{R}^d)$, for certain values of p_1 and p_2, where the mild solution $(\rho, \nabla c)$ is such that $\rho \in C\big([0, T], L^p(\mathbb{R}^d)\big)$ and $\nabla c \in C\big([0, T], L^r(\mathbb{R}^d)\big)$ for certain values of p and r. These parameters (p, r, $\wp$, p_1 and p_2) defining the Lebesgue spaces depend on the values of α and β, and the inequality $\alpha \neq \beta$ adds complexity to the problem by preventing certain simplifications possible for the case $\alpha = \beta$ or for the parabolic-elliptic case (see also Section 1.5). The number of parameters and conditions to ensure the boundedness of the mild solution is larger than in the classical case. Additionally, we examine the asymptotic behavior of solutions, which offers insights into the long-term evolution of the system. Subsequently, in this section, we explore further results and some implications arising from the theorem of global-in-time existence. Next, in Section 3.4, we establish conditions under which the solutions remain nonnegative, thereby ensuring the biological relevance of the model. The chapter concludes by addressing local well-posedness in weighted spaces, proving the stability of the system in spaces that account for fractional diffusion.

Chapter 4 focuses on the interaction between chemotaxis and biological reactions, by studying an extension of the parabolic-elliptic fractional Keller-Segel system that incorporates a reaction term into the chemotaxis model (for example and context on this type of system, refer to Section 1.7). This extension enables the analysis of how chemotactic movement influences biological processes such as broadcast spawning – a reproductive strategy used by many aquatic invertebrates. **Local well-posedness (Theorem 4.3)** is established via the fixed-point theorem, similar to the analysis in Chapter 3, as implied by Lemma 4.2, and **global well-posedness (Theorem 4.9)** by constraining the growth of the solution norms in a specific space, which is achieved through a bound on the $L^\infty(\mathbb{R}^d)$ norm of the solution (Lemma 4.6). The control on the $L^\infty(\mathbb{R}^d)$ norm arises from the balance between the chemotactic term, responsible for population aggregation and possible singularities formation, and the reaction term (see Section 1.2), which prevents the solution from losing regularity in finite time. Thus, the time existence of the solution is extended through iterative applications of local results. The chapter's main focus is the analyses of how chemotaxis enhances biological responses, particularly overall reaction rates, in systems where organisms or cells display superdiffusion behavior (see Section 1.3.1 for a discussion of superdiffusion in biological processes). By comparing environments with and without chemotaxis, insights are gained into the role of directed movement in enhancing biological reactions. To achieve this, bounds for the $L^1(\mathbb{R}^d)$ norm of the solution ρ, which describes the fraction of unfertilized eggs ($m(t)$), are established in both chemotactic and chemotactic-free environments. These bounds indicate the effectiveness of chemotaxis in ensuring a higher fertilization rate, a crucial metric in reproductive biology. The results in this chapter pave the way for further investigations into chemotaxis in more complex reaction environments.

The results in Chapters 3 and 4 represent new contributions to the study of chemotaxis with fractional diffusion. In Chapter 3, the rigorous analysis of the generalized fractional Keller-Segel model provides novel findings on well-posedness, especially for the case of fractional diffusion with different exponents, a topic not extensively covered in prior research. In Chapter 4, the analyses provided reveal original insights into the interplay between chemotaxis and biological processes in environments characterized by superdiffusion. These contributions advance the mathematical understanding of chemotaxis models involving the fractional Laplacian, offering new perspectives on the behavior of such systems.

Appendix A presents key mathematical properties of the Fractional Laplacian, an operator that extends the classical one and accounts for nonlocal interactions, crucial for modeling superdiffusion (see Section 1.3).

Appendix B is dedicated to the study of the parameters defining Lebesgue spaces in Theorems 3.4 and 3.12. In this appendix, the existence of these parameters is established, and it is demonstrated that the constraints they satisfy are precisely the conditions required for applying Hölder's inequality, as well as some other key estimates from Chapter 2 that allows us to use the Fixed Point Theorem and to prove estimates for the solution established in Chapter 3.

Chapter 1

Literature Review

In this chapter, we begin by introducing the foundational concepts that underpin this thesis, focusing on chemotaxis, mathematical models of chemotaxis, and fractional diffusion processes. The chapter offers a detailed review of relevant mathematical results, including key advancements related to the Keller-Segel model and its fractional variations. The background provided here aims to equip the reader with a solid understanding of these models and their implications in biological systems, particularly in the context of anomalous diffusion. If the reader is already familiar with these preliminary aspects, it is possible to skip directly to Chapter 2.

1.1 Chemotaxis

Organisms and cells don't move randomly, rather, their movement is influenced by external stimulants/signals that determine both the direction and distance of their motion. This controlled movement, known as taxis, plays a critical role in various aspects of an organism's behavior, such as finding food, avoiding predators, and attracting mates. Essentially, organisms and cells sense their environment and respond to it [3]. Taxis can be triggered by different types of stimuli, each eliciting a specific response: phototaxis (light), chemotaxis (chemicals), thermotaxis (temperature changes), electrotaxis (electric fields or galvanotaxis), and gravitaxis (gravity). Among these, chemotaxis is particularly significant, as it involves the movement of cells in response to chemical signals.

Chemotaxis is the process by which cells detect and move along concentration gradients of specific chemicals, allowing organisms to position themselves optimally within their environments. For example, male moths follow pheromone gradients released by females to locate mates, while fruit flies navigate toward attractive odors when searching for food and away from repellent substances. In this regard, the process is called positive chemotaxis when the movement is directed toward a higher concentration of a chemical substance. Conversely, movement toward a lower concentration is known as negative chemotaxis. The chemicals that induce these movements are called chemoattractants and repellents, respectively [3, 41, 66].

Beyond these examples, chemotaxis plays a crucial role in cellular communication, influencing how cells arrange and organize themselves. It is fundamental in coordinating cell migration during organogenesis in embryonic development and tissue homeostasis in adults. In multicellular organisms, chemotaxis is fundamental at all stages of the life cycle: During fertilization, sperm cells are drawn to chemical substances released from the egg's outer layer. Throughout embryonic development, chemotaxis is crucial in organizing cell positioning, fundamental for processes such as gastrulation and patterning of the nervous system. In immune responses, chemotaxis guides immune cell migration to sites of inflammation and fibroblasts into wounded regions to initiate healing. Chemotaxis also plays a significant role in cancer progression. Solid tumors release chemical signals, such as Vascular Endothelial Growth Factors (VEGFs), that induce chemotactic responses within the body. This triggers angiogenesis, the development of a capillary network directed toward the tumor, which enhances the tumor's blood supply and supports its growth [40, 41, 66].

1.2 Mathematical Models of Chemotaxis

The mathematical modeling of chemotaxis in cellular systems has its origins in the pioneering works of Keller and Segel (1970) [45] and Patlak (1953) [65], with the Keller-Segel model standing out as the most extensively studied one. This model consists of a system of partial differential equations describing the chemically induced movement of cells (or organism) density ρ towards increasing concentrations of a chemical substance, a chemoattractant, of density c. In its general form, the Keller-Segel model can be expressed as [3, 40]:

$$\begin{cases} \partial_t \rho = \nabla \cdot (\phi(\rho, c)\nabla \rho - \psi(\rho, c)\nabla c) + f(\rho, c) & x \in \mathbb{R}^d, \quad t > 0, \\ \tau \partial_t c = \kappa \Delta c + g(\rho, c) - h(\rho, c)c & x \in \mathbb{R}^d, \quad t > 0, \\ \rho(t=0) = \rho_0, \quad c(t=0) = c_0 & x \in \mathbb{R}^d. \end{cases} \tag{1.1}$$

where $\phi = \phi(\rho, c)$ describes cell diffusivity (also referred to as motility), $\psi = \psi(\rho, c)$ represents the chemotactic sensitivity (determining the intensity of the chemotactic flux in response to the chemical gradient), $f = f(\rho, c)$ accounts for the growth and death of cells. The parameter κ is the diffusion coefficient of the chemoattractant, while $g = g(\rho, c)$ and $h = h(\rho, c)$ are kinetic functions describing the production and degradation of the chemical signal, respectively.

This system of equations is classified as a parabolic-parabolic system when both $\tau > 0$ and $\kappa > 0$. Depending on the properties of the chemoattractant, the system can be simplified. For chemoattractants with rapid diffusion, the system reduces to a parabolic-elliptic system (when $\tau = 0$ and $\kappa > 0$). Alternatively, for non-diffusive or slowly diffusing chemoattractants, it simplifies to a parabolic-hyperbolic (or parabolic-ODE) system (when $\tau > 0$ and $\kappa = 0$) [3].

1.2.1 The Minimal Model

The **minimal Keller-Segel model**, as named by Childress and Percus [28], is the simplest and most classical model used to describe the collective motion of cells in response to chemical signals [66]. This model is a simplified version of system (1.1), where the functions ϕ, ψ, f, g and h are linear. Specifically, $\psi(\rho, c) = \chi\rho$ with $\chi > 0$, $f(\rho, c) = 0$, $g(\rho, c) = \zeta\rho$ with $\zeta > 0$, and $h(\rho, c) = \gamma$ with $\gamma \geq 0$ (which is biologically relevant when $\gamma > 0$). The model can thus be expressed as:

$$\begin{cases} \partial_t \rho = \Delta\rho - \chi\nabla\cdot(\rho\nabla c) & x \in \mathbb{R}^d, \quad t > 0, \\ \tau\partial_t c = \kappa\Delta c + \zeta\rho - \gamma c & x \in \mathbb{R}^d, \quad t > 0, \\ \rho(t=0) = \rho_0, \quad c(t=0) = c_0 & x \in \mathbb{R}^d. \end{cases} \tag{1.2}$$

As the biological background of (1.2) suggests, the solution to this system remains nonnegative, provided that the initial data is nonnegative and sufficiently regular. Moreover, since this model does not account for birth or death processes, it focuses solely on cell motion, the total number of cells, given by the $L^1(\mathbb{R}^d)$ norm of ρ, is conserved over time: $\|\rho_0\|_{L^1} = \|\rho(\cdot, t)\|_{L^1}$ [3, 31, 41, 66, 75].

The Minimal Keller-Segel Model is a foundational framework for more complex chemotaxis models [3]. It is also a paradigm for studying pattern formation in various cellular processes, including meiosis, embryogenesis, angiogenesis, Balo disease, and bio-convection. Additionally, it can be viewed as an initial step towards comprehending the evolutionary transition from unicellular organisms to more complex structures [17].

This model has been extensively studied in mathematical biology due to its rich mathematical structure and its ability to capture essential features of biological systems [3]. The model's qualitative behavior includes phenomena such as finite-time blow-up solutions and pattern formation, where cells exhibit strong density variations in their spatial distribution, as positive chemotaxis leads to cell aggregation, resulting in high-density clusters.

The model mainly describes the tendency of cells to aggregate in response to a chemical signal, where the aggregation, modeled by the term $\nabla \cdot (\chi \rho \nabla c)$, is balanced by cell diffusion. The boundedness of a solution indicates that the power of diffusion is stronger than that of chemotaxis. Conversely, aggregation can manifest mathematically as a finite-time blow-up, where cell density ρ becomes unbounded in finite time. Such a phenomenon to occur requires a certain threshold number of individuals. For example, in some species of myxamoebae, such as *Dictyostelium discoideum (Dd)*, experimental observations show that aggregation occurs only if the total number of myxamoebae exceeds a certain threshold; otherwise, the cells continue to spread [3, 17, 41, 66].

1.2.2 Literature Review of Key Mathematical Results for the Classical Keller-Segel Model

System (1.2) is also referred to as the Classical Parabolic-Parabolic Keller-Segel Model, or simply the Keller-Segel model. A parabolic-elliptic version of this model arises, as mentioned before, from the assumption that the production and diffusion of the chemical occur much faster than other time scales in the problem ($\kappa \sim \zeta \gg \tau$). Under this condition, the second equation in (1.2) simplifies to an elliptic equation. Consequently, this model reduces to a single equation, as the gradient of the chemical concentration can be expressed as a function of ρ [8, 46, 48]. Specifically, by setting $\tau = 0$ and $\kappa = \zeta = 1$, the second equation of (1.2) leads $\nabla c(x,t) = \nabla (-\Delta + \gamma I)^{-1} \rho(x,t)$, which further reduces to $\nabla c(x,t) = \nabla (-\Delta)^{-1} \rho(x,t)$ when $\gamma = 0$. This leads to the following form of the classical parabolic-elliptic Keller-Segel Model:

$$\begin{cases} \partial_t \rho = \Delta \rho - \chi \nabla \cdot \left(\rho \nabla (-\Delta + \gamma I)^{-1} \rho \right) & x \in \mathbb{R}^d, \quad t > 0, \\ \rho(x,0) = \rho_0(x) & x \in \mathbb{R}^d. \end{cases} \tag{1.3}$$

Below, we summarize key mathematical results for the parabolic-elliptic and parabolic-parabolic Keller-Segel models on the domain $\mathbb{R}^d$. The global existence of solutions for these systems is influenced by the parameters τ, κ, γ, and χ, as the competition between diffusion and chemotaxis drives the possibility of blow-up. Therefore, to focus on the core dynamics, we retain the parameter γ in systems (1.2) and (1.3), while normalizing all other biological and chemical constants to 1.

- **The parabolic-elliptic Keller-Segel model** with $\gamma = 0$ exhibits the following behavior:
  - In one dimension, this system has globally smooth and unique solutions that remain uniformly bounded for all $t \geq 0$ [3, 41].
  - In two dimensions, solutions exist globally in time if the total mass $m_0 = \|\rho_0\|_{L^1}$ satisfies $m_0 < 8\pi/\chi$. If $m_0 > 8\pi/\chi$, the solution blows up in finite time, meaning the solution becomes unbounded, i. e., at the blow-up time $T > 0$, $\limsup_{t\to T}\|\rho(\cdot,t)\|_{L^\infty} = \infty$ [18]. For bounded domain see [60, 61, 63].
  - For $d \geq 2$, the Keller-Segel system is critical in $L^{d/2}(\mathbb{R}^d)$, meaning that a small initial condition in $L^{d/2}(\mathbb{R}^d)$ ensures global well-posedness in time, whereas a large mass leads to blow-up. There exists a unique local-in-time mild solution where $\rho \in C([0,T), L^p(\mathbb{R}^d))$ for every $\rho_0 \in L^p(\mathbb{R}^d)$ with $p > d/2$. Additionally, there is a constant C, small enough, such that when $\left(\chi \int_{\mathbb{R}^d} \frac{|x|^2}{2}\rho_0(x)\mathrm{d}x\right)^{\frac{d-2}{d}} \leq C\,\chi m_0$ there is no global smooth solution, with enough decay in x at infinity, to the system [5, 66].

- **The parabolic-parabolic Keller-Segel system** is characterized by the following features:
  - In one dimension, it has a global solution in time that converges to a stationary solution as $t \to \infty$ [41].
  - In dimension two:
    - Mizoguchi [59] proved that if $m_0 < 8\pi/\chi$, then for $\rho_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $c_0 \in L^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$, the solution exists globally in time. For the critical case $m_0 = 8\pi/\chi$, the solution may either exist globally or blow up, depending on additional conditions on the initial data.
    - Mizoguchi [59] pointed out that the meaning of "critical" mass, which separates global existence and blow-up, differs between the parabolic-parabolic and parabolic-elliptic systems. According to the author, $8\pi/\chi$ is the critical mass in the sense that all solutions to the parabolic-parabolic system with $m_0 < 8\pi/\chi$ exist globally in time, while there exists a solution with $m_0 > 8\pi/\chi$ that blows up in finite time.

  - Biler et al. [10] constructed a forward self-similar nonnegative solutions and prove that, in some cases, such solutions exist globally even when their total mass is above $8\pi/\chi$, contrasting with the parabolic-elliptic case.
  - We highlight the contributions of Biler [6], Calvez et al. [27], and Nagai [62] in the context of earlier results. For example, Nagai [62] established global existence for masses smaller than $4\pi/\chi$, provided the initial conditions $(\rho_0, c_0) \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$.

- In dimensions $d \geq 3$, Corrias et al. [30] proved global existence when the initial density is small in $L^q(\mathbb{R}^d)$ with $q > d/2$, and the initial chemical gradient is small in $L^d(\mathbb{R}^d)$. This solution vanishes as the heat equation for large times and exhibits a regularizing effect of hypercontractivity type.
- Biler et al. [7] proved that global solutions of the parabolic-parabolic Keller–Segel system in $\mathbb{R}^2$, $d \geq 2$, can be obtained from initial data of arbitrary size, given that parameter τ in (1.2) is large enough.

Remark 1.1. *Below, we highlight some key aspects of the Keller-Segel system's behavior and the corresponding mathematical results, which will be revisited and analyzed throughout this chapter.*

(a) Threshold Values: *Arumugam et al. [3] emphasized that the threshold values for global existence and blow-up of solutions are well-established only for the classical parabolic-elliptic and parabolic-parabolic Keller-Segel models.*

(b) Impact of γ: *Triebel [75] pointed out that while the value of γ (whether $\gamma > 0$ or $\gamma = 0$) does not affect significantly the behavior of the parabolic-parabolic classical Keller-Segel model, it has a significant impact on the parabolic-elliptic version.*

(c) The parabolic-elliptic Keller-Segel system with $\gamma > 0$: *Biler et al. [9] proved the existence of a unique global-in-time solution for the classical parabolic-elliptic Keller-Segel (1.3) in two dimensions, even for $m_0 > 8\pi/\chi$, if γ is sufficiently large. Their results show that for any initial condition $\rho_0 \in L^1(\mathbb{R}^2)$, there exists $\gamma(\rho_0) > 0$ such that for all $\gamma \geq \gamma(\rho_0)$ the solution remains global-in-time. Thus, for each ρ_0 (not necessarily nonnegative) and γ large enough depending $\rho_0 \in L^1(\mathbb{R}^2)$ equation (1.3) has no critical value of mass which leads to a blow-up of solutions. On the other hand, if $m_0 > 8\pi/\chi$ and $0 \leq \gamma << 1$, the solutions blow up in a finite time.*

(d) Blow-up solution for parabolic-elliptic Keller-Segel model: *The viral argument used to demonstrate blow-up in the classical parabolic-elliptic Keller-Segel system involves studying the evolution of the second-order moment, $m_2(t) = \int_{\mathbb{R}^d} \frac{|x|^2}{2} \rho(x,t)\, dx$, showing that $m_2(t)$ vanishes at some time $T > 0$, leading to a contradiction. Biler et al. [9] formulated a criteria for blow-up of nonnegative solutions based on the local concentration of the initial data, which does* *not require supplementary properties of ρ_0 such as the argument using moment that assumes $\rho_0 \in L^1(\mathbb{R}^d, (1+|x|^2)\mathrm{d}x)$.*

1.2.3 Chemotactic Collapse in Keller-Segel Systems

The potential for chemotactic equations to produce singular solutions was first proposed by Nanjundiah (1973) and later examined by Childress and Percus (Childress and Percus (1981); Childress (1984)). Nanjundiah [64] suggested that aggregation proceeds to the formation of $\delta-$functions in cell density – a phenomenon later termed "chemotactic collapse" by Childress et al. [28]. Childress et al. [28] argued that although this outcome might seem to contradict the diffusive nature of the model, the fact that chemotaxis has some features of “negative diffusion” suggests the possibility of singular behavior. They showed that the chemotactic collapse cannot occur in the case of one-dimensional and further argued the possibility that, in a two-dimensional system, a threshold number of cells is required for such a collapse. The physical mechanism behind these singularities is that the generation of attractants is increased by the concentration of cells, which in turn further increases the cells' density [3, 21].

As mentioned, in one-dimensional systems, solutions to the minimal model remain globally bounded over time, as diffusion dominates aggregation, preventing blow-up. However, in higher dimensions, the balance between diffusion and aggregation becomes more delicate, with the global behavior of solutions depending on the initial conditions [19].

The fact that solutions in one dimension always exist globally in time crucially affects the types of patterns that can form in higher-dimensional biological systems. For instance, in two-dimensional systems, while a collapse to an infinite-density point can occur, a collapse can't result in an infinite-density line due to the inherent restriction of one-dimensional collapse. Correspondingly, in a three-dimensional system, collapse to infinite-density lines and points can occur, but a collapse to an infinite-density sheet is mathematically impossible. Consequently, in a three-dimensional system, a mass of cells cannot collapse to a two-dimensional plane in finite time, whereas a two-dimensional collapse (a cylindrical mass of cells contracting to a line) and a three-dimensional collapse (a spherical mass of cells contracting to a point) are admissible [21, 33].

However, this is not the general case, since one-dimensional collapse was shown experimentally. Escudero [33] mentioned an in vitro experiment with *mesenchymal cells*, where a two-dimensional system of these cells chemotactically aggregated into one-dimensional structures. This discrepancy is attributed to the unrealistic nature of diffusion in the Keller-Segel model for *mesenchymal* cells. In reality, these cells exhibit a sort of nonlocal diffusion, better modeled by the operator $\Delta/(1-\Delta)$, defined via its Fourier transform as

$$\left(\widehat{\frac{\Delta}{1-\Delta}f}\right)(\xi) = \frac{-\xi^2}{1+\xi^2}\hat{f}(\xi), \quad \forall \xi \in \mathbb{R}^2,$$

instead of the standard Laplacian. Carrying out this substitution, it was proven that the resulting nonlocal Keller-Segel system experiences blow-up in one dimension for a sufficiently large initial condition, allowing the type of chemotactic aggregation observed in the experiments.

Inspired by these findings, Escudero [33] modified the Keller-Segel system to model situations where cellular dispersion is better described by a fractional Laplacian operator $\Lambda^\alpha = (-\Delta)^{\alpha/2}$ for $1 < \alpha < 2$, defined via its Fourier transform as

$$\widehat{(\Lambda^\alpha f)}(\xi) = \left[\widehat{(-\Delta)^{\alpha/2}}\, u\right](\xi) = (2\pi|\xi|)^\alpha\, \widehat{f}(\xi), \quad \forall \xi \in \mathbb{R}^d, \tag{1.4}$$

for $\alpha > 0$. The study investigated whether *Escherichia coli* colonies could also undergo one-dimensional collapse. He emphasized that the cellular motion's diffusive nature is one of the Keller-Segel model's strongest assumptions, which does not hold in many biological contexts.

Next, we explored the mathematical description of some anomalous diffusion processes.

1.3 Anomalous Diffusion Process

The diffusion equation is a partial differential equation that models density fluctuations in a material undergoing diffusion. The classical diffusion equation is expressed as $\partial_t u = \kappa \Delta u$ (also known as the heat equation), where u represents the density of the diffusing material and κ is the diffusion coefficient, assumed to be independent of u. In this context of classical diffusion, the Laplace operator (Laplacian) $\Delta = \partial^2_{x_1} + \cdots + \partial^2_{x_d}$ in a d-dimensional domain, along with the first-order time derivative ∂_t, are employed under the assumption of Gaussian process for the particle motion. At the microscopic level, this corresponds to the **Brownian motion** of individual particles, characterized by diffusion with a mean free path, or flight (the longest straight-line trajectory a particle follows without changing direction or pausing), and a mean pause time between flights. Furthermore, the mean squared displacement (MSD), which is a measure of the average displacement of a given object from the origin, is a linear function of time, i. e., $\langle x^2(t)\rangle \sim K_\vartheta t^\vartheta$, with $\vartheta = 1$. Einstein first demonstrated that the probability of a particle being at a certain distance from its initial position after a given time follows a Gaussian distribution.

However, numerous studies have shown that anomalous diffusion models under certain circumstances provide a superior fit to experimental data. In these types of diffusion, the MSD grows differently from that in Gaussian processes: in a subdiffusion process (or dispersive, slow diffusion process), the MSD grows slower following a power-law with an exponent $0 < \vartheta < 1$; in a superdiffusion process (or fast diffusion process), the MSD grows faster, following a power-law with an exponent $\vartheta > 1$. At a macroscopic level, these behaviors are typically captured by fractional differential equations, incorporating fractional derivatives in time and/or space.

Subdiffusion processes can be described, at the microscopic level, by a **continuous-time random walk (CTRW)**, where particles experience random waiting times between jumps that follow a heavy-tailed distribution. This results in periods of delay between movements, effectively slowing down the diffusion process, causing a cloud of particles to spread more slowly than in classical diffusion. These behaviors are captured and described by differential equations featuring a fractional derivative in time. A typical equation is $\partial_t^\alpha u = \Delta u$, for $0 < \alpha < 1$, where ∂_t^α is the Caputo fractional derivative given by $\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{\partial u(x,\eta)}{\partial \eta}\frac{\mathrm{d}\eta}{(t-\eta)^\alpha}$. This fractional derivative can describe processes featuring history dependencies, such as materials with memory (e.g. viscoelastic materials), and heterogeneous media.

On the other hand, the superdiffusion process can be described at a microscopic level by **Lévy flights** or **Lévy walk**, where the length of particle jumps (flights) follow a heavy-tailed distribution, reflecting the long-range interactions among particles (a super-diffusive nature). This dynamic results in a scenario where occasional large jumps dominate the more common smaller jumps, leading to a more rapid spread of particles. At a macroscopic level, these processes are described by differential equations with a fractional derivative in space, such as $\partial_t u = (-\Delta)^\alpha u$ for $0 < \alpha < 1$, where $(-\Delta)^\alpha$ is the fractional laplacian given by $(-\Delta)^\alpha u(x) = \frac{4^\alpha \Gamma(d/2+\alpha)}{\pi^{d/2}|\Gamma(-\alpha)|}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d\setminus B_\varepsilon(0)}\frac{u(x)-u(y)}{|x-y|^{d+2\alpha}}\mathrm{d}y$ (see Appendix A). This fractional derivative can describe long-range interactions of particle motions at a microscopic level.

These fractional derivatives are integro-differential operators, and thus nonlocal - it does not act through pointwise differentiation, but rather through integration with respect to a singular kernel. This nonlocality complicates the mathematical and numerical analysis, raising questions about the influence of the inherent nonlocal physics of anomalous diffusion processes on solution behavior, including uniqueness and stability.

The detailed mathematical treatment of these processes, including the introduction of fractional derivatives, is covered in the works [39, 44, 57], which provide a comprehensive framework for understanding the unique behaviors and challenges posed by anomalous diffusion. Additionally, the works of [34, 68] further support the concepts presented in this text, upon which much of this discussion is based.

1.3.1 Superdiffusion Process in Biology

According to Burczak et al. [25], since the 1990s, substantial theoretical and empirical evidence has emerged to support the replacement of classical diffusion (modeled by $-\Delta = \Lambda^2$) with a fractional one (modeled by $\Lambda^\alpha = (-\Delta)^{\alpha/2}$, $0 < \alpha < 2$) in Keller-Segel equations. In a biological context, Lévy walk (superdiffusion) is frequently adopted as an efficient search method for organisms, especially when navigating sparse or rare resources like chemoattractants or food. The non-negligible probability for long positional jumps in the Lévy walk process corresponds in that context that cells/organisms persist in a single direction of motion for a substantially longer time than in typical random walks [25, 33, 34].

To illustrate this, we describe two systems highlighted by [34], where organisms are suggested to exhibit Lévy walk behavior in their chemotactic responses. For more detailed information, see [34]. Moreover, to explore additional biological examples where organisms exhibit chemotactic responses characterized by Lévy walk behavior we refer to [25, 34] and the references therein.

• **Escherichia coli (E. Coli):** This bacterium swims due to their flagellums that act as a propeller when rotated counterclockwise, resulting in straight runs. When the flagella bundle separates due to clockwise rotation, the bacterium undergoes a tumble, reorienting itself without significant displacement. The direction of rotation is controlled by chemical signals binding to membrane-bound receptors, inducing signaling to the flagellum's rotational machinery. As a result, the bacterium moves along straight lines, suddenly stops to choose a new direction, and then continues moving in this new direction [3, 52, 66].

Experimental and theoretical studies suggest that E. coli switches from local (Brownian) to nonlocal (Lévy) search strategies, influenced by the activity of **CheR**, a cytoplasmic signaling protein regulating receptor activity. Fluctuating CheR levels lead to power-law distribution in E. coli's running behavior, emphasizing Lévy walk characteristics, while constant CheR levels lead to a Brownian motion. Simulations showed that for the case of fluctuating CheR, the dynamic adaptation enables bacteria to locate food sources more efficiently by balancing short and long runs [34].

• **Dictyostelium Discoideum (Dd):** This species of myxamoebae grows by cell division as long as sufficient nourishment is available. When the food resources are exhausted, they spread over their available domain. After a while, some cells – the founder cell – begin to emit a chemical signal that attracts other cells towards a higher concentration of this signal. These recruited cells, in turn, release the same signal, leading to aggregation at multiple collection points or centers (it is worth noting that Keller et al. [45] aimed to describe this aggregation process in cellular slime molds like *Dd*).

At the end of this aggregation process, a multicellular organism, known as a slug, is formed. This slug then migrates toward environmental attractants such as light, heat, and humidity. Subsequently, the slug differentiates into distinct cell types – prestalk and prespore cells – and ultimately forms a fruiting body, a multicellular structure, that releases spores into the environment. These spores can be blown away by the wind to colonize new areas, allowing the survival of the population afterward, and restarting the life cycle as unicellular myxamoebae [17, 41, 45].

In addition to its life cycle transitions, chemotaxis is a crucial mechanism for *Dd* to find food during its unicellular phase. For instance, chemotaxis to folic acid enables *Dd* to locate bacteria that secrete folate. Studies revealed that the mean squared displacement of *Dd* can be characterized by a power-law distribution, implying the suitability of the Lévy walk model to describe the movement of these organisms in a nutrient-depleted environment. The cells alter their strategy from making very localized searches to expanding their search area by persistently moving in a single direction. This behavior suggests that cells explore larger regions not by increasing their speed but by exhibiting a bias towards very long runs.

We now turn our attention to mathematical chemotaxis models within the context of anomalous diffusion dynamics. Specifically, starting from the minimal model, we modify both equations in (1.2) by replacing the Laplacian with a fractional Laplacian, resulting in the following system with $\alpha \in (1, 2]$ and $\beta \in (1, d]$:

$$\begin{cases} \partial_t \rho = -\Lambda^\alpha \rho - \nabla \cdot (\rho \nabla c) & x \in \mathbb{R}^d, \quad t > 0, \\ \tau \partial_t c = -\Lambda^\beta c + \rho - \gamma c & x \in \mathbb{R}^d, \quad t > 0, \\ \rho(t = 0) = \rho_0, \quad c(t = 0) = c_0 & x \in \mathbb{R}^d, \end{cases} \tag{1.5}$$

where all constants are normalized to 1, except for the parameters τ and γ. We refer to (1.5) as the **Fractional Keller-Segel Model**.

The parabolic-elliptic version of this model is obtained by setting $\tau = 0$ in (1.5), simplifying the system to:

$$\begin{cases} \partial_t \rho = -\Lambda^\alpha \rho - \nabla \cdot (\rho \nabla c) & x \in \mathbb{R}^d, \quad t > 0, \\ \Lambda^\beta c = \rho - \gamma c & x \in \mathbb{R}^d, \quad t > 0, \\ \rho(t = 0) = \rho_0, & x \in \mathbb{R}^d, \end{cases} \tag{1.6a}$$

or to one equation, which includes the case of no chemoattractant consumption effect $\gamma = 0$,

$$\begin{cases} \partial_t \rho = -\Lambda^\alpha \rho - \nabla \cdot \left(\rho \nabla \left((-\Delta)^{\beta/2} + \gamma I \right)^{-1} \rho \right) & x \in \mathbb{R}^d, \quad t > 0, \\ \rho(x, 0) = \rho_0(x) & x \in \mathbb{R}^d. \end{cases} \tag{1.6b}$$

Before exploring the key mathematical results available in the literature for this model, we first review some essential mathematical tools that aid in understanding this system.

1.4 Partial Differential Equations and Fourier Analysis

Fourier analysis is a powerful tool for converting linear partial differential equations into simpler equations, allowing one to find a formal integral formulation for it. In this section, we briefly discuss and present some applications of Fourier analysis to the theory of partial differential equations that will be used in the next chapter. We start by defining the Fourier transform and the convolution between two functions.

Definition 1.2 (Fourier transform). *If $u \in L^1\left(\mathbb{R}^d\right)$, we define its Fourier transform, $\mathcal{F}u(\xi) = \hat{u}(\xi)$, and its inverse Fourier transform, $\mathcal{F}^{-1}u(\xi) = \check{u}(\xi)$, respectively, as*

$$\hat{u}(\xi) := \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} u(x)\, \mathrm{d}x, \tag{1.7}$$

and

$$\check{u}(\xi) := \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} u(x)\,\mathrm{d}x, \tag{1.8}$$

for $\xi \in \mathbb{R}^d$*. Note that, since* $\left|e^{\pm 2\pi i x\cdot\xi}\right| = 1$ *and* $u \in L^1\left(\mathbb{R}^d\right)$*, these integrals converge for each* $\xi \in \mathbb{R}^d$*.*

Definition 1.3 (convolution). *Let* u *and* v *be measurable functions from* $\mathbb{R}^d$ *to* $\mathbb{R}^m$*. The convolution of* u *and* v *is the function* $u * v$ *defined as*

$$u * v(x) = \int_{\mathbb{R}^d} u(x-y)\cdot v(y)\,\mathrm{d}y$$

where $u(x-y)\cdot v(y)$ *is the inner product between* $u(x-y)$ *and* $v(y)$ *in* $\mathbb{R}^m$*.*

1.4.1 Applications

Fourier analysis is often used to obtain solutions or perform theoretical analysis of partial differential equations. It converts differentiation into multiplication, and, with the inverse Fourier transform, it converts multiplication into convolution between functions, as stated in the next theorem.

Theorem 1.4 (Properties of Fourier transform). *Assume* $u, v \in \mathcal{S}(\mathbb{R}^d)$*. Then*

(i) $\widehat{D^{\varsigma} u} = (2\pi i\xi)^{\varsigma}\hat{u}$ *for each multi-index ;*

(ii) $(u * v)^{\wedge} = \hat{u}\hat{v}$;

(iii) $u = (\hat{u})^{\vee}$.

Therefore, by applying the Fourier Transform to both sides of a partial differential equation, it can be transformed into an algebraic equation or an ordinary differential equation. This transformation allows us to derive its integral representation, expressed as convolutions between a function and the fundamental solution of the partial differential equation. For example, by applying Theorem 1.4, we can construct the fundamental solution to several classical differential equations, such as the Poisson equation, the heat equation, the wave equation, and others [35, 36]. Note that, from Theorem 1.4, $\widehat{(-\Delta u)}(\xi) = 4\pi^2|\xi|^2\widehat{u}(\xi)$ holds for all $\xi \in \mathbb{R}^d$ and $u \in \mathcal{S}(\mathbb{R}^d)$. Similarly, the Fractional Laplacian $\Lambda^\alpha = (-\Delta)^{\alpha/2}$ can be defined naturally via the Fourier transform as (1.4).

Remark 1.5. *Consider the parabolic-elliptic fractional Keller-Segel model* (1.6) *with* $\gamma = 0$ *and* $\beta = 2$*. In this case, the second equation simplifies to the Poisson equation* $-\Delta c = \rho$*, which has a well-known solution expressed as the convolution of* ρ *with the fundamental solution* Φ*. The* *fundamental solution, also referred to as the Green function, is defined as follows:*

$$\Phi(x) := \begin{cases} |x| & \text{if} \quad d = 1, \\ -\frac{1}{2\pi} \ln |x| & \text{if} \quad d = 2, \\ \frac{1}{d(d-2)\omega(d)} \frac{1}{|x|^{d-2}} & \text{if} \quad d \geq 3, \end{cases} \tag{1.9}$$

for $x \neq 0$, where $\omega_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$ represents the volume of the unit ball in $\mathbb{R}^d$.

Since c does not appear directly in the first equation of (1.6a), it is customary to consider ∇c instead, which can be expressed (assuming the integral exists) as

$$\nabla c(x,t) = \int_{\mathbb{R}^d} \nabla\Phi(x-y)\rho(y,t)\mathrm{d}y = (\nabla\Phi * \rho)(x,t), \tag{1.10}$$

where $\nabla\Phi$ is given by

$$\nabla\Phi(x) = -\frac{1}{d\omega_d}\frac{x}{|x|^d}. \tag{1.11}$$

Therefore, this parabolic-elliptic fractional model reduces to the single parabolic equation:

$$\begin{cases} \partial_t \rho = -\Lambda^\alpha \rho + \chi\nabla\cdot(\rho\nabla\Phi * \rho) & t > 0, \quad x \in \mathbb{R}^d, \\ \rho(x,0) = \rho_0(x), \end{cases} \tag{1.12}$$

where ρ_0 is a nonnegative initial condition.

By applying the Fourier Transform, we can construct solutions to differential equations involving the fractional Laplacian. For that, consider first the following lemma:

Lemma 1.6. *[36, 71] Let $P_k(x)$ be a homogeneous harmonic polynomial of degree k, $k \geq 0$. Then, for $0 < \alpha < d$,*

$$\left(\frac{P_k(x)}{|x|^{k+d-\alpha}}\right)^\wedge = \gamma_{k,\alpha}\frac{P_k(x)}{|x|^{k+\alpha}}$$

with $\gamma_{k,\alpha} = i^k\pi^{d/2-\alpha}\frac{\Gamma(k/2+\alpha/2)}{\Gamma(k/2+d/2-\alpha/2)}$, where this identity holds in the sense that

$$\int_{\mathbb{R}^d}\frac{P_k(x)}{|x|^{k+d-\alpha}}\hat{\varphi}(x)\mathrm{d}x = \gamma_{k,\alpha}\int_{\mathbb{R}^d}\frac{P_k(x)}{|x|^{k+\alpha}}\varphi(x)\mathrm{d}x$$

for every φ which is sufficiently rapidly decreasing at ∞, and whose Fourier transform has the same property.

Based on Theorem 1.4 and Lemma 1.6, we obtain the following proposition:

Proposition 1.7. *Let $u \in \mathcal{S}(\mathbb{R}^d)$. Then, we have, for $\alpha > 0$*

$$\Lambda^\alpha(\Phi_\alpha * u) = u, \quad \text{with} \quad \alpha < d \tag{1.13}$$

$$\partial_t \left(K_t^\alpha * u\right) = \Lambda^\alpha \left(K_t^\alpha * u\right), \tag{1.14}$$

$$(\Lambda^\alpha + \gamma \, \mathrm{I}) \left(\Phi_\gamma * u\right) = u, \tag{1.15}$$

where

$$\Phi_\alpha(x) = \frac{\Gamma((d-\alpha)/2)}{\pi^{d/2} 2^\alpha \Gamma(\alpha/2)} \frac{1}{|x|^{d-\alpha}}, \tag{1.16}$$

$$K_t^\alpha(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{-t|\xi|^\alpha} \mathrm{e}^{ix\cdot\xi} \mathrm{d}\xi, \tag{1.17}$$

$$\Phi_{\alpha,\gamma}(x) = \int_0^{+\infty} K_s^\alpha(x) e^{-\gamma s} \mathrm{d}s. \tag{1.18}$$

Proof. The proof involves applying the Fourier Transform on both sides of the correspondent differential equation and using Theorem 1.4. In addition, for (1.13), we can use Lemma 1.6, and for (1.15), the identity $\frac{1}{a} = \int_0^\infty e^{-sa} \mathrm{d}s$. The proof is straightforward and thus omitted. □

Remark 1.8. *Note that $\Phi_\alpha * u(x)$ in (1.13) corresponds to the Riesz potential, I_α, introduced in Definition A.3. As mentioned in Appendix A, for $0 < \alpha < d$, the operator $I_\alpha = (-\Delta)^{-\alpha/2}$ is the inverse of $(-\Delta)^{\alpha/2}$ and is expressed through a convolution: $(-\Delta)^{-\alpha/2} u(x) = \Phi_\alpha * u(x)$.*

Remark 1.9. *The function $K_t^\alpha(x)$ given by (1.17) is defined via the Fourier transform as follows:*

$$\begin{aligned} K_t^\alpha(x) &= (2\pi)^{-d} \int_{\mathbb{R}^d} \mathrm{e}^{-t|\xi|^\alpha} \mathrm{e}^{ix\cdot\xi} \mathrm{d}\xi \\ &= (2\pi)^{-d} t^{-\frac{d}{\alpha}} \int_{\mathbb{R}^d} \mathrm{e}^{-|\xi|^\alpha} \mathrm{e}^{i\frac{x}{t^{1/\alpha}}\cdot\xi} \mathrm{d}\xi \\ &= t^{-\frac{d}{\alpha}} K^\alpha \left(\frac{x}{t^{1/\alpha}}\right), \end{aligned}$$

where the kernel function $K^\alpha(x)$ is defined as

$$K^\alpha(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \mathrm{e}^{ix\cdot\xi} \mathrm{e}^{-|\xi|^\alpha} \mathrm{d}\xi. \tag{1.19}$$

Remark 1.10. *Let $\mathcal{B}$ be the linear vector operator formally defined as*

$$\mathcal{B}(u) = \nabla \left((-\Delta)^{-\beta/2} u \right),$$

for some $\beta > 0$. From Theorem 1.4 and Lemma 1.6, for $1 < \beta < d+1$, $\mathcal{B}(u)$ can be expressed as the following integral

$$\mathcal{B}(u) = C_{d,\beta} \int_{\mathbb{R}^d} \frac{x-y}{|x-y|^{d-\beta+2}} u(y) \mathrm{d}y, \tag{1.20}$$

where $C_{d,\beta} = -\frac{\Gamma((d-\beta+2)/2)}{\pi^{d/2} 2^{\beta-1} \Gamma(\beta/2)}$.

Moreover, by the Hardy-Littlewood-Sobolev inequality, if $u \in L^p(\mathbb{R}^d)$ *for* $\frac{d}{\beta-1+d} < p < \frac{d}{\beta-1}$ *and* $1 < \beta < d+1$, *we obtain that*

$$\|\mathcal{B}(u)\|_{L^q} \leq C_{d,\beta} \left\| |\cdot|^{-d+\beta-1} * u \right\|_{L^q} \leq C\|u\|_{L^p}$$

for every $1 < p < q < \infty$ *satisfying* $\frac{1}{p} - \frac{\beta-1}{d} = \frac{1}{q}$. *Additionally, if* $u \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ *and* $1 < \beta < d+1$, *by decomposing* $\mathcal{B}(u)$ *into two parts, we find that*

$$\|\mathcal{B}(u)\|_{L^\infty} \leq C_{d,\beta} \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} \left(\int_{B_1(x)} \frac{|u(y)|}{|x-y|^{d-\beta+1}}\, \mathrm{d}y + \int_{\mathbb{R}^d \setminus B_1(x)} \frac{|u(y)|}{|x-y|^{d-\beta+1}}\, \mathrm{d}y \right) \leq C\left(\|u\|_{L^\infty} + \|u\|_{L^1}\right).$$

Thus, for $1 < \beta < d+1$, (1.20) *is a bounded linear operator from* $L^p(\mathbb{R}^d)$ *into* $L^q(\mathbb{R}^d)$ *for every* $1 < p < q < \infty$ *satisfying* $\frac{1}{p} - \frac{\beta-1}{d} = \frac{1}{q}$ *and from* $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ *into* $L^\infty(\mathbb{R}^d)$.

Remark 1.11. *Consider the parabolic-elliptic fractional Keller-Segel model* (1.6a) *with* $\gamma = 0$. *Similar to Remark 1.5,* ∇c *can be expressed using an integral formulation, and* (1.6a) *can be rewritten as the single parabolic equation* (1.12). *Indeed, in this case, the second equation becomes* $(-\Delta)^{\beta/2} c = \rho$, *for* $0 < \beta \leq d$, *and formally we obtain* $\nabla c = \nabla\left((-\Delta)^{-\beta/2}\rho\right)$. *Then, from Remark 1.10, for* ρ *with enough regularity, we can write* (1.10) *redefining* (1.11) *as*

$$\nabla\Phi(x) = -\frac{\beta\omega_\beta}{(2\pi)^{\beta-1}} \frac{1}{(d-\beta+2)\,\omega_{d-\beta+2}} \frac{x}{|x|^{d-\beta+2}}, \tag{1.21}$$

since this is well-defined for $0 < \beta \leq d$, *and* $C_{d,\beta}$ *in* (1.20) *can be rewritten as*

$$C_{d,\beta} = -\frac{\beta\omega_\beta}{(2\pi)^{\beta-1}} \frac{1}{(d-\beta+2)\,\omega_{d-\beta+2}}, \tag{1.22}$$

where ω_β *and* $\omega_{d-\beta+2}$ *represent the volumes of the unit balls in* $\mathbb{R}^\beta$ *and* $\mathbb{R}^{d-\beta+2}$, *respectively. Note that* (1.21) *generalizes the definition provided in* (1.11).

1.5 Self-Similar Solutions and Critical spaces

A system of equations admits self-similar solutions when any solution $u(x,t)$, with $(x,t) \in \mathbb{R}^d \times (0,\infty]$, generates a family of solutions of the form $\lambda^a u(\lambda^b x, \lambda^c t)$ for all $\lambda > 0$ and suitably real numbers $a,\ b,\ c$. In such cases, the system exhibits scaling invariance.

The process of scaling heuristics is a form of determining whether a system is scaling invariance and identifying some possible values for a, b, and c. Through this procedure, one can identify the **critical spaces** for the system, which are function spaces where the norm of a solution remains unchanged under scaling transformations.

The concept of critical spaces is based on the idea of identifying the space of initial data for which the given partial differential equation (PDE) is well-posed. Thus, the scaling heuristics procedure and the identification of critical spaces provide an understanding of which spaces are particularly well-suited for the equations under consideration. Understanding these spaces is crucial, as the local existence of solutions in a critical space can often be extended to establish global existence [75].

1.5.1 Scaling Heuristics

We apply the scaling heuristics procedure to determine whether the solutions of the system (1.5) with $\gamma = 0$ are invariant under scaling. For that, we define the functions ρ_λ and c_λ for any nonzero real number λ as follows:

$$\begin{cases} \rho_\lambda(x,t) = \lambda^{m_1}\rho(\lambda^{m_2}x, \lambda^{m_3}t) \\ c_\lambda(x,t) = \lambda^{m_4}c(\lambda^{m_2}x, \lambda^{m_3}t) \end{cases} \tag{1.23}$$

where $m_1, m_2, m_3, m_4 \in \mathbb{R} \setminus \{0\}$.

From this definition, we obtain

$$\begin{aligned} \Lambda^\alpha \rho_\lambda(x,t) &= \lambda^{m_1} \int_{y\in\mathbb{R}^d} \frac{\rho(\lambda^{m_2}y, \lambda^{m_3}t) - \rho(\lambda^{m_2}x, \lambda^{m_3}t)}{|x-y|^{d+\alpha}} \mathrm{d}y \\ &= \lambda^{m_1} \int_{z\in\mathbb{R}^d} \frac{\rho(z, \lambda^{m_3}t) - \rho(\lambda^{m_2}x, \lambda^{m_3}t)}{\lambda^{-m_2(d+\alpha)}|\lambda^{m_2}x - z|^{d+\alpha}} \lambda^{-m_2 d}\mathrm{d}z \\ &= \lambda^{m_1+m_2\alpha}(\Lambda^\alpha\rho)(\lambda^{m_2}x, \lambda^{m_3}t), \end{aligned}$$

and the first equation in (1.5) become

$$\begin{aligned} \partial_t\rho_\lambda(x,t) &= \lambda^{m_1+m_3}(\partial_t\rho)(\lambda^{m_2}x, \lambda^{m_3}t) \\ &= \lambda^{m_1+m_3}\left[-(\Lambda^\alpha\rho)(\lambda^{m_2}x, \lambda^{m_3}t) - \chi\left((\nabla\rho)(\lambda^{m_2}x, \lambda^{m_3}t)\cdot(\nabla c)(\lambda^{m_2}x, \lambda^{m_3}t)\right.\right. \\ &\qquad \left.\left. + \rho(\lambda^{m_2}x, \lambda^{m_3}t)(\Delta c)(\lambda^{m_2}x, \lambda^{m_3}t)\right)\right] \\ &= \lambda^{m_1+m_3}\left[-\lambda^{-m_1-m_2\alpha}\Lambda^\alpha\rho_\lambda(x,t) - \lambda^{-m_1-m_4-2m_2}\chi\left(\nabla\rho_\lambda\cdot\nabla c_\lambda(x,t) + \rho_\lambda\Delta c_\lambda(x,t)\right)\right] \\ &= -\lambda^{-m_2\alpha+m_3}\Lambda^\alpha\rho_\lambda(x,t) - \lambda^{-m_4-2m_2+m_3}\chi\nabla\cdot\left(\rho_\lambda(x,t)\nabla c_\lambda(x,t)\right). \end{aligned}$$

Next, if $\tau > 0$, the second equation in (1.5) becomes

$$\begin{aligned} \tau\partial_t c_\lambda(x,t) &= \lambda^{m_4+m_3}(\partial_t c)(\lambda^{m_2}x, \lambda^{m_3}t) \\ &= \lambda^{m_4+m_3}\left(-(\Lambda^\beta c)(\lambda^{m_2}x, \lambda^{m_3}t) + \rho(\lambda^{m_2}x, \lambda^{m_3}t)\right) \\ &= \lambda^{m_4+m_3}\left(-\lambda^{-m_4-m_2\beta}\Lambda^\beta c_\lambda(x,t) + \lambda^{-m_1}\lambda^{m_1}\rho(\lambda^{m_2}x, \lambda^{m_3}t)\right) \\ &= -\lambda^{m_3-m_2\beta}(\Lambda^\beta c_\lambda)(x,t) + \lambda^{m_4+m_3-m_1}\rho_\lambda(x,t). \end{aligned}$$

Therefore, $(\rho_\lambda, c_\lambda)$ satisfies the equations in (1.5) if we can find m_1, m_2, m_3, and $m_4 \in \mathbb{R}\setminus\{0\}$ such that $A \cdot m = 0$, where

$$A = \begin{bmatrix} 0 & -\alpha & 1 & 0 \\ 0 & -2 & 1 & -1 \\ -1 & 0 & 1 & 1 \\ 0 & -\beta & 1 & 0 \end{bmatrix},$$

$m = (m_1, m_2, m_3, m_4)^T$ and $0 = (0, 0, 0, 0)^T$. Since $\det A = \alpha - \beta$, system (1.5) with $\tau > 0$ can only exhibit self-similar solutions if $\alpha = \beta$. In that case, the scaling transformation from (ρ, c) to $(\rho_\lambda, c_\lambda)$, preserves system (1.5), allowing us to rewrite (1.23) as

$$\begin{cases} \rho_\lambda(x,t) = \lambda^{2\alpha-2}\rho(\lambda x, \lambda^\alpha t) \\ c_\lambda(x,t) = \lambda^{\alpha-2}c(\lambda x, \lambda^\alpha t). \end{cases} \tag{1.24}$$

For the parabolic-elliptic system, we set $\tau = 0$ in (1.5) to obtain for its second equation

$$\Lambda^\beta c_\lambda(x,t) = \lambda^{m_4+m_2\beta}(\Lambda^\beta c)(\lambda^{m_2}x, \lambda^{m_3}t) = \lambda^{m_4+m_2\beta}\rho(\lambda^{m_2}x, \lambda^{m_3}t) = \lambda^{m_4-m_1+m_2\beta}\rho_\lambda(x,t),$$

and the system reduces to $A \cdot m = 0$, where A is given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\alpha & 1 & 0 \\ 0 & -2 & 1 & -1 \\ -1 & \beta & 0 & 1 \end{bmatrix}.$$

As this system is over-determined, the parabolic-elliptic equation (1.6) is scaling invariant regardless of the values of α and β. Therefore, since (1.6b) is a PDE only on ρ, we can express the rescaled function (1.23) simply as

$$\rho_\lambda(x,t) = \lambda^{\alpha+\beta-2}\rho(\lambda x, \lambda^\alpha t). \tag{1.25}$$

Remark 1.12. *As mention in Section 1.2.2, in dimension two, for the classical Keler-Segel model (1.3), with $\gamma = 0$, the $L^1(\mathbb{R}^d)$ is a critical space. Thus, the existence or nonexistence of a global solution in time depended on the value of $m_0 = \int_{\mathbb{R}^d}\rho_0(x)\mathrm{d}x = \int_{\mathbb{R}^d}\rho(x,t)\mathrm{d}x$, in which the critical value $m_0 = 8\pi/\chi$ decides whether a nonnegative integrable initial datum leads to a global-in-time solution or not. In the case of $\alpha \neq d$, as pointed out by Biler et al. [9, 14], mass m_0 cannot play such a role anymore because the scaling $\rho_\lambda(x,t) = \lambda^\alpha\rho(\lambda x, \lambda^\alpha t)$ for each $\lambda > 0$ leads to the* *equality*

$$\int_{\mathbb{R}^d} \rho_\lambda(x,t)\mathrm{d}x = \lambda^{\alpha-d}\int_{\mathbb{R}^d} \rho(x,t)\mathrm{d}x = \lambda^{\alpha-d} m_0,$$

which implies that the total mass of a rescaled solution ρ_λ can be chosen arbitrarily with suitable $\lambda > 0$. Thus, system (1.6) in two-dimension with $\gamma = 0$, $\beta = 2$ and $\alpha < 2$ is insensitive to the value of m_0.

1.5.2 Analyses of Critical spaces

It is clear from the previous section that the classical Keller-Segel models (both parabolic-parabolic system (1.2) and parabolic-elliptic system (1.3), with $\gamma = 0$) are scaling invariant. Moreover, the model admits self-similar solution, as pointed out in the following propositions [75].

Proposition 1.13. *[75, Proposition 2.1] Assume that $\lambda > 0$ and $\rho_0 = \rho_0(x)$, for $x \in \mathbb{R}^d$. Let $\rho^\lambda = \rho^\lambda(x,t)$ (and $c^\lambda = c^\lambda(x,t)$) in $x \in \mathbb{R}^d$ and $0 \le t < T$, be a solution of (1.3) with $\gamma = 0$ and initial condition $\rho^\lambda(x,0) = \lambda^{-2}\rho_0\left(\lambda^{-1}x\right)$, for $x \in \mathbb{R}^d$. Then, $\rho_\lambda(x,t) = \lambda^2\rho^\lambda\left(\lambda x, \lambda^2 t\right)$ (and $c_\lambda(x,t) = c^\lambda\left(\lambda x, \lambda^2 t\right)$) in $x \in \mathbb{R}^d$ and $0 \le t < \lambda^{-2}T$, is a solution to (1.3) with $\gamma = 0$ and initial condition $\rho_\lambda(x,0) = \rho_0(x)$, $x \in \mathbb{R}^d$.*

Proposition 1.14. *[75, Proposition 2.3] Assume that $\lambda > 0$ and consider $\rho_0 = \rho_0(x)$ and $c_0 = c_0(x)$, $x \in \mathbb{R}^d$. Moreover, let $\rho^\lambda = \rho^\lambda(x,t)$, $c^\lambda = c^\lambda(x,t)$ in $x \in \mathbb{R}^d$ and $0 \le t < T$, be a solution of (1.2) with $\gamma = 0$ and initial condition $\rho^\lambda(x,0) = \lambda^{-2}\rho_0\left(\lambda^{-1}x\right)$ and $c^\lambda(x,0) = c_0\left(\lambda^{-1}x\right)$, $x \in \mathbb{R}^d$. Then, $\rho_\lambda(x,t) = \lambda^2\rho^\lambda\left(\lambda x, \lambda^2 t\right)$ and $c_\lambda(x,t) = c^\lambda\left(\lambda x, \lambda^2 t\right)$ in $x \in \mathbb{R}^d$ and $0 \le t < \lambda^{-2}T$ is a solution of (1.2) with $\gamma = 0$ and initial condition $\rho_\lambda(x,0) = \rho_0(x)$, and $c_\lambda(x,0) = c_0(x)$ in $x \in \mathbb{R}^d$.*

Now, assume that there exist positive numbers δ and T such that ρ in $\mathbb{R}^d \times (0,T)$ is a solution of (1.3) with $\gamma = 0$, $d \ge 2$ and initial data $\rho(x,0) = \rho_0(x)$ for $x \in \mathbb{R}^d$ if

$$\rho_0 \in L^p(\mathbb{R}^d) \quad \text{with} \quad \|\rho_0\|_{L^p} \le \delta. \tag{1.26}$$

We then ask whether ρ_λ, given by (1.25), is also a solution of this system with the same initial data ρ_0 on $\mathbb{R}^d \times \left(0, \lambda^{-2}T\right)$. For this purpose consider first the solution ρ^λ assumed by Proposition 1.13 under the condition (1.26), that is

$$\|\rho^\lambda(\cdot,0)\|_{L^p} = \lambda^{-2}\|\rho_0\left(\lambda^{-1}\cdot\right)\|_{L^p} = \lambda^{-2+\frac{d}{p}}\|\rho_0\|_{L^p} \le \delta.$$

Thus, if the condition $\|\rho_0\|_{L^p} \le \delta\lambda^{2-\frac{d}{p}}$ holds for $\lambda > 0$, then ρ_λ, given by Proposition 1.13, solves (1.3). This suggests that the space $L^{d/2}(\mathbb{R}^d)$ is well-suited to the system. Since, if parabolic-elliptic system in $\mathbb{R}^d \times (0,T)$ has a solution for any initial data ρ_0 that satisfies condition (1.26) with $\delta > 0$ and $p = d/2$, then the parabolic-elliptic system in $\mathbb{R}^d \times (0, \lambda^{-2}T)$ also has a solution for any $\lambda > 0$, with the same initial data. In particular, for initial data satisfying (1.26), the system has solutions over any time interval, as γ can be made arbitrarily small. Then, assuming uniqueness, this yields global well-posedness for system (1.3). In this case ρ_λ in $\mathbb{R}^d \times (0,\infty)$ is called self-similar solution, and $L^{d/2}(\mathbb{R}^d)$ is the **critical** space for the system.

The spaces $L^p(\mathbb{R}^d)$ with $p > d/2$ are known as **supercritical** for the classical parabolic-elliptic Keller-Segel system, while those with $p < d/2$ are called **subcritical**. If a solution exists in $\mathbb{R}^d \times (0,T)$ for any initial data $\rho(x,0) = \rho_0(x)$ in a supercritical space $L^p(\mathbb{R}^d)$, with $p > d/2$, satisfying (1.26), then, as $\lambda \to \infty$, it follows that for arbitrarily large initial data $\rho_0 \in L^p(\mathbb{R}^d)$, a solution exists in $\mathbb{R}^d \times \left(0, \lambda^{-2}T\right)$, which corresponds to shrinking time intervals. On the other hand, for an initial data ρ_0 in a subcritical space $L^p(\mathbb{R}^d)$, with $p < d/2$, letting $\lambda \to 0$ implies that a solution would exist globally in time for arbitrarily large $\rho_0 \in L^p(\mathbb{R}^d)$, which is unlikely.

As pointed out by Triebel [75] for the classical Keller-Segel systems, the scaling heuristics below serve as a guideline for identifying which spaces (critical and supercritical) are naturally adapted to the related problems. However, it does not provide rigorous barriers for other spaces. In summary, understanding which spaces are critical, supercritical, and subcritical for the system under analysis is crucial, as it allows us to trade time of existence for the size of initial data (measured with respect to the norm space) and vice versa.

Remark 1.15. *The classification of functional spaces reveals important aspects of solution behavior and well-posedness of the PDE:*

*- **Subcritical spaces:** If the solution lies in a subcritical space with respect to scaling, we can often establish local well-posedness for large initial data by shrinking the time intervals. This is because, in subcritical spaces, the norms controlling the solution become smaller as we zoom in on shorter time scales.*

*- **Supercritical spaces:** If the solution lies in a supercritical space, we can use scaling to convert bad behavior arising from large initial data at some time $T > 0$ to bad behavior arising from small initial data at some time less or equal to T. This suggests the existence of a significant barrier to developing a well-posedness theory below a certain regularity level.*

*- **Critical spaces:** In a critical space, establishing local well-posedness also leads to global well-posedness for the same initial data. This is because the size of the initial data remains*

unchanged under the scaling, allowing time intervals to be expanded. Thus, if we can control the solution locally, this control extends globally without the norms changing under scaling.

1.6 Literature Review of Key Mathematical Results for the Fractional Keller-Segel Model

In this section, we provide a comprehensive review of key mathematical results related to the fractional Keller-Segel model, systems (1.5) and (1.6). The review begins with an exploration of the parabolic-elliptic fractional Keller-Segel model (1.6). It is worth emphasizing that a significant portion of the existing literature addresses the case where the fractional exponent $\beta = 2$, which corresponds to classical diffusion in the chemoattractant equation, while the cell density equation incorporates fractional diffusion with $\alpha \in (1, 2]$.

Next, we explore a generalized parabolic-elliptic fractional model, which further extends the classical framework by introducing more generalized forms of chemical signal diffusion (interaction kernels).

Finally, the review turns to the parabolic-parabolic fractional Keller-Segel model, where the cell density and chemical concentration evolve according to parabolic fractional equations. To the best of our knowledge, despite the progress made, the fundamental question regarding global and local wellposedness for the fully parabolic-parabolic fractional Keller-Segel system (1.5) in $\mathbb{R}^d$, with distinct exponents α and β, remained unanswered prior to this thesis. In Chapter 3, we address this particular system in detail. As Burczak et al. [23] highlighted, it is appealing to determine the minimum diffusion strength that dominates the chemotactic forces, demonstrating the existence of regular solutions globally, or to investigate the maximum diffusion strength that does not inhibit blow-up.

The literature on the fractional case review that a substantial portion of the mathematical results have been centered around the parabolic-elliptic formulation (1.6). However, the exploration of these results has provided us insights for tackling the more general parabolic-parabolic fractional Keller-Segel model (1.5).

1.6.1 The Parabolic-elliptic Fractional Keller-Segel Model

The pioneering work of Escudero [33] extended classical parabolic-elliptic Keller-Segel model results driven by Brownian motion to those driven by α Lévy processes with $1 < \alpha < 2$ in one dimension, by considering system (1.6) with $\beta = 2$ and $d = 1$. According to Escudero [33], as the fractional Laplacian operator (equation (1.4)) with $1 < \alpha < 2$ imparts less regularization than the Laplacian, it is possible that the solution of system (1.6) with $\beta = 2$ blows up in finite time. In fact, as pointed out by Burczak et al. [25], for any dimension, $-\Lambda^{\alpha}$ provides, for $\alpha < 2$, a weaker dissipation than the classical one, making the system more prone to exhibiting a blow-up. This outcome would imply a significant change in the collapsed structure that can form in many situations where the classical Keller-Segel system does not apply. Since, as discussed in Section 1.2.3, for the case of the minimal system, one-dimensional global existence implies that chemotactic collapse to a line is impossible in two dimensions, and chemotactic collapse to a sheet cannot occur in a three-dimensional setting.

Escudero [33] established global boundedness in time of the $L^{\infty}(\mathbb{R})$ norm of ρ and c, for initial condition $\rho_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\mathrm{d}\rho_0/\mathrm{d}x \in L^2(\mathbb{R})$, . Hence, similar to the standard Keller-Segel model, it was established that solutions to the fractional Keller-Segel model exist globally in time in one dimension.

Subsequently, Bournaveas et al. [19] considered system (1.6) with $\beta = 2$ in one space dimension with $\alpha \in (0, 2]$ and simplified the problem by setting $\gamma = 0$. The authors asserted that adopting $\gamma > 0$ would not significantly affect the outcomes of global existence, although blow-up results would be slightly modified due to the fast decay of the interacting kernel at infinity (as $\gamma > 0$ plays a damping role; see Chapter 2 for further discussions). They stated that one can think of this solution as the limit of c_γ as $\gamma \to 0$ where c_γ is the solution to the elliptic problem: $-\partial^2_{xx} c_\gamma + \gamma c_\gamma = \rho$. Moreover, they proved that in the case $1 < \alpha \leq 2$, assuming $\rho_0 \in L^{p_0}(\mathbb{R})$ for some $p_0 > 1$, the solutions to the parabolic-elliptic fractional Keller-Segel model are global in time and belong to any $L^p(\mathbb{R})$ space for all positive time $t > 0$. The authors mentioned that their methods could be used to improve the results of Escudero [33] by relaxing the regularity hypotheses on the initial data.

In the case $0 < \alpha < 1$, Bournaveas et al. [19] proved that solutions may exist globally or blow up depending on the initial data. Specifically, if $\rho_0 \in L^{p_0}(\mathbb{R})$ for some $p_0 > 1/\alpha$, there exists a constant $K_1(\alpha)$ such that the condition $\|\rho_0\|_{L^{1/\alpha}} \leq K_1(\alpha)$ ensures the existence of global weak solutions. Moreover, the density belongs to any $L^p(\mathbb{R})$ space for any positive time $t > 0$ (these results extend to $\alpha = 1$ as well). On the other hand, assuming an even $\rho_0 \in L^1(\mathbb{R}, (1 + |x|)\mathrm{d}x)$, there exists a constant $K_2(\alpha)$ such that the condition

$$\left(\int_{\mathbb{R}} |x|\rho_0(x)\mathrm{d}x \right)^{1-\alpha} < K_2(\alpha) m_0^{2-\alpha} \tag{1.27}$$

excludes the global existence of a regular solution: a singularity must appear in finite time.

The authors did not describe the behavior of solutions for a large initial mass when $\alpha = 1$. This omission was attributed to the fact that, in this case, their strategy fails, since the constant $K_2(\alpha)$ in (1.27) diverges as α approaches 1. Then, guided by numerical evidence, Bournaveas et al. [19] conjectured that solutions might blow up for sufficiently large initial data. However, Burczak et al. [22] later refuted this conjecture, showing that, at least in the periodic setting, the system exhibits global-in-time behavior regardless of the size of initial data. Additionally, for $\alpha = 1$, Ascasibar et al. [4] explicitly estimated the constant $K_1(1)$ as $(2\pi)^{-2}$, for $\chi = 1$ and within the periodic torus setting.

Biler et al. [15] presented conditions for the global-in-time existence of solutions versus finite time blow-up for $d \geq 2$, $\alpha = 2$, $\gamma = 0$ and $1 < \beta \leq d$. The authors analyzed the evolution of the second moment of a solution, $m_2(t) = \int_{\mathbb{R}^d} \frac{|x|^2}{2} \rho(x,t)\,\mathrm{d}x$, demonstrating the contradiction that $m_2(t)$ vanishes for some $t > 0$. This result established the nonexistence of global-in-time, nonnegative, nontrivial solutions for $\alpha = 2$ under some conditions on ρ_0. As mentioned in Remark 1.1, a similar approach can be found in the study of the classical Keller-Segel model (see [66]). Moreover, they present result on local-in-time solutions to (1.6) for $d \geq 2$ and $\alpha \in (1,2)$ in Lebesgue, that can be continued to a global-in-time solution for $\beta + \alpha > d + 1$.

Huang et al. [42] also considered the system in dimension $d \geq 2$ with $\gamma = 0$, $1 < \alpha < 2$, $\beta = 2$. They proved that for small initial density $\rho_0 \in L^1(\mathbb{R}^d) \cap L^{d/\alpha}(\mathbb{R}^d)$, the system admits a global and bounded weak solution. They also provided decay estimates of the solution in different function spaces. Furthermore, they proved local existence for a more general nonnegative initial density $\rho_0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

Additionally, Li et al. [55] consider the same problem in $d = 2$, and proved, for any $\rho_0 \in H^s(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$ with $s > 3$ and $1 < q < 2$, local existence and uniqueness of solutions $\rho \in C\big([0,T), H^s(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)\big)$. Moreover $\rho \in C^1\big([0,T), H^{s_0}(\mathbb{R}^2)\big)$, where $s_0 = \min\{s - \alpha, s - 1\} > 1$, and, if $\rho_0 \geq 0$, ρ remains nonnegative for any $0 \leq t < T$. If in addition $\rho_0 \in L^1(\mathbb{R}^2)$, then

$\|\rho(\cdot,t)\|_{L^1} = \|\rho_0\|_{L^1}$ for any $0 \le t < T$. For T, the maximal-lifespan solution, either $T = +\infty$, in which case there is a global solution, or $T < \infty$, and $\lim_{t\to T}\int_0^t \|\rho(s)\|_{L^\infty} ds = +\infty$. They also showed the existence of a blow-up solution for the system.

Biler et al. [11] analyzed system (1.6) in $d \ge 2$, with parameters $\gamma = 0$, $\alpha \in (1,2]$ and $\beta \in (1,d]$. They established the global-in-time existence in the critical space $L^{d/(\alpha+\beta-2)}(\mathbb{R}^d)$ and formulate a condition on the initial data which leads to the blow-up in a finite time of the corresponding solution. For that, they extended the classical method to prove the existence of blowing-up solutions with $\alpha = 2$ by studying moments of lower order $\vartheta \in (1,2)$, since for $0 < \alpha < 2$ the second moment cannot be finite. Then, they showed that a sufficient condition for blow-up is that ρ_0 is well concentrated, namely

$$\left(\frac{\int_{\mathbb{R}^d} |x|^\vartheta \rho_0(x)\mathrm{d}x}{m_0}\right)^{\frac{d+2-\alpha-\beta}{\vartheta}} \le c\, m_0$$

for some $\vartheta \in (1,\alpha)$, α and β satisfying $\alpha + \beta < d + 2$, $\rho_0 \in L^1(\mathbb{R}^d, \left(1 + |x|^\vartheta\right) \mathrm{d}x)$ and $c > 0$, a sufficiently small constant independent of ρ_0. Further details about this study are discussed in Section 3.1, where we detail their findings on the existence of both global and local-in-time solutions, and Section 4.3.2.2, where their results on the blow-up of solutions are explored.

Note that the results in [11, 15] have been formulated in terms of global quantities like the moment. In contrast, Biler et al. [9, 13, 14] formulated their results in terms of local properties of initial data instead of a comparison of the total mass and moments of the initial data as done previously. Their approach did not require additional assumptions about ρ_0, such as $\rho_0 \in L^1(\mathbb{R}^d, (1+|x|^\vartheta)\mathrm{d}x)$, which were necessary in the earlier works by Biler et al. [11, 15]. The blow-up results of [55] were also in terms of local properties of the initial condition.

Specifically, Biler et al. [13] presented criteria for blow-up of solutions to system (1.6) with parameters $\gamma = 0$, $\alpha \in (1,2]$ and $\beta = 2$, in $d = 2$. They established that the solution ρ ceases to exists in a finite time if there exist $x_0 \in \mathbb{R}^2$ and $R > 0$ such that

$$R^{\alpha-2}\int_{\{|x-x_0|<R\}} \rho_0(x)\,\mathrm{d}x > C \qquad \text{and} \qquad \int_{\{|x-x_0|\ge R\}} \rho_0(x)\,\mathrm{d}x < \nu$$

for a small $\nu > 0$ and a large $C > 0$ constants. This criteria for a blow-up of solutions with large concentration can be expressed using Morrey space $M^{2/\alpha}(\mathbb{R}^2)$ norms and the size of such a norm is critical for the global-in-time existence versus finite time blow-up. Additionally, in $d \ge 2$, they derived, for radially symmetric solutions, a criteria for blow-up of solutions in terms of the radial initial concentrations. They showed that there exists a critical constant $C_{\alpha,d} > 0$ such that if

$$R_0^{\alpha-d} \int_{\{|x|<R_0\}} u_0(x)\mathrm{d}x > C_{\alpha,d}$$

for some $R_0 > 0$, then the corresponding solution could not be global-in-time. Next, Biler et al. [9] extended these results in $d = 2$ to include cases with $\gamma > 0$ (see Remark 1.1(c) and (d) for more details on this work).

Biler et al. [13] pointed out that these conditions were, in a sense, complementary to those ensuring the global-in-time existence of solutions, where it imposed smallness of initial conditions in the Morrey space norm. Exemplars of such results can be found in [11, Remark 2.7] and [54, Theorem 2], where global-in-time solutions were proven for $d \geq 1$ and $\alpha = \beta \in (1, 2]$ (futher details on this work see Section 1.6.3). Note that these results contrast sharply with the case $\alpha = 2$, where, as previously mentioned, the blow-up condition depends solely on the initial mass size (see [50, 66] and Remark 1.12).

Biler et al. [14] tried to identify the threshold for size and for a singularity of an initial condition such that the corresponding solution of the problem in $d \geq 2$ for $\alpha \in (0, 2)$ and $\beta = 2$ is still regular and global-in-time. They focused on radially symmetric solutions and presented some results in global-in-time solutions in Sobolev spaces for $\alpha \in (0, 1)$ and Morrey spaces for $\alpha \in (1, 2)$ and blow-up of solutions for $\alpha \in (0, 2]$. They extended the work of [13] in radially symmetric solutions for $\alpha \in (0, 2]$.

In addition, Biler et al. [14] pointed out that the parabolic-elliptic system (1.6) with $\gamma = 0$ and $\beta = 2$ in $d \geq 2$ classify into subcritical case for $\alpha \in (1, 2)$ and supercritical case for $\alpha \in (0, 1]$. Moreover, they emphasized that various results on local-in-time and global-in-time solutions to this system with $\alpha \in (1, 2)$ – such as those found in [15, Theorem 2.2], [16, Theorem 1.1], [11, Theorem 2.1], [54, Theorem 2] and [14, Section 7] – in various functional spaces (Lebesgue, Besov, Morrey) are, broadly speaking, analogous to those for $\alpha = 2$. In [16], Biler and Wu presented results on local-in-time and blow-up solutions to (1.6) in $d = 2$ for initial data in critical Besov spaces $\dot{B}^{1-\alpha}_{2,r}(\mathbb{R}^2)$ with $r \in [1, \infty]$.

Sugiyama et al. [73] mentioned an expanded classification that categorizes this system as subcritical for $\alpha > 1$, critical for $\alpha = 1$, and supercritical for $\alpha < 1$. The authors presented results on local and global existence and uniqueness of solutions of (1.6) with $\gamma = 0$ and $\beta = 2$

for $\alpha \in (0,1)$ and $d \geq 3$ and $\alpha = 1$ and $d \geq 2$ in the Besov space $\dot{B}_{p,q}^{\frac{d}{p}-\alpha}(\mathbb{R}^d)$, which is a critical space for the problem.

1.6.2 Generalized Parabolic-elliptic fractional Keller-Segel model

Consider the following parabolic fractional equation:

$$\begin{cases} \partial_t \rho = -\Lambda^\alpha \rho + \chi \nabla \cdot (\rho \mathcal{B}(\rho)) & t > 0, \quad x \in \mathbb{R}^d, \\ \rho(x,0) = \rho_0(x), \end{cases} \tag{1.28}$$

where $\rho_0 \geq 0$, and $\mathcal{B}(\rho)(x) = C_\vartheta (K * \rho)(x)$, with $C_\vartheta > 0$ being a constant and the attractive kernel K defined by

$$K(x) := \frac{x}{|x|^\vartheta}, \qquad x \in \mathbb{R}^d \setminus \{0\} \quad \text{and} \quad \vartheta > 0. \tag{1.29}$$

Note that for $\vartheta = d$, K corresponds to the Newtonian kernel in $\mathbb{R}^d$.

According to Lafleche et al. [52], this equation can be seen as a generalization of the classical parabolic-elliptic Keller-Segel equation (see also Remarks 1.5 and 1.11). Observe that system (1.28) reduces to the classical parabolic-elliptic Keller-Segel model, represented by (1.3) with $\gamma = 0$, when $\alpha = 2$ and $\vartheta = d$ (which correspond to $\beta = 2$ in system (1.6)).

Moreover, the results presented by Biler et al. [11, 15], discussed in the previous section, correspond to (1.28) with $\alpha \in (1,2)$ and $\vartheta \in [2, \ d+1)$. Additionally, Biler et al. [15] explored the local well-posedness, global well-posedness and finite time blow-up of solutions for parabolic equations (1.28) in arbitrary dimension $d \geq 2$, with $\alpha \in (0,2]$ and potential kernels satisfying $|K(x)| \leq C|x|^\vartheta$, where $1 < \vartheta \leq d$. They determined threshold conditions on the values of α, ϑ, and d that determine whether solutions can be extended indefinitely in time or blow up in finite time for suitable initial data. Furthermore, they highlighted various physical phenomena involving diffusion and particle interactions that can be described by equation (1.28) for an appropriate integral operator $\mathcal{B}$ and specific values of α. For additional examples and detailed results see [12, 15, 24, 25, 55].

To conclude, we highlight the work of Lafleche et al. [52], where the authors examined system (1.28) for $d \geq 1$ and $(\alpha, \ \vartheta) \in \mathbb{R}_+^2$, with $\vartheta < d$. The objective of this study was to evaluate how diffusion competes with an aggregation field. To establish local and global well-posedness, the authors considered the parameter range $(\alpha, \vartheta) \in [0,2) \times [0,d)$ with the condition

$\vartheta + \alpha > 1$ and $k \in [(1-\vartheta)_+, \alpha)$. They then organized their findings into the following distinct cases.

- **Diffusion dominated case ($\vartheta < \alpha$):** global well-posedness for any initial condition in $L^1_k(\mathbb{R}^d)$, where $L^p_k(\mathbb{R}^d) := \left\{\rho \in L^p(\mathbb{R}^d), \left(\sqrt{1+|x|^2}\right)^k \rho \in L^p(\mathbb{R}^d)\right\}$ is a weighted space;

- **Fair competition case ($\vartheta = \alpha$):** global well-posedness for an initial condition in $L^1_k(\mathbb{R}^d) \cap L\ln L(\mathbb{R}^d)$ with small mass, where $L\ln L(\mathbb{R}^d)$ is the space of functions with finite entropy, defined as $L\ln L(\mathbb{R}^d) := \left\{\rho \in L^1(\mathbb{R}^d), \rho\ln(\rho) \in L^1(\mathbb{R}^d)\right\}$;

- **Aggregation dominated case ($\vartheta > \alpha$):** global or local well-posedness for an initial condition in $L^1_k(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ with $p \in \left(\frac{d}{d+\alpha-\vartheta}, \frac{d}{d-\vartheta}\right)$, depending on the smallness condition on the $L^p(\mathbb{R}^d)$ norm of the initial data.

In the case $\vartheta \le \alpha$, their finds expand the existing result by Biler et al. [15], where global existence was proved for $d = 2, 3$ in the case $\alpha \le \frac{d}{2}$. Lafleche et al. [52] also examined the conditions under which solutions to the equation exhibit finite-time blow-up. They showed that, for $(\alpha, \vartheta) \in [0,2) \times [1, d)$ such that $\alpha < \vartheta$ and $k \in (0, \alpha)$, an even nonnegative weak solution to equation (1.28) with initial condition $\rho_0 \in L^1_k(\mathbb{R})$ ceases to exist in finite time if

$$\int_{\mathbb{R}^d} \rho_0(x)\left(\sqrt{1+|x|^2}\right)^k \,\mathrm{d}x \le C_1 \lambda^{\frac{k}{2(\vartheta-k)}} m_0^{\frac{2\vartheta-k}{2(\vartheta-k)}}, \quad \text{for } \alpha > 1$$
$$\int_{\mathbb{R}^d} \rho_0(x)|x|^k \,\mathrm{d}x \le C_2 m_0 \quad \text{and} \quad \lambda m_0 \ge C_3, \quad \text{for } \alpha < 1,$$

where C_1, C_2 and C_3 are constants depending only on d, ϑ, α and k.

The proof of this result relies on the time differentiation of an adequate moment, which leads to a contradiction. This approach is the same as the one followed to prove the existence of a blow-up solution to the parabolic-elliptic Keller-Segel model without chemoattractant consumption ($\gamma = 0$); for instance: the classical Keller-Segel model (1.3) is addressed in [66]; the fractional version (1.6) with $\alpha = 2$ and $1 < \beta \le d$ is discussed in [15]; and the more general fractional system (1.6) with $1 < \alpha \le 2$ and $1 < \beta \le d$ is treated in [11].

Lafleche et al. [52] pointed out that one of the strengths of the blow-up result, even if it deals only with even solutions, is that it applies to weakly singular interactions, i. e., $\vartheta < 2$, which corresponds to considering $\beta > d$ in (1.6).

1.6.3 The Parabolic-parabolic Fractional Keller-Segel Model

The doubly parabolic case with fractional operators, system (1.5), has been addressed for example by Biler et al. [16], Burczak et al. [23], Jiang et al. [43], X. Wang et al. [77], and Wu et al. [78], however for the case where $\alpha = \beta$.

Biler et al. [16] and Wu et al. [78] used the same techniques to analyze the doubly parabolic model. Biler et al. [16] considered the model in $\mathbb{R}^2$ with $\alpha = \beta \in (1, \ 2)$ and initial condition in critical Besov spaces, and proved the existence of a locally unique solution. Wu et al. [78] considered the model in $\mathbb{R}^d$ with $\alpha = \beta \in (1, \ 2]$ in the critical Fourier-Herz spaces, and proved local well-posedness and a global well-posedness with a small initial data.

Besides to prove global-in-time solutions to parabolic-elliptic case, P. G. Lemarié-Rieusset [54] established result on global-in-time solutions to parabolic-parabolic case for $d \geq 1$ and $\alpha = \beta \in (1, 2]$ under smallness condition on initial data ρ_0 in Morrey space and $\nabla c_0 = 0$. Moreover, they proved that when τ goes to 0 in (1.5) the solution of the parabolic-parabolic problem converges in certain space to the solution of the parabolic-elliptic problem.

X. Wang et al. [77] studied a doubly parabolic system with fractional Laplacian $\alpha \in (4/3, \ 2)$ for $d \geq 2$ in Sobolev space. They proved the existence and the uniqueness of a global classical solution, assuming that the initial data are small enough. They also showed the asymptotic decay behaviors of the $W^{m-d-3,\infty}(\mathbb{R}^d)$-norm of ρ and ∇c, for $m \geq d + 4$.

Jiang et al. [43] studied a doubly parabolic model with signal-dependent sensitivity and the source term, on $\mathbb{R}^d$ with $d \geq 2$, in Sobolev space, with $\alpha = \beta \in (0, \ 2)$. They showed the existence, uniqueness, and temporal decay of the global classical solutions to the problem under the assumption of small initial data.

Once again, it is important to emphasize that, to the best of our knowledge, the fundamental question concerning the existence of a solution to fully parabolic fractional Keller-Segel system (1.5) in $\mathbb{R}^d$ with distinct exponents α and β remains unanswered. In Chapter 3, we address this specific system in detail.

1.7 Further PDE models for chemotaxis

Besides the fractional Keller-Segel models, system (1.5) and (1.6), the classical Keller-Segel equations as described in Section 1.2.1 have been under numerous modifications. Several variations of the classical Keller-Segel models have been developed to incorporate additional biological realism and to address specific biological problems. The surveys [3, 40, 41] and the books [66, 67, 75] provide an overview of some of the available adaptations of the Keller-Segel chemotaxis model. They also highlight certain findings from the existing literature, including discussions on aspects like existence, boundedness, blowup of solutions, and numerical analysis. We also mention the following works with fractional Laplacian [23, 38, 43, 80, 81].

A variation of the classical Keller-Segel model that we address here involves incorporating an additional source function, $f(\rho, \nabla\rho)$, into (1.2):

$$\begin{cases} \partial_t \rho = \Delta\rho - \chi\nabla\cdot(\rho\nabla c) + f(\rho,\nabla\rho) & x \in \mathbb{R}^d, \quad t > 0, \\ \tau\partial_t c = \Delta c + \rho - \gamma c & x \in \mathbb{R}^d, \quad t > 0, \\ \rho(t=0) = \rho_0, \quad c(t=0) = c_0 & x \in \mathbb{R}^d. \end{cases} \tag{1.30}$$

where for simplicity, besides retaining the constants τ and γ, all other constants in the system have been normalized to 1.

For instance, f can incorporate logistic terms into the model: $f(\rho) = \nu\rho - \mu\rho^2$, where $\nu\rho$ represents the birth (reproduction) rate of cells, and $-\mu\rho^2$ represents the death rate of cells [75]. Burczak et al. [25] introduces a logistic term into the minimal system by prescribing $f(\rho) = r\rho(1-\rho)$ with $r > 0$. They emphasized that the parabolic-elliptic system is less prone to admitting solutions that exhibit blowup for $r > 0$ compared to the case when $r = 0$. The authors also mention that blowups are excluded for any initial mass, regardless of the relationship between the mass and the parameters r and χ.

Moreover, system (1.30) can model chemotaxis in reaction-diffusion processes [46–48, 67]. Kiselev et al. [48] considered a model with $\tau = 0$ and $f(\rho) = -u\cdot\nabla\rho - \epsilon\rho^q$, where u is divergence free, regular and prescribed independent of ρ. This model is related to the phenomenon known as broadcast spawning, an external fertilization strategy employed by various benthic invertebrates, such as sea urchins, anemones, corals, and jellyfish. As with the previous model, Kiselev et al. [48] proved that the blowup in the parabolic-elliptic system, with $\epsilon > 0$, is excluded for any initial mass. This model is explored in detail in Chapter 4.

Chapter 2

Preparations

With system (1.30) as a starting point, consider the prototype of Keller-Segel equations with nonlocal diffusion terms in dimension $d \geq 2$, with $\alpha \in (1,2]$ and $\beta \in (1,d]$, given by

$$\begin{cases} \partial_t \rho = -\Lambda^\alpha \rho - \chi \nabla \cdot (\rho \nabla c) + f(\rho, \nabla \rho) & x \in \mathbb{R}^d, \quad t > 0, \\ \tau \partial_t c = -\Lambda^\beta c + \rho - \gamma c & x \in \mathbb{R}^d, \quad t > 0, \\ \rho(t=0) = \rho_0, \quad c(t=0) = c_0 & x \in \mathbb{R}^d. \end{cases} \tag{2.1}$$

In this chapter, we aim to establish foundational results and a set of essential tools for investigating the existence, uniqueness, and properties of the solutions to (2.1). We define the specific type of solution under consideration and introduce key notations. Furthermore, we provide a general discussion on some properties of the solutions.

Note that throughout this thesis, constants, which may change from line to line, are denoted by the same letter C, and are independent of the variables ρ, c, x, and t. The notation $C = C(*)$ emphasizes the dependency of C on a parameter represented by "$*$".

2.1 Introduction

System (2.1) models the chemotactic movement of cells, represented by their density $\rho = \rho(x,t)$, toward higher concentrations of a chemical substance $c = c(x,t)$, i. e., ρ moves along the gradient of the chemical. Since ρ and c represent densities, nonnegative initial data is expected to result in nonnegative solutions to the system.

As outlined in Section 1.2, system (2.1) incorporates, in the right-hand side of the first equation, the diffusion of cells (representing random microscopic movements that result in observable macroscopic motion), a chemotactic flux of advective type (where χ stands for the chemotactic sensitivity), and a source term f. The second equation accounts for the diffusion, production, and consumption (or degradation) of the chemical signal, known as the chemoattractant. This signal is emitted by the cells, diffused in the environment, and degraded at a rate proportional to its local concentration, with the nonnegative constant γ describing this degradation rate [3, 19, 66, 75]. Furthermore, the parabolic-elliptic version of (2.1) arises from the assumption that the production and diffusion of the chemical occur much faster than other time scales in the problem, leading the second equation of (2.1) to become $\Lambda^{\beta} c = \rho - \gamma c$. Thus, the parabolic-elliptic model takes the following form

$$\begin{cases} \partial_t \rho = -\Lambda^{\alpha}\rho - \chi\nabla\cdot\left(\rho\nabla\left((-\Delta)^{\beta/2} + \gamma I\right)^{-1}\rho\right) + f(\rho,\nabla\rho), & x\in\mathbb{R}^d, \quad t>0, \\ \rho(x,0) = \rho_0(x) & x\in\mathbb{R}^d. \end{cases} \tag{2.2}$$

which includes the case of $\gamma = 0$.

In Chapter 3, we focus on the parabolic-parabolic model (2.1) with $f(\rho,\nabla\rho) = 0$ and, as discussed in Section 1.2.2 for the Classical Keller-Segel model, we demonstrate that the value of γ (whether $\gamma > 0$ or $\gamma = 0$) does not significantly impact our results on well-posedness. In Chapter 4, we explore the parabolic-elliptic model (2.2) with $f(\rho,\nabla\rho) \neq 0$, extending the study by Kiselev et al. [48] mentioned in Section 1.7.

Remark 2.1. *To study the chemotactic response, it is important to analyze the behavior of the total number of cells over time t given by*

$$m(t) \equiv \int_{\mathbb{R}^d} \rho(x,t)\,\mathrm{d}x. \tag{2.3}$$

If the model is proved to ensure nonnegative solutions for cell density and chemical concentration under nonnegative initial conditions, then (2.3) *is given by the* $L^1(\mathbb{R}^d)$ *norm of* ρ.

Note that, through a formal calculation, considering that ρ *vanishes when* $|x| \to \infty$ *and* $\widehat{(\Lambda^{\alpha}\rho)}(0)$ *is well defined, we obtain*

$$\widehat{(\Lambda^{\alpha}\rho)}(0) = \int_{\mathbb{R}^d} \Lambda^{\alpha}\rho\,\mathrm{d}x = 0,$$

and a direct application of the boundary conditions yields

$$\frac{\mathrm{d}}{\mathrm{d}t} m(t) = \int_{\mathbb{R}^d} f(\rho,\nabla\rho)\,\mathrm{d}x. \tag{2.4}$$

This means that, for $f = 0$, $m(t) = m_0$ *over time. This further implies that, for a nonnegative solution with a nonnegative initial data* $\rho_0 \in L^1(\mathbb{R}^d)$, *the* $L^1(\mathbb{R}^d)$ *norm of* ρ *is conserved over time:*

$$m(t) \equiv \|\rho(\cdot,t)\|_{L^1} = \|\rho_0\|_{L^1} \equiv m_0.$$

That is then total number of cells does not vary in time, which is expected as system (2.1) or (2.2), in that case, models only the motion of the cells (there are no birth or death processes).

Moreover, using the same approach as before, for the parabolic-parabolic model (2.1), again with $f = 0$ and assuming nonnegative initial condition $c_0 \in L^1(\mathbb{R}^d)$ and a nonnegative solution, the concentration of the chemical over time t, $\|c(\cdot,t)\|_{L^1}$, grows proportionally to $\|\rho_0\|_{L^1}$:

$$\|c(\cdot,t)\|_{L^1} = \|\rho_0\|_{L^1}\left(\frac{1-e^{-\frac{\gamma}{\tau}t}}{\gamma}\right) + \|c_0\|_{L^1}e^{-\frac{\gamma}{\tau}t} \tag{2.5}$$

and

$$\|c(\cdot,t)\|_{L^1} \xrightarrow[t\to\infty]{} \frac{\|\rho_0\|_{L^1}}{\gamma}, \tag{2.6}$$

which includes $\gamma = 0$ by taking $\gamma \to 0$.

*Note that, for small t, this growth is approximately linear in time, and for $\gamma > 0$, the influence of c_0 is reduced over time and the growth of the chemical concentration is bounded. This explains why γ is called **damping constant**.*

2.2 Mild solution to Keller-Segel system

Based on Section 1.4, we turn (2.1) and (2.2) into an integral equation on ρ depending on ∇c. That is,

$$\rho(x,t) = K_t^\alpha * \rho_0(x) - \chi\int_0^t \nabla K_{t-s}^\alpha * (\rho\nabla c)\,\mathrm{d}s + \int_0^t K_{t-s}^\alpha * f(\rho,\nabla\rho)\mathrm{d}s, \tag{2.7}$$

where ∇c given by:

-for the parabolic-parabolic system (2.1),

$$\nabla c(x,t) = e^{-\frac{\gamma}{\tau}t}K_{\frac{t}{\tau}}^\beta * \nabla c_0(x) + \int_0^t \frac{1}{\tau}e^{\gamma\left(\frac{s-t}{\tau}\right)}\nabla K_{\frac{t-s}{\tau}}^\beta * \rho\,\mathrm{d}s; \tag{2.8}$$

-for the parabolic-elliptic equation (2.2) with $\gamma = 0$,

$$\nabla c(x,t) = \nabla\Phi * \rho(x) = -\frac{\Gamma\left((d-\beta+2)/2\right)}{\pi^{d/2}2^{\beta-1}\Gamma\left(\beta/2\right)}\int_{\mathbb{R}^d}\frac{x-y}{|x-y|^{d-\beta+2}}\rho(y)\,\mathrm{d}y; \tag{2.9}$$

-for the parabolic-elliptic equation (2.2) with $\gamma > 0$,

$$\nabla c(x,t) = \nabla\Phi_{\beta,\gamma} * \rho(x) = \int_0^\infty e^{-\gamma s}\nabla K_s^\beta * \rho(x,t)\mathrm{d}s, \tag{2.10}$$

with the kernels K_t^α, K_t^β and $\Phi_{\beta,\gamma}$ as defined in Proposition 1.7 and $\nabla\Phi$ as described in Remark 1.11.

Remark 2.2. *Note that, if* $\nabla c_0 = 0$ *in* (2.1)*, each of the models can be considered as a single nonlinear parabolic equation for* ρ *with a nonlocal (either in* x *or in* (x,t)*) nonlinearity since the term* ∇c *can be expressed as a linear integral operator acting on* ρ*.*

Remark 2.3. *It is usual to specify* ∇c *since* c *itself does not appear in the first equation of the parabolic-parabolic system* (2.1)*. Therefore, in the existence theorem in Chapter 3, the initial condition and the results are specified for* ∇c*.*

Similarly, as pointed out in Remarks 1.5 and 1.11, for the parabolic-elliptic case (2.2)*,* ∇c *is considered instead of* c*. However, the theorem concerning the solution to this equation generally specifies only* ρ *for the initial condition and the results, as, in this case,* ∇c *is solely a function of* ρ *(in addition to* x *and* t*).*

Remark 2.4. *Note that, as mentioned before, for* $\gamma > 0$*,* γc *plays a role of damping in* (2.8)*.*

This general method for writing solutions to partial differential equations via integral formulation is also known as Duhamel's principle, and (2.7) with the correspondent ∇c is known as Duhamel's formula.

Using this integral formulation, we define a mild solution to systems (2.1) and (2.2) as:

Definition 2.5 (Mild solution). *A solution to* (2.7) *is called the mild solution to system* (2.1) *or* (2.2)*, with the correspondent definition of* ∇c *above.*

2.2.1 Contraction Mapping Principle

Through the integral formulation, we can convert the problem of finding solutions to system (2.1) or (2.2) into a fixed point problem in a Banach space $(\mathbf{X}, \|\cdot\|_{\mathbf{X}})$. With that in mind, let us recall the Banach Fixed Point Theorem:

Theorem 2.6 (Banach Fixed Point Theorem). *Let* $(\mathbf{X}, \|\cdot\|_{\mathbf{X}})$ *be a Banach space, and assume that* $\mathcal{T} : \mathbf{X} \to \mathbf{X}$ *is a bounded linear operator and a contraction on* $\mathbf{X}$*. That is, there exists a constant* $\Theta \in (0, 1)$ *such that*

$$\|\mathcal{T}(u) - \mathcal{T}(v)\|_{\mathbf{X}} \le \Theta \|u - v\|_{\mathbf{X}} \tag{2.11}$$

for all $u, v \in \mathbf{X}$*. Then, there exists a unique* $u \in \mathbf{X}$ *such that* $\mathcal{T}(u) = u$*, i. e., the linear operator* $\mathcal{T}$ *has a unique fixed point in* $\mathbf{X}$*.*

Now, using Theorem 2.6, we can transform the problem of finding the solution to the parabolic-elliptic system (2.2) in a space $\mathbf{X}$ into a fixed point problem. This is achieved by considering the mapping defined through (2.7):

$$\mathcal{T}(\rho) \equiv K_t^\alpha * \rho_0 + \int_0^t K_{t-s}^\alpha * (\chi\nabla\cdot(\rho(s)\nabla c(s)) + f(\rho,\nabla\rho))\,\mathrm{d}s. \tag{2.12}$$

This is equivalent to showing that the initial data ρ_0 is such that $K_t^\alpha * \rho_0 \in \mathbf{X}$ and there exists a constant $\Theta \in (0,1)$ for which the inequality (2.11) holds for the linear operator $\mathcal{B}$, defined as

$$\mathcal{B}(\rho) \equiv \int_0^t K_{t-s}^\alpha * (\chi\nabla\cdot(\rho(s)\nabla c(s)) + f(\rho,\nabla\rho))\,\mathrm{d}s. \tag{2.13}$$

This approach ensures that $\mathcal{T}$, as defined, is a contraction mapping in $\mathbf{X}$, thereby guaranteeing the existence of a unique fixed point.

To convert the problem of finding the solution of the parabolic-parabolic model (2.1) into a fixed point problem, let us consider a corollary of Theorem 2.6 based on a result from [53].

Corollary 2.7. *Let $(\mathbf{X},\|\cdot\|_{\mathrm{X}})$ be a Banach space, $\mathcal{A} : \mathbf{X}\times\mathbf{X} \to \mathbf{X}$ a bounded bilinear form satisfying*

$$\|\mathcal{A}(u,v)\|_{\mathrm{X}} \le C_{\mathcal{A}}\|u\|_{\mathrm{X}}\|v\|_{\mathrm{X}} \quad \textit{for all } u,\ v \in \mathbf{X},\ \textit{and a constant } C_{\mathcal{A}} > 0, \tag{2.14}$$

and $\mathcal{L} : \mathbf{X} \to \mathbf{X}$ a bounded linear form satisfying

$$\|\mathcal{L}(u)\|_{\mathrm{X}} \le C_{\mathcal{L}}\|u\|_{\mathrm{X}} \quad \textit{for all } u \in \mathbf{X},\ \textit{and a constant } C_{\mathcal{L}} > 0. \tag{2.15}$$

Then, if $0 < \delta < \frac{1-2C_{\mathcal{L}}}{4C_{\mathcal{A}}}$ and $u_1 \in \mathbf{X}$ is such that $\|u_1\|_{\mathrm{X}} < \delta$, the equation $u = u_1 + \mathcal{A}(u,u) + \mathcal{L}(u)$ has a solution in $\mathbf{X}$ such that $\|u\|_{\mathrm{X}} \le 2\delta$. This solution is the only one in the ball $\bar{B}(0,2\delta)$.

Proof. Starting from u_1, we define the sequence $u_{n+1} = u_1 + \mathcal{A}(u_n,u_n) + \mathcal{L}(u_n)$. By induction, we assume that $\|u_n\|_{\mathrm{X}} \le 2\delta$. Then

$$\|u_{n+1}\|_{\mathrm{X}} \le \|u_1\|_{\mathrm{X}} + C_{\mathcal{A}}\|u_n\|_{\mathrm{X}}^2 + C_{\mathcal{L}}\|u_n\|_{\mathrm{X}} \le \delta + 4C_{\mathcal{A}}\delta^2 + 2C_{\mathcal{L}}\delta = \delta + \delta(4C_{\mathcal{A}}\delta + 2C_{\mathcal{L}}) \le 2\delta,$$

since $4\delta C_{\mathcal{A}} + 2C_{\mathcal{L}} < 1$. Moreover, we have

$$\begin{aligned}
\|u_{n+1} - u_n\|_{\mathrm{X}} &= \|\mathcal{A}(u_n,u_n) - \mathcal{A}(u_{n-1},u_{n-1}) + \mathcal{L}(u_n) - \mathcal{L}(u_{n-1})\|_{\mathrm{X}} \\
&= \|\mathcal{A}(u_n - u_{n-1},u_n) + \mathcal{A}(u_{n-1},u_n - u_{n-1}) + \mathcal{L}(u_n) - \mathcal{L}(u_{n-1})\|_{\mathrm{X}} \\
&\le \|\mathcal{A}(u_n - u_{n-1},u_n)\|_{\mathrm{X}} + \|\mathcal{A}(u_{n-1},u_n - u_{n-1})\|_{\mathrm{X}} + \|\mathcal{L}(u_n) - \mathcal{L}(u_{n-1})\|_{\mathrm{X}} \\
&\le C_{\mathcal{A}}\|u_n\|_{\mathrm{X}}\|u_n - u_{n-1}\|_{\mathrm{X}} + C_{\mathcal{A}}\|u_{n-1}\|_{\mathrm{X}}\|u_n - u_{n-1}\|_{\mathrm{X}} + C_{\mathcal{L}}\|u_n - u_{n-1}\|_{\mathrm{X}} \\
&\le [C_{\mathcal{A}}(\|u_n\|_{\mathrm{X}} + \|u_{n-1}\|_{\mathrm{X}}) + C_{\mathcal{L}}]\,\|u_n - u_{n-1}\|_{\mathrm{X}},
\end{aligned}$$

and, with the previous conclusion, it follows that

$$\begin{aligned}
\|u_{n+1} - u_n\|_{\mathrm{X}} &\le (4\delta C_{\mathcal{A}} + C_{\mathcal{L}})\,\|u_n - u_{n-1}\|_{\mathrm{X}} \\
&< (4\delta C_{\mathcal{A}} + 2C_{\mathcal{L}})\,\|u_n - u_{n-1}\|_{\mathrm{X}} \\
&\le C\|u_n - u_{n-1}\|_{\mathrm{X}},
\end{aligned}$$

where $C < 1$. Hence $\|u_{n+1} - u_n\|_{\mathrm{X}} \le C^n \|u_2 - u_1\|_{\mathrm{X}}$.

Therefore, u_n converges to a limit u, which is the required solution. The uniqueness of u in $\bar{B}(0, 2\delta)$ is then trivial. □

Now, in view of (2.7), we can define u_1 as

$$u_1(x,t) = K_t^\alpha * \rho_0(x), \tag{2.16}$$

the bilinear form $\mathcal{A} : \mathrm{X} \times \mathrm{X} \to \mathrm{X}$ as

$$\mathcal{A}(u,v)(t) = -\int_0^t \left[\chi \nabla K_{t-s}^\alpha * \left(u(s) \int_0^s \frac{1}{\tau} e^{\gamma\left(\frac{w-s}{\tau}\right)} \nabla K^\beta_{\frac{s-w}{\tau}} * v(w) \mathrm{d}w \right) + K_{t-s}^\alpha * f(u, \nabla u) \right] \mathrm{d}s, \tag{2.17}$$

and the linear form $\mathcal{L} : \mathrm{X} \to \mathrm{X}$ as

$$\mathcal{L}(u)(t) = -\int_0^t \nabla K_{t-s}^\alpha * \left[u(s) e^{-\frac{\gamma}{\tau} s} K^\beta_{\frac{s}{\tau}} * \nabla c_0 \right] \mathrm{d}s. \tag{2.18}$$

Then, we transform the problem of finding the solution to the parabolic-parabolic model (2.1) in a space X into a fixed point problem, through (2.7), by applying Corollary 2.7.

Note that this is equivalent to considering the fixed point problem for the mapping

$$\mathcal{T}(\rho) \equiv u_1 + \mathcal{A}(\rho,\rho) + \mathcal{L}(\rho) \tag{2.19}$$

through the application of Theorem 2.6.

Remark 2.8. *Sugiyama et al. [73] point out that the application of this method is effective in some cases where the order of the derivative of fractional dissipation is larger than that of the nonlinear term. They analyzed the parabolic-elliptical case where $\gamma = 0$ and $\beta = 2$. The authors also mentioned that this method is difficult to apply to the case $0 \le \alpha \le 1$ since the dissipation balances nonlinearity when $\alpha = 1$ (it is difficult to show that the continuity holds for the integral operator of type (2.13)).*

2.3 Preliminary

Here, we present key results crucial for establishing the existence and properties of solutions to systems of type (2.1) and (2.2). We start by presenting important properties and estimates regarding the kernel functions K_t^α and K^α, defined in (1.17) and (1.19) respectively.

Lemma 2.9. *Consider the functions K_t^α and K^α defined by (1.17) and (1.19), respectively. Then,*

1. *it follows that $K^\alpha \in C_0^\infty(\mathbb{R}^d)$;*

2. *$K^\alpha, \nabla K^\alpha \in L^p(\mathbb{R}^d)$ and, for $0 < t < \infty$, $K_t^\alpha, \nabla K_t^\alpha \in L^p(\mathbb{R}^d)$, for any $1 \le p \le \infty$ [58];*

3. *it follows that*

$$\int_{\mathbb{R}^d} K^\alpha(x)\,\mathrm{d}x = 1, \qquad \int_{\mathbb{R}^d} K_t^\alpha(x)\,\mathrm{d}x = 1, \quad \textit{and} \quad \int_{\mathbb{R}^d} \nabla K^\alpha(x)\,\mathrm{d}x = 0;$$

4. *for every $u \in L^1(\mathbb{R}^d)$, it follows that*

$$\int_{\mathbb{R}^d} K_t^\alpha * u(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} u(x)\,\mathrm{d}x \quad \textit{and} \quad \int_{\mathbb{R}^d} \nabla K_t^\alpha * u(x)\,\mathrm{d}x = 0.$$

Proof. **1.** Note that $\widehat{K^\alpha}(\xi) = \mathrm{e}^{-|2\pi\xi|^\alpha} \in L^1(\mathbb{R}^d)$. Thus, from the Fourier transform, $K^\alpha \in L^\infty(\mathbb{R}^d) \cap C(\mathbb{R}^d)$. Moreover, from the Riemann Lebesgue Lemma, $\mathfrak{F}\left(L^1(\mathbb{R}^d)\right) \subset C_0(\mathbb{R}^d)$, $K^\alpha \in L^\infty(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$. In an analogous way, we have $\nabla K^\alpha \in L^\infty(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$, since $2\pi \mathrm{i}\xi \mathrm{e}^{-|2\pi\xi|^\alpha} \in \left(L^1(\mathbb{R}^d)\right)^d$. Additionally, $\partial^\eta K^\alpha \in L^\infty(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$, since $(2\pi\xi)^\eta \mathrm{e}^{-|2\pi\xi|^\alpha} \in L^1(\mathbb{R}^d)$, where for higher-order derivatives we use multi-index notation (see Index of Notation). Therefore, we conclude that $K^\alpha \in C_0^\infty(\mathbb{R}^d)$.

2. See [58].

3. According to the Fourier Inversion Theorem, if f and its Fourier transform $\widehat{f}$ lie in $L^1(\mathbb{R}^d)$, then f agrees almost everywhere with a continuous function f_0, and $(\widehat{f})^\vee = (f^\vee)^\wedge = f_0$. Thus, since K^α and $\widehat{K^\alpha}$ are in $L^1(\mathbb{R}^d)$, we have $\int_{\mathbb{R}^d} K^\alpha(x)\,\mathrm{d}x = \widehat{K^\alpha}(0) = 1$.

Furthermore, for the transformation $T(x) = t^{-1}x$ $(t > 0)$, we have $(f \circ T)\widehat{\ }(\xi) = t^d \widehat{f}(t\xi)$. Applying this to $K_t^\alpha(x) = t^{-\frac{d}{\alpha}} K^\alpha\left(\frac{x}{t^{1/\alpha}}\right)$, we see that $\int_{\mathbb{R}^d} K_t^\alpha(x)\,\mathrm{d}x = \widehat{K_t^\alpha}(0) = \widehat{K^\alpha}(0) = 1$.

Moreover, utilizing the property $(\nabla f)\widehat{\ }(\xi) = 2\pi i\xi \widehat{f}(\xi)$ for $f \in C^1(\mathbb{R}^d)$ with $\nabla f \in L^1(\mathbb{R}^d)$, we derive the final equality in 3.

4. Set $F(x,y) = K_t^\alpha(x-y)u(y)$. Then, from Lemma 2.9.2, for a.e. $y \in \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} |F(x,y)|\mathrm{d}x = |u(y)| \int_{\mathbb{R}^N} |K_t^\alpha(x-y)|\mathrm{d}x = \|K_t^\alpha\|_{L^1}|u(y)| < \infty$$

and, moreover,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |F(x,y)|\mathrm{d}x\mathrm{d}y = \|K_t^\alpha\|_{L^1}\|u\|_{L^1} < \infty,$$

since $u \in L^1(\mathbb{R}^d)$. Then, from Tonelli's theorem we see that $F \in L^1\left(\mathbb{R}^d \times \mathbb{R}^d\right)$. Applying Fubini's theorem and Lemma 2.9.3, we have that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_t^\alpha(x-y)u(x)\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^d} u(x)\left(\int_{\mathbb{R}^d} K_t^\alpha(x-y)\mathrm{d}x\right)\mathrm{d}y = \int_{\mathbb{R}^d} u(y)\mathrm{d}y.$$

Similarly, we prove

$$\int_{\mathbb{R}^d} \nabla K_t^\alpha * v(x)\mathrm{d}x = \int_{\mathbb{R}^d} v(y)\left(\int_{\mathbb{R}^d} \nabla K_t^\alpha(x-y)\mathrm{d}x\right)\mathrm{d}y = 0.$$

□

Lemma 2.10. *Let $1 \le q \le p \le \infty$ and let $\varphi \in L^q(\mathbb{R}^d)$. For $t > 0$, we have*

$$\left\|K_t^\alpha * \varphi\right\|_{L^p} \le C t^{-\frac{d}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)} \|\varphi\|_{L^q}, \tag{2.20}$$

$$\left\|\nabla K_t^\alpha * \varphi\right\|_{L^p} \le C t^{-\frac{d}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{\alpha}} \|\varphi\|_{L^q}. \tag{2.21}$$

Proof. From Lemma 2.9.2, the kernel function, $K^\alpha(x)$, defined in (1.19), and $\nabla K^\alpha(x)$ lie in $L^r(\mathbb{R}^d)$ for any $1 \le r \le \infty$. Therefore, as $K_t^\alpha(x) = t^{-\frac{d}{\alpha}} K^\alpha\left(\frac{x}{t^{1/\alpha}}\right)$ for $t > 0$, we have

$$\begin{aligned}
\|K_t^\alpha\|_{L^r} &= t^{-\frac{d}{\alpha}} \left(\int_{\mathbb{R}^d} \left| K^\alpha\left(\frac{x}{t^{1/\alpha}}\right)\right|^r \, \mathrm{d}x \right)^{1/r} \\
&= t^{-\frac{d}{\alpha}\left(1-\frac{1}{r}\right)} \left(\int_{\mathbb{R}^d} |K^\alpha(z)|^r \, \mathrm{d}z \right)^{1/r} \\
&= t^{-\frac{d}{\alpha}\left(1-\frac{1}{r}\right)} \|K^\alpha\|_{L^r}.
\end{aligned}$$

Then, by Young's Inequality, for $\frac{1}{p} + 1 = \frac{1}{r} + \frac{1}{q}$, with $1 \le r,\ p,\ q \le \infty$, and $\varphi \in L^q(\mathbb{R}^d)$, we have

$$\left\|K_t^\alpha * \varphi\right\|_{L^p} \le \left\|K_t^\alpha\right\|_{L^r} \|\varphi\|_{L^q} \le t^{-\frac{d}{\alpha}\left(1-\frac{1}{r}\right)} \|K^\alpha\|_{L^r} \|\varphi\|_{L^q} \le C t^{-\frac{d}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)} \|\varphi\|_{L^q},$$

where $C = \|K^\alpha\|_{L^r}$. In the same way, since

$$\nabla K_t^\alpha(x) = t^{-\frac{d}{\alpha}} (\nabla K^\alpha)\left(\frac{x}{t^{1/\alpha}}\right) t^{-1/\alpha} = t^{-\frac{d}{\alpha}-\frac{1}{\alpha}} (\nabla K^\alpha)\left(\frac{x}{t^{1/\alpha}}\right),$$

we have $\|\nabla K_t^\alpha\|_{L^r} = t^{-\frac{d}{\alpha}\left(1-\frac{1}{r}\right)-\frac{1}{\alpha}} \|\nabla K^\alpha\|_{L^r}$. Then, the results follow from Young's Inequality. □

Lemma 2.11. *Let $1 \le q \le p \le \infty$ and $\varphi \in W^{k,q}(\mathbb{R}^d)$, where $k > 0$ is an integer. For $t > 0$, we have*

$$\left\|K_t^\alpha * \varphi\right\|_{W^{k,p}} \le C t^{-\frac{d}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)} \|\varphi\|_{W^{k,q}}, \tag{2.22}$$

$$\left\|\nabla K_t^\alpha * \varphi\right\|_{W^{k,p}} \le C t^{-\frac{d}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{\alpha}} \|\varphi\|_{W^{k,q}}. \tag{2.23}$$

In particular, for Sobolev spaces $H^k(\mathbb{R}^d)$, we obtain

$$\|K_t^\alpha * \varphi\|_{H^k} \le C \|\varphi\|_{H^k}, \tag{2.24}$$

$$\|\nabla K_t^\alpha * \varphi\|_{H^k} \le C t^{-1/\alpha} \|\varphi\|_{H^k}. \tag{2.25}$$

Proof. It follows from Lemma 2.10 by replacing φ with $D^\nu \varphi$, where ν is a multiindex. □

Definition 2.12 (Weighted space $L^\infty_{\alpha+d}(\mathbb{R}^d)$). *The space $L^\infty_{\alpha+d}(\mathbb{R}^d)$ is a weighted $L^\infty(\mathbb{R}^d)$ space defined as*

$$L^\infty_{\alpha+d}(\mathbb{R}^d) = \left\{ \varphi \in L^\infty(\mathbb{R}^d) \ : \ \|\varphi\|_{L^\infty_{\alpha+d}} \equiv \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} (1 + |x|)^{\alpha+d} |\varphi(x)| < \infty \right\}.$$

where α is the order of the fractional differential operator in (2.1) or (2.2).

Remark 2.13. *In general, we can define the spaces $L^\infty_\vartheta(\mathbb{R}^d)$ for a fixed $\vartheta \geq 0$ as weighted $L^\infty(\mathbb{R}^d)$ spaces by replacing $\alpha + d$ with ϑ in the definition above.*

Note that $L^\infty_\vartheta(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ and, for $\vartheta > d/p$, $L^\infty_\vartheta(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$. Indeed, let $\varphi \in L^\infty_\vartheta(\mathbb{R}^d)$, since $1^\vartheta \leq (1 + |x|)^\vartheta$ and $\|\varphi\|_{L^\infty} = \operatorname{ess\,sup}_{x\in\mathbb{R}^d} |\varphi(x)|$, we have $\|\varphi\|_{L^\infty} \leq \|\varphi\|_{L^\infty_{\alpha+d}}$. Moreover, since $(1 + |x|)^{-\vartheta} \in L^p(\mathbb{R}^d)$ for $\vartheta > d/p$, we obtain $\|\varphi\|_{L^p} \leq \left\|(1 + |\cdot|)^{-\vartheta}\right\|_{L^p} \left\|(1 + |\cdot|)^\vartheta \varphi\right\|_{L^\infty} = C \|\varphi\|_{L^\infty_\vartheta}$. Therefore, we can infer that the weighted space $L^\infty_{\alpha+d}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, since $\alpha > 0$.

Lemma 2.14. *[11, Lemma 3.3] Let $\varphi \in L^\infty_{\alpha+d}(\mathbb{R}^d)$ and $t > 0$. There exists $C > 0$ independent of φ and t such that*

$$\left\|K^\alpha_t * \varphi\right\|_{L^\infty_{\alpha+d}} \leq C(1 + t) \|\varphi\|_{L^\infty_{\alpha+d}}, \tag{2.26}$$

$$\left\|\nabla K^\alpha_t * \varphi\right\|_{L^\infty_{\alpha+d}} \leq Ct^{-1/\alpha} \|\varphi\|_{L^\infty_{\alpha+d}} + Ct^{1-1/\alpha} \|\varphi\|_{L^1}. \tag{2.27}$$

Definition 2.15. *We define the Banach space $E_{\alpha+d}$ as*

$$E_{\alpha+d} = \left\{\varphi \in W^{1,\infty}_{loc}(\mathbb{R}^d) \,:\, \|\varphi\|_{E_{\alpha+d}} \equiv \|\varphi\|_{L^\infty_{\alpha+d}} + \|\nabla\varphi\|_{L^\infty_{\alpha+d}} < \infty\right\}.$$

where α is the order of the fractional differential operator in (2.1) or (2.2).

Lemma 2.16. *[20, Lemma 3.2] Assume $\varphi \in E_{\alpha+d}(\mathbb{R}^d)$ and $t > 0$. There exists $C > 0$ independent of φ and t such that*

$$\left\|K^\alpha_t * \varphi\right\|_{E_{\alpha+d}} \leq C(1 + t) \|\varphi\|_{E_{\alpha+d}}, \tag{2.28}$$

$$\left\|\nabla K^\alpha_t * \varphi\right\|_{E_{\alpha+d}} \leq Ct^{-1/\alpha} \|\varphi\|_{E_{\alpha+d}} + Ct^{1-1/\alpha} \|\varphi\|_{L^1}. \tag{2.29}$$

Lemma 2.17. *[70, Lemma 6] Let γ, ϑ be multi-indices, $|\vartheta| < |\gamma| + \alpha \max(j, 1)$, $j = 0, 1, 2, \ldots$, $1 \leq p \leq \infty$, and $\alpha \in (0, 2]$. Then,*

$$\|x^\vartheta D^j_t D^\gamma K^\alpha_t\|_{L^p} = Ct^{\frac{|\vartheta|-|\gamma|}{\alpha} - j - \frac{d(p-1)}{\alpha p}} \tag{2.30}$$

for C a constant depending only on α, γ, ϑ, j, p, and the space dimension d.

Definition 2.18. *Let $\vartheta \geq 0$. We define the Banach space $M_\vartheta(\mathbb{R}^d)$ as*

$$M_\vartheta = \left\{\varphi \in W^{1,1}(\mathbb{R}^d) \,:\, \|\varphi\|_{M_\vartheta} \equiv \int_{\mathbb{R}^d} (|\varphi(x)| + |\nabla\varphi(x)|) \left(1 + |x|^\vartheta\right) \,dx < \infty\right\}.$$

Lemma 2.19. *Assume $\varphi \in M_\vartheta(\mathbb{R}^d)$, with $0 \leq \vartheta < \alpha$, and $t > 0$. There exists $C > 0$ independent of φ and t such that*

$$\|K^\alpha_t * \varphi\|_{M_\vartheta} \leq C\left(1 + t^{\vartheta/\alpha}\right) \|\varphi\|_{M_\vartheta}, \tag{2.31}$$

$$\|\nabla K^\alpha_t * \varphi\|_{M_\vartheta} \leq C\left(t^{-1/\alpha} + t^{(\vartheta-1)/\alpha}\right) \|\varphi\|_{M_\vartheta}. \tag{2.32}$$

Proof. Note that $\left(1 + |x|^\vartheta\right) \leq C\left(1 + |x - y|^\vartheta\right)\left(1 + |y|^\vartheta\right)$ and $\left(1 + |x|^\vartheta\right) \leq 1 + \sum_{i=1}^d |x_i|^\vartheta$. Moreover, from Lemma 2.17, as $\vartheta < \alpha$, we have K^α_t and $x^\vartheta_i K^\alpha_t \in L^1(\mathbb{R}^d)$, for $1 \leq i \leq d$ and $t > 0$.

Then, considering $\varphi \in M_\vartheta(\mathbb{R}^d)$ and employing Fubini's theorem, we have

$$\begin{aligned}
\|K_t^\alpha * \varphi\|_{M_\vartheta} &\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(1+|x|^\vartheta\right)|K_t^\alpha(x-y)|\left(|\varphi(y)|+|\nabla\varphi(y)|\right)\,\mathrm{d}y\,\mathrm{d}x\\
&\le C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(1+|x-y|^\vartheta\right)\left(1+|y|^\vartheta\right)|K_t^\alpha(x-y)|\left(|\varphi(y)|+|\nabla\varphi(y)|\right)\,\mathrm{d}y\,\mathrm{d}x\\
&\le C\|(1+|\cdot|^\vartheta)K_t^\alpha\|_{L^1}\|\varphi\|_{M_\vartheta}\\
&\le C\left\|\left(1+\sum_{i=1}^d |x_i|^\vartheta\right)K_t^\alpha\right\|_{L^1}\|\varphi\|_{M_\vartheta}\\
&\le C\left(\|K_t^\alpha\|_{L^1}+\sum_{i=1}^d\|x_i^\vartheta K_t^\alpha\|_{L^1}\right)\|\varphi\|_{M_\vartheta}\\
&\le C(1+t^{\frac{\vartheta}{\alpha}})\|\varphi\|_{M_\vartheta},
\end{aligned}$$

where the last line is due to application of Lemma 2.17 as $\vartheta < \alpha$.

Similarly, from Lemma 2.17, as $\vartheta < \alpha+1$, we see that ∇K_t^α and $x_i^\vartheta \nabla K_t^\alpha \in L^1(\mathbb{R}^d)$ for $1 \le i \le d$ and $t > 0$, and

$$\|\nabla K_t^\alpha * \varphi\|_{M_\vartheta} \le C\left(\|\nabla K_t^\alpha\|_{L^1}+\sum_{i=1}^d \|x_i^\vartheta \nabla K_t^\alpha\|_{L^1}\right)\|\varphi\|_{M_\vartheta} \le C(t^{-\frac{1}{\alpha}}+t^{\frac{\vartheta-1}{\alpha}})\|\varphi\|_{M_\vartheta}.$$

□

Remark 2.20. *The constraint on the value of ϑ in Lemma 2.19 arises from the distinctive behavior of non-Gaussian Lévy α-stable semigroups, for $0 < \alpha < 2$. Unlike Gaussian kernel ($\alpha = 2$), which exhibits exponential decay, these semigroups decay only at an algebraic rate, specifically $|x|^{-d-\alpha}$ as $|x| \to \infty$. This difference in decay behavior explains the imposed constraint on ϑ.*

Remark 2.21. *Note that, for $0 \le \vartheta < \alpha$, $E_{\alpha+d}(\mathbb{R}^d) \subset M_\vartheta(\mathbb{R}^d)$. To see this, observe that since $\alpha + d - \vartheta > d$, we have*

$$\begin{aligned}
\|\varphi\|_{M_\vartheta} &\le C\int_{\mathbb{R}^d}\left(|\varphi(x)|+|\nabla\varphi(x)|\right)(1+|x|)^\vartheta\,\mathrm{d}x\\
&\le C\int_{\mathbb{R}^d}(|\varphi(x)|+|\nabla\varphi(x)|)(1+|x|)^{-\alpha-d+\vartheta}(1+|x|)^{\alpha+d}\,\mathrm{d}x\\
&\le C\|\varphi\|_{E_{\alpha+d}}\int_{\mathbb{R}^d}(1+|x|)^{-\alpha-d+\vartheta}\,\mathrm{d}x\\
&= C\|\varphi\|_{E_{\alpha+d}}.
\end{aligned}$$

Lemma 2.22. *Assume $b+1>0$, $a+1>0$. Then the following inequality holds*

$$\int_0^t (t-s)^a s^b ds \le Ct^{a+b+1}, \quad \textit{for all } \; t>0 \tag{2.33}$$

where C is a positive constant independent of t.

Proof. We have

$$\begin{aligned}\int_0^t (t-s)^a s^b ds &= t^{a+b} \int_0^t \left(1-\frac{s}{t}\right)^a \left(\frac{s}{t}\right)^b ds \\ &= t^{a+b+1} \int_0^1 (1-z)^{(1+a)-1} z^{(1+b)-1} dz \\ &= \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} t^{a+b+1}.\end{aligned}$$

□

Lemma 2.23. *Let q and s be positive integers such that $s > d/2+1$, and $\vartheta \geq 0$. Consider $f, g \in H^s(\mathbb{R}^d) \cap M_\vartheta(\mathbb{R}^d)$. Then, the following estimates hold:*

$$\|f^q - g^q\|_{H^s} \leq C\left(\|f\|_{H^s}^{q-1} + \|g\|_{H^s}^{q-1}\right)\|f-g\|_{H^s}, \tag{2.34}$$

$$\|f^q - g^q\|_{M_\vartheta} \leq C\left(\|f\|_{H^s}^{q-1} + \|g\|_{H^s}^{q-1}\right)\|f-g\|_{M_\vartheta}, \tag{2.35}$$

$$\|f\nabla\Delta^{-1}f - g\nabla\Delta^{-1}g\|_{H^s} \leq C\left(\|f\|_{H^s} + \|f\|_{M_\vartheta} + \|g\|_{H^s}\right)\left(\|f-g\|_{M_\vartheta} + \|f-g\|_{H^s}\right), \tag{2.36}$$

$$\|f\nabla\Delta^{-1}f - g\nabla\Delta^{-1}g\|_{M_\vartheta} \leq C\left(\|f\|_{H^s} + \|f\|_{M_\vartheta} + \|g\|_{H^s} + \|g\|_{M_\vartheta}\right)\left(\|f-g\|_{M_\vartheta} + \|f-g\|_{H^s}\right). \tag{2.37}$$

Proof. Estimate (2.34) follows from writing $f^q - g^q = (f-g)\left(\sum_{j=0}^{q-1} f^{q-1-j} g^j\right)$ and the fact that $H^s(\mathbb{R}^d)$ is an algebra when $s > d/2$. For (2.35), a similar approach is adopted, using the Sobolev embedding theorem for $s > d/2+1$: $\|\varphi\|_{L^\infty} + \|\nabla\varphi\|_{L^\infty} \leq C\|\varphi\|_{H^s}$, for any $\varphi \in H^s(\mathbb{R}^d)$. This leads to

$$\|f^q - g^q\|_{M_\vartheta} \leq \left(\|f^{q-1} + \ldots + g^{q-1}\|_{L^\infty} + \|\nabla\left(f^{q-1} + \ldots + g^{q-1}\right)\|_{L^\infty}\right)\|f-g\|_{M_\vartheta}.$$

Now, consider $\varphi \in H^s(\mathbb{R}^d)$. By decomposing $\nabla\Delta^{-1}\varphi$ into two parts, we find that

$$\|\nabla\Delta^{-1}\varphi\|_{L^\infty} \leq C \operatorname{ess\,sup}_{x\in\mathbb{R}^d}\left(\int_{B_1(x)} \frac{|\varphi(y)|}{|x-y|^{d-1}}\,\mathrm{d}y + \int_{\mathbb{R}^d\setminus B_1(x)} \frac{|\varphi(y)|}{|x-y|^{d-1}}\,\mathrm{d}y\right),$$

leading to the estimate

$$\|\nabla\Delta^{-1}\varphi\|_{L^\infty} \leq C\left(\|\varphi\|_{L^\infty} + \|\varphi\|_{L^1}\right), \tag{2.38}$$

which further implies

$$\|\nabla\Delta^{-1}\varphi\|_{L^\infty} \leq C\left(\|\varphi\|_{H^s} + \|\varphi\|_{M_\vartheta}\right). \tag{2.39}$$

Next, since s is an integer, using $\|\varphi\|_{H^s} = \left(\sum_{|\zeta|\leq s} \|\partial^\zeta\varphi\|_2^2\right)^{1/2}$, we have for $\varphi, \psi \in H^s(\mathbb{R}^d)$

$$\|\psi\nabla\Delta^{-1}\varphi\|_{H^s} \leq \sum_{|\zeta|\leq s} \|\partial^\zeta\psi\nabla\Delta^{-1}\varphi\|_{L^2} + \sum_{\substack{|\zeta+\gamma|\leq s\\ |\gamma|=1}} \|\partial^\zeta\left(\psi\partial^\gamma\left(\nabla\Delta^{-1}\varphi\right)\right)\|_{L^2} = I_1 + I_2.$$

Using (2.39), we estimate I_1:

$$I_1 \leq \sum_{|\zeta|\leq s} \|\partial^\zeta\psi\|_{L^2}\|\nabla\Delta^{-1}\varphi\|_{L^\infty} \leq \sum_{|\zeta|\leq s} C\|\partial^\zeta\psi\|_{L^2}\left(\|\varphi\|_{H^s} + \|\varphi\|_{M_\vartheta}\right).$$

By the Sobolev embedding theorem and [49, Lemma A.1], we have $I_2 \leq C\|\psi\|_{H^s}\|\varphi\|_{H^s}$. Thus,

$$\|\psi\nabla\Delta^{-1}\varphi\|_{H^s} \leq C\left(\|\varphi\|_{H^s} + \|\varphi\|_{M_\vartheta}\right)\|\psi\|_{H^s}. \tag{2.40}$$

Now, to establish (2.36), we write

$$f\nabla\Delta^{-1}f - g\nabla\Delta^{-1}g = (f-g)\nabla\Delta^{-1}f + g\left(\nabla\Delta^{-1}f - \nabla\Delta^{-1}g\right), \tag{2.41}$$

and apply (2.40) to obtain

$$\begin{aligned}\|f\nabla\Delta^{-1}f - g\nabla\Delta^{-1}g\|_{H^s} &\leq C\left(\|f\|_{H^s} + \|f\|_{M_\vartheta}\right)\|f-g\|_{H^s} + C\left(\|f-g\|_{H^s} + \|f-g\|_{M_\vartheta}\right)\|g\|_{H^s} \\ &\leq C\left(\|f\|_{H^s} + \|f\|_{M_\vartheta} + \|g\|_{H^s}\right)\left(\|f-g\|_{M_\vartheta} + \|f-g\|_{H^s}\right).\end{aligned}$$

For (2.37), note that

$$\begin{aligned}\nabla\cdot\left(f\nabla\Delta^{-1}f - g\nabla\Delta^{-1}g\right) &= \left(\nabla f\cdot\nabla\Delta^{-1}f - \nabla g\cdot\nabla\Delta^{-1}g\right) + \left(f^2 - g^2\right) \\ &= \nabla(f-g)\cdot\nabla\Delta^{-1}f + \nabla g\cdot\nabla\Delta^{-1}(f-g) + (f-g)(f+g).\end{aligned} \tag{2.42}$$

Then, by summing (2.41) and (2.42), integrating it against $\left(1+|x|^\vartheta\right)$, and applying estimate (2.39) along with the Sobolev embedding theorem, we see that (2.37) does not exceed

$$\begin{aligned}&\|f-g\|_{M_\vartheta}\|\nabla\Delta^{-1}f\|_{L^\infty} + \|g\|_{M_\vartheta}\|\nabla\Delta^{-1}(f-g)\|_{L^\infty} + \|f-g\|_{M_\vartheta}\|f+g\|_{L^\infty} \\ &\leq C\left(\|f\|_{H^s} + \|f\|_{M_\vartheta} + \|g\|_{H^s} + \|g\|_{M_\vartheta}\right)\left(\|f-g\|_{M_\vartheta} + \|f-g\|_{H^s}\right).\end{aligned}$$

□

2.4 Properties of Solutions

In this section, we explore the key properties of solutions to systems (2.1) and (2.2), providing an analysis of the system's dynamics.

Therefore, we assume that $(\rho, \nabla c)$ is a mild solution to the parabolic-parabolic system (2.1) on $[0, T]$ with initial condition $(\rho_0, \nabla c_0)$. As noted in Remark 2.3, we specify ∇c instead of c because c itself does not appear in the first equation of the model. For the parabolic-elliptic system (2.2) with initial condition ρ_0, we assume that ρ is its mild solution on $[0, T]$. In this formulation, it is sufficient to specify only ρ, as the variable c does not explicitly appear in (2.2). This is because ∇c is purely a function of ρ, along with x and t (see Remark 2.3 for further details).

With these assumptions in place, we start by analyzing the behavior of the total number of cells and the concentration of the chemical signal over time. Building on this, we enhance Remark 2.1 by providing a rigorous justification for this observation. Additionally, we focus on the study of nonnegative solutions, demonstrating that for nonnegative initial conditions, sufficiently regular solutions to the system remain nonnegative over time. This analysis ensures that the model remains consistent with the biological phenomena it is intended to represent.

Proposition 2.24. *Assume $d \geq 2$, $\alpha \in (1,2]$ and $\beta \in (1,d]$. Let $(\rho, \nabla c)$ be a mild solution to system (2.1) on $[0,T]$ with initial condition $(\rho_0, \nabla c_0)$, where $\rho_0 \in L^1(\mathbb{R}^d)$ and $\rho\nabla c$, and $f(\rho, \nabla\rho)$ lie in $L^\infty\big([0,T], L^1(\mathbb{R}^d)\big)$. Then, as long as the solution is well-defined, the following conditions hold:*

(i) *$\rho(\cdot,t) \in L^1(\mathbb{R}^d)$ for every $t \in [0,T]$ and*

$$\int_{\mathbb{R}^d} \rho(x,t)\mathrm{d}x = \int_{\mathbb{R}^d} \rho_0(x)\mathrm{d}x + \int_0^t \int_{\mathbb{R}^d} f(\rho,\nabla\rho)\mathrm{d}x\mathrm{d}s, \tag{2.43}$$

or, equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \rho(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^d} f(\rho,\nabla\rho)\,\mathrm{d}x; \tag{2.44}$$

(ii) *if $c_0 \in L^1(\mathbb{R}^d)$, then, $c(\cdot,t) \in L^1(\mathbb{R}^d)$ for every $t \in [0,T]$ and*

$$\int_{\mathbb{R}^d} c(x,t)\mathrm{d}x = e^{-\frac{\gamma}{\tau}t}\int_{\mathbb{R}^d} c_0(x)\mathrm{d}x + \int_0^t \frac{1}{\tau}e^{\gamma\left(\frac{s-t}{\tau}\right)}\int_{\mathbb{R}^d} \rho_0(x) + \left(\int_0^s f(\rho,\nabla\rho)\mathrm{d}\varsigma\right)\mathrm{d}x\mathrm{d}s \tag{2.45}$$

Remark 2.25. *For $f = 0$, (2.43) describes the conservation of the total number of cells, i. e., $m(t) = m_0$, and (2.45) turns into*

$$\int_{\mathbb{R}^d} c\,(x,t)\,\mathrm{d}x = \left(\int_{\mathbb{R}^d} \rho_0(x)\mathrm{d}x\right)\left(\frac{1-e^{-\frac{\gamma}{\tau}t}}{\gamma}\right) + \left(\int_{\mathbb{R}^d} c_0(x)\mathrm{d}x\right)e^{-\frac{\gamma}{\tau}t} \tag{2.46}$$

and

$$\int_{\mathbb{R}^d} c\,(x,t)\,\mathrm{d}x \xrightarrow[t\to\infty]{} \frac{\int_{\mathbb{R}^d}\rho_0(x)\mathrm{d}x}{\gamma}, \tag{2.47}$$

where, for $\gamma = 0$, the result follows by letting $\gamma \to 0$.

Proof. ***(i)*** Applying Lemma 2.10 to (2.7), since $\alpha > 1$ and $\rho\nabla c$, $f \in L^\infty\big([0,T], L^1(\mathbb{R}^d)\big)$, we obtain

$$\begin{aligned}
\|\rho(\cdot,t)\|_{L^1} &\leq \|\rho_0\|_{L^1} + \int_0^t \left\|\nabla K^\alpha_{t-s} * [\rho(s)\nabla c(s)]\right\|_{L^1} + \left\|K^\alpha_{t-s} * f(\rho,\nabla\rho)\right\|_{L^1}\mathrm{d}s\\
&\leq \|\rho_0\|_{L^1} + C\int_0^t (t-s)^{-\frac{1}{\alpha}}\left\|\rho(s)\nabla c(s)\right\|_{L^1} + \left\|f(\rho,\nabla\rho)\right\|_{L^1}\mathrm{d}s\\
&\leq \|\rho_0\|_{L^1} + Ct\left(t^{-\frac{1}{\alpha}}\sup_{t\in[0,T]}\left\|(\rho\nabla c)(\cdot,t)\right\|_{L^1} + \sup_{t\in[0,T]}\left\|f(\rho,\nabla\rho)\right\|_{L^1}\right).
\end{aligned}$$

Thus $\rho(\cdot,t) \in L^1(\mathbb{R}^d)$ for every $t \in [0,T]$.

Next, integrating both sides of equation (2.7) and applying Lemma 2.9.4, since $\rho_0 \in L^1(\mathbb{R}^d)$ and $\rho\nabla c$ and $f(\rho,\nabla\rho) \in L^\infty\big([0,T], L^1(\mathbb{R}^d)\big)$, we obtain (2.43). Indeed, we establish

$$\begin{aligned}
\int_{\mathbb{R}^d} \rho(x,t)\,\mathrm{d}x &= \int_{\mathbb{R}^d} \rho_0(x)\,\mathrm{d}x + \int_0^t \left(\int_{\mathbb{R}^d} \nabla K^\alpha_{t-s} * (\chi\rho(s)\nabla c(s))\,\mathrm{d}x + \int_{\mathbb{R}^d} K^\alpha_{t-s} * f(\rho,\nabla\rho)\,\mathrm{d}x\right)\mathrm{d}s\\
&= \int_{\mathbb{R}^d} \rho_0(x)\,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^d} f(\rho,\nabla\rho)\,\mathrm{d}x\mathrm{d}s
\end{aligned}$$

for every $t \in [0,T]$. Thus, by applying the derivative in both sides of (2.43), (2.44) follows.

(ii) Note that, from equation (2.8), we can obtain the correspondent equation for c. Moreover, since from ***(i)*** we have $\rho(\cdot,t) \in L^1(\mathbb{R}^d)$ for every $t \in [0,T]$, we can apply Lemma 2.9.4. Then, proceeding the same way as before, we establish

$$\begin{aligned}\int_{\mathbb{R}^d} c(x,t)\,\mathrm{d}x &= e^{-\frac{\gamma}{\tau}t}\int_{\mathbb{R}^d} K^{\beta}_{\frac{t}{\tau}} * c_0(x)\,\mathrm{d}x + \int_{\mathbb{R}^d}\int_0^t \frac{1}{\tau}e^{\gamma\left(\frac{s-t}{\tau}\right)}K^{\beta}_{\frac{t-s}{\tau}} * \rho(x,s)\,\mathrm{d}s\,\mathrm{d}x \\ &= e^{-\frac{\gamma}{\tau}t}\int_{\mathbb{R}^d} c_0(x)\,\mathrm{d}x + \int_0^t \frac{1}{\tau}e^{\gamma\left(\frac{s-t}{\tau}\right)}\int_{\mathbb{R}^d}\rho(x,s)\,\mathrm{d}x\,\mathrm{d}s \\ &= e^{-\frac{\gamma}{\tau}t}\int_{\mathbb{R}^d} c_0(x)\,\mathrm{d}x + \int_0^t \frac{1}{\tau}e^{\gamma\left(\frac{s-t}{\tau}\right)}\left(\int_{\mathbb{R}^d}\rho_0(x)\,\mathrm{d}x + \int_0^s\int_{\mathbb{R}^d} f(\rho,\nabla\rho)\,\mathrm{d}x\mathrm{d}\varsigma\right)\mathrm{d}s,\end{aligned}$$

and

$$\begin{aligned}\|c(\cdot,t)\|_{L^1} &\le e^{-\frac{\gamma}{\tau}t}\left\|K^{\beta}_{\frac{t}{\tau}} * c_0\right\|_{L^1} + \int_0^t \frac{1}{\tau}e^{\frac{\gamma}{\tau}(s-t)}\left\|K^{\beta}_{\frac{t-s}{\tau}} * \rho(s)\right\|_{L^1}\mathrm{d}s \\ &\le Ce^{-\frac{\gamma}{\tau}t}\|c_0\|_{L^1} + C\int_0^t \frac{1}{\tau}e^{\frac{\gamma}{\tau}(s-t)}\|\rho(s)\|_{L^1}\mathrm{d}s \\ &\le Ce^{-\frac{\gamma}{\tau}t}\|c_0\|_{L^1} + C\left(\frac{1-e^{-\frac{\gamma}{\tau}t}}{\gamma}\right)\sup_{t\in[0,T]}\|\rho(\cdot,t)\|_{L^1}.\end{aligned}$$

□

Proposition 2.26. *Assume $d \ge 2$, $\alpha \in (1,2]$ and $\beta \in (1,d]$. Let ρ be a mild solution to system (2.2) on $[0,T]$ with initial condition ρ_0, where $\rho_0 \in L^1(\mathbb{R}^d)$ and $\rho\nabla c$, and $f(\rho,\nabla\rho)$ lie in $L^\infty([0,T], L^1(\mathbb{R}^d))$. Then, as long as the solution is well-defined, $\rho(\cdot,t) \in L^1(\mathbb{R}^d)$ for all $t \in [0,T]$ and (2.43) and (2.44) still hold. Moreover, if $c_0 \in L^1(\mathbb{R}^d)$ and $\gamma > 0$, then $c(\cdot,t) \in L^1(\mathbb{R}^d)$ for all $t \in [0,T]$ and*

$$\int_{\mathbb{R}^d} c(x,t)\mathrm{d}x = \frac{\int_{\mathbb{R}^d}\rho(x,t)\mathrm{d}x}{\gamma}.$$

Proof. The proof here is analogous to the previous one. Thus, we only show that for the case $\gamma > 0$ the following hold

$$\int_{\mathbb{R}^d} c(x,t)\mathrm{d}x = \int_{\mathbb{R}^d}\int_0^\infty e^{-\gamma s}K^{\beta}_s * \rho(x,t)\mathrm{d}s\mathrm{d}x = \frac{\int_{\mathbb{R}^d}\rho(x,t)\mathrm{d}x}{\gamma},$$

and

$$\|c(x,t)\|_{L^1} \le \int_0^\infty e^{-\gamma s}\|K^{\beta}_s * \rho(\cdot,t)\|_{L^1}\mathrm{d}s = \frac{\|\rho(\cdot,t)\|_{L^1}}{\gamma}.$$

□

As mentioned before, systems (2.1) and (2.2) describe the density of cells ρ and the concentration of the chemical signal c. Therefore, given nonnegative initial densities, $\rho_0 \ge 0$ and $c_0 \ge 0$, it is biologically relevant to ensure that the solutions remain nonnegative over time.

Thus, we prove that, for a nonnegative initial condition, a sufficiently regular solution remains nonnegative. For that, consider the following lemma based on Definition A.6.

Lemma 2.27. *Let $w \in C^1\left([0,T], W^{1,p}(\mathbb{R}^d)\right)$ and $p \geq 2$. Then we obtain that*

$$\int_{\mathbb{R}^d} w_-|w_-|^{p-2}\partial_t w \mathrm{d}x = \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|w_-(\cdot,t)\|_{L^p}^p. \tag{2.48}$$

Moreover, for $v \in W^{2,\infty}(\mathbb{R}^d)$, we obtain

$$-\int_{\mathbb{R}^d} w_-|w_-|^{p-2}\nabla\cdot(w\nabla v)\,\mathrm{d}x \leq \frac{p-1}{p}\|\Delta v\|_{L^\infty}\|w_-(\cdot,t)\|_{L^p}^p, \tag{2.49}$$

for every $t \in [0,T]$.

Proof. For $w \in C^1\left((0,T), W^{1,p}(\mathbb{R}^d)\right)$ and $p \geq 2$, we have, in the weak sense,

$$w_-|w_-|^{p-2}\partial_t w = w_-|w_-|^{p-2}\partial_t w_- = \frac{1}{p}\partial_t|w_-|^p, \tag{2.50}$$

and

$$w\nabla\left(w_-|w_-|^{p-2}\right) = (p-1)\left(|w_-|^{p-2}\nabla w_-\right)w = (p-1)\,w_-|w_-|^{p-2}\nabla w_- = \frac{p-1}{p}\nabla|w_-|^p. \tag{2.51}$$

Then, (2.48) follows from (2.50), and assuming $v \in W^{2,\infty}(\mathbb{R}^d)$, we obtain from (2.51) that

$$\begin{aligned}-\int_{\mathbb{R}^d} w_-|w_-|^{p-2}\nabla\cdot(w\nabla v)\,\mathrm{d}x &= -\frac{p-1}{p}\int_{\mathbb{R}^d}|w_-|^p\,\nabla\cdot\nabla v\mathrm{d}x\\ &\leq \frac{p-1}{p}\|\Delta v\|_{L^\infty}\|w_-(\cdot,t)\|_{L^p}^p.\end{aligned}$$

□

Proposition 2.28. *Assume $d \geq 2$, $\alpha \in (1,2]$ and $\beta \in (1,d]$. Let $(\rho,\nabla c)$ be a mild solution to system (2.1) on $[0,T]$ with initial condition $(\rho_0,\nabla c_0)$. Moreover, let ρ_0 and c_0 be nonnegative functions, and*

- *$\rho \in C\big((0,T), L^p(\mathbb{R}^d)\big)$ and $\rho(\cdot,t) \in W^{2,p}(\mathbb{R}^d)$ for all $t \in [0,T]$, where $2 \leq p < \infty$;*
- *$\operatorname{ess\,sup}_{t\in[0,T]}\|\Delta c\|_{L^\infty} < \infty$;*
- *$\||\rho_-|^{p-2}\rho_- f\|_{L^p} \leq C\|\rho_-(\cdot,t)\|_{L^p}$, for all $t \in [0,T]$, where $2 \leq p < \infty$ and $C \geq 0$ is a constant;*
- *$\nabla c(\cdot,t) \in W^{2\lceil\beta/2\rceil-1,r}(\mathbb{R}^d)$ for all $t \in [0,T]$, where $1 < r < \infty$.*

Then, ρ and c remain nonnegative as long as the solution is well-defined.

Proof. Multiplying the first equation of (2.1) by $|\rho_-|^{p-2}\rho_-$ and integrating over $\mathbb{R}^d$, we obtain from (2.48), (2.49), and Lemma A.8

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho_-(\cdot,t)\|_{L^p}^p \leq \frac{p-1}{p}\|\Delta c(\cdot,t)\|_{L^\infty}\|\rho_-(\cdot,t)\|_{L^p}^p + C\|\rho_-(\cdot,t)\|_{L^p}^p.$$

Let $\eta(t) = \|\rho_-(\cdot,t)\|_{L^p}^p$ and $\phi(t) = (p-1)\,\|\Delta c(\cdot,t)\|_{L^\infty} + C$, where $C \geq 0$ is a constant. The previous inequality becomes

$$\eta'(t) \leq \phi(t)\eta(t),$$

where $\phi(t)$ is nonnegative, summable functions on $[0,T]$, since by assumption $\|\Delta c(\cdot,t)\|_{L^\infty} < \infty$ for all $t \in [0,T]$. Then, by Gronwall's inequality, we obtain

$$\eta(t) \leq e^{\int_0^t \phi(s)ds}\eta(0)$$

for all $0 \leq t \leq T$. Since $\eta(0) = 0$, as $\rho_0 \geq 0$, we conclude that $\eta \equiv 0$ on $[0,T]$. Thus, $\rho_- = 0$ and, consequently, $\rho \geq 0$.

Next, we multiply the second equation of (2.1) by $|c_-|^{r-2}c_-$ and integrate over $\mathbb{R}^d$. From (2.48) and Lemma A.8, we obtain

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\|c_-(\cdot,t)\|_{L^r}^r \leq \int_{\mathbb{R}^d} |c_-(x,t)|^{r-2}c_-(x,t)\rho(x,t)\mathrm{d}x - \gamma\,\|c_-(\cdot,t)\|_{L^r}^r \leq 0,$$

since $\rho \geq 0$ implies $\int_{\mathbb{R}^d} |c_-(x,t)|^{r-2}c_-(x,t)\rho(x,t)\mathrm{d}x \leq 0$. Now, $\|c_-(\cdot,0)\|_{L^r} = 0$, since $c_0 \geq 0$. Therefore, we conclude that $c_- = 0$ and, consequently, $c \geq 0$. □

Proposition 2.29. *Assume $d \geq 2$, $\alpha \in (1,2]$ and $\beta \in (1,d]$. Let ρ be a mild solution to system (2.2) on $[0,T]$ with initial condition ρ_0. Moreover, let ρ_0 be nonnegative function, and*

- *$\rho \in C\big((0,T), L^p(\mathbb{R}^d)\big)$ and $\rho(\cdot,t) \in W^{2,p}(\mathbb{R}^d)$ for all $t \in [0,T]$, where $2 \leq p < \infty$;*
- *$\operatorname{ess\,sup}_{t\in[0,T]} \|\Delta c\|_{L^\infty} < \infty$;*
- *$\||\rho_-|^{p-2}\rho_- f\|_{L^p} \leq C\|\rho_-(\cdot,t)\|_{L^p}$, for all $t \in [0,T]$, where $2 \leq p < \infty$ and $C \geq 0$ is a constant.*

Then, ρ remains nonnegative as long as the solution is well-defined.

Proof. The result can be obtained by following the same approach as in the first part of the previous proposition. □

Next, we establish conditions on the solution ρ that ensure $\operatorname{ess\,sup}_{t\in[0,T]} \|\Delta c\|_{L^\infty} < \infty$, which is crucial for proving that the solutions remain nonnegative over time.

Proposition 2.30. *Assume $d \geq 2$, $\alpha \in (1,2]$ and $\beta \in (1,d]$. Let $(\rho, \nabla c)$ be a mild solution to system (2.1) on $[0,T]$ with initial condition $(\rho_0, \nabla c_0)$. Suppose that $\nabla c_0 \in L^q(\mathbb{R}^d)$ and $\rho \in L^\infty\big([0,T], W^{2,p}(\mathbb{R}^d)\big)$ for $1 \leq p \leq q \leq \infty$, where $p > d/\beta$. Then, as long as the solution is well-defined, $\Delta c(\cdot,t) \in L^\infty(\mathbb{R}^d)$ for all $t \in (0,T]$.*

Proof. From (2.8), we establish for $0 < t \le T$ that

$$
\begin{aligned}
\|\Delta c(\cdot,t)\|_{L^\infty} &\le e^{-\frac{\gamma}{\tau}t}\left\|\nabla K^\beta_{\frac{t}{\tau}} * \nabla c_0\right\|_{L^\infty} + \int_0^t \frac{1}{\tau} e^{\frac{\gamma}{\tau}(s-t)}\left\|K^\beta_{\frac{t-s}{\tau}} * \Delta\rho(\cdot,s)\right\|_{L^\infty} \mathrm{d}s \\
&\le C e^{-\frac{\gamma}{\tau}t} t^{-\frac{1}{\beta}\left(\frac{d}{q}+1\right)}\|\nabla c_0\|_{L^q} + C\int_0^t \frac{1}{\tau} e^{\frac{\gamma}{\tau}(s-t)}\left(\frac{s-t}{\tau}\right)^{-\frac{d}{\beta}\frac{1}{p}}\|\Delta\rho(\cdot,s)\|_{L^p}\mathrm{d}s \\
&\le C e^{-\frac{\gamma}{\tau}t} t^{-\frac{1}{\beta}\left(\frac{d}{q}+1\right)}\|\nabla c_0\|_{L^q} + C\left(\int_0^t \frac{1}{\tau} e^{\frac{\gamma}{\tau}(s-t)}\left(\frac{s-t}{\tau}\right)^{-\frac{d}{\beta}\frac{1}{p}}\mathrm{d}s\right)\sup_{t\in[0,T]}\|\rho(\cdot,t)\|_{W^{2,p}}.
\end{aligned}
$$

Then, since $\frac{d}{\beta}\frac{1}{p} < 1$, we estimate the inner integration as, for $\gamma \neq 0$,

$$
\begin{aligned}
\int_0^t \frac{1}{\tau} e^{\frac{\gamma}{\tau}(s-t)}\left(\frac{t-s}{\tau}\right)^{-\frac{d}{\beta}\frac{1}{p}}\mathrm{d}s &= \gamma^{-\left(1-\frac{d}{\beta}\frac{1}{p}\right)}\int_0^{\gamma t/\tau} e^{-u} u^{\left(1-\frac{d}{\beta}\frac{1}{p}\right)-1}\mathrm{d}u \\
&\le C\Gamma\left(1-\frac{d}{\beta}\frac{1}{p}\right) < \infty,
\end{aligned}
$$

and, for $\gamma = 0$.

$$
\int_0^t \frac{1}{\tau}(t-s)^{-\frac{d}{\beta}\frac{1}{p}}\mathrm{d}s \le C t^{1-\frac{d}{\beta}\frac{1}{p}}.
$$

Therefore, since $\rho \in L^\infty\big([0,T], W^{2,p}(\mathbb{R}^d)\big)$, we conclude. □

Proposition 2.31. *Assume $d \ge 2$, $\alpha \in (1,2]$ and $\beta \in (1,d]$. Let ρ be a mild solution to system (2.2) on $[0,T]$ with initial condition ρ_0. Assume $\rho \in L^\infty\big([0,T], W^{2,p}(\mathbb{R}^d)\big)$. Then, as long as the solution is well-defined, $\Delta c(\cdot,t) \in L^\infty(\mathbb{R}^d)$ a.e. $t \in (0,T]$.*

Proof. To prove this assertion we can apply similar steps as in the previous proposition. Therefore, we omit the proof here. □

Remark 2.32. *For (2.2) with $\gamma = 0$, if the solution is such that $\rho \in L^\infty\big([0,T], L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\big)$, we can apply the Hardy-Littlewood-Sobolev inequality as done in Remark 1.10 to prove Proposition 2.31.*

Chapter 3

Generalized Keller-Segel model

3.1 Introduction

In this chapter, we explore the existence, uniqueness, and analytical properties of nonnegative solutions for the system (2.1) in the particular case of $f(\rho, \nabla\rho) = 0$ and $\chi = 1$. That is, our focus is on the following generalized Keller-Segel model with nonlocal diffusion terms in dimension $d \geq 2$

$$\begin{cases} \partial_t \rho = -\Lambda^\alpha \rho - \nabla\cdot(\rho\nabla c) & x \in \mathbb{R}^d, \quad t > 0, \quad \alpha \in (1,2] \\ \tau\partial_t c = -\Lambda^\beta c + \rho - \gamma c & x \in \mathbb{R}^d, \quad t > 0, \quad \beta \in (1,d] \\ \rho(t=0) = \rho_0, \quad c(t=0) = c_0 & x \in \mathbb{R}^d. \end{cases} \tag{3.1}$$

We consider system (3.1) in its integral form using the Duhamel principle, which involves equation (2.7) with $f(\rho, \nabla\rho) = 0$ and $\chi = 1$, and equation (2.8). We will establish both local and global well-posedness of mild solution (see Definition 2.5) by converting system (3.1) into a fixed point problem in a suitable function space $\mathbf{X}$ for the mapping $\mathcal{T}(\rho) \equiv u_1 + \mathcal{A}(\rho,\rho) + \mathcal{L}(\rho)$, where u_1, the bilinear operator $\mathcal{A} : \mathbf{X}\times\mathbf{X} \to \mathbf{X}$, and the linear operator $\mathcal{L} : \mathbf{X} \to \mathbf{X}$ are defined in (2.16), (2.17) and (2.18), respectively.

For the parabolic-elliptic case with $\gamma = 0$ (system (3.1) with $\tau = 0$), Biler et al. [11] demonstrated the existence of a unique local-in-time mild solution $\rho \in C\big([0,T], L^p(\mathbb{R}^d)\big)$ for arbitrary initial condition $\rho_0 \in L^p(\mathbb{R}^d)$, where $T = T\big(\|\rho_0\|_{L^p}\big)$ and $\max\left\{\frac{d}{\alpha+\beta-2}, \frac{2d}{d-\beta-1}\right\} < p \leq d$. Additionally, they proved the existence of a unique global-in-time mild solution $\rho \in C\big([0,\infty), L^p(\mathbb{R}^d)\big)$ for any initial condition $\rho_0 \in L^{d/(\alpha+\beta-2)}(\mathbb{R}^d)$ provided that $\|\rho_0\|_{L^{d/(\alpha+\beta-2)}}$ is sufficiently small. Furthermore, in both local and global cases, Biler et al. [11] proved that the solution is nonnegative if it is initially nonnegative $\rho_0 \geq 0$. Moreover, if $\rho_0 \in L^1(\mathbb{R}^d)$, then the corresponding solution conserves mass. Their proof followed the standard contraction mapping principle discussed in Section 2.2.1. Here, we extend this approach to address the parabolic-parabolic version of the system. Note that the results presented by Biler et al. [11] also hold for $\gamma > 0$, as γ acts as a damping constant.

Biler et al. [11] point out that the assumption $\alpha > 1$ allows local control of the nonlinearity in the parabolic-elliptic system by the linear term. We will demonstrate that this control extends to the parabolic-parabolic system as well. Let us point out that the main goal of Biler et al. [11] was to prove the finite time blow-up of solutions to system (2.2) with $\gamma = 0$. We present and discuss these results in the next chapter (Section 4.3.2.2).

3.2 Local existence of solutions

To prove the local existence of solutions to the parabolic-parabolic system (3.1) consider the following assumptions:

(A1) the parameters of the system are such that $d \geq 2$, $\alpha \in (1, 2]$ and $\beta \in (1, d]$;

(A2) parameters p and r satisfy

- $\max\left\{\frac{2d}{d+\beta-1}, \frac{d}{\alpha+\beta-2}\right\} < p \leq \frac{d}{\beta-1}$ and $\max\left\{p, \frac{p}{p-1}, \frac{d}{\alpha-1}\right\} < r < \frac{pd}{d-p(\beta-1)}$, or
- $p > \frac{d}{\beta-1}$ and $r > \max\left\{p, \frac{p}{p-1}, \frac{d}{\alpha-1}\right\}$,

where, in both cases, the equality $r = \max\left\{p, \frac{p}{p-1}\right\}$ is possible if $\max\left\{p, \frac{p}{p-1}\right\} > \frac{d}{\alpha-1}$;

(A3) the parameter $\wp$ satisfies

$$\wp \in \left[\frac{d}{\alpha - 1}, r\right] \qquad \text{if} \quad \alpha \leq \beta,$$

$$\wp = r \qquad \text{if} \quad \alpha > \beta;$$

(A4) the Banach space $\mathbf{X}$ is defined as $\mathbf{X} = C\big([0, T], L^p(\mathbb{R}^d)\big)$ with the usual norm $\|u\|_{\mathbf{X}} = \sup_{t\in[0,T]} \|u(t)\|_{L^p}$.

Next, we first establish some necessary estimates.

Lemma 3.1. *Assume that (A1), (A2) and (A4) are in force, and $T > 0$. Then,*

$$\|u_1\|_{\mathbf{X}} \leq C_1 \|\rho_0\|_{\mathbf{X}}, \tag{3.2}$$

where $u_1(\cdot,t) = K_t^\alpha * \rho_0$ *for* $t > 0$, *with* K_t^α *given by* (1.17), *and* C_1 *is a constant depending on parameters* α, p, *and* d.

Proof. For $0 < t < T$, using u_1 into (2.20) with $q = p$ and the definition of $\mathbf{X}$, the result follows as $\|u_1\|_{L^p} = \|K_t^\alpha * \rho_0\|_{L^p} \leq C_1\|\rho_0\|_{L^p}$. □

Lemma 3.2. *Assume that (A1), (A2) and (A4) are in force, and* $T > 0$. *Then,*

$$\|\mathcal{A}(u,v)\|_{\mathbf{X}} \leq C_2 T^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)} \|u\|_{\mathbf{X}} \|v\|_{\mathbf{X}} \quad \textit{for } \gamma \neq 0, \tag{3.3}$$

and

$$\|\mathcal{A}(u,v)\|_{\mathbf{X}} \leq C_2 T^{2-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}} \|u\|_{\mathbf{X}} \|v\|_{\mathbf{X}} \quad \textit{for } \gamma = 0, \tag{3.4}$$

where $\mathcal{A}$ *is given by* (2.17) *with* $f(\rho,\nabla\rho) = 0$ *and* $\chi = 1$, *and* C_2 *is a constant depending on parameters* α, β, p, r, *and* d.

Proof. Let $1 \leq q \leq p \leq r \leq \infty$ be such that $1/q = 1/p + 1/r$. For $0 < t < T$, applying estimate (2.21) and the definition of $\mathbf{X}$ into (2.17), we have

$$\begin{aligned}
\|\mathcal{A}(u,v)\|_{L^p} &\leq \int_0^t \left\| \nabla K_{t-s}^\alpha * \left[u(s) \int_0^s \frac{1}{\tau} e^{\gamma\left(\frac{w-s}{\tau}\right)} \nabla K_{\frac{s-w}{\tau}}^\beta * v(w) \mathrm{d}w \right] \right\|_{L^p} \mathrm{d}s \\
&\leq C \int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{\alpha}} \left\| u(s) \int_0^s \frac{1}{\tau} e^{\gamma\left(\frac{w-s}{\tau}\right)} \nabla K_{\frac{s-w}{\tau}}^\beta * v(w) \mathrm{d}w \right\|_{L^q} \mathrm{d}s \\
&\leq C \int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{\alpha}} \|u(s)\|_{L^p} \left\| \int_0^s \frac{1}{\tau} e^{\gamma\left(\frac{w-s}{\tau}\right)} \nabla K_{\frac{s-w}{\tau}}^\beta * v(w) \mathrm{d}w \right\|_{L^r} \mathrm{d}s \\
&\leq C \left(\int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{\alpha}} \left(\int_0^s \frac{1}{\tau} e^{\frac{\gamma}{\tau}(w-s)} \left(\frac{s-w}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}} \mathrm{d}w \right) \mathrm{d}s \right) \|u\|_{\mathbf{X}} \|v\|_{\mathbf{X}}.
\end{aligned}$$

Then, for $\gamma \neq 0$, provided that $\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right) + \frac{1}{\beta} < 1$, which, from Lemma B.3, is fulfilled, the inner integral above is bounded, since

$$\begin{aligned}
\int_0^s \frac{1}{\tau} e^{\frac{\gamma}{\tau}(w-s)} \left(\frac{s-w}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}} \mathrm{d}w &= \gamma^{-\left(1-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}\right)} \int_0^{\gamma s/\tau} e^{-u} u^{\left(1-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}\right)-1} \mathrm{d}u \\
&\leq C\Gamma\left(1 - \frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right) - \frac{1}{\beta}\right) < \infty.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{\alpha}} \left(\int_0^s \frac{1}{\tau} e^{\frac{\gamma}{\tau}(w-s)} \left(\frac{s-w}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}} \mathrm{d}w \right) \mathrm{d}s &\leq C \int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{\alpha}} \mathrm{d}s \\
&\leq C T^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)},
\end{aligned}$$

since $\frac{1}{\alpha}\left(\frac{d}{r}+1\right) < 1$ again from Lemma B.3. Therefore, estimate (3.3) holds.

Now, considering the case $\gamma = 0$, we estimate the inner integration as

$$\int_0^s \frac{1}{\tau} (s-w)^{-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}} \mathrm{d}w \leq C s^{1-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}},$$

since $\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)+\frac{1}{\beta}<1$. Then, applying (2.33), we get

$$\begin{aligned}\int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{\alpha}}\left(\int_0^s \frac{1}{\tau}\left(\frac{s-w}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}} \mathrm{d}w\right)\mathrm{d}s &\le C\int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{\alpha}} s^{1-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}}\mathrm{d}s\\ &= C\int_0^t (t-s)^{-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)} s^{1-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}}\mathrm{d}s\\ &\le Ct^{-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}+2},\end{aligned}$$

since $\frac{1}{\alpha}\left(\frac{d}{r}+1\right)<1$ and $\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)+\frac{1}{\beta}<2$. Hence, we obtain estimate (3.4). □

Lemma 3.3. *Assume that (A1), (A2), (A3) and (A4) are in force, and $T>0$. Then,*

$$\|\mathcal{L}(u)\|_{\mathrm{X}} \le C_3 T^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)-\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)}\|\nabla c_0\|_{L^{\wp}}\|u\|_{\mathrm{X}}, \tag{3.5}$$

where $\mathcal{L}$ is given by (2.18) and C_3 is a constant depending on parameters α, β, p, r, $\wp$, and d.

Proof. Let $1\le q\le p\le r\le\infty$ be such that $1/q = 1/p + 1/r$. For $0<t<T$, applying estimates (2.20) and (2.21), and the definition of $\mathbf{X}$ into (2.18) as $1\le\wp\le r$, we have

$$\begin{aligned}\|\mathcal{L}(u)\|_{L^p} &\le \int_0^t \left\|\nabla K^{\alpha}_{t-s} * \left[u(s)e^{-\frac{\gamma}{\tau}s}K^{\beta}_{\frac{s}{\tau}} * \nabla c_0(x)\right]\right\|_{L^p}\mathrm{d}s\\ &\le C\int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{\alpha}}\left\|u(s)e^{-\frac{\gamma}{\tau}s}K^{\beta}_{\frac{s}{\tau}} * \nabla c_0(x)\right\|_{L^q}\mathrm{d}s\\ &\le C\int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{\alpha}}\|u(s)\|_{L^p}\left\|e^{-\frac{\gamma}{\tau}s}K^{\beta}_{\frac{s}{\tau}} * \nabla c_0(x)\right\|_{L^r}\mathrm{d}s\\ &\le C\int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{\alpha}}\|u(s)\|_{L^p} e^{-\frac{\gamma}{\tau}s}\left(\frac{s}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)}\|\nabla c_0\|_{L^{\wp}}\mathrm{d}s\\ &\le C\left(\int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{\alpha}} e^{-\frac{\gamma}{\tau}s}\left(\frac{s}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)}\mathrm{d}s\right)\|\nabla c_0\|_{L^{\wp}}\|u\|_{\mathrm{X}}.\end{aligned}$$

Now, applying (2.33) to estimate the above integration, we obtain

$$\begin{aligned}\int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{\alpha}} e^{-\frac{\gamma}{\tau}s}\left(\frac{s}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)}\mathrm{d}s &\le C\int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{\alpha}} s^{-\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)}\mathrm{d}s\\ &\le Ct^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)-\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)}\end{aligned}$$

for $\frac{1}{\alpha}\left(\frac{d}{r}+1\right)<1$ and $\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)<1$, which, from Lemma B.6, are fulfilled. Moreover,

$$\int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{\alpha}} e^{-\frac{\gamma}{\tau}s}\left(\frac{s}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)}\mathrm{d}s \le CT^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)-\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)}$$

provided that $\frac{1}{\alpha}\left(\frac{d}{r}+1\right)+\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)\le 1$, which, again from Lemma B.6, is satisfied. Therefore, (3.5) holds. □

Theorem 3.4 (Local existence of solutions). *Assume that (A1), (A2) and (A4) are in force.*

Case (a): *Then, for every initial condition* $\rho_0 \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ *and* $\nabla c_0 \in L^r(\mathbb{R}^d)$*, there exist* $T = T(\|\rho_0\|_{L^p}, \|\nabla c_0\|_{L^r})$ *and a unique local mild solution* (ρ, c) *to system* (3.1) *in* $[0,T]$*, such that* $\rho \in C\big([0,T], L^p(\mathbb{R}^d)\big)$, $\nabla c \in C\big([0,T], L^r(\mathbb{R}^d)\big)$*, and*

$$\sup_{t\in[0,T]} t^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)} \|\nabla c(\cdot,t)\|_{L^r} < C\left(T, \|\rho_0\|_{L^p}, \|\nabla c_0\|_{L^r}\right). \tag{3.6}$$

Moreover,

(i) $\rho \in L^\infty\left((0,T); L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)\right)$*, with*

$$\|\rho(\cdot,t)\|_{L^p} \le C\|\rho_0\|_{L^p}, \tag{3.7}$$

and the total mass is conserved;

(ii) $\rho \in L^1\left((0,T); L^q(\mathbb{R}^d)\right)$ *for all* $1 \le q \le p$*;*

(iii) *if* $c_0 \in W^{1,r}(\mathbb{R}^d)$*, then* $c \in C\big([0,T], W^{1,r}(\mathbb{R}^d)\big)$*, and*

$$\sup_{t\in[0,T]} t^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)} \|c(\cdot,t)\|_{L^r} < C\left(T,\ \|\rho_0\|_{L^p},\ \|c_0\|_{L^q}\right); \tag{3.8}$$

(iv) *if* $\int_{\mathbb{R}^d} c_0(x)\mathrm{d}x < \infty$*, the chemical concentration, c, grows as* (2.46) *and* (2.47)*, that is,*

$$\int_{\mathbb{R}^d} c(x,t)\,\mathrm{d}x = \left(\int_{\mathbb{R}^d} \rho_0(x)\mathrm{d}x\right)\left(\frac{1-e^{-\frac{\gamma}{\tau}t}}{\gamma}\right) + \left(\int_{\mathbb{R}^d} c_0(x)\mathrm{d}x\right)e^{-\frac{\gamma}{\tau}t}$$

and

$$\int_{\mathbb{R}^d} c(x,t)\,\mathrm{d}x \xrightarrow[t\to\infty]{} \frac{\int_{\mathbb{R}^d}\rho_0(x)\mathrm{d}x}{\gamma},$$

which include $\gamma = 0$ *by taking* $\gamma \to 0$.

Assume $\beta \in (1,2]$*. We also obtain*

(v) $\rho \in L^1\left((0,T); L^q(\mathbb{R}^d)\right)$ *for all* $q \ge 1$*, and* $\rho \in L^\infty_{\mathrm{loc}}\left((0,T]; L^q(\mathbb{R}^d)\right)$ *for all* $q > p$.

(vi) $\nabla c \in C\big([0,T], L^q(\mathbb{R}^d)\big)$ *for* $q > \frac{d}{\alpha-1}$*, and*

$$\sup_{t\in[0,T]} t^{1-\frac{1}{\alpha}\left(\frac{d}{q}+1\right)} \|\nabla c(\cdot,t)\|_{L^q} < C\left(T, \|\rho_0\|_{L^p}, \|\nabla c_0\|_{L^r}\right). \tag{3.9}$$

Case (b): *Let* $\alpha \le \beta$ *and the initial condition be such that* $\rho_0 \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ *and* $\nabla c_0 \in L^{\wp}(\mathbb{R}^d)$*, where*

$$\wp \in \left[\frac{d}{\alpha-1}, r\right), \tag{3.10}$$

and

$$\frac{1}{\alpha}\left(\frac{d}{r}+1\right) + \frac{d}{\beta}\left(\frac{1}{\wp} - \frac{1}{r}\right) = 1. \tag{3.11}$$

Then, there exist $\epsilon > 0$ and $T = T(\|\rho_0\|_{L^p}, \|\nabla c_0\|_{L^\wp}, \epsilon)$ such that, if $\|\nabla c_0\|_{L^\wp} < \epsilon$, there exists a unique local mild solution $(\rho, \nabla c)$ to system (3.1), with $\rho \in C\big([0,T], L^p(\mathbb{R}^d)\big)$ and $\nabla c \in C\big([0,T], L^r(\mathbb{R}^d)\big)$. Moreover,

$$\sup_{t\in[0,T]} t^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)} \|\nabla c(\cdot,t)\|_{L^r} < C\left(T,\ \|\rho_0\|_{L^p},\ \|\nabla c_0\|_{L^\wp}\right), \tag{3.12}$$

and (i), (ii), (iii), (iv), and (v) are still satisfied. Additionally, (vi) is also satisfied by replacing $\|\nabla c_0\|_{L^r}$ with $\|\nabla c_0\|_{L^\wp}$ in (3.9). On the other hand, if $\wp$ satisfies

$$\frac{1}{\alpha}\left(\frac{d}{r}+1\right) + \frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right) < 1, \tag{3.13}$$

the smallness condition on $\|\nabla c_0\|_{L^\wp}$ is not necessary, and $T = T(\|\rho_0\|_{L^p}, \|\nabla c_0\|_{L^\wp})$.

Remark 3.5. *Assumption (A3) was not explicitly mentioned in Theorem 3.4, as it is incorporated into the theorem through its cases. Note that, in case (a), $\wp$ is equal to r, since no restrictions on the relationship between α and β are imposed. In case (b), where $\alpha \le \beta$, $\wp$ is given by (3.10), as $\wp = r$ is already addressed in case (a). Furthermore, the constraints imposed on $\wp$ that divided case (b) into two parts, specified in (3.11) and (3.13), are satisfied by $\wp$ when (A3) is in force (Lemma B.6).*

Remark 3.6. *For $d \ge 3$ and $\alpha = \beta = 2$, Theorem 3.4 recovers and extends the result established by Corrias et al. [30]. Specifically, in Theorem 2.1, Corrias et al. [30] established the existence of a local solution for the classical parabolic-parabolic Keller-Segel model with initial data $\rho_0 \in L^p(\mathbb{R}^d)$ and $\nabla c_0 \in L^\wp(\mathbb{R}^d)$ such that $p > d/2$, and $\wp = d$. In contrast, in Theorem 3.4 $\wp$ can vary within the range $[d, r]$, where r satisfies $d < r < \frac{pq}{d-p}$ for $p \in \left(\frac{d}{2}, d\right)$, or $r > p$ for $p > d$.*

Proof. We establish the existence and uniqueness of local solution to system (3.1) by applying the fixed point theorem through Corollary 2.7. Specifically, we construct the unique mild solution, with u_1, $\mathcal{A}$, and $\mathcal{L}$ as described in Section 2.2.1, in the Banach space $\mathbf{X} = C\big([0,T], L^p(\mathbb{R}^d)\big)$ equipped with the usual norm, and show that the hypotheses of Corollary 2.7 are satisfied.

From Lemmas 3.2 and 3.3, we can set the constants $C_{\mathcal{A}} = C_2 T^{\vartheta_2}$ and $C_{\mathcal{L}} = C_3 \|\nabla c_0\|_{L^q} T^{\vartheta_1}$, respectively, where $q = r$ for case (a) and $q = \wp$ for case (b), C_2 and C_3 are constants depending on α, β, q, r and p, $T > 0$, and $\vartheta_2 \ge \vartheta_1 \ge 0$. Additionally, if condition (3.11) is satisfied, then $\vartheta_2 > 0$ and $\vartheta_1 = 0$; otherwise, $\vartheta_2 \ge \vartheta_1 > 0$.

Next, let $\delta > 0$ be such that $\delta = C_1\|\rho_0\|_{\mathrm{X}}$. From Lemma 3.1, this implies that $\|u_1\|_{\mathrm{X}} \le C_1\|\rho_0\|_{\mathrm{X}} = \delta$. Thus, to fall into the hypotheses of Corollary 2.7, $F(T)$ defined as

$$F(T) = \frac{1 - 2C_{\mathcal{L}}}{4C_{\mathcal{A}}} = \frac{1 - 2C_3 \|\nabla c_0\|_{L^q} T^{\vartheta_1}}{4C_2 T^{\vartheta_2}}$$

must satisfy $\delta < F(T)$, i. e., $F(T) > C_1\|\rho_0\|_{L^p}$. Note first that, since it is required that $F(T) > 0$, for $\vartheta_1 > 0$ we have

$$T < T_0 \equiv \left(\frac{1}{2C_3 \|\nabla c_0\|_{L^q}}\right)^{1/\vartheta_1}.$$

Alternatively, for $\vartheta_1 = 0$, $F(T) > 0$ implies that $1 - 2C_3 \|\nabla c_0\|_{L^q} > 0$. Then, it must exist $\epsilon > 0$ such that this condition is met if $\|\nabla c_0\|_{L^q} < \epsilon$. In that case, $T_0 = \infty$. In addition, since $T > 0$, $F(T)$ is a continuous function on T and, as

$$\lim_{T\to 0} F(T) = \infty \quad \text{and} \quad \lim_{T\to T_0} F(T) = 0,$$

by the intermediate value theorem, for any nontrivial $\rho_0 \in L^p(\mathbb{R}^d)$there is $T^* \in (0, T_0)$ such that $F(T^*) > C_1\|\rho_0\|_{L^p}$. Hence, choosing this value of T, we prove the local existence of solutions by applying Corollary 2.7. Notice that T^* depends on the values of $\|\rho_0\|_{L^p}$ and $\|\nabla c_0\|_{L^q}$. Moreover, as a conclusion of Corollary 2.7, we obtain (3.7). Then, as ρ is given by (2.7), $\rho \in C\big([0,T], L^p(\mathbb{R}^d)\big)$ (renaming T^* by T).

Throughout this proof, consider the parameter $\wp$, where $\nabla c_0 \in L^{\wp}(\mathbb{R}^d)$, such that $\wp \leq r$, with r defined as in (A2). Note that for any α and β, setting $\wp = r$ places us in case (a), whereas case (b) corresponds to $\wp \in [d/(\alpha-1), r)$ for $\alpha \leq \beta$.

To prove the estimate of ∇c, that is, (3.6) for case (a), and (3.12) for case (b), we consider integral formulation (2.8). Then,

$$\|\nabla c(\cdot,t)\|_{L^r} \leq e^{-\frac{\gamma}{\tau}t} \left\|K^{\beta}_{\frac{t}{\tau}} * \nabla c_0\right\|_{L^r} + \int_0^t \frac{1}{\tau} e^{\frac{\gamma}{\tau}(s-t)} \left\|\nabla K^{\beta}_{\frac{t-s}{\tau}} * \rho(s)\right\|_{L^r} \mathrm{d}s, \tag{3.14}$$

and applying (2.20) and (2.21) to (3.14), we get

$$\begin{aligned}\|\nabla c(\cdot,t)\|_{L^r} &\leq Ce^{-\frac{\gamma}{\tau}t}\left(\frac{t}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)}\|\nabla c_0\|_{L^{\wp}} + C\int_0^t \frac{1}{\tau}e^{\frac{\gamma}{\tau}(s-t)}\left(\frac{t-s}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}}\|\rho(s)\|_{L^p}\mathrm{d}s\\ &\leq Ce^{-\frac{\gamma}{\tau}t}\left(\frac{t}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)}\|\nabla c_0\|_{L^{\wp}} + C\left(\int_0^t \frac{1}{\tau}e^{\frac{\gamma}{\tau}(s-t)}\left(\frac{t-s}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}}\mathrm{d}s\right)\|\rho\|_{\mathrm{X}}.\end{aligned}$$

Notice that hypothesis (A2) implies $\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right) < 1$. Hence if $\gamma > 0$, the integral in the last term above satisfies

$$\int_0^t \frac{1}{\tau}e^{\frac{\gamma}{\tau}(s-t)}\left(\frac{t-s}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}}\mathrm{d}s \leq C\Gamma\left(1-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}\right) < \infty,$$

and we obtain $\|\nabla c(\cdot,t)\|_{L^r} \leq Ct^{-\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)}\|\nabla c_0\|_{L^{\wp}} + C\|\rho\|_{\mathrm{X}}$. Therefore,

$$\begin{aligned}t^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)}\|\nabla c(\cdot,t)\|_{L^r} &\leq Ct^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)-\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)}\|\nabla c_0\|_{L^{\wp}} + Ct^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)}\|\rho\|_{\mathrm{X}}\\ &\leq CT^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)-\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)}\|\nabla c_0\|_{L^{\wp}} + 2CT^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)}\|\rho_0\|_{L^p}\\ &< C\left(T,\ \|\rho_0\|_{L^p},\ \|\nabla c_0\|_{L^{\wp}}\right),\end{aligned}$$

since $\frac{1}{\alpha}\left(\frac{d}{r}+1\right) + \frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right) \leq 1$, which follows from the definition of r and by setting $\wp = r$ (case (a)) or $\wp \in [d/(\alpha-1), r)$ (case (b)). On the other hand, if $\gamma = 0$, we have

$$\int_0^t \frac{1}{\tau}(t-s)^{-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}}\mathrm{d}s \leq Ct^{1-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}},$$

and, consequently, $\|\nabla c(\cdot,t)\|_{L^r} \leq Ct^{-\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)}\|\nabla c_0\|_{L^\wp} + Ct^{1-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}}\|\rho\|_{\mathrm{X}}$. Therefore, again we obtain

$$
\begin{aligned}
t^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)}\|\nabla c(\cdot,t)\|_{L^r} &\leq Ct^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)-\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)}\|\nabla c_0\|_{L^\wp} + Ct^{2-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}}\|\rho\|_{\mathrm{X}} \\
&\leq CT^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)-\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)}\|\nabla c_0\|_{L^\wp} + CT^{2-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}}\|\rho\|_{\mathrm{X}} \\
&\leq CT^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)-\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)}\|\nabla c_0\|_{L^\wp} + 2CT^{2-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}}\|\rho_0\|_{L^p} \\
&< C\left(T,\ \|\rho_0\|_{L^p},\ \|\nabla c_0\|_{L^\wp}\right),
\end{aligned}
$$

since $\frac{1}{\alpha}\left(\frac{d}{r}+1\right)+\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right) \leq 1$. Thus, considering $\wp = r$ or $\wp$ as defined in (3.10), the estimate (3.6) and (3.12), respectively, hold.

Proof of (i): Applying Lemma 2.10 to (2.7) for $t > 0$ and $1 \leq \left(\frac{r}{p+r}\right)p \leq p_1 < p$, we obtain

$$
\begin{aligned}
\|\rho(t)\|_{L^{p_1}} &\leq \|K_t^\alpha * \rho_0\|_{L^{p_1}} + \left\|\int_0^t \nabla K_{t-s}^\alpha * [\rho(s)\nabla c(s)]\,\mathrm{d}s\right\|_{L^{p_1}} \\
&\leq \|\rho_0\|_{L^{p_1}} + \int_0^t \left\|\nabla K_{t-s}^\alpha * [\rho(s)\nabla c(s)]\right\|_{L^{p_1}} \mathrm{d}s \\
&\leq \|\rho_0\|_{L^{p_1}} + C\int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{p}+\frac{1}{r}-\frac{1}{p_1}\right)-\frac{1}{\alpha}}\|\rho(s)\|_{L^p}\|\nabla c(s)\|_{L^r}\,\mathrm{d}s.
\end{aligned}
\tag{3.15}
$$

Since $\frac{d}{\alpha}\left(\frac{1}{p}+\frac{1}{r}-\frac{1}{p_1}\right)+\frac{1}{\alpha} < 1$, $\frac{1}{\alpha}\left(\frac{d}{r}+1\right) > 0$ and $\frac{d}{\alpha}\left(\frac{1}{p_1}-\frac{1}{p}\right) > 0$, we can apply Lemma 2.22 and use estimate (3.6) for case (a) and (3.12) for case (b) to obtain

$$
\begin{aligned}
&\int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{p}+\frac{1}{r}-\frac{1}{p_1}\right)-\frac{1}{\alpha}}\|\rho(s)\|_{L^p}\|\nabla c(s)\|_{L^r}\,\mathrm{d}s \\
&\qquad\leq \int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{p}+\frac{1}{r}-\frac{1}{p_1}\right)-\frac{1}{\alpha}} s^{-1+\frac{1}{\alpha}\left(\frac{d}{r}+1\right)}\|\rho(s)\|_{L^p}\, s^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)}\|\nabla c(s)\|_{L^r}\,\mathrm{d}s \\
&\qquad\leq C\left(\int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{p}+\frac{1}{r}-\frac{1}{p_1}\right)-\frac{1}{\alpha}} s^{-1+\frac{1}{\alpha}\left(\frac{d}{r}+1\right)}\mathrm{d}s\right)\|\rho_0\|_{L^p}\left(\|\rho_0\|_{L^p}+\|\nabla c_0\|_{L^\wp}\right) \\
&\qquad\leq CT^{\frac{d}{\alpha}\left(\frac{1}{p_1}-\frac{1}{p}\right)}\|\rho_0\|_{L^p}\left(\|\rho_0\|_{L^p}+\|\nabla c_0\|_{L^\wp}\right),
\end{aligned}
$$

where $\wp = r$ for case (a), $\wp$ is in the range defined by (3.10) for case (b), and the constant C depends on T. Thus, from (3.15), we have

$$
\|\rho(t)\|_{L^{p_1}} < C\left(T,\ \|\rho_0\|_{L^p},\ \|\rho_0\|_{L^{p_1}},\ \|\nabla c_0\|_{L^\wp}\right). \tag{3.16}
$$

Note that if $1/p + 1/r = 1$, we can take $p_1 = 1$. Otherwise, we have $1/p + 1/r < 1$, and we can recalculate (3.15) using the new information stated in (3.16). Specifically, we can recalculate (3.15) for $p_2 \geq 1$ such that $\left(\frac{r}{p_1+r}\right)p_1 \leq p_2 < p_1$ using (3.16), and continue iteratively for $p_n \geq 1$ such that $\left(\frac{r}{p_{n-1}+r}\right)p_{n-1} \leq p_n < p_{n-1}$, using

$$
\|\rho(t)\|_{L^{p_n}} < C\left(T,\ \|\rho_0\|_{L^p},\ \|\rho_0\|_{L^{p_1}},\ \|\rho_0\|_{L^{p_2}},\ \cdots,\ \|\rho_0\|_{L^{p_n}},\ \|\nabla c_0\|_{L^\wp}\right),
$$

until $\left(\frac{r}{p_n+r}\right) p_n \leq 1$, at which point we can choose $p_{n+1} = 1$. Therefore, we conclude that $\rho \in L^\infty\left((0,T); L^1 \cap L^p\right)$.

Next, to prove the mass conservation, we can apply Proposition 2.24 since $(\rho\nabla c)(\cdot,t) \in L^1$. Indeed, with the above calculation, we can choose p_1 so that $\rho \in L^{p_1}$ and $1/p_1 + 1/r = 1$. Then, $\|(\rho\nabla c)(\cdot,t)\|_{L^1} \leq C\|\rho(\cdot,t)\|_{L^{p_1}} \|\nabla c(\cdot,t)\|_{L^r} < \infty$, and we establish that $\int_{\mathbb{R}^d} \rho(x,t)\mathrm{d}x = \int_{\mathbb{R}^d} \rho_0(x)\mathrm{d}x$ for every $t \geq 0$ where the solution exists. Therefore, we conclude ***(i)***.

Proof of (ii): It follows from (3.15).

Proof of (iii): Analogously to the way we proved the behavior of ∇c and obtained estimates (3.6) and (3.12), we now prove the estimate of c. From integral formulation, as $\wp \leq r$, with r defined as in (A2), we obtain

$$
\begin{aligned}
\|c(\cdot,t)\|_{L^r} &\leq e^{-\frac{\gamma}{\tau}t}\left\|K^\beta_{\frac{t}{\tau}} * c_0\right\|_{L^r} + \int_0^t \frac{1}{\tau} e^{\frac{\gamma}{\tau}(s-t)}\left\|K^\beta_{\frac{t-s}{\tau}} * \rho(s)\right\|_{L^r}\mathrm{d}s \\
&\leq Ce^{-\frac{\gamma}{\tau}t}\left(\frac{t}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)}\|c_0\|_{L^\wp} + C\int_0^t \frac{1}{\tau}e^{\frac{\gamma}{\tau}(s-t)}\left(\frac{t-s}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)}\|\rho(s)\|_{L^p}\mathrm{d}s \\
&\leq Ce^{-\frac{\gamma}{\tau}t}\left(\frac{t}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)}\|c_0\|_{L^\wp} + C\left(\int_0^t \frac{1}{\tau}e^{\frac{\gamma}{\tau}(s-t)}\left(\frac{t-s}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)}\mathrm{d}s\right)\|\rho\|_{\mathrm{X}}.
\end{aligned}
$$

Then, if $\gamma > 0$, the integral in the last term above satisfies

$$
\int_0^t \frac{1}{\tau}e^{\frac{\gamma}{\tau}(s-t)}\left(\frac{t-s}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)}\mathrm{d}s \leq C\Gamma\left(1-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)\right),
$$

since $\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)+\frac{1}{\beta} < 1$, and we obtain $\|c(\cdot,t)\|_{L^r} \leq Ct^{-\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)}\|c_0\|_{L^\wp} + C\|\rho\|_{\mathrm{X}}$. Therefore,

$$
\begin{aligned}
t^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)}\|c(\cdot,t)\|_{L^r} &\leq Ct^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)-\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)}\|c_0\|_{L^\wp} + Ct^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)}\|\rho\|_{\mathrm{X}} \\
&\leq CT^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)-\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)}\|c_0\|_{L^\wp} + CT^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)}\|\rho_0\|_{L^p} \\
&< C\left(T,\ \|\rho_0\|_{L^p},\ \|c_0\|_{L^\wp}\right).
\end{aligned}
$$

On the other hand, if $\gamma = 0$, we have

$$
\int_0^t \frac{1}{\tau}(t-s)^{-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)}\mathrm{d}s \leq Ct^{1-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)},
$$

since $\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)+\frac{1}{\beta} < 1$, and, consequently, $\|c(\cdot,t)\|_{L^r} \leq Ct^{-\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)}\|c_0\|_{L^\wp} + Ct^{1-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)}\|\rho\|_{\mathrm{X}}$. Therefore,

$$
\begin{aligned}
t^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)}\|c(\cdot,t)\|_{L^r} &\leq Ct^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)-\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)}\|c_0\|_{L^\wp} + Ct^{2-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)}\|\rho\|_{\mathrm{X}} \\
&\leq CT^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)-\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)}\|c_0\|_{L^\wp} + CT^{2-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)}\|\rho_0\|_{L^p} \\
&< C\left(T,\ \|\rho_0\|_{L^p},\ \|c_0\|_{L^\wp}\right),
\end{aligned}
$$

where again $\wp = r$ for case (a), $\wp$ is in the range defined by (3.10) for case (b).

Proof of (iv): we can apply Proposition 2.24.

Proof of (v): Let $q_0 > p$ be such that

$$\frac{1}{p} + \frac{1}{r} - \frac{\alpha - 1}{d} < \frac{1}{q_0} < \frac{1}{p}. \tag{3.17}$$

Then, for $t > 0$, with estimate (2.21) and Hölder's inequality, we can bound the integral

$$\begin{aligned}
\|\rho(t)\|_{L^{q_0}} &\le \|K_t^\alpha * \rho_0\|_{L^{q_0}} + \left\| \int_0^t \nabla K_{t-s}^\alpha * [\rho(s)\nabla c(s)]\, \mathrm{d}s \right\|_{L^{q_0}} \\
&\le C t^{-\frac{d}{\alpha}\left(\frac{1}{p} - \frac{1}{q_0}\right)} \|\rho_0\|_{L^p} + \int_0^t \left\| \nabla K_{t-s}^\alpha * [\rho(s)\nabla c(s)] \right\|_{L^{q_0}} \mathrm{d}s \\
&\le C t^{-\frac{d}{\alpha}\left(\frac{1}{p} - \frac{1}{q_0}\right)} \|\rho_0\|_{L^p} + C \int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{p} + \frac{1}{r} - \frac{1}{q_0}\right) - \frac{1}{\alpha}} \|\rho(s)\|_{L^p} \|\nabla c(s)\|_{L^r}\, \mathrm{d}s \\
&\le C t^{-\frac{d}{\alpha}\left(\frac{1}{p} - \frac{1}{q_0}\right)} \|\rho_0\|_{L^p} + C\left(T,\ \|\rho_0\|_{L^p},\ \|\nabla c_0\|_{L^\wp}\right) \int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{p} + \frac{1}{r} - \frac{1}{q_0}\right) - \frac{1}{\alpha}} s^{-1 + \frac{1}{\alpha}\left(\frac{d}{r} + 1\right)} \mathrm{d}s \\
&\le C\left(T,\ \|\rho_0\|_{L^p},\ \|\nabla c_0\|_{L^\wp}\right) t^{-\frac{d}{\alpha}\left(\frac{1}{p} - \frac{1}{q_0}\right)},
\end{aligned} \tag{3.18}$$

since, from (3.17), $\frac{d}{\alpha}\left(\frac{1}{p} + \frac{1}{r} - \frac{1}{q_0}\right) + \frac{1}{\alpha} < 1$. Now, let $q_1 > q_0$ be such that

$$\frac{1}{q_0} + \frac{1}{r} - \frac{\alpha - 1}{d} < \frac{1}{q_1} < \frac{1}{q_0}. \tag{3.19}$$

Moreover, note that, from (A2), we obtain $\frac{1}{p} - \frac{1}{r} < \frac{\beta - 1}{d}$, which, given $\beta \in (1, 2]$, implies

$$\frac{d}{\alpha}\left(\frac{1}{p} - \frac{1}{r}\right) - \frac{1}{\alpha} < \frac{d}{\alpha}\left(\frac{\beta - 1}{d}\right) - \frac{1}{\alpha} = \frac{\beta - 2}{\alpha} \le 0. \tag{3.20}$$

Then, using (3.18) and estimate (3.6) for case (a), or (3.12) for case (b), we obtain

$$\begin{aligned}
\|\rho(t)\|_{L^{q_1}} &\le \|K_t^\alpha * \rho_0\|_{L^{q_1}} + \left\| \int_0^t \nabla K_{t-s}^\alpha * [\rho(s)\nabla c(s)]\, \mathrm{d}s \right\|_{L^{q_1}} \\
&\le C t^{-\frac{d}{\alpha}\left(\frac{1}{p} - \frac{1}{q_1}\right)} \|\rho_0\|_{L^p} + C \int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{q_0} + \frac{1}{r} - \frac{1}{q_1}\right) - \frac{1}{\alpha}} \|\rho(s)\|_{L^{q_0}} \|\nabla c(s)\|_{L^r}\, \mathrm{d}s \\
&\le C t^{-\frac{d}{\alpha}\left(\frac{1}{p} - \frac{1}{q_1}\right)} \|\rho_0\|_{L^p} + C\left(T,\ \|\rho_0\|_{L^p},\ \|\nabla c_0\|_{L^\wp}\right) \int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{q_0} + \frac{1}{r} - \frac{1}{q_1}\right) - \frac{1}{\alpha}} s^{-\frac{d}{\alpha}\left(\frac{1}{p} - \frac{1}{r} - \frac{1}{q_0}\right) - 1 + \frac{1}{\alpha}} \mathrm{d}s \\
&\le C\left(T,\ \|\rho_0\|_{L^p},\ \|\nabla c_0\|_{L^\wp}\right) t^{-\frac{d}{\alpha}\left(\frac{1}{p} - \frac{1}{q_1}\right)},
\end{aligned}$$

since, from (3.19) and (3.20), we have $\frac{d}{\alpha}\left(\frac{1}{q_0} + \frac{1}{r} - \frac{1}{q_1}\right) + \frac{1}{\alpha} < 1$ and $\frac{d}{\alpha}\left(\frac{1}{p} - \frac{1}{r} - \frac{1}{q_0}\right) - \frac{1}{\alpha} < 0$, respectively.

By induction, let $q_{n+1} > q_n$ be such that

$$\frac{1}{q_n} + \frac{1}{r} - \frac{\alpha - 1}{d} < \frac{1}{q_{n+1}} < \frac{1}{q_n}, \tag{3.21}$$

and assume

$$\|\rho(t)\|_{L^{q_n}} \le C\left(T,\ \|\rho_0\|_{L^p},\ \|\nabla c_0\|_{L^\wp}\right) t^{-\frac{d}{\alpha}\left(\frac{1}{p} - \frac{1}{q_n}\right)}. \tag{3.22}$$

Then, for $t > 0$, with estimates (2.21) and (3.22), we obtain

$$
\begin{aligned}
&\|\rho(t)\|_{L^{q_{n+1}}} \\
&\quad \le \|K_t^\alpha * \rho_0\|_{L^{q_{n+1}}} + \left\| \int_0^t \nabla K_{t-s}^\alpha * [\rho(s)\nabla c(s)]\, \mathrm{d}s \right\|_{L^{q_{n+1}}} \\
&\quad \le Ct^{-\frac{d}{\alpha}\left(\frac{1}{p}-\frac{1}{q_{n+1}}\right)}\|\rho_0\|_{L^p} + C\int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{q_n}+\frac{1}{r}-\frac{1}{q_{n+1}}\right)-\frac{1}{\alpha}} \|\rho(s)\|_{L^{q_n}}\, s^{-1+\frac{1}{\alpha}\left(\frac{d}{r}+1\right)} s^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)} \|\nabla c(s)\|_{L^r}\, \mathrm{d}s \\
&\quad \le Ct^{-\frac{d}{\alpha}\left(\frac{1}{p}-\frac{1}{q_{n+1}}\right)}\|\rho_0\|_{L^p} + C\left(T,\ \|\rho_0\|_{L^p},\ \|\nabla c_0\|_{L^\wp}\right)\int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{q_n}+\frac{1}{r}-\frac{1}{q_{n+1}}\right)-\frac{1}{\alpha}} s^{-\frac{d}{\alpha}\left(\frac{1}{p}-\frac{1}{r}-\frac{1}{q_n}\right)-1+\frac{1}{\alpha}}\mathrm{d}s \\
&\quad \le Ct^{-\frac{d}{\alpha}\left(\frac{1}{p}-\frac{1}{q_{n+1}}\right)}\|\rho_0\|_{L^p} + C\left(T,\ \|\rho_0\|_{L^p},\ \|\nabla c_0\|_{L^\wp}\right) t^{-\frac{d}{\alpha}\left(\frac{1}{p}-\frac{1}{q_{n+1}}\right)},
\end{aligned}
\tag{3.23}
$$

since, from (3.21) and (3.20), we have $\frac{d}{\alpha}\left(\frac{1}{q_n}+\frac{1}{r}-\frac{1}{q_{n+1}}\right)+\frac{1}{\alpha} < 1$ and $\frac{d}{\alpha}\left(\frac{1}{p}-\frac{1}{r}-\frac{1}{q_n}\right)-\frac{1}{\alpha} < 0$, respectively.

Next, from (3.23), we infer that

$$
\|\rho(t)\|_{L^q} \le C\left(T,\ \|\rho_0\|_{L^p},\ \|\nabla c_0\|_{L^\wp}\right) t^{-\frac{d}{\alpha}\left(\frac{1}{p}-\frac{1}{q}\right)} \tag{3.24}
$$

for all $q > p$. Thus, we conclude that $\rho \in L^\infty_{\mathrm{loc}}\left((0,T]; L^q(\mathbb{R}^d)\right)$ for all $q > p$.

Moreover, notice that $0 < \frac{d}{\alpha}\left(\frac{1}{p}-\frac{1}{q_n}\right) < 1$, since $q_n > p$ for all n and p in the range defined by (A2). Then, from (3.24), it is easy to prove that $\rho \in L^1\left((0,T); L^q(\mathbb{R}^d)\right)$ for all $q > p$. Additionally, we have already proven in (ii) that $\rho \in L^1\left((0,T); L^q(\mathbb{R}^d)\right)$ for all $1 \le q \le p$. Therefore, we obtain $\rho \in L^1\left((0,T); L^q(\mathbb{R}^d)\right)$ for all $q \ge 1$.

Proof of (vi): We now prove the estimate of ∇c, as before, by considering the integral formulation (2.8) and applying estimates (2.20) and (2.21)to obtain

$$
\begin{aligned}
\|\nabla c(\cdot,t)\|_{L^q} &\le e^{-\frac{\gamma}{\tau}t}\left\|K^\beta_{\frac{t}{\tau}} * \nabla c_0\right\|_{L^q} + \int_0^t \frac{1}{\tau} e^{\frac{\gamma}{\tau}(s-t)} \left\|\nabla K^\beta_{\frac{t-s}{\tau}} * \rho(s)\right\|_{L^q} \mathrm{d}s \\
&\le Ce^{-\frac{\gamma}{\tau}t}\left(\frac{t}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{\wp}-\frac{1}{r}\right)} \|\nabla c_0\|_{L^\wp} + C\int_0^t \frac{1}{\tau} e^{\frac{\gamma}{\tau}(s-t)}\left(\frac{t-s}{\tau}\right)^{-\frac{1}{\beta}} \|\rho(s)\|_{L^q}\mathrm{d}s.
\end{aligned}
$$

From ***(v)*** we know that $\rho \in L^1\left((0,T); L^q(\mathbb{R}^d)\right)$ and $\|\rho\|_{L^\infty\left([t_1,t_2]; L^q(\mathbb{R}^d)\right)} < \infty$ for $t_1,\ t_2 \in (0,T]$, as $\rho \in L^\infty_{\mathrm{loc}}\left((0,T]; L^q(\mathbb{R}^d)\right)$. Then, taking into account ***(v)*** and the fact that $\beta > 1$, we obtain

$$
\begin{aligned}
&\int_0^t \frac{1}{\tau} e^{\frac{\gamma}{\tau}(s-t)}\left(\frac{t-s}{\tau}\right)^{-\frac{1}{\beta}} \|\rho(s)\|_{L^q}\mathrm{d}s \\
&\qquad = \int_0^{t_1} \frac{1}{\tau} e^{\frac{\gamma}{\tau}(s-t)}\left(\frac{t-s}{\tau}\right)^{-\frac{1}{\beta}} \|\rho(s)\|_{L^q}\mathrm{d}s + \int_{t_1}^t \frac{1}{\tau} e^{\frac{\gamma}{\tau}(s-t)}\left(\frac{t-s}{\tau}\right)^{-\frac{1}{\beta}} \|\rho(s)\|_{L^q}\mathrm{d}s \\
&\qquad \le C(T)\|\rho\|_{L^1\left((0,T); L^q(\mathbb{R}^d)\right)} + C(T)\|\rho\|_{L^\infty\left([t_1,t_2]; L^q(\mathbb{R}^d)\right)}.
\end{aligned}
$$

Therefore, (3.9) follows from setting $\wp = r$. For case (b), $\wp$ can be set in the range defined by (3.10). □

Remark 3.7 (Higher-order regularity in space of local solutions). *Higher regularity of the solution in space – implying $\rho \in C\big((0,T), W^{N,p}(\mathbb{R}^d)\big)$, and $\nabla c \in C\big((0,T), W^{N,r}(\mathbb{R}^d)\big)$ for $N > 0$ – follows from Theorem 3.4, Lemma 2.11 and the fact that in the Sobolev spaces, assuming $\varphi \in W^{N,p}(\mathbb{R}^d)$ and $\psi \in W^{N,r}(\mathbb{R}^d)$, where $1/p + 1/r = 1/q \leq 1$, we have $\varphi\psi \in W^{N,q}(\mathbb{R}^d)$ and $\|\varphi\psi\|_{W^{N,q}} \leq C\|\varphi\|_{W^{N,p}}\|\psi\|_{W^{N,r}}$. Indeed, we can consider the Banach space $\mathbf{X} = C\big([0,T], W^{N,p}(\mathbb{R}^d)\big)$ with the usual norm: $\|u\|_{\mathbf{X}} = \sup_{t\in[0,T]}\|u(t)\|_{W^{N,p}}$. Then, by replacing $L^v(\mathbb{R}^d)$ with $W^{N,v}(\mathbb{R}^d)$ (and $W^{1,r}(\mathbb{R}^d)$ with $W^{N+1,r}(\mathbb{R}^d)$), where v can be equal to p, r, q or $\wp$, we obtain the version of Theorem 3.4 for higher regularity.*

3.3 Global in time solutions

To prove the global existence of solutions to the parabolic-parabolic system (3.1) consider (A1), (A2) and the following assumptions:

(A5) in addition to **(A2)**, if $2\beta(\alpha-1) \geq \alpha$, parameters p and r satisfy $p < \frac{d\alpha}{2\beta(\alpha-1)-\alpha}$;

(A6) the Banach space $\mathbf{X}$ is defined as $\mathbf{X} = \{u \in C\big([0,\infty), L^p(\mathbb{R}^d)\big) \;:\; \|u\|_{\mathbf{X}} \equiv \sup_{t>0} t^{\sigma}\|u(t)\|_{L^p} < \infty\}$, where $\sigma = 2 - \frac{1}{\alpha}\left(\frac{d}{r}+1\right) - \frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right) - \frac{1}{\beta}$.

Now, before stating the global existence of solutions, we proceed by establishing some estimates.

Lemma 3.8. *Assume that (A1), (A5) and (A6) are in force. Then*

$$\|u_1\|_{\mathbf{X}} \leq C_1\|\rho_0\|_{L^{p_1}}, \tag{3.25}$$

where $u_1(\cdot,t) = K_t^{\alpha} * \rho_0$ for $t > 0$, K_t^{α} is given by (1.17), $p_1 = \frac{pd}{\alpha\sigma p + d}$, and C_1 is a time-independent constant.

Proof. For $t > 0$, we apply (2.20) to (2.16) and, in view of Lemma B.7, use $1 \leq p_1 < p$ to obtain

$$\|u_1\|_{L^p} = \left\|K_t^{\alpha}(x) * \rho_0\right\|_{L^p} \leq C_1 t^{-\frac{d}{\alpha}\left(\frac{1}{p_1}-\frac{1}{p}\right)}\|\rho_0\|_{L^{p_1}} = C_1 t^{-\sigma}\|\rho_0\|_{L^{p_1}}.$$

Therefore, (3.25) follows from the definitions of $\mathbf{X}$ and p_1. ☐

Lemma 3.9. *Assume that (A1), (A5) and (A6) are in force. Then*

$$\|\mathcal{A}(u,v)\|_{\mathbf{X}} \leq C_2\|u\|_{\mathbf{X}}\|v\|_{\mathbf{X}}, \tag{3.26}$$

where $\mathcal{A}$ is given by (2.17) with $f(\rho,\nabla\rho) = 0$ and $\chi = 1$, and C_2 is a time-independent constant.

Proof. Let $1 \le q \le p \le r \le \infty$ be such that $1/q = 1/p + 1/r$. For $t > 0$, applying estimate (2.21) to (2.17), we obtain

$$\begin{aligned}\|\mathcal{A}(u,v)\|_{L^p} &\le C \int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{\alpha}} \|u(s)\|_{L^p} \left(\int_0^s \frac{1}{\tau} e^{\frac{\gamma}{\tau}(w-s)} \left(\frac{s-w}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}} \|v(w)\|_{L^p} \mathrm{d}w \right) \mathrm{d}s \\ &\le C \left(\int_0^t (t-s)^{-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)} s^{-\sigma} \int_0^s \frac{1}{\tau} e^{\frac{\gamma}{\tau}(w-s)} \left(\frac{s-w}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}} w^{-\sigma} \mathrm{d}w \mathrm{d}s \right) \|u\|_{\mathbf{X}} \|v\|_{\mathbf{X}}.\end{aligned}$$

Since, by Lemma B.3, $\sigma < 1$ and $\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)+\frac{1}{\beta} < 1$, we can apply Lemma 2.22 to estimate the inner integral above as follows:

$$\begin{aligned}\int_0^s \frac{1}{\tau} e^{\frac{\gamma}{\tau}(w-s)} \left(\frac{s-w}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}} w^{-\sigma} \mathrm{d}w &\le C \int_0^s (s-w)^{-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}} w^{-\sigma} \mathrm{d}w \\ &\le C s^{1-\sigma-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}}.\end{aligned}$$

Then, applying again Lemma 2.22 together with the last estimate, from the definition of σ, we obtain

$$\begin{aligned}&\int_0^t (t-s)^{-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)} s^{-\sigma} \int_0^s \frac{1}{\tau} e^{\frac{\gamma}{\tau}(w-s)} \left(\frac{s-w}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}} w^{-\sigma} \mathrm{d}w \, \mathrm{d}s \\ &\qquad \le C \int_0^t (t-s)^{-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)} s^{-\sigma} s^{1-\sigma-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}} \mathrm{d}s \\ &\qquad \le C \int_0^t (t-s)^{-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)} s^{1-2\sigma-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}} \mathrm{d}s \\ &\qquad \le C t^{-\sigma},\end{aligned}$$

since, in view of Lemma B.3, we have $\frac{1}{\alpha}\left(\frac{d}{r}+1\right) < 1$ and $\sigma + \frac{1}{2}\left[\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)+\frac{1}{\beta}\right] < 1$.

Therefore, (3.26) follows from the definition of $\mathbf{X}$. □

Lemma 3.10. *Assume that (A1), (A5) and (A6) are in force. Then*

$$\|\mathcal{L}(u)\|_{\mathbf{X}} \le C_3 \|\nabla c_0\|_{L^{p_2}} \|u\|_{\mathbf{X}}, \tag{3.27}$$

where $\mathcal{L}$ is given by (2.18), $p_2 = \frac{dr\alpha}{\beta(r(\alpha-1)-d)+d\alpha}$, and C_3 is a time-independent constant.

Proof. Let $1 \le q \le p$ be such that $1/q = 1/p + 1/r$. For $t > 0$, applying estimates (2.20) and (2.21) to (2.18), and using $1 \le p_2 \le r$ (Lemma B.8), we obtain

$$\begin{aligned}\|\mathcal{L}(u)\|_{L^p} &\le C \int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{\alpha}} \|u(s)\|_{L^p} e^{-\frac{\gamma}{\tau}s} \left\| K^{\beta}_{\frac{s}{\tau}}(x) * \nabla c_0(x) \right\|_{L^r} \mathrm{d}s \\ &\le C \int_0^t (t-s)^{-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)} \|u(s)\|_{L^p} e^{-\frac{\gamma}{\tau}s} \left(\frac{s}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{p_2}-\frac{1}{r}\right)} \|\nabla c_0\|_{L^{p_2}} \mathrm{d}s \\ &\le C \left(\int_0^t (t-s)^{-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)} s^{-\sigma-\frac{d}{\beta}\left(\frac{1}{p_2}-\frac{1}{r}\right)} e^{-\frac{\gamma}{\tau}s} \mathrm{d}s \right) \|\nabla c_0\|_{L^{p_2}} \|u\|_{\mathbf{X}}.\end{aligned}$$

Since, by Lemmas B.3 and B.8, we have $\frac{1}{\alpha}\left(\frac{d}{r}+1\right) < 1$ and $\sigma + \frac{d}{\beta}\left(\frac{1}{p_2}-\frac{1}{r}\right) < 1$, we can apply Lemma 2.22 to estimate the inner integral above:

$$\begin{aligned}\int_0^t (t-s)^{-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)} s^{-\sigma-\frac{d}{\beta}\left(\frac{1}{p_2}-\frac{1}{r}\right)} e^{-\frac{\gamma}{\tau}s}\mathrm{d}s &= \int_0^t (t-s)^{-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)} s^{-\sigma-\frac{d}{\beta}\left(\frac{1}{p_2}-\frac{1}{r}\right)} e^{-\frac{\gamma}{\tau}s}\mathrm{d}s \\ &\leq \int_0^t (t-s)^{-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)} s^{-\sigma-\frac{d}{\beta}\left(\frac{1}{p_2}-\frac{1}{r}\right)}\mathrm{d}s \\ &\leq Ct^{-\sigma},\end{aligned}$$

as the definition of p_2 implies that $\frac{1}{\alpha}\left(\frac{d}{r}+1\right) + \frac{d}{\beta}\left(\frac{1}{p_2}-\frac{1}{r}\right) = 1$.

Therefore, (3.27) follows from the definition of $\mathbf{X}$. □

Remark 3.11. *It is important to note that the parameters σ, p_1 and p_2 are chosen to ensure that the constants C_1, $C_{\mathcal{A}}$ and $C_{\mathcal{L}}$, in estimates (3.25), (2.14) and (2.15), respectively, are time-independent. This choice is crucial for the validity of the global solution estimates.*

Theorem 3.12 (Global in time solutions). *Assume that (A1), (A5) and (A6) are in force. Then, there exists $\epsilon > 0$ such that, if*

$$\|\rho_0\|_{L^{p_1}} + \|\nabla c_0\|_{L^{p_2}} < \epsilon, \tag{3.28}$$

there is a unique global mild solution $(\rho, \nabla c)$ to system (3.1), $\rho \in C\big((0,\infty), L^p(\mathbb{R}^d)\big) \cap \mathbf{X}$, $\nabla c \in C\big((0,\infty), L^r(\mathbb{R}^d)\big)$, with initial condition $\rho_0 \in L^{p_1}(\mathbb{R}^d)$ and $\nabla c_0 \in L^{p_2}(\mathbb{R}^d)$, where

$$p_1 = \frac{pd}{\alpha\sigma p + d} \qquad \text{and} \qquad p_2 = \frac{dr\alpha}{\beta\left(r(\alpha-1)-d\right)+d\alpha}. \tag{3.29}$$

Moreover,

$$\sup_{t\geq 0} t^{\sigma}\|\rho(\cdot,t)\|_{L^p} \leq C\|\rho_0\|_{L^{p_1}}, \tag{3.30}$$

and

$$\sup_{t\geq 0} t^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)} \|\nabla c(\cdot,t)\|_{L^r} < C\|\nabla c_0\|_{L^{p_2}} + C\|\rho_0\|_{L^{p_1}}. \tag{3.31}$$

Furthermore, if, in addition to (A5) (or equivalently, in that case, (A2)), p satisfies

$$p \leq \frac{2d}{(\alpha-1)+2(\beta-1)} \tag{3.32}$$

or p and r satisfy

$$\frac{2d}{(\alpha-1)+2(\beta-1)} < p < \frac{\alpha d}{\max\left\{2\beta(\alpha-1)-\alpha,\ \alpha(\alpha-2)+\beta\right\}} \tag{3.33}$$

and

$$r \leq \frac{(2\beta-\alpha)pd}{[\beta(\alpha-1)+\alpha(\beta-1)]p-\alpha d}, \tag{3.34}$$

then $\rho \in L^{\infty}\left((0,\infty), L^{p_1}(\mathbb{R}^d)\right)$.

Proof. To establish the existence and uniqueness of global solution to system (3.1), we apply the fixed point theorem as articulated in Corollary 2.7, with $\mathbf{X} = \{\rho \in C\big((0,\infty), L^p(\mathbb{R}^d)\big) : \sup_{t>0} t^\sigma \|\rho(t)\|_{L^p(\mathbb{R}^d)} < \infty\}$. Then, we construct the unique mild solution using u_1, $\mathcal{A}$, and $\mathcal{L}$ as defined in Section 2.2.1 and demonstrate that the hypotheses of Corollary 2.7 are satisfied.

In view of Lemmas 3.8, 3.9 and 3.10 we set δ and the constants $C_{\mathcal{A}}$ and $C_{\mathcal{L}}$ from Corollary 2.7 as $\delta = C_1 \|\rho_0\|_{L^{p_1}}$, $C_{\mathcal{A}} = C_2$, and $C_{\mathcal{L}} = C_3 \|\nabla c_0\|_{L^{p_2}}$. Then, choosing $\epsilon = \frac{1}{2\max\{4C_1C_2, 2C_3\}}$, condition (3.28) implies that

$$C_1 \|\rho_0\|_{L^{p_1}} = \delta < \frac{1-2C_{\mathcal{L}}}{4C_{\mathcal{A}}} = \frac{1-2C_3\|\nabla c_1\|_{L^{p_2}}}{4C_2}.$$

Consequently, as a result of Corollary 2.7, we prove the global existence of solutions and establish estimate (3.30). Therefore, $\rho \in C((0,\infty), L^p) \cap \mathbf{X}$.

Next note that estimate (3.31) follows from integral formulation (2.8). Indeed, we have

$$\|\nabla c(\cdot,t)\|_{L^r} \le e^{-\frac{\gamma}{\tau}t} \left\| K^\beta_{\frac{t}{\tau}} * \nabla c_0 \right\|_{L^r} + \int_0^t \frac{1}{\tau} e^{\frac{\gamma}{\tau}(s-t)} \left\| \nabla K^\beta_{\frac{t-s}{\tau}} * \rho(s) \right\|_{L^r} \mathrm{d}s.$$

Now, from Lemma B.3, we have $\sigma < 1$ and $\frac{d}{\beta}\left(\frac{1}{p} - \frac{1}{r}\right) + \frac{1}{\beta} < 1$. Then, we can apply Lemmas 2.10 and 2.22 to obtain

$$\begin{aligned}
\|\nabla c(\cdot,t)\|_{L^r} &\le Ce^{-\frac{\gamma}{\tau}t} \left(\frac{t}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{p_2}-\frac{1}{r}\right)} \|\nabla c_0\|_{L^{p_2}} + C\int_0^t \frac{1}{\tau} e^{\frac{\gamma}{\tau}(s-t)} \left(\frac{t-s}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}} \|\rho(s)\|_{L^p} \mathrm{d}s \\
&\le C\left(\frac{t}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{p_2}-\frac{1}{r}\right)} \|\nabla c_0\|_{L^{p_2}} + C\int_0^t \frac{1}{\tau}\left(\frac{t-s}{\tau}\right)^{-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}} s^{-\sigma} \mathrm{d}s \, \|\rho\|_{\mathrm{X}} \\
&\le Ct^{-\frac{d}{\beta}\left(\frac{1}{p_2}-\frac{1}{r}\right)} \|\nabla c_0\|_{L^{p_2}} + Ct^{1-\sigma-\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{\beta}} \|\rho\|_{\mathrm{X}}.
\end{aligned}$$

Therefore, by multiplying both sides of the inequality above by $t^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)}$ and using the definitions of σ and p_2, as well as estimate (3.30), inequality (3.31) is obtained.

To prove that $\rho \in L^\infty\left((0,\infty), L^{p_1}(\mathbb{R}^d)\right)$, consider p satisfying (3.32) or p and r satisfying (3.33) and (3.34). Note that, since from Lemma B.9 we have $\frac{\alpha}{d}\sigma \le \frac{1}{r}$, q defined as $1/q = 1/p + 1/r$ is such that $q \le p_1$. Indeed, $\frac{1}{p_1} = \frac{\alpha}{d}\sigma + \frac{1}{p} \le \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$. Then, for $t > 0$, we can apply Lemma 2.10 to obtain

$$\begin{aligned}
\|\rho(t)\|_{L^{p_1}} &\le \|K^\alpha_t * \rho_0\|_{L^{p_1}} + \left\| \int_0^t \nabla K^\alpha_{t-s} * [\rho(s)\nabla c(s)] \, \mathrm{d}s \right\|_{L^{p_1}} \\
&\le C\|\rho_0\|_{L^{p_1}} + \int_0^t \left\| \nabla K^\alpha_{t-s} * [\rho(s)\nabla c(s)] \right\|_{L^{p_1}} \mathrm{d}s \\
&\le C\|\rho_0\|_{L^{p_1}} + C\int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{q}-\frac{1}{p_1}\right)-\frac{1}{\alpha}} \|\rho(s)\nabla c(s)\|_{L^q} \, \mathrm{d}s.
\end{aligned}$$

Then, from Lemma B.3, $\frac{1}{\alpha}\left(\frac{d}{r}+1\right)-1<\sigma<\frac{1}{\alpha}\left(\frac{d}{r}+1\right)$, and we can apply (3.30), (3.31), and Lemma 2.22 to estimate the integral above as follows:

$$\begin{aligned}\|\rho(t)\|_{L^{p_1}} &\le C\|\rho_0\|_{L^{p_1}} + C\int_0^t (t-s)^{-\frac{d}{\alpha}\left(\frac{1}{p}+\frac{1}{r}-\frac{\alpha}{d}\sigma-\frac{1}{p}\right)-\frac{1}{\alpha}} \|\rho(s)\|_{L^p}\|\nabla c(s)\|_{L^r}\,\mathrm{d}s \\ &\le C\|\rho_0\|_{L^{p_1}} + C\int_0^t (t-s)^{-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)+\sigma} s^{-\sigma-1+\frac{1}{\alpha}\left(\frac{d}{r}+1\right)} s^{\sigma}\|\rho(s)\|_{L^p}\, s^{1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)}\|\nabla c(s)\|_{L^r}\,\mathrm{d}s \\ &\le C\|\rho_0\|_{L^{p_1}} + C\left(\int_0^t (t-s)^{-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)+\sigma} s^{-\sigma-1+\frac{1}{\alpha}\left(\frac{d}{r}+1\right)}\mathrm{d}s\right)\|\rho_0\|_{L^{p_1}}\left(\|\rho_0\|_{L^{p_1}}+\|\nabla c_0\|_{L^{p_2}}\right) \\ &\le C\|\rho_0\|_{L^{p_1}}\left(1+\|\rho_0\|_{L^{p_1}}+\|\nabla c_0\|_{L^{p_2}}\right).\end{aligned}$$

□

Remark 3.13 (Higher-order regularity in space of global solutions). *As discussed in Remark 3.7, higher regularity of the solution in space (implying $\rho \in C\big((0,\infty), W^{N,p}(\mathbb{R}^d)\big)$ and $\nabla c \in C\big((0,\infty), W^{N,r}(\mathbb{R}^d)\big)$, for $N>0$) follows from Theorem 3.12 and properties of the Sobolev spaces $W^{N,p}(\mathbb{R}^d)$ (see Remark 3.7). Indeed, for the global existence of solutions, consider the Banach space $\mathbf{X}=\{u\in C\big((0,\infty),W^{N,p}(\mathbb{R}^d)\big)\ :\ \|u\|_{\mathbf{X}}=\sup_{t>0} t^{\sigma}\|u(t)\|_{W^{N,p}}<\infty\}$. Then, by replacing $L^q(\mathbb{R}^d)$ with $W^{N,q}(\mathbb{R}^d)$, where q can be equal to p_1, p_2, p, or r, we obtain a version of Theorem 3.12 for higher regularity.*

Theorem 3.14. *Let $d=2$, and consider system (3.1) with $\alpha=\beta\in(5/3,\ 2]$. Assume that (A2) and (A5) are in force and, in addition*

$$r\le\min\left\{\frac{p}{p(\alpha-1)-1},\ \frac{1}{2-\alpha}\right\}. \tag{3.35}$$

Then, there exists $\epsilon>0$ such that, if

$$\|\rho_0\|_{L^{\frac{1}{\alpha-1}}}+\|\nabla c_0\|_{L^{\frac{2}{\alpha-1}}}<\epsilon, \tag{3.36}$$

there exists a unique global mild solution $\rho\in C((0,\infty),L^p(\mathbb{R}^d))\cap\mathbf{X}$, $\nabla c\in C((0,\infty),L^r(\mathbb{R}^d))$, with initial condition $\rho_0\in L^{\frac{1}{\alpha-1}}(\mathbb{R}^d)$, $\nabla c_0\in L^{\frac{2}{\alpha-1}}(\mathbb{R}^d)$. Moreover, $\rho\in L^\infty\left((0,\infty),L^{\frac{1}{\alpha-1}}(\mathbb{R}^d)\right)$,

$$\sup_{t\ge0} t^{\sigma}\|\rho(\cdot,t)\|_{L^p}\le C\|\rho_0\|_{L^{\frac{1}{\alpha-1}}}, \tag{3.37}$$

and

$$\sup_{t\ge0} t^{1-\frac{1}{\alpha}\left(\frac{2}{r}+1\right)}\|\nabla c(\cdot,t)\|_{L^r}<C\left(\|\nabla c_0\|_{L^{\frac{2}{\alpha-1}}}+\|\rho_0\|_{L^{\frac{1}{\alpha-1}}}\right). \tag{3.38}$$

Furthermore, if $\rho_0\in L^1(\mathbb{R}^d)$, then, $\rho\in L^1(\mathbb{R}^d)$, the corresponding solution conserves mass and, for $\int_{\mathbb{R}^d} c_0\mathrm{d}x<\infty$, the chemical concentration, c, grows as

$$\int_{\mathbb{R}^d} c(x,t)\,\mathrm{d}x=\left(\int_{\mathbb{R}^d}\rho_0(x)\mathrm{d}x\right)\left(\frac{1-e^{-\frac{\gamma}{\tau}t}}{\gamma}\right)+\left(\int_{\mathbb{R}^d}c_0(x)\mathrm{d}x\right)e^{-\frac{\gamma}{\tau}t}$$

and $\displaystyle\int_{\mathbb{R}^d} c(x,t)\,\mathrm{d}x \xrightarrow[t\to\infty]{} \frac{\int_{\mathbb{R}^d}\rho_0(x)\mathrm{d}x}{\gamma}$, which include $\gamma=0$ by taking $\gamma\to0$.

Remark 3.15. *Note that, for $d = 2$ and $\alpha = \beta \in (5/3, 2]$, assumption **(A2)** becomes*

$$\frac{4}{\alpha+1} < p < \frac{2}{\alpha-1} \qquad \text{and} \qquad \max\left\{\frac{2}{\alpha-1}, \frac{p}{p-1}\right\} < r < \frac{2p}{2-p(\alpha-1)},$$

*where $r = \frac{p}{p-1}$ is also possible if $\frac{p}{p-1} > \frac{d}{\alpha-1}$. Moreover, assumption **(A3)** is reduced to $\sigma = 2\left(\frac{\alpha-1}{\alpha}\right) - \frac{2}{\alpha p}$.*

Note also that additional condition (3.34) in Theorem 3.12 becomes $r \leq \frac{p}{p(\alpha-1)-1}$ in (3.35).

In more detail, the ranges of p and r defined in Theorem 3.14 can be written as

$$\begin{cases} \dfrac{4}{\alpha+1} < p \leq \dfrac{2}{3}\dfrac{2}{\alpha-1} \quad \text{and} \quad \max\left\{\dfrac{2}{\alpha-1}, \dfrac{p}{p-1}\right\} < r < \min\left\{\dfrac{2p}{2-p(\alpha-1)}, \dfrac{1}{2-\alpha}\right\}, \\ \text{or} \\ \dfrac{2}{3}\dfrac{2}{\alpha-1} < p < \dfrac{2}{\alpha-1} \quad \text{and} \quad \max\left\{\dfrac{2}{\alpha-1}, \dfrac{p}{p-1}\right\} < r \leq \min\left\{\dfrac{p}{p(\alpha-1)-1}, \dfrac{1}{2-\alpha}\right\}. \end{cases}$$

In the first line, r can be equal to $\frac{1}{2-\alpha}$ if $\frac{1}{2-\alpha} < \frac{2p}{2-p(\alpha-1)}$ and, in both cases, it can be equal to $\frac{p}{p-1}$ if $\frac{p}{p-1} > \frac{d}{\alpha-1}$.

Proof. We apply Theorem 3.12 to prove the existence, uniqueness, and asymptotic behavior of the solution described in (3.37) and (3.38).

To prove that $\rho \in L^1(\mathbb{R}^d)$ and to verify the mass conservation and chemical concentration behavior, we apply Proposition 2.24. Then, it suffices to show that $(\rho\nabla c)(\cdot,t) \in L^1(\mathbb{R}^d)$ for all $t \geq 0$. Note that, as $\rho \in L^\infty\left((0,\infty), L^{\frac{1}{\alpha-1}}(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)\right)$, we get $\rho \in L^\varsigma(\mathbb{R}^d)$, $\varsigma \in \left[\frac{1}{\alpha-1}, p\right]$, for all $t \geq 0$, and from (3.35), we obtain $\frac{1}{r} + \alpha - 1 \geq 1$. As a result, there is $q \in \left[\frac{1}{\alpha-1}, p\right]$ such that $\frac{1}{q} + \frac{1}{r} = 1$. Therefore, $\|(\rho\nabla c)(\cdot,t)\|_{L^1}(t) \leq C\|\rho(\cdot,t)\|_{L^q}\|\nabla c(\cdot,t)\|_{L^r} < \infty$, and we conclude. □

Remark 3.16. *In Section 1.5.2, the importance of determining the critical space to establish global well-posedness was discussed. Through the analysis of self-similar solutions, it was shown that when $\alpha = \beta$, system (3.1) has $L^{\frac{1}{\alpha-1}}(\mathbb{R}^d)$ and $L^{\frac{2}{\alpha-1}}(\mathbb{R}^d)$ as critical spaces. Note that, as demonstrated in Theorem 3.14, these spaces are crucial for ensuring the existence of solutions to system (3.1) with $\alpha = \beta$, as evidenced by the condition for solution existence specified in (3.36).*

Remark 3.17. *If we assume $\alpha = \beta = 2$, then system (3.1) is the classical one. In that case, for $d = 2$, Theorem 3.12 becomes:*

> *Let $\rho_0 \in L^1(\mathbb{R}^d)$ and $\nabla c_0 \in L^2(\mathbb{R}^d)$. Then, there exists $\delta > 0$ such that, if $\|\rho_0\|_{L^1} + \|\nabla c_0\|_{L^2} < \delta$, there is a unique global mild solution $\rho \in C\left((0,\infty), L^p(\mathbb{R}^d)\right) \cap \mathbf{X}$, $\nabla c \in C\left((0,\infty), L^r(\mathbb{R}^d)\right)$, where $\frac{4}{3} < p < 2$ and $r = \frac{p}{p-1}$ (i. e., $2 < r < 4$). Moreover, $\rho \in L^\infty\left((0,\infty), L^1(\mathbb{R}^d)\right)$ conserves mass, $\sup_{t\geq 0} t^{1-\frac{1}{p}}\|\rho(\cdot,t)\|_{L^p} \leq C\|\rho_0\|_{L^1}$, and $\sup_{t\geq 0} t^{\frac{1}{p}-\frac{1}{2}}\|\nabla c(\cdot,t)\|_{L^r} < C\|\nabla c_0\|_{L^2} + C\|\rho_0\|_{L^1}$. In addition, if $\int_{\mathbb{R}^d} c_0(x)\mathrm{d}x < \infty$, the concentration of the chemical, c, grows as in (2.46) and (2.47).*

3.4 Nonnegative Solution

In Section 2.4, we prove that given nonnegative initial conditions, the solution of (3.1) with sufficient regularity remains nonnegative. In this section, we demonstrate that, under certain assumptions on the initial conditional, the solution given by Theorem 3.4 or Theorem 3.12 satisfies the hypotheses of Proposition 2.28.

Proposition 3.18. *Consider system (3.1) with $\rho_0 \in W^{N,p_0}(\mathbb{R}^d)$, and $\nabla c_0 \in W^{N,q_0}(\mathbb{R}^d)$, where $N = \max\{2, 2\lceil \beta/2 \rceil - 1\}$. Moreover, assume that ρ_0 and c_0 are nonnegative. Then, the solution ρ and c, given by Theorem 3.4 and Remark 3.7 for $0 < t < T$ (or Theorem 3.12 and Remark 3.13 with $q_0 \neq \frac{\alpha}{\beta}\frac{d}{\alpha-1}$ for $\alpha \neq \beta$ and $0 < t < T = \infty$) are also nonnegative, as long as the solution is well-defined.*

Proof. From Theorem 3.4 (Theorem 3.12 with $T = \infty$) and Remark 3.7 (Remark 3.13), we know that the solutions of (3.1) are such that $\rho(\cdot,t) \in W^{N,p_s}(\mathbb{R}^d)$ and $\nabla c(\cdot,t) \in W^{N,q_s}(\mathbb{R}^d)$ for $0 < t < T$ and for some values of $1 \leq p_s < \infty$ and $1 \leq q_s < \infty$. Then, since $N = \max\{2, 2\lceil \beta/2 \rceil - 1\}$, we have $\rho(\cdot,t) \in W^{2,p}(\mathbb{R}^d)$. Moreover, since p satisfies (A2), $p > \frac{d}{\beta}$, and from Proposition 2.30 it follows that $\nabla c(\cdot,t) \in W^{1,\infty}(\mathbb{R}^d)$ for $0 < t < T$. Therefore, by applying Proposition 2.28 we ensure the nonnegativity of ρ and c for $0 \leq t < T$. □

3.5 Well-posedness in Weighted spaces

In this section, we establish the existence and uniqueness of solutions for system (3.1) in weighted spaces by applying the fixed point theorem. To begin, we consider the weighted L^∞ spaces introduced in Definition 2.12. For this, we prove the following proposition:

Proposition 3.19. *Let $d \geq 2$, $\alpha \in (1,2]$ and $\beta \in (1,d]$. Then, for every initial condition $\rho_0 \in L^\infty_{\alpha+d}(\mathbb{R}^d)$ and $\nabla c_0 \in L^\infty(\mathbb{R}^d)$, there exist $T = T\left(\|\rho_0\|_{L^\infty_{\alpha+d}}, \|\nabla c_0\|_{L^\infty}\right)$ and a unique local mild solution $(\rho, \nabla c)$ to system (3.1) in $[0,T]$, such that $\rho \in C\big([0,T], L^\infty_{\alpha+d}(\mathbb{R}^d)\big)$ and $\nabla c \in C\big([0,T], L^\infty(\mathbb{R}^d)\big)$. Moreover,*

- ***(i)*** *if $\nabla c_0 \in L^r(\mathbb{R}^d)$, then $\nabla c \in C\big([0,T], L^r(\mathbb{R}^d)\big)$, for $1 \leq r \leq \infty$;*
- ***(ii)*** *if $c_0 \in W^{1,r}(\mathbb{R}^d)$, then $c \in C\big([0,T], W^{1,r}(\mathbb{R}^d)\big)$, for $1 \leq r \leq \infty$;*
- ***(iii)*** *$\int_{\mathbb{R}^d} |x|^\vartheta \rho(x,t) dx < \infty$ for any $0 \leq \vartheta < \alpha$.*

Proof. We prove the existence and uniqueness of local solution to system (3.1) by employing the fixed point theorem through Corollary 2.7. In particular, we consider the Banach space $\mathbf{X} = C\big([0,T], L^\infty_{\alpha+d}(\mathbb{R}^d)\big)$ equipped with the usual norm, where T depends on $\|\rho_0\|_{L^\infty_{\alpha+d}}$ and $\|\nabla c_0\|_{L^\infty}$.

We then construct the unique mild solution, with u_1, $\mathcal{A}$, and $\mathcal{L}$ given by (2.16), (2.17) and (2.18) respectively. To show that the premises of Corollary 2.7 are satisfied, we use Lemmas 2.10, 2.14 and 2.22 and Remark 2.13 to obtain the following estimates for $T > 0$:

$$\begin{aligned}
\|u_1(\cdot,t)\|_{L^\infty_{\alpha+d}} &\le C(1+T)\|\rho_0\|_{L^\infty_{\alpha+d}},\\
\|\mathcal{A}(u,v)\|_{\mathrm{X}} &\le CT^{2-\frac{1}{\alpha}-\frac{1}{\beta}}\left(1+T\right)\|u\|_{\mathrm{X}}\|v\|_{\mathrm{X}},\\
\|\mathcal{L}(u)\|_{\mathrm{X}} &\le CT^{1-\frac{1}{\alpha}-\frac{1}{\beta}}\left(1+T\right)\|\nabla c_0\|_{L^\infty}\|u\|_{\mathrm{X}},
\end{aligned}$$

which follow from $\|u_1(\cdot,t)\|_{L^\infty_{\alpha+d}} = \|K^\alpha_t * \rho_0\|_{L^\infty_{\alpha+d}} \le C(1+t)\|\rho_0\|_{L^\infty_{\alpha+d}}$,

$$\begin{aligned}
&\|\mathcal{A}(u,v)\|_{L^\infty_{\alpha+d}}\\
&\le \int_0^t \left\|\nabla K^\alpha_{t-s} * \left(u(s)\int_0^s \nabla K^\beta_{s-w} * v(\cdot,w)\mathrm{d}w\right)\right\|_{L^\infty_{\alpha+d}} \mathrm{d}s\\
&\le C\int_0^t (t-s)^{-\frac{1}{\alpha}}\left(\left\|u(s)\int_0^s \nabla K^\beta_{s-w} * v(\cdot,w)\mathrm{d}w\right\|_{L^\infty_{\alpha+d}} + (t-s)\left\|u(s)\int_0^s \nabla K^\beta_{s-w} * v(\cdot,w)\mathrm{d}w\right\|_{L^1}\right)\mathrm{d}s\\
&\le C\int_0^t (t-s)^{-\frac{1}{\alpha}}\left\|\int_0^s \nabla K^\beta_{s-w} * v(\cdot,w)\mathrm{d}w\right\|_{L^\infty}\left(\|u(s)\|_{L^\infty_{\alpha+d}} + (t-s)\|u(s)\|_{L^1}\right)\mathrm{d}s\\
&\le C\int_0^t (t-s)^{-\frac{1}{\alpha}}(1+t-s)\left(\int_0^s (s-w)^{-\frac{1}{\beta}}\|v(w)\|_{L^\infty_{\alpha+d}}\mathrm{d}w\right)\mathrm{d}s\,\|u\|_{\mathrm{X}}\\
&\le C\left(t^{2-\frac{1}{\alpha}-\frac{1}{\beta}} + t^{3-\frac{1}{\alpha}-\frac{1}{\beta}}\right)\|u\|_{\mathrm{X}}\|v\|_{\mathrm{X}}\\
&\le Ct^{2-\frac{1}{\alpha}-\frac{1}{\beta}}(1+t)\|u\|_{\mathrm{X}}\|v\|_{\mathrm{X}}, \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
\|\mathcal{L}(u)\|_{L^\infty_{\alpha+d}} &\le \int_0^t \left\|\nabla K^\alpha_{t-s} * \left[u(s)e^{-\frac{\gamma}{\tau}s}K^\beta_{\frac{s}{\tau}} * \nabla c_0\right]\right\|_{L^\infty_{\alpha+d}}\mathrm{d}s\\
&\le C\int_0^t (t-s)^{-\frac{1}{\alpha}}\left(\left\|u(s)K^\beta_{\frac{s}{\tau}} * \nabla c_0\right\|_{L^\infty_{\alpha+d}} + (t-s)\left\|u(s)K^\beta_{\frac{s}{\tau}} * \nabla c_0\right\|_{L^1}\right)\mathrm{d}s\\
&\le C\int_0^t (t-s)^{-\frac{1}{\alpha}}\left(\|u(s)\|_{L^\infty_{\alpha+d}} + (t-s)\|u(s)\|_{L^1}\right)\left\|K^\beta_{\frac{s}{\tau}} * \nabla c_0\right\|_{L^\infty}\mathrm{d}s\\
&\le C\int_0^t (t-s)^{-\frac{1}{\alpha}}(1+(t-s))\|u(s)\|_{L^\infty_{\alpha+d}}s^{-\frac{1}{\beta}}\|\nabla c_0\|_{L^\infty}\mathrm{d}s\\
&\le C\left(\int_0^t (t-s)^{-\frac{1}{\alpha}}s^{-\frac{1}{\beta}}\mathrm{d}s + \int_0^t (t-s)^{1-\frac{1}{\alpha}}s^{-\frac{1}{\beta}}\mathrm{d}s\right)\|\nabla c_0\|_{L^\infty}\|u\|_{\mathrm{X}}\\
&\le C\left(t^{1-\frac{1}{\alpha}-\frac{1}{\beta}} + t^{2-\frac{1}{\alpha}-\frac{1}{\beta}}\right)\|\nabla c_0\|_{L^\infty}\|u\|_{\mathrm{X}}\\
&\le Ct^{1-\frac{1}{\alpha}-\frac{1}{\beta}}(1+t)\|\nabla c_0\|_{L^\infty}\|u\|_{\mathrm{X}}.
\end{aligned}$$

Then, it suffices to use the same argument from the proof of Theorem 3.4 to construct solutions to system (3.1) and obtain $\rho \in C\big([0,T], L^\infty_{\alpha+d}(\mathbb{R}^d)\big)$.

Proof of (i): Now, in order to prove the estimate of ∇c, we use the integral formulation (2.8) and Remark 2.13 and apply Lemma 2.10 to obtain

$$\begin{aligned}
\|\nabla c(\cdot,t)\|_{L^r} &\leq e^{-\frac{\gamma}{\tau}t} \left\| K^{\beta}_{\frac{t}{\tau}} * \nabla c_0 \right\|_{L^r} + \int_0^t \frac{1}{\tau} e^{\frac{\gamma}{\tau}(s-t)} \left\| \nabla K^{\beta}_{\frac{t-s}{\tau}} * \rho(s) \right\|_{L^r} \mathrm{d}s \\
&\leq C \|\nabla c_0\|_{L^r} + C \int_0^t (t-s)^{-\frac{1}{\beta}} \|\rho(s)\|_{L^r} \mathrm{d}s \\
&\leq C \|\nabla c_0\|_{L^r} + C t^{1-\frac{1}{\beta}} \|\rho\|_{\mathrm{X}},
\end{aligned}$$

which proves that $\nabla c \in C\big([0,T], L^r(\mathbb{R}^d)\big)$, for $1 \leq r \leq \infty$.

Proof of (ii): Also, from integral formulation for c, we obtain

$$\|c(\cdot,t)\|_{L^r} \leq C \|c_0\|_{L^r} + C \int_0^t \|\rho(s)\|_{L^r} \mathrm{d}s \leq C \|c_0\|_{L^r} + Ct \|\rho\|_{\mathrm{X}}.$$

Proof of (iii): To prove that $\int_{\mathbb{R}^d} |x|^{\vartheta} \rho(x,t)\mathrm{d}x < \infty$ for $0 \leq \vartheta < \alpha$, note that $|x|^{\vartheta} \leq (1+|x-y|)^{\vartheta} (1+|y|)^{\vartheta}$ and $(1+|x|)^{\vartheta} \leq 1 + \sum_{i=1}^d |x_i|^{\vartheta}$. Moreover, from Lemma 2.17, as $\vartheta < \alpha$, we have K^{α}_t and $x_i^{\vartheta} K^{\alpha}_t \in L^1(\mathbb{R}^d)$, for $1 \leq i \leq d$ and $t > 0$. Then, as $\rho_0 \in L^{\infty}_{\alpha+d}(\mathbb{R}^d)$ we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d} |x|^{\vartheta} (K^{\alpha}_t * \rho_0)(x)\, \mathrm{d}x &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x|^{\vartheta} K^{\alpha}_t(x-y) \rho_0(y)\, \mathrm{d}y\, \mathrm{d}x \\
&\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (1+|x-y|)^{\vartheta} (1+|y|)^{\vartheta} K^{\alpha}_t(x-y)\rho_0(y)\, \mathrm{d}y\, \mathrm{d}x \\
&\leq \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (1+|x-y|)^{\vartheta} K^{\alpha}_t(x-y) (1+|y|)^{-d-\alpha+\vartheta}\ \mathrm{d}y\, \mathrm{d}x \right) \|\rho_0\|_{L^{\infty}_{\alpha+d}} \\
&\leq \left(\left\| (1+|\cdot|)^{\vartheta} K^{\alpha}_t \right\|_{L^1} \left\| (1+|\cdot|)^{-d-\alpha+\vartheta} \right\|_{L^1} \right) \|\rho_0\|_{L^{\infty}_{\alpha+d}} \\
&= C \left\| (1+|\cdot|)^{\vartheta} K^{\alpha}_t \right\|_{L^1} \|\rho_0\|_{L^{\infty}_{\alpha+d}} \\
&\leq C \left\| \left(1 + \sum_{i=1}^d |x_i|^{\vartheta} \right) K^{\alpha}_t \right\|_{L^1} \|\rho_0\|_{L^{\infty}_{\alpha+d}} \\
&\leq C \left(\|K^{\alpha}_t\|_{L^1} + \sum_{i=1}^d \|x_i^{\vartheta} K^{\alpha}_t\|_{L^1} \right) \|\rho_0\|_{L^{\infty}_{\alpha+d}} \\
&\leq C \left(1 + t^{\frac{\vartheta}{\alpha}} \right) \|\rho_0\|_{L^{\infty}_{\alpha+d}},
\end{aligned} \tag{3.39}$$

where the last line is due to application of Lemma 2.17 as $\vartheta < \alpha$.

Moreover, from Lemma 2.17, as $\vartheta < \alpha + 1$, we see that ∇K^{α}_t and $x_i^{\vartheta} \nabla K^{\alpha}_t \in L^1(\mathbb{R}^d)$ for $1 \leq i \leq d$ and $t > 0$. Then, similarly to the calculation of (3.39), as $(\rho\nabla c)(\cdot,t) \in L^{\infty}_{\alpha+d}(\mathbb{R}^d)$ for $t \geq 0$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d} |x|^{\vartheta} \nabla K^{\alpha}_{t-s} * [\rho(s)\nabla c(s)]\mathrm{d}x &\leq C \left(\|\nabla K^{\alpha}_t\|_{L^1} + \sum_{i=1}^d \|x_i^{\vartheta} \nabla K^{\alpha}_t\|_{L^1} \right) \|\rho(s)\nabla c(s)\|_{L^{\infty}_{\alpha+d}} \\
&\leq C(t^{-\frac{1}{\alpha}} + t^{\frac{\vartheta-1}{\alpha}}) \|\rho(s)\|_{L^{\infty}_{\alpha+d}} \|\nabla c(s)\|_{L^{\infty}}.
\end{aligned}$$

Therefore, from the integral formulation (2.7) and prior estimates the result follows. □

Now consider the space $E_{\alpha+d}$ introduced in Definition 2.15, we establish the following proposition:

Proposition 3.20. *Let $d \geq 2$, $\alpha \in (1,2]$ and $\beta \in (1,d]$. Then, for every initial condition $\rho_0 \in E_{\alpha+d}(\mathbb{R}^d)$ and $\nabla c_0 \in L^\infty(\mathbb{R}^d)$, there exist $T = T\left(\|\rho_0\|_{E_{\alpha+d}}, \|\nabla c_0\|_{L^\infty}\right)$ and a unique local mild solution $(\rho, \nabla c)$ to system (3.1) in $[0,T]$, such that $\rho \in C\big([0,T], E_{\alpha+d}(\mathbb{R}^d)\big)$ and $\nabla c \in C\big([0,T], L^\infty(\mathbb{R}^d)\big)$. Moreover, if $\nabla c_0 \in L^r(\mathbb{R}^d)$, then $\nabla c \in C\big([0,T], L^r(\mathbb{R}^d)\big)$, for $1 \leq r \leq \infty$. Furthermore, if $c_0 \in W^{1,r}(\mathbb{R}^d)$, then $c \in C\big([0,T], W^{1,r}(\mathbb{R}^d)\big)$.*

Proof. The proof follows the same steps of Proposition 3.19. □

Remark 3.21. *Note that, from Remark 2.21, the solution provided by Proposition 3.20 belongs to the space M_ϑ, introduced in Definition 2.18.*

Chapter 4

Chemotaxis and Reactions

This chapter is part of the research internship conducted at Duke University under the supervision of Professor Dr. Alexander Kiselev. During this period, we focused on the analytical study of partial differential equations modeling chemotaxis and reactions. These mechanisms are fundamental biological processes and are often intricately interconnected, with chemotaxis playing a crucial role in maintaining and accelerating reactions. The research aimed to extend the investigation initiated in the work of Kiselev et al. [48] by examining the impact of chemotactic attraction on reproduction and other processes. For that, we considered a partial differential equation, with a single density function, that includes advection, chemotaxis, absorbing reaction, and diffusion, incorporating the fractional Laplacian Λ^{α}. Thus, we delve into the model describe by (2.2), focusing on the specific case where $f(\rho, \nabla\rho) = -u \cdot \nabla\rho - \epsilon\rho^{q}$, $\gamma = 0$ and $\beta = 2$. The following introduction provides a biological and mathematical interpretation of this model.

4.1 Introduction

Several variations of the Keller-Segel model have been proposed to explore chemotaxis in conjunction with other phenomena, such as biological reactions. In that context, Kiselev et al. [48] investigated the role of chemotaxis in enhancing biological reactions, focusing on coral broadcast spawning, a fertilization strategy adopted by various benthic invertebrates, such as sea urchins, anemones, and corals. In this process, males and females release sperm and egg gametes into the surrounding flow, and the chemotaxis appears as the eggs release a chemical that attracts sperm.

The mathematical model analyzed by Kiselev et al. [48] is a modification of the minimal model (see Section 1.2.1), including advection and absorbing reaction, in which the approximation to one equation was based on the assumption that the chemical diffusion is much faster than the diffusion of gamete densities. Another simplification in this model is the assumption that sperm and egg gametes have identical densities, leading to a single density function $\rho \geq 0$ as follows:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = \Delta \rho - \chi \nabla \cdot \left(\rho \nabla (-\Delta)^{-1} \rho \right) - \epsilon \rho^q, & x \in \mathbb{R}^d, \quad t > 0, \\ \rho(x,0) = \rho_0(x), & x \in \mathbb{R}^d, \quad d \geq 2, \end{cases} \tag{4.1}$$

where $u = u(x,t)$ is a given vector field modeling the ambient ocean flow (u is divergence-free, regular, and prescribed, independent of ρ), $\nabla (-\Delta)^{-1} \rho$ is given by (2.9), and the term $(-\epsilon \rho^q)$ models the reaction (fertilization), with the parameter ϵ regulating the strength of the fertilization process. They pointed out that the value of ϵ is small because an egg gets fertilized only if a sperm attaches to a certain limited area on its surface. The model does not track the reaction production – fertilized eggs.

We extend the investigation initiated by Kiselev et al. [48] by examining the impact of chemotactic attraction on reproduction and other processes in the context of anomalous diffusion of gamete densities. To this purpose, we consider a modified single partial differential equation modeling a fertilization process,

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = -\Lambda^\alpha \rho - \chi \nabla \cdot \left(\rho \nabla (-\Delta)^{-1} \rho \right) - \epsilon \rho^q, & \alpha \in (1,2] \\ \rho(x,0) = \rho_0(x), & x \in \mathbb{R}^d \quad d \geq 2. \end{cases} \tag{4.2}$$

As mentioned in Section 1.3, the inclusion of the fractional Laplacian, $\Lambda^\alpha = (-\Delta)^{\alpha/2}$, is motivated by experimental evidence supporting the efficacy of anomalous diffusion models, particularly in scenarios with sparse targets. The fractional Laplacian accommodates the nonlocal nature of superdiffusion processes, providing, in some cases, a more accurate representation than traditional diffusion models.

To study the chemotactic attraction effect on reproduction and other biological processes, we analyze the behavior of the total fraction of unfertilized eggs by time t given by equation (2.3). From Remark 2.1, we can see that, if $\rho \geq 0$, $m(t)$ is a monotone decreasing function, as (2.4) turns into

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\rho(\cdot,t)\|_{L^1} = -\epsilon \int_{\mathbb{R}^d} \rho^q \, \mathrm{d}x. \tag{4.3}$$

We seek to understand how fertilization can be efficient in chemotactic and chemotaxis-free scenarios. Note that high-efficiency fertilization corresponds to $m(t)$ becoming small over time, as almost all egg gametes are fertilized.

Furthermore, we prove the existence of solution for any initial condition sufficiently regular, emphasizing that the presence of an additional negative reaction term $-\epsilon\rho^q$ with $q > 2$ prevents the solution from losing regularity in finite time, as proved by Kiselev et al. [48] for model (4.1). It is important to note that the classical parabolic-elliptic Keller-Segel system (1.3) with $\gamma = 0$, for $d \geq 2$, does not follow the same dynamics. As mentioned in Section 1.2.2, this system exhibits critical behavior in $L^{d/2}(\mathbb{R}^d)$, i. e., a small initial condition in the $L^{d/2}(\mathbb{R}^d)$ space ensures global well-posedness, whereas a large initial condition leads to blow-up in finite time. In particular, for $d = 2$, solutions exist globally in time for $m_0 < 8\pi/\chi$, whereas blow-up occurs in finite time for $m_0 > 8\pi/\chi$ [18, 66].

This critical behavior also holds for the nonlocal parabolic-elliptic Keller-Segel system (1.6) with $\gamma = 0$ and $1 < \alpha < 2$, for which the existence of blowing-up solutions was proved (see [11] for $\beta \in (1, d]$, and [9, 55] for $\beta = 2$). In this case, Biler et al. [11] proved that the system exhibits critical behavior in $L^{d/(\alpha+\beta-2)}(\mathbb{R}^d)$ space.

4.2 Global Existence of smooth solutions

As in Chapter 3, we use the Duhamel's principle to write system (4.2) in its integral form. Specifically, we look at equation (2.7) with $f(\rho, \nabla\rho) = -u \cdot \nabla\rho - \epsilon\rho^q$, and equation (2.9) with $\gamma = 0$ and $\beta = 2$. This leads to the following formulation:

$$\rho(x,t) = K_t^\alpha * \rho_0(x) + \int_0^t K_{t-r}^\alpha * \left(-\nabla \cdot (u\rho) - \epsilon\rho^q + \chi\nabla \cdot \left(\rho\nabla\Delta^{-1}\rho\right)\right) \mathrm{d}r, \tag{4.4}$$

where $K_t^\alpha(x)$ is defined by (1.17). As highlighted in Definition 2.5, a solution to (4.4) is referred to as the mild solution to the system (4.2).

The approach employed in this section to establish the global existence of smooth solutions closely follows the methodology outlined by Kiselev et al. [48]. We first establish local well-posedness of a mild solution using the standard fixed-point procedure for the mapping (2.13), which takes the form

$$\mathcal{B}_t(\rho) \equiv \int_0^t K_{t-r}^\alpha * \left(-\nabla \cdot (u\rho) - \epsilon\rho^q + \chi\nabla \cdot \left(\rho\nabla\Delta^{-1}\rho\right)\right) \mathrm{d}r$$

in the Banach space $X^T_{s,\vartheta}(\mathbb{R}^d) \equiv C\big([0,T], K_{s,\vartheta}(\mathbb{R}^d)\big)$, where $T > 0$ is chosen sufficiently small and $K_{s,\vartheta}(\mathbb{R}^d)$ is the Banach space defined by the norm $\|\varphi\|_{K_{s,\vartheta}} = \|\varphi\|_{M_\vartheta} + \|\varphi\|_{H^s}$, with $s \geq 0$, $\alpha \in (1,2]$, $0 \leq \vartheta < \alpha$, $H^s(\mathbb{R}^d)$ the standard Sobolev space, and $M_\vartheta(\mathbb{R}^d)$ introduced in Definition 2.18.

Subsequently, we demonstrate control over the growth of solution norms, leading to the proof of global well-posedness.

Remark 4.1. *Throughout this chapter, the vector field $u = u(x,t)$, modeling the ambient ocean flow, is assumed to belong to $C^\infty\big([0,\infty) \times \mathbb{R}^d\big)$. By this, we mean that u is not only infinitely differentiable on $[0,\infty) \times \mathbb{R}^d$ but also that u and all its derivatives are uniformly bounded over $(x,t) \in \mathbb{R}^d \times [0,T]$, for every $T > 0$. This definition is tailored for our purposes and differs from the standard interpretation of a function on $C^\infty\big([0,\infty) \times \mathbb{R}^d\big)$. For clarity, we define the norm $\|u\|_{C^s}$ as*

$$\|u\|_{C^s} = \sum_{|\kappa| \leq s} \sup_{(x,t) \in \mathbb{R}^d \times [0,T]} |\partial^\kappa u(x,t)|,$$

where κ is a multi-index, with this definition holding for every $T > 0$.

Lemma 4.2. *Assume $u \in C^\infty\big([0,\infty) \times \mathbb{R}^d\big)$ satisfies $\nabla \cdot u = 0$. Let s and q be positive integers, where $s > d/2 + 1$, and let α and ϑ be such that $\alpha \in (1,2]$, and $0 \leq \vartheta < \alpha$. Then, for any $f, g \in X^T_{s,\vartheta}(\mathbb{R}^d)$, we have*

$$\|\mathcal{B}_T(f) - \mathcal{B}_T(g)\|_{X^T_{s,\vartheta}} \leq \Theta \|f - g\|_{X^T_{s,\vartheta}}, \tag{4.5}$$

where, for $T \leq 1$, we have

$$\Theta \leq C(d,q,\vartheta,\epsilon,\chi) \max_{0 \leq t \leq T} \left(\|u(\cdot,t)\|_{C^s} + \|f(\cdot,t)\|^{q-1}_{K_{s,\vartheta}} + \|g(\cdot,t)\|^{q-1}_{K_{s,\vartheta}} + \|f(\cdot,t)\|_{K_{s,\vartheta}} + \|g(\cdot,t)\|_{K_{s,\vartheta}} \right) T^{\frac{\alpha-1}{\alpha}}.$$

Proof. Consider

$$\mathcal{B}_t(f) - \mathcal{B}_t(g) = \int_0^t K^\alpha_{t-r} * \left(\nabla(u(f-g)) - \epsilon\,(f^q - g^q) + \chi \nabla \cdot \left(f \nabla \Delta^{-1} f - g \nabla \Delta^{-1} g \right) \right) \mathrm{d}r.$$

Using Lemmas 2.19, 2.11 and 2.23, and since $\alpha > 1$, we derive (4.5) as follows

$$\begin{aligned}
\|\mathcal{B}_t(f) - \mathcal{B}_t(g)\|_{K_{s,\vartheta}} &\leq C \int_0^t \Big[\left((t-r)^{-1/\alpha} + (t-r)^{(\vartheta-1)/\alpha} \right) \left(\|u\|_{C^s} + \|f\|_{K_{s,\vartheta}} + \|g\|_{K_{s,\vartheta}} \right) \\
&\qquad + \left(1 + (t-r)^{\vartheta/\alpha} \right) \left(\|f\|^{q-1}_{K_{s,\vartheta}} + \|g\|^{q-1}_{K_{s,\vartheta}} \right) \Big] \|f - g\|_{K_{s,\vartheta}} \mathrm{d}r \\
&\leq C \Bigg[\left(t^{\frac{\alpha-1}{\alpha}} + t^{\frac{\alpha-1+\vartheta}{\alpha}} \right) \max_{0 \leq r \leq t} \left(\|u\|_{C^s} + \|f\|_{K_{s,\vartheta}} + \|g\|_{K_{s,\vartheta}} \right) \\
&\qquad + \left(t + t^{\frac{\vartheta+\alpha}{\alpha}} \right) \max_{0 \leq r \leq t} \left(\|f\|^{q-1}_{K_{s,\vartheta}} + \|g\|^{q-1}_{K_{s,\vartheta}} \right) \Bigg] \max_{0 \leq r \leq t} \|f - g\|_{K_{s,\vartheta}},
\end{aligned}$$

and considering $T \leq 1$ we can neglect the higher powers of t, yielding the estimate for Θ. □

In a standard way, the existence of a local solution is implied by Lemma 4.2 through the contraction mapping principle (see Section 2.2.1). This leads to the following theorem.

Theorem 4.3. *Consider positive integers q and s such that $s > d/2+1$. Let $\alpha \in (1,2]$, $0 \leq \vartheta < \alpha$, and assume $u \in C^\infty\big([0,\infty)\times\mathbb{R}^d\big)$ with $\nabla\cdot u = 0$. Additionally, suppose $\rho_0 \in K_{s,\vartheta}(\mathbb{R}^d)$. Then, there exists $T = T\left(q,d,u,s,\epsilon,\chi,\alpha,\|\rho_0\|_{K_{s,\vartheta}}\right)$ such that equation (4.2) has a unique solution $\rho \in X^T_{s,\vartheta}$ satisfying $\rho(x,0) = \rho_0(x)$.*

Remark 4.4. *Higher regularity of the solution in space and time — implying, in particular, $\rho \in C\big((0,T],H^m(\mathbb{R}^d)\big)$ for every $m > 0$ — follows from Theorem 4.3 and parabolic regularity estimates applied iteratively [26, 37].*

Remark 4.5. *Note that, from Lemma 4.2, the value of Θ depends on the $H^s(\mathbb{R}^d)$ and $M_\vartheta(\mathbb{R}^d)$ norms of functions in $X^T_{\vartheta,s}(\mathbb{R}^d)$. Therefore, under the conditions of Theorem 4.3, if there is a control over the growth of the $H^s(\mathbb{R}^d)$ and $M_\vartheta(\mathbb{R}^d)$ norms of the solution, the time existence of the solution can be extended through iterative applications of local results. Hence, to prove global existence of smooth solutions, we can establish a global a priori estimate on $\|\rho(\cdot,t)\|_{H^s}$ and $\|\rho(\cdot,t)\|_{M_\vartheta}$, and then the local solution can be extended globally to $X^T_{\vartheta,s}(\mathbb{R}^d)$ with arbitrary T.*

To prove the bounds on $M_\vartheta(\mathbb{R}^d)$ and $H^s(\mathbb{R}^d)$ norms of the solution we first establish control of the $L^\infty(\mathbb{R}^d)$ norm.

Lemma 4.6. *Assume that ρ is the local solution guaranteed by Theorem 4.3, and $q > 2$. Then,*

$$\|\rho(\cdot,t)\|_{L^\infty} \leq N_0 \equiv \max\left((\chi/\epsilon)^{\frac{1}{q-2}}, \|\rho_0\|_{L^\infty}\right) \tag{4.6}$$

for all $0 \leq t \leq T$.

Proof. For the sake of completeness, we reproduce the proof provided in [48, Lemma 5.6], adapting it to address the fractional Laplacian. Assume, for contradiction, that the statement does not hold. Then, there exist constants $N_1 > N_0$ and $0 < t_1 \leq T$ such that $\|\rho(x,t_1)\|_{L^\infty} = N_1$ is achieved for the first time, i. e., $|\rho(x,t)| \leq N_1$ holds for every x and $0 \leq t \leq t_1$.

In this case, there exists a point x_0 such that $\rho(x_0,t_1) = N_1$. Indeed, consider a sequence x_k such that $\rho(x_k,t_1) \to N_1$ as $k \to \infty$. If x_k has finite accumulation points, one of them can be labeled x_0 and, by continuity, $\rho(x_0,t_1) = N_1$. Alternatively, if $x_k \to \infty$, we can consider a subsequence and assume the unit balls $B_1(x_k)$ around these points are disjoint. Then, using a version of Poincare inequality (e.g., [82]), we find

$$\|\rho - \bar{\rho}\|^2_{L^\infty(B_1(x_k))} \leq C\|\rho\|^2_{H^s(B_1(x_k))}$$

and, since $\sum_k \|\rho\|^2_{H^s(B_1(x_k))} \leq C(t_1) < \infty$, it follows that

$$\bar{\rho}_k \equiv \frac{1}{|B_1(x_k)|}\int_{B_1(x_k)} \rho dx \xrightarrow{k\to\infty} N_1.$$

However, this contradicts the integrability condition $\int_{\mathbb{R}^d} |\rho(x)|\left(1+|x|^\beta\right)dx \leq C(t_1)$.

Consequently, there exists x_0 such that $\rho(x_0, t_1) = N_1$, representing a maximum (the case of a minimum is treated similarly). Evaluating the fractional Laplacian at this point, we obtain

$$\Lambda^\alpha \rho(x_0, t_1) = c_{d,\alpha} P.V. \int_{\mathbb{R}^d} \frac{\rho(x_0, t_1) - \rho(y, t_1)}{|x_0 - y|^{\alpha+d}} \, \mathrm{d}y \geq 0 \ (\leq 0) \quad \text{for all } y \in \mathbb{R}^d,$$

which implies

$$\begin{aligned} \partial_t \rho(x_0, t)|_{t=t_1} =& (u \cdot \nabla)\rho(x_0, t_1) - \Lambda^\alpha \rho(x_0, t_1) + \chi \nabla \rho(x_0, t_1) \cdot \nabla \Delta^{-1} \rho(x_0, t_1) \\ &+ \chi \rho(x_0, t_1)^2 - \epsilon \rho(x_0, t_1)^q \leq \rho(x_0, t_1)^2 \left(\chi - \epsilon \rho(x_0, t_1)^{q-2}\right). \end{aligned}$$

Therefore, based on the assumption about N_1, we deduce that $\partial_t \rho(x_0, t_1) < 0$, contradicting the definition of t_1. □

Next, we prove an upper bound on the growth of the $M_\vartheta(\mathbb{R}^d)$ norm of the solution.

Lemma 4.7. *Assume that ρ is the local solution guaranteed by Theorem 4.3. Then, on the interval of existence, we have the following bound for the growth of the $M_\vartheta(\mathbb{R}^d)$ norm of the solution*

$$\begin{aligned} \|\rho(\cdot, t)\|_{M_\vartheta} \leq C \left(1 + t^{\frac{\vartheta}{\alpha}}\right) \|\rho_0\|_{M_\vartheta} \exp \Bigg(C \int_0^t \Big[\epsilon \left(1 + (t-r)^{\frac{\vartheta}{\alpha}}\right) \|\rho(\cdot, r)\|_{L^\infty}^{q-1} \\ + \left((t-r)^{-\frac{1}{\alpha}} + (t-r)^{\frac{\vartheta-1}{\alpha}}\right) (\|u(\cdot, r)\|_{C^1} + \|\rho(\cdot, r)\|_{L^\infty} + \|\rho(\cdot, r)\|_{L^1}) \Big] \, \mathrm{d}r \Bigg). \end{aligned} \tag{4.7}$$

Proof. Note that the following estimates hold:

$$\|u\rho\|_{M_\vartheta} \leq \|u\|_{C^1} \|\rho\|_{M_\vartheta} \qquad \text{and} \qquad \|\rho^q\|_{M_\vartheta} \leq C \|\rho\|_{L^\infty}^{q-1} \|\rho\|_{M_\vartheta}. \tag{4.8}$$

Moreover, from

$$\begin{aligned} \|\rho \nabla \Delta^{-1} \rho\|_{M_\vartheta} &= \int_{\mathbb{R}^d} \left[|\nabla \Delta^{-1} \rho| (|\rho(x)| + |\nabla \rho(x)|) + |\rho(x)|^2 \right] \left(1 + |x|^\vartheta\right) \, \mathrm{d}x \\ &\leq \left(\|\nabla \Delta^{-1} \rho\|_{L^\infty} + \|\rho\|_{L^\infty} \right) \|\rho\|_{M_\vartheta} \end{aligned}$$

and (2.38), we obtain

$$\|\rho \nabla \Delta^{-1} \rho\|_{M_\vartheta} \leq C (\|\rho\|_{L^\infty} + \|\rho\|_{L^1}) \|\rho\|_{M_\vartheta}. \tag{4.9}$$

Then, applying estimates (2.31) and (2.32) to (4.4), and using (4.8) and (4.9) , we obtain

$$\begin{aligned} \|\rho(\cdot, t)\|_{M_\vartheta} \leq C \left(1 + t^{\frac{\vartheta}{\alpha}}\right) \|\rho_0\|_{M_\vartheta} + C \int_0^t \Big[\epsilon \left(1 + (t-r)^{\frac{\vartheta}{\alpha}}\right) \|\rho(\cdot, r)\|_{L^\infty}^{q-1} \\ + \left((t-r)^{-\frac{1}{\alpha}} + (t-r)^{\frac{\vartheta-1}{\alpha}}\right) (\|u(\cdot, r)\|_{C^1} + \|\rho(\cdot, r)\|_{L^\infty} + \|\rho(\cdot, r)\|_{L^1}) \Big] \|\rho(\cdot, r)\|_{M_\vartheta} \mathrm{d}r. \end{aligned} \tag{4.10}$$

Now, applying Gronwall's inequality to (4.10), noticing that $(1 + t^{\vartheta/\alpha})$ is a non-decreasing function on t, we obtain (4.7). □

Next, we prove uniform in time bounds on the $H^s(\mathbb{R}^d)$ norm of the solution.

Lemma 4.8. *Let ρ be the local solution whose existence is guaranteed by Theorem 4.3 and suppose $\|\rho(\cdot,t)\|_{L^\infty}$ does not exceed N_0 for all $0 \le t \le T$. Then, for s even, we have*

$$\|\rho(\cdot,t)\|_{H^s} \le \max\left(\|\rho_0\|_{H^s}, C\left(u,d,q,s,\chi,\epsilon,N_0\right)\right).$$

Proof. Here we follow the same steps as in [48, Lemma 5.8]. Applying $\Delta^{s/2}$ to (4.2), multiplying by $\Delta^{s/2}\rho$, and integrating, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|^2_{\dot{H}^s} = \int_{\mathbb{R}^d} \Delta^{s/2}\left[(u\cdot\nabla)\rho\right]\left(\Delta^{s/2}\rho\right)\,\mathrm{d}x - \int_{\mathbb{R}^d}\left(\Delta^{(s+\alpha)/2}\rho\right)\left(\Delta^{s/2}\rho\right)\,\mathrm{d}x$$
$$- \epsilon\int_{\mathbb{R}^d}\left(\Delta^{s/2}\rho^q\right)\left(\Delta^{s/2}\rho\right)\,\mathrm{d}x + \chi\int_{\mathbb{R}^d}\left[\nabla\cdot\Delta^{s/2}\left(\rho\nabla\Delta^{-1}\rho\right)\right]\left(\Delta^{s/2}\rho\right)\,\mathrm{d}x. \quad (4.11)$$

Now, using the fact that $\nabla\cdot u = 0$, for the first integral on the right-hand side of (4.11), we obtain

$$\left|\int_{\mathbb{R}^d}\Delta^{s/2}\left[(u\cdot\nabla)\rho\right]\left(\Delta^{s/2}\rho\right)\,\mathrm{d}x\right| \le C\|u\|_{C^s}\|\rho\|^2_{\dot{H}^s}.$$

Next, for the second one, we see that

$$\int_{\mathbb{R}^d}\left(\Delta^{(s+\alpha)/2}\rho\right)\left(\Delta^{s/2}\rho\right)\,\mathrm{d}x = \|\rho\|^2_{\dot{H}^{s+\frac{\alpha}{2}}}.$$

Subsequently, the third integral can be written as a sum of a finite number of terms of the form $\int_{\mathbb{R}^d} D^s\rho\,\prod_{i=1}^q D^{s_i}\rho\;\;\mathrm{d}x$, $s_1+\cdots+s_q = s$, $s_i \ge 0$, where D^l denotes any partial derivative operator of the lth order. By Hölder's inequality, we have

$$\left|\int_{\mathbb{R}^d} D^s\rho\prod_{i=1}^q D^{s_i}\rho\,\mathrm{d}x\right| \le \|D^s\rho\|_{L^2}\prod_{i=1}^q\|D^{s_i}\rho\|_{L^p_i},$$

$\sum_{i=1}^q 1/p_i = 1/2$. Then, taking $p_i = 2s/s_i$ and using Gagliardo-Nirenberg inequality, we obtain

$$\|D^{s_i}\rho\|_{L^{2s/s_i}} \le C\|\rho\|^{1-\frac{s_i}{s}}_{L^\infty}\|D^s\rho\|^{\frac{s_i}{s}}_{L^2}, \quad (4.12)$$

and hence

$$\left|\int_{\mathbb{R}^d}\Delta^{s/2}\rho^q\Delta^{s/2}\rho\,\mathrm{d}x\right| \le C\|\rho\|^{q-1}_{L^\infty}\|\rho\|^2_{\dot{H}^s}.$$

Additionally, the fourth integral can be written as a sum of a finite number of terms of the form $\int_{\mathbb{R}^d} D^s\rho D^k\rho D^{s+2-k}\Delta^{-1}\rho\,\mathrm{d}x$, where $k = 0,\ldots,s$. The only term one gets from the direct differentiation that does not appear to be of this form is $\int_{\mathbb{R}^d}\Delta^{s/2}\rho\nabla\Delta^{s/2}\rho\nabla\Delta^{-1}\rho\,\mathrm{d}x$. However, we find that this term is equal to $-\frac{1}{2}\int_{\mathbb{R}^d}\rho\left|\Delta^{s/2}\rho\right|^2\,\mathrm{d}x$ through integration by parts. Now,

$$\left|\int_{\mathbb{R}^d} D^s\rho D^k\rho D^{s+2-k}\Delta^{-1}\rho\,\mathrm{d}x\right| \le C\|D^s\rho\|_{L^2}\|D^k\rho\|_{L^{p_1}}\|D^{s-k}\rho\|_{L^{p_2}},$$

$1/p_1 + 1/p_2 = 1/2$, $p_2 < \infty$, where we used boundedness of Riesz transforms on $L^{p_2}(\mathbb{R}^d)$, $p_2 < \infty$. Then, setting $p_1 = \frac{2s}{k}$, $p_2 = \frac{2s}{s-k}$, and using Gagliardo-Nirenberg inequality (4.12) with

$s_i = k, s-k$, we get

$$\left| \int_{\mathbb{R}^d} D^s \rho D^k \rho D^{s+2-k} \Delta^{-1} \rho \, \mathrm{d}x \right| \le C \|\rho\|_{L^\infty} \|\rho\|^2_{\dot{H}^s}.$$

Thus, putting all the estimates into (4.11), we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|^2_{\dot{H}^s} \le C\|\rho\|_{L^\infty}\|\rho\|^2_{\dot{H}^s} - \|\rho\|^2_{\dot{H}^{s+\frac{\alpha}{2}}}, \tag{4.13}$$

and from Hölder's inequality we obtain $\|\rho\|_{\dot{H}^s} \le C\|\rho\|^{\frac{s}{s+\alpha/2}}_{\dot{H}^{s+\frac{\alpha}{2}}}\|\rho\|^{\frac{\alpha/2}{s+\alpha/2}}_{L^2}$. Then, we can rewrite (4.13) as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|^2_{\dot{H}^s} \le C\|\rho\|_{L^\infty}\|\rho\|^{\frac{2s}{s+\alpha/2}}_{\dot{H}^{s+\frac{\alpha}{2}}}\|\rho\|^{\frac{\alpha}{s+\alpha/2}}_{L^2} - \|\rho\|^2_{\dot{H}^{s+\frac{\alpha}{2}}}. \tag{4.14}$$

Note that, as $\|\rho\|_{L^2} \le \|\rho\|^{1/2}_{L^1}\|\rho\|^{1/2}_{L^\infty} < \infty$, the differential inequality (4.14) implies the result of the lemma, since if $C\|\rho\|_{L^\infty}\|\rho\|^{\frac{2s}{s+\alpha/2}}_{\dot{H}^{s+\frac{\alpha}{2}}}\|\rho\|^{\frac{\alpha}{s+\alpha/2}}_{L^2} - \|\rho\|^2_{\dot{H}^{s+\frac{\alpha}{2}}} > 0$, then

$$\|\rho\|_{\dot{H}^s} \le \|\rho\|_{\dot{H}^{s+\frac{\alpha}{2}}} < \|\rho\|^{\frac{s-\alpha/2}{s}}_{L^\infty}\|\rho\|_{L^2} \le C\|\rho\|_{L^1}N_0^{\frac{3}{2}-\frac{\alpha}{2s}},$$

whereas if $C\|\rho\|_{L^\infty}\|\rho\|^{\frac{2s}{s+\alpha/2}}_{\dot{H}^{s+\frac{\alpha}{2}}}\|\rho\|^{\frac{\alpha}{s+\alpha/2}}_{L^2} - \|\rho\|^2_{\dot{H}^{s+\frac{\alpha}{2}}} \le 0$, then $\|\rho\|_{\dot{H}^s} \le \|\rho_0\|_{\dot{H}^s}$. □

Given the $M_\vartheta(\mathbb{R}^d)$ and $H^s(\mathbb{R}^d)$ norms bounds, we proved that the local solution can now be continued globally, establishing the following theorem:

Theorem 4.9 (Global existence of smooth solutions). *Let $q > 2$, $s > d/2+1$ be integers, $\alpha \in (1,2]$, and ϑ satisfy $0 \le \vartheta < \alpha$. Assume $u \in C^\infty\big([0,\infty)\times\mathbb{R}^d\big)$ is divergence free, and $\rho_0 \in K_{s,\vartheta}(\mathbb{R}^d)$. Then, there exists a unique solution ρ to equation (4.2) in $C\left([0,\infty), K_{s,\vartheta}(\mathbb{R}^d)\right) \cap C^\infty\left((0,\infty)\times\mathbb{R}^d\right)$.*

4.2.1 Nonnegative Solution

Since ρ in equation (4.2) describes the densities of sperm and egg gametes, given a nonnegative initial condition, the biological interest lies in nonnegative solutions. In Proposition 2.29, we prove for a more general system, equation (2.2), with nonnegative initial conditions $\rho_0 \ge 0$, the solution ρ remains nonnegative, provided they have sufficient regularity. Note that the solutions guaranteed by Theorem 4.3 or Theorem 4.9 possess the required regularity. Therefore, we can restate Proposition 2.29 for system (4.2) as follows:

Theorem 4.10. *Let $q > 2$, $s > d/2+1$ be integers, $\alpha \in (1,2]$, and ϑ satisfy $0 \le \vartheta < \alpha$. Suppose that $u \in C^\infty\big([0,\infty)\times\mathbb{R}^d\big)$ is divergence free and $\rho_0 \in K_{s,\vartheta}(\mathbb{R}^d)$ is nonnegative. Then, the solution ρ guaranteed by Theorem 4.3 or Theorem 4.9 remains nonnegative for all x and t.*

Proof. Consider Definition A.6, and note that by replacing ∇v in (2.49) with the divergence-free vector field u we obtain that

$$\int_{\mathbb{R}^d} \rho_- |\rho_-|^{p-2} u \cdot \nabla \rho \mathrm{d}x = 0.$$

Moreover, for $q \geq 1$ integer, we obtain

$$\begin{cases} 0 \leq -\int_{\mathbb{R}^d} \rho_- |\rho_-|^{p-2}\, \rho^q \,\mathrm{d}x \leq \|\rho(\cdot,t)\|_{L^\infty}^{q-1}\, \|\rho_-(\cdot,t)\|_{L^p}^p & \text{if } q \text{ is even,} \\ -\int_{\mathbb{R}^d} \rho_- |\rho_-|^{p-2} \rho^q \,\mathrm{d}x \leq 0 & \text{if } q \text{ is odd.} \end{cases} \tag{4.15}$$

Therefore, by setting $p = 2$ the results follows from Proposition 2.29 and Proposition 2.31 (or Remark 2.32). □

4.3 Reaction efficiency

In this section, we aim to analyze the dynamics of the total fraction of unfertilized eggs, denoted as $m(t)$ and given by (2.3), in both chemotactic and chemotaxis-free scenarios. Our investigation delves into understanding the impact of chemotaxis on reaction efficiency by comparing these specific cases.

It is important to emphasize that, for a nonnegative initial condition, the quantity $m(t)$ corresponds to the $L^1(\mathbb{R}^d)$ norm of ρ (Theorem 4.10). We also highlight that $m(t)$ exhibits a monotone decreasing behavior. Indeed, by applying Proposition 2.26 we can provide a rigorous justification of this observation.

Proposition 4.11. *Consider the nonnegative initial condition $\rho_0 \geq 0$, and let ρ be the solution provided by Theorem 4.3 or Theorem 4.9. Then, the total fraction of unfertilized eggs by time t, $m(t)$, exhibits a monotone decreasing behavior.*

Proof. From direct application of Proposition 2.26 we need to prove that $\rho^q(\cdot,t) \in L^1(\mathbb{R}^d)$ and $v = -u\rho + \chi \rho \nabla \Delta^{-1} \rho \in L^1(\mathbb{R}^d)$. Note that, $\|\rho^q(\cdot,t)\|_{L^1} \leq \|\rho(\cdot,t)\|_{L^\infty}^{q-1} \|\rho(\cdot,t)\|_{L^1}$ and $\|v\|_{L^1} \leq \|u\|_{C^1} \|\rho\|_{L^1} + \|\nabla \Delta^{-1} \rho\|_{L^\infty} \|\rho\|_{L^1} \leq C\,(\|u\|_{C^1} + \|\rho\|_{L^\infty} + \|\rho\|_{L^1})\, \|\rho\|_{L^1}$.

Therefore, we establish that

$$\int_{\mathbb{R}^d} \rho(x,t)\, \mathrm{d}x = \int_{\mathbb{R}^d} \rho_0(x)\, \mathrm{d}x - \epsilon \int_0^t \int_{\mathbb{R}^d} \rho^q(x,s)\, \mathrm{d}x\mathrm{d}s$$

for every $t \geq 0$ where the solution exists. Equivalently, we can write equation (4.3). □

The following lemma provides an inequality that will be important in establishing a lower bound for the $L^1(\mathbb{R}^d)$ norm of ρ in both chemotactic and chemotaxis-free scenarios.

Lemma 4.12. *[29, 39] Let $\alpha \in [0,2]$ and $f, \Lambda^\alpha f \in L^p(\mathbb{R}^d)$. Then, for any $p \geq 2$, it is true that*

$$\int_{\mathbb{R}^d} |f|^{p-2} f \Lambda^\alpha f \, \mathrm{d}x \geq \frac{2}{p} \int_{\mathbb{R}^d} \left(\Lambda^{\frac{\alpha}{2}} |f|^{\frac{p}{2}} \right)^2 \mathrm{d}x. \tag{4.16}$$

4.3.1 Reaction in a Chemotaxis-Free Environment

Here, we establish that, in the chemotaxis-free scenario, there exists a constant C_0 depending on ϵ, q, d, α, and ρ_0 and independent of u, such that $m(t) \geq C_0$ for all $t \geq 0$. Additionally, when ρ_0, u and q are fixed, the quantity $m(t)$ converges to m_0 as $\epsilon \to 0$. To demonstrate this, consider equation (4.2) with $\chi = 0$:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = -\Lambda^\alpha \rho - \epsilon \rho^q, & \alpha \in (1,2] \\ \rho(x,0) = \rho_0(x), \quad x \in \mathbb{R}^d & d \geq 2. \end{cases} \tag{4.17}$$

Note that, by the comparison principle, $\rho \leq b$, where

$$\begin{cases} \partial_t b + u \cdot \nabla b = -\Lambda^\alpha b, & \alpha \in (1,2] \\ b(x,0) = \rho_0(x), \quad x \in \mathbb{R}^d & d \geq 2. \end{cases} \tag{4.18}$$

Also, since $\rho \geq 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\rho(\cdot,t)\|_{L^1} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \rho(x,t) \, \mathrm{d}x = -\epsilon \int_{\mathbb{R}^d} \rho^q(x,t) \, \mathrm{d}x \geq -\epsilon \int_{\mathbb{R}^d} b^q(x,t) \, \mathrm{d}x.$$

The behavior of the $L^q(\mathbb{R}^d)$ norm of b can be used for estimating the decay of the $L^1(\mathbb{R}^d)$ norm of ρ.

Lemma 4.13. *There exists $C = C(d,\alpha)$ (independent of the flow u) such that*

$$\|b(\cdot,t)\|_{L^2} \leq \min\left(\|b_0\|_{L^2}, Ct^{-d/2\alpha} \|b_0\|_{L^1} \right), \quad \|b(\cdot,t)\|_{L^\infty} \leq \min\left(\|b_0\|_{L^\infty}, Ct^{-d/\alpha} \|b_0\|_{L^1} \right). \tag{4.19}$$

Proof. Consider the Riesz potential $I_v = (-\Delta)^{-v/2}$ on $\mathbb{R}^d$ and let r be defined as $\frac{1}{r} = \frac{1}{p} - \frac{v}{d}$, where p and r are related by the Hardy-Littlewood-Sobolev fractional integration theorem. This theorem, applicable for $0 < v < d$ and $1 < p < r < \infty$, establishes the existence of a constant C (dependent solely on p) such that $\|I_v f\|_{L^r} \leq C \|f\|_{L^p}$ is satisfied for suitable functions. Setting $v = \alpha/2$, considering $f = \Lambda^{\frac{\alpha}{2}} b$, and choosing $p = 2$, we derive the inequality $\|b\|_{L^{\frac{2d}{d-\alpha}}} \leq C \|\Lambda^{\frac{\alpha}{2}} b\|_{L^2}$ for $\alpha \neq d$. Moreover, by interpolation inequality for $L^p(\mathbb{R}^d)$ norms, we see that $\|b\|_{L^2}^{1+\frac{\alpha}{d}} \leq \|b\|_{L^1}^{\frac{\alpha}{d}} \|b\|_{L^{\frac{2d}{d-\alpha}}}$. Therefore,

$$\|b\|_{L^2}^{1+\frac{\alpha}{d}} \leq C \|b\|_{L^1}^{\alpha/d} \|\Lambda^{\frac{\alpha}{2}} b\|_{L^2}. \tag{4.20}$$

For $\alpha = d = 2$, Nash's inequality yields (4.20) directly: $\|b\|_{L^2}^{1+\frac{2}{d}} \le C\|b\|_{L^1}^{2/d}\|\nabla b\|_{L^2}$. Then, from Plancherel theorem and (4.20), we have, for a constant C,

$$\int_{\mathbb{R}^d} b\Lambda^\alpha b\,\mathrm{d}x = \|\Lambda^{\frac{\alpha}{2}} b\|_{L^2}^2 \ge C\frac{\|b\|_{L^2}^{2+\frac{2\alpha}{d}}}{\|b\|_{L^1}^{2\frac{\alpha}{d}}}. \tag{4.21}$$

Now, we proceed by multiplying equation (4.18) by the variable b, integrating the resulting equation, and using the incompressibility of the velocity field u and inequality (4.21) to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|b\|_{L^2}^2 \le -\|\Lambda^{\frac{\alpha}{2}} b\|_{L^2}^2 \le -C\frac{\|b\|_{L^2}^{2+\frac{2\alpha}{d}}}{\|b\|_{L^1}^{\frac{2\alpha}{d}}} = -C\frac{\|b\|_{L^2}^{2+\frac{2\alpha}{d}}}{\|b_0\|_{L^1}^{\frac{2\alpha}{d}}},$$

with the final step obtained from the conservation of the $L^1(\mathbb{R}^d)$ norm of b. Now, setting $z(t) = \|b(\cdot,t)\|_{L^2}^2$, we have

$$z'(t) \le -Cz(t)^{1+\frac{\alpha}{d}}\|b_0\|_{L^1}^{-\frac{2\alpha}{d}},$$

and solving this differential inequality, we find that

$$z(t) \le \left(\frac{\alpha C t}{d\|b_0\|_{L^1}^{2\alpha/d}} + \frac{1}{\|b_0\|_{L^2}^{2\alpha/d}}\right)^{-d/\alpha},$$

implying the first inequality in (4.19).

The second inequality in equation (4.19) emerges via a duality argument using the incompressibility of the velocity field u. This approach involves considering $\theta(x,s)$, which is a solution to

$$\partial_s\theta + u(x,t-s)\cdot\nabla\theta = -\Lambda^\alpha\theta, \quad \theta(x,0) = \theta_0(x) \in \mathcal{S}(\mathbb{R}^d).$$

Note that $\theta(x,s)$ also satisfies the first estimate in (4.19), and we have

$$\frac{d}{ds}\int_{\mathbb{R}^d} b(x,s)\theta(x,t-s)\,\mathrm{d}x = 0, \tag{4.22}$$

which stems from the self-adjoint property of Λ^α leading to the equivalence of integrals

$$\int_{\mathbb{R}^d} \Lambda^\alpha b(x,s)\theta(x,t-s)\,\mathrm{d}x = \int_{\mathbb{R}^d} b(x,s)\Lambda^\alpha\theta(x,t-s)\,\mathrm{d}x.$$

Then, considering first $s = t$ and then $s = t/2$ on (4.22), we obtain

$$\begin{aligned}
\|b(\cdot,t)\|_{L^\infty} &= \sup_{\|\theta_0\|_{L^1}=1}\left|\int_{\mathbb{R}^d} b(x,t)\theta_0(x)\,\mathrm{d}x\right| \\
&= \sup_{\|\theta_0\|_{L^1}=1}\left|\int_{\mathbb{R}^d} b(x,t/2)\theta(x,t/2)\,\mathrm{d}x\right| \\
&\le \sup_{\|\theta_0\|_{L^1}=1} \|b(\cdot,t/2)\|_{L^2}\|\theta(\cdot,t/2)\|_{L^2} \\
&\le \sup_{\|\theta_0\|_{L^1}=1} C(t/2)^{-d/2\alpha}\|b_0\|_{L^1}(t/2)^{-d/2\alpha}\|\theta_0\|_{L^1} \\
&\le C(d,\alpha)\, t^{-d/\alpha}\|b_0\|_{L^1}.
\end{aligned}$$

Here we used the first inequality in (4.19) and adjusted $C(d,\alpha)$. Finally, from the weak parabolic maximum principle, we have $\|b(\cdot,t)\|_{L^\infty} \le \|b_0\|_{L^\infty}$ [37, 72, 76]. □

Lemma 4.14. *Assume that ρ solves (4.17) with a smooth, bounded, and divergence-free vector u, and $\rho_0 \in \mathcal{S}(\mathbb{R}^d)$. Then, for every $t > 0$, we have*

$$\frac{\|\rho(\cdot,t)\|_{L^p}}{\|\rho(\cdot,t)\|_{L^1}} \le \frac{\|\rho_0\|_{L^p}}{\|\rho_0\|_{L^1}} \quad \textit{for all} \quad 1 \le p \le \infty.$$

Proof. Here, for the sake of completeness, we provide a proof similar to the one presented by Kiselev et al. [48], but considering the Fractional Laplacian. To start, note that for $p = 1$ the result is immediate. Moving on to the case of $1 < p < \infty$, we use (4.3) to find

$$\|\rho(\cdot,t)\|_{L^1}^2 \frac{\mathrm{d}}{\mathrm{d}t}\frac{\|\rho(\cdot,t)\|_{L^p}}{\|\rho(\cdot,t)\|_{L^1}} = \|\rho(\cdot,t)\|_{L^1}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho(\cdot,t)\|_{L^p} + \epsilon\|\rho(\cdot,t)\|_{L^q}^q\|\rho(\cdot,t)\|_{L^p},$$

and from Lemma 4.12, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho(\cdot,t)\|_{L^p} &= \|\rho(\cdot,t)\|_{L^p}^{1-p}\int_{\mathbb{R}^d}\rho^{p-1}\left(-u\cdot\nabla\rho - \Lambda^\alpha\rho - \epsilon\rho^q\right)\,\mathrm{d}x \\ &\le \|\rho(\cdot,t)\|_{L^p}^{1-p}\left(-\frac{2}{p}\|\Lambda^{\alpha/2}\rho^{p/2}\|_{L^2}^2 - \epsilon\int_{\mathbb{R}^d}\rho^{q+p-1}\,\mathrm{d}x\right).\end{aligned} \tag{4.23}$$

Therefore,

$$\begin{aligned}\|\rho(\cdot,t)\|_{L^1}^2\frac{\mathrm{d}}{\mathrm{d}t}\frac{\|\rho(\cdot,t)\|_{L^p}}{\|\rho(\cdot,t)\|_{L^1}} \le \|\rho(\cdot,t)\|_{L^p}^{1-p}\Bigg[&-\frac{2}{p}\|\rho(\cdot,t)\|_{L^1}\|\Lambda^{\alpha/2}\rho^{p/2}\|_{L^2}^2 \\ &+ \epsilon\left(\|\rho(\cdot,t)\|_{L^q}^q\|\rho(\cdot,t)\|_{L^p}^p - \|\rho(\cdot,t)\|_{L^1}\int_{\mathbb{R}^d}\rho^{q+p-1}\,\mathrm{d}x\right)\Bigg].\end{aligned} \tag{4.24}$$

Now, by applying Hölder's inequality, we can observe that the expression

$$\int_{\mathbb{R}^d}\rho^q\,\mathrm{d}x\int_{\mathbb{R}^d}\rho^p\,\mathrm{d}x - \int_{\mathbb{R}^d}\rho\,\mathrm{d}x\int_{\mathbb{R}^d}\rho^{q+p-1}\,\mathrm{d}x$$

is less than or equal to zero. The case $p = \infty$ follows from a limiting procedure, since $\rho(\cdot,t) \in \mathcal{S}(\mathbb{R}^d)$ for all t. □

Theorem 4.15. *Let ρ solve (4.17) with a divergence-free vector $u \in C^\infty\big([0,\infty)\times\mathbb{R}^d\big)$ and initial data $\rho_0 \ge 0 \in \mathcal{S}(\mathbb{R}^d)$. Assume that $qd > d + \alpha$ and the chemotaxis is absent: $\chi = 0$. Then there exists a constant C_0 depending only on ϵ, q, d, α, and ρ_0 but not on u such that $m(t) \ge C_0$ for all $t \ge 0$. Moreover, $C_0 \to m_0$ as $\epsilon \to 0$ while ρ_0, u and q are fixed.*

Proof. As in [48], the idea here is to show that if the $L^1(\mathbb{R}^d)$ norm of ρ at some time t_0 is sufficiently small then, for all times $t > t_0$, the $L^1(\mathbb{R}^d)$ norm of ρ cannot drop below $\|\rho(\cdot,t_0)\|_{L^1}/2$. This shows that ρ cannot tend to zero as $t \to +\infty$. For this, recall that, for every t,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\rho(x,t)\,\mathrm{d}x = -\epsilon\int_{\mathbb{R}^d}\rho(x,t)^q\,\mathrm{d}x \ge -\epsilon\int_{\mathbb{R}^d}b(x,t)^q\,\mathrm{d}x,$$

where b is given by (4.18). From Lemma 4.13 and Hölder's inequality,

$$\int_{\mathbb{R}^d} b(x,t)^q \, dx \le C \min\left(\|\rho_0\|_{L^\infty}^{q-2} \|\rho_0\|_{L^2}^2, t^{-\frac{d(q-1)}{\alpha}} \|\rho_0\|_{L^1}^q \right) \le C \min\left(\|\rho_0\|_{L^\infty}^{q-1} \|\rho_0\|_{L^1}, t^{-\frac{d(q-1)}{\alpha}} \|\rho_0\|_{L^1}^q \right).$$

Thus, for every $\tau > 0$,

$$\begin{aligned} \int_{t_0}^{\infty} \int_{\mathbb{R}^d} b(x,t)^q \, dxdt &= \int_{t_0}^{t_0+\tau} \int_{\mathbb{R}^d} b(x,t)^q \, dxdt + \int_{t_0+\tau}^{\infty} \int_{\mathbb{R}^d} b(x,t)^q \, dxdt \\ &\le C(d,\alpha) \left(\|\rho(\cdot,t_0)\|_{L^\infty}^{q-1} \|\rho(\cdot,t_0)\|_{L^1} \tau + \|\rho(\cdot,t_0)\|_{L^1}^q \int_{t_0+\tau}^{\infty} (t-t_0)^{-\frac{d(q-1)}{\alpha}} \, dt \right), \end{aligned}$$

and, as by assumption $qd > d + \alpha$, we have

$$\int_{t_0}^{\infty} \int_{\mathbb{R}^d} b(x,t)^q \, dxdt \le C(d,\alpha,q) \left(\|\rho(\cdot,t_0)\|_{L^\infty}^{q-1} \|\rho(\cdot,t_0)\|_{L^1} \tau + \|\rho(\cdot,t_0)\|_{L^1}^q \tau^{\frac{d+\alpha-qd}{\alpha}} \right). \tag{4.25}$$

Assume, on the contrary, that the $L^1(\mathbb{R}^d)$ norm of ρ does go to zero for some u and ρ_0. Then, consider some time $t_0 > 0$ when $\|\rho(\cdot,t_0)\|_{L^1}$ is sufficiently small. By Lemma 4.14 and (4.25), we see that further decrease of the $L^1(\mathbb{R}^d)$ norm from that level is bounded as

$$\|\rho(\cdot,t_0)\|_{L^1} - \|\rho(\cdot,t)\|_{L^1} \le C(d,\alpha,q)\epsilon \left(\frac{\|\rho_0\|_{L^\infty}^{q-1}}{\|\rho_0\|_{L^1}^{q-1}} \|\rho(\cdot,t_0)\|_{L^1}^q \tau + \|\rho(\cdot,t_0)\|_{L^1}^q \tau^{\frac{d+\alpha-qd}{\alpha}} \right) \tag{4.26}$$

for all $t > t_0$, $\tau > 0$. Choosing τ to minimize expression (4.26), for every $t > t_0$, we find that

$$\|\rho(\cdot,t_0)\|_{L^1} - \|\rho(\cdot,t)\|_{L^1} \le C(d,\alpha,q)\epsilon \|\rho(\cdot,t_0)\|_{L^1}^q \left(\frac{\|\rho_0\|_{L^\infty}}{\|\rho_0\|_{L^1}} \right)^{\frac{qd-d-\alpha}{d}}.$$

If $\|\rho(\cdot,t)\|_{L^1} \to 0$ as $t \to +\infty$, we may choose t_0 so that

$$C(d,\alpha,q)\epsilon \|\rho(\cdot,t_0)\|_{L^1}^{q-1} \left(\frac{\|\rho_0\|_{L^\infty}}{\|\rho_0\|_{L^1}} \right)^{\frac{qd-d-\alpha}{d}} \le \frac{1}{2}. \tag{4.27}$$

Then we get a contradiction to the assumption that $\|\rho(\cdot,t)\|_{L^1} \to 0$ as $t \to +\infty$, as

$$\|\rho(\cdot,t)\|_{L^1} \ge \frac{1}{2} \|\rho(\cdot,t_0)\|_{L^1} \ge C_0\,(q,d,\epsilon,\alpha,\rho_0)$$

for every $t > t_0$, where C_0 in the statement of the theorem can be defined as

$$C_0\,(q,d,\epsilon,\alpha,\rho_0) \equiv \min\left(\frac{1}{2}\|\rho_0\|_{L^1}, \frac{1}{2^{\frac{q}{q-1}} \epsilon^{\frac{1}{q-1}} C(q,d,\alpha)^{\frac{1}{q-1}}} \left(\frac{\|\rho_0\|_{L^1}}{\|\rho_0\|_{L^\infty}} \right)^{1-\frac{\alpha}{d(q-1)}} \right).$$

Note that, if $\epsilon \to 0$ while ρ_0, u and q are fixed, we can replace condition $\le \frac{1}{2}$ in (4.27) with $\le \kappa$, where κ can be taken as small as desired, proving the last statement of the theorem. $\square$

4.3.2 Reaction in a Chemotactic Environment

In the chemotactic environment, we prove that the large time limit of the $L^1(\mathbb{R}^d)$ norm of ρ tends to zero as chemotaxis coupling increases, with an upper bound independent of ϵ (Theorem 4.22). However, we establish lower bounds for the $L^1(\mathbb{R}^d)$ norm of the solution, showing that, for each fixed coupling, $\|\rho(\cdot,t)\|_{L^1}$ does not go to zero as $t\to\infty$, as outlined next.

4.3.2.1 Lower bound for the total fraction of unfertilized eggs

We proceed by setting a lower bound for the $L^1(\mathbb{R}^d)$ norm of ρ.

Theorem 4.16. *Let $d=2$, $q,s>2$ be integers, $\alpha\in(1,2)$, and ϑ satisfy $0\le\vartheta<\alpha$. Suppose that $u\in C^\infty\big([0,\infty)\times\mathbb{R}^d\big)$ is divergence free, and ρ solves (4.2) with $\rho_0\in K_{s,\vartheta}(\mathbb{R}^d)$ and $\rho_0\ge 0$. Then,* $\lim_{t\to\infty}\|\rho(\cdot,t)\|_{L^1}>0$.

Proof. In order to establish lower bounds on the $L^1(\mathbb{R}^2)$ norm of the solution ρ, let us deduce estimates on $\|\rho(\cdot,t)\|_{L^q}$. By multiplying (4.2) by ρ^{q-1} and integrating, we obtain

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\rho^q\mathrm{d}x=-\int_{\mathbb{R}^2}\rho^{q-1}\Lambda^\alpha\rho\mathrm{d}x+\chi\int_{\mathbb{R}^2}\rho^{q-1}\nabla\cdot\left(\rho\nabla\Delta^{-1}\rho\right)\mathrm{d}x-\epsilon\int_{\mathbb{R}^2}\rho^{2q-1}\mathrm{d}x. \tag{4.28}$$

Note that, from Lemma 4.12, we have

$$-\int_{\mathbb{R}^2}\rho^{q-1}\Lambda^\alpha\rho\,\mathrm{d}x\le-\frac{2}{q}\|\Lambda^{\alpha/2}\rho^{q/2}\|_{L^2}^2, \tag{4.29}$$

and using integration by parts, we find

$$\int_{\mathbb{R}^2}\rho^{q-1}\nabla\cdot\left(\rho\nabla\Delta^{-1}\rho\right)\,\mathrm{d}x=-(q-1)\int_{\mathbb{R}^2}\rho^{q-1}\nabla\rho\cdot\nabla\Delta^{-1}\rho\,\mathrm{d}x=\frac{q-1}{q}\int_{\mathbb{R}^2}\rho^{q+1}\,\mathrm{d}x. \tag{4.30}$$

Furthermore, from interpolation inequality for $L^p(\mathbb{R}^2)$ norms, we see that

$$\int_{\mathbb{R}^2}\rho^{q+1}\,\mathrm{d}x=\|\rho^{q/2}\|_{L^{2+\frac{2}{q}}}^{\frac{2(q+1)}{q}}\le C(q,\alpha)\|\rho^{q/2}\|_{L^{\frac{4}{2-\alpha}}}^{\frac{4q}{2q-2+\alpha}}\|\rho^{q/2}\|_{L^{\frac{2}{q}}}^{\frac{2}{q}\left(\frac{\alpha q-2+\alpha}{2q-2+\alpha}\right)}. \tag{4.31}$$

Now, employing Hardy-Littlewood-Sobolev fractional integration theorem, we can derive that

$$\|\rho^{q/2}\|_{L^{\frac{4}{2-\alpha}}}\le C\|\Lambda^{\frac{\alpha}{2}}\rho^{q/2}\|_{L^2}. \tag{4.32}$$

Moreover, by standard interpolation inequality for $L^p(\mathbb{R}^2)$ norms, we obtain

$$\|\rho^{q/2}\|_{L^{\frac{4}{2-\alpha}}}^{\frac{2}{q}}=\|\rho\|_{L^{\frac{2q}{2-\alpha}}}\le\|\rho\|_{L^1}^{\frac{2-\alpha}{2q}}\|\rho\|_{L^\infty}^{\frac{2q-2+\alpha}{2q}}. \tag{4.33}$$

Then, from (4.32) and (4.33), we establish that

$$\|\rho^{q/2}\|_{L^{\frac{4}{2-\alpha}}}^{\frac{4q}{2q-2+\alpha}}=\|\rho^{q/2}\|_{L^{\frac{4}{2-\alpha}}}^2\|\rho^{q/2}\|_{L^{\frac{4}{2-\alpha}}}^{\frac{2(2-\alpha)}{2q-2+\alpha}}\le C\|\Lambda^{\frac{\alpha}{2}}\rho^{q/2}\|_{L^2}^2\|\rho\|_{L^1}^{\frac{(2-\alpha)^2}{2(2q-2+\alpha)}}\|\rho\|_{L^\infty}^{\frac{2-\alpha}{2}}.$$

Applying this to (4.31), we can write

$$\int_{\mathbb{R}^2} \rho^{q+1}\,\mathrm{d}x \le C(q,\alpha)\|\Lambda^{\frac{\alpha}{2}}\rho^{q/2}\|_{L^2}^2 \|\rho\|_{L^1}^{\frac{(2-\alpha)^2}{2(2q-2+\alpha)}} \|\rho\|_{L^\infty}^{\frac{2-\alpha}{2}} \|\rho^{q/2}\|_{L^{\frac{2}{q}}}^{\frac{2}{q}\left(\frac{\alpha q-2+\alpha}{2q-2+\alpha}\right)}$$
$$\le C(q,\alpha)\|\Lambda^{\frac{\alpha}{2}}\rho^{q/2}\|_{L^2}^2 \|\rho\|_{L^1}^{\frac{(2-\alpha)^2}{2(2q-2+\alpha)}+\frac{\alpha q-2+\alpha}{2q-2+\alpha}} \|\rho\|_{L^\infty}^{\frac{2-\alpha}{2}}.$$

Thus, from (4.29), (4.30), and the inequality above, we obtain

$$q\left(-\int_{\mathbb{R}^2}\rho^{q-1}\Lambda^\alpha\rho\mathrm{d}x + \chi\int_{\mathbb{R}^2}\rho^{q-1}\nabla\cdot\left(\rho\nabla\Delta^{-1}\rho\right)\mathrm{d}x\right) \le$$
$$\|\Lambda^{\frac{\alpha}{2}}\rho^{q/2}\|_{L^2}^2\left(C(q,\alpha)\chi\|\rho\|_{L^1}^{\frac{\alpha}{2}\left(1+\frac{2}{2q-2+\alpha}\right)}\|\rho\|_{L^\infty}^{\frac{2-\alpha}{2}}-2\right).$$

Therefore, we can rewrite (4.28) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\rho^q\mathrm{d}x \le N_0^{\frac{2-\alpha}{2}}\|\Lambda^{\frac{\alpha}{2}}\rho^{q/2}\|_{L^2}^2\left(C(q,\alpha)\chi\|\rho\|_{L^1}^{\frac{\alpha}{2}\left(1+\frac{2}{2q-2+\alpha}\right)}-2N_0^{-\frac{2-\alpha}{2}}\right)-q\epsilon\int_{\mathbb{R}^2}\rho^{2q-1}\mathrm{d}x, \qquad (4.34)$$

where N_0 is a uniform in time upper bound of $\|\rho(\cdot,t)\|_{L^\infty}$, i. e., $\|\rho\|_{L_t^\infty L_x^\infty} := \operatorname{ess\,sup}_{t\in[0,\infty)}\|\rho(\cdot,t)\|_{L^\infty} \le N_0$.

Note that $N_0 = \max\left((\chi/\epsilon)^{\frac{1}{q-2}}, \|\rho_0\|_{L^\infty}\right) < \infty$ according to Lemma 4.6, and $\|\rho(\cdot,t)\|_{L^1}$ is non-increasing in time. Now supposing that, at some time t_0, $C(q,\alpha)\chi\|\rho\|_{L^1}^{\frac{\alpha}{2}\left(1+\frac{2}{2q-2+\alpha}\right)}$ drops below $2N_0^{-\frac{2-\alpha}{2}}$, we also have

$$C(q,\alpha)\chi\|\rho(\cdot,t)\|_{L^1}^{\frac{\alpha}{2}\left(1+\frac{2}{2q-2+\alpha}\right)} < 2N_0^{-\frac{2-\alpha}{2}}, \qquad \forall\, t > t_0. \qquad (4.35)$$

Then, for all later times, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\rho^q\mathrm{d}x \le -N_0^{\frac{2-\alpha}{2}}\|\Lambda^{\frac{\alpha}{2}}\rho^{q/2}\|_{L^2}^2, \qquad (4.36)$$

and, again from interpolation inequality for $L^p(\mathbb{R}^2)$ norms, we obtain

$$\|\rho^{q/2}\|_{L^2} \le C(q,\alpha)\|\rho^{q/2}\|_{L^{\frac{4}{2-\alpha}}}^{\frac{2(q-1)}{2q-2+\alpha}}\|\rho^{q/2}\|_{L^{\frac{2}{q}}}^{\frac{\alpha}{2q-2+\alpha}}.$$

Hence, from this and (4.32), we have

$$\|\rho^{q/2}\|_{L^2}^{1+\frac{\alpha}{2(q-1)}} \le C(q,\alpha)\|\Lambda^{\frac{\alpha}{2}}\rho^{q/2}\|_{L^2}\|\rho^{q/2}\|_{L^{\frac{2}{q}}}^{\frac{\alpha}{2(q-1)}},$$

which, applied to (4.36), leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\rho^q\,\mathrm{d}x \le -C(q,\alpha)\,N_0^{\frac{2-\alpha}{2}}\left(\int_{\mathbb{R}^2}\rho^q\,\mathrm{d}x\right)^{1+\frac{\alpha}{2}\left(\frac{1}{q-1}\right)}\left(\int_{\mathbb{R}^2}\rho\,\mathrm{d}x\right)^{-\frac{\alpha}{2}\left(\frac{q}{q-1}\right)}. \qquad (4.37)$$

Then, using the fact that $\int_{\mathbb{R}^2}\rho(x,t)\,\mathrm{d}x$ is monotone decreasing and introducing $z(t) = \|\rho(\cdot,t)\|_{L^q}^q$, we can represent (4.37) as

$$z'(t) \le -C(q,\alpha)\,N_0^{\frac{2-\alpha}{2}}z(t)^{1+\frac{\alpha}{2}\left(\frac{1}{q-1}\right)}\|\rho(\cdot,t_*)\|_{L^1}^{-\frac{\alpha}{2}\left(\frac{q}{q-1}\right)}$$

for $t_* \geq t_0$. Solving this differential inequality, we find

$$z(t) \leq \left(\frac{C(q,\alpha)\, N_0^{\frac{2-\alpha}{2}} (t-t_*)}{\|\rho(\cdot,t_*)\|_{L^1}^{\frac{\alpha}{2}\left(\frac{q}{q-1}\right)}} + \frac{1}{\|\rho(\cdot,t_*)\|_{L^q}^{\frac{\alpha}{2}\left(\frac{q}{q-1}\right)}} \right)^{-\frac{2}{\alpha}(q-1)},$$

implying

$$\|\rho(\cdot,t)\|_{L^q}^q \leq \min\left(\|\rho(\cdot,t_*)\|_{L^q}^q, C(q,\alpha)\, N_0^{-\frac{2-\alpha}{\alpha}(q-1)} (t-t_*)^{-\frac{2}{\alpha}(q-1)} \|\rho(\cdot,t_*)\|_{L^1}^q \right). \tag{4.38}$$

Next, setting $p = q$ and $t > t_0$ as in the proof of Lemma 4.14, we can certify that

$$\frac{\|\rho(\cdot,t)\|_{L^q}}{\|\rho(\cdot,t)\|_{L^1}} \leq \frac{\|\rho(\cdot,t_0)\|_{L^q}}{\|\rho(\cdot,t_0)\|_{L^1}}. \tag{4.39}$$

Indeed, with chemotaxis, an inequality equivalent to (4.23) can be deduced from (4.34). This leads to an inequality equivalent to (4.24) whose expression inside square brackets turns out to be

$$q^{-1}\|\rho\|_{L^1}\|\Lambda^{\frac{\alpha}{2}}\rho^{q/2}\|_{L^2}^2 \left(C(q,\alpha)\chi\|\rho\|_{L^1}^{\frac{\alpha}{2}\left(1+\frac{2}{2q-2+\alpha}\right)} \|\rho\|_{L^\infty}^{\frac{2-\alpha}{2}} - 2 \right) + \epsilon \left(\|\rho\|_{L^q}^{2q} - \|\rho\|_{L^1} \int_{\mathbb{R}^2} \rho^{2q-1}\, \mathrm{d}x \right),$$

which, based on assumption (4.35) and Hölder's inequality, is negative for all $t > t_0$.

Now, for every $\tau > 0$ and $t_* \geq t_0$, we obtain

$$\begin{aligned}
\int_{t_*}^{\infty}\int_{\mathbb{R}^2} \rho^q \,\mathrm{d}x\mathrm{d}t &= \int_{t_*}^{t_*+\tau}\int_{\mathbb{R}^2} \rho^q \,\mathrm{d}x\mathrm{d}t + \int_{t_*+\tau}^{\infty}\int_{\mathbb{R}^2} \rho^q \,\mathrm{d}x\mathrm{d}t \\
&\leq C(q,\alpha) \left(\|\rho(\cdot,t_*)\|_{L^q}^q \tau + N_0^{-\frac{2-\alpha}{\alpha}(q-1)} \|\rho(\cdot,t_*)\|_{L^1}^q \int_{t_*+\tau}^{\infty} (t-t_*)^{-\frac{2(q-1)}{\alpha}}\, \mathrm{d}t \right) \\
&\leq C(q,\alpha) \left(\|\rho(\cdot,t_*)\|_{L^q}^q \tau + N_0^{-\frac{2-\alpha}{\alpha}(q-1)} \|\rho(\cdot,t_*)\|_{L^1}^q \tau^{\frac{2+\alpha-2q}{\alpha}} \right),
\end{aligned}$$

where we use (4.38) and the fact that $q > 1 + \frac{\alpha}{2}$. Thus, using (4.39) and Hölder's inequality, we see that

$$\|\rho(\cdot,t_*)\|_{L^1} - \|\rho(\cdot,t)\|_{L^1} \leq C(q,\alpha)\epsilon\|\rho(\cdot,t_*)\|_{L^1}^q \left(\frac{\|\rho(\cdot,t_0)\|_{L^\infty}^{q-1}}{\|\rho(\cdot,t_0)\|_{L^1}^{q-1}} \tau + N_0^{-\frac{2-\alpha}{\alpha}(q-1)} \tau^{\frac{2+\alpha-2q}{\alpha}} \right) \tag{4.40}$$

for all $t > t_* > t_0$, and $\tau > 0$. Subsequently, choosing τ to minimize expression (4.40), we find, for every $t > t_* > t_0$, that

$$\|\rho(\cdot,t_*)\|_{L^1} - \|\rho(\cdot,t)\|_{L^1} \leq C(q,\alpha)\epsilon N_0^{-\frac{2-\alpha}{2}} \|\rho(\cdot,t_*)\|_{L^1}^q \left(\frac{\|\rho(\cdot,t_0)\|_{L^\infty}}{\|\rho(\cdot,t_0)\|_{L^1}} \right)^{\frac{2(q-1)-\alpha}{2}}.$$

Then, employing the same argument used in the proof of Theorem 4.15, we obtain

$$\inf_t \int_{\mathbb{R}^2} \rho(x,t)\, \mathrm{d}x \geq \min\left(\frac{1}{2}\|\rho(\cdot,t_0)\|_{L^1}, C(q,\alpha)\, N_0^{\frac{2-\alpha}{2(q-1)}} \epsilon^{-\frac{1}{q-1}} \left(\frac{\|\rho(\cdot,t_0)\|_{L^1}}{\|\rho(\cdot,t_0)\|_{L^\infty}} \right)^{\frac{q-(1+\alpha/2)}{q-1}} \right). \tag{4.41}$$

Observe that (4.41) implies $\lim_{t\to\infty} \|\rho(\cdot,t)\|_{L^1} > 0$, as global well-posedness of the solution has been established and, from Lemma 4.6,

$$N_0^{\frac{2-\alpha}{2(q-1)}} \|\rho(\cdot,t_0)\|_{L^\infty}^{-\frac{q-(1+\alpha/2)}{q-1}} \geq N_0^{-\frac{q-2}{q-1}} > 0.$$

Additionally, if there exists ρ such that (4.41) is not satisfied, it implies that there is no $t > 0$ for which (4.35) holds. Consequently, once again, $\lim_{t\to\infty} \|\rho(\cdot,t)\|_{L^1} > 0$. □

4.3.2.2 Upper bound for the total fraction of unfertilized eggs

To establish the existence of solutions that blow up for the classical parabolic-elliptic Keller-Segel model (1.3) with no chemoattractant consumption effect, a virial argument involves analyzing the evolution of the second-order moment $m_2(t) = \int_{\mathbb{R}^d} \frac{|x-x_0|^2}{2}\rho(x,t)\mathrm{d}x$, where $x_0 \in \mathbb{R}^d$, and showing that the ordinary differential equation (ODE) corresponding to its evolution,

$$\frac{\mathrm{d}}{\mathrm{d}t} m_2(t) \leq h(m(t), m_2(t)), \tag{4.42}$$

generates a negative solution in finite time given sufficiently large initial data, since, in that scenario, $\frac{\mathrm{d}m_2(t)}{\mathrm{d}t} < 0$. In essence, the objective is to demonstrate that the second-order moment, initially positive, vanishes for some $t > 0$ [11, 55, 66]. For this classical model, the function h in (4.42) is explicitly defined as

$$h(m_0, m_2(t)) = \begin{cases} 2m_0\left(1 - \frac{\chi}{8\pi} m_0\right), & d = 2, \\ m_0\left(-1 + C\frac{\chi m_2(t)}{(\chi m_0)^{\frac{d}{d-2}}}\right), & d > 2, \end{cases} \tag{4.43}$$

where $m(t) = m_0$, since the total mass $m(t)$ is conserved over time.

Thus, as mentioned in the literature review (Section 1.2.2), for $d = 2$, if m_0 exceeds $8\pi/\chi$, it would imply $m_2(t)$ becomes negative in finite time, contradicting the non-negative nature of ρ. Analogously, for $d > 2$, $m_2(t)$ will decrease for all t if $\chi m_2(0)$ is sufficiently small compared to $(\chi m_0)^{\frac{d}{d-2}}$, leading to a contradiction. It is important to observe that this smallness condition on $\chi m_2(0)$ is inconsistent with the assumption of a sufficiently small norm $\|\rho_0\|_{L^{d/2}}$ required for the global well-posedness outcome.

Kiselev et al. [48] adapted this virial method to establish an upper bound for $m(t)$ in the modified model (4.1) when $d = 2$. They observed the following key points:

1. The ODE associated with the evolution of $m_2(t)$ is intricately linked to the behavior of $m(t)$ (note that h in (4.42) is a function of $m(t)$ and $m_2(t)$). Given that $m(t)$ is a monotone decreasing function, it introduces a dynamic element into the ODE, as this may imply that the ODE undergoes a change in sign as $m(t)$ decreases. Consequently, despite the ODE being negative for a sufficiently large m_0, there exists a possibility for it to transition to a positive value, preventing $m_2(t)$ from vanishing;

2. The moment of second order, $m_2(t)$, cannot vanish, as Kiselev et al. [48] established global regularity for solutions to (4.1) for any initial data.

Hence, assuming $\|\rho(\cdot,t)\|_{L^1} \geq Y$ for all $t \in [0,\tau]$, in order to prevent the contradiction of m_2 vanishing, the value of Y had to be restricted to a threshold, providing an upper bound for $m(t)$. This threshold did not depend quantitatively on the reaction term, and the only role the reaction term played was making sure that a smooth decaying solution would occur. Kiselev et al. [48] highlighted that, on a qualitative level, chemotaxis without reaction would lead $\|\rho(\cdot,t)\|_{L^\infty}$ to exhibit a δ function profile blow-up. In contrast, with a reaction in place, the growth in the $L^\infty(\mathbb{R}^2)$ norm of the solution was controlled by the balance between the chemotaxis and the reaction term, as corroborated by the same result described in Lemma 4.6.

The case $0 < \alpha < 2$ makes this argument inapplicable, as the second moment of a typical solution to an evolution equation with fractional Laplacian cannot be finite. Li et al. [55] pointed out that the weight function $|x|^2$ makes the linear term too strong to be controlled by the nonlinear part. To overcome this limitation and prove the existence of blowing-up solutions for a nonlocal Keller-Segel equation (1.6) with no chemoattractant consumption effect, $1 < \alpha < 2$ and $\beta \in (1, d]$, Biler et al. [11] extended the classical method by studying moments of lower order $\gamma \in (1, 2)$:

$$m_\gamma(t) = \int_{\mathbb{R}^d} |x - x_0|^\gamma \rho(x,t) \mathrm{d}x, \quad x_0 \in \mathbb{R}^d. \tag{4.44}$$

They emphasized that, for $\alpha < 2$, the existence of higher-order moments m_γ with $\gamma \geq \alpha$ cannot be expected. Even for the linear equation $\partial_t \rho + (-\Delta)^{\alpha/2} \rho = 0$, the fundamental solution $p_\alpha(x,t)$ behaves like $p_\alpha(x,t) \sim \left(t^{d/\alpha} + |x|^{d+\alpha}/t\right)^{-1}$, forcing moments with $\gamma \geq \alpha$ to be infinite. Biler et al. [11] then demonstrated the existence of blowing-up solutions by showing the finite-time extinction of the function (setting $x_0 = 0$ for simplicity)

$$w(t) \equiv \int_{\mathbb{R}^d} \varphi(x) \rho(x,t) \mathrm{d}x, \tag{4.45}$$

where $\varphi(x)$ is a smooth nonnegative weight function on $\mathbb{R}^d$ defined as

$$\varphi(x) \equiv \left(1 + |x|^2\right)^{\gamma/2} - 1, \tag{4.46}$$

with $\gamma \in (1, \alpha)$. Biler et al. [11] highlighted that the quantity $w(t)$, defined in (4.45), is essentially equivalent to the moment m_γ of order γ of the solution, as for every $\varepsilon > 0$, a suitably chosen $C(\varepsilon) > 0$ ensures that

$$\varphi(x) \leq |x|^\gamma \leq \varepsilon + C(\varepsilon)\varphi(x) \qquad \forall x \in \mathbb{R}^d. \tag{4.47}$$

Note that, in our study, Lemma 4.6 guarantees, as far as $q > 2$, that the balance between chemotaxis and the reaction term results in a smooth decaying solution, leading to global regularity of the solutions (Theorem 4.9). As a result, the quantity $m_\gamma(t)$ cannot vanish, even when the initial condition $\|\rho_0\|_{L^1}$ is sufficiently large and well-concentrated around a point, i. e., $m_\gamma(0)$ is small enough. Analysis of the nonlocal Keller-Segel model (1.6) with no chemoattractant consumption, $1 < \alpha < 2$ and $\beta = 2$ shows a behavior contrary to this, as presented in Section 1.6.1 or in the works of Biler et al. [9, 11].

Therefore, building on the insights of Kiselev et al. [48] and Biler et al. [11], we study the ODE governing the evolution of $w(t)$ to establish an upper bound for $m(t)$ (Theorem 4.22), which holds regardless of changes in the coupling of the reaction term, ϵ. Unlike the approach in [48], we explicitly focus on analyzing the sign of $\mathrm{d}w/\mathrm{d}t$ for the initial condition (m_0, w_0). To start, consider the two following lemmas from [11]:

Lemma 4.17. *[11] Let $\alpha \in (1, 2)$, $\gamma \in (1, \alpha)$, and φ be defined as in (4.46). Then, $(-\Delta)^{\alpha/2}\varphi \in L^\infty(\mathbb{R}^d)$.*

Lemma 4.18. *[11] For every $\gamma \in (1, 2]$, the function φ defined in (4.46) is locally uniformly convex on $\mathbb{R}^d$. Moreover, there exists $K = K(\gamma)$ such that the inequality*

$$(\nabla\varphi(x) - \nabla\varphi(y)) \cdot (x - y) \geq \frac{K|x - y|^2}{1 + |x|^{2-\gamma} + |y|^{2-\gamma}} \tag{4.48}$$

holds true for all $x, y \in \mathbb{R}^d$.

Remark 4.19. *The derivation of inequality (4.48) hinges on the condition $\gamma > 1$, and consequently $\alpha > 1$, since m_γ with $\gamma \geq \alpha$ cannot be expected to be finite. Furthermore, it can be established that $K(\gamma) = \gamma - 1$.*

Lemma 4.20. *Let $\alpha \in (1, 2)$, $\gamma \in (1, \alpha)$, and φ be defined as in (4.46). Then, the inequality*

$$\left|\frac{\nabla\varphi(x)}{\gamma}\right|^{\frac{\gamma}{\gamma-1}} \leq \frac{2}{\gamma}\varphi(x) \tag{4.49}$$

holds for all $x \in \mathbb{R}^d$.

Proof. Note that $\nabla\varphi(x) = \gamma \left(1 + |x|^2\right)^{\frac{\gamma}{2}-1} x$ and both $|\nabla\varphi(x)|^{\frac{\gamma}{\gamma-1}}$ and $\varphi(x)$ are radially symmetric functions. Then, the functions $H,\ G : \mathbb{R}^d\backslash\{0\} \to \mathbb{R}$ defined as

$$H(x) = \frac{|\gamma^{-1}\nabla\varphi(x)|^{\frac{\gamma}{\gamma-1}}}{\varphi(x)}, \quad \text{and} \quad G(x) = \frac{|\gamma^{-1}\nabla\varphi(x)||x|}{\varphi(x)}$$

are equivalent to the functions $h,\ g : \mathbb{R}_+^* \to \mathbb{R}$, defined

$$h(x) = \frac{\left[\left(1 + x^2\right)^{\frac{\gamma}{2}-1} x\right]^{\frac{\gamma}{\gamma-1}}}{(1 + x^2)^{\gamma/2} - 1}, \quad \text{and} \quad g(x) = \frac{\left(1 + x^2\right)^{\frac{\gamma}{2}-1} x^2}{(1 + x^2)^{\gamma/2} - 1},$$

respectively.

We observe that $h(x) < g(x)$. Indeed, this inequality can be expressed as

$$\left[\left(1 + x^2\right)^{\frac{\gamma}{2}-1} x\right]^{\frac{\gamma}{\gamma-1}} < \left(1 + x^2\right)^{\frac{\gamma}{2}-1} x^2$$

which simplifies to $\left(1 + x^2\right)^{1/2} > x$, a statement valid for all $x \in \mathbb{R}_+^*$. Furthermore,

$$g'(x) = \frac{2x \left(1 + x^2\right)^{\frac{\gamma-4}{2}} \left(\left(1 + x^2\right)^{\frac{\gamma}{2}} - \frac{\gamma}{2}x^2 - 1\right)}{\left((1 + x^2)^{\frac{\gamma}{2}} - 1\right)^2} < 0 \quad \forall\ x > 0,$$

since

$$\left(1 + x^2\right)^{\frac{\gamma}{2}} - \frac{\gamma}{2}x^2 - 1 = \left(1 + x^2\right) \left(\left(1 + x^2\right)^{\frac{\gamma}{2}-1} - \frac{\gamma}{2}\right) + \left(\frac{\gamma}{2} - 1\right) < 0 \quad \forall\ x > 0,$$

implying $g(x) \leq \lim_{x\to 0} g(x)$ for all $x \in \mathbb{R}_+^*$. Finally, since

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \left(\frac{|x|^2}{(1 + |x|^2)^{\frac{\gamma}{2}} - 1}\right) \overset{\text{L'H}}{=} \lim_{x\to 0} \left(\frac{2x}{\gamma x \left(1 + |x|^2\right)^{\frac{\gamma}{2}-1}}\right) = \frac{2}{\gamma},$$

we conclude that $h(x) \leq \frac{2}{\gamma}$ holds for all $x \in \mathbb{R}_+^*$, completing the proof. □

Proposition 4.21. *Let q and s be integers such that $q,\ s > 2$, $d = 2$, $\alpha \in (1, 2)$, ϑ satisfy $0 \leq \vartheta < \alpha$, and $\gamma \in (1, \vartheta]$. Assume $u \in C^\infty\left([0, \infty) \times \mathbb{R}^d\right)$ is divergence free, and ρ solves (4.2) with $\rho_0 \geq 0 \in K_{s,\vartheta}(\mathbb{R}^d)$. Then the time derivative of w, defined in (4.45), can be expressed as*

$$\frac{\mathrm{d}}{\mathrm{d}t} w(t) \leq 2\delta_{(u)}\chi^{-\mu} w(t) + h_{(u)}(m, w), \tag{4.50}$$

where $\delta_{(u)}$ equals zero when $u = 0$ and one otherwise, and

$$h_{(u)}(m, w) = m(t) \left(C_1 + \chi^{\mu(\gamma-1)} \|u\|_{L^\infty}^\gamma - \chi C_2 m(t)^{\frac{2}{\gamma}} \left(m(t) + 2w(t)\right)^{-\frac{2-\gamma}{\gamma}}\right), \tag{4.51}$$

with C_1 and C_2 being functions of γ, and $\mu \geq 0$ a parameter to be chosen.

Proof. By differentiating equation (4.45), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}w(t) = \int_{\mathbb{R}^2} \varphi(x)(u\cdot\nabla)\rho \,\mathrm{d}x - \int_{\mathbb{R}^2} \varphi(x)\,(-\Delta)^{\alpha/2}\,\rho\,\mathrm{d}x + \chi\int_{\mathbb{R}^2}\varphi(x)\nabla\cdot\left(\rho\nabla\Delta^{-1}\rho\right)\,\mathrm{d}x - \epsilon\int_{\mathbb{R}^2}\varphi(x)\rho^q\,\mathrm{d}x. \tag{4.52}$$

Notice that all the upcoming integrations by parts are justified for all $t \geq 0$ by Theorem 4.9.

Then, since u is an smooth divergence-free vector field, we have

$$\int_{\mathbb{R}^2}\varphi(x)(u\cdot\nabla)\rho\,\mathrm{d}x = -\int_{\mathbb{R}^2}\nabla\varphi(x)\cdot u\rho\,\mathrm{d}x,$$

and applying Young's inequality and Lemma 4.20, we see that

$$\left|\frac{\nabla\varphi(x)}{\gamma}\right||u| \leq \chi^{-\mu}\left|\frac{\nabla\varphi(x)}{\gamma}\right|^{\frac{\gamma}{\gamma-1}} + \chi^{\mu(\gamma-1)}\left((\gamma-1)^{\gamma-1}\gamma^{-\gamma}\right)|u|^\gamma \leq \frac{2}{\gamma}\chi^{-\mu}\varphi(x) + \frac{\chi^{\mu(\gamma-1)}}{\gamma}|u|^\gamma,$$

with $\mu > 0$ to be chosen later. Therefore, we can bound the first term on (4.52) as follows

$$\left|\int_{\mathbb{R}^2}\nabla\varphi(x)\cdot u\rho\,\mathrm{d}x\right| \leq 2\chi^{-\mu}w(t) + \chi^{\mu(\gamma-1)}\|u\|_{L^\infty}^\gamma m(t).$$

Next, from Lemma 4.17, we find

$$-\int_{\mathbb{R}^2}(-\Delta)^{\alpha/2}\rho(x,t)\varphi(x)\,\mathrm{d}x = -\int_{\mathbb{R}^2}\rho(x,t)\,(-\Delta)^{\alpha/2}\varphi(x)\,\mathrm{d}x \leq \|\,(-\Delta)^{\alpha/2}\varphi\|_{L^\infty}\int_{\mathbb{R}^2}\rho(x,t)\,\mathrm{d}x,$$

and for the chemotaxis term, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2}\varphi(x)\nabla\cdot\left(\rho\nabla\Delta^{-1}\rho\right)\,\mathrm{d}x &= -\int_{\mathbb{R}^2}\nabla\varphi(x)\cdot\left(\rho\nabla\Delta^{-1}\rho\right)\,\mathrm{d}x\\ &= -\int_{\mathbb{R}^2\times\mathbb{R}^2}\nabla\varphi(x)\cdot\frac{x-y}{|x-y|^2}\rho(x,t)\rho(y,t)\,\mathrm{d}y\,\mathrm{d}x \\ &= -\frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}^2}(\nabla\varphi(x)-\nabla\varphi(y))\cdot(x-y)\,\frac{\rho(x,t)\rho(y,t)}{|x-y|^2}\,\mathrm{d}y\,\mathrm{d}x,\end{aligned}\tag{4.53}$$

where in the last step we used the symmetry in the variables x and y. Now observe that

$$\begin{aligned}m^2 &= \int_{\mathbb{R}^2\times\mathbb{R}^2}\rho(x,t)\rho(y,t)\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{\mathbb{R}^2\times\mathbb{R}^2}\rho(x,t)\rho(y,t)\frac{1}{(1+|x|^{2-\gamma}+|y|^{2-\gamma})^{\gamma/2}}\left(1+|x|^{2-\gamma}+|y|^{2-\gamma}\right)^{\gamma/2}\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{\mathbb{R}^2\times\mathbb{R}^2}\left(\frac{\rho(x,t)\rho(y,t)}{1+|x|^{2-\gamma}+|y|^{2-\gamma}}\right)^{\gamma/2}(\rho(x,t)\rho(y,t))^{\frac{2-\gamma}{2}}\left(1+|x|^{2-\gamma}+|y|^{2-\gamma}\right)^{\gamma/2}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Then, by applying the Hölder's inequality, we obtain

$$m^2 \leq \left(\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{\rho(x,t)\rho(y,t)}{1+|x|^{2-\gamma}+|y|^{2-\gamma}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{\gamma}{2}} \left(\int_{\mathbb{R}^2\times\mathbb{R}^2}\rho(x,t)\rho(y,t)\left(1+|x|^{2-\gamma}+|y|^{2-\gamma}\right)^{\frac{\gamma}{2-\gamma}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{2-\gamma}{2}}. \tag{4.54}$$

From the properties of convex functions, as $\gamma/(2-\gamma) > 1$, and utilizing the inequality $|x|^\gamma \le 1 + \varphi(x)$, we have

$$\left(1 + |x|^{2-\gamma} + |y|^{2-\gamma}\right)^{\frac{\gamma}{2-\gamma}} \le C\left(1 + |x|^\gamma + |y|^\gamma\right) \le C\left(1 + \varphi(x) + \varphi(y)\right),$$

for C a function of γ. This implies that

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} \rho(x,t)\rho(y,t)\left(1 + |x|^{2-\gamma} + |y|^{2-\gamma}\right)^{\frac{\gamma}{2-\gamma}} \,\mathrm{d}x\,\mathrm{d}y \le C\left[m^2 + 2m\left(\int_{\mathbb{R}^2} \varphi(x)\rho(x,t)\,\mathrm{d}x\right)\right].$$

Thus, back to (4.54), we have

$$\left(\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\rho(x,t)\rho(y,t)}{1 + |x|^{2-\gamma} + |y|^{2-\gamma}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{\gamma}{2}} \ge C^{-\frac{2-\gamma}{2}} \frac{m^2}{(m^2 + 2mw(t))^{\frac{2-\gamma}{2}}}. \tag{4.55}$$

Now, considering Lemma 4.18, observe that

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\rho(x,t)\rho(y,t)}{1 + |x|^{2-\gamma} + |y|^{2-\gamma}}\,\mathrm{d}x\,\mathrm{d}y \le \frac{1}{K}\int_{\mathbb{R}^2\times\mathbb{R}^2} (\nabla\varphi(x) - \nabla\varphi(y))\cdot(x-y)\frac{\rho(x,t)\rho(y,t)}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y.$$

Then, from (4.55),

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} (\nabla\varphi(x) - \nabla\varphi(y))\cdot(x-y)\frac{\rho(x,t)\rho(y,t)}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y \ge KC^{-\frac{2-\gamma}{\gamma}} \frac{m^{\frac{4}{\gamma}}}{(m^2 + 2mw(t))^{\frac{2-\gamma}{\gamma}}}.$$

Therefore, the chemotaxis term (4.53) can be estimated as

$$\begin{aligned}\int_{\mathbb{R}^2} \varphi(x)\nabla\cdot\left(\rho\nabla\Delta^{-1}\rho\right)\,\mathrm{d}x &\le -C_2\frac{m(t)^{\frac{4}{\gamma}}}{(m(t)^2 + 2m(t)w(t))^{\frac{2-\gamma}{\gamma}}} \\ &\le -C_2 m(t)^{1+\frac{2}{\gamma}}\left(m(t) + 2w(t)\right)^{-\frac{2-\gamma}{\gamma}},\end{aligned}$$

and we can rewrite (4.52) as

$$\frac{\mathrm{d}w}{\mathrm{d}t} \le m(t)\Big(C_1 + \chi^{\mu(\gamma-1)}\|u\|_{L^\infty}^\gamma - \chi C_2 m(t)^{\frac{2}{\gamma}}\left(m(t) + 2w(t)\right)^{-\frac{2-\gamma}{\gamma}}\Big) + 2\chi^{-\mu}w(t) - \epsilon\int_{\mathbb{R}^2}\varphi(x)\rho^q\,\mathrm{d}x,$$

which can be simplified to (4.50). □

Theorem 4.22. *Let q and s be integers such that $q, s > 2$, $d = 2$, $\alpha \in (1,2)$, ϑ satisfy $0 \le \vartheta < \alpha$, and $\gamma \in (1, \vartheta]$. Suppose $u \in C^\infty\big([0,\infty)\times\mathbb{R}^d\big)$ is divergence free, and ρ solves (4.2) with $\rho_0 \ge 0 \in K_{s,\vartheta}(\mathbb{R}^d)$. Then, for all $\tau > 0$, we have*

$$\|\rho(\cdot,\tau)\|_{L^1} < \max\left\{\psi^{\frac{\gamma}{2+\gamma}} w_0^{\frac{2}{2+\gamma}},\ \psi\right\}, \tag{4.56}$$

where

$$\psi(\chi,u,\tau) = \frac{C_1 + \|u\|_{L^\infty}^\gamma}{\chi C_2}\left(\frac{\Theta(\tau)}{C_1 + \|u\|_{L^\infty}^\gamma} + 1\right), \qquad \Theta(\tau) = \begin{cases}\tau^{-1} & \text{if } u = 0\\ 2\left(1 - e^{-2\tau}\right)^{-1} & \text{if } u \ne 0,\end{cases} \tag{4.57}$$

$w_0 = w(0)$, and C_1, C_2 are functions of γ.

Remark 4.23. *Here C_2 is obtained by multiplying the constant C_2 from (4.51) with $3^{-\frac{2-\gamma}{\gamma}}$.*

Remark 4.24. *Notice that when $u = 0$, if $w_0 < \frac{C_1}{\chi C_2}\left(\frac{1}{C_1\tau} + 1\right)$, the level $\|\rho(\cdot,\tau)\|_{L^1} \sim \chi^{-1}$ will be attained in at most $\tau \sim 1$. Otherwise, if this condition is not met, the level $\|\rho(\cdot,\tau)\|_{L^1} \sim \chi^{-1}$ will be reached in at most $\tau \sim \chi^{\frac{2}{\gamma}}$, while the level $\sim \chi^{\frac{\gamma}{2+\gamma}}$ $(1/3 < \gamma/(2+\gamma) < 1/2)$ in at most $\tau \sim 1$. Moreover, for $u \neq 0$, if $w_0 < \frac{C_1}{\chi C_2}\left(\frac{1}{C_1\tau} + 1\right)$, on the time scale $\tau \sim 1$ the level $\|\rho(\cdot,\tau)\|_{L^1}$ is $\sim \chi^{-1}$, and otherwise, the level $\|\rho(\cdot,\tau)\|_{L^1}$ is $\sim \chi^{\frac{\gamma}{2+\gamma}}$.*

Proof. Observe that, since $\rho_0 \in K_{s,\vartheta}(\mathbb{R}^d)$ and $\gamma \leq \vartheta$, the integral $w_0 = \int_{\mathbb{R}^2} \varphi(x)\rho_0(x)\,\mathrm{d}x$ is well-defined and finite. Furthermore, from the ordinary differential inequality (4.50), we establish that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-2\delta_{(u)}\chi^{-\mu}t}w(t)\right) \leq e^{-2\delta_{(u)}\chi^{-\mu}t}h_{(u)}(m,w), \tag{4.58}$$

where $h_{(u)}$ and $\delta_{(u)}$ are defined in Proposition 4.21.

To proceed, we divide the analysis into two cases.

- **Case 1** ($u = 0$)**:** Substituting $u = 0$ into (4.58) simplifies the differential inequality to

$$\frac{\mathrm{d}}{\mathrm{d}t}w(t) \leq h(m,w), \tag{4.59}$$

where $h(m,w) \equiv h_{(u=0)}(m,w)$.

Next, assume the initial condition (m_0, w_0) satisfies $h(m_0,w_0) < 0$. By the continuity of h with respect to m and w, and the continuity of $w(t)$ and $m(t)$, there exists an open neighborhood U_1 of $t = 0$ such that, for all $t \in U_1$,

$$|h(m(t),w_0) - h(m_0,w_0)| < -\frac{1}{2}h(m_0,w_0),$$

implying that $h(m(t),w_0) < h(m_0,w_0)/2$ in U_1, and open neighborhood U_2 of $t = 0$ such that $h(m(t),w(t)) < 0$ for all $t \in U_2$. Choosing $\tau \in U_1 \cap U_2$ ensures $h(m(t),w(t)) < 0$ for all $t \in [0,\tau]$ and $h(m(\tau),w_0) < 0$.

Since $h(m(t),w(t)) < 0$ on $[0,\tau]$, it follows from (4.59) that $w(t)$ decreases, implying $w_0 \geq w(t)$ on this interval. Moreover, as $m(t)$ is monotonically decreasing, $m(\tau) \leq m(t)$ for every $t \in [0,\tau]$. As a result,

$$h(m(t),w(t)) \leq h(m(\tau),w_0),$$

for all $t \in [0,\tau]$. To substantiate this, we analyze the partial derivatives of $h(m,w)$:

$$\partial_m h(m,w) = \frac{h(m,w)}{m} - C_2\chi m^{\frac{2}{\gamma}}(m+2w)^{-\frac{2}{\gamma}}\left(m + \frac{4w}{\gamma}\right), \tag{4.60}$$

$$\partial_w h(m,w) = 2\gamma^{-1}C_2\chi m^{\frac{2}{\gamma}+1}(2-\gamma)(m+2w)^{-\frac{2}{\gamma}}. \tag{4.61}$$

Notably, $\partial_w h(m,w) > 0$ and, as $h(m,w) < 0$ holds for all $t \in [0,\tau]$, $\partial_m h(m,w) < 0$ in this interval. These derivative signs indicate that h decreases as w decreases, and, over the interval $[0,\tau]$, h increases as m decreases. Thus, the inequalities $w_0 \geq w(t)$ and $m(\tau) \leq m(t)$ for every $t \in [0,\tau]$ imply $h(m(t),w(t)) \leq h(m(\tau),w_0)$. In a more direct argument, we can compare $-h(m(t), w(t))$ and $-h(m(\tau), w_0)$ for $t \in [0, \tau]$ using the relations $-h(m(t), w(t)) > 0$, $w_0 \geq w(t)$ and $m(\tau) \leq m(t)$ in this interval:

$$\begin{aligned}-h\left(m(t), w(t)\right) &= m(t)\left(\chi C_2 m(t)\left(1 + \frac{2w(t)}{m(t)}\right)^{-\frac{2-\gamma}{\gamma}} - C_1\right)\\ &\geq m(\tau)\left(\chi C_2 m(\tau)\left(1 + \frac{2w_0}{m(\tau)}\right)^{-\frac{2-\gamma}{\gamma}} - C_1\right) = -h\left(m(\tau), w_0\right).\end{aligned}$$

Next, integrating equation (4.59) yields the following estimate:

$$w_0 - w(\tau) \geq -\int_0^\tau h\left(m(\tau), w_0\right) \mathrm{d}t = -\tau h\left(m(\tau), w_0\right).$$

Thus, we must have $-\tau h\left(m(\tau), w_0\right) < w_0$ to avoid the contradiction that $w(\tau)$ vanishes, that is,

$$\tau m(\tau)\left[\chi C_2 m(\tau)^{\frac{2}{\gamma}}\left(m(\tau) + 2w_0\right)^{-\frac{2-\gamma}{\gamma}} - C_1\right] < w_0, \tag{4.62}$$

which can be rewritten as

$$m(\tau)^{1+\frac{2}{\gamma}} < \frac{C_1}{\chi C_2}\left(\frac{w_0}{C_1\tau} + m(\tau)\right)\left(m(\tau) + 2w_0\right)^{\frac{2-\gamma}{\gamma}}.$$

From this, observe that if $m(\tau) < w_0$, we have

$$m(\tau) < \left[\frac{C_1}{\chi C_2}\left(\frac{1}{C_1\tau} + 1\right)\right]^{\frac{\gamma}{2+\gamma}} w_0^{\frac{\gamma}{2+\gamma}}\left(3w_0\right)^{\frac{2-\gamma}{2+\gamma}} = \left[\frac{C_1}{\chi C_2}\left(\frac{1}{C_1\tau} + 1\right)\right]^{\frac{\gamma}{2+\gamma}} w_0^{\frac{2}{2+\gamma}},$$

where $3^{-\frac{2-\gamma}{\gamma}}$ is incorporated into C_2.

On the other hand, if $m(\tau) \geq w_0$, the inequality becomes

$$m(\tau)^{1+\frac{2}{\gamma}} < \left[\frac{C_1}{\chi C_2}\left(\frac{1}{C_1\tau} + 1\right)\right] m(\tau)\left(3m(\tau)\right)^{\frac{2-\gamma}{\gamma}} = \left[\frac{C_1}{\chi C_2}\left(\frac{1}{C_1\tau} + 1\right)\right] m(\tau)^{\frac{2}{\gamma}},$$

where $3^{-\frac{2-\gamma}{\gamma}}$ is again absorbed into C_2. Consequently,

$$m(\tau) < \frac{C_1}{\chi C_2}\left(\frac{1}{C_1\tau} + 1\right).$$

Finally, observe that if the initial assumption, $h(m_0, w_0) < 0$, is not satisfied, the result still follows since $m(t) \leq m_0$ for all $t > 0$, and $h(m_0, w_0) \geq 0$ implies $m_0^{1+\frac{2}{\gamma}} \leq \frac{C_1}{\chi C_2} m_0\left(m_0 + 2w_0\right)^{\frac{2-\gamma}{\gamma}}$. Then, repeating the previous steps and incorporating $3^{-\frac{2-\gamma}{\gamma}}$ into C_2, we find that if $m_0 < w_0$, then $m_0 < \left(\frac{C_1}{\chi C_2}\right)^{\frac{\gamma}{2+\gamma}} w_0^{\frac{2}{2+\gamma}}$. Otherwise, if $m_0 \geq w_0$, we conclude that $m_0 \leq \frac{C_1}{\chi C_2}$.

- **Case 2** ($u \neq 0$)**:** Assume the initial condition (m_0, w_0) satisfies $h_{(u)}(m_0, w_0) < 0$. As proved in **Case 1**, it follows that there exists a time $\tau > 0$ such that

$$h_{(u)}\left(m(\tau), w_0\right) < 0 \qquad \text{and} \qquad h_{(u)}\left(m(t), w(t)\right) < 0,$$

for all $t \in [0, \tau]$.

Moreover, since the partial derivatives of $h_{(u)}$ with respect to m and w are given by equations (4.60) and (4.61), with C_1 replaced by $C_1 + \chi^{\mu(\gamma-1)}\|u\|_{L^\infty}^{\gamma}$, it follows that

$$h_{(u)}(m(t), w(t)) \leq h_{(u)}(m(\tau), w_0)$$

throughout the interval $[0, \tau]$. This conclusion is justified as in **Case 1**.

Next, since $\delta_{(u)} = 1$ in (4.58), the time integral of the right-hand side of (4.58) over $[0, \tau]$ satisfies

$$\int_0^\tau e^{-2\chi^{-\mu}t} m(\tau)\left[C_1 + \chi^{\mu(\gamma-1)}\|u\|_{L^\infty}^{\gamma} - \chi C_2 m(\tau)^{\frac{2}{\gamma}} (m(\tau) + 2w_0)^{-\frac{2-\gamma}{\gamma}}\right] \, dt = \frac{\left(1 - e^{-2\chi^{-\mu}\tau}\right)}{2}\chi^\mu m(\tau)\left[C_1 + \chi^{\mu(\gamma-1)}\|u\|_{L^\infty}^{\gamma} - \chi C_2 m(\tau)^{\frac{2}{\gamma}} (m(\tau) + 2w_0)^{-\frac{2-\gamma}{\gamma}}\right]. \quad (4.63)$$

To avoid contradiction, the following condition must be satisfied:

$$\left(1 - e^{-2\chi^{-\mu}\tau}\right) \chi^\mu m(\tau)\left[\chi C_2 m(\tau)^{\frac{2}{\gamma}} (m(\tau) + 2w_0)^{-\frac{2-\gamma}{\gamma}} - \left(C_1 + \chi^{\mu(\gamma-1)}\|u\|_{L^\infty}^{\gamma}\right)\right] < 2w_0. \quad (4.64)$$

Proceeding similarly to **Case 1**, inequality (4.64) can be expressed as

$$m(\tau)^{1+\frac{2}{\gamma}} < \left(\frac{C_1 + \chi^{\mu(\gamma-1)}\|u\|_{L^\infty}^{\gamma}}{\chi C_2}\right) \left(\frac{2w_0}{\left(C_1 + \chi^{\mu(\gamma-1)}\|u\|_{L^\infty}^{\gamma}\right)\left(1 - e^{-2\chi^{-\mu}\tau}\right)\chi^\mu} + m(\tau)\right)(m(\tau) + 2w_0)^{\frac{2-\gamma}{\gamma}}.$$

Then, incorporating $3^{-\frac{2-\gamma}{\gamma}}$ into C_2, as before, we obtain the following bounds:

1. For $m(\tau) < w_0$

$$m(\tau) < \left[\left(\frac{C_1 + \chi^{\mu(\gamma-1)}\|u\|_{L^\infty}^{\gamma}}{\chi C_2}\right)\left(\frac{2}{\left(C_1 + \chi^{\mu(\gamma-1)}\|u\|_{L^\infty}^{\gamma}\right)\left(1 - e^{-2\chi^{-\mu}\tau}\right)\chi^\mu} + 1\right)\right]^{\frac{\gamma}{2+\gamma}} w_0^{\frac{2}{2+\gamma}}, \quad (4.65)$$

2. For $m(\tau) \geq w_0$,

$$m(\tau) < \left(\frac{C_1 + \chi^{\mu(\gamma-1)}\|u\|_{L^\infty}^{\gamma}}{\chi C_2}\right)\left(\frac{2}{\left(C_1 + \chi^{\mu(\gamma-1)}\|u\|_{L^\infty}^{\gamma}\right)\left(1 - e^{-2\chi^{-\mu}\tau}\right)\chi^\mu} + 1\right). \quad (4.66)$$

We observe that the optimal choice for the parameter μ – which is independent of any specific quantities within the problem – minimizes $m(\tau)$ is zero. Finally, as in **Case 1**, if the initial condition does not satisfy the assumption $h_{(u)}(m_0, w_0) < 0$, the result still holds due to the inequality $m(t) \leq m_0$ for all $t > 0$. □

4.3.3 Chemotaxis Impact on Reaction Efficiency

In examining the interplay between chemotaxis, reaction, and anomalous diffusion in the context of reproduction processes, several key insights have emerged.

First, in both chemotactic and chemotaxis-free scenarios, it was shown that the lower bound for the total fraction of unfertilized eggs, $m(t)$, remains above a limit C for all $t \geq 0$, irrespective of the influence of the flow field u. This observation implies a capped threshold for the reaction rate, beyond which it cannot be increased, regardless of the strength or form of the flow. Furthermore, the value of C depends on various parameters, including those governing the reaction term (fertilization), such as ϵ and q, where ϵ regulates the strength of the fertilization process. Additionally, it is influenced by the initial condition ρ_0, d, and the diffusion term, as C is a function of α. In scenarios involving chemotaxis, the chemotactic sensitivity χ also contributes to shaping C.

In chemotaxis-free settings, the function $C(\epsilon, q, \alpha, d, \rho_0)$ tends to m_0 as $\epsilon \to 0$, given fixed parameters ρ_0, u, and q. Additionally, the quantity that reacts, $m(0) - \lim_{t\to\infty} m(t)$, exhibits a decrease of order ϵ (Theorem 4.15).

In the presence of chemotaxis, particularly for $d = 2$, Kiselev et al. [48] established, for $\alpha = 2$, the existence of solutions where the lower bound of its $L^1(\mathbb{R}^2)$ norm is independent of the reaction term, and thus independent of the coupling of the reaction term ϵ. However, for $1 < \alpha < 2$, the possible dependence of condition (4.35) on ϵ challenges this assertion. Therefore, we can not affirm that the lower bound of the $L^1(\mathbb{R}^2)$ norm does not depend on ϵ.

Also in the chemotactic environment, concerning the upper bound for $m(t)$, note that the reaction term does not quantitatively impact its estimates, that is, all estimates of the $L^1(\mathbb{R}^2)$ norm are independent of the coupling of the reaction term, ϵ. This implies that the amount of the density that reacts, $m(0) - \lim_{t\to\infty} m(t)$, satisfies a lower bound independent of ϵ. While the reaction term does not directly impact the total fraction of reacting density, it plays a crucial role in controlling the growth of the solution's $L^\infty(\mathbb{R}^2)$ norm, determined by the balance between chemotaxis and the reaction term. For instance, in a two-dimensional space without the reaction term (classical parabolic-elliptic Keller-Segel), for an initial mass large enough, chemotaxis alone would result in a blow-up with a δ function profile.

Additionally, it is important to note that the chemotactic term, in contrast to the flow and diffusion alone, plays a crucial role in achieving highly efficient fertilization rates, as in the chemotactic scenario, both the upper and lower bounds of $m(t)$ decrease as the chemotaxis strength, χ, increases.

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Appendix A

Fractional Laplacian

In this appendix, we present and prove essential properties of the Fractional Laplacian for completeness. We begin by introducing a definition of the nonlocal operator $(-\Delta)^s$ for any $s \in \mathbb{R}_+ \setminus \mathbb{N}$.

Definition A.1. *Let $s = m + \sigma$, where $\sigma \in (0, 1)$ and $m \in \mathbb{N}_0$. Then, the operator $(-\Delta)^s$ can be defined as [1, 2, 69]*

$$(-\Delta)^s u(x) := \frac{c_{d,2s}}{2} P.V. \int_{\mathbb{R}^d} \frac{\delta_{m+1} u(x, y)}{|y|^{d+2s}} dy, \quad x \in \mathbb{R}^d, \tag{A.1}$$

where $P.V.$ stands for the Cauchy principal value, $c_{d,s}$ is the positive normalization constant

$$c_{d,2s} := \frac{4^s \Gamma\left(\frac{d}{2} + s\right)}{\pi^{\frac{d}{2}} \Gamma(-s)} \left(\sum_{k=1}^{m+1} (-1)^k \begin{pmatrix} 2(m+1) \\ m+1-k \end{pmatrix} k^{2s} \right)^{-1}, \tag{A.2}$$

and $\delta_{m+1} u$ is the finite difference of order $2(m + 1)$ of u

$$\delta_{m+1} u(x, y) := \sum_{k=-m-1}^{m+1} (-1)^k \begin{pmatrix} 2(m+1) \\ m+1-k \end{pmatrix} u(x + ky) \quad \text{for } x, y \in \mathbb{R}^d. \tag{A.3}$$

Note that, by a change of variables, it is easy to see that (A.1) can be rewritten as

$$(-\Delta)^s u(x) = c_{d,2s} P.V. \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x - y|^{d+2s}} \mathrm{d}y, \quad x \in \mathbb{R}^d, \tag{A.4}$$

where the normalization constant is given by

$$c_{d,2s} := \frac{4^s s \Gamma(d/2 + s)}{\pi^{d/2} |\Gamma(1 - s)|} = \frac{4^s \Gamma(d/2 + s)}{\pi^{d/2} |\Gamma(-s)|}. \tag{A.5}$$

Remark A.2. *In literature, the definition of the fractional Laplacian, $\Lambda^{2s} = (-\Delta)^s$, generally applies for $0 < s < 1$ and can be characterized in multiple equivalent ways [see 51, 56]. One such characterization is as the singular integral operator (A.4) [37, 39].*

Next, consider the definition of the operator I_α for $0 < \alpha < d$, known as the Riesz potential.

Definition A.3 (Riesz potential). *For any $d \in \mathbb{N}$, let $0 < \alpha < d$. The operator I_α is the Riesz potential of order α given by [37]*

$$I_\alpha u(x) = c_{d,-\alpha}|x|^{-d+\alpha} * u(x) = c_{d,-\alpha}\int_{\mathbb{R}^d} |x-y|^{-d+\alpha}u(y)\mathrm{d}y, \tag{A.6}$$

where $c_{d,-\alpha} = \frac{\Gamma((d-\alpha)/2)}{\pi^{d/2}2^\alpha\Gamma(\alpha/2)}$.

For $0 < \alpha < 2$, the operator I_α is defined to be the inverse of $(-\Delta)^{\alpha/2}$, and the constant $c_{d,-\alpha}$ is chosen to ensure the validity of $I_\alpha(-\Delta)^{\alpha/2}u = (-\Delta)^{\alpha/2}I_\alpha u = u$ in $\mathcal{S}'(\mathbb{R}^d)$ for any $u \in \mathcal{S}(\mathbb{R}^d)$ [37, 39].

Now, we will show that $\Lambda^\alpha = (-\Delta)^{\alpha/2}$ maps $W^{k,p}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ for certain values of p and k. For that purpose, consider the following lemma.

Lemma A.4. *[32] Let $u \in W^{k,p}(\mathbb{R}^d)$, for some $1 \le p \le \infty$, and k be a nonnegative integer. Then, the following estimate holds*

$$\|u(\cdot+y) - P_k u(\cdot,y)\|_{L^p} \le C\|u\|_{W^{k,p}}|y|^k, \tag{A.7}$$

where $C > 0$ depends only on d and k, and $P_k u(x,\cdot)$ denotes the Taylor polynomial of order k, centered at x, of the function u, i. e., using the standard multi-index notation (see Index of Notation),

$$P_k u(x,y) := \sum_{|\zeta|\le k} \frac{D^\zeta u(x)}{\zeta!} y^\zeta.$$

Proposition A.5. *Let $\alpha > 0$. If $u \in W^{2\lceil\alpha/2\rceil,p}(\mathbb{R}^d)$, then $\Lambda^\alpha u \in L^p(\mathbb{R}^d)$ for $1 < p < \infty$. Specifically, the following inequality holds*

$$\|\Lambda^\alpha u\|_{L^p} \le C\|u\|_{W^{2\lceil\alpha/2\rceil,p}},$$

where $C > 0$ depends only on d, α, and p.

Proof. The cases where $\alpha/2 \in \mathbb{N}$ can be checked directly, as Λ^α is a local differential operator, representing a power of the standard Laplacian. For $\alpha/2 \in \mathbb{R}_+ \setminus \mathbb{N}$, from (A.4), we write

$$\begin{aligned}\left\|\int_{\mathbb{R}^d}\frac{u(y)-u(x)}{|x-y|^{d+\alpha}}\,\mathrm{d}y\right\|_{L^p} &= \left\|\int_{|x-y|<1}\frac{u(y)-u(x)}{|x-y|^{d+\alpha}}\,\mathrm{d}y\right\|_{L^p} + \left\|\int_{|x-y|\ge 1}\frac{u(y)-u(x)}{|x-y|^{d+\alpha}}\,\mathrm{d}y\right\|_{L^p}\\ &= I_1 + I_2.\end{aligned}$$

To compute I_1 let us write $u \in W^{k,p}(\mathbb{R}^d)$, where $k = 2\lceil\alpha/2\rceil$, using the Taylor expansion centered at x, i. e., $u(x+y) = P_k u(x,y) + r(x,y)$, where $P_k u(x,y)$ is defined in Lemma A.4. From Minkowski's inequality for integrals, since $\chi_{[-1,1]}(y)\left(\frac{u(\cdot+y)-P_k(\cdot,y)}{|y|^{d+\alpha}}\right) \in L^p(\mathbb{R}^d)$ for a.e. y, and $\chi_{[-1,1]}(y)\left\|\frac{u(\cdot+y)-P_k(\cdot,y)}{|y|^{d+\alpha}}\right\|_{L^p} \in L^1(\mathbb{R}^d)$, we have

$$\begin{aligned}
\left\|\int_{|y|<1} \frac{r(x,y)}{|y|^{d+\alpha}}\,\mathrm{d}y\right\|_{L^p} &\leq C\int_{|y|<1} \frac{\|u(\cdot+y)-P_k(\cdot,y)\|_{L^p}}{|y|^{d+\alpha}}\,\mathrm{d}y \\
&\leq C\int_{|y|<1} \frac{\|u\|_{W^{k,p}}|y|^k}{|y|^{d+\alpha}}\,\mathrm{d}y \\
&\leq \left(\int_{|y|<1} \frac{1}{|y|^{d+\alpha-k}}\,\mathrm{d}y\right)\|u\|_{W^{k,p}} \\
&\leq C\|u\|_{W^{k,p}},
\end{aligned}$$

where we used (A.7) and the fact that $d+\alpha-k<d$. Next, note that the finite difference of order k of u can be expressed as

$$\delta_{k/2}u(x,y) = -\sum_{|\zeta|=k} y^\zeta D^\zeta u + \sum_{l=-k/2}^{k/2}(-1)^l\begin{pmatrix} k \\ k/2-l\end{pmatrix} r(x+ly) \quad \text{for } x,y\in\mathbb{R}^d.$$

Therefore, from the above, we obtain

$$\begin{aligned}
I_1 &= C\left\|\int_{|y|<1} \frac{\delta_{k/2}u(x,y)}{|y|^{d+\alpha}}\,\mathrm{d}y\right\|_{L^p} \\
&\leq C\left\|\int_{|y|<1} \sum_{|\zeta|=k}\frac{y^\zeta D^\zeta u}{|y|^{d+\alpha}} + \sum_{l=-k/2}^{k/2}(-1)^l\begin{pmatrix} k \\ k/2-l\end{pmatrix}\frac{r(x+ly)}{|y|^{d+\alpha}}\,\mathrm{d}y\right\|_{L^p} \\
&\leq C\left\|D^k u\right\|_{L^p}\left(\int_{|y|<1}\frac{1}{|y|^{d+\alpha-k}}\,\mathrm{d}y\right) + C\|u\|_{w^{k,p}} \\
&\leq C\|u\|_{w^{k,p}},
\end{aligned}$$

where $\left|D^k u\right| = \left(\sum_{|\zeta|=k}\left|D^\zeta u\right|^2\right)^{1/2}$, and we used the fact that $1/|y|^{d+\alpha-k}$ is integrable as $k-\alpha>0$.

Now applying again Minkowski's inequality, we obtain

$$\begin{aligned}
I_2 &= \left\|\int_{|z|\geq 1}\frac{u(x+z)-u(x)}{|z|^{d+\alpha}}\,\mathrm{d}z\right\|_{L^p} \\
&\leq \int_{|z|\geq 1}\frac{\|u(\cdot+z)-u(\cdot)\|_{L^p}}{|z|^{d+\alpha}}\,\mathrm{d}z \\
&\leq 2\left(\int_{|z|\geq 1}\frac{1}{|z|^{d+\alpha}}\,\mathrm{d}z\right)\|u\|_{L^p} \\
&\leq C\,\|u\|_{L^p},
\end{aligned}$$

where we used the fact that $1/|z|^{d+\alpha}$ is integrable, as $d+\alpha>d$ and $|z|\geq 1$. □

Now consider the following definition:

Definition A.6. *For a real Lebesgue-measurable function u in $\mathbb{R}^d, d \in \mathbb{N}$, we define $u \mapsto u_+$ and $u \mapsto u_-$, bounded maps in $L^p(\mathbb{R}^d)$ spaces, with $1 \leq p \leq \infty$, as*

$$u_+(x) = \max\{u(x), 0\}, \quad \text{and} \quad u_-(x) = \min\{u(x), 0\}, \quad x \in \mathbb{R}^d. \tag{A.8}$$

Thus, $u(x) = u_+(x) + u_-(x)$ a.e. x. We also define $\partial_i u_+$ and $\partial_i u_-$, for $u \in W^{1,p}(\mathbb{R}^d)$, as

$$\partial_i u_+(x) = \begin{cases} \partial_i u(x) & \text{if } u(x) > 0 \\ 0 & \text{if } u(x) \leq 0 \end{cases} \quad \text{and} \quad \partial_i u_-(x) = \begin{cases} \partial_i u(x) & \text{if } u(x) \leq 0 \\ 0 & \text{if } u(x) > 0. \end{cases} \tag{A.9}$$

Then, we can prove for u, u_- and u_+ the following:

Lemma A.7. *Let $\alpha \geq 0$ and $u \in \mathcal{S}(\mathbb{R}^d)$. Then,*

$$u_-\Lambda^\alpha u \geq u_-\Lambda^\alpha u_- \quad \text{and} \quad u_+\Lambda^\alpha u \geq u_+\Lambda^\alpha u_+. \tag{A.10}$$

Proof. The cases where $\alpha/2 \in \mathbb{N}$ can be verified directly, as Λ^α acts as a local differential operator, specifically a power of the Laplacian. For $\alpha/2 \in \mathbb{R}_+ \setminus \mathbb{N}$, from (A.4), we obtain

$$\begin{aligned} u_-\Lambda^\alpha u(x) &= c_{d,\alpha} P.V. \int_{\mathbb{R}^d} \frac{u_-(x)u(x) - u_-(x)u(y)}{|x-y|^{d+\alpha}} \, dy \\ &= c_{d,\alpha} P.V. \int_{\mathbb{R}^d} \frac{u_-^2(x) - u_-(x)\left(u_-(y) + u_+(y)\right)}{|x-y|^{d+\alpha}} \, dy \\ &\geq c_{d,\alpha} P.V. \int_{\mathbb{R}^d} \frac{u_-^2(x) - u_-(x)u_-(y)}{|x-y|^{d+\alpha}} \, dy \qquad (-u_-(x)u_+(y) \geq 0) \\ &= u_-\Lambda^\alpha u_-(x). \end{aligned}$$

□

Lemma A.8. *Let $\alpha \geq 0$, and suppose that $u \in W^{2\lceil \alpha/2 \rceil, p}(\mathbb{R}^d)$, with $1 < p < \infty$. Then,*

$$\int_{\mathbb{R}^d} |u|^{p-2} u \Lambda^\alpha u \, dx \geq 0. \tag{A.11}$$

and

$$\int_{\mathbb{R}^d} |u_\pm|^{p-2} u_\pm \Lambda^\alpha u \, dx \geq 0. \tag{A.12}$$

Proof. The proof of (A.11) with $\alpha \in [0, 2]$ is available in Córdoba et al. [29] for $x \in \mathbb{R}^2$ or $\mathbb{T}^2$. Interestingly, the same proof extends seamlessly to $x \in \mathbb{R}^d$ and $\alpha > 2$. This extension holds, particularly for $\alpha > 2$, owing to the equivalence between the fractional Laplacian expressions (A.1) and (A.4). Note that (A.12) follows from (A.10) and (A.11). Indeed,

$$\int_{\mathbb{R}^d} |u_\pm|^{p-2} u_\pm \Lambda^\alpha u \, dx \geq \int_{\mathbb{R}^d} |u_\pm|^{p-2} u_\pm \Lambda^\alpha u_\pm \, dx \geq 0.$$

□

Appendix B

Parameters for Lebesgue Spaces

In Sections 3.2 and 3.3, suitable restrictions are imposed on the parameters of the system, d, α and β, as well as on the parameters related to the space of initial data and solutions: p, r and $\wp$ for the local-in-time results, and p, r, p_1 and p_2 for the global-in-time results.

In this Appendix, we present the proof of the existence of p and r satisfying these restrictions and show that these imply (B.4) and (B.15) for the local results, and (B.4), (B.5), (B.16), $1 \leq p_1 < p$ and $1 < p_2 < r$ for the global results. Such conditions constitute the hypotheses necessary to apply Hölder's inequality, Lemmas 2.10 and 2.22 to ensure that the mild solution belongs to the appropriate function spaces.

Remark B.1. *Throughout this appendix, consider* $\alpha \in (1, 2]$, $\beta \in (1, d]$ *and* $d \geq 2$.

Definition B.2. *Let* $1 < p \leq r \leq \infty$. *We define* σ *as*

$$\sigma = 2 - \frac{1}{\alpha}\left(\frac{d}{r} + 1\right) - \frac{d}{\beta}\left(\frac{1}{p} - \frac{1}{r}\right) - \frac{1}{\beta}. \tag{B.1}$$

Lemma B.3. *Consider* p *and* r *satisfying*

$$\max\left\{\frac{2d}{d+\beta-1}, \frac{d}{\alpha+\beta-2}\right\} < p \leq \frac{d}{\beta-1} \quad \text{and}$$
$$\max\left\{p, \frac{p}{p-1}, \frac{d}{\alpha-1}\right\} < r < \frac{pd}{d-p(\beta-1)}, \quad \text{or} \tag{B.2a}$$

$$p > \frac{d}{\beta-1} \qquad \text{and} \qquad r > \max\left\{p, \frac{p}{p-1}, \frac{d}{\alpha-1}\right\}, \tag{B.2b}$$

where, in both cases, the equality $r = \max\left\{p, \frac{p}{p-1}\right\}$ *is possible if* $\max\left\{p, \frac{p}{p-1}\right\} > \frac{d}{\alpha-1}$.

Part 1. *Assume* $2\beta(\alpha-1) - \alpha \geq 0$. *Then, there exist* p *and* r *satisfying* (B.2) *with*

$$p < \frac{d\alpha}{2\beta(\alpha-1)-\alpha}. \tag{B.3}$$

Part 2: *Consider p and r satisfying (B.2). Then $p > 1$, and the following conditions hold*

$$\textit{(a)}\quad \frac{1}{\alpha}\left(\frac{d}{r}+1\right) < 1, \qquad \textit{(b)}\quad \frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)+\frac{1}{\beta} < 1, \qquad \textit{(c)}\quad \frac{1}{p}+\frac{1}{r} \le 1. \tag{B.4}$$

Additionally, if $2\beta(\alpha-1)-\alpha < 0$ or p also satisfies (B.3), then

$$\textit{(d)}\quad \sigma + \frac{1}{2}\left[\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)+\frac{1}{\beta}\right] < 1, \qquad \textit{(e)}\quad \frac{1}{\alpha}\left(\frac{d}{r}+1\right)-1 < \sigma < \frac{1}{\alpha}\left(\frac{d}{r}+1\right). \tag{B.5}$$

Proof. **Part 1.** Note that, since by assumption $2\beta(\alpha-1)-\alpha \ge 0$, we have

$$\frac{d\alpha}{2\beta(\alpha-1)-\alpha} \ge \frac{d}{\beta-1} \tag{B.6}$$

as $\alpha \le 2$. Therefore, for $\alpha < 2$, there exists p in the range defined by (B.2b) and (B.3), i. e., such that $\frac{d}{\beta-1} < p < \frac{d\alpha}{2\beta(\alpha-1)-\alpha}$. Then, for $\alpha < 2$, there exist numbers p and r in the ranges defined by (B.2b) where p satisfies (B.3).

Note that (regardless of whether $2\beta(\alpha-1)-\alpha \ge 0$) there also exist numbers p and r in the ranges defined in (B.2a). Indeed, for $p \le \frac{d}{\beta-1}$, we have

$$\frac{d}{\beta-1} > \frac{2d}{d+\beta-1} \qquad \text{as} \qquad \beta < d+1,$$
$$\frac{d}{\beta-1} > \frac{d}{\alpha+\beta-2} \qquad \text{as} \qquad \alpha > 1.$$

Moreover, since $d-p(\beta-1) \ge 0$,

$$\frac{pd}{d-p(\beta-1)} > \frac{p}{p-1} \qquad \text{as} \qquad p > \frac{2d}{d+\beta-1},$$
$$\frac{pd}{d-p(\beta-1)} > \frac{d}{\alpha-1} \qquad \text{as} \qquad p > \frac{d}{\alpha+\beta-2},$$
$$\frac{pd}{d-p(\beta-1)} > p \qquad \text{as} \qquad \beta > 1.$$

Then, for $\alpha < 2$, there are p and r, with p satisfying (B.3), in the ranges defined by either cases in (B.2). On the other hand, for $\alpha = 2$, such numbers exist only in the ranges defined by (B.2a), and we must have $p < \frac{d}{\beta-1}$ so that (B.3) also holds. Therefore, it is always possible to find numbers p and r in the ranges defined in (B.2a) or (B.2b) where p also satisfies (B.3).

Part 2. Note that $p > 1$ follows from the fact that $\frac{2d}{d+\beta-1} > 1$, since $1 < \beta < d+1$, and $p > \frac{2d}{d+\beta-1}$.

Now, from either restrictions in (B.2), we obtain

$$\frac{1}{r} < \frac{\alpha-1}{d} \quad \text{and} \quad \frac{1}{r} \le 1-\frac{1}{p}, \tag{B.7}$$

and conditions (B.4(a)) and (B.4(c)) follow immediately.

Next, assuming (B.2a), we see that $r < \frac{pd}{d-p(\beta-1)}$ and $d - p(\beta-1) \leq 0$, and assuming (B.2b), we have $\frac{1}{p} - \frac{\beta-1}{d} < 0$ and $\frac{1}{r} \geq 0$. Then, from either cases, it follows that

$$\frac{1}{p} - \frac{\beta-1}{d} < \frac{1}{r}, \tag{B.8}$$

which implies (B.4(b)).

To prove (B.5(d)), note that if $2\beta(\alpha-1) - \alpha < 0$, $\frac{\beta-1}{d} + \frac{\beta(\alpha-2)}{d\alpha} < 0$. Otherwise, p satisfies (B.3). Then, it follows from both scenarios that $\frac{1}{p} > \frac{\beta-1}{d} + \frac{\beta(\alpha-2)}{d\alpha}$, implying $\alpha\left(\frac{\beta-1}{d} - \frac{1}{p}\right) + \frac{\beta(\alpha-2)}{d} < 0$. Hence, as $\frac{1}{r} \geq 0$ and $2\beta - \alpha > 0$, we have

$$\frac{1}{r}(2\beta-\alpha) > \alpha\left(\frac{\beta-1}{d} - \frac{1}{p}\right) + \frac{\beta(\alpha-2)}{d},$$

which leads to

$$1 - \frac{1}{\alpha}\left(\frac{d}{r}+1\right) - \frac{1}{2}\left[\frac{d}{\beta}\left(\frac{1}{p} - \frac{1}{r}\right) + \frac{1}{\beta}\right] < 0,$$

proving (B.5(d)) .

Finally, from (B.4(a)) and (B.4(b)), we obtain

$$\begin{aligned} \frac{1}{\alpha}\left(\frac{d}{r}+1\right) - 1 &< \sigma - 3 + \frac{2}{\alpha}\left(\frac{d}{r}+1\right) + \frac{d}{\beta}\left(\frac{1}{p} - \frac{1}{r}\right) + \frac{1}{\beta} \\ &< \sigma + 2\left[\frac{1}{\alpha}\left(\frac{d}{r}+1\right) - 1\right] + \left[\frac{d}{\beta}\left(\frac{1}{p} - \frac{1}{r}\right) + \frac{1}{\beta} - 1\right] < \sigma, \end{aligned}$$

and from (B.5(d)),

$$\begin{aligned} \sigma - \frac{1}{\alpha}\left(\frac{d}{r}+1\right) &= 2\left\{1 - \frac{1}{\alpha}\left(\frac{d}{r}+1\right) - \frac{1}{2}\left[\frac{d}{\beta}\left(\frac{1}{p} - \frac{1}{r}\right) + \frac{1}{\beta}\right]\right\} \\ &= 2\left\{\sigma - 1 + \frac{1}{2}\left[\frac{d}{\beta}\left(\frac{1}{p} - \frac{1}{r}\right) + \frac{1}{\beta}\right]\right\} \\ &= 2\left\{\sigma + \frac{1}{2}\left[\frac{d}{\beta}\left(\frac{1}{p} - \frac{1}{r}\right) + \frac{1}{\beta}\right]\right\} - 2 < 0, \end{aligned}$$

which proves (B.5(e)). □

Remark B.4. *The lower bound on p in Lemma B.3 is the same as that in [11, Theorem 2.1] used to establish the local and global existence of mild solutions for the parabolic-elliptic form of* (3.1).

Remark B.5. *Consider σ defined in* (B.1) *and p and r in the ranges defined in Lemma B.3.*

1. ***(a)*** *We have $p < 2$ iff $\frac{p}{p-1} > 2$ (since this is true for any number greater than zero).*

(b) *From the fact that $\alpha \in (1,2]$, $\beta \in (1,d]$, $d \geq 2$, the following relations are established:*

- $\dfrac{d}{\alpha-1} \geq 2, \quad$ *thus* $\quad r > 2;$
- $\dfrac{p}{p-1} > \dfrac{d}{\alpha-1}, \quad$ *iff* $\quad 1 < p < \dfrac{d}{d+1-\alpha};$

- $p < 2$ *if* $p < \dfrac{d}{d+1-\alpha}$, *as* $\dfrac{d}{d+1-\alpha} \leq 2$ *for any* α *and* d;
- $\dfrac{d}{d+1-\alpha} > \dfrac{2d}{d+\beta-1}$ *iff* $2\alpha+\beta > d+3$;
- $\dfrac{2d}{d+\beta-1} > \dfrac{d}{\alpha+\beta-2}$ *iff* $2\alpha+\beta > d+3$.

(c) *If* $r = \frac{p}{p-1}$, *we can state that* $\frac{d}{\alpha-1} < r < \frac{2d}{d-\beta+1} < \frac{pd}{d-p(\beta-1)}$.

2. *In more detail, from Remark B.5.1, (B.2) can be rewritten as*

(i) *for* $2\alpha+\beta \leq d+3$,

$$\max\left\{\frac{d}{\alpha-1},\ p\right\} < r < \frac{pd}{d-p(\beta-1)} \qquad \text{if} \qquad \frac{d}{\alpha+\beta-2} < p \leq \frac{d}{\beta-1}, \tag{B.9}$$

$$r > \max\left\{\frac{d}{\alpha-1},\ p\right\} \qquad \text{if} \qquad p > \frac{d}{\beta-1}, \tag{B.10}$$

(ii) *otherwise, for* $2\alpha+\beta > d+3$,

$$\frac{p}{p-1} \leq r < \frac{pd}{d-p(\beta-1)} \qquad \text{if} \qquad \frac{2d}{d+\beta-1} < p < \frac{d}{d+1-\alpha}, \tag{B.11}$$

$$\max\left\{\frac{d}{\alpha-1},\ p\right\} < r < \frac{pd}{d-p(\beta-1)} \qquad \text{if} \qquad \frac{d}{d+1-\alpha} \leq p \leq \frac{d}{\beta-1}, \tag{B.12}$$

$$r > \max\left\{\frac{d}{\alpha-1},\ p\right\} \qquad \text{if} \qquad p > \frac{d}{\beta-1}, \tag{B.13}$$

with, in both cases, the equality $r = p$ *being possible if* $p > d/(\alpha-1)$.

3. *Adding (B.3), for* $2\beta(\alpha-1)-\alpha \geq 0$, *Remark B.5.2 incorporates the following changes:*

- ***Case A.*** $\alpha < 2$

  addition of condition (B.3) to lines (B.10) and (B.13), i. e.,

$$r > \max\left\{\frac{d}{\alpha-1},\ p\right\} \qquad \text{if} \qquad \frac{d}{\beta-1} < p < \frac{d\alpha}{2\beta(\alpha-1)-\alpha}.$$

- ***Case B.*** $\alpha = 2$

  exclusion of (B.10) and (B.13); replacement of $p \leq \frac{d}{\beta-1}$ *with* $p < \frac{d}{\beta-1}$ *in (B.9) and (B.12).*

4. *From (B.1), (B.4(a)) and (B.4(c)), we see that the parameter* σ *is such that* $\sigma > 0$, *which is a better lower bound than the one given by (B.5(e)). Moreover, (B.5(e)) ensures that* $\sigma < 1$. *Then, seeking a better upper bound for* σ *that has the advantage over the one in (B.5(e)) of* *depending only on α and β, we find, when $\alpha, \beta \in (1,2)$, that*

$$\sigma < \frac{\max\{\alpha,\beta\} - 2}{\min\{\alpha,\beta\}} + 1.$$

Indeed, we can rewrite σ defined in (B.1) as $\sigma = 2 - \frac{1}{\alpha\beta}\left[d\left(\frac{1}{r}(\beta-\alpha) + \frac{\alpha}{p}\right) + \beta + \alpha\right]$. Then, due to relation (B.7), if $\alpha \geq \beta$, we obtain

$$\sigma < 2 - \frac{1}{\alpha\beta}\left[d\left(\frac{\alpha-1}{d}(\beta-\alpha) + \frac{\alpha}{p}\right) + \beta + \alpha\right] = \frac{\alpha-1}{\beta} + \frac{\beta-1}{\beta} - \frac{d}{\beta p} < \frac{\alpha-2}{\beta} + 1;$$

and, due to relation (B.8), if $\alpha < \beta$, we obtain

$$\sigma < 2 - \frac{1}{\alpha\beta}\left\{d\left[\left(\frac{1}{p} - \frac{\beta-1}{d}\right)(\beta-\alpha) + \frac{\alpha}{p}\right] + \beta + \alpha\right\} = \frac{\alpha-1}{\alpha} + \frac{\beta-1}{\alpha} - \frac{d}{\alpha p} < \frac{\beta-2}{\alpha} + 1.$$

Therefore, the estimate follows.

Lemma B.6. *Consider p and r in the ranges defined in (B.2) and let $\wp$ be given by*

$$\begin{aligned} &\wp \in \left[\frac{d}{\alpha-1}, r\right] && \text{if} \quad \alpha \leq \beta, \\ &\wp = r && \text{if} \quad \alpha > \beta. \end{aligned} \tag{B.14}$$

Then, $2 \leq \wp \leq r$ and

$$\frac{1}{\alpha}\left(\frac{d}{r} + 1\right) + \frac{d}{\beta}\left(\frac{1}{\wp} - \frac{1}{r}\right) \leq 1. \tag{B.15}$$

Moreover, for some values of α, β, d and r, there exist $\wp$ defined in (B.14) such that the equality in (B.15) holds.

Proof. It is easy to see that $2 \leq \wp \leq r$ from the definition of $\wp$ and the fact that $\frac{d}{\alpha-1} \geq 2$ (Remark B.5.1). Next note that, if $\alpha > \beta$, condition (B.15) falls into (B.4(a)). If $\alpha \leq \beta$, then $\frac{1}{\wp} \leq \frac{\alpha-1}{d}$, and from condition (B.4(a)), we have

$$\begin{aligned} \frac{1}{\alpha}\left(\frac{d}{r} + 1\right) + \frac{d}{\beta}\left(\frac{1}{\wp} - \frac{1}{r}\right) &\leq \frac{1}{\alpha}\left(\frac{d}{r} + 1\right) + \frac{d}{\beta}\left(\frac{\alpha-1}{d} - \frac{1}{r}\right) \\ &= \frac{1}{\alpha}\left(\frac{d}{r} + 1\right)\left(1 - \frac{\alpha}{\beta}\right) + \frac{\alpha}{\beta} \\ &< 1 - \frac{\alpha}{\beta} + \frac{\alpha}{\beta} = 1. \end{aligned}$$

Note that for $\beta > \alpha$ and $r \leq \frac{d\alpha}{\alpha-1}$ there exist $\wp$ defined in (B.14) such that the equality in (B.15) holds. Moreover, for $\alpha = \beta$ the equality in (B.15) holds for $\wp = \frac{d}{\alpha-1}$. □

The following lemmas are pertinent to Section 3.3. They address the proof of some inequalities that must be satisfied by p_1 and p_2, the parameters defining the space of the initial data: $\rho_0 \in L^{p_1}(\mathbb{R}^d)$ and $\nabla c_0 \in L^{p_2}(\mathbb{R}^d)$.

Lemma B.7. *Consider p and r as in (B.2), with p satisfying (B.3) if $2\beta(\alpha-1)-\alpha \geq 0$, and p_1 given by $p_1 = \frac{pd}{\alpha\sigma p+d}$. Then, $1 \leq p_1 < p$.*

Proof. Note that $p_1 \geq 1$ if and only if $\left[\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)+\frac{1}{\beta}\right](\beta-\alpha) \leq d+2(1-\alpha)$. Moreover, as $\alpha \leq 2$ and $d \geq 2$, we see that $d+2(1-\alpha) \geq 0$. Therefore, if $\beta \leq \alpha$, the inequality is automatically verified. Otherwise, from (B.4(b)) we obtain $\left[\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)+\frac{1}{\beta}\right](\beta-\alpha) < \beta-\alpha$, and, as $\beta \leq d$ and $\alpha \leq 2$, it follows that $\beta-\alpha \leq d-\alpha+2-\alpha = d+2(1-\alpha)$; thus, $p_1 \geq 1$. Furthermore, $p_1 < p$, as $\alpha\sigma p > 0$ implies $\frac{d}{\alpha\sigma p+d} < 1$. □

Lemma B.8. *Consider p and r as in (B.2), with p satisfying (B.3) if $2\beta(\alpha-1)-\alpha \geq 0$, and p_2 given by*

$$p_2 = \frac{dr\alpha}{\beta(r(\alpha-1)-d)+d\alpha}. \tag{B.16}$$

Then, $1 < p_2 < r$ and the following condition is satisfied

$$\sigma + \frac{d}{\beta}\left(\frac{1}{p_2}-\frac{1}{r}\right) < 1. \tag{B.17}$$

Proof. Note first that $\beta(r(\alpha-1)-d)+d\alpha > 0$ as $r > \frac{d}{\alpha-1}$. Then, (B.16) can be rewritten as

$$\frac{1}{p_2} = \frac{\beta}{d}\left(1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)\right)+\frac{1}{r}, \tag{B.18}$$

and it follows from condition (B.4(a)) that $\frac{1}{p_2} > \frac{1}{r}$. Thus, $r > p_2$. Moreover, $p_2 > 1$. Indeed, from (B.18), conditions (B.5(d)) and (B.4(c)) and the fact that $d \geq 2$, we obtain

$$\begin{aligned}
\frac{1}{p_2} &= \frac{\beta}{d}\left[\sigma+\frac{1}{2}\left(\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)+\frac{1}{\beta}\right)-1+\frac{1}{2}\left(\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)+\frac{1}{\beta}\right)\right]+\frac{1}{r} \\
&\leq \frac{1}{2}\left(\frac{1}{p}+\frac{1}{r}+\frac{1}{d}\right) \\
&\leq \frac{3}{4} < 1.
\end{aligned}$$

Next, note that (B.18) leads to $\frac{d}{\beta}\left(\frac{1}{p_2}-\frac{1}{r}\right) = 1-\frac{1}{\alpha}\left(\frac{d}{r}+1\right)$. Therefore, condition (B.17) is equivalent to $\sigma < \frac{1}{\alpha}\left(\frac{d}{r}+1\right)$, which is true from estimate (B.5(e)). □

In the following lemma, new ranges for the parameters p and r are defined to satisfy the supplementary condition: $r\sigma \leq d/\alpha$. This condition is essential for proving that the global mild solution ρ belongs to the space $L^{p_1}(\mathbb{R}^d)$, where the parameter p_1 (defined in Lemma B.7) specifies the space of the initial data $\rho_0 \in L^{p_1}(\mathbb{R}^d)$.

Lemma B.9. *Consider p and r satisfying*

$$\max\left\{\frac{2d}{d+\beta-1},\ \frac{d}{\alpha+\beta-2}\right\} < p \leq \frac{2d}{(\alpha-1)+2(\beta-1)} \quad \text{and}$$
$$\max\left\{p,\ \frac{p}{p-1},\ \frac{d}{\alpha-1}\right\} < r < \frac{pd}{d-p(\beta-1)}, \quad \text{or} \tag{B.19a}$$

$$\frac{2d}{(\alpha-1)+2(\beta-1)} < p < \frac{\alpha d}{\max\{2\beta(\alpha-1)-\alpha,\ \alpha(\alpha-2)+\beta\}} \quad \text{and}$$
$$\max\left\{p,\ \frac{p}{p-1},\ \frac{d}{\alpha-1}\right\} < r \leq \frac{(2\beta-\alpha)pd}{[\beta(\alpha-1)+\alpha(\beta-1)]p-\alpha d}, \tag{B.19b}$$

where, in both cases, the equality $r = \max\left\{p,\ \frac{p}{p-1}\right\}$ *is possible if* $\max\left\{p,\ \frac{p}{p-1}\right\} > \frac{d}{\alpha-1}$.

Part 1: *There exist p and r such that (B.19) holds.*

Part 2: *Consider p and r satisfying constraint (B.19). Then $p > 1$, and the following conditions are satisfied:*

(a) $\frac{1}{\alpha}\left(\frac{d}{r}+1\right) < 1,$

(b) $\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)+\frac{1}{\beta} < 1,$

(c) $\frac{1}{p}+\frac{1}{r} \leq 1,$

(d) $\sigma + \frac{1}{2}\left[\frac{d}{\beta}\left(\frac{1}{p}-\frac{1}{r}\right)+\frac{1}{\beta}\right] < 1,$ (B.20)

(e) $-1 < \sigma - \frac{1}{\alpha}\left(\frac{d}{r}+1\right) < 0,$

(f) $r\sigma \leq d/\alpha.$

Remark B.10. *Consider p and r satisfying (B.19b). Note that, since $\beta > \alpha(2-\alpha)$ for any $\alpha \in (1,2]$, $\max\{2\beta(\alpha-1)-\alpha,\ \alpha(\alpha-2)+\beta\} > 0$. Moreover, as $\beta > \alpha/2$ implies $\frac{2d}{(\alpha-1)+2(\beta-1)} > \frac{\alpha d}{\beta(\alpha-1)+\alpha(\beta-1)}$, we have $p > \frac{\alpha d}{\beta(\alpha-1)+\alpha(\beta-1)}$, and hence $[\beta(\alpha-1)+\alpha(\beta-1)]p-\alpha d > 0$. Therefore,*

$$\frac{\alpha d}{\max\{2\beta(\alpha-1)-\alpha,\ \alpha(\alpha-2)+\beta\}} > 0 \quad \text{and} \quad \frac{(2\beta-\alpha)pd}{[\beta(\alpha-1)+\alpha(\beta-1)]p-\alpha d} > 0.$$

Proof. **Part 1:** Consider first the restriction given by (B.19a). For $(\alpha,\beta) \neq (2,d)$, we obtain

$$\frac{2d}{(\alpha-1)+2(\beta-1)} > \frac{2d}{d+\beta-1} \qquad \text{as} \qquad \alpha+\beta < d+2,$$
$$\frac{2d}{(\alpha-1)+2(\beta-1)} > \frac{d}{\alpha+\beta-2} \qquad \text{as} \qquad \alpha > 1.$$

Moreover, note that

$$\frac{2d}{(\alpha-1)+2(\beta-1)} < \frac{d}{\beta-1} \tag{B.21}$$

as $\alpha > 1$. Then, $p \leq \frac{2d}{(\alpha-1)+2(\beta-1)}$ is also less then $\frac{d}{\beta-1}$. Thus, $d - p(\beta-1) \geq 0$ and $\frac{pd}{d-p(\beta-1)} > \max\left\{\frac{p}{p-1},\ \frac{d}{\alpha-1},\ p\right\}$, as we have already shown in Lemma B.3. Therefore, there exist numbers p and r in the ranges defined in (B.19a) for $(\alpha,\beta) \neq (2,d)$.

Now, consider the restriction given by (B.19b) in the case $(\alpha, \beta) = (2, d)$. In this scenario, (B.19b) becomes

$$\frac{2d}{2d-1} < p < \frac{2d}{2d-2} \quad \text{and} \quad \frac{p}{p-1} \leq r \leq \frac{2(d-1)pd}{2(d-1)p + dp - 2d}, \tag{B.22}$$

since $\frac{2d}{2d-2} \leq d < \frac{p}{p-1} < 2d$. Notice that there exists number r in the range defined by (B.22), since $\frac{2d}{2d-1} < p < 2$ implies $-\frac{1}{2} < \frac{d}{2(d-1)} - \frac{2d}{2(d-1)p} < 0$, which leads to

$$\frac{p}{p-1} \leq \frac{2(d-1)pd}{2(d-1)p + dp - 2d} = \frac{d}{1 + \left(\frac{d}{2(d-1)} - \frac{2d}{2(d-1)p}\right)}.$$

Moreover, it is trivial to see that there exists a number p in the range defined by (B.22). Thus, we obtain that there exist numbers p and r in the ranges defined by (B.19b) for $(\alpha, \beta) = (2, d)$.

Part 2: To prove (B.20(a)-(e)), first we assume (B.19a). In that case, note that p also satisfies (B.3) if $2\beta(\alpha - 1) - \alpha \geq 0$, since from (B.21) and (B.6) it follows that

$$\frac{2d}{(\alpha-1) + 2(\beta-1)} < \frac{\alpha d}{2\beta(\alpha-1) - \alpha}.$$

Moreover, from (B.21), we see that p and r also satisfy (B.2a). Therefore, from Lemma B.3, (B.20(a)-(e)) follows.

On the other hand, if we assume (B.19b), we have, for $d - p(\beta - 1) \geq 0$,

$$\frac{(2\beta - \alpha)pd}{[\beta(\alpha-1) + \alpha(\beta-1)]p - \alpha d} < \frac{pd}{d - p(\beta-1)},$$

since $p > \frac{2d}{(\alpha-1)+2(\beta-1)}$. Moreover, p satisfies (B.3) if $2\beta(\alpha - 1) - \alpha \geq 0$, since in that case

$$\frac{\alpha d}{\max\{2\beta(\alpha-1) - \alpha,\ \alpha(\alpha-2) + \beta\}} \leq \min\left\{\frac{\alpha d}{2\beta(\alpha-1) - \alpha}, \frac{\alpha d}{\alpha(\alpha-2) + \beta}\right\}.$$

Then, again from Lemma B.3, (B.20(a)-(e)) follows.

To prove (B.20(f)), note that this is equivalent to

$$r \leq \frac{(2\beta - \alpha)pd}{[\beta(\alpha-1) + \alpha(\beta-1)]p - \alpha d}.$$

Then, assuming (B.19a), we have

$$\frac{pd}{d - p(\beta-1)} \leq \frac{(2\beta - \alpha)pd}{[\beta(\alpha-1) + \alpha(\beta-1)]p - \alpha d}, \tag{B.23}$$

since $p \leq \frac{2d}{(\alpha-1)+2(\beta-1)}$, and (B.20(f)) follows. On the other hand, assuming (B.19b), (B.20(f)) follows directly from (B.23). □