RELATÓRIO DE PESQUISA 1994

OPTIMAL STOPPING TIME FOR A POISSON POINT PROCESS

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Maio RP 18/94

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ABSTRACT – A plane dropping leaflets passes through a point and looking sequentially at regions in the plane, the estimation of the direction followed by the plane is desired. This is an example of a non-homogeneous Poisson process and the maximum likelihood estimator can be obtained explicitly when search regions are circles. In this case, the existence of optimal stopping time, when the cost function is the sum of the cost of sampling plus the cost of estimating error, can be established but numerical results are hard to be obtained. Suboptimal stopping times are proposed.

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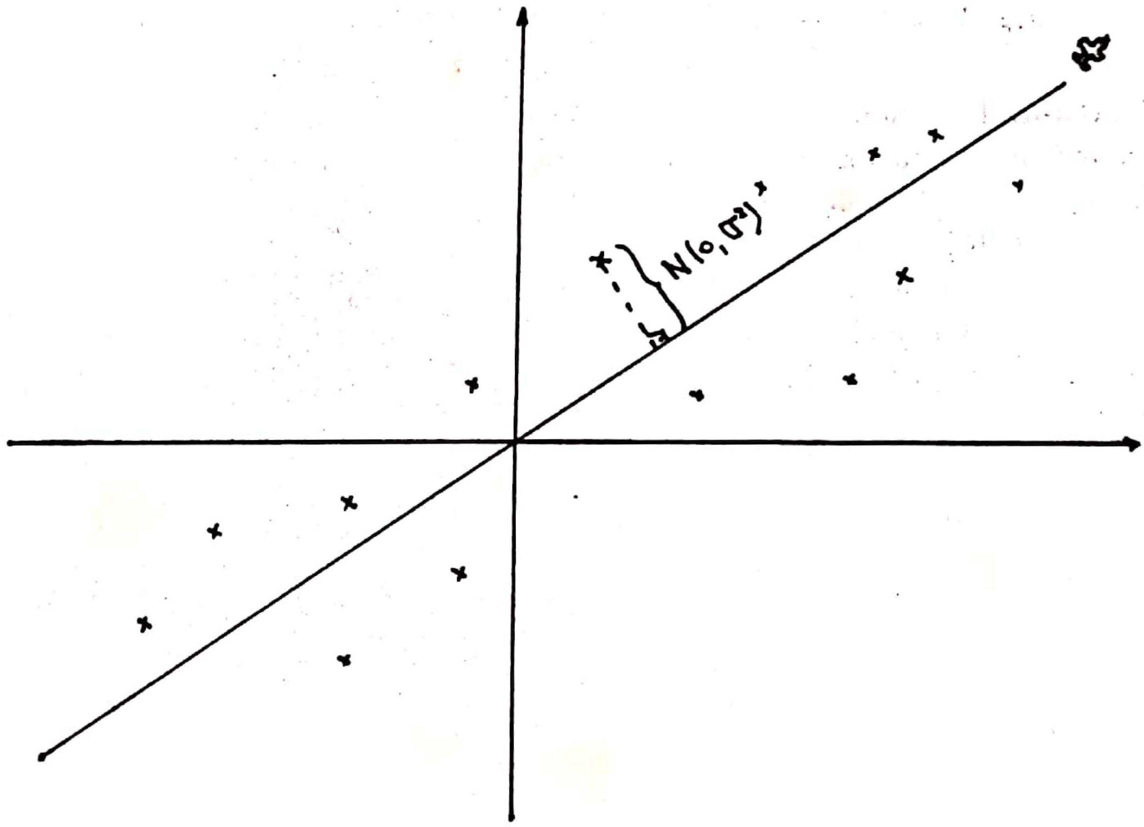
Maio – 1994

1 Optimal and Suboptimal Stopping times for a Poisson Point Process

1.1 Introduction

Suppose the following situation: In the middle of the night, you are awaken by a plane dropping leaflets right above you. In the following morning looking sequentially at regions Γ_t you want to estimate which direction the plane flew.



Let θ be the direction of the plane and λ the intensity at which the leaflets were thrown. Assume the distance from a leaflet to the line followed by the plane ($y = \alpha x, \alpha = \tan\theta$) is normally distributed with mean zero and variance σ^2, i.e., $r_i \sim N(0, \sigma^2)$.

Let N be the point process obtained by the positions of the leaflets. Therefore, N is a non-homogeneous Poisson process with mean measure determined by $(\sigma^2, \lambda, \theta)$.

Let $\hat{\theta}_t$ be the maximum likelihood estimator of θ based on L_{Γ_t}, where Γ_t is a random set chosen sequentially in order to minimize

$$\mathbf{E}[m(\Gamma_t) + C(\hat{\theta}_t - \theta)^2] \tag{1}$$

where $m(A)$ = Lebesgue measure of A.

1.2 Mean Measure

The process N is a Poisson process on $(\Omega, \mathcal{F}, \mathbf{P}_\theta)$ with parameter measure M_θ defined by

$$M_\theta(A) = \lambda \int_A \frac{1}{\sqrt{2\pi\sigma}} e^{-(x\sin\theta - y\cos\theta)^2/2\sigma^2}\, dx\, dy, \qquad A \in \mathcal{B}(\mathbf{R}^2)$$

Justification: It is enough to prove the result for $\theta = 0$ since for $\theta \neq 0$ the process can be obtained by a rotation of a process generated by $\theta = 0$. Let A be any bounded set and $(X_i, Y_i), i = 1, 2, \ldots$ be any enumeration of the points of N. Define $A_X = \{x; (x, y) \in A, \text{for some } y \in \mathbf{R}\}$.
Let N_X be the random variable $N_X = \#\{i; X_i \in A_X\}$ and $I = \{i_1, i_2, \ldots, i_{N_X}\}$ be such that $X_k \in A_X$ if $k \in I$. In this case, from the properties of the Poisson process, we know N_X is a Poisson random variable with mean $\lambda m(A_X)$. Then, define $A(x) = \{y; (x, y) \in A\}$. Therefore,

$$N(A) = \sum_{j=1}^{N_X} \mathbf{I}_{\{Y_{i_j} \in A(X_{i_j})\}}$$

Given N_X, we have $X_{i_1}, \ldots, X_{i_{N_X}}$ i.i.d. with common distribution $U(A_X)$ and

$\mathbf{E}[\mathrm{N(A)} \mid X_{i_1}, \ldots, X_{i_{N_X}}] = \sum_{j=1}^{N_X} \int_{A(X_{i_j})} \frac{1}{\sqrt{2\pi\sigma}} e^{-y^2/2\sigma^2} dy$
Consequently,

$$\begin{aligned}
\mathbf{E}[N(A)] &= \mathbf{E}[N_X] \int_{A_X} \int_{A(x)} \frac{1}{m(A_X)} \frac{1}{\sqrt{2\pi\sigma}} e^{-y^2/2\sigma^2}\, dy\, dx \\
&= \lambda m(A_X) \int_{A_X} \int_{A(x)} \frac{1}{m(A_X)} \frac{1}{\sqrt{2\pi\sigma}} e^{-y^2/2\sigma^2}\, dy\, dx \\
&= \lambda \int_A \frac{1}{\sqrt{2\pi\sigma}} e^{-y^2/2\sigma^2}\, dy\, dx
\end{aligned}$$

1.3 Maximum Likelihood Estimation

Let A be a bounded subset belonging to $\mathcal{B}(\mathbf{R}^2)$ (Borel σ-algebra of $\mathbf{R}^2$). Let $\mathcal{F}_A = \sigma(N(B); B \subset A)$ and $\mathbf{P}_\theta^A = \mathbf{P}_\theta | \mathcal{F}_A$. For A bounded, $\mathbf{P}_\theta^A \ll \mathbf{P}_0^A$. Moreover, since $M_\theta \ll m$, $\mathbf{P}_\theta^A \ll \mathbf{P}^m | \mathcal{F}_A$, where m is the Lebesgue measure and under $\mathbf{P}^m$, N is a Poisson process with Lebesgue mean measure. Also, denote by $\mu_\theta(x,y)$ the density of M_θ with respect to the Lebesgue measure.

We want to find the likelihood ratio

$$L_A = \frac{d\mathrm{P}_\theta|\mathcal{F}_A}{d\mathrm{P}^m|\mathcal{F}_A}$$

Let $(X_i, Y_i), i = 1, 2, \ldots, N(A)$ denote the points of N in A. Then, by Daley and Vere-Jones (1988), the *likelihood function* can be written as

$$L_A(\theta) = \exp\{\sum_{i=1}^{N(A)} \log \mu_\theta(X_i, Y_i) - \int_A (\mu_\theta(x,y) - 1)\, dx\, dy\}$$

In our case,

$$\begin{aligned}
\log L_A(\theta) &= \sum_{i=1}^{N(A)} \log(\frac{\lambda}{\sqrt{2\pi}\sigma} e^{-(X_i \sin\theta - Y_i \cos\theta)^2/2\sigma^2}) \\
&\quad - \int_A (\frac{\lambda}{\sqrt{2\pi}\sigma} e^{-(x\sin\theta - y\cos\theta)^2/2\sigma^2} - 1)\, dx\, dy \\
&= \sum_{i=1}^{N(A)} \log(\frac{\lambda}{\sqrt{2\pi}\sigma}) - \sum_{i=1}^{N(A)} \frac{(X_i \sin\theta - Y_i\cos\theta)^2}{2\sigma^2} \\
&\quad - \int_A (\frac{\lambda}{\sqrt{2\pi}\sigma} e^{-(x\sin\theta - y\cos\theta)^2/2\sigma^2} - 1)\, dx\, dy \qquad (2)
\end{aligned}$$

Assume σ and λ are known

$$\begin{aligned}
\frac{\partial \log L_A(\theta)}{\partial\theta} &= \\
&= -\sum_{i=1}^{N(A)} \frac{2(X_i\sin\theta - Y_i\cos\theta)}{2\sigma^2}(X_i\cos\theta + Y_i\sin\theta) \\
&\quad - \int_A \frac{\lambda}{\sqrt{2\pi}\sigma}\frac{1}{2\sigma^2} e^{-(x\sin\theta - y\cos\theta)^2/2\sigma^2}[-2(x\sin\theta - y\cos\theta)(x\cos\theta + y\sin\theta)]\, dx\, dy \\
&= -\sum_{i=1}^{N(A)} \frac{(X_i\sin\theta - Y_i\cos\theta)(X_i\cos\theta + Y_i\sin\theta)}{\sigma^2} \\
&\quad + \int_A \frac{\lambda}{\sqrt{2\pi}\sigma^3} e^{-(x\sin\theta - y\cos\theta)^2/2\sigma^2}(x\sin\theta - y\cos\theta)(x\cos\theta + y\sin\theta)\, dx\, dy \\
&= -\sum_{i=1}^{N(A)} \frac{(X_i^2 - Y_i^2)}{2\sigma^2}\sin 2\theta + \frac{X_iY_i}{\sigma^2}\cos 2\theta
\end{aligned}$$

$$+\underbrace{\int_A \frac{\lambda}{\sqrt{2\pi\sigma^3}} e^{-(x\sin(\theta-y\cos(\theta))^2/2\sigma^2}[\frac{x^2-y^2}{2}\sin 2\theta - xy\cos 2\theta]\,dx\,dy}_{(*)}$$

Letting $u = x\sin\theta - y\cos\theta$ and $v = x\cos\theta + y\sin\theta$ in the integral above we have:

$$\begin{aligned}(*) &= \int_{T(A)} \frac{\lambda}{\sqrt{2\pi\sigma^3}} e^{-u^2/2\sigma^2}[-\frac{(u^2-v^2)\cos 2\theta\sin 2\theta}{2} + \frac{2uv\sin^2 2\theta}{2} \\ &\quad + \frac{(u^2-v^2)\sin 2\theta\cos 2\theta}{2} + uv\cos^2 2\theta]\,du\,dv \\ &= \int_{T(A)} \frac{2\lambda}{\sqrt{2\pi\sigma^3}} e^{-u^2/2\sigma^2} uv(\sin^2 2\theta + \cos^2 2\theta)\,du\,dv \\ &= \int_{T(A)} \frac{2\lambda}{\sqrt{2\pi\sigma^3}} uve^{-u^2/2\sigma^2}\,du\,dv\end{aligned}$$

If A is a disk with center at origin and radius r, $T(A) = A$ and $(*) = 0$.

Therefore,

$$\frac{\partial \log L_A(\theta)}{\partial\theta} = -\frac{\sin 2\theta}{2\sigma^2}\sum_{i=1}^{N(A)}(X_i^2 - Y_i^2) + \frac{\cos 2\theta}{\sigma^2}\sum_{i=1}^{N(A)} X_iY_i$$

Consequently, if A is a disk of center at origin and radius r, the maximum likelihood estimator $\hat{\theta}_t$ is the value of θ that maximizes the expression above. Note that solving $\frac{\partial \log L_A(\theta)}{\partial\theta} = 0$, we have

$$\tilde{\theta} = \frac{1}{2}\arctan\frac{2\sum_{i=1}^{N(A)} X_iY_i}{\sum_{i=1}^{N(A)}(X_i^2 - Y_i^2)} \tag{3}$$

Note that the value of $\tilde{\theta}$ above does not always give the maximum value; sometimes it gives the minimum value. Using the second derivative to find the maximum we have that $\tilde{\theta}$ is the maximum likelihood estimator if $\sum(X_i^2 - Y_i^2) > 0$ and it gives the minimum if $\sum(X_i^2 - Y_i^2) < 0$. In this case $\tilde{\theta} \pm \pi/2$ gives the maximum likelihood estimator of θ. Note that we are only interested in values of θ in the interval $[-\pi/2, \pi/2]$. Consequently, the maximum likelihood estimator of θ based on observations made in the disk Γ_t of center at the origin and radius t is

$$\hat{\theta}_t = \frac{1}{2}\arctan\frac{2\sum_{\Gamma_t} X_iY_i}{\sum_{\Gamma_t}(X_i^2 - Y_i^2)} + \frac{\pi}{2}\mathrm{sgn}(\sum_{\Gamma_t} X_iY_i)\mathrm{I}(\sum_{\Gamma_t}(X_i^2 - Y_i^2) < 0) \tag{4}$$

1.3.1 Consistency of the Maximum Likelihood Estimator

In this section, without loss of generality we can assume $\theta = 0$, since for $\theta \neq 0$ the process can be obtained by a rotation of a process generated by $\theta = 0$.

We want to prove that $\hat{\theta}_t \to 0$ as $t \to \infty$.
In order to do this we need a continuous version of **Kronecker's Lemma**

Kronecker's Lemma: If $\sum_k (b_k/a_k) < \infty$ and $a_k \to \infty$, as $k \to \infty$, then

$$\lim_n \frac{1}{a_n} \sum_{k=1}^{n} b_k = 0$$

Continuous Version: Let A be an increasing process, $A(t) \to \infty$ as $t \to \infty$, if

$$\int_0^\infty \frac{1}{A(s-)} dB(s) < \infty$$

then

$$\lim_{t\to\infty} \frac{B(t)}{A(t)} = 0$$

Proof: Let $dH(s) = \frac{1}{A(s-)} dB(s)$. Then,

$$\begin{aligned} B(t) &= \int_0^t A(s-) \frac{1}{A(s-)} dB(s) = \int_0^t A(s-) dH(s) \\ &= A(t)H(t) - \int_0^t H(s-) dA(s) - [H, A] \quad \text{(by Ito's Formula)} \\ &= A(t)H(t) - \int_0^t H(s) dA(s) \end{aligned}$$

Consequently, $\frac{B(t)}{A(t)} = H(t) - \frac{1}{A(t)} \int_0^t H(s) dA(s)$. Since $H(t) = \int_0^t \frac{1}{A(s-)} dB(s)$ converges as $t \to \infty$, we have

$$H(t) - \frac{1}{A(t)} \int_0^t H(s) dA(s) \to 0$$

as $t \to \infty$, if $A(t) \to \infty$ as $t \to \infty$.
In fact, let C and t_0 be such that, $|H(t) - C| < \epsilon$ for $t > t_0$, then

$$\begin{aligned} &|H(t) - \frac{1}{A(t)} \int_0^t H(s) dA(s)| = \\ &= |H(t) - C - \frac{1}{A(t)} \int_0^{t_0} (H(s) - C) dA(s) + \frac{1}{A(t)} \int_{t_0}^t (H(s) - C) dA(s)| \end{aligned}$$

$$
\begin{aligned}
&\leq |H(t) - C| + \frac{1}{A(t)} \int_0^{t_0} |H(s) - C| dA(s) + \frac{1}{A(t)} \int_{t_0}^{t} |H(s) - C| dA(s) \\
&\leq \epsilon + \frac{\epsilon(A(t) - A(t_0))}{A(t)} + \frac{1}{A(t)} \underbrace{\sup_{s \leq t_0} |H(s) - C| A(t_0)}_{D} \\
&\leq 2\epsilon + \frac{(\epsilon + D)A(t_0)}{A(t)}
\end{aligned}
$$

Therefore,

$$\frac{B(t)}{A(t)} \to 0 \quad \text{as } t \to \infty.$$

Consistency of the maximum likelihood estimator: To prove that $\hat{\theta}_t \to 0$ a.s., it is enough to prove that:

1. $\frac{1}{t^3} \sum_{\Gamma_t} X_i Y_i \to 0$ a.s.
2. $\frac{1}{t^3} \sum_{\Gamma_t} (X_i^2 - Y_i^2) \to C$, $C > 0$ a.s.
3. $\mathrm{I}(\sum_{\Gamma_t} (X_i^2 - Y_i^2) < 0) \to 0$ a.s.

Note that 3 follows immediately from 2.

Note: The normalizing constant was chosen to be t^3 because, if $S(t) = \{(x, y); -t \leq x \leq t\}$, then

$$\mathrm{E}(\sum_{i=1}^{N(\Gamma_t)} X_i^2) \leq \mathrm{E}(\sum_{i=1}^{N(S_t)} X_i^2) = (2/3)\lambda t^3.$$

$$\mathrm{E}(\sum_{i=1}^{N(\Gamma_t)} Y_i^2) \leq \mathrm{E}(\sum_{i=1}^{N(S_t)} Y_i^2) = 2\lambda\sigma^2 t$$

($X_1, X_2, \ldots X_{N(S_t)}$ are iid random variables with common distribution $U(-t, t)$,$Y_1, Y_2, \ldots, Y_{S_t}$ are iid random variables with common distribution $N(0, \sigma^2)$. Also, $\mathrm{E}(N(S_t)) = 2\lambda t$.)

In order to prove 1. and 2. above note the following fact: for any function $f(.,.)$,

$$M(t) = \int_{\Gamma_t} f(x, y) N(dx, dy) - \int_{\Gamma_t} f(x, y) \mu(dx, dy)$$

is a martingale.

We want to prove that $\frac{M(t)}{t^3} \to 0$ a.s. as $t \to \infty$ (equivalently, $\frac{M(t)}{(t+1)^3} \to 0$ a.s as $t \to \infty$). By the continuous version of the Kronecker's lemma, it is enough to prove

$$\int_0^\infty \frac{1}{(s+1)^3} dM(s) < \infty \qquad \text{a.s.}$$

which follows from

$$\mathbf{E}[(\int \frac{1}{(s+1)^3} dM(s))^2] = \mathbf{E}[\int \frac{1}{(s+1)^3} d[M]_s] < \infty$$

But

$$[M]_t = \int_{\Gamma_t} f^2(x,y) N(dx,dy)$$

However,

$$[M]_t - \int_{\Gamma_t} f^2(x,y) \mu(dx,dy)$$

is a martingale and we have to prove

$$\int_0^\infty \frac{1}{(s+1)^6} dA(s) < \infty$$

where

$$A(s) = \int_{\Gamma_s} f^2(x,y) \mu(dx,dy)$$

Particular Case 1: $f(x,y) = xy$

$$\int_{\Gamma_t} xy\mu(dx,dy) = \frac{\lambda}{\sqrt{2\pi}\sigma} \int_{-t}^{t} \int_{-\sqrt{t^2-y^2}}^{\sqrt{t^2-y^2}} xye^{-y^2/2^2}\, dx\, dy = 0$$

Therefore,

$$M(t) = \int_{\Gamma_t} xyN(dx,dy) = \sum_{\Gamma_t} X_i Y_i$$

In this case,

$$\begin{aligned}
\lim_{t\to\infty} \int_0^t \frac{1}{(s+1)^6} dA(s) &= \lim_{t\to\infty} \int_{\Gamma_t} \frac{x^2y^2}{(\sqrt{x^2+y^2}+1)^6} \mu(dx,dy) \\
&= \lim_{t\to\infty} \int_{\Gamma_t} \frac{x^2y^2}{(\sqrt{x^2+y^2}+1)^6} \frac{\lambda}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2}\, dx\, dy \\
&\le \lim_{t\to\infty} \int_{-t}^{t} \int_{-\infty}^{\infty} \frac{x^2y^2}{(|x|+1)^6} \frac{\lambda}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2}\, dx\, dy \\
&= \int_{-\infty}^{\infty} \lambda\sigma^2 \frac{x^2}{(|x|+1)^6}\, dx < \infty
\end{aligned}$$

Particular Case 2: $f(x,y) = x^2 - y^2$

Let

$$G(t) = \int_{\Gamma_t}(x^2-y^2)\mu(dx,dy) = \frac{\lambda}{\sqrt{2\pi}\sigma}\int_{\Gamma_t} x^2 e^{-y^2/2\sigma^2}\,dx\,dy - \frac{\lambda}{\sqrt{2\pi}\sigma}\int_{\Gamma_t} y^2 e^{-y^2/2\sigma^2}\,dx\,dy$$

Note that,

$$\begin{aligned}
\frac{1}{t^3}\int_{\Gamma_t} y^2 e^{-y^2/2\sigma^2}\,dx\,dy &= \frac{1}{t^3}\int_{-t}^{t}\sqrt{t^2-y^2}y^2 e^{-y^2/2\sigma^2}\,dy \\
&\leq \frac{1}{t^3}\int_{-t}^{t} t^2 y^2 e^{-y^2/2\sigma^2}\,dy \\
&\leq \frac{1}{t}\int_{-\infty}^{\infty} y^2 e^{-y^2/2\sigma^2}\,dy \to 0 \qquad \text{as } t\to\infty
\end{aligned}$$

Also, let

$$I(t) = \int_{\Gamma_t} x^2\frac{\lambda}{\sqrt{2\pi}\sigma}e^{-y^2/2\sigma^2}\,dx\,dy - \int_{-t}^{t}\lambda x^2 dx$$

We claim that $I(t)/t^3 \to 0$ as $t\to\infty$. Note that,

$$\begin{aligned}
I(t) &= \int_{-t}^{t}\lambda x^2 [\int_{-\sqrt{t^2-x^2}}^{\sqrt{t^2-x^2}}\frac{1}{\sqrt{2\pi}\sigma}e^{-y^2/2\sigma^2}dy - 1]dx \\
&= -4[\int_0^t \frac{\lambda}{\sqrt{2\pi}\sigma}e^{-y^2/2\sigma^2}\int_{\sqrt{t^2-y^2}}^{t} x^2 dx + \int_t^\infty \frac{\lambda}{\sqrt{2\pi}\sigma}e^{-y^2/2\sigma^2}\int_0^t x^2 dx]\,dy\,dx \\
&= -4[\int_0^t \frac{\lambda}{\sqrt{2\pi}\sigma}e^{-y^2/2\sigma^2}(\frac{t^3-(t^2-y^2)^{\frac{3}{2}}}{3}) + \int_t^\infty \frac{\lambda}{\sqrt{2\pi}\sigma}e^{-y^2/2\sigma^2}\frac{t^3}{3}] \\
&= -4[\int_0^\infty \frac{\lambda}{\sqrt{2\pi}\sigma}e^{-y^2/2\sigma^2}(\frac{t^3}{3} - \frac{(t^2-y^2)^{\frac{3}{2}}}{3}I_{(y<t)})dy]
\end{aligned}$$

Therefore,

$$\frac{|I(t)|}{t^3} = 4[\int_0^\infty \frac{\lambda}{\sqrt{2\pi}\sigma}e^{-y^2/2\sigma^2}(\frac{1}{3} - \frac{(1-y^2/t^2)^{\frac{3}{2}}}{3})I_{(y<t)}dy]$$

But,

$$\frac{4\lambda}{\sqrt{2\pi}\sigma}|e^{-y^2/2\sigma^2}(\frac{1}{3} - \frac{(1-y^2/t^2)^{\frac{3}{2}}}{3})I_{(y<t)}| \leq \frac{8\lambda}{3\sqrt{2\pi}\sigma}e^{-y^2/2\sigma}$$

and by the Dominated Convergence Theorem,

$$\lim_{t\to\infty}\frac{|I(t)|}{t^3} = 4[\int_0^\infty \frac{\lambda}{\sqrt{2\pi}\sigma}e^{-y^2/2\sigma^2}\underbrace{\lim_{t\to\infty}(\frac{1}{3} - \frac{(1-y^2/t^2)^{\frac{3}{2}}}{3}I_{(y<t)})}_{=0}\,dy] = 0$$

Therefore,

$$\lim_{t\to\infty}\frac{G(t)}{t^3}=\lim_{t\to\infty}\frac{1}{t^3}\int_{-t}^{t}\lambda x^2dx=\frac{2}{3}>0$$

If we prove that,

$$\frac{M(t)}{(t+1)^3}=\frac{\sum_{\Gamma_t}(X_i^2-Y_i^2)-G(t)}{(t+1)^3}\to 0$$

Then

$$\lim_{t\to\infty}\frac{1}{t^3}\sum_{\Gamma_t}(X_i^2-Y_i^2)=\lim_{t\to\infty}\frac{G(t)}{t^3}>0$$

In this case take

$$A(s)=\underbrace{\int_{\Gamma_s}x^4\mu(dx,dy)}_{A_1(s)}-2\underbrace{\int_{\Gamma_s}x^2y^2\mu(dx,dy)}_{A_2(s)}+\underbrace{\int_{\Gamma_s}y^4\mu(dx,dy)}_{A_3(s)}$$

then

$$\int_0^\infty\frac{1}{(s+1)^6}dA(s)=\int_0^\infty\frac{1}{(s+1)^6}dA_1(s)-2\int_0^\infty\frac{1}{(s+1)^6}dA_2(s)+\int_0^\infty\frac{1}{(s+1)^6}dA_3(s)$$

We proved before that

$$\int_0^\infty\frac{1}{(s+1)^6}dA_2(s)<\infty$$

Note that

$$\begin{aligned}\int_0^\infty\frac{1}{(s+1)^6}dA_1(s)&=\int_{\Gamma_t}\frac{x^4}{(\sqrt{x^2+y^2}+1)^6}\frac{\lambda}{\sqrt{2\pi}\sigma}e^{-y^2/2\sigma^2}\,dx\,dy\\&\le\int_{-t}^{t}\int_{-\infty}^{\infty}\frac{x^4}{(|x|+1)^6}\frac{\lambda}{\sqrt{2\pi}\sigma}e^{-y^2/2\sigma^2}\,dy\,dx\\&=\int_{-t}^{t}\lambda\frac{x^4}{(|x|+1)^6}dx\overset{t\to\infty}{\longrightarrow}\int_{-\infty}^{\infty}\lambda\frac{x^4}{(|x|+1)^6}dx<\infty\end{aligned}$$

Analogously,

$$\begin{aligned}\int_0^t\frac{1}{(s+1)^6}dA_3(s)&=\int_{\Gamma_t}\frac{y^4}{(\sqrt{x^2+y^2}+1)^6}\frac{\lambda}{\sqrt{2\pi}\sigma}e^{-y^2/2\sigma^2}\,dx\,dy\\&\le\int_{-t}^{t}\int_{-\infty}^{\infty}\frac{y^4}{(|x|+1)^6}\frac{\lambda}{\sqrt{2\pi}\sigma}e^{-y^2/2\sigma^2}\,dy\,dx\\&=\int_{-t}^{t}3\lambda\sigma^4\frac{1}{(|x|+1)^6}dx\overset{t\to\infty}{\longrightarrow}\int_{-\infty}^{\infty}3\lambda\sigma^4\frac{1}{(|x|+1)^6}dx<\infty\end{aligned}$$

Conclusion: The maximum likelihood estimator $\hat{\theta}_t$ is strongly consistent.

1.3.2 Central Limit Theorem for the Maximum Likelihood Estimator

We have that

$$\frac{\partial}{\partial\theta}\log L_{\Gamma_t}(\theta) = -\sum_{\Gamma_t}\left[\frac{X_i^2 - Y_i^2}{2\sigma^2}\sin 2\theta - \frac{X_iY_i}{\sigma^2}\cos 2\theta\right]$$

if $\Gamma_t = \{(x, y); x^2 + y^2 \leq t^2\}$.

Then,

$$M_\theta(t) = \sum_{\Gamma_t}\left[\frac{X_i^2 - Y_i^2}{2\sigma^2}\sin 2\theta - \frac{X_iY_i}{\sigma^2}\cos 2\theta\right]$$

is a martingale with respect to $\{\mathcal{F}_t\}$ when θ is the true parameter value. Let

$$M_n^\theta(t) = \frac{1}{n^{3/2}}M_\theta(nt)$$

Claim 1: For each $T > 0$, $\lim_n \mathbf{E}[\sup_{t\leq T}|M_n^\theta(t) - M_n^\theta(t-)|^2] = 0$.
Proof: Let $U_i = X_i \sin\theta - Y_i\cos\theta$ and $V_i = X_i\cos\theta + Y_i\sin\theta$. Denote by N_0 the process formed by the points (U_i, V_i). Then N_0 is a rotation of the original process and it is a Poisson process with mean measure

$$M_0(A) = \int_A \frac{\lambda}{\sqrt{2\pi}\sigma}e^{-y^2/2\sigma^2}\,dx\,dy$$

And, since Γ_{nt} is invariant under rotation,

$$M_n^\theta(t) = \frac{1}{n^{3/2}}\sum_{\Gamma_{nt}} U_iV_i$$

Consequently, if there is a point in the circle of radius nt,

$$|M_n^\theta(t) - M_n^\theta(t-)|^2 = \frac{1}{n^3}\frac{|U_iV_i|^2}{\sigma^4} \leq \frac{1}{n^3}n^2T^2\frac{|U_i|^2}{\sigma^4}$$

Let $S_t = \{(x, y); |x| \leq t\}$, then $N_0(\Gamma_{nt}) \leq N_0(S_{nt})$. Also, $N_0(S_{nt})$ and $\{U_i\}$ are independent. Therefore,

$$\{\mathbf{E}[\sup_{t\leq T}|M_n^\theta(t) - M_n^\theta(t-)|^2] \quad = \quad \mathbf{E}[\sup_{i\leq N_0(S_{nT})}\frac{1}{n}T^2\frac{U_i^2}{\sigma^4}]$$

$$
\begin{aligned}
&\le \frac{T^2}{\sigma^4}\sqrt{\mathbf{E}[\sup_{i\le N_0(S_{nT})}\frac{1}{n^2}U_i^4]}\\
&\le \frac{T^2}{\sigma^4}\sqrt{\mathbf{E}[\sum_{i=1}^{N_0(S_{nT})}\frac{1}{n^2}U_i^4]}\\
&\le \frac{T^2}{\sigma^4}\sqrt{\frac{1}{n^2}\mathbf{E}[\sum_{i=1}^{N_0(S_{nT})}U_i^4]}\\
&\le \frac{T^2}{\sigma^4}\sqrt{\frac{1}{n^2}\mathbf{E}[N_0(S_{nT})]\mathbf{E}[U_i^4]}\\
U_i \sim N(0,\sigma^2)\ &\le \frac{T^2}{\sigma^4}\sqrt{\frac{1}{n^2}2\lambda nT3\sigma^4} \to 0 \text{ as } n\to\infty
\end{aligned}
$$

Claim 2: Let $A_n(t) = [M_n^\theta]_t$, then $\lim_{n\to\infty}\mathbf{E}[\sup_{t\le T}|A_n(t) - A_n(t-)|] = 0$.
Proof: Since

$$M_n^\theta(t) = \frac{1}{n^{3/2}}\int_{\Gamma_{nt}}(\frac{(x^2-y^2)}{2\sigma^2}\sin 2\theta - \frac{xy}{\sigma^2}\cos 2\theta)\, N(dx,dy)$$

then

$$A_n(t) = \frac{1}{n^3}\sum_{\Gamma_{nt}}\frac{U_i^2V_i^2}{\sigma^4}$$

Therefore, claim 2 is equivalent to claim 1.

Claim 3: $M_n^\theta(t)^2 - A_n(t)$ is martingale with respect to $\{\mathcal{F}_t^n\}$.
Proof: By Proposition 2.6.1 (Ethier & Kurtz (1986)), it is enough to prove that $M_n^\theta(t)$ is square integrable martingale. In fact,

$$
\begin{aligned}
\mathbf{E}[M_n^\theta(t)^2] &= \frac{1}{n^3}\mathbf{E}[(\sum_{\Gamma_{nt}}\frac{(X_i^2-Y_i^2)}{2\sigma^2}\sin 2\theta - \frac{X_iY_i}{\sigma^2}\cos 2\theta)^2]\\
&= \frac{1}{n^3}\mathbf{E}[(\sum_{\Gamma_{nt}}\frac{U_iY_i}{\sigma^2})^2]\\
&= \frac{1}{n^3}\mathbf{E}[(\int_{\Gamma_{nt}} f(u,v)N_0(du,dv))^2] \quad \text{where } f(u,v) = uv\\
\text{by Campbell's equations} &= \frac{1}{n^3}\int_{\Gamma_{nt}} f^2(u,v)\mu_0(du,dv)\\
&\quad +\frac{1}{n^3}\int_{\Gamma_{nt}}\int_{\Gamma_{nt}} f(u,v)f(z,w)\mu_0(du,dv)\mu_0(dz,dw)\\
&\le \int_{-nt}^{nt}\lambda x^2\sigma^2\,dx = \frac{2\lambda n^3t^3}{n^3}\sigma^2 = 2\lambda t^3
\end{aligned}
$$

Since,

$$\int_{\Gamma_{nt}} u^2v^2\frac{\lambda}{\sqrt{2\pi}\sigma}e^{-v^2/2\sigma^2}\,du\,dv \leq \int_{S_{nt}} u^2v^2\frac{\lambda}{\sqrt{2\pi}\sigma}e^{-v^2/2\sigma^2}\,du\,dv$$
$$= \int_{-nt}^{nt} u^2\int_{-\infty}^{\infty} v^2\frac{\lambda}{\sqrt{2\pi}\sigma}e^{-v^2/2\sigma^2}\,dv\,du$$

And,

$$\frac{1}{n^3}\int_{\Gamma_{nt}}\int_{\Gamma_{nt}} uvzw(\frac{\lambda}{\sqrt{2\pi}\sigma})^2e^{-v^2/2\sigma^2}e^{-w^2/2\sigma^2}\,du\,dv\,dz\,dw =$$
$$= \frac{1}{n^3}(\int_{\Gamma_{nt}} uv\frac{\lambda}{\sqrt{2\pi}\sigma}e^{-u^2/2\sigma^2}\,du\,dv)^2 = 0$$

Claim 4: $A_n(t) \rightarrow \frac{2\lambda t^3}{3\sigma^2}$ a.s. as $n \rightarrow \infty$.
Proof: Let

$$M(t) = \sum_{\Gamma_t} U_i^2V_i^2 - \int_{\Gamma_t} u^2v^2\mu_0(du,dv)$$

therefore, M(t) is a martingale and we want to prove that

$$\frac{M(t)}{(t+1)^3} \rightarrow 0 \qquad \text{as } t \rightarrow \infty$$

Let

$$A(s) = \int_{\Gamma_s} u^4v^4\mu_0(du,dv).$$

It is enough to prove that

$$\int_0^{\infty}\frac{1}{(s+1)^6}dA(s) < \infty.$$

In our case,

$$\frac{\int_{\Gamma_t} u^2v^2\mu_0(du,dv)}{t^3} = \frac{\lambda}{t^3}\int_{\Gamma_t} u^2v^2\frac{1}{\sqrt{2\pi}\sigma}e^{-u^2/2\sigma^2}\,du\,dv$$
$$= \frac{\lambda}{t^3}\int_{-t}^{t}(\int_{-\sqrt{t^2-u^2}}^{\sqrt{t^2-u^2}} v^2dv)u^2\frac{e^{-u^2/2\sigma^2}}{\sqrt{2\pi}\sigma}du$$
$$= \frac{2\lambda}{3}\int_{-t}^{t}(1-u^2/t^2)^{3/2}u^2\frac{e^{-u^2/2\sigma^2}}{\sqrt{2\pi}\sigma}du$$
$$\text{DCT} \rightarrow \frac{2\lambda}{3}\int_{-\infty}^{\infty}u^2\frac{e^{-u^2/2\sigma^2}}{\sqrt{2\pi}\sigma}du = \frac{2\lambda\sigma^2}{3}$$

Also,

$$\begin{aligned}\int_0^t \frac{1}{(s+1)^6} dA(s) &= \lambda \int_{\Gamma_t} \frac{u^4 v^4}{(\sqrt{u^2+v^2}+1)^6} \frac{1}{\sqrt{2\pi}\sigma} e^{-u^2/2\sigma^2} du\, dv \\ &\le \lambda \int_{-t}^{t} \frac{v^4}{(|v|+1)^6} dv \int_{-\infty}^{\infty} \frac{u^4}{\sqrt{2\pi}\sigma} e^{-u^2/2\sigma^2} du \\ &= 3\sigma^4 \lambda \int_{-t}^{t} v^6 (|v|+1)^6 dv\end{aligned}$$

Consequently,

$$\frac{1}{n^3} \sum_{\Gamma_{nt}} \frac{U_i^2 V_i^2}{\sigma^4} \to \frac{2\lambda t^3}{3\sigma^2} \quad \text{a.s.}$$

Theorem: $\frac{4}{3}\lambda t^3 n^{3/2}(\hat{\theta}_{nt} - \theta) \Rightarrow B$
where B is a process with independent Gaussian increments and

$$B^2 - \frac{2\lambda t^3 \sigma^2}{3}$$

is a martingale.

Proof: By claims 1, 2, 3 and 4 proved above, we can apply the Central Limit Theorem for martingales(Ethier & Kurtz (1986), Theorem 7.1.4). Let

$$M_n^*(t,\theta) = \frac{1}{n^{3/2}} \sum_{\Gamma_{nt}} \frac{(X_i^2 - Y_i^2)}{2} \sin 2\theta - X_i Y_i \cos 2\theta$$

Then,

$$M_n^* \Rightarrow X$$

where X is a process with independent Gaussian increments,

$$X^2(t) - \frac{2\lambda t^3 \sigma^2}{3}$$

is a martingale with respect to $\{\mathcal{F}_t^X\}$.

Note that by the definition of M_n^*, we have $M_n^*(t,\hat{\theta}) = 0$. Therefore, when θ is the true parameter value, we have

$$M_n^*(\cdot,\theta) - M_n^*(\cdot,\hat{\theta}) \Rightarrow X$$

By Taylor's expansion, for some ξ between θ and $\hat{\theta}$ we have,

$$M_n^*(t,\theta) - M_n^*(t,\hat{\theta}) = \frac{(\theta-\hat{\theta})}{n^{3/2}}[\cos 2\theta \sum_{\Gamma_{nt}} (X_i^2 - Y_i^2) + 2\sin 2\theta \sum_{\Gamma_{nt}} X_i Y_i] + R_n(\theta)$$

where

$$R_n(\theta) = [-4\sin 2\xi \sum_{\Gamma_{nt}} \frac{(X_i^2 - Y_i^2)}{2} + 4\cos 2\xi \sum_{\Gamma_{nt}} X_i Y_i]\frac{(\theta - \hat{\theta})^2}{2n^{3/2}}$$

and $R_n(\theta) \to 0$ a.s. as $n \to \infty$. In fact,

$$\frac{1}{n^3}\mathbf{E}[-4\sin 2\xi \sum_{\Gamma_{nt}} \frac{(X_i^2 - Y_i^2)}{2} + 4\cos 2\xi \sum_{\Gamma_{nt}} X_i Y_i]^2 < \infty$$

and $(\theta - \hat{\theta}) \to 0$ a.s as $n \to \infty$. Using the process$\{(U_i, V_i)\}$, with true parameter value equals to zero, we have

$$M_n^*(t,\theta) - M_n^*(t,\hat{\theta}_{nt}) = \frac{(\hat{\theta}_{nt} - \theta)}{n^{3/2}}2\sum_{\Gamma_{nt}}(U_i^2 - V_i^2) + R_n(\theta)$$

and we have proved that

$$\frac{1}{s^3}\sum_{\Gamma_s}(U_i^2 - V_i^2) \to \frac{2\lambda}{3} \qquad \text{as } s \to \infty$$

Then,

$$\underbrace{M_n^*(t,\theta) - M_n^*(t,\hat{\theta}_{nt})}_{\Rightarrow X} = (\hat{\theta}_{nt} - \theta)2t^3 n^{3/2} \underbrace{\frac{1}{n^3 t^3}\sum_{\Gamma_{nt}}(U_i^2 - V_i^2)}_{\to \frac{2\lambda}{3}} + \underbrace{R_n}_{\to 0}$$

Therefore,

$$\frac{4}{3}\lambda t^3 n^{3/2}(\hat{\theta}_{nt} - \theta) \Rightarrow X(t)$$

1.4 Optimal Stopping Time

Let $X(t) = (t, \sum_{\Gamma_t} X_i Y_i, \sum_{\Gamma_t} X_i^2, \sum_{\Gamma_t} Y_i^2)$. Then, X(t) is a Markov process and also a piecewise deterministic Markov process. Looking at the expression for the likelihood function (see (2)) we can see that the distribution of the process N depends only on the following four functions of the data when a disc of radius t is searched, this means that $X(t)$ is the vector of sufficient statistics for the parameter θ.

We want to find the optimal stopping time in order to minimize the expected cost

$$\begin{aligned} g(X(t)) &= \min\{\pi t^2 + C\int_{-\pi/2}^{\pi/2}(\frac{1}{2}\arctan\frac{2\sum_{\Gamma_t} X_i Y_i}{\sum_{\Gamma_t} X_i^2 - \sum_{\Gamma_t} Y_i^2} \\ &+ \frac{\pi}{2}(\text{sgn}\sum_{\Gamma_t} X_i Y_i) I_{(\sum_{\Gamma_t} X_i^2 - \sum_{\Gamma_t} Y_i^2 < 0)} - \theta)^2 g(\theta|\mathcal{F}_t)d\theta, A\} \end{aligned}$$

where

A = maximum cost allowed;

$g(\theta|\mathcal{F}_t)$ = conditional distribution of the parameter θ given the knowledge of the point process over the circle Γ_t given by

$$g(\theta|\mathcal{F}_t) \propto \exp\{-\frac{1}{2\sigma^2}(\cos^2\theta\sum_{\Gamma_t} Y_i^2 + \sin^2\theta\sum_{\Gamma_t} X_i^2 - 2\sin\theta\cos\theta\sum_{\Gamma_t} X_iY_i)\} \tag{5}$$

1.4.1 Piecewise Deterministic Processes Method

Let $X(t)$ be the vector of sufficient statistics, i.e.

$$X(t) = (t, \sum_{\Gamma_t} X_iY_i, \sum_{\Gamma_t} X_i^2, \sum_{\Gamma_t} Y_i^2).$$

Then $(X(t), \mathcal{F}_t, \mathbf{P}_x)_{t\leq 0}, x \in (0,\infty)\times\mathbf{R}\times[0,\infty)\times[0,\infty)$ is a piecewise-deterministic process (PDP) (see Davis(1984) and Gugerli(1986)).

Let the initial state $Z_0 = x = (r, \alpha, \beta, \gamma)$ and set $T_0 = 0$. The process follows a deterministic trajectory except at random times $0 < T_1 < T_2, \ldots$ when it jumps off the trajectory and immediately starts anew at a randomly chosen state $Z_1, Z_2, \ldots$ respectively

$$X(t) = (t, \sum_{\Gamma_{T_i}} X_iY_i, \sum_{\Gamma_{T_i}} X_i^2, \sum_{\Gamma_{T_i}} Y_i^2), \quad T_i \leq t < T_{i+1}$$

Let $N^*(t)$ be the inhomogeneous Poisson process with mean

$$M(t) = \frac{4\lambda}{\sqrt{2\pi}\sigma}(\int_0^t e^{-u^2/2\sigma^2}\sqrt{t^2 - u^2 du}).$$

Then $T_k = \inf\{t : N^*(t) = k\}$ is the time of the k-th jump.

The probability law $\mathbf{P}_x$ of the PDP $X(t)$ is determined by: (i) a deterministic drift ϕ; (ii) a **jump rate** λ; (iii) a **transition measure** Q.

Drift: $\phi : \mathbf{R}\times E \rightarrow E \quad \phi(t,(r,\alpha,\beta,\gamma)) = (r+t,\alpha,\beta,\gamma)$
Intensity: $\lambda : E \rightarrow [0,\infty)$

$$\begin{aligned}\lambda(t,\alpha,\beta,\gamma) = \lambda(t) &= \frac{\partial}{\partial t}M(t) \\ &= \frac{\lambda}{\sqrt{2\pi}\sigma}\int_0^{2\pi} te^{-t^2\cos^2\phi/2\sigma^2}d\phi\end{aligned}$$

$$\Lambda : [0,\infty] \times E \rightarrow [0,\infty]$$

$$\begin{aligned}\Lambda(t,(r,\alpha,\beta,\gamma)) = \Lambda(t,r) &= \int_0^t \lambda(s+r,\alpha,\beta,\gamma))ds = \int_0^t \lambda(s+r)ds \\ &= M(t+r) - M(t) = M_\theta(\Gamma_{t+r} \setminus \Gamma_t)\end{aligned}$$

Also,

$$F(t,(r,\alpha,\beta,\gamma)) = 1 - \exp(-\int_0^t \lambda(s+r)ds)$$

For any initial state $x \in E, F(.,x)$ is the pdf of T_1, i.e., $\mathbf{P}_x(T_1 \leq t) = F(t,x)$ and $F(.,x)$ has density

$$f(t,(r,\alpha,\beta,\gamma)) = f(t,r) = \lambda(r+t)\exp\{-\int_0^t \lambda(r+s)ds\}$$

Transition Probability: $Q : \bar{E} \times \mathcal{B} \rightarrow [0,1]$, where $\mathcal{B}$ is the Borel σ-algebra of E. For any bounded, measurable function $w : E \rightarrow \mathbf{R}$

$$\begin{aligned}&Qw(r,\alpha,\beta,\gamma) = \\ &= \int_{-\pi/2}^{\pi/2}\int_{-r}^{r} p_1 w(r, y\sqrt{r^2-y^2}+\alpha, r^2-y^2+\beta, y^2+\gamma)\Pi_\theta(r^2,dy)g(\theta|r,\alpha,\beta,\gamma)d\theta \\ &+ \int_{-\pi/2}^{\pi/2}\int_{-r}^{r} p_2 w(r, -y\sqrt{r^2-y^2}+\alpha, r^2-y^2+\beta, y^2+\gamma)\Pi_\theta(r^2,dy)g(\theta|r,\alpha,\beta,\gamma)d\theta\end{aligned}$$

where
$p_1 = p_1(\theta,y,r)$ = conditional probability of getting the point $(\sqrt{r^2-y^2},y)$ given that the point (X,Y) lies in the circle $\{(x,y) : x^2+y^2 = r^2\}$ and Y=y, given by (8);
$p_2 = p_2(\theta,y,r) = 1 - p_1(\theta,y,r)$;
$\Pi_\theta(r^2,dy)$ = conditional density of Y given that there is a point in the circle $\{(x,y) : x^2+y^2 = r^2\}$, given by (7);
$g(\theta|r,\alpha,\beta,\gamma)$ = conditional distribution of the parameter θ given the knowledge that the vector of sufficient statistics $X(r)$ at time r has the value (r,α,β,γ), given by

$$g(\theta|r,\alpha,\beta,\gamma) \propto \exp\{-\frac{1}{2\sigma^2}(\gamma\cos^2\theta + \beta\sin^2\theta - 2\alpha\sin\theta\cos\theta)\} \qquad (6)$$

Conditional distribution of Y_i given $X_i^2 + Y_i^2 = z^2$. Note that, in order to find the conditional distribution of Y_i given $X_i^2 + Y_i^2 = z^2$, we want Π_θ such that

$$\int f(y)\Pi_\theta(y|z)\,dy = \mathbf{E}[f(Y)|R^2 = z^2]$$

We have,

$$\begin{aligned}
\mathbf{E}[\int f(y)\Pi(y|R)g(R^2)\,dy] &= \mathbf{E}[f(Y)g(X^2+Y^2)] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)g(x^2+y^2)\frac{\lambda}{\sqrt{2\pi}\sigma}e^{-(x\sin\theta-y\cos\theta)^2/2\sigma^2}\,dy\,dx
\end{aligned}$$

Let $f(x,y)=(x^2+y^2,y)$, f is not a 1-1 function from $\mathbf{R}^2$ to $[0,\infty)\times\mathbf{R}$. Let $f_1:[0,\infty)\times\mathbf{R}\to\{(u,v);v\in\mathbf{R},u\geq v^2\}$, defined by $f_1(x,y)=(x^2+y^2,y)$ and $f_2:(-\infty,0)\times\mathbf{R}\to\{(u,v);v\in\mathbf{R},u\geq v^2\}$, defined by $f_2(x,y)=(x^2+y^2,y)$. Then, f_1 and f_2 are 1-1 functions and

$$f_1^{-1}(u,v)=(\sqrt{u-v^2},v)$$
$$f_2^{-1}(u,v)=(-\sqrt{u-v^2},v)$$

Then,

$$|J_{f_1}(u,v)|=\begin{vmatrix}\frac{1}{2}(u-v^2)^{-\frac{1}{2}} & 0\\ -v(u-v^{2-\frac{1}{2}} & 1\end{vmatrix}=\frac{1}{2}(u-v^2)^{-\frac{1}{2}}$$

and

$$|J_{f_2}(u,v)|=\begin{vmatrix}-\frac{1}{2}(u-v^2)^{-\frac{1}{2}} & 0\\ v(u-v^{2-\frac{1}{2}} & 1\end{vmatrix}=\frac{1}{2}(u-v^2)^{-\frac{1}{2}}$$

Thus, the joint density of Y and R^2 is

$$\begin{aligned}
&f_{R^2,Y}(u,v)= \\
&= \frac{1}{2}(u-v^2)^{-1/2}\mu_\theta(f_1^{-1}(u,v))+\frac{1}{2}(u-v^2)^{-1/2}\mu_\theta(f_2^{-1}(u,v)) \\
&= \frac{1}{2}(u-v^2)^{-1/2}\frac{\lambda}{\sqrt{2\pi}\sigma}e^{-(\sqrt{u-v^2}\sin\theta-v\cos\theta)/2\sigma^2} \\
&\quad+\frac{1}{2}(u-v^2)^{-1/2}\frac{\lambda}{\sqrt{2\pi}\sigma}e^{-(-\sqrt{u-v^2}\sin\theta-v\cos\theta)/2\sigma^2} \\
&= \frac{1}{2}(u-v^2)^{-1/2}\frac{\lambda}{\sqrt{2\pi}\sigma}(e^{-(\sqrt{u-v^2}\sin\theta-v\cos\theta)/2\sigma^2}+e^{-(-\sqrt{u-v^2}\sin\theta-v\cos\theta)/2\sigma^2})
\end{aligned}$$

Consequently,

$$\Pi_\theta(y|z)= \\
=\frac{\frac{1}{2}(z^2-y^2)^{-1/2}\frac{\lambda}{\sqrt{2\pi}\sigma}(e^{-(\sqrt{z^2-y^2}\sin\theta-y\cos\theta)/2\sigma^2}+e^{-(-\sqrt{z^2-y^2}\sin\theta-y\cos\theta)/2\sigma^2})}{\int_{-z}^{z}\frac{1}{2}(z^2-y^2)^{-1/2}\frac{\lambda}{\sqrt{2\pi}\sigma}(e^{-(\sqrt{z^2-y^2}\sin\theta-y\cos\theta)/2\sigma^2}+e^{-(-\sqrt{z^2-y^2}\sin\theta-y\cos\theta)/2\sigma^2})dy} \quad (7)$$

Also,

$$p_1(\theta, y, z) = \frac{e^{-(\sqrt{z^2-y^2}\sin\theta - y\cos\theta)^2/2\sigma^2}}{(e^{-(\sqrt{z^2-y^2}\sin\theta - y\cos\theta)^2/2\sigma^2} + e^{-(-\sqrt{z^2-y^2}\sin\theta - y\cos\theta)^2/2\sigma^2})} \tag{8}$$

Using the notation from Gugerli (1986), define $J : B_\infty(E) \times B^c_\infty(E) \to ([0,\infty) \times E)$ as
$J(w,h)(t,(r,\alpha,\beta,\gamma)) := Iw(t,(r\alpha,\beta,\gamma)) + Hh(t,(r\alpha,\beta,\gamma))$
where
$Iw(t,(r,\alpha,\beta,\gamma)) = \int_0^t Qw(r+s,\alpha,\beta,\gamma)\lambda(r+s)\exp\{-\int_0^s \lambda(r+u)\,du\}ds$
$Hh(t,(r,\alpha,\beta,\gamma)) = h(t+r,\alpha,\beta,\gamma)\exp\{-\int_0^t \lambda(s+r)ds\}$

Define also, $K : B_\infty(E) \to B_\infty(E)$ as
$Kw(r,\alpha,\beta,\gamma) = \int_0^\infty Qw(r+s,\alpha,\beta,\gamma)\lambda(r+s)\exp\{-\int_0^s \lambda(r+u)du\}ds$

The operator $L : B_\infty(E) \times B^c_\infty(E) \to B_\infty(E)$ is defined as:
$L(w,h)(r,\alpha,\beta,\gamma) := \min\{J_0(w,h)(r,\alpha,\beta,\gamma), Kw(r,\alpha,\beta,\gamma)\}$
where
$J_t(w,h)(r,\alpha,\beta,\gamma) = \inf_{s\geq t} J(w,h)(s,(r,\alpha,\beta,\gamma))$

In our case, the cost function

$$g(r,\alpha,\beta,\gamma) = \min\{\pi r^2 + C\int_{-\pi/2}^{\pi/2}(\frac{1}{2}\arctan\frac{2\alpha}{\beta-\gamma} + \frac{\pi}{2}(\text{sgn}\alpha)I_{(\beta-\gamma<0)} - \theta)^2 g(\theta|\mathcal{F}_r)d\theta, A\}$$

belongs to $B^c_\infty(E)$ and to find the pay-off function $s(r,\alpha,\beta,\gamma)$ for the optimal stopping problem, let

$$s_0 = g$$

$$s_{n+1} = L(s_n, g) \tag{9}$$

and

$$s = \lim_n s_n$$

and the optimal stopping time is given by

$$\tau = \inf\{t \geq 0; g(X(t)) = s(X(t))\} \tag{10}$$

1.4.2 Shyriayev's Approach

Let

$$X(t) = (t, \sum_{\Gamma_t} X_i Y_i, \sum_{\Gamma_t} X_i^2, \sum_{\Gamma_t} Y_i^2)$$

We Apply the the approach described in Shyriayev (1981) with cost function given by

$g(t,\alpha,\beta,\gamma) =$

$$= \min\{\pi t^2 + C\int_{\pi/2}^{\pi/2}(\frac{1}{2}\arctan\frac{2\alpha}{\beta-\gamma}+\frac{\pi}{2}(\text{sgn}\alpha)I_{(\beta-\gamma<0)} - \theta)^2 g(\theta|t,\alpha,\beta,\gamma)d\theta, A\} \quad (11)$$

where $g(\theta|t,\alpha,\beta,\gamma)$ is given by (6).

Let Q_N be the operator:
$Q_N g(t,\alpha,\beta,\gamma) = \min\{g(t,\alpha,\beta,\gamma), S(2^{-N})g(t,\alpha,\beta,\gamma)\}$
where

$$S(s)g(t,\alpha,\beta,\gamma) = \mathbf{E}_{(t,\alpha,\beta,\gamma)}[X(s)] \quad (12)$$

In this case, the **pay-off** function $s(t,\alpha,\beta,\gamma)$ is given by

$$s(t,\alpha,\beta,\gamma) = \lim_n \lim_N Q_N^n g(t,\alpha,\beta,\gamma)$$

where $Q_N^{n+1}g = Q_N(Q_N^n g)$. And the **optimal stopping time** is given by

$$\tau = \inf\{t \geq 0; g(X(t)) = s(X(t))\} \quad (13)$$

Determination of S(s) h(t,α,β,γ)

Given a function $h \in B_\infty^c(E)$, where $E = [0,\infty) \times \mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}^+$, we want to compute $S(s)h(t,\alpha,\beta,\gamma) = \mathbf{E}_{(t,\alpha,\beta,\gamma)}[h(X(s))] = \mathbf{E}[h(X(t+s))|X(t) = (t,\alpha,\beta,\gamma)]$. In order to compute this, define the event:
$A(t,s)$ = there is some point in the ring $\Gamma_{t+s}\backslash\Gamma_t = \{(x,y); t^2 < x^2+y^2 \leq (t+s)^2\}$.

We have that,

$$P(A(t,s)) = 1 - e^{-\mu(\Gamma_{t+s}\backslash\Gamma_t)}$$

where

$$\begin{aligned}
\mu(\Gamma_{t+s}\backslash\Gamma_t) &= \frac{\lambda}{\sqrt{2\pi}\sigma}\int_{\Gamma_{t+s}\backslash\Gamma_t} e^{-(x\sin\theta - y\cos\theta)^2/2\sigma^2}\,dx\,dy \\
\text{(using polar coordinates)} &= \frac{\lambda}{\sqrt{2\pi}\sigma}\int_0^{2\pi}\int_t^{t+s} re^{-r^2\sin^2\phi/2\sigma^2}\,dr\,d\phi \\
&= \frac{\lambda\sigma}{\sqrt{2\pi}}\int_0^{2\pi}\frac{1}{\sin^2\phi}[e^{-t^2\sin^2\phi/2\sigma^2} - e^{-(t+s)^2\sin^2\phi/2\sigma^2}]d\phi
\end{aligned}$$

We have by the previous equation,

$$\lim_{s\to 0}\frac{\mu(\Gamma_{t+s}\backslash\Gamma_t)}{s} = \frac{\lambda}{\sqrt{2\pi}\sigma}\int_0^{2\pi} te^{-t^2\sin^2\phi/2\sigma^2}\,d\phi$$

Consequently, as $s \to 0$,

$$P(A(t,s)) \approx \frac{\lambda s}{\sqrt{2\pi}\sigma} \int_0^{2\pi} te^{-t^2 \sin^2 \phi / 2\sigma^2} d\phi \tag{14}$$

Therefore, we can write $S(s)h(t,\alpha,\beta,\gamma)$ as

$$\begin{aligned} S(s)h(t,\alpha,\beta,\gamma) &= \mathrm{E}[h(X(t+s))|X(t)=(t,\alpha,\beta,\gamma), A(t,s)]\mathrm{P}(A(t,s)) \\ &\quad + \mathrm{E}[h(X(t+s))|X(t)=(t,\alpha,\beta,\gamma), A^c(t,s)]\mathrm{P}(A^c(t,s)) \end{aligned} \tag{15}$$

Let s be small enough so that, there is at most one point in the ring $\Gamma_{t+s}\backslash\Gamma_t$ and its distance to the origin is $(t+s)^2$, then, we have

$$\begin{aligned} &\mathrm{E}[h(X(t+s))|X(t)=(t,\alpha,\beta,\gamma), A(t,s)] = \\ &= \int_{-\pi/2}^{\pi/2} [\int_{-(t+s)}^{(t+s)} p_1(\theta,y,t)h_1(s,t,\alpha,\beta,\gamma,y)\Pi_\theta(y|(t+s))]g(\theta|t,\alpha,\beta,\gamma)d\theta\, dy \\ &+ \int_{-\pi/2}^{\pi/2} [\int_{-(t+s)}^{(t+s)} p_2(\theta,y,t)h_2(s,t,\alpha,\beta,\gamma,y)\Pi_\theta(r|(t+s))]g(\theta|t,\alpha,\beta,\gamma)d\theta\, dy \end{aligned} \tag{16}$$

where

$$h_1(s,t,\alpha,\beta,\gamma,y) = h(t+s, y\sqrt{(t+s)^2-y^2}+\alpha, (t+s)^2-y^2+\beta, y^2+\gamma)$$

$$h_2(s,t,\alpha,\beta,\gamma,y) = h(t+s, -y\sqrt{(t+s)^2-y^2}+\alpha, (t+s)^2-y^2+\beta, y^2+\gamma)$$

and $\Pi_\theta(y|z)$ is given by (7), $p_1(\theta,y,t)$ is given by (8), $p_2(\theta,y,t) = 1 - p_1(\theta,y,t)$ and $g(\theta|t,\alpha,\beta,\gamma)$ is given by (6).

1.4.3 Generalized Stefan Problem

The approach used in the subsection(1.4.2) can essentially be described as: Given a standard Markov process $X(t)$ and a reward function $g(t,\alpha,\beta,\gamma)$, find the Snell envelope s. This approach is equivalent to solving the Generalized Stefan Problem. We want to find a function s that satisfies

$$s \le g$$

$$As(t,\alpha,\beta,\gamma) \ge 0$$

$$As(t,\alpha,\beta,\gamma) = 0, \qquad \text{if } s(t,\alpha,\beta,\gamma) < g(t,\alpha,\beta,\gamma)$$

where A is the generator of the process $X(t)$.

In our case

$$X(t) = (t, \sum_{\Gamma_t} X_i Y_i, \sum_{\Gamma_t} X_i^2, \sum_{\Gamma_t} Y_i^2)$$

and

$$g(t,\alpha,\beta,\gamma) = \min\{\pi t^2 + C\int_{-\pi/2}^{\pi/2} (\frac{1}{2}\arctan\frac{2\alpha}{\beta-\gamma} + \frac{\pi}{2}(\text{sgn}\alpha)I_{(\beta-\gamma<0)} - \theta)^2 g(\theta|t,\alpha,\beta,\gamma_t)d\theta, A\}$$

Note that the process $X(t)$ has one deterministic component plus a jump process and the generator of the process will consist of the derivative with respect to the deterministic variable plus the jump component.

To find the generator of the process $X(t)$ we want an operator A such that

$$f(X(t)) - \int_0^t Af(X(s))ds$$

is a martingale for all $f \in Dom(A)$ (see Ethier & Kurtz (1986)). That is, we want a process $Z(t)$, $\{\mathcal{F}_t\}$-adapted such that,

$$f(X(t)) - \int_0^t Z(s)ds$$

is a martingale.

In our case, for each θ we can find $Z(\theta,t)$ a $\{\mathcal{F}_t \bigvee \sigma(\theta)\}$-adapted process such that

$$f_\theta(X(t)) - \int_0^t Z(\theta,s)ds$$

is a martingale with respect to $\mathcal{F}_t \bigvee \sigma(\theta)$, where $f(X(t)) = \mathbf{E}[f_\theta(X(t))|\mathcal{F}_t]$. Then we take

$$Z(s) = \mathbf{E}[Z(\theta,s)|\mathcal{F}_s]$$

That is, for each fixed θ we find the generator A_θ of the process $X(t)$ and

$$Af(X(t)) = \mathbf{E}[A_\theta f_\theta(X(t))|\mathcal{F}_t]$$

By definition of generator we can find $A_\theta f_\theta$ by

$$A_\theta f_\theta(t,\alpha,\beta,\gamma) = \lim_{s\searrow 0}\frac{1}{s}\mathbf{E}[f_\theta(X(t+s)) - f_\theta(X(t))|X(t) = (t,\alpha,\beta,\gamma)]$$

Defining the event $A(t,s)$ as there is some point in the ring $\Gamma_{t+s}\backslash\Gamma_t$, note that given $A^c(t,s)$ we have $X(t+s) = (s,0,0,0) + X(t)$, that is, if $X(t) = (t,\alpha,\beta,\gamma)$ then $X(t+s) = (t+s,\alpha,\beta,\gamma)$. Therefore,

$$
\begin{aligned}
&A_\theta f_\theta(t,\alpha,\beta,\gamma) = \\
&= \lim_{s\searrow 0}\frac{1}{s}\mathbf{E}[f_\theta(X(t+s)) - f_\theta(X(t))|X(t) = (t,\alpha,\beta,\gamma)] \\
&= \lim_{s\searrow 0}\frac{1}{s}\mathbf{E}[f_\theta(X(t+s)) - f_\theta(X(t))|X(t) = (t,\alpha,\beta,\gamma), A(t,s)]\mathbf{P}(A(t,s)) \\
&\quad + \lim_{s\searrow 0}\frac{1}{s}\mathbf{E}[f_\theta(X(t+s)) - f_\theta(X(t))|X(t) = (t,\alpha,\beta,\gamma), A^c(t,s)]\mathbf{P}(A^c(t,s)) \\
&= \lim_{s\searrow 0}\frac{1}{s}\mathbf{E}[f_\theta(X(t+s)) - f_\theta(X(t))|X(t) = (t,\alpha,\beta,\gamma), A(t,s)]\mathbf{P}(A(t,s)) \\
&\quad + \lim_{s\searrow 0}\frac{1}{s}(f_\theta(t+s,\alpha,\beta,\gamma) - f_\theta(t,\alpha,\beta,\gamma)\mathbf{P}(A^c(t,s))
\end{aligned}
$$

Therefore, if $f_\theta(t,\alpha,\beta,\gamma) = f_1(t) + f_2(\alpha,\beta,\gamma)$

$$
\begin{aligned}
&A_\theta f_\theta(t,\alpha,\beta,\gamma) = \\
&= \frac{\partial}{\partial t}f_\theta(t,\alpha,\beta,\gamma) \\
&+\mathbf{P}(A(t))[\int_{-t}^{t} p_1(f_\theta(t, y\sqrt{t^2-y^2}+\alpha, r^2-y^2+\beta, y^2+\gamma) - f_\theta(t,\alpha,\beta,\gamma))]\Pi_\theta(y|t)dy \\
&+\int_{-t}^{t} p_2(f_\theta(t, -y\sqrt{t^2-y^2}+\alpha, r^2-y^2+\beta, y^2+\gamma) - f_\theta(t,\alpha,\beta,\gamma))]\Pi_\theta(y|t)dy] \qquad (17)
\end{aligned}
$$

where $\mathbf{P}(A(t)) = \lim_{s\searrow 0}\mathbf{P}(A(t,s))$,$\mathbf{P}(A(t,s))$ is given by (14), $p_2 = p_2(\theta,y,t) = 1 - p_1(\theta,y,t)$, $p_1 = p_1(\theta,y,t)$ and $\Pi_\theta(y|t)$ are given by (8) and (7), respectively. Consequently,

$$
Af(t,\alpha,\beta,\gamma) = \int_{-\pi/2}^{\pi/2} A_\theta f_\theta(r,\alpha,\beta,\gamma)g(\theta|t,\alpha,\beta,\gamma)d\theta
$$

where $g(\theta|r,\alpha,\beta,\gamma)$ is given by (6).

1.5 Numerical Results

The PDP method and Shyriayev's approach give us that the optimal stopping time exists and in both cases we have mathematical expressions that theoretically give us a method on how to find the optimal stopping time through functional interaction. However, these expressions are not of practical value and implementing a procedure that utilizes either formulas (9) and (10) or equations (12) and (13) is not feasible, since it would require huge amount of computer time and computer memory to run it.

One option is to work with some suboptimal stopping time. One method is suggested by the corresponding Generalized Stefan Problem (Section 1.4.3). Use as a suboptimal stopping time

$$\tau_1 = \inf\{t : Ag(X(t)) \geq 0\}$$

Note that $\tau_1 \leq \tau$ and since the cost is the sum of the cost due to sampling plus the cost due to estimation error, we have that τ_1 is the first time that the gain obtained by finding one more observation in an infinitesimal ring after time t does not compensate for the cost of sampling for one more observation. Several examples were carried out and the suboptimal stopping time based on the generator is not very good, usually it stops after just one or two observations, even when the first observations are not taken into account the suboptimal time still does not have a good performance.

Another option is suggested by the comment above. Let Δ_r be the time after r until finding another point of the process. Then, an suboptimal stopping time could be

$$\tau_2 = \inf\{r : \mathbf{E}g(X(r + \Delta_r)) > g(X(r))\}$$

Note that,

$$\mathbf{P}(\Delta_r > t) = \mathbf{P}(N(\Gamma_{r+t} \setminus \Gamma_r) = 0) = e^{-M_\theta(\Gamma_{r+t} \setminus \Gamma_r)}$$

From section 1.4.2, we have,

$$M_\theta(\Gamma_{r+t} \setminus \Gamma_r) = \frac{\lambda\sigma}{\sqrt{2\pi}} \int_0^{2\pi} \frac{1}{\cos^2\phi}[e^{-r^2\cos^2\phi/2\sigma^2} - e^{-(r+t)^2\cos^2\phi/2\sigma^2}]d\phi$$

Then, the density of Δ_r is given by

$$f_{\Delta_r}(t) = e^{-M_\theta(\Gamma_{r+t} \setminus \Gamma_r)} \frac{\partial}{\partial t} M_\theta(\Gamma_{r+t} \setminus \Gamma_r)$$

Therefore,

$$\mathbf{E}[\pi(r + \Delta_r)^2 - \pi r^2] = \mathbf{E}[2\pi r\Delta_r + \pi\Delta_r^2] \quad (18)$$

$$= 2\pi r \int_0^\infty t f_{\Delta_r}(t)dt + \pi \int_0^\infty t^2 f_{\Delta_r}(t)dt \quad (19)$$

Also,

$$\mathbf{E}[(\hat\theta_{r+\Delta_r} - \theta)^2|\mathcal{F}_r] - \mathbf{E}[(\hat\theta_r - \theta)^2|\mathcal{F}_r] = \mathbf{E}[\hat\theta^2_{r+\Delta_r} - \hat\theta_r^2 - 2\theta(\hat\theta_{r+\Delta_r} - \hat\theta_r)|\mathcal{F}_r]$$

Therefore,

$$\mathbf{E}[(\hat{\theta}_{r+\Delta_r} - \theta)^2 | X(r) = (r, \alpha, \beta, \gamma)] =$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{\infty} \int_{-(z+r)}^{(z+r)} [(\hat{\theta}_1(r,y,z) - \theta)^2 p_1(\theta, y, r+z) + (\hat{\theta}_2(r,y,z) - \theta)^2 p_2(\theta, y, r+z)] \times$$
$$\Pi_\theta(y|z+r) f_{\Delta_r}(z) g(\theta | r, \alpha, \beta, \gamma)\, dy\, dz\, d\theta \qquad (20)$$
$$(21)$$

where

$$\hat{\theta}_1(r,y,z) = \frac{1}{2}\arctan\frac{2(\alpha + y\sqrt{(r+z)^2 - y^2})}{\beta + (r+z)^2 - \gamma - y^2}$$
$$+\frac{\pi}{2}\mathrm{sgn}(\alpha + y\sqrt{(r+z)^2 - y^2})\mathrm{I}(\beta + (r+z)^2 - \gamma - y^2 < 0)$$

and

$$\hat{\theta}_2(r,y,z) = \frac{1}{2}\arctan\frac{2(\alpha - y\sqrt{(r+z)^2 - y^2})}{\beta + (r+z)^2 - \gamma - y^2}$$
$$+\frac{\pi}{2}\mathrm{sgn}(\alpha - y\sqrt{(r+z)^2 - y^2})\mathrm{I}(\beta + (r+z)^2 - \gamma - y^2 < 0)$$

The idea above could be pursued further, that is, instead of looking at the expected cost for one observation ahead, we could look at the expected cost function for 2 observations ahead. Let Δ_r be the time after r until finding another point of the process and $\Delta_r^{(2)}$ be the time after Δ_r until finding another point of the process. Then, an suboptimal stopping time could be

$$\tau_3 = \inf\{r : \mathbf{E}g(X(r+\Delta_r)) > g(X(r)) \text{ and} \mathbf{E}g(X(r+\Delta_r+\Delta_r^{(2)})) > \mathbf{E}g(X(r))\}$$

However, this would be computer intensive to calculate and that is not the idea behind finding suboptimal stopping times.

Looking at the cost function we can see that basically it is the posterior risk for the maximum likelihood estimator plus a sampling cost, if our intention is to minimize the cost, maybe it would be a better idea to use the conditional expectation as an estimator instead of the maximum likelihood estimator. However, in all our examples we found that the conditional expectation and the maximum likelihood estimator are very close after a few observations are sampled, and the resulting cost is almost the same. Since the conditional expectation can only be calculated numerically through numerical integration, the use of the conditional expectation would increase the computer time to be used in the computation of the suboptimal stopping times. This additional computation can be avoided by using the maximum likelihood estimator that can be calculated explicitly.

Due to the computer intensive routines being used, instead of working with simulations, we are going to present some examples comparing the suboptimal stopping time τ_2 with the true minimum value of the reward function. Note that this is not the optimal stopping value since it is found on the ground that we can see all the history of the process (past and future).

	λ	σ	θ	Time	Cost	MLE
Minimum	1.0	1.0	0.0	13.0906	1176.408	0.00986
Suboptimal				20.6156	1541.286	-0.00491
Minimum	1.0	1.0	$\pi/4$	15.4490	1071.205	0.78717
Suboptimal				20.3512	1450.404	0.77943
Minimum	1.0	1.0	$-\pi/4$	16.0464	1208.367	-0.79591
Suboptimal				22.9431	1794.118	-0.79758
Minimum	1.0	3.0	0.0	25.8631	3045.464	-0.03740
Suboptimal				44.1161	6324.750	-0.01969
Minimum	1.0	3.0	$\pi/4$	23.7833	2741.874	0.73949
Suboptimal				32.9034	3863.179	0.75224
Minimum	1.0	3.0	$-\pi/4$	23.7363	2696.709	-0.75290
Suboptimal				39.5677	5113.177	-0.77810
Minimum	2.0	1.0	0.0	13.6945	877.254	0.00725
Suboptimal				20.9030	1349.634	0.00075
Minimum	2.0	1.0	$\pi/4$	13.4085	822.934	0.81418
Suboptimal				13.4085	822.934	0.81418
Minimum	2.0	1.0	$-\pi/4$	11.7957	821.304	-0.81113
Suboptimal				16.1311	964.445	-0.80369

Table 1: Time of minimum cost and suboptimal stopping time

By Table1 and Figure 1 we can note that the suboptimal time estimates the true time quite well, in these examples it is always bigger than the true value, but not very much. As expected as σ grows the stopping time increases and also the suboptimal stopping time diverges from the minimizing value.

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