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Topics on the Quantum Dynamics of Fields and Accelerated Systems

Tópicos na Dinâmica Quântica de Campos e Sistemas Acelerados

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Abstract

This thesis explores key aspects of quantum field theory in accelerated systems, focusing on the thermodynamic effects of acceleration and causal horizons. It is known since the 1970s that a deep connection exists between acceleration and thermodynamics, but direct observations of its consequences are still elusive. Here, we focus on the observable consequences in two distinct fronts: strong-field electrodynamics (QED), where the typical systems are represented by electrons accelerated by classical potentials, and topologically non-trivial quantum fields, where the typical representatives are given by non-abelian gauge theories such as quantum chromodynamics (QCD).

First we investigate the thermality of the quantum vacuum from an accelerated observer's perspective, using the Unruh effect as a foundation. Through a study of fluctuation-dissipation dynamics, we analyze observables such as the electron's transverse momentum fluctuations, deriving predictions for thermalization timescales and their dependence on acceleration. The results indicate significant quantum electrodynamical corrections at higher accelerations, but good agreement with classical and semiclassical results for moderately-low regimes of acceleration. We discuss experimental implications for probing such effects.

Second we address the impact of causal horizons on gauge theory vacua, particularly in relation to the strong CP problem in QCD. We propose that the presence of horizons causes decoherence in topologically non-trivial gauge theories, potentially providing a new perspective on the absence of observable CP violation. A horizon-induced dissipation model is introduced, offering a mechanism for the incoherent summation of topological sectors and a pathway to understanding vacuum structure in non-abelian gauge fields.

Overall, this work advances our understanding of the thermodynamical consequences of acceleration in both electrodynamics and non-abelian gauge theories, suggesting experimental and theoretical avenues for further exploration.

Resumo

Esta tese explora aspectos fundamentais da teoria quântica de campos em sistemas acelerados, com foco nos efeitos termodinâmicos da aceleração e dos horizontes causais. Desde a década de 1970, é sabido que existe uma conexão profunda entre aceleração e termodinâmica, mas as observações diretas de suas consequências ainda são difíceis de obter. Aqui, focamos nas consequências observáveis em dois âmbitos distintos: eletrodinâmica quântica (QED) de campos fortes, onde os sistemas típicos são representados por elétrons acelerados por potenciais clássicos, e campos quânticos topologicamente não triviais, onde os exemplos típicos são dados por teorias de gauge não abelianas, como a cromodinâmica quântica (QCD).

Primeiro, investigamos a termalidade do vácuo quântico do ponto de vista de um observador acelerado, utilizando o efeito Unruh como base. Através de um estudo das dinâmicas de flutuação-dissipação, propomos e analisamos observáveis como as flutuações do momento transversal do elétron, derivando previsões para as escalas de tempo de termalização e sua dependência da aceleração. Os resultados indicam correções significativas da eletrodinâmica quântica em acelerações mais altas, mas mostram boa concordância com resultados clássicos e semiclassicos para regimes de aceleração moderadamente baixos. Discutimos as implicações experimentais para a investigação desses efeitos.

Em seguida, abordamos o impacto dos horizontes causais nos vácuos de teorias de gauge, particularmente em relação ao problema de CP forte na QCD. Propomos que a presença de horizontes causa decoerência em teorias de gauge topologicamente não triviais, potencialmente fornecendo uma nova perspectiva sobre a ausência de violação de CP observável. Um modelo de dissipação induzida por horizontes é introduzido, oferecendo um mecanismo para a soma incoerente dos setores topológicos e um caminho para entender a estrutura do vácuo em campos de gauge não abelianos.

No geral, este trabalho avança nosso entendimento das consequências termodinâmicas da aceleração tanto na eletrodinâmica quanto em teorias de gauge não abelianas, sugerindo caminhos experimentais e teóricos para futuras explorações.

Contents

Chapter 1

Introduction

Among the various surprising predictions of quantum field theory (QFT), the connection between uniform acceleration and thermodynamics is one of the most puzzling ones. This connection was first discovered by Unruh in the mid 1970s [1], inspired by the work of Hawking on black hole thermodynamics [2] and based on the previous works of Fulling [3] and Davies [4] on quantization in accelerated Rindler spacetimes. Mathematically, this connection is given by the observation that the usual Poincaré-invariant vacuum of inertial observers, when restricted to the causally-connected region of an accelerated observer, is a thermal state with respect to said observer's proper-time translations [5, 6]. Physically, this means that uniformly accelerated observers can associate to the inertial ("Minkowski") vacuum state $|0_M\rangle$ a temperature given by

$$T_U = \frac{\hbar a}{2\pi c k_B} \simeq 4 \times 10^{-21} \left(\frac{a}{\mathrm{m/s^2}} \right) \mathrm{K}, \tag{1.1}$$

where a is the (constant) proper-acceleration of the observer. This connection between acceleration and temperature is called *the Unruh effect*, and the accelerated temperature T_U is called *the Unruh temperature*.

At first glance, this effect seems almost paradoxical: how can an observer associate a temperature and a thermal state to the vacuum, which is defined as the "no-particle" state? It gets even more surprising when we realize, from the expression of the Unruh temperature T_U in Eq. (1.1), that the Unruh effect is a non-trivial result of the combination between three of the most important areas of modern physics: special relativity (c), quantum mechanics ($\hbar$), and statistical physics (k_b). In fact, if we count the fact that the Unruh effect was first worked out as a way to get better insight into the physics of the Hawking effect for black hole radiation and evaporation at temperatures $T_H = \hbar c^3 / 8\pi k_B G M$, we also can say that the Unruh effect also has deep ties to general relativity (G) as well, which provides a link with another one of the major areas of research in fundamental physics.

Moreover, it can be argued that the thermality of acceleration is in fact weirder than its black hole counterpart. Black holes are these exotic astrophysical systems that are way out of direct human grasp. It is easy to visualize and accept that quantum mechanics under such extreme conditions, such as the vicinity of black hole event horizons, can lead to weird and unexpected results. Acceleration, however, is very much under the realm of human experience. It is much more mundane, every-day occurrence, and, more importantly, it is more suitable and directly accessible to experimental scrutiny in a laboratory. The idea that we can somehow gain insight on the physics of exotic astrophysical objects like black holes by simply considering how quantum mechanics interplays with accelerated observers is nothing short of extraordinary.

For these reasons and more, it is no wonder why the Unruh effect has been the target of heavy theoretical investigation and philosophical debates ever since it was first discovered.

It is not only its connection to black hole radiation physics—and its potential usefulness in the investigation of how eventual quantum theories of gravity should look like—that makes the effect so important. It is also of great theoretical and experimental interest because it directly involved with various counter-intuitive and nuanced subtleties in the standard quantum theory of fields. Perhaps the most important of these nuances is the fact that quantum field theory is, fundamentally, a true theory of fields, and not a theory of particles [7]. That is one of the insights about quantum field theory that the Unruh effect really highlights: the concept of "particle" is *not* a fundamental concept in the theory. In the standard approach to QFT, this particular subtlety is often overlooked because in flat spacetime there exists a prefered class of observers—the inertial observers—who can unambiguously agree on a definition of particles, but this is no longer the case when we leave the realm of flat spacetime (or leave the realm of inertial reference frames, such as in the case of the accelerated observers in flat spacetime).

This fact was known ever since Fulling [3] discovered that quantum field theory allowed for infinitely-many unitarily-inequivalent representations of the canonical commutation relations (CCR). That means there is no unique way of constructing, in general, the Fock-space

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n\in\mathbb{N}} S_\sigma(\mathcal{H}^{\otimes n}) \tag{1.2}$$

of a theory from a "single-particle" Hilbert space $\mathcal{H}$. In the end, this means that the "particle content" of states does not possess an objective, fundamental meaning in QFT. This is true even in standard QFT in inertial coordinates: states, as elements on the Hilbert space that represent the configurations of the theory, are never observables themselves. The only observables in QFT are self-adjoint operators constructed from the field distributions smeared by appropriate test functions. That is true even for observables such as S-matrix elements $\langle \mathbf{p_1} \ldots \mathbf{p_n} | S | \mathbf{k_1} \ldots \mathbf{k_m} \rangle$, which can be reduced in terms of correlation functions by means of the LSZ reduction formula [8].

This fact is what explains the apparent paradox of accelerated observers to perceive the "zero-temperature" Minkowski vacuum $|0_M\rangle$ as a thermal state: the accelerated observer simply constructs their Fock-space of states from a different "single-particle" Hilbert space that defines what are the excitations of the field. In other words, we can say that the "particle content" of a quantum field theory is not generally covariant, depending on the observer's particular choice of construction of the space of states of the theory.

In fact, this non-uniqueness behavior is exclusive to quantum field theories, where we have infinitely-many degrees of freedom. In ordinary quantum mechanics, where the number of degrees of freedom is finite, the uniqueness of the representation of the canonical commutation relations is guaranteed by the Stone-von Neumann theorem [9]. The Unruh effect is just a prime example of the consequences due to the break down of this theorem in field systems.

Another point of tension between the Unruh effect and our intuition based on the usual quantum mechanics is on the precise meaning of the term "thermal state". In ordinary quantum mechanics, we can simply define thermal states (with respect to a Hamiltonian $\hat{H}$) as the mixed states whose density matrices take the Gibbs form

$$\rho = \frac{1}{Z}\exp(-\beta H), \tag{1.3}$$

where $\beta \equiv T_U^{-1}$ is the inverse temperature associated with the mixed-state ρ. In QFT, however, it is necessary to generalize the definition of what we mean by "thermal state" beyond the quantum mechanical counterpart of Eq. (1.3). That is because the exponential defining the Gibbs state ρ becomes quite an ill-defined object if the Hamiltonian H has a continuous part in its spectrum [7, 10], which is the case in the overwhelming majority of the QFTs of interest. The generalization of "thermal states" that can account for these scenarios is given by the famous Kubo-Martin-Schwinger (KMS) condition [11, 12, 13]. The KMS condition establishes a criterion for defining the thermal equilibrium of a state of temperature $T_U = \beta^{-1}$ with respect to a generator of "time-translations" H [14]: for any two operators A and B and any real number τ, we have

$$\langle A(\tau)B\rangle_\beta = \langle BA(\tau + i\beta)\rangle_\beta, \tag{1.4}$$

where $A(\tau) = e^{iH\tau}A\,e^{-iH\tau}$ is the time-translation of the operator A with respect to the flow of H, and $\langle\mathcal{O}\rangle_\beta$ is the expectation value of the operator $\mathcal{O}$ on the "thermal bath". In fewer words, the KMS condition defines thermal states as those states that are "periodic with respect to time-translations in imaginary time".

That is precisely what the more rigorous approaches to the Unruh effect establish [5, 6]: the restriction of the inertial, Minkowski vacuum $|0_M\rangle$ to the right Rindler wedge where the accelerated observer lives (see Fig. 1) is a KMS state with respect to the

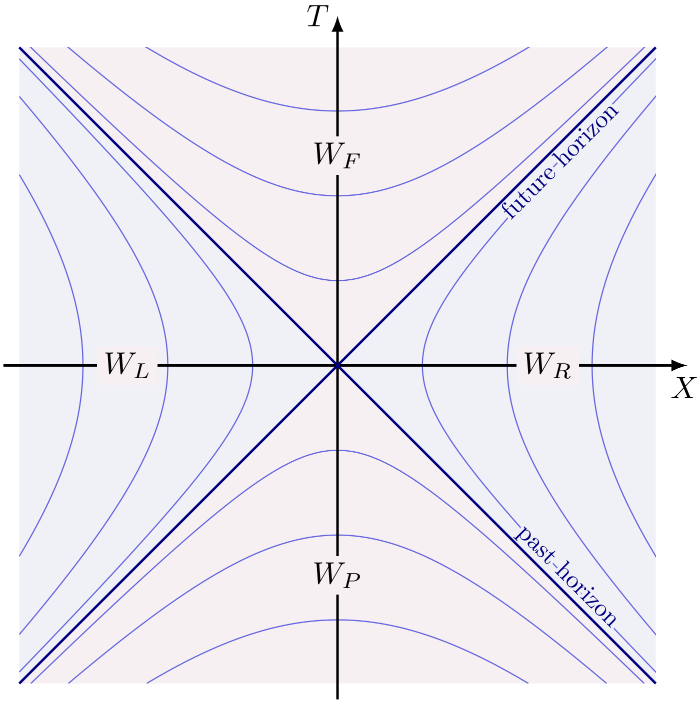

Figure 1 – The spacetime diagram for Minkowski space and its associated right, left, future, and past Rindler wedges $W_{R/L/F/P}$. Drawn in blue we have the flow lines for the Killing-field in Eq. (1.5), which generate proper-time translations for the accelerated observers in both W_R and W_L. Shown in the $T \pm X = 0$ lines are the past and future causal horizons for the right Rindler wedge W_R, which represent the boundaries of spacetime that separate the regions where the accelerated observers in W_R can send and receive light signals. Note that W_R and W_L are completely causally disconnected.

proper-time translations $\tau \to \tau + i\beta$ generated by the boost's Killing field

$$K = X\left(\frac{\partial}{\partial T}\right) + T\left(\frac{\partial}{\partial X}\right), \tag{1.5}$$

where (T, X) are global, inertial coordinates for flat Minkowski spacetime and a is the acceleration parameter of the observers that enters on the temperature $\beta^{-1} = a/2\pi$ (in natural units). Being the generator of proper-time τ-translations for accelerated observers, the Killing field $aK \equiv \partial/\partial\tau$ is also called in the literature the "Rindler Hamiltonian" H_R.

This mathematical subtlety in the choice of criterion for what we mean by thermal state is at the core of the tension between "standard intuition" and the Unruh effect we mentioned before. In the context of the Unruh effect, acceleration temperature has a very precise meaning: it is the temperature associated with the accelerated observer's own notion of time-translations, which is generated by the Lorentz boosts along the acceleration axis that are generated by the Killing vector-field K given in Eq. (1.5).

This means that, in principle, there may be differences between detailed properties of the "thermal bath" seen by the accelerated observer and the usual "thermal bath" described by an inertial observer in a thermodynamic medium with the same temperature. In fact, several differences between the thermal behavior of field observables in the accelerated and inertial "thermal baths" can be established (for a comprehensive review of the differences, see [15]).

Among those differences, perhaps the most notable is the difference between the vacuum noise spectrum F of a field along the trajectory $\gamma_a(\tau) = (\sinh a\tau, \cosh a\tau, 0, \dots)$ of an uniformly accelerated particle in n-dimensions. This is given in terms of the Fourier transform of the positive-frequency Wightman's function G^s_+ for a massless field [15]:

$$F^s_{\text{accel}}(\omega) = \int d\tau\, e^{-i\omega\tau} G^s_+(\gamma_a(\tau), \gamma_a(0)) \propto r_{\beta,n+2s}(\omega) F^s_{\text{mink}}(\omega), \tag{1.6}$$

where s is the spin of the field, β is the inverse Unruh temperature T_U^{-1}, n is the number of spacetime dimensions $\dim(\mathcal{M})$, and $r_{\beta,n+2s}(\omega)$ is a temperature-dependent factor that quantifies the difference in spectral noise between the "acceleration-induced thermal bath" and the usual inertial thermal noise $F^s_{\text{mink}}(\omega)$. While the exact general expression for the ratio $r_{\beta,n+2s} = F^s_{\text{accel}}/F^s_{\text{mink}}$ is unimportant for the present discussion[1], the really surprising observation comes from the fact that $r_{\beta,2}(\omega) = r_{\beta,4}(\omega) = 1$. This means that for massless scalar fields in both $n = 2$ and $n = 4$ dimensions, the response of the accelerated thermal noise is exactly the same as for the usual thermal noise felt by an inertial observer at finite temperature.

At first, one can think that all the worry about defining precisely what we mean call a thermal state might be just some non-essential mathematical detail. That however is not the case, as much effort has been spent in debating the existence and the "observability" of the "particles" that comprise the Unruh thermal bath [17, 18, 19, 20, 21]. In fact, the usual derivations of the Unruh effect based on the representation of the Minkowski vacuum $|0_M\rangle$ in terms of Rindler n-particle states $|n_i\rangle$ corresponding to the solutions to the wave equation in the accelerated frame,

$$\rho = \text{Tr}_{\text{Horizons}} |0_M\rangle\langle 0_M| \propto \prod_i \left(\sum_{n\in\mathbb{N}} e^{-n\beta\omega_i} |n_i\rangle\langle n_i| \right), \tag{1.7}$$

which is normally used to claim that the Minkowski vacuum is composed of a "thermal bath of Rindler particles", have been criticized in the literature [22, 23, 24, 25] for precisely the reasons regarding the mathematical difficulties of constructing and comparing two distinct Fock-space representations of the theory based on different "single particle" spaces of states. In fact, the expression for the reduced density matrix in Eq. (1.7) representing

[1] This factor is responsible for the famous statistics-inversion behavior for the Rindler noise in odd spacetime dimensions [15], where bosonic fields show fermionic noise and vice versa, depending on the dimensions of spacetime. This statistics-inversion behavior is another example of the differences between an "inertial" and an "accelerated" thermal states. For more information about the statistics-inversion and how it is connected to the lack of a Huygens principle in odd spacetime dimensions, see [16].

the restriction of the Minkowski vacuum $|0_M\rangle$ in terms of the accelerated observer's "single-particle" states $|n_i\rangle$ can only be interpreted as a formal expression, since the $|n_i\rangle$ states are eigenstates of the Rindler Hamiltonian (1.5) and possess a continuous spectrum parametrized by $i = (\omega, \mathbf{k}_\perp) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^2$.

That is all to say that in the end, we are bound to encounter controversies regarding the "reality" and the physical meaning of "Rindler particles" if we try too hard to hold on to the concept of "particles" in order to make sense of the Unruh effect. Thus, in the classical statement of the Unruh effect of the form "the inertial vacuum appears as a thermal bath of particles from the point of view of an accelerated observer" should be understood, in light of the discussion above, as a shorthand for the more precise formulation of the result in terms of the KMS condition for observables living in the causally accessible region available to the accelerated observer.

All this nuances puts a premium on understanding how the Unruh temperature, and consequently the thermodynamical effects of acceleration, appear in actual physical systems. In this thesis, we shall be mainly interested in two aspects of the thermality of the "accelerated" vacuum.

The first aspect is about the experimental observability of the Unruh effect. As we can see from the expression for the Unruh temperature (1.1), a direct observation of the thermality of acceleration is not so easy: to obtain temperatures of the order of 1 K, it is necessary accelerations of the order of $10^{20}\mathrm{m/s}^2$ are necessary. Not only that, but we also need to be able to maintain said linear acceleration for long enough durations in order for the accelerated probe to equilibrate with the field and thermalize its own state.

Naturally, the first candidate to serve as a probe to the Unruh "thermal bath" is the electron. Being the lightest charged particle, electrons are capable of achieving the highest possible acceleration given a particular electromagnetic potential, and therefore capable of experiencing the highest Unruh temperature.

The first proposal to use electrons as a "thermometer" for the Unruh temperature came from Bell and Leinaas [26, 27], where they proposed a reinterpretation of the depolarization of the electrons accelerating inside a storage ring in terms of a Unruh-like analogous effect for a circular trajectory. In the standard analysis of the phenomena from the point of view of inertial observers stationary in the laboratory, this effect known as the Sokolov-Ternov effect [28, 29, 30]. Bell and Leinaas studied this effect from the point of view of an observer comoving with the electron beam and concluded that both the comoving and inertial analysis mainly agree with each other. This proposal of accepting the depolarization of electrons as a signal of the "Unruh" effect has been discarted mainly because of two reasons: it does not correspond to the same conditions of the classical Unruh prediction, namely linear uniform-acceleration, and it does not yield an "exact" thermal response for the electron's spin distribution [31].

An alternative, more recent proposal to probe the Unruh effect come from Chen and Tajima [32], where they suggest the use of ultraintense laser pulses as a method of accelerating the electron and propose the observation of emitted photons due to the electron's quivering, "Brownian" response to the Unruh temperature.

In Chapter 4 we present our published work [33], where we discuss and construct potential electron observables that can be used as probes for the acceleration temperature in the same context of Chen and Tajima's proposal. We focus on charged particles coupled to uniform, background electric fields, and we make general arguments for the response of the particle's transverse-momentum fluctuations due to its interactions with vacuum fluctuations in its comoving frame. We study and compare these observables in the cases of classical electromagnetic theory, pure photodynamics coupled to a semiclassical point-like charge, and full quantum electrodynamics coupled to a background electric field, discussing them in light of the current laser technology.

The second aspect of the Unruh effect we will be interested in this thesis is the connection between the accelerated observer's horizon and its interplay with topologically non-trivial quantum field theories. It is a well-known fact that in certain gauge theories, such as in quantum chromodynamics, a family of vacua $|\theta\rangle$ can be constructed from a coherent superposition of topologically inequivalent configurations $|n\rangle$ [34],

$$|\theta\rangle = \sum_{i\in\mathbb{Z}} e^{in\theta}|n\rangle, \tag{1.8}$$

where $\theta \in \mathbb{R}$ is a number parametrizing the superposition and $n \in \mathbb{Z}$ is the so-called called the winding number of the field configuration: the number of ways a gauge field can twist and turn at the asymptotic regions $A(r \to \infty)$ in Euclidean four-space. This introduces in the gauge theory's effective Lagrangian for the field the famous CP-violating θ-term

$$\mathcal{L}_\theta = \frac{\theta}{16\pi^2}\,\mathrm{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}), \tag{1.9}$$

and this introduction leads to several observable consequences, such as a θ-dependent contribution to the neutron's electric dipole moment (EDM). Experimental measurements on the neutron's EDM constraint θ to be very close to zero, $|\theta| < 10^{-10}$, and a first-principles explanation for why is the value of θ so small is a famous unsolved problems in physics: *the strong CP problem.*

In Chapter 5 we present our published work [35], where we discuss the strong CP problem in the context of causal horizons. We speculate on how general covariance may be able to constraint the topological structure of non-trivial gauge theories, and discuss the possible consequences for the CP-violating θ-term of Eq. (1.9). We draw a an analogy between the topology-induced θ-vacua of gauge fields and the well-known model of a quantum particle in a periodic potential to argue how dissipative effects can decohere the θ-vacua superposition in Eq. (1.8), suggesting a possible dynamical mechanism that can solve the strong CP problem.

Chapter 2 gives a quick overview of the Unruh effect, introducing what is understood by acceleration temperature and thermodynamics, extending some results found in the literature, as well as setting the context for the rest of the thesis, while Chapter 3 gives a quick overview of the various ways of treating the electromagnetic radiation emitted by an accelerated electron that sets the context for Chapter 4. Finally, in Chapter 6, we discuss the results obtained and the arguments made in the previous chapters, taking a bird's eye view on the subjects and highlighting the important takeaways. In Chapter 7 we conclude by summarizing the main results and pointing to future directions.

Chapter 2

Review: The Unruh Effect

The purpose of this chapter is to give a brief overview of the Unruh effect and to establish the connection between Rindler KMS-states and the "inertial vacuum" seen by an inertial observer in th laboratory. We give a quick review of uniform acceleration for classical relativistic particles in order to establish notation, introduce the accelerated frames associated with an observer comoving with the accelerated particle, and setup the general framework to construct QFT from an accelerated point of view. Then we discuss one of the ways of constructing finite-temperature, KMS-states in QFT, based on the real (proper-)time-path formalism [36, 37], and use it to show that in the accelerated observer's accessible region of spacetime, the KMS state with respect to its proper-time yields the same correlation functions of the Minkowski vacuum of inertial observers. We finish by discussing how the finite-temperature fluctuations of the "accelerated thermal vacuum" can be understood from the point of view of broken correlations between regions outside the Rindler horizon, and make the connection with the usual formulation of the Unruh effect based on the common formal density matrix approach.

2.1 Uniformly-accelerated particle

In classical relativistic mechanics, the uniformly accelerated particle is an idealization of an infinitesimally small body that is subjected to a constant accelerating force. By this we mean a body whose dimensions are so small that we can neglect its dimensions when describing its motion, which can be mathematically represented by a smooth mapping $\gamma : \mathbb{R} \to \mathcal{M}$ of the real line into a chosen configuration space $\mathcal{M}$ that represents the spacetime that the particle inhabits. When $\mathcal{M}$ is a differentiable manifold with a given connection ∇, we can say that the particle has constant acceleration when the covariant derivative $|\nabla_{\dot\gamma}\dot\gamma|$ of the proper four-velocity $\dot\gamma$ is constant.

The classical description of the accelerated point-particle starts with the choice of time-oriented pseudo-Riemannian manifold $(\mathcal{M}, g)$ that acts as the background spacetime for the particle's trajectory γ. The standard choice from classical relativistic mechanics is the Minkowski spacetime $\mathcal{M} = (R^4, \eta)$, where $\eta = \text{diag}(1, -1, -1, -1)$ is the standard Lorentzian metric. From the condition of constant acceleration $|\nabla_{\dot{\gamma}}\dot{\gamma}|$ with respect to the metric η, we get

$$\eta_{\mu\nu}\frac{d\dot{\gamma}^\mu}{d\tau}\frac{d\dot{\gamma}^\nu}{d\tau} = -a^2, \tag{2.1}$$

where τ is the proper-time of the trajectory γ and the negative sign is due to the convention used for the signs of the metric for space-like four-vectors.

For simplicity (and without loss of generality), we can choose as the initial conditions for the differential equation the conditions $\gamma^\mu(0) = a^{-1}\delta^{\mu 3}$, and $\dot{\gamma}^\mu(0) = \delta^{\mu 0}$, which chooses the Z-axis as the direction of acceleration and determines that the turning-point of the trajectory in the direction of acceleration takes place at the proper-time $\tau = 0$. These initial conditions yield a worldline γ_a given by

$$\begin{aligned} \gamma_a^0(\tau) &= a^{-1}\sinh a\tau, \\ \gamma_a^1(\tau) &= \gamma_a^2(\tau) = 0, \\ \gamma_a^3(\tau) &= a^{-1}\cosh a\tau, \end{aligned} \tag{2.2}$$

which represents a hyperbola in the $T - Z$ plane. We call this solution the uniformly-accelerated worldline, and it is represented in Figure 2.

2.2 Accelerated reference frames

Together with the accelerated particle's worldline γ_a from Eq. (2.2), we can define an accelerated observer by choosing an orthonormal basis $\{e_\mu^R(\tau)\}$ on each tangent space $T_{\gamma(\tau)}\mathcal{M}$ that the observer's worldline passes through [38]. However, in order to simplify the discussion (and later make the connection with field theory), we instead introduce the reference frame of an accelerated observer by directly choosing comoving coordinates with respect to γ_a.

One such choice consists of coordinates $(\tau, \rho, \mathbf{x}_\perp)$, $\rho > 0$, defined by the transformations

$$T = \rho\sinh a\tau, \quad X = \rho\cosh a\tau, \quad Y = y, \quad Z = z \tag{2.3}$$

with respect to the usual global coordinates (T, X, Y, Z) that cover the configuration space $\mathcal{M}$. We shall call these coordinates *polar Rindler coordinates*[1], and the observer defined by the accelerated worldline γ together with such choice of reference frame is called an accelerated (or Rindler) observer, and it is also shown in Figure 2.

Another closely related coordinate system that will be very useful is the coordinates $(\tau, \xi, \mathbf{x}_\perp) \in \mathbb{R}^4$, defined by the relationship $\rho = a^{-1}e^{a\xi}$ to the polar Rindler

[1] We call these coordinates polar Rindler coordinates because when performing a Wick rotation $\tau \to i\tau_E$, these coordinates take the form of proper polar coordinates in the Euclidean section of the manifold.

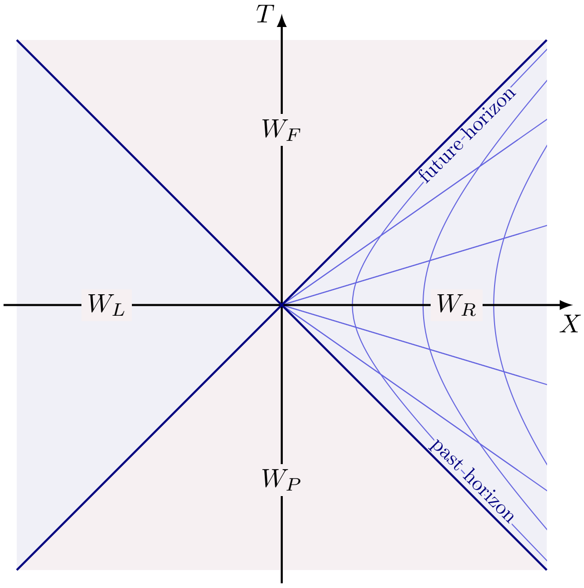

Figure 2 – Representation of an accelerated (Rindler) observer. The equal-time coordinate lines $\tau = c^{\text{st}}$ are straight lines that begin at the origin $(t, \mathbf{x}) = 0$, while $\rho = c^{\text{st}}$ lines are hyperbolas asymptotic to the lightcones $t \pm x = 0$ based on the origin. The observer's original accelerated worldline is given by the coordinate line $\rho = a^{-1}$.

coordinates. With respect to the usual inertial Minkowski coordinates, we have

$$T = a^{-1} e^{a\xi} \sinh a\tau, \quad X = a^{-1} e^{a\xi} \cosh a\tau, \quad \mathbf{X}_\perp = \mathbf{x}_\perp. \tag{2.4}$$

We shall call this coordinate system *(conformal) Rindler coordinates*[2]. One of the advantages of the conformal Rindler coordinates $(\tau, \xi, \mathbf{x}_\perp)$ is the simplified expression for the metric tensor

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = e^{2a\xi}(d\tau^2 - d\xi^2) - d\mathbf{x}_\perp^2 \tag{2.5}$$

in the region covered by the coordinate chart in Eq. (2.4).

One important characteristic of accelerated observers comes from causality. Due to their state of motion, only a part of the entire spacetime is completely available for observation and probing by an accelerated observer. By this statement we mean that there are events in spacetime that cannot be connected to the accelerated worldline γ via time-like or light-like curves in $\mathcal{M}$.

The region where the accelerated observer has complete information—that is, the region where all events can both be seen and be probed by an accelerated observer—is defined by the condition $W_R = \{x > |t|\}$, and we call W_R the *right Rindler wedge*. Any event outside this region is, in some way, shape, or form, disconnected from the worldline of such observer. The accelerated observer perceives all events in the "past" wedge $W_P = \{t < -|x|\}$ as unreachable, all the events in the "future" wedge $W_F = \{t > |x|\}$ as

2 The name conformal comes from their behavior when we deal with the two-dimensional spacetime Mink_2, where they represent a true conformal symmetry of the metric.

unobservable, and all the events in the left Rindler wedge $W_L = \{x < -|t|\}$ as completely inaccessible. (See Figure 2 for the geometry of all the wedges.)

Another important characteristic of uniformly accelerated observers is their behavior under the action of a Lorentz boost in the direction of acceleration. Lorentz boosts are symmetries of Minkowski spacetime, given by the the transformations induced by the vector field (1.5),

$$K \equiv X\left(\frac{\partial}{\partial T}\right) + T\left(\frac{\partial}{\partial X}\right),$$

and by treating the right Rindler wedge W_R as a submanifold of $\mathcal{M}$, we can restrict their action of K to W_R, which can be re-written in terms of Rindler coordinates as $aK = \dfrac{\partial}{\partial \tau}$. That means boosts are symmetries of the uniformly accelerated trajectories and act as a global, timelike Killing vector field for the entire Rindler space.

2.3 Quantum fields in Rindler space

Now, given that the causally accessible region of an uniformly accelerated observer is given by the right Rindler wedge $W_R \subset \mathcal{M}$, we can treat the sub-manifold $(W_R, \eta|_{W_R})$ of Minkowski space $(\mathcal{M}, \eta)$ as its own globally hyperbolic spacetime, which we shall call the Rindler space. As a globally hyperbolic spacetime, we can study field theories in this space without having to refer back to the theory in its "parent" manifold $\mathcal{M}$. In this section, we shall be considering a free scalar field theory in Rindler space, given by the action functional S of the form

$$S[\varphi, \partial\varphi] = \frac{1}{2}\int d^n x \sqrt{|\det g_{\mu\nu}|}\left(g^{\mu\nu}\partial_\mu\varphi(x)\partial_\nu\varphi(x) - m^2\varphi^2(x)\right), \tag{2.6}$$

where m is the bare-mass of the field and $n = \dim W_R$.

The variational principle $\delta S[\varphi, \partial\varphi] = 0$ applied to the scalar field yields the classical equations of motion $(\Box_g + m^2)\varphi(x) = 0$ for the field, where

$$\Box_g = |\det g_{\mu\nu}|^{-1/2}\,\partial_\mu(\sqrt{|\det g_{\mu\nu}|}g^{\mu\nu}\partial_\nu) \tag{2.7}$$

is the d'Alembertian operator in terms of the Rindler metric $g = \eta|_{W_R}$. In Rindler coordinates, this equation of motion takes the form

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\xi^2} + e^{2a\xi}(-\nabla_\perp^2 + m^2)\right)\varphi(x) = 0, \tag{2.8}$$

which is a type of modified Bessel equation and can be solved exactly. The general solutions to Eq. (2.6) can be written in terms of a complete set of solutions $\left\{f_{\omega\mathbf{k}_\perp}, f^*_{\omega\mathbf{k}_\perp}\right\}$ constructed from the solution

$$f_{\omega\mathbf{k}_\perp}(\tau, \xi, \mathbf{x}_\perp) = \left(\frac{\sinh(\pi\omega/a)}{4\pi^4 a}\right)^{1/2} K_{i\omega/a}\left(\frac{\mu}{a}e^{a\xi}\right)e^{-i\omega\tau + i\mathbf{k}_\perp\cdot\mathbf{x}_\perp}, \tag{2.9}$$

where $\omega > 0$ is the frequency of the Rindler modes $f_{\omega \mathbf{k}_\perp}$ and $\mu \equiv \sqrt{\mathbf{k}_\perp^2 + m^2}$. Thus the general solution to the classical equation of motion for the scalar field is given by

$$\varphi(x) = \int_0^\infty d\omega \, d^{n-2}\mathbf{k}_\perp \left(a_{\omega \mathbf{k}_\perp} f_{\omega \mathbf{k}_\perp}(x) + a^*_{\omega \mathbf{k}_\perp} f^*_{\omega \mathbf{k}_\perp}(x) \right), \tag{2.10}$$

where the Rindler mode $f_{\omega \mathbf{k}_\perp}$ is a normal mode with positive frequency with respect to the timelike Killing field K in Eq. (1.5).

The usual prescription to canonically quantize the field is to promote the coefficients $\{a_{\omega \mathbf{k}_\perp}, a^*_{\omega \mathbf{k}_\perp}\}$ on the normal-mode expansion above into operators $\{\hat{a}_{\omega \mathbf{k}_\perp}, \hat{a}^\dagger_{\omega \mathbf{k}_\perp}\}$ acting on a Hilbert space of states $\mathcal{H}_R$, constructed from the action of the creation operator $a^\dagger_{\omega \mathbf{k}_\perp}$ on a vacuum state $|0_{W_R}\rangle$:

$$|n_{\omega \mathbf{k}_\perp}\rangle \propto (\hat{a}^\dagger_{\omega \mathbf{k}_\perp})^n |0_{W_R}\rangle. \tag{2.11}$$

The states $|n_{i_1}, n_{i_2}, \ldots\rangle$, $i = (\omega, \mathbf{k}_\perp)$, are called the "Rindler n-particle states". These are the states that are usually used in the derivation of the Unruh effect, which we discussed on Chapter 1, and in particular the states that constitute the formal expression for the density matrix in Eq. (1.7),

$$\rho = \mathrm{Tr}_{\text{Horizons}} |0_\mathcal{M}\rangle\langle 0_\mathcal{M}| \propto \prod_i \left(\sum_{n \in \mathbb{N}} e^{-n\beta\omega_i} |n_i\rangle\langle n_i| \right). \tag{2.12}$$

This expression for the Rindler Gibbs state is obtained by interpreting the inertial vacuum state $|0_\mathcal{M}\rangle \in \mathcal{H}_\mathcal{M}$ of the usual quantum field in Minkowski space in terms of the Hilbert space $\mathcal{H}_{W_L} \otimes \mathcal{H}_{W_R}$ associated with the quantization of both the right and the left Rindler wedges. Here, the Hilbert space $\mathcal{H}_{W_L}$ is obtained from the canonical quantization of the scalar field in the left Rindler wedge W_L in the exact same way as we've done for the right wedge W_R. Since the left wedge W_L is completely hidden from the accelerated observer by the presence of a causal horizon, the trace over "horizons" in Eq. (2.12) is in fact a trace over the left wedge sector $\mathcal{H}_{W_L}$ of the entire space $\mathcal{H}_{W_L} \otimes \mathcal{H}_{W_R}$.

By now, it is clear how the expression for the reduced density matrix for $|0_\mathcal{M}\rangle$ in the right Rindler wedge W_R can only be a formal one, given that the index i in the product must take values in the continuous range $i = (\omega, \mathbf{k}_\perp)$ given by the spectrum of the modes $f_{\omega \mathbf{k}_\perp}$. As we've discussed before, trying to interpret physical phenomena that occurs for accelerated systems coupled to $|0_\mathcal{M}\rangle$ in terms of its "Rindler particle content" does not involve the actual physics behind the phenomena, since the only thing with physical meaning are the correlators of the fields in a given state, instead of the states themselves.

2.4 Finite-temperature in QFT

We've discussed in the previous section that trying to use canonical quantization methods to describe the inertial vacuum in terms of "Rindler particles" leads to a generally ill-defined expression for the reduced density matrix representing $|0_{\mathcal{M}}\rangle\langle 0_{\mathcal{M}}|$ in the Rindler wedge W_R. One way of circumventing this problem is to find a way of determining the time-ordered thermal n-point functions $\langle \mathcal{T}[\varphi(x_1)\ldots\varphi(x_n)]\rangle_\beta$ directly, instead of relying on the particular representation of $\rho = Z^{-1}\exp(-\beta H)$ in terms of the eigenstates of the Hamiltonian H of the field.

This can be done by working within the path-integral formulation of QFT. In the standard Gibbs representation of a quantum field at finite temperature $T = \beta^{-1}$, the expectation values of time-ordered field operators are given by the formal expression

$$D_F^\beta(x_1, x_2, \ldots, x_n)\text{“} = \text{”}\frac{1}{Z}\operatorname{Tr}\left(e^{-\beta H}\mathcal{T}\left[\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)\right]\right), \tag{2.13}$$

where $Z = \operatorname{Tr} e^{-\beta H}$ is a normalization factor for the state, and $\mathcal{T}$ indicates time-ordering with respect to time-translations associated with H. In the zero-temperature ($\beta = \infty$) theory, we know that these n-point functions can be obtained from the generating functional $Z[J]$,

$$Z[J] = \int [\mathcal{D}\varphi]\,[\mathcal{D}\pi]\,\exp\left(i\int d^n X\,(\pi\dot{\varphi} - H + J\varphi)\right), \tag{2.14}$$

by means of functional differentiation with respect to the sources $J : \mathcal{M} \to \mathbb{R}$:

$$D_F(x_1, x_2, \ldots, x_n) = (-i)^n \frac{\delta^n \ln Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}\Big|_{J=0}. \tag{2.15}$$

The idea now is to generalize the zero-temperature procedure above for the finite-temperature case, where the thermal Feynman n-point functions $D_F^\beta(x_1, \ldots, x_n)$ described in Eq. (2.13). This can be done by generalizing the zero-temperature generating functional $Z[J]$ to the complex T-plane [37],

$$Z_\beta[J] = \int_{\mathcal{C}} [\mathcal{D}\varphi]\,[\mathcal{D}\pi]\,\exp\left(i\int_{\mathcal{C}} d^n x\,(\pi\dot{\varphi} - H + J\varphi)\right), \tag{2.16}$$

where now the timelike coordinate $T \in \mathcal{C} \subseteq \mathbb{C}$ lives in the contour in the complex T-plane that runs along the segment $\mathcal{C}_1 = (-T \to +T)$ on the real line, continues along $\mathcal{C}_2 = (+T \to +T - i\beta/2)$downwards in the imaginary time direction, winds back around along $\mathcal{C}_3 = (+T - i\beta/2 \to -T - i\beta/2)$ parallel to the real axis, and finally descends down along $\mathcal{C}_4 = (-T - i\beta/2 \to -T - i\beta)$. The contour $\mathcal{C}$ is represented in Figure 3. The fields φ and π are defined on the complex extension of $\mathcal{M}$ determined by $\mathcal{C}$, and φ satisfies the periodic boundary condition

$$\varphi(-T) = \varphi(-T - i\beta). \tag{2.17}$$

Functional derivatives of $Z_{\mathcal{C}}[J]$ with respect to the source J gives the time-ordered, thermal n-point functions $D_F^\beta(x_1, \ldots, x_n)$,

$$\langle \mathcal{T}_{\mathcal{C}}[\varphi(x_1)\cdots\varphi(x_n)]\rangle = (-i)^n \frac{\delta^n \ln Z_{\mathcal{C}}[J]}{\delta J(x_1)\cdots\delta J(x_n)}\Big|_{J=0}, \tag{2.18}$$

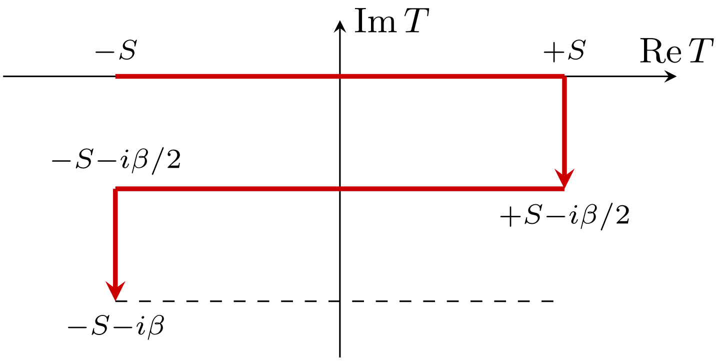

Figure 3 – Contour in the complex T-plane used to construct the finite-temperature generating functional $Z_{\mathcal{C}}[J]$ in Eq. (2.16). For the parallel segments $\mathcal{C}_{1,3}$, a small slope in the negative imaginary T direction is understood.

but now the time ordering in the complex T-plane is given by the ordering on the contour $\mathcal{C}$: given $\tau, \tau' \in \mathcal{C}$, we have

$$\theta_{\mathcal{C}}(\tau - \tau') = \int_{-\infty}^{\tau} d\lambda\, \delta_{\mathcal{C}}(\lambda - \tau'), \tag{2.19}$$

where $\delta_{\mathcal{C}}$ is the delta function on the contour $\mathcal{C}$, defined by

$$f(\tau) = \int_{\mathcal{C}} d\lambda\, \delta_{\mathcal{C}}(\lambda - \tau) f(\lambda). \tag{2.20}$$

The construction of the generating functional in Eq. (2.16) is the real-time analog of the Matsubara formalism in Euclidean path-integrals [39], and entirely equivalent to the Schwinger-Keldysh formalism [40, 36] (modulo the choice of contour $\mathcal{C}$).

2.5 The Unruh effect

Having constructed a way to generate time-ordered n-point functions for a thermal state, we will now show that the KMS state corresponding to the generating functional $Z_{\mathcal{C}}[J]$ in Rindler space yields the same correlators as the usual Minkowski inertial vacuum $|0_{\mathcal{M}}\rangle$. This is a generalization of the construction by Horibe et al. [41], where this approach has been used to show the equivalency for massless scalar fields in $(1+1)$ dimensions. Here we extend that result to the case of n-dimensional scalar fields with possible non-zero masses.

First, we integrate over the canonical field momenta π, which is a gaussian integral and can be carried on exactly to yield

$$Z_{\beta}[J] = \int [\mathcal{D}_{\mathcal{C}}\varphi] \exp\left[i \int_{\mathcal{C}} d^n x \, \left(\mathcal{L}[\varphi, \partial_\mu \varphi] + J \cdot \varphi\right)\right], \tag{2.21}$$

where $\mathcal{L}$ is the Lagrangian density for the scalar field φ, given by

$$\mathcal{L}[\varphi, \partial_\mu \varphi](x) = -\frac{1}{2} \left|\det g_{\mu\nu}\right|^{1/2} \varphi(x)(\Box_g + m^2)\varphi(x), \tag{2.22}$$

where $x \equiv (\tau, \xi, \mathbf{x}_\perp)$ is written in terms of the Rindler coordinates on W_R. By defining the thermal Feynman propagator D^F_β on the countour $\mathcal{C}$ by

$$(\Box_g + m^2) D^F_\beta(x, x') = -\frac{\delta_{\mathcal{C}}(x - x')}{\sqrt{\det g_{\mu\nu}}}, \tag{2.23}$$

we can change the integration variable φ in the path integral (2.21) to

$$\varphi(x) \to \varphi(x) + \int_{\mathcal{C}} d^n x \, D^F_\beta(x, y) J(y) \tag{2.24}$$

and get the generalization of the known form of the generating functional

$$Z_\beta[J] = Z_\beta[0] \exp\left[-\frac{i}{2}\int_{\mathcal{C}} d^n x \, d^n y \, J(x) D^F_\beta(x, y) J(y)\right]. \tag{2.25}$$

To solve Eq. (2.25), we make the ansatz

$$D^F_\beta(x, y) = D^>_\beta(x, y)\theta_{\mathcal{C}}(\Delta\tau) + D^<_\beta(x, y)\theta_{\mathcal{C}}(-\Delta\tau), \tag{2.26}$$

where $\Delta\tau \equiv \tau_x - \tau_y$. The change of variables (2.24) together with the periodic boundary conditions (2.17) on the contour $\mathcal{C}$ yields the KMS condition on the propagators $D^{\lessgtr}_\beta$,

$$D^>_\beta(\tau_x - i\beta, \tau_y) = D^<_\beta(\tau_x, \tau_y), \tag{2.27}$$

where we've omited the spatial coordinates for brevity. Remembering that the propagator equation (2.23) can be solved in terms of modified Bessel functions $K_{i\omega/a}(a^{-1}\mu e^{a\xi})$, we can find the solutions $D^{\lessgtr}_\beta(x, y)$ of Eq. (2.23) subjected to the KMS boundary conditions:

$$D^>_\beta(x, y) = -\frac{i}{\pi^2 a}\int_0^\infty d\omega \int \frac{d^{n-2}\mathbf{k}_\perp}{(2\pi)^{n-2}} \left(\frac{\sinh(\beta\omega/2)}{e^{\beta\omega} - 1}\right) \mathcal{G}_\omega(\xi_x, \mathbf{x}_\perp; \xi_y, \mathbf{y}_\perp) \mathcal{F}_{\mathcal{C}}(\tau_x - \tau_y), \tag{2.28}$$

where the functions $\mathcal{G}_\omega$ $\mathcal{F}_{\mathcal{C}}$ are given by

$$\mathcal{G}_\omega(\xi_x, \mathbf{x}_\perp; \xi_y, \mathbf{y}_\perp) = K_{i\omega/a}\left(\frac{\mu}{a}e^{a\xi_x}\right) K_{i\omega/a}\left(\frac{\mu}{a}e^{a\xi_y}\right) e^{i\mathbf{k}_\perp(\mathbf{x}_\perp - \mathbf{y}_\perp)}, \tag{2.29}$$

$$\mathcal{F}_{\mathcal{C}}(\tau_x - \tau_y; \omega) = \left(e^{-i\omega(\tau_x - \tau_y) + \beta\omega} + e^{i\omega(\tau_x - \tau_y)}\right), \tag{2.30}$$

where $\mu \equiv (\mathbf{k}^2 + m^2)^{1/2}$ and, finally, $D^<_\beta(x, y) = D^>_\beta(y, x)$.

Now we can calculate the τ-ordered two-point function Eq. (2.26) obtained from the thermal generating functional $Z_\beta[J]$ in Rindler space. Events in the Rindler wedge $x, y \in W_R$ are represented on the countour $\mathcal{C}$ by the segment $\mathcal{C}_1$ (see Fig. 3). In this segment, the time-ordering of the contour coincides with the time-ordering of the normal real line $\mathbb{R}$. Assuming for simplicity $\Delta\tau \equiv \tau_x - \tau_y > 0$, and noting we can combine the terms in Eq. (2.28) and extend the integration region, we can write

$$D^F_\beta(x, y) = -\frac{i}{\pi^2 a}\int_{-\infty}^{\infty} d\omega \int \frac{d^{n-2}\mathbf{k}_\perp}{(2\pi)^{n-2}} \mathcal{G}_\omega(\xi_x, \mathbf{x}_\perp; \xi_y, \mathbf{y}_\perp) e^{-i\omega\Delta\tau + \frac{1}{2}\beta\omega} \tag{2.31}$$

Using the integral representation for the product of modified Bessel functions $K_{i\nu}(z)K_{i\nu}(z')$ [42],

$$\int_{-\infty}^{+\infty} d\nu\, e^{i\nu\tau} K_{i\nu}(\alpha_1)K_{i\nu}(\alpha_2) = \pi K_0\left(\sqrt{\alpha_1^2+\alpha_2^2+2\alpha_1\alpha_2\cosh\tau}\right), \tag{2.32}$$

valid for $|\arg\alpha_1| + |\arg\alpha_2| + |\Im\tau| < \pi$, we obtain

$$D_\beta^F(x,y) = -\frac{i}{2\pi}\int \frac{d^{n-2}\mathbf{k}_\perp}{(2\pi)^{n-2}} e^{i\mathbf{k}_\perp\Delta\mathbf{x}_\perp} K_0\left(\frac{\mu}{a}\gamma\right), \tag{2.33}$$

where

$$\gamma^2 \equiv e^{2a\xi_x} + e^{2a\xi_y} + 2e^{a(\xi_x+\xi_y)}\cosh(a\Delta\tau + i(\pi - 0^+)). \tag{2.34}$$

By using the integral representation for the zeroth-order Bessel function K_0,

$$K_0(z) = \int_0^\infty \frac{dx}{2x}\exp\left[-\frac{1}{2}\left(x+\frac{z}{x}\right)\right], \tag{2.35}$$

we can swap the order of integration on (2.33) and complete the $(n-2)$ gaussian integrals with respect to the transverse momenta $\mathbf{k}_\perp$ left, and obtain

$$\int \frac{d^{n-2}\mathbf{k}_\perp}{(2\pi)^{n-2}} e^{i\mathbf{k}_\perp\Delta\mathbf{x}_\perp} K_0\left(\frac{\mu}{a}\gamma\right) = \int_0^\infty \frac{dx}{2x}\left(\frac{mx}{2\pi\Upsilon}\right)^{\frac{n}{2}-1}\exp\left[-\frac{m}{2}\Upsilon\left(x-\frac{1}{x}\right)\right] \tag{2.36}$$

$$= \left(\frac{m}{2\pi\Upsilon}\right)^{\frac{n}{2}-1} K_{\frac{n}{2}-1}(m\Upsilon), \tag{2.37}$$

where $\Upsilon^2 = \gamma^2 + \Delta\mathbf{x}_\perp^2$.

The final expression for the Feynman propagator $D_\beta^F(x,y)$ for the Rindler KMS state represented by $Z_\beta[J]$ is given by

$$D_\beta^F(x,y) = \frac{i}{2\pi}\left(\frac{m}{2\pi\Upsilon}\right)^{\frac{n}{2}-1} K_{\frac{n}{2}-1}(m\Upsilon). \tag{2.38}$$

By noting that $\Upsilon^2 = e^{2a\xi_x} + e^{2a\xi_y} - 2e^{a(\xi_x+\xi_y)}\cosh(a\Delta\tau) + \Delta\mathbf{x}_\perp^2 + i0^+$ is simply the geodesic distance between x and y with the correct sign for the $i0^+$ prescription in the imaginary-time plane, we identify (2.38) as simply usual zero-temperature Feynman propagator of inertial observers, $D_F(x,y)$, for points inside the Rindler wedge W_R of the accelerated observer. This establishes the previously mentioned result that the inertial vacuum is indeed a thermal state—as understood by the KMS condition—with respect to accelerated observers.

Chapter 3

Review: Classical and quantum dynamics of accelerated electrons

The purpose of this chapter is to give a brief overview of various approaches to calculating the electromagnetic radiation emitted by an uniformly accelerated electron. Following the discussion about the Unruh effect in Chapter 2, we expect that there may be thermal-like signatures in observables that are sensitive to the n-point functions of the quantized electromagnetic field. As we will discuss in Chapter 4, one such observable is the fluctuations around the electron's momentum transverse to the acceleration axis. The measure for the transverse-momentum fluctuations of the electron is given by the mean-squared transverse-momentum transfer $\kappa \propto \frac{d}{d\tau}\langle \Delta \mathbf{p}_\perp^2 \rangle$, which can be obtained from the photon emission rates $dN_\gamma / d\tau d^2\mathbf{k}_\perp$ by the integration over the photon transverse-momentum:

$$\frac{d}{d\tau}\langle \Delta \mathbf{p}_\perp^2 \rangle = \int d^2\mathbf{k}_\perp \frac{dN_\gamma}{d\tau d^2\mathbf{k}_\perp} \Delta \mathbf{p}_\perp^2. \tag{3.1}$$

In this chapter, we shall consider the more realistic case where the charged particles are accelerated by a constant background electric field, where we get a more general expression for the accelerated trajectory (2.2) in terms of the electromagnetic field strength $F_{\mu\nu}$. We shall focus on calculating the photon emission differentials dN_γ for three cases that will be important in Chapter 4: the classical electromagnetic field interacting with a classical electron current, and the quantized electromagnetic field interacting with the same classical source, and the full quantum electrodynamical calculation assuming the fermionic field of the electron is coupled to a background electric field.

3.1 Classical electron accelerated by a constant electric field

In the last chapter, the acceleration of the classical point-like particle was introduced by hand in the conditions of constant proper acceleration in Eq. (2.1). Now, however, we shall be interested in the actual mechanics of how we accelerate the charged particles in our models. In this section, we will consider the dynamics of electrically charged particles when coupled to a background electromagnetic potential B_μ, corresponding to a constant electric field $\mathbf{E} = E_z\mathbf{z}$, that will be responsible for accelerating the charge along the $\mathbf{z}$ axis.

The action for a charged particle coupled to a background field B_μ is given by

$$S_B[\gamma] = -\int d\tau \sqrt{\eta_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu} - \int d^4x\, B_\mu(x)J^\mu_\gamma(x), \tag{3.2}$$

where τ is the proper-time of the particle and the coupling of the charge to the background electromagnetic potential is given in terms of the four-vector current

$$J^\mu_\gamma(x) = e\int d\tau\, \dot{\gamma}^\mu(\tau)\delta^4(x-\gamma(\tau)). \tag{3.3}$$

where we take, for convenience, the parameter τ as the proper-time for the electron's trajectory γ^μ. Considering just the cases where the background field configuration has a constant field-strength tensor $\partial_\mu F_{\alpha\beta} = 0$, the classical equations of motion corresponding to the action $S_B[\gamma]$ are given by

$$\frac{\delta S_B[\gamma]}{\delta\gamma_\mu(\tau)} = m\frac{d^2}{d\tau^2}\gamma^\mu(\tau) - \int d^4x\, \frac{J^\nu(x)}{\delta\gamma_\mu(\tau)}B_\nu(x) = 0 \iff \frac{dp^\mu(\tau)}{d\tau} = \left(\frac{e}{m}\right) F^\mu{}_\nu\, p^\nu(\tau), \tag{3.4}$$

where $p^\mu(\tau) \equiv m\dot{\gamma}^\mu(\tau)$ is the four-momentum of the particle and $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ is the field-strength tensor corresponding to the background field configuration B_μ. For a constant electric field in the $\mathbf{z}$ direction, we have a background potential $B_\mu = E_z t\delta_{\mu 3}$, and an associated $F_{\mu\nu}$ given by

$$F^\mu{}_\nu = E_z(\delta^{\mu 0}\delta_{\nu 3} + \delta^{\mu 3}\delta_{\nu 0}). \tag{3.5}$$

For a constant electromagnetic field-strength F, we can integrate the classical equations of motion (3.4) directly. The solution to the particle's four-momentum in the presence of a constant electromagnetic field is given by

$$p^\mu(\tau) = \exp\left(\frac{eF^\mu{}_\nu\,\Delta\tau}{m}\right)p^\nu(\tau_0), \tag{3.6}$$

where $p^\mu(\tau_0)$ is the initial condition on the momentum of the charge at the instant τ_0. Even though we will take $\tau_0 = 0$ and $p^\mu(0) = m\delta^{\mu 0}$ as the initial conditions at the end, we shall keep the expressions general throughout the calculation.

The exponential of the field-strength tensor is given by

$$\exp\left(\frac{eF^\mu{}_\nu\,\Delta\tau}{m}\right) - I = \frac{1}{E_z^2}\left(\cosh\left(\frac{eE_z\Delta\tau}{m}\right) - 1\right)F^2 + \frac{1}{E_z}\sinh\left(\frac{eE_z\Delta\tau}{m}\right)F, \tag{3.7}$$

and the explicit solution for the four-momentum vector $p^\mu(\tau)$ is

$$p^\mu(\tau) - p^\mu(\tau_0) = \left(\cosh\left(\frac{eE_z\Delta\tau}{m}\right) - 1\right)\frac{((eF)^2\cdot p_0)^\mu}{(eE_z)^2} + \sinh\left(\frac{eE_z\Delta\tau}{m}\right)\frac{(eF\cdot p_0)^\mu}{(eE_z)}, \tag{3.8}$$

where the contraction with F^2 is understood as $F^2 \cdot p_0 \equiv F^{\alpha}{}_{\beta} F^{\beta}{}_{\gamma} p^{\gamma}(\tau_0)$. The trajectory of the particle is obtained by integrating once again Equation (3.8),

$$\begin{aligned}\gamma^{\mu}(\tau) - \gamma^{\mu}(\tau_0) = \frac{1}{m}\int_{\tau_0}^{\tau} d\lambda\, p^{\mu}(\lambda) = \frac{1}{m}\left(p_0^{\mu} - \frac{((eF)^2 \cdot p_0)^{\mu}}{(eE_z)^2}\right)\Delta\tau \\ + \sinh(a\Delta\tau)\frac{((eF)^2 \cdot p_0)^{\mu}}{(eE_z)^3} + (\cosh(a\Delta\tau) - 1)\frac{(eF \cdot p_0)^{\mu}}{(eE_z)^2},\end{aligned} \tag{3.9}$$

where $a \equiv eE_z/m$ is the acceleration parameter of the trajectory.

By evaluating the products $(eF \cdot p_0)$ and $((eF)^2 \cdot p_0)$ with respect to the field-strength form of Eq. (3.5), we can calculate magnitude of the four-acceleration of the particle in a constant electric field background. It is given by

$$\ddot{\gamma}^2(\tau) = -\left(\frac{eE_z}{m}\right)^2\left(1 + \frac{\mathbf{p}_{\perp}^2}{m^2}\right), \tag{3.10}$$

where $\mathbf{p}_{\perp} = \mathbf{p}_{\perp}(0)$ is the initial momentum of the particle in the plane transverse to the electric field axis $\mathbf{z}$.

3.2 Classical electromagnetic radiation from an accelerated electron

With the general expression for the trajectory γ of an electron accelerated by a constant electric field in Eq. (3.9), we can study the radiation emission by a point-like charge moving along this worldline[1]. The action that defines the interaction between the charged particle with the (now dynamical) electromagnetic potential A_{μ} can be written as

$$S[\gamma, A] = S_B[\gamma] + S_{\text{EM}}[A] + S_I[\gamma, A], \tag{3.11}$$

where $S_B[\gamma]$ is the relativistic free-particle contribution under a fixed background field B_{μ}, given by Eq. (3.2), $S_{\text{EM}}[A]$ is the free electromagnetic field action for the electromagnetic potential A,

$$S_{\text{EM}}[A] = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu}, \tag{3.12}$$

where $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$, and $S_I[\gamma, A]$ is the field-charge interaction term given by

$$S_I[\gamma, A] = -e\int d^4x\, A_{\mu}(x)J_{\gamma}^{\mu}(x). \tag{3.13}$$

The classical equations of motion for the electromagnetic potential A_{μ} associated with the action $S[\gamma, A]$ in the Lorenz gauge $\partial_{\mu}A^{\mu} = 0$ is given by the inhomogeneous wave equation

$$\partial^2 A^{\mu}(x) = J_{\gamma}^{\mu}(x). \tag{3.14}$$

[1] It is worth mentioning that the emission of radiation by uniformly accelerated charges as predicted by classical electrodynamics—particularly in the context of different coordinate systems—has been a source of debate and controversy during the first part of the twentieth century. This discussion at the classical level was settled by the works of Fulton, Rohrlich, and Boulware [43, 44], to which we refer the reader for more information about the topic.

The solution for the radiated field $A_\mu^{\text{rad}}(x)$ can be written as [45]

$$A^\mu_{\text{rad}}(x) = \int d^4x\, G^{(-)}(x - x')J^\mu_\gamma(x'), \tag{3.15}$$

where $G^{(-)} \equiv G_{\text{ret}} - G_{\text{adv}}$ is the difference between the retarded and advanced Green's function of the homogeneous Klein-Gordon wave equation (3.14),

$$G^{(-)}(x) = i\int \frac{d^4k}{(2\pi)^3} e^{-ikx}\,\text{sign}(k_0)\delta(k^2). \tag{3.16}$$

The radiated four-momentum by the field, P^μ_{rad}, is given by the integral over the electromagnetic energy-momentum tensor $T^{\mu\nu} = \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + F^{\mu\alpha}F_\lambda{}^\nu$ over a space-like surface of spacetime Σ,

$$P^\mu_{\text{rad}} = \int_\Sigma d^3\mathbf{x}\, T^{\mu 0}_{\text{rad}}(x), \tag{3.17}$$

which can be expressed, in momentum space, as the differential

$$dP^\mu_{\text{rad}} = -\frac{1}{2}\,\text{sign}(k_0)\delta(k^2)k^\mu(A^\lambda(k)A^*_\lambda(k))_{\text{rad}}\frac{d^4k}{(2\pi)^5}. \tag{3.18}$$

Taking into account the form of $G^{(-)}(k) = 2\pi i\,\text{sign}(k_0)\delta(k^2)$ in Fourier space, we can write the radiating field A^μ_{rad} as

$$A^\mu_{\text{rad}}(k) = 2\pi i\,\text{sign}(k_0)\delta(k^2)J^\mu_\gamma(k). \tag{3.19}$$

Integrating over dk^0 and interpreting the "classical photon number" by dividing the radiated energy by the photon energy $k^0 \equiv \hbar\,|\mathbf{k}|$ (recovering the $\hbar$ momentarily for clarity), we can write the number of emitted photons emitted in a phase space element $d^3\mathbf{k}$ as

$$dN^{\text{cl}}_{\text{photon}} = \frac{dP^0_{\text{rad}}}{k^0} = -\frac{1}{(2\pi)^3}J^\lambda(k)J^*_\lambda(k)\frac{d^3\mathbf{k}}{2k^0}, \tag{3.20}$$

where $J^\mu_\gamma(k)$ is the Fourier transform of the classical electron current

$$J^\mu_\gamma(k) = e\int d\tau\,\dot\gamma^\mu(\tau)\exp\left(ik\cdot\gamma(\tau)\right). \tag{3.21}$$

The current product $J(k)\cdot J^*(k)$ can be obtained from Equation (3.3) and the expression trajectory (3.9) for the trajectory γ. The final expression is given by

$$\eta_{\mu\nu}J^\mu_\gamma(k)J^\nu_\gamma(k)^* = e^2\left[\left(1 - \frac{E_\perp^2}{m^2}\right)|\mathcal{I}_0|^2 - \frac{E_\perp^2}{m^2}\left(|\mathcal{I}_1|^2 - |\mathcal{I}_2|^2\right)\right], \tag{3.22}$$

where $E_\perp^2 \equiv \mathbf{p}_\perp^2 + m^2$ is the charge's "transverse" energy on the plane perpendicular to $\mathbf{E}$, and the auxiliary functions $\mathcal{I}_i$ are given by

$$\begin{aligned}
\mathcal{I}_0 &= \frac{2m}{eE_z}e^{\pi\kappa_\perp/2}K_{i\kappa_\perp}(\kappa_\parallel),\\
\mathcal{I}_1 &= \frac{2m}{eE_z}e^{\pi\kappa_\perp/2}\left(\frac{i\kappa_\perp}{\kappa_\parallel}K_{i\kappa_\perp}(\kappa_\parallel)\sinh\chi - K'_{i\kappa_\perp}(\kappa_\parallel)\cosh\chi\right)\\
\mathcal{I}_2 &= \frac{2m}{eE_z}e^{\pi\kappa_\perp/2}\left(-\frac{i\kappa_\perp}{\kappa_\parallel}K_{i\kappa_\perp}(\kappa_\parallel)\cosh\chi + K'_{i\kappa_\perp}(\kappa_\parallel)\sinh\chi\right),
\end{aligned} \tag{3.23}$$

where $\kappa_\perp$ and $\kappa_\parallel$ are parameters related to the photon wave-vector k^μ and the electron's four-momentum p^μ by

$$\kappa_\parallel = \frac{|\mathbf{k}_\perp|\, E_\perp}{eE_z} \quad \text{and} \quad \kappa_\perp = \frac{\mathbf{p}_\perp \cdot \mathbf{k}_\perp}{eE_z}, \tag{3.24}$$

and χ is a parameter whose particular form is unimportant for the calculation of the electron's current product (3.22) [46].

The complete expression for the number of photons emitted by the accelerated charge on the phase space element $d^3\mathbf{k}$ is

$$dN^{\rm cl}_{\rm photon} = \frac{m^2 e^{\pi\kappa_\perp}}{4\pi^3 E_z^2}\left[\left(\frac{E_\perp^2}{m^2}\left(1-\frac{\kappa_\perp^2}{\kappa_\parallel^2}\right)-1\right)|K_{i\kappa_\perp}(\kappa_\parallel)|^2 + \frac{E_\perp^2}{m^2}|K'_{i\kappa_\perp}(\kappa_\parallel)|^2\right]\frac{d^3\mathbf{k}}{k^0}, \tag{3.25}$$

and for the particular case of a charged particle with vanishing transverse momentum $\mathbf{p}_\perp = 0$, we get the simplified expression

$$dN^{\rm cl}_{\rm photon} = \frac{m^2}{4\pi^3 E^2}|K_1(k_\perp/a)|^2\frac{d^3\mathbf{k}}{k^0}, \tag{3.26}$$

where we have used the modified Bessel function property $K_0'(z) = -K_1(z)$.

In Chapter 4, we discuss how we can understand the integration over longitudinal momenta as the proper-time integration over the classical electron's trajectory, $dk^z/k^0 = (eE_z/m)d\tau$. (In particular, for a scalar radiation field, this relation can be obtained directly from the integration of the Fourier-transformed electron current (3.21). See Appendix C of [33], shown in Chapter 4). Using this relation, we can finally write the expression for the photon emition amplitude per unit proper-time,

$$\frac{dN^{\rm cl}_{\rm photon}}{d\tau\, d^2\mathbf{k}_\perp} = \frac{e^2}{4\pi^3 a}|K_1(k_\perp/a)|^2. \tag{3.27}$$

3.3 Quantized electromagnetic radiation from an accelerated electron

In the last section, we calculated the "classical" photon emition rate by interpreting the total energy emitted by the electron onto the field in terms of the photon dispersion relation $E_\gamma = \hbar\,|\mathbf{k}|$. To proceed with the actual quantum mechanical calculation, we now proceed to quantize the electromagnetic field A_μ following the usual canonical procedure [45]. We keep working in the Lorenz gauge, where the equations of motion are given by Eq. (3.14). The solutions to Eq. (3.14) are still given by

$$A^\mu(x) = A_0^\mu + \int d^4x\, G(x-x')J^\mu_\gamma(x'), \tag{3.28}$$

where A_0^μ is a solution to the homogeneous version of the wave-equation (3.14), and G is some Green function of the same homogeneous equation depending on the choice of boundary conditions. In terms of the asymptotic solutions[2]

$$\begin{aligned} A^\mu_{\text{in}}(x) &= \lim_{t\to-\infty} A^\mu(x), \\ A^\mu_{\text{out}}(x) &= \lim_{t\to+\infty} A^\mu(x), \end{aligned} \tag{3.29}$$

where the retarded and advanced Green functions are given by

$$G_{\substack{\text{ret}\\ \text{adv}}}(x) = -\int \frac{d^4p}{(2\pi)^4} \frac{e^{-ipx}}{(p_0 \pm i0^+) - \mathbf{p}^2} = \frac{1}{2\pi}\theta(\pm x_0)\delta(x^2) \tag{3.30}$$

and the general solution of the free homogeneous wave equation can be written as both

$$\begin{aligned} A^\mu(x) &= A^\mu_{\text{in}} + \int d^4x\, G_{\text{ret}}(x-x')J^\mu_\gamma(x') \\ &= A^\mu_{\text{out}} + \int d^4x\, G_{\text{adv}}(x-x')J^\mu_\gamma(x'). \end{aligned} \tag{3.31}$$

The "in" and "out" solutions can be connected by the equation

$$\begin{aligned} A^\mu_{\text{out}}(x) &= A^\mu_{\text{in}}(x) + \int d^4x\, G^{(-)}(x-x')J^\mu_\gamma(x') \\ &= A^\mu_{\text{in}}(x) + A^\mu_{\text{rad}}(x), \end{aligned} \tag{3.32}$$

where A^μ_{rad} is simply the c-number quantity corresponding to the classical radiating field defined in Eq. (3.15). Here we can see that $G^{(-)}$ is the Green function that takes free-field solutions in the asymptotic past and evolves them into free-field solutions in the asymptotic future, "after" the source J_γ had its effect on the field.

The canonical quantization of the free fields $A^\mu_{\text{in/out}}$ proceed in the standard manner: associating to the negative and positive frequency modes of the homogeneous wave-equation with creation and annihilation operators $\hat{a}^\dagger_\lambda(\mathbf{k})_{\text{in/out}}, \hat{a}_\lambda(\mathbf{k})_{\text{in/out}}$,

$$A^\mu_{\text{in/out}}(x) = \int \frac{d^3\mathbf{k}}{2k_0(2\pi)^3} \sum_{\lambda=0}^{3} \left[\hat{a}_\lambda(\mathbf{k})_{\text{in/out}}\epsilon^\mu_\lambda(\mathbf{k})e^{-ikx} + \hat{a}^\dagger_\lambda(\mathbf{k})_{\text{in/out}}\epsilon^{\mu *}_\lambda(\mathbf{k})e^{ikx}\right], \tag{3.33}$$

acting on vacuum states $|0\rangle_{\text{in/out}}$ satisfying $a_\lambda(\mathbf{k})_{\text{in}}|0\rangle_{in} = a_\lambda(\mathbf{k})_{\text{out}}|0\rangle_{\text{out}} = 0$. The four-vectors $\epsilon^\mu_\lambda(\mathbf{k})$ are the polarization states of the photons, labeled by λ, and $k_0 \equiv |\mathbf{k}|$ is the dispersion relation for the massless photons. Of course, due to gauge-invariance, only two polarization states $\epsilon^\lambda_\mu(\mathbf{k})$ are physical: any other polarization states that are not transverse to the photon momentum $\mathbf{k}$ are cancelled in the calculations of physical observables.

From Eq. (3.32), we can read the relationship between the creation and annihilation operators in the asymptotic past and future,

$$\hat{a}_\lambda(\mathbf{k})_{\text{out}} = \hat{a}_\lambda(\mathbf{k})_{\text{in}} - i\eta_{\lambda\mu}J^\mu_\gamma(\mathbf{k}), \tag{3.34}$$

[2] For a more nuanced discussion on the subtleties on the interpretation of these asymptotic field solutions, see the discussion on [47].

and we can write the photon Hamiltonian in terms of both the "in" and "out" fields,

$$\begin{aligned} H(A_{\rm out}) &= \int d^3x \; :\pi_\mu \dot{A}^\mu_{\rm out} - \mathcal{L}(A_{\rm out}): = -\frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\eta^{\mu\nu} a^\dagger_\mu(\mathbf{k})_{\rm out} a_\nu(\mathbf{k})_{out} \\ &= H(A_{in}) + \frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\eta_{\mu\nu} J^\mu_\gamma(\mathbf{k}) J^\nu_\gamma(\mathbf{k}). \end{aligned} \tag{3.35}$$

The expression for the Hamiltonian shows that the average value of the energy emitted in the process of photon emission by the classical source J_γ is given by

$$\mathscr{E}_{\rm tot} = {}_{\rm in}\langle 0|H(A_{\rm out})|0\rangle_{\rm in} = \frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\eta_{\mu\nu} J^\mu_\gamma(\mathbf{k}) J^\nu_\gamma(\mathbf{k}), \tag{3.36}$$

which agrees with the classical calculation (3.20). Thus, by using the dispersion relation for the photon, we arrive that quantized photodynamics yields the same prediction for the rate of emitted photons as the classical calculation,

$$\frac{dN^{\rm quant}_{\rm photon}}{d\tau\, d^2\mathbf{k}_\perp} = \frac{e^2}{4\pi^3 a}|K_1(k_\perp/a)|^2. \tag{3.37}$$

Despite being two qualitatively distinct calculations, the equality between the classical radiation and the quantized photon emission, espressed in Eq. (3.37), can be expected a priori. One possible argument for this is mathematical, since the responses of the field to the classical point-like charge in both the classical and quantized radiation cases are given entirely in terms of the two-point correlations of the electromagnetic field evaluated along the uniformly accelerated trajectory, and thus can be expected to match. The more physical explanation for this equality comes from the fact that the quantized radiation calculation with a classical source is bounded to its domain-of-validity regime, where the momentum of the emitted photon is not high enough to probe the internal structure of the electron. This scale separation argument constrains the photon emission process to the momentum scale where classical radiation behavior is expected to dominate, and thus it is not a surprise that the rate of emitted photons match in both cases.

3.4 Quantum electrodynamics in a constant electrical field

When we consider higher accelerations, we are considering strong background electric fields that couple to the charged particle. Naturally, the previous point particle model we used to describe the charge can only be used up to a certain energy scale. The most well known of these scales is the the critical field $E_{\rm crit} = \dfrac{m^2c^3}{e\hbar}$, corresponding to the Schwinger limit for pair production by the electric field. When the background electric field gets close to that scale, the potential clearly begins to probe into the internal structure of the electron field and, at that point, we must describe the electron as a true quantum field. The action that describes such a system is given by the QED Lagrangian

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\gamma^\mu D_\mu - m)\psi, \tag{3.38}$$

where $D_\mu = \partial_\mu + iqA_\mu + ieA^{\text{bg}}_\mu$ is the gauge-covariant derivative that couples the fermionic field ψ with both the dynamical electromagnetic field A_μ and the background potential A^{bg}_μ.

To calculate the amplitude for photon emission by an electron, we need first to quantize the theory defined by the Lagrangian (3.38). To do this, we need to proceed like in Chapter 2, finding the positive and negative frequency modes of the fermion field ψ, but now over the constant electromagnetic background A^{bg}_μ [48, 49, 46, 50].

The equation of motion for the fermion field ψ given by

$$[-i\gamma^\mu\partial_\mu + q\gamma^\mu A_\mu(x) + m]\psi(x) = 0, \tag{3.39}$$

where the field is understood as a four-component spinor field, and $m \equiv m\mathbb{I}_{4\times4}$ is a four-by-four matrix. Making the ansatz

$$\psi(x) = [i\gamma^\mu\partial_\mu - q\gamma^\mu A_\mu(x) + m]\varphi(x), \tag{3.40}$$

we can substitute the Dirac equation 3.39 for a second-order equation for the bispinor field $\varphi(x)$:

$$\left[(i\partial_\mu - qA_\mu(x))^2 - m^2 - \frac{1}{2}qF_{\mu\nu}\sigma^{\mu\nu}\right]\varphi(x) = 0, \tag{3.41}$$

where we have defined[3] $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ and used the well-known identities of Dirac-matrix calculations

$$\not{\alpha}\cdot\not{\beta} = \frac{1}{2}\{\alpha_\mu, \beta_\nu\}\eta^{\mu\nu} - \frac{i}{2}[\alpha_\mu, \beta_\nu]\sigma^{\mu\nu}, \tag{3.42}$$

$$[i\partial_\mu - qA_\mu, i\partial_\nu - qA_\nu]\,\psi = -iqF_{\mu\nu}\psi, \tag{3.43}$$

with $\not{\alpha} \equiv \gamma^\mu\alpha_\mu$ the contraction of α with the Dirac gamma matrices γ^μ.

Choosing the chiral representation for the Dirac matrices

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}, \quad \sigma^\mu = (1, \sigma^i), \quad \bar{\sigma}^\mu = (1, -\sigma^i), \tag{3.44}$$

we find that the matrix structure of the interaction term $\frac{1}{2}qF_{\mu\nu}\sigma^{\mu\nu}$ for the bispinor field φ is given in block-diagonal form,

$$\frac{1}{2}qF_{\mu\nu}\sigma^{\mu\nu} = \frac{1}{2}qF_{\mu\nu}\,\text{diag}(s^{\mu\nu}, \bar{s}^{\mu\nu}), \tag{3.45}$$

where the 2×2 matrices $s^{\mu\nu}$, $\bar{s}^{\mu\nu}$ are combinations of products between the two-dimensional Pauli matrices σ and $\bar{\sigma}$. Choosing the temporal gauge $A_\mu(x) = (-E_z t)\delta_{\mu 3}$, $E_z > 0$, for the

[3] These are the generators of Lorentz transformations on spinors. Under this prefactor convention, the spinors ψ transform as

$$\psi(x) \to \exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\psi(\Lambda^{-1}x).$$

See for example [8, Equation (3.30)].

background potential corresponding to an electric field $\mathbf{E} = E_z\hat{\mathbf{z}}$, we get an electromagnetic field tensor

$$F_{\mu\nu} = -E_z(\delta_{\mu 0}\delta_{\nu 3} - \delta_{\mu 3}\delta_{\nu 0}), \tag{3.46}$$

and the interaction term takes the exact form

$$\frac{1}{2}qF_{\mu\nu}\sigma^{\mu\nu} = iqE_z \,\mathrm{diag}(\sigma^3, -\sigma^3) = iqE_z \,\mathrm{diag}(1, -1, -1, 1). \tag{3.47}$$

Now we proceed to find solutions to bispinor field (3.41) associated with the interaction term (3.47) of the form

$$\varphi = \mathcal{N}_\psi e^{i\mathbf{p}\cdot\mathbf{x}} f_{\mathbf{p}\sigma}(t) u_\sigma, \tag{3.48}$$

where u_σ are two eigenvectors of the interaction term $\frac{1}{2}qF_{\mu\nu}\sigma^{\mu\nu}$ with eigenvalue iqE_z:

$$\frac{1}{2}qF_{\mu\nu}\sigma^{\mu\nu}u_\sigma = (iqE_z)u_\sigma. \tag{3.49}$$

Specializing all the equations and relations obtained so far for the electron's case $q = -e$, where $e > 0$ is the magnitude of the electron charge, and applying the ansatz (3.48) to the second-order Dirac equation, we get the parabolic cylinder equation

$$\left[\frac{d^2}{d\xi^2} - \left(i\lambda + \frac{1}{2}\right) - \frac{\xi^2}{4}\right] f_{\mathbf{p}\sigma}(\xi) = 0, \tag{3.50}$$

$$\xi = e^{i\pi/4}\sqrt{\frac{2}{eE_z}}(p_z - eE_z t), \quad \lambda = \frac{p_\perp^2 + m^2}{2eE_z}. \tag{3.51}$$

Two sets of linearly independent solutions ${}_{\pm}f_{\mathbf{p}\sigma}(\xi)$ and ${}^{\pm}f_{\mathbf{p}\sigma}(\xi)$ can be found for this equation in terms of Weber's parabolic cylinder functions $D_\nu(z)$. These sets are given by [42, Eq. 9.255]

$${}_{+}f_{\mathbf{p}\sigma} = D_{i\lambda}(\xi^*), \quad {}_{-}f_{\mathbf{p}\sigma} = D_{-i\lambda-1}(\xi), \tag{3.52}$$

and

$${}^{+}f_{\mathbf{p}\sigma} = D_{-i\lambda-1}(-\xi), \quad {}^{-}f_{\mathbf{p}\sigma} = D_{i\lambda}(-\xi^*). \tag{3.53}$$

The two sets ${}_{\pm}f_{\mathbf{p}\sigma}(t)$ and ${}^{\pm}f_{\mathbf{p}\sigma}(t)$ differ in their behavior in the asymptotic past ($t \to -\infty$) and future ($t \to \infty$) regions. From the asymptotic expansion of the parabolic cylinder functions [42, Eq. 9.246],

$$D_\nu(z) \sim e^{-z^2/4}z^\nu\left(1 + \mathcal{O}\left(1/z^2\right)\right), \quad |\arg z| < \frac{3\pi}{4}, \tag{3.54}$$

we find that as $t \to \infty$, the complex phases on ${}^{\pm}f_{\mathbf{p}\sigma}(t)$ go as

$${}^{\pm}f_{\mathbf{p}\sigma}(t) \to \exp\left(\mp i\Theta(t)\right), \tag{3.55}$$

where $\Theta(t)$ is a positive, monotonically increasing function of t [49]. That relation singles out the positive and negative frequency solutions on the asymptotic future as ${}^+f_{\mathbf{p}\sigma}$ and ${}^-f_{\mathbf{p}\sigma}$ respectively, as suggested by the notation. Similar arguments hold for ${}_\pm f_{\mathbf{p}\sigma}$, which are the positive and negative frequency solutions in the asymptotic past.

Now we find the complete spinors which are solutions to the Dirac equation (3.39). To do this, we plug the solutions to the second-order equation in the ansatz (3.40):

$$ {}^\pm\psi_{\mathbf{p}\sigma}(x) = {}^\pm\mathcal{N}_{\mathbf{p}\sigma}[i\gamma^\mu\partial_\mu - q\gamma^\mu A_\mu(x) + m]\left({}^\pm f_{\mathbf{p}\sigma}(t)e^{i\mathbf{p}\cdot\mathbf{x}}u_\sigma\right), \tag{3.56} $$

$$ {}_\pm\psi_{\mathbf{p}\sigma}(x) = {}_\pm\mathcal{N}_{\mathbf{p}\sigma}[i\gamma^\mu\partial_\mu - q\gamma^\mu A_\mu(x) + m]\left({}_\pm f_{\mathbf{p}\sigma}(t)e^{i\mathbf{p}\cdot\mathbf{x}}u_\sigma\right). \tag{3.57} $$

The matrix form of the operator in the ansatz is given by

$$ i\gamma^\mu\Pi_\mu + m = \begin{bmatrix} m & 0 & \Pi_0+\Pi_3 & p_1 - ip_2 \\ 0 & m & p_1+ip_2 & \Pi_0-\Pi_3 \\ \Pi_0-\Pi_3 & -p_1+ip_2 & m & 0 \\ -p_1-ip_2 & \Pi_0+\Pi_3 & 0 & m \end{bmatrix}, \tag{3.58} $$

where we have defined, for convenience,

$$ \Pi_0 = i\frac{d}{dt}, \quad \text{and} \quad \Pi_3 = (p_3 - eE_z t). \tag{3.59} $$

The operators $\Pi_0 \pm \Pi_3$ can be written in terms of the parabolic cylinder's argument $\xi = e^{i\pi/4}\sqrt{2/eE_z}(p_3 - eE_z t)$, and they take the forms

$$ \Pi_0 \pm \Pi_3 = e^{-i\pi/4}\sqrt{2eE_z}\left(\frac{d}{d\chi} \pm \frac{\chi}{2}\right). \tag{3.60} $$

These operators act on the parabolic cylinder functions $D_\nu(z)$, which satisfy the relations [50, 42]

$$ \left(\frac{d}{dz} - \frac{z}{2}\right)D_\nu(z) = -D_{\nu+1}(z), \quad \left(\frac{d}{dz} + \frac{z}{2}\right)D_\nu(z) = \nu D_{\nu-1}(z), \tag{3.61} $$

and thus help us trade the time derivatives for other parabolic cylinder functions.

Now we are free to choose the two eigenvectors u_σ of the bispinor field in ansatz (3.48),

$$ u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad u_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \tag{3.62} $$

obtaining the two spin states for the incoming ${}_\pm\psi_{\mathbf{p}\sigma}$ and outgoing ${}^\pm\psi_{\mathbf{p}\sigma}$ electron/positron states. We show the spinors for the positive frequencies ${}_+\psi_{\mathbf{p}\sigma}$, ${}^+\psi_{\mathbf{p}\sigma}$ on the remote past and future, since the procedure is analogous to the negative frequencies[4]

$$ {}_{+}\psi_{\mathbf{p},1}(x) = (eE_z\lambda)^{-1/2} e^{-\frac{\pi\lambda}{4}+i\mathbf{p}\cdot\mathbf{x}} \begin{bmatrix} mD_{i\lambda}(\xi^*) \\ 0 \\ \lambda\, e^{-i\pi/4}\sqrt{2eE_z} D_{i\lambda-1}(\xi^*) \\ -(p_1+ip_2)D_{i\lambda}(\xi^*) \end{bmatrix}, \tag{3.63} $$

$$ {}_{+}\psi_{\mathbf{p},2}(x) = (eE_z\lambda)^{-1/2} e^{-\frac{\pi\lambda}{4}+i\mathbf{p}\cdot\mathbf{x}} \begin{bmatrix} (p_1-ip_2)D_{i\lambda}(\xi^*) \\ \lambda\, e^{-i\pi/4}\sqrt{2eE_z} D_{i\lambda-1}(\xi^*) \\ 0 \\ mD_{i\lambda}(\xi^*) \end{bmatrix}, \tag{3.64} $$

$$ {}^{+}\psi_{\mathbf{p},1}(x) = (2eE_z)^{-1/2} e^{-\frac{\pi\lambda}{4}+i\mathbf{p}\cdot\mathbf{x}} \begin{bmatrix} mD_{-i\lambda-1}(-\xi) \\ 0 \\ e^{-i\pi/4}\sqrt{2eE_z} D_{-i\lambda}(-\xi) \\ -(p_1+ip_2)D_{-i\lambda-1}(-\xi) \end{bmatrix}, \tag{3.65} $$

$$ {}^{+}\psi_{\mathbf{p},2}(x) = (2eE_z)^{-1/2} e^{-\frac{\pi\lambda}{4}+i\mathbf{p}\cdot\mathbf{x}} \begin{bmatrix} (p_1-ip_2)D_{-i\lambda-1}(-\xi) \\ e^{-i\pi/4}\sqrt{2eE_z} D_{-i\lambda}(-\xi) \\ 0 \\ mD_{-i\lambda-1}(-\xi) \end{bmatrix}, \tag{3.66} $$

where we introduce the normalization with respect to the usual inner product in the space of spinors,

$$ (\psi, \psi') = \int d^3\mathbf{x}\, \overline{\psi}(x)\gamma^0\psi'(x). \tag{3.67} $$

These solutions are the same found in [50], and equivalent to those found in [46] by means of a unitary transformation

$$ U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \tag{3.68} $$

on the spinors. The differences are in the chosen representations of Dirac gamma matrices γ^μ; we use the chiral representation, while [46] uses the Dirac representation.

From the linear relations between parabolic cylinder functions [42, Eq. 9.248],

$$ D_\nu(z) = e^{-i\nu\pi} D_\nu(-z) + \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{i\pi(\nu+1)/2} D_{-\nu-1}(iz), \tag{3.69} $$

we can establish a relationship between the positive and negative frequencies in the asymptotic past and asymptotic future,

$$ {}_{+}\psi_{\mathbf{p}\sigma} = i\sqrt{\frac{2\pi}{\lambda}} \frac{e^{-\pi\lambda/2}}{\Gamma(-i\lambda)} {}^{+}\psi_{\mathbf{p}\sigma} + e^{-\pi\lambda} {}^{-}\psi_{\mathbf{p}\sigma}, \tag{3.70} $$

[4] Here we use the convenient relationship $\xi^* = -i\xi$, valid for the variable ξ defined in (3.51).

$$ {}_{-}\psi_{\mathbf{p}\sigma} = -e^{-\pi\lambda}{}^{+}\psi_{\mathbf{p}\sigma} - i\sqrt{\frac{2\pi}{\lambda}}\frac{e^{-\pi\lambda/2}}{\Gamma(i\lambda)}{}^{-}\psi_{\mathbf{p}\sigma}. \tag{3.71} $$

From these relationships, we can read the Bogoliubov transformation coefficients that connect the modes:

$$ \begin{aligned} C_{\mathbf{p}}({}_{+}|{}^{+}) &= \overline{C_{\mathbf{p}}({}_{-}|{}^{-})} = i\sqrt{\frac{2\pi}{\lambda}}\frac{e^{-\pi\lambda/2}}{\Gamma(-i\lambda)}, \\ C_{\mathbf{p}}({}_{+}|{}^{-}) &= -C_{\mathbf{p}}({}_{-}|{}^{+}) = e^{-\pi\lambda}, \end{aligned} \tag{3.72} $$

where we've used the notation ${}_{\pm}\psi_{\mathbf{p}\sigma} = C_{\mathbf{p}}({}_{\pm}|{}^{+}){}^{+}\psi_{\mathbf{p}\sigma} + C_{\mathbf{p}}({}_{\pm}|{}^{-}){}^{-}\psi_{\mathbf{p}\sigma}$.

The Green's function $G(x', x)$, responsible for evolving the states to the asymptotic regions $t \to \pm\infty$, can be expressed in terms of the exact solutions ${}_{\pm}\psi_{\mathbf{p}\sigma}$ and ${}^{\pm}\psi_{\mathbf{p}\sigma}$:

$$ G(x', x) = \sum_{\mathbf{p},\sigma} N_{\mathbf{p}\sigma}\left[\theta(t'-t){}^{+}\psi_{\mathbf{p}\sigma}(x'){}_{+}\overline{\psi}_{\mathbf{p}\sigma}(x) - \theta(t-t'){}_{-}\psi_{\mathbf{p}\sigma}(x'){}^{-}\overline{\psi}_{\mathbf{p}\sigma}(x)\right], \tag{3.73} $$

where

$$ N_{\mathbf{p}\sigma} = iG_{\mathbf{p}}^{-1}({}_{-}|{}^{-}) = -\sqrt{\frac{\lambda}{2\pi}}\Gamma(i\lambda)e^{\pi\lambda/2}, \quad |N_{\mathbf{p}\sigma}|^2 = (1-e^{-2\pi\lambda})^{-1}. \tag{3.74} $$

From the exact solutions, we can calculate the amplitude for a photon to be emited by the scattering of an electron by the external field. In first-order perturbation in the radiative interaction,

$$ S_I = \int d^4x\, J^{\mu}_{\psi}(x)A_{\mu}(x) = -e\int d^4x\,\overline{\psi}(x)\gamma^{\mu}A_{\mu}(x)\psi(x), \tag{3.75} $$

the amplitude can be written as

$$ -i\mathcal{M}\left[e_{\mathbf{p}\sigma} \to e_{\mathbf{p}'\sigma'}\,\gamma_{\mathbf{k}\epsilon}\right] = -i\left\langle e_{\mathbf{p}'\sigma'}(\infty),\,\gamma_{\mathbf{k}\epsilon}(\infty)|S_{\mathrm{I}}|e_{\mathbf{p}\sigma}(-\infty)\right\rangle, \tag{3.76} $$

which is the amplitude for an incoming electron with momentum $\mathbf{p}$ and spin u_σ to scatter from the external electromagnetic potential, changing its momentum and spin to $\mathbf{p}'$ and $u_{\sigma'}$ and emitting a photon with wave vector $\mathbf{k}$ and polarization ϵ. In terms of the exact solutions for the fermion field in the background electric field, we get [49]

$$ -i\mathcal{M} = -\frac{ie}{\sqrt{(2\pi)^3 2k_0}}\int d^4x\,{}^{+}\overline{\Phi}_{\mathbf{p}'\sigma'}(x)\gamma^{\mu}\epsilon_{\mu}(\mathbf{k}){}_{+}\Phi_{\mathbf{p}\sigma}(x), \tag{3.77} $$

where the electron's wavefunctionals are given by

$$ \begin{aligned} {}^{+}\overline{\Phi}_{\mathbf{p}'\sigma'}(x) &= \int_{t_{\mathrm{out}}\to\infty} d^3\mathbf{x}_{\mathrm{out}}\,{}^{+}\overline{\psi}_{\mathbf{p}'\sigma'}(x_{\mathrm{out}})\gamma^0 G(x_{\mathrm{out}}, x) = N_{\mathbf{p}'\sigma'}{}_{+}\psi_{\mathbf{p}'\sigma'}(x), \\ {}_{+}\Phi_{\mathbf{p}\sigma}(x) &= \int_{t_{\mathrm{in}}\to-\infty} d^3\mathbf{x}_{\mathrm{in}}\,G(x, x_{\mathrm{in}})\gamma^0{}_{+}\overline{\psi}_{\mathbf{p}\sigma}(x_{\mathrm{in}}) = N_{\mathbf{p}\sigma}{}^{+}\psi_{\mathbf{p}\sigma}(x). \end{aligned} \tag{3.78} $$

The differential probability amplitude for a photon emission event from a one electron initial state (without pair creation), averaged over the initial spin states and insensitive to the final electron spin, is given by [46]

$$ dW = \frac{1}{2}\sum_{\sigma,\sigma',\epsilon}\int\frac{d^3\mathbf{k}\,d^3\mathbf{p}'}{(2\pi)^6}\left|\mathcal{M}\left[e_{\mathbf{p}\sigma} \to e_{\mathbf{p}'\sigma'}\,\gamma_{\mathbf{k}\epsilon}\right]\right|^2 = \mathcal{N}_W\,w(\mathbf{p},\mathbf{k})\frac{d^3k}{(2\pi)^3}, \tag{3.79} $$

where $\mathcal{N}_W$ is a normalization factor given by

$$\mathcal{N}_W = \frac{e^2 \exp\left(-\pi(\lambda+\lambda')/2\right)}{k_0 \lambda' (eE_z)^2 (1-e^{-2\pi\lambda})(1-e^{-2\pi\lambda'})}, \tag{3.80}$$

and $w(\mathbf{p},\mathbf{k})$ is the momentum distribution function given by

$$\begin{aligned} w(\mathbf{p},\mathbf{k}) = \pi e^{3\pi\nu/2} \Big\{ (\mathbf{p}_\perp^2 + \mathbf{p}_\perp'^2) \left|\Psi\right|^2 + [2(\mathbf{p}_\perp \cdot \mathbf{p}'_\perp + m^2) + eE_z \rho^2] \left(\frac{\rho^2}{2\lambda}\right) \left|\Psi'\right|^2 \\ -[\mathbf{p}_\perp^2 \rho^2 + 2\nu(\mathbf{p}_\perp \cdot \mathbf{p}'_\perp + m^2)] \left(\frac{1}{\lambda}\right) \Re(\Psi' \Psi^*) \Big\}, \end{aligned} \tag{3.81}$$

where $\rho^2 \equiv \mathbf{k}_\perp^2 / eE_z$, $\nu \equiv \lambda - \lambda'$. The function Ψ is defined by the combination of the confluent hypergeometric functions ${}_1F_1(a,b;z)$ [42, Eq. 9.210],

$$\Psi(a,b;z) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} {}_1F_1(a,b;z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} {}_1F_1(a-b+1, 2-b; z), \tag{3.82}$$

evaluated at

$$\Psi = \Psi\left(i\lambda, 1+i\nu; -i\rho^2/2\right). \tag{3.83}$$

The prime Ψ' refers to the derivative of the hypergeometric with respect to the last argument $\Psi'(a,b;z) = d\Psi/dz$. All expressions are evaluated over the momentum conserving relation

$$\mathbf{p} = \mathbf{p}' + \mathbf{k}. \tag{3.84}$$

Again, just as the classical electromagnetic radiation case, the differential probability of photon emission does not have an explicit dependence on the initial longitudinal momenta of the electron, p_z, and only depends trivially on the emitted photon longitudinal momentum k_z via the integration measure $(2k_0)^{-1} dk_z$. Such divergence is expected since the classical background field acts on the system for an infinitely long period of time, and is equivalent to the divergence we get from the classical radiation analysis. By using the same relationship between the longitudinal momentum and the proper-time of the classical electron's worldline,

$$\frac{dk_z}{k_0} = \frac{eE_z}{m} d\tau, \tag{3.85}$$

we can finally write the probability of emitting photons per unit proper-time and unit transverse momentum $\mathbf{k}_\perp$, given an initial electron transverse momentum of $\mathbf{p}_\perp = 0$:

$$\frac{dN_\gamma^{\mathrm{QED}}}{d\tau d^2\mathbf{k}_\perp} = \frac{dW}{d\tau d^2\mathbf{k}_\perp} = \frac{\alpha}{m} w(\mathbf{k}_\perp^2), \tag{3.86}$$

where the specialized function $w(\mathbf{k}_\perp^2) \equiv (2\pi)^{-3} \alpha^{-1} \mathcal{N}_W w(\mathbf{p},\mathbf{k})|_{\mathbf{p}_\perp=0}$ is given by

$$w(\mathbf{k}_\perp^2) = \left(\frac{1}{4\pi}\right) \operatorname{csch}\left(\frac{\pi m^2}{2eE_z}\right) f(E_\perp)\, e^{-\frac{\pi \mathbf{k}_\perp^2}{eE_z}} \left(\frac{\mathbf{k}_\perp^2}{\mathbf{k}_\perp^2 + m^2}\right) \left[\left|\Psi\right|^2 + \left(2 + \frac{\mathbf{k}_\perp^2}{m^2}\right) \left|\Psi'\right|^2 + 2\Re(\Psi'\Psi^*) \right], \tag{3.87}$$

where we defined for convenience

$$E_\perp \equiv \mathbf{k}_\perp^2 + m^2, \qquad f(E_\perp) = \frac{1}{1 - \exp\left(-\pi E_\perp / eE_z\right)}, \tag{3.88}$$

and the confluent hypergeometric functions are evaluated at

$$\Psi = \Psi\left(\frac{im^2}{2eE_z}, 1 - \frac{i\mathbf{k}_\perp^2}{2eE_z}; -\frac{i\mathbf{k}_\perp^2}{2eE_z}\right). \tag{3.89}$$

Chapter 4

Electron response to radiation under linear acceleration

In this chapter, we present our article discussing the electron's response to radiation due to the thermality of the vacuum as seen by uniformly accelerated observers. In it, we examine how the interaction between an accelerated electron with the quantum field vacuum can be understood as a stochastic "Brownian motion" type of problem. We derive observables related to the fluctuating and dissipative behavior of the electron under the action of the Unruh temperature, and quantitatively evaluate the predictions for these observables in the classical, semiclassical, and quantum regimes. We compare the results between various regimes, and interpret them in the light of possible experimental setups that may be capable of detecting signals of the vacuum's acceleration temperature.

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Electron response to radiation under linear acceleration: classical, QED and accelerated frame predictions

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A model detector undergoing constant, infinite-duration acceleration converges to an equilibrium state described by the Hawking-Unruh temperature $T_a = (a/2\pi)(\hbar/c)$. To relate this prediction to experimental observables, a point-like charged particle, such as an electron, is considered in place of the model detector. Instead of the detector's internal degree of freedom, the electron's low-momentum fluctuations in the plane transverse to the acceleration provide a degree of freedom and observables which are compatible with the symmetry and thermalize by interaction with the radiation field. General arguments in the accelerated frame suggest thermalization and a fluctuation-dissipation relation but leave underdetermined the magnitude of either the fluctuation or the dissipation. Lab frame analysis reproduces the radiation losses, described by the classical Lorentz-Abraham-Dirac equation, and reveals a classical stochastic force. We derive the fluctuation-dissipation relation between the radiation losses and stochastic force as well as equipartitation $\langle p_\perp^2 \rangle = 2mT_a$ from classical electrodynamics alone. The derivation uses only straightforward statistical definitions to obtain the dissipation and fluctuation dynamics. Since high accelerations are necessary for these dynamics to become important, we compare classical results for the relaxation and diffusion times to strong-field quantum electrodynamics results. We find that experimental realization will require development of more precise observables. Even wakefield accelerators, which offer the largest linear accelerations available in the lab, will require improvement over current technology as well as high statistics to distinguish an effect.

I. INTRODUCTION

The study of detectors in accelerated states was inspired by the quest to understand Hawking's prediction of thermal radiation from a black hole [1]. A detector undergoing constant acceleration exhibits a thermal excitation spectrum at temperature [2, 3]

$$T_a = \frac{a}{2\pi}\frac{\hbar}{c} = \frac{a}{\mathrm{m/s^2}} 3.5 \times 10^{-25}\ \mathrm{eV}. \tag{1}$$

In each case, the detector is coupled to a massless field which is quantized in the classical spacetime. The thermal spectrum is manifestly associated with the wavefunctions of the quantized field and can be factored out from the transition probability of the detector. For this reason, it is frequently said that the massless field viewed by the accelerated detector is in a thermal state [4–6], as appears to be the case for the massless field in a black hole spacetime [7, 8]. The apparent finding of thermalized behavior in hadronic collisions, including very small systems, has added a phenomenological dimension to these speculations, as the Unruh effect has been advocated as a mechanism under which a coherent classical field configuration dissipates into a thermal distribution in a time-scale parametrically shorter than perturbative expectations [9–12].

To understand the apparent thermal state better, we consider a concrete realization: a specific accelerated detector and a consequence of the detector's thermalization that is measurable in the laboratory inertial frame. Most proposals for experiments involve accelerating electrons [13–18], which as the lightest charged particles achieve the highest accelerations. The problem then is to derive the electron's response to the predicted thermal excitation as well as a dynamical observable measurable in the lab frame.

As a charged particle undergoing high acceleration, the electron radiates electromagnetically. The massless photon field should exhibit a thermal distribution in the rest frame of an electron in constant acceleration. Therefore the electron might reveal an imprint of this thermal bath in some characteristic of its radiation distribution. This is the basic idea behind two proposals for experiments, based either on the stochastic recoil of the probe particle from the radiation in the accelerated frame [15] or on correlations in 2-photon emission processes [17].

Nonequilibrium quantum theory methods were developed to analyze the real-time dynamics of a classical detector or particle coupled to a quantum field [19–23] and from the dynamics compute the radiation [24–26] laying to rest questions raised about whether any radiation survives in the far-field. These real-time calculations are also extended to nonconstant accelerations to test approximations and assumptions of the previous proposals [27]. Perhaps most interesting for experimental observation, the electron transverse momentum "thermalizes", i.e. after a sufficiently long time satisfies equipartition at the temperature Eq. (1) [28],

$$\frac{1}{2m}\langle p_\perp^i p_\perp^j\rangle = \frac{1}{2}T_a\delta^{ij} + \mathcal{O}\left(\frac{a}{m}\right)^2 \tag{2}$$

This equipartition relation is an element of a fluctuation-dissipation relation, apparently consistent with the hypothesis of coupling the transverse momentum fluctuations to a thermal bath at temperature T_a, as we discuss below.

Ultimately, progress on understanding the physics content of Eq. (1) must be compared to experiment. We show that equipartition for transverse dynamics arises in a consistent expansion for small accelerations ($a\hbar/c \ll mc^2$) and small transverse fluctuations ($|\vec{p}_\perp| \ll mc$) around the approximately constant longitudinal acceleration. The last condition is the experimental challenge: the acceleration should be approximately constant long enough for transient effects and initial state information to be erased, that is several times longer than the dissipation time. We obtain the dissipation time from classical radiation theory, finding agreement with previous calculations. Classical radiation theory also yields the correct noise, proving the fluctuation-dissipation and equipartition theorems. Since the classical radiation calculation involves only a single scale a, $\langle p_\perp^2\rangle$ proportional to T_a is inevitable. What is nontrivial is the correct numerical factor for the equipartiation relation. The $\hbar$ in Eq. (2) arises from the conversion of the classical wave number k of the radiation to the momentum it imparts to the electron/detector upon emission. On the other hand as acceleration approaches $a\hbar/mc^3 \to 1$, quantum electrodynamics, can be applied to determine the radiation emitted by the electron. We evaluate the dissipation time, noise and mean-square transverse momentum using strong-field quantum electrodynamics to quantify the high-acceleration departure from classical predictions of radiation response and the thermal fluctuation-dissipation and equipartition relations. Before closing, we discuss the timescales in the context of linear accelerator technology and find that both conventional radio-frequency accelerators and wakefield accelerators currently provide gradients that are too small and over too short times to access directly the "thermalized" state of an accelerating particle.

II. ACCELERATED FRAME ANALYSIS

Supposing horizons imply a thermodynamic description of the vacua of a massless field [29–32], we examine the implications for the dynamics of a probe coupled to such a massless field. More specifically, lab frame analysis of the two-point correlation function of the radiation field proves it equivalent to a thermal two-point correlator [4, 19]. Concretely of course, we are thinking of describing the dynamics of the electron coupled to the massless photon field in the accelerated, co-moving frame, but the inferences should be applicable more generally. We refer to the massless field as the radiation field, as in later sections, it is identical with the radiation component of the electromagnetic field.

The simplest consequence is that the expectation value of the energy of the probe degree of freedom should equilibrate at T_a,

$$\langle E \rangle = T_a. \tag{3}$$

This result is straightforwardly applied to models in which the probe degree of freedom is an "internal" state Q to which the radiation field couples, as in the Unruh-DeWitt detector [3]. In these models, the probability of excitation to an internal state with energy E is given by the usual Bose or Fermi statistics distribution with temperature T_a, which implies Eq. (3).

However, most experimental proposals using electrons and electromagnetic radiation involve phase space dynamics in response to the radiation field (electron spin is a notable exception [33]). Involving phase space dynamics poses a potential difficulty in that radiation dynamics generally change the acceleration. Lab frame analysis (Sec. III) shows that radiation causes the acceleration in a general state to decrease to a well-defined non-zero minimum. This dynamic will shortly be derived in the accelerated state as well. Clearly we must assume for the moment–and justify *a posteriori*–that the accelerated state can be treated as quasi-stationary, so that the decay is much slower than the dynamics we are considering and the acceleration and temperature can be considered approximately constant. Without the quasi-stationary approximation, applying a thermodynamic description would be nonsense.

Additionally, for the interaction of the probe (electron) with an accelerated frame radiation field to be described by classical thermodynamics, the temperature must be much less than the mass of the probe, $T_a \ll m$. Otherwise, the radiation field would have enough energy to probe the internal structure of the probe and create electron-positron pairs. This condition is equivalent to the lab frame condition that the probe particle must have negligible recoil from interactions with the radiation field and supports the *a posteriori* justification that the accelerated state is at least quasi-stationary.

To use the accelerated electron as the probe and its radiation as a signal accessible in the lab frame, we need a degree of freedom which interacts with the radiation field in such a way that the dynamics can be computed in both the lab frame and the accelerated frame. The simplest choice, if it exists, is an observable invariant under the change in frame. Since any point on the accelerated trajectory is related to the lab frame by a boost (and the accelerated trajectory itself is boost invariant), we are looking for an observable invariant under boosts along the direction of the acceleration. Such longitudinal boosts leave the transverse directions invariant, so observables describing dynamics in the transverse plane should be equal whether computed in lab or accelerated frame. Equality of observables has been verified explicitly for the probability of photon emission per unit transverse momentum by Refs. [14].

Therefore, we can investigate $(\vec{x}_\perp, \vec{p}_\perp)$ dynamics of the probe to seek effects of the thermal state of the radiation field. The first inference is that equipartition Eq. (3) should be applicable to the transverse kinetic energy. Since we are limited to the locally nonrelativistic regime $T_a \ll m$ (in the instantaneous co-moving frame the motion is non-relativistic for much longer than equilibration time defined below), we have $E_\perp \simeq p_\perp^2/2m$

$$\frac{1}{2m}\langle p_\perp^i p_\perp^j \rangle = \frac{1}{2} T_a \delta^{ij}, \tag{4}$$

The difference between this statement and Eq. (2) is that this has been obtained from general reasoning about the accelerated state, whereas Eq. (2) was obtained from a lab frame calculation [28]. The relativistic correction to the kinetic energy would imply a correction to the right hand side of $T^2/4m^2$, which we can compare to T/m corrections from other calculations.

A second inference is to recall that under these conditions the dynamics of a heavy probe coupled a thermal bath are described by Brownian motion. Specifically, according to Eq. (4) we have a heavy particle with momentum $p_\perp \sim \sqrt{mT_a}$ and velocity $v_\perp \sim \sqrt{T_a/m} \ll 1$.

Since $p_\perp \gg T$ and collisions with momentum transfer $\Delta p_\perp \sim T$ are rare, many collisions are required to significantly change the momentum. Therefore, we can model the interaction as dominated by dissipation and uncorrelated kicks. The dynamics are then described by a (macroscopic) Langevin equation, defined for the transverse momentum [34],

$$\frac{dp^i}{ds} = -\frac{1}{\tau_D}p^i + \xi^i, \qquad \langle \xi^i(s)\xi^j(s')\rangle = \kappa\delta(s-s')\delta^{ij}, \tag{5}$$

where τ_D is the dissipation (or relaxation) time and ξ^i is a classical random variable describing the stochastic force. The time variable s in the accelerated frame is the proper time of the accelerated probe. The dissipation time τ_D is the timescale for the exponential decay of correlations, including initial data. For a thermal bath, the dynamics of ξ^i are completely determined by its 2-point function, which being a δ-function in time represents white noise and has no higher order correlations. $N_d\kappa$ is the mean-square momentum transfer per unit time. The number of spatial dimensions $N_d = 2$ in our case but we keep it as an explicit factor to highlight how various thermodynamic relations are affected by the conversion from usual 3-dimensional dynamics to 2 dimensions.

The relationship between momentum loss and diffusion is described by a fluctuation-dissipation theorem, which follows from the general analysis of thermal equilibrium between the probe and the thermal bath [34]. Integrating Eq. (5) leads to the mean square momentum

$$\langle p_\perp^2\rangle \underset{t\gg\tau_D}{\longrightarrow} \frac{N_d}{2}\tau_D\kappa \tag{6}$$

Since equilibration in the long time limit $t \gg \tau_D$ requires Eq. (4), we obtain the fluctuation dissipation relation

$$2mT_a = \kappa\tau_D \tag{7}$$

which is independent of N_d. Since τ_D is the timescale to erase initial conditions, it is also the minimum (proper) duration of the quasi-constant period of acceleration required for these thermal dynamics to become dominant (see Ref [27] for calculations of equilibration times for nonconstant acceleration).

Integrating the momentum to obtain the mean square transverse displacement yields

$$\langle \Delta x_\perp(t)^2\rangle = 2N_d\frac{T_a}{m}\tau_D t \tag{8}$$

and comparison to the definition of the diffusion constant

$$\langle \Delta x_\perp^i(t)\Delta x_\perp^j(t)\rangle = 2Dt\delta^{ij} \tag{9}$$

shows that

$$D = \frac{\kappa}{2m^2}\tau_D^2 = \frac{T_a}{m}\tau_D. \tag{10}$$

The latter equality has the form of an Einstein relation $D \propto T$, modulo temperature dependence of τ_D, which we will find is essential.

Thus we have 3 characteristic quantities for the fluctuation and dissipation dynamics, and 2 relations determined by thermodynamics. We need to compute at least one of these from the microscopic theory describing collisions between the probe and the thermalized particles. Naively, it appears we could compute the mean square momentum transfer per unit time from a standard finite temperature field theory in the limit of a heavy scatterer (e.g. as in Ref. [35]), but as we discuss below, such calculations will appear in disagreement with the present results since they results in $\kappa \propto e^4$.

III. LAB FRAME ANALYSIS

From the lab frame, the electron is undergoing constant acceleration. Fluctuations in the transverse momentum converge to a steady state in which the mean square momentum is proportional the temperature T_a, as would be expected for thermalization [28]. Verifying this steady state would provide evidence for the thermal character of the interaction of the electron with the radiation field. In this section, we show this apparently thermal character is derived from classical electromagnetic theory. We compare the classical approach to the quantum dynamical formalisms of Refs. [22, 23, 28]. As the effect of the accelerated state thermalization is expected to become more important for high accelerations, we compute the same observables in quantum electrodynamics in order to obtain corrections proportional to $T/m \sim a/m$.

A. Classical electrodynamics

Classical electrodynamics predicts that any accelerating charged particle radiates, in general causing the particle to lose energy. We recall some of the basic equations here for comparison to the approaches below. The starting point, the classical action, is

$$S = -m\int \sqrt{u^\mu u_\mu}d\tau - \int \frac{1}{4}F^{\mu\nu}F_{\mu\nu}d^4x - \int j^\mu(x)A_\mu(x)d^4x \tag{11}$$

where the classical point-particle current is

$$j^\mu = -eu^\mu \delta^4 \left(x - \xi(\tau)\right) \tag{12}$$

with $u^\mu = p^\mu/m$ the electron 4-velocity and $\xi(\tau)$ its trajectory. Constant, linear acceleration is provided by a homogeneous and static electric field, and as usual we are implicitly splitting the electromagnetic field into a classical, external field, A_μ^{cl} which is not perturbed by the probe particle, and a dynamic radiation field A_μ^{rad} which is sourced by the particle dynamics. Integrating the Lorentz force equation for a general electron momentum, we find the 4-velocity u^μ and trajectory ξ^μ recalled in Appendix A, and the magnitude acceleration in a constant electric field is

$$a^\mu a_\mu = -\frac{|eE|^2}{m^2}\frac{p_\perp^2 + m^2}{m^2}, \tag{13}$$

which is equal to $|eE|/m$ only for $p_\perp = 0$. The minus sign is due to the 4-acceleration being spacelike. Any nonvanishing transverse momentum perturbs the acceleration from the naive value. However, even as $p_\perp^2$ acquires a nonvanishing expectation value due to radiation, its magnitude is consistent with the implicit expansion in $a/m \sim T_a/m$.

Computing the momentum flux of the A_μ^{rad} field through a sphere at infinity provides the rate of 4-momentum radiated by the electron [36, 37],

$$dP_{\text{rad}}^\mu = -\frac{1}{2}\text{sgn}(k_0)\delta(k^2)k^\mu j(k)\cdot j(k)^* \frac{d^4k}{(2\pi)^3}, \tag{14}$$

where $k^\mu = (\omega_k, \vec{k})$ with $|\vec{k}| = 2\pi/\lambda$ is the wave 4-vector of the radiation field. After inserting the classical trajectory and integrating, one finds the usual Larmor formula,

$$\frac{dP_{\text{rad}}^\mu}{d\tau} = -\frac{e^2}{6\pi}a^\nu a_\nu u^\mu = -\frac{dp_{\text{loss}}^\mu}{d\tau} \tag{15}$$

which is manifestly positive. The trajectory and other supporting calculations are found in Appendix. In this construction, this momentum loss is not incorporated in the solution of the trajectory entering the current. It is added to the Lorentz force equation to obtain a radiation-corrected equation of motion, known as the Lorentz-Abraham-Dirac (LAD) equation,

$$\frac{dp^\mu}{d\tau} = F_{\text{ext}}^\mu + \frac{e^2}{6\pi m}\left(p^\mu \left(\frac{du^\mu}{d\tau}\right)^2 + \frac{da^\mu}{d\tau}\right), \tag{16}$$

where $F^\mu_{\rm ext}$ is the driving force, here the Lorentz force $F^\mu_{\rm ext} \to qF^{\mu\nu}u_\nu$. The damping timescale due to radiation emission is derived from

$$\frac{1}{\tau_D} \simeq \frac{1}{E}\frac{dP^0_{\rm rad}}{d\tau} = \frac{e^2}{6\pi m}a^2 = \tau_e \frac{a^2}{c^2}, \tag{17}$$

restoring powers of c in the last equality. τ_e is the timescale arising with the LAD,

$$\tau_e = \frac{e^2}{6\pi\epsilon_0 mc^3} \simeq 6.24 \times 10^{-24}\,\text{s}, \tag{18}$$

related in turn to the Larmor radiation rate, but is not the timescale associated with the dissipation of the charged particle's energy. As expected, the dissipation time τ_D is inversely proportional to the acceleration and is classical.

Considering the acceleration exactly constant $da^\mu/d\tau = 0$ and ignoring the second term in parentheses in Eq. (16) leaves an equation of the Langevin form $\frac{dp^\mu}{d\tau} = F^\mu_{\rm ext} - p^\mu/\tau_D$. However, the second term is required in the equation of motion to conserve the norm of the 4-momentum $p^2 = m^2$, and therefore arises in any consistent derivation of dynamics from the classical electrodynamic action. Consequently the complete two-term LAD correction is obtained from a more rigorous linearization of the response of the particle to its radiation field [20, 22, 23] together with Eq. (17) [28].

Now by interpreting the classical results in terms of photon emission, we can compute higher order moments of the radiation, such as the mean square momentum transfer, for comparison to the accelerated frame. To start, the number of photons emitted is determined(estimated) from the radiated 4-momentum as

$$dN^{\rm cl}_\gamma = \frac{dP^0_{\rm rad}}{k^0} = -\frac{1}{2}\text{sgn}(k_0)\delta(k^2)j(k)\cdot j(k)^* \frac{d^4k}{(2\pi)^3}. \tag{19}$$

To determine how fluctuations in the radiation contribute to the electron dynamics, we need the mean square transverse momentum transfer from photon emission

$$N_d\kappa_{\rm cl} = \frac{d}{d\tau}\langle\Delta p_\perp^2\rangle = \int d^2k_\perp \frac{dN^{\rm cl}_\gamma}{d\tau d^2k_\perp}\Delta p_\perp^2. \tag{20}$$

where $\Delta p_\perp$ is the momentum transfer during the radiation process. Clearly the δ function in $dN^{\rm cl}_\gamma$ Eq. (19) reduces one of the k integrals but to obtain a rate per unit (proper) time $d\tau$, we must convert from the longitudinal momentum dk_z.

There are two ways to obtain the emission rate differential in time and transverse momentum. The first method is to calculate from first principles. The Fourier transformed current $j^\mu(k)$ in Eq. (19) involves an integral over t, but instead of evaluating each Fourier integral individually (as in Refs. [12, 14]) the current correlator $j(k) \cdot j(k)^*$ can be written in terms of average and relative electron rapidity y, related to proper time by $y = a\tau/c$. Due to the boost invariance of the source, emitted photon rapidity is determined only by the average rapidity. Integrating over photon rapidity therefore eliminates dependence on average rapidity, yielding the emission rate per unit transverse momentum per unit rapidity of the source. This procedure is described in detail in Appendix A.

The second method is perhaps more transparent and utilizes the same symmetry of the problem, but relies on a semiclassical estimate of the region of the t integration contributing for each photon wavenumber k. Due to the boost invariance of the source, the fully differential emission probability

$$dN_{\rm cl} = \frac{dP^0_{\rm rad}}{k_0} = \frac{e^2 m^2 e^{\pi\kappa_\perp}}{4\pi^3 (eE)^2}\left(\left(\frac{E_\perp^2}{m^2}\left(1 - \frac{\kappa_\perp^2}{\kappa_\parallel^2}\right) - 1\right) K_{i\kappa_\perp}(\kappa_\parallel)^2 + \frac{E_\perp^2}{m^2}\left(K'_{i\kappa_\perp}(\kappa_\parallel)\right)^2\right)\frac{d^3k}{2k_0}, \tag{21}$$

depends on the photon longitudinal wavenumber k_z only in the phase space factor $dk_z/2k_0$. Consequently, the k_z integral diverges logarithmically, as evidenced by the result for a finite interval,

$$\int_{-k_z^{\rm max}}^{k_z^{\rm max}} \frac{dk_3}{2k_0} = {\rm asinh}\frac{k_z^{\rm max}}{k_\perp}. \tag{22}$$

Now saddlepoint analysis of the Fourier integral of the current correlator $j(k) \cdot j(k)^*$ corroborates the reasoning in the previous paragraph: the dominant contribution to probability comes from a region of the source's trajectory determined by its average momentum, $\tau_{\rm s.p.} = (p_z + p'_z)/2eE$, with width $\delta\tau_{\rm s.p.} = |p_z - p'_z|/eE = |k_z|/eE$. It follows that the integrations over τ and k_z are equivalent as they are for spontaneous pair creation [38], with the interval of k_z covered corresponding (up to scaling) to the interval of τ covered,

$${\rm asinh}\frac{k_z^{\rm max}}{k_\perp} \simeq \ln\frac{2k_z^{\rm max}}{k_\perp} = \ln\frac{eEt}{m} + {\rm const.} \tag{23}$$

As the dependence is logarithmic, the differential relation is known only up to a constant scaling,

$$\frac{dk_z}{k_0} = C\frac{eE}{mc^2}d\tau. \tag{24}$$

No $\hbar$ appears since eE/m has units of acceleration. Comparison with the first-principles calculation (Appendix A) checks that the constant scaling factor is $C = 1$.

Applying the variable change Eq. (24), we obtain in the limit of zero electron transverse momentum

$$\frac{dN_\gamma^{(\mathrm{cl})}}{d^2k_\perp d\tau} = \frac{e^2}{4\pi^3\epsilon_0}\frac{1}{a}\left(K_1(k_\perp/a)\right)^2, \tag{25}$$

where $K_\nu(z)$ is a modified Bessel function of the second kind. No $\hbar$ appears in the classical emission probability $N_\gamma^{\mathrm{cl}}/d\tau d^2k_\perp$. By itself, the second moment of the transverse wave number, $\langle k_\perp^2\rangle = \int k_\perp^2(dN/dtd^2k_\perp)d^2k_\perp$, also remains a classical quantity. However to obtain the mean square momentum transfer to the electron per unit time, we must multiply the wave vector k by $\hbar$ to obtain the correct units, $\Delta p_\perp = \hbar k_\perp$. In fact, we need only one power of $\hbar$ since $kdN \propto dE$ Eq. (19). The modified Bessel function diverges like $K_\nu(z) \sim z^{-\nu}$ for small z, so the transverse wavenumber approaches the conformal limit at small $k_\perp$, like that of a free unaccelerated charge. The distribution Eq. (25) is exponentially suppressed at high $k_\perp$, with a temperature-like parameter proportional but not equal to T_a [12] (because a is the only scale in the classical radiation problem). The integal of the modified Bessel functions is analytic and yields a constant with the result

$$\begin{aligned}\kappa_{\mathrm{cl}} = \frac{d\langle\Delta p_\perp^2\rangle}{d\tau} &= \frac{1}{\hbar}\int d^2k_\perp(\hbar k_\perp)^2\frac{dN_\gamma^{\mathrm{cl}}}{d\tau d^2k_\perp}\\ &= \frac{e^2}{6\pi^2\epsilon_0}\frac{\hbar}{c^6}a^3.\end{aligned} \tag{26}$$

These properties of the emission probability support a picture of the radiation dynamics like that in the accelerated frame, even without the hypothesis of a thermal bath. Specifically, since collisions with small momentum transfer are frequent, causing dissipation known as radiation reaction Eq. (16), and collisions with momentum transfer $\Delta p_\perp \sim T$ are rare, many collisions are required to significantly change the momentum and we might model the interaction as dominated by dissipation and uncorrelated kicks. We could therefore hypothesize a generalized Langevin equation for the transverse dynamics, with the LAD radiation loss term replacing the dissipation term $-p^i/\tau_D$ in Eq. (5),

$$\frac{dp^i}{d\tau} = F^i_{\mathrm{ext}} + \frac{e^2}{6\pi m}\left(p^i\left(\frac{du^i}{d\tau}\right)^2 + \frac{da^i}{d\tau}\right) + \xi^i, \quad \langle\xi^i(\tau)\xi^j(\tau')\rangle = \kappa_{\mathrm{cl}}\delta(\tau-\tau')\delta^{ij}. \tag{27}$$

The stochastic force ξ^i has the same form as for the Langevin equation because the kicks are assumed to be uncorrelated. In principle, computing higher order correlation functions of the radiation, we should find higher order correlations in the noise, but these are suppressed by the coupling. Combining Eq. (26) with Eq. (17), we find

$$\kappa_{\rm cl}\tau_D = 2m\frac{\hbar a}{2\pi c} = 2mT_a, \tag{28}$$

and integrating Eq. (27) would lead to $\langle p_\perp^2\rangle = 2mT_a$ upon using Eq. (28). According to Eq. (10) the diffusion constant would be

$$D_{\rm cl} = \frac{\kappa_{\rm cl}\tau_D^2}{2m^2} = \frac{3\epsilon_0}{e^2 a}\hbar c^4, \tag{29}$$

with the $\hbar$ coming from κ. While the justification for Eq. (27) is a bit hand-waving at this point, we can derive it rigorously with guidance from a different but closely related approach to the electron-radiation interaction, namely considering $A_\mu^{\rm rad}$ as a quantized photon field.

B. Quantized photon dynamics

The original black hole and accelerated detector problems were formulated as the interaction of a classical object or detector with a quantized field, and therefore it has been natural for most authors to study the dynamics of the quantized radiation field, which is easily compared between frames. However for the massless and uncharged photon field, it turns out that calculations of the radiation distribution with a quantized radiation field from a classical point source are equivalent to calculations within classical radiation theory [39].

The equivalence is highlighted by computing the probability of photon emission. The action is the same as the classical action Eq. (11), modulo a gauge fixing term which we do not need for the tree-level calculations here. Only the photon is quantized. The probability of photon emission differential in photon wave number is related to the squared matrix element for photon emission,

$$dW = \sum_{\epsilon,\epsilon'} |\mathcal{M}|^2 \frac{d^3k}{(2\pi)^3 2|\vec{k}|} \tag{30}$$

$$\mathcal{M} = \int d^4x \langle \vec{k},\vec{\epsilon}| j(x)\cdot \hat{A}(x)|0\rangle = \int d^4x (j(x)\cdot\epsilon) e^{-i|\vec{k}|t+\vec{k}\cdot\vec{x}}. \tag{31}$$

The current is classical, so the matrix element is straightforwardly evaluated in terms of plane waves and the polarization vector ϵ^μ of the photon field, which satisfies $k\cdot\epsilon = 0$. Rewriting the photon wavenumber phase space using a $\delta(k^2)$, we have

$$dW = \sum_{\epsilon,\epsilon'} \int d^4x \int d^4x' (\epsilon\cdot j(x))(\epsilon'\cdot j(x')) e^{-ik(x-x')}] \frac{1}{2}{\rm sgn}(k_0)\delta(k^2)\frac{d^4k}{(2\pi)^3}. \tag{32}$$

Then using the usual polarization sum identity $\sum_{\epsilon,\epsilon'} \epsilon_\mu \epsilon'_\nu = -g_{\mu\nu}$ and the definition of the Fourier transform, we are back to the classically obtained expression Eq. (19).

Neither classical radiation theory nor the quantized radiation field have the power to compute all observables. While Eq. (19) or Eq. (30) can be used to compute the spectrum and moments of the photon distribution, they cannot compute the radiation intensity, which relies on considering the emission as a continuous process and the radiation as a continuous field. Extensions of the quantized photon approach using nonequilibrium quantum theory methods enable investigation of the system-environment separation and the conditions and dynamics of decoherence. Such more powerful methods are necessary to determine more quantitatively when the intuitive picture of dynamics obtained here is valid.

Sacrificing some rigor for clarity, we can simplify the calculation of the feedback of the radiation on the classical source to obtain a generalized Langevin equation of the form Eq. (27). The leading order equation of motion for the current is the Lorentz force,

$$\frac{dp^\mu}{d\tau} = qF^{\mu\nu}u_\nu, \tag{33}$$

which if we separate $F^{\mu\nu}$ into an external field and the photon field, $F^{\mu\nu} = F^{\mu\nu}_{\text{ext}} + \hat{F}^{\mu\nu}$, can be rewritten

$$\frac{dp^\mu}{d\tau} = F^\mu_{\text{ext}} + q\hat{F}^{\mu\nu}u_\nu, \qquad F^\mu_{\text{ext}} \equiv qF^{\mu\nu}_{\text{ext}}u_\nu. \tag{34}$$

The external field generates the leading order classical trajectory, around which we will perturb. From the action, we construct an iterative solution for the photon field $\hat{A}^\mu$. With the Lorenz gauge condition

$$\partial_\mu \hat{A}^\mu = 0, \tag{35}$$

the equation of motion for A^μ is the Maxwell equation,

$$j^\nu = \partial_\mu F^{\mu\nu} = \partial^2 A^\nu \tag{36}$$

with j^ν the classical current Eq. (12).

The general solution to Eq. (36) is $A^\mu(x) = A^\mu_{\text{h}}(x) + A^\mu_{\text{inh}}(x)$, the sum of a homogeneous solution A^μ_{h}, which brings in the vacuum (free-field) dynamics of the photon, and an inhomogeneous solution A^μ_{inh}, which brings in the excitation of the photon field by the classical source current. Assuming the initial state of the radiation field is gaussian, consistent with a free field state uncoupled to the charge, the homogeneous solution contributes a stochastic field with a nominally classical probability distribution, whereas the inhomogeneous solution contributes the history-dependent dissipation [20, 40]. The reason for this separation is analyticity: the propagator for the radiation field can be separated into real and imaginary parts, which under causal construction devolve respectively to the Hadamard and retarded propagators.

Formally, we obtain the same result by inserting the homogeneous solution and inhomogeneous solution into Eq. (34) [28]. The homogeneous solution, solving $\partial^2 A = 0$, is a complete set of plane waves,

$$A^\mu_{\mathrm{h}}(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2k^0}}\left(\epsilon^\mu_k a_k e^{-ik^\nu x_\nu} + \epsilon^{*\mu}_k a^\dagger_k e^{ik^\nu x_\nu}\right), \tag{37}$$

satisfying the usual on-shell condition $k^0 = |\vec{k}|$. The polarization vectors satisfy $k \cdot \epsilon_k = 0$ and the mode functions $a_k, a^\dagger_k$ are classical amplitudes. The inhomogeneous solution is constructed from the retarded Green's function,

$$A^\mu_{\mathrm{inh}}(x) = \int d^4x' G_R(x,x') j^\mu(x') \tag{38}$$

where the Green's function satisfies

$$\partial^2_x G_R(x,x') = \delta^4(x-x') \tag{39}$$

and. With this Ansatz for $A^\mu(x)$, using the δ functions in Eq. (12) to reduce the x' integral and regularizing the singular contributions from the $\tau' \to \tau$ limit [22, 23], we obtain

$$\frac{dp^\mu}{d\tau} = F^\mu_{\mathrm{ext}} + q(\partial^\mu \hat{A}^\nu_{\mathrm{h}} - \partial^\nu \hat{A}^\mu_{\mathrm{h}})u_\nu + \frac{e^2}{6\pi m}\left(p^\mu\left(\frac{du^\mu}{d\tau}\right)^2 + \frac{da^\mu}{d\tau}\right). \tag{40}$$

Like the Langevin equation, this equation describes the dynamics of an observable; physical quantities are expectation values of the observable and its moments. The expectation value defines the contribution of the stochastic field $\hat{A}_{\mathrm{h}}$, which has the properties of a noise field $\langle \hat{A}_{\mathrm{h}}(x)\rangle = 0$ and must be symmetrized before evaluating the two-point function $\langle \hat{A}_{\mathrm{h}}(x)\hat{A}_{\mathrm{h}}(y)\rangle \to \frac{1}{2}\langle\{\hat{A}_{\mathrm{h}}(x), \hat{A}_{\mathrm{h}}(y)\}\rangle$ corresponding to the Hadamard propagator arising in the more rigorous derivation.

To investigate small transverse fluctuations, we linearize around the zeroth order solution, $p^\mu = p^\mu_{(0)} + \delta p^\mu$, that satisfies the external force $dp^\mu_{(0)}/d\tau - F^\mu_{\mathrm{ext}} = 0$. In agreement with the classical estimate, the solution to the stochastic equation of motion for transverse motion shows the damping time scale for transverse dynamics to be $\tau_D = c^2/a^2\tau_e$ identical to Eq. (17). Further, it is verified by explicit calculation that mean square momentum converges after long times $\tau \gg \tau_D$ to (Eq. 5.15 of Ref. [28])

$$\frac{1}{2m}\langle \delta p^i \delta p^j \rangle = \frac{1}{2} T_U \delta^{ij} \left(1 + \mathcal{O}\left(\frac{a^2}{m^2}\right)\right). \tag{41}$$

By analysis similar to the Langevin dynamics, we obtain the diffusion constant from the long time dynamics of the mean square transverse displacement. The result is

$$D = \frac{3}{e^2 a} \tag{42}$$

in agreement with Eq. (29) [41]. The mean square momentum transfer κ is not explicitly defined as such in this approach, but it can be read off from from the calculation of the field correlator (Eq. 3.11 of [28]) and multiplying by factors of e^2 (for the coupling) and 2 (for the 2 polarizations of the photon)

$$\kappa = \frac{e^2 a^3}{6\pi^2} \tag{43}$$

in agreement with Eq. (26).

Although this approach yields the same observable results as classical radiation theory, it provides a more rigorous basis for introducing the Langevin dynamics and understanding its origin in neglecting higher order correlations in the radiation field.

C. Quantum electrodynamics

To obtain corrections at high acceleration $a/m \to 1$ we must start from a theory that accounts for recoil from photon emission. The electron must be quantized in order to conserve momentum at each emission. As the constant electric field generates dynamics identical to uniform acceleration, we quantize the electron in the classical gauge potential $A^\mu_{\mathrm{cl}} = -eEt\delta^\mu_3$ corresponding to a homogeneous and static electric field in the $\hat{z}$ direction. The time-dependent gauge is chosen for this time-dependent problem. The hard work of constructing wavefunctions and simplifying the matrix element has been done [37] and salient aspects of the calculation reviewed in Appendix B. The fully differential probability, at

$p_\perp = 0$, is

$$dW = \frac{d^3k}{(2\pi)^3 2k_0}\frac{1}{2}\sum_{\substack{\sigma,\sigma'\\ \epsilon,\epsilon'}}\int\frac{d^3p'}{(2\pi)^3 2E_{p'}}\left|\mathcal{M}[e_{\vec{p}}\to e_{\vec{p}'}\gamma_{\vec{k}}]\right|^2$$

$$\equiv \frac{d^3k}{k_0}\frac{1}{|eE|}w(k_\perp^2,|eE|) \tag{44}$$

$$w(k_\perp^2,|eE|) = \frac{\alpha}{2\pi}\frac{e^{-\pi\frac{k_\perp^2}{eE}}}{(1-e^{-\pi\frac{k_\perp^2+m^2}{eE}})2\sinh(\frac{\pi m^2}{eE})}\frac{k_\perp^2}{k_\perp^2+m^2}\left[\left(2+\frac{k_\perp^2}{m^2}\right)|\Psi'|^2+|\Psi|^2+2\mathrm{Re}[\Psi'\Psi^*]\right]$$

$$\Psi = \Psi\left(\frac{im^2}{2eE}, 1-\frac{ik_\perp^2}{2eE};\frac{-ik_\perp^2}{2eE}\right) \tag{45}$$

where $\Psi(a,b;z)$ is the second confluent hypergeometric (see Eq. (B15)) and the prime denotes differentiation with respect to the argument z, $\Psi'(a,b;z) = d\Psi/dz$. For notational brevity, we have suppressed the $\hbar$s in this expression. From this, we need to compute two quantities for comparison, the dissipation time τ_D and the mean-squared momentum transfer per unit time κ.

The first, τ_D encounters the difficulty pointed out in the previous section: in quantized radiation dynamics, we do not have a definition of continuous momentum flux in the radiation field, since it is composed of the probabilities of finding quanta in a given mode. To obtain a definition of the energy loss rate, we extend the semiclassical analysis of Sec. III A. The discussion above Eq. (24) showed that the probability of emission in a given k_z mode is dominated by a saddle-point on the electron's trajectory determined by the electron's momentum. Therefore we can say that the energy lost over a given finite interval is given by integrating over the corresponding k_z (and all $k_\perp$) and dividing by the duration of the interval [37],

$$\frac{\Delta E}{\Delta t} = \int d^2k_\perp \frac{1}{\Delta t_{\text{s.p.}}}\int_{-k_z^{\max}}^{k_z^{\max}} dk_z \frac{1}{|eE|}w(k_\perp^2,|eE|) \tag{46}$$

$$\Delta t_{\text{s.p.}} = \frac{m}{eE}\frac{2k_z^{\max}}{k_\perp}.$$

Since this is an estimate expected to be valid to within a constant of order unity, we introduce a constant in the time interval $\Delta t \to C_t\Delta t$ with which we match to the classical result. Since $k^0 dN_\gamma$ is independent of k_z, the integral yields $2k_z^{\max}$, which cancels with the same factor in $\Delta t_{\text{s.p.}}$. The result is

$$\left.\frac{\Delta E}{\Delta t}\right|_{\text{QED}} = \frac{1}{C_t m}\int d^2k_\perp k_\perp w(k_\perp^2,|eE|). \tag{47}$$

To determine the constant C_t, we take the classical limit $\hbar \to 0$. The limit is clarified by writing all parameters in terms of the dimensionless quantities $k_\perp \ell_a$ and λ_e/ℓ_a where $\ell_a = c^2/a = m_e c^2/eE$ is the length scale associated with the classical acceleration and $\lambda_e = \hbar/m_e c$. Thus the $\hbar \to 0$ limit is manifestly the limit of a point-like electron, i.e. the Compton wavelength vanishes relative to the acceleration length scale, $\lambda_e/\ell_a = (\hbar/m_e c)/(m_e c^2/eE) \to 0$. As expected from the Euler-Heisenberg effective action, quantum effects become important as $a/m \sim 1$ [42], which is equivalent to the electric field approaching the "critical field" $eE \sim m_e^2 c^3/\hbar$. Using Eq 8.14 of Ref. [37], the limit is

$$\lim_{\hbar\to 0} \frac{\Delta E}{\Delta t}\bigg|_{\text{QED}} = \frac{e^2}{2\pi^2}\frac{1}{C_t a}\int_0^\infty \left(K_1(k_\perp/a)\right)^2 k_\perp^2 dk_\perp = \frac{9\pi}{32 C_t}\frac{e^2}{6\pi}a^2, \tag{48}$$

which fixes $C_t = 9\pi/32$. The relaxation time is then defined paralleling the classical estimate Eq. (17),

$$\tau_{Dq}^{-1} = \frac{1}{E}\frac{\Delta E}{\Delta t}\bigg|_{\text{QED}}, \tag{49}$$

which we evaluate numerically below.

Second, to evaluate the mean-square transverse momentum transfer, we need $dN/d\tau d^2k_\perp$. The derivation proceeds in parallel to the previous. We use the change of variables described in the classical case Eq. (24). We keep the scaling constant C, this time determining its value by taking the classical limit with the result that $C = 1$ (again). Thus we obtain

$$\frac{dW}{dt d^2k_\perp} = \frac{1}{m} w(k_\perp^2, |eE|) \tag{50}$$

Then the mean-square transverse momentum transfer is simply

$$2\kappa_{\text{q}} = \int \frac{dW}{dt d^2k_\perp} k_\perp^2 d^2k_\perp. \tag{51}$$

The classical limit commutes with the small $k_\perp$ limit, which could also be used to determine the scaling constant. In the small $k_\perp$ region, $k_\perp^2 \ll m^2, eE$, we find that QED predicts greater emission probability,

$$\frac{dW/dt d^2k_\perp}{dN_\gamma^{\text{cl}}/dt d^2k_\perp} = \frac{1}{1 - e^{-\pi m^2 c^3/eE\hbar}}\left(1 + ...\right), \tag{52}$$

which is a quantum effect (disappearing with $\hbar \to 0$) and only becomes significant for $eE\hbar/m^2c^3 = a/m \sim 1$. Similar to the Bose factor in the accelerated detector calculations

[3], it arises from the normalization of the wavefunctions which in turn is related to the hyperbolic functions in the classical particle action as recognized in analysis of spontaneous pair production [43, 44].

As we shall see in numerical evaluations of the differential emission rate, the phenomenology of photon emission does not change qualitatively with inclusion of electron recoil in QED. As $a \sim m$, the rate of small $k_\perp$ emission is slightly enhanced Eq. (52). For this reason–and ignoring the novel phenomena at strong fields $E \simeq m_e^2 c^3/e\hbar$ especially pair creation–we argue that a Langevin equation should continue to model the electron-radiation dynamics. We define the diffusion constant from the Langevin relation,

$$D_{\mathrm{q}} = \frac{\kappa_{\mathrm{q}} \tau_{Dq}^2}{2m^2}. \tag{53}$$

IV. COMPARISON OF RESULTS AND DISCUSSION

We now make quantitative comparisons of the observables computed in the previous section. To establish intuition for the diffusion-related observables, we start with the photon emission rate differential in transverse momentum. As shown in Figure 1, the small $k_\perp$ behaviour is the same $dN/dtd^2k \perp\sim k_\perp^{-2}$ for classical and QED calculations, with the normalization of the QED result enhanced by the Bose-like factor Eq. (52) visible for larger acceleration $a/m > 1$. However for high $k_\perp \gtrsim 1/\ell_a$, QED predicts a significantly lower emission probability especially for $a/m \gtrsim 0.1$.

In classical calculations, the acceleration is the only variable scale and quantities such as the rate of energy loss and transverse momentum transfer should vanish as $a \to 0$. The only other scale that can be involved is the LAD time scale τ_e Eq. (18). Considering first the damping time τ_D in Figure 2, we find that QED predicts an enhancement from the classical result for $a/m < 40$ and a suppression for $a/m \gtrsim 40$. Since the differential emission rate Eq. (44) is isotropic in transverse wavenumber, $d^2k_\perp = 2\pi k_\perp dk_\perp$, and the resulting $k_\perp^2$ weight in the integrand cancels the $1/k_\perp^2$ divergence of the emission rate at small $k_\perp$. This increases the importance of larger $k_\perp$ to the integral, where the QED differential probability is smaller, thus decreasing the energy loss rate. The keen reader may notice small variations in the calculated value of τ_D around $a/m \simeq 0.1$ and later in κ and derived quantities; these are numerical artifacts that seem to arise from challenges in finding a sufficiently accurate representation of the confluent hypergeometric functions in the differential QED emission

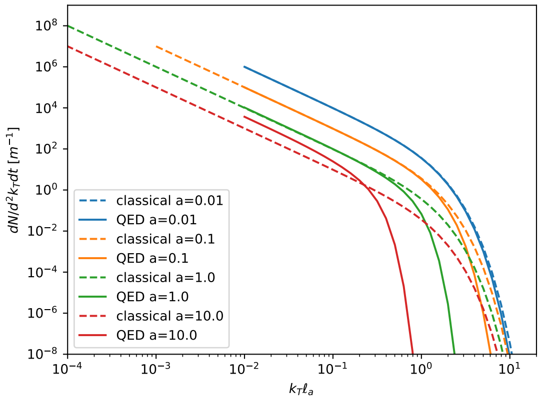

FIG. 1. The rate of photon emission per unit transverse momentum. The wavenumber is normalized to the acceleration length scale $\ell_a = c^2/a = m/eE$, with curves comparing different magnitude of acceleration, normalized to m.

rate.

In dimensionful units, the damping time is of order 1 femtosecond for an acceleration $a/m \simeq 0.01$ corresponding to an electric field $|E| \simeq 10^{16}$ V/m. As observed in Ref. [28], this is the timescale and therefore the electric field strength that would be required if thermalization were desired within a single cycle of a laser pulse, as proposed by Ref. [15]. However, more recent calculations for oscillating trajectories show that a model detector does not converge to equilibrium at the temperature T_a [27]. Laser wakefield acceleration utilizes (co-moving) quasi-stationary longitudinal electric fields, which persist over ∼10 cm of propagation or 0.3 ns. If we require thermalization within half of that acceleration time (150 picoseconds), the electric field should be $|E| \simeq 2.4 \times 10^{13}$ V/m. The longitudinal fields generated during laser wakefield acceleration $\sim 10^{11}$ V/m remain orders of magnitude lower. Conversely, for $|\vec{E}| \simeq 10^{11}$ V/m, the acceleration would have to persist for ~ 10 microseconds to exceed the dissipation time, corresponding to an acceleration length of 3 km. Conventional radio-frequency accelerators that are actually 3 km long fare worse, with maximum accelerating gradients of $\sim 10^8$ V/m, which due to the a^{-2} scaling of τ_D would require an acceleration time of 10 seconds or length of 3×10^6 km. This estimate obviously assumes that focusing elements interspersed between ∼ 1-2 m accelerator chambers do not interfere with considering the acceleration approximately constant, and every accelerator chamber provides the same accelerating gradient.

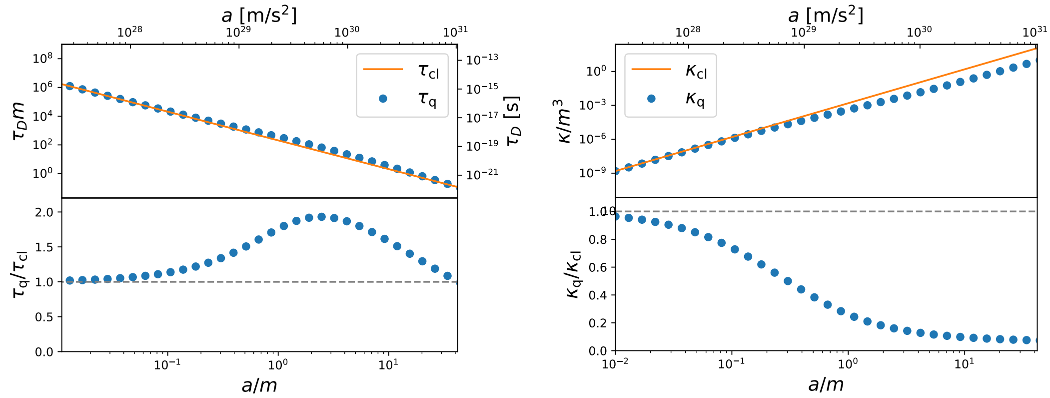

FIG. 2. Left: The dissipation time τ_D as a function of acceleration, classical radiation Eq. (17) and QED Eq. (49) predictions. Right: The mean-square momentum transfer to the electron obtained from classical Eq. (26) and QED Eq. (51).

In the classical limit, the mean-square momentum transfer per unit time is a function of only a. In the comparison to QED, the $k_\perp^3$ weight in the integrand ensures that the high-$k_\perp$ region is still more important in determining the integral and the QED result $\kappa_{\rm q}$ is less than the classical result $\kappa_{\rm cl}$ for all values of a.

Aside from the dissipation time setting the scale for the required duration of the acceleration, the diffusion constant is next most important step toward a measurement. For a heavy particle in a thermal bath, the diffusion constant describes the linear growth of the mean square displacement in time. In the present dynamics, it describes the linear growth of the transverse size of a hypothetical electron beam being accelerated. However in accelerator physics the mean square displacement alone is typically not measured, and the calculation here should be consider a stepping-stone to more specialized observables.

The diffusion constant is a combination of τ_D and κ, and since $\tau_D \propto a^{-2}$ and $\kappa \propto a^3$ the diffusion constant $D \sim a^{-1} = T^{-1}$. This inverse proportionality contrasts with diffusion associated with nonrelativistic Brownian motion but is typical for diffusion in massless gauge theories. An intuitive reason for this inverse proportionality is that, as massless particles, the number density of photons increases with temperature. Therefore the density of scatterers rises with temperature and increases the rate of soft, largely dissipative scattering events. This picture is consistent with the finding that QED further enhances the emission rate at small $k_\perp$ and results in a smaller diffusion constant, shown in Figure 3.

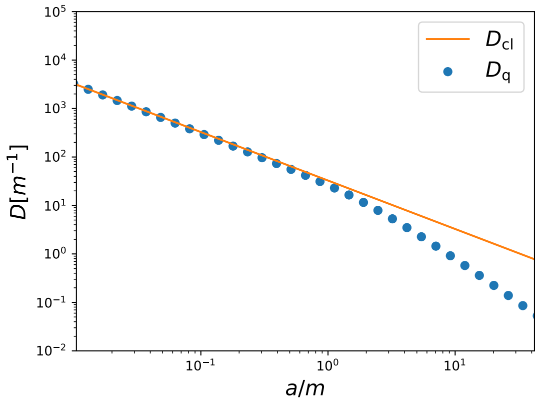

FIG. 3. Diffusion constant derived from classical Eq. (29) and QED Eq. (53) radiation dynamics.

However electron diffusion in a low temperature ($T \ll m_e$) QED plasma or heavy quark diffusion in a QCD plasma ($\Lambda_{QCD} \ll T \ll m_Q$) differ from the results for constant acceleration in their manifest dependence on the coupling constant e^2. Statistical definitions of the dissipation time and mean-square momentum transfer involve squared matrix elements (as they did implicitly in Sec. III B Sec. III C), schematically [35, 45, 46]

$$\frac{1}{\tau_D} = \frac{1}{|\vec{v}|}\frac{dE}{dt} = \int [dk][dk'][dp'](p_0' - p_0)|\mathcal{M}|^2 n_b(\vec{k}_\perp)\big(1 + n_b(\vec{k}'_\perp)\big) \tag{54}$$

$$N_d\kappa = \frac{1}{2m}\int [dk][dk'][dp'](\vec{p}'_\perp - \vec{p}_\perp)^2|\mathcal{M}|^2 n_b(\vec{k}_\perp)\big(1 + n_b(\vec{k}'_\perp)\big) \tag{55}$$

where the phase space integrals $[dk] \equiv d^3k/(2\pi)^3$ come also with momentum conserving δ functions. The matrix elements are $2 \to 2$ scattering amplitudes, e.g. linear Compton scattering for an electron in a QED plasma. The phase space integrals therefore involve an incoming photon momentum k and outgoing photon momentum k', each matrix element is proportional to e^2, and the observables τ_D^{-1}, κ are proportional to α^2. In Eq. (29), one power of e is hidden in the acceleration, $D \propto (e^2 a)^{-1} \sim (e^3 E)^{-1}$, and one might argue that the missing power of e would be restored on considering the source of the $\vec{E}$ field from Maxwell's equation $\partial_\mu F^{\mu\nu} = j^\nu \sim enu^\nu$.

Last, we plot the product of the damping time and mean-square momentum transfer, $\tau_D\kappa/2mT_a$. In the classical limit, this combination is a constant equal to 1 Eq. (28). Combining the QED results, we find that the ratio is suppressed from the classical value for all values of a, approaching zero for $a \gg m$. This combination of observables, related by the Langevin dynamics to the mean-squared transverse momentum in equilibrium $\langle p_\perp^2\rangle$, shows

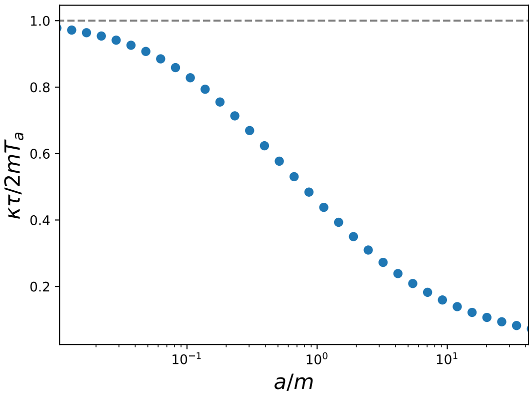

FIG. 4. The product $\kappa_q \tau_q$ normalized to its classical value $2mT_a$.

the fastest deviation from the classical result as a increases.

The mean-square transverse momentum Figure 4 or the transverse diffusion Figure 3 likely provide the most useful observables to study experimentally. Though we have found quite small QED corrections, we could with sufficient statistics and precise control at least verify the classical radiation predictions. An experiment based on laser wakefield acceleration requires substantial improvements in the control and consistency of the acceleration dynamics to be successful. Transverse momentum oscillations, which can approach $|p_\perp| \sim m$ in magnitude, will have to be accounted for, though it is possible that radiation reaction Eq. (16) gradually suppresses the oscillations in the absence of a driving force.

The description here of particle dynamics in strong-field QED regime is of course incomplete. The characteristic timescale for the dissipation of field energy into electron-positron pairs is exponential in the electric field strength, with a field providing an acceleration $a \gtrsim 0.2m$ decaying on the order of 3 ps [47]. Higher order in α processes, such as the direct bremsstrahlung of a pair by the electron are not likely to be important until $a \sim m$. These dynamics are expected to correct the calculations here in the $a \gtrsim m$ regime.

In summary, we have found that thermalization of a probe particle (electron) undergoing constant acceleration is due to its classical radiation. Nonzero variance in the mean-square transverse momentum (chosen for being invariant under boosts compatible with the symmetry of constant acceleration) is explained by computing the second momentum of the radiation distribution, and $\hbar$ only enters as a matter of converting units of photon wavenumber to electron momentum. We expect that the diffusion-related observables obtained here by way of the classical photon number can also be obtained from the appropriate correlator of the classical radiation field, similar to QED and QCD calculations [48]. Such a calculation would be interesting in revealing how $\hbar$ enters. Building on the work of Refs. [22, 28], our discussion emphasizes the origin of the characteristic features of a thermal system in the model of the radiation dynamics. Specifically, the uncorrelated nature of the noise is valid in the classical regime where most emission is soft and dissipative while rarer hard emissions drive the momentum fluctuations. It follows that any more nuanced description of the radiation dynamics, e.g. bringing in higher order correlations from the trajectory, will generally break the perfectly thermal relations obtained here. The quantitative results give an idea of the experimental challenge in observing effects of the acceleration temperature. Laser wakefield accelerators provide the best combination of field strength and acceleration length, but are still a factor ~ 100 too weak field or too short duration. Although some increase of both may be possible in wakefield accelerators e.g. by using "flying focus" laser wakefield schemes or a combination of laser and beam-driven wakefields, these numbers suggest that we will require more precise calculations of well-defined electron beam observables and high-statistics measurements to distinguish the impact of this "thermalization" effect for constant acceleration.

ACKNOWLEDGMENTS

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Appendix A: Transverse photon emission rate: classical calculation

The calculation of the photon emission rate is available from many references [12, 14, 37], so we here just highlight the small refinements in our derivations with respect to present goals. For an electron in a constant electric field $\vec{E} = |\vec{E}|\hat{z}$, the 4-velocity u^μ and trajectory

ξ^μ is equivalent to that under constant acceleration,

$$u^\mu = \big(\cosh(a\tau/c), u_x(0), u_y(0), c\sinh(a\tau/c)\big) \tag{A1}$$

$$\xi^\mu(\tau) = \big((c/a)\sinh(a\tau/c), u_x(0)\tau, u_y(0)\tau, (c^2/a)\cosh(a\tau/c)\big). \tag{A2}$$

For notational simplicity we continue with the electron $p_\perp = 0$ case. We start from the classical formula for the emitted photon number [49]

$$dN_\gamma^{\rm cl} = \frac{e^2}{8\pi^2 c|\vec{k}|^2}|\vec{A}(\vec{k})|^2 d^3k \tag{A3}$$

with the Fourier transformed vector potential determined by the Lienard-Wiechert potentials,

$$A(\vec{k}) = \int e^{i\varphi(t)}\frac{d}{dt}\left[\frac{\vec{n}\times(\vec{n}\times\vec{\beta})}{1-\vec{n}\cdot\vec{\beta}}\right]dt \tag{A4}$$

$$\varphi(t) = k(ct - \vec{n}\cdot\vec{\xi}), \qquad c\vec{\beta} = \frac{d\vec{\xi}}{dt},$$

where $\vec{\beta} = \vec{u}/u^0$ is the normalized 3-velocity of the electron, $\vec{n}$ is the unit vector in the direction of the emission and $k = |\vec{k}|$ is the magnitude of the wave vector. It is convenient to change variables to electron rapidity y, linearly related to the proper time τ, and photon rapidity η, related to the angle of emission,

$$\tanh y = |\vec{\beta}| = \tanh(a\tau/c), \tag{A5}$$

$$\tanh\eta = \vec{n}\cdot\vec{\beta} = k_z/k \tag{A6}$$

so that the phase factor in Eq. (A4) reads simply

$$\varphi(y) = \frac{c^2 k_\perp}{a}\sinh(y-\eta). \tag{A7}$$

Changing integration variables $dt \to dy$, the vector cross product in the integrand is written in terms of (constant) transverse polarization vectors on the unit sphere $\vec{\epsilon}_\Omega$, such that vector in square brackets in Eq. (A4) is

$$\frac{d}{dy}\left[\frac{\vec{n}\times(\vec{n}\times\vec{\beta})}{1-\vec{n}\cdot\vec{\beta}}\right] = \vec{\epsilon}_\Omega\frac{\cosh\eta}{\cosh^2(y-\eta)}. \tag{A8}$$

Rather than using these two expressions to evaluate the integral in Eq. (A4), we write out the squared vector potential in Eq. (A3),

$$dN_\gamma^{\rm cl} = \frac{e^2}{4\pi^2 c|\vec{k}|^2}d^3k\int dy dy'\frac{\exp\left(i(k_\perp/a)(\sinh(y-\eta)-\sinh(y'-\eta)\right))}{\cosh^2(y-\eta)\cosh^2(y'-\eta)} \tag{A9}$$

and change variables to average $2\bar{y} = y + y'$ and relative rapidity $r = y - y'$. After some algebra, the integrand depends only $\bar{y} - \eta$,

$$dN_\gamma^{\rm cl} = \frac{e^2}{4\pi^2 c|\vec{k}|^2} d^2k_\perp dk_z \int d\bar{y} dr \frac{\exp\left(2i(k_\perp/a)\sinh(r/2)\cosh(\bar{y}-\eta)\right)}{(\cosh(2(\bar{y}-\eta)) + \cosh(r))^2}, \tag{A10}$$

where $k_z = k_\perp \sinh\eta$. Changing the integration variable for the photon longitudinal wavenumber to the photon rapidity, $dk_z = ck_\perp \cosh\eta d\eta$, we integrate over η first, shifting $\eta \to \bar{y} - \eta$ with no change to the integrand since the integration domain is $(-\infty, \infty)$. Having eliminated dependence on $\bar{y}$, we undo much of the algebra and change variables $(r, \eta) \mapsto (z = \eta + r/2, z' = \eta - r/2)$ to obtain two decoupled complex conjugate integrals. The result is

$$dN_\gamma^{\rm cl} = \frac{e^2}{4\pi^2} d^2k_\perp d\bar{y} \left| \int_{-\infty}^{\infty} dz \frac{\exp\left(i(k_\perp/a)\sinh z\right)}{\cosh^2(z)} \right|^2 \tag{A11}$$

The integral then yields the modified Bessel function $K_0'(k_\perp/a) = -K_1(k_\perp/a)$.

The mean square momentum transfer integral is made dimensionless by scaling $k_\perp = |\vec{k}_\perp| \to k_\perp/a$,

$$2\kappa_{\rm cl} = \frac{1}{\hbar}\int d^2k_\perp (\hbar k_\perp)^2 \frac{dN_\gamma^{\rm cl}}{d\tau d^2k_\perp} = \frac{2\alpha}{\pi} a^3 \frac{\hbar^2}{c} \int_0^\infty x^3 |K_1(x)|^2 dx \tag{A12}$$

and evaluated using Eq. 6.576 of Ref. [50],

$$\begin{aligned}\int_0^\infty x^{-\lambda} K_\mu(ax) K_\nu(bx) dx =& \frac{2^{-2-\lambda} a^{-\nu+\lambda-1} b^\nu}{\Gamma(1-\lambda)} \Gamma\left(\frac{1-\lambda+\mu+\nu}{2}\right) \Gamma\left(\frac{1-\lambda-\mu+\nu}{2}\right) \\ &\times \Gamma\left(\frac{1-\lambda+\mu-\nu}{2}\right) \Gamma\left(\frac{1-\lambda-\mu-\nu}{2}\right) \\ &\times\ {}_2F_1\left(\frac{1-\lambda+\mu+\nu}{2}, \frac{1-\lambda-\mu+\nu}{2}; 1-\lambda; 1-\frac{b^2}{a^2}\right) \\ &\mathrm{Re}(a+b) > 0 \qquad \mathrm{Re}\lambda < 1 - |\mathrm{Re}\mu| - |\mathrm{Re}\nu| \end{aligned} \tag{A13}$$

Since the $a = b = 1$ in our case, the confluent hypergeometric function is evaluated at 0, which for all values of the parameters reduces to 1. The product of $\Gamma(z)$ functions and 2^{-1} reduces to the constant 2/3, to arrive at the result quoted in the text Eq. (26).

Appendix B: Transverse photon emission rate: QED calculation

We wish to compute the emitted photon distribution fully differential in photon momentum,

$$dW = \frac{d^3k}{(2\pi)^3 2k_0} \frac{1}{2} \sum_{\sigma,\sigma'} \sum_{\epsilon,\epsilon'} \int \frac{d^3p'}{(2\pi)^3 2E_{p'}} \left|\mathcal{M}[e_{\vec{p}} \to e_{\vec{p}'}\gamma_{\vec{k}}]\right|^2, \tag{B1}$$

summed over final electron spin and photon polarization and averaged over initial electron spin. The matrix element is

$$- i\mathcal{M}[e_p \to e_{p'}\gamma_k] = -ie \int d^4x \bar{\psi}^{(+)}_{\sigma',p'}(x) \not{\epsilon}^* \frac{e^{ikx}}{\sqrt{2k_0}} \psi_{\sigma,p(+)}(x). \tag{B2}$$

where $\psi_{\sigma,p(+)}(x)$ is the incoming electron wavefunction and $\bar{\psi}^{(+)}_{\sigma',p'}(x)$ is the outgoing electron wavefunction. The wavefunctions are solutions to the Dirac equation with a classical external vector potential corresponding to an electric field in the $\hat{z}$ direction,

$$\left(i\not{\partial}_x - e\not{A}_{\rm cl}(x) - m\right)\psi(x) = 0, \qquad A^\mu_{\rm cl}(x) = \delta^\mu_3 Et. \tag{B3}$$

Going to the second order equation with the Ansatz $\psi(x) = \left(i\not{\partial} - e\not{A}_{\rm cl} + m\right)\psi^{(2)}(x)$ and changing variables to $u = \sqrt{2/eE}(p_z - eEt)$ leads the parabolic cylinder differential equation

$$\left(\partial_u^2 + \lambda \pm \frac{i}{2} + \frac{u^2}{4}\right) f_\lambda(u) = 0. \tag{B4}$$

The complete set of solutions is $D_{i\lambda}(-e^{-i\pi/4}u), D_{i\lambda-1}(e^{-i\pi/4}u), D_{-i\lambda}(-e^{i\pi/4}u), D_{-i\lambda-1}(e^{i\pi/4}u)$. A detailed derivation of the wavefunctions with updated notation in Ref. [51] and the results are [37]

$$\psi_{\sigma\lambda(\pm)}(x) = N_\lambda \sqrt{2eE} e^{-\pi\lambda/4 \pm i\zeta_\lambda} e^{i\vec{p}\cdot\vec{x}} \chi_{\sigma\lambda(\pm)}(u), \tag{B5}$$

$$\sqrt{2eE}\chi_{\lambda,1(+)}(x) = e^{i\pi/4}(i\lambda) D_{i\lambda-1}(-\xi)u_2 + \frac{p_1u_3 + (m - ip_2)u_1}{\sqrt{2eE}} D_{i\lambda}(-\xi), \tag{B6}$$

$$\sqrt{2eE}\chi_{\lambda,2(+)}(x) = -e^{i\pi/4} D_{i\lambda}(-\xi)u_1 + \frac{p_1u_4 + (m + ip_2)u_2}{\sqrt{2eE}} D_{i\lambda-1}(-\xi), \tag{B7}$$

$$\sqrt{2eE}\chi_{\lambda,1(-)}(x) = e^{-i\pi/4} D_{-i\lambda}(-\xi^*)u_2 + \frac{p_1u_3 + (m - ip_2)u_1}{\sqrt{2eE}} D_{-i\lambda-1}(-\xi^*), \tag{B8}$$

$$\sqrt{2eE}\chi_{\lambda,2(-)}(x) = e^{-i\pi/4} i\lambda D_{-i\lambda-1}(-\xi^*)u_1 + \frac{p_1u_4 + (m + ip_2)u_2}{\sqrt{2eE}} D_{-i\lambda}(-\xi^*). \tag{B9}$$

where we have defined for notational simplicity, $\xi = e^{-i\pi/4}u$, $\zeta_\lambda = (\lambda/2)(1 - \ln\lambda)$ and an orthogonal and complete basis of spinors,

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}. \tag{B10}$$

Using that the outgoing electron solution is equivalent to the time-reversed incoming positron solution, $\psi^{(+)}(t,\vec{x}) = \psi_{(-)}(-t,\vec{x})$, we have

$$-i\mathcal{M} = \frac{-ie}{\sqrt{2k_0}} N_\lambda N^*_{\lambda'}(2eE)e^{-(\lambda+\lambda')\pi/4+i(\zeta_{\lambda'}+\zeta_\lambda)} \int d^4x e^{ikx} e^{i(\vec{p'}-\vec{p})\cdot\vec{x}} \chi^\dagger_{\sigma'\lambda'(-)}(-u)\gamma^0 \not\epsilon^* \chi_{\sigma,\lambda(+)}(u) \tag{B11}$$

The spatial integrals can be done immediately to yield 3-momentum conservation $\vec{p'} = \vec{p} - \vec{k}$. Integrating over the final state momentum with the δ function, and after extensive algebra to reduce the remaining t integral, the fully differential rate is [37]

$$\begin{aligned} dW &= \frac{d^3k}{(2\pi)^3 2k_0} \int \frac{d^3p'}{(2\pi)^3 2p'_0} \frac{1}{2} \sum_{\epsilon,\epsilon',\sigma,\sigma'} |\mathcal{M}|^2 \\ &= \frac{d^3k}{(2\pi)^3 2k_0} \mathcal{N}\pi e^{-\frac{3\pi}{4}\frac{k_\perp^2 - 2p_\perp\cdot k_\perp}{eE}} \left\{ \left(2E_\perp^2 + k_\perp^2 - 2p_\perp\cdot k_\perp\right) \frac{k_\perp^2}{E_\perp^2} |\Psi'|^2 \right. \\ &\left. +(2p_\perp^2 + k_\perp^2 - 2p_\perp\cdot k_\perp)|\Psi|^2 - \left(\frac{2p_\perp^2 k_\perp^2}{E_\perp^2} + \frac{2(2p_\perp\cdot k_\perp - k_\perp^2)}{E_\perp^2}(E_\perp^2 - p_\perp\cdot k_\perp)\right) \mathrm{Re}[\Psi'\Psi^*] \right\} \end{aligned} \tag{B12}$$

where $E_\perp^2 = p_\perp^2 + m^2$. The wavefunction normalizations have been combined into

$$\mathcal{N} = \frac{2e^2 \exp\big(-\pi(\lambda+\lambda')/2)\big)}{2\lambda'(eE)^2(1-e^{-2\pi\lambda})(1-e^{-2\pi\lambda'})}, \tag{B13}$$

and Ψ is the confluent hypergeometric of the second kind, evaluated at

$$\Psi \equiv \Psi\left(\frac{iE_\perp^2}{2eE}, 1 - \frac{i(k_\perp^2 - 2p_\perp\cdot k_\perp)}{2eE}; \frac{-ik_\perp^2}{2eE}\right), \tag{B14}$$

which is related to the confluent hypergeometric ${}_1F_1(a,b;z)$ by

$$\Psi(a,b;z) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)}\,{}_1F_1(a,b;z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b}\,{}_1F_1(a-b+1, 2-b; z) \tag{B15}$$

The $\hbar \to 0$ limit yields the classical result [37].

Appendix C: Results for a scalar radiation field

The number of particles emitted by a classical source $J(x)$ on a general scalar field φ can be found in standard textbooks [52, Chapter 2], and is given by

$$\int dN = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |J(p)|^2. \tag{C1}$$

For a classical charged particle source following an accelerated trajectory Eq. (A1), we have

$$J(x;\xi) = e\int d\tau \sqrt{u^2(\tau)}\delta^4(x - \xi(\tau)), \tag{C2}$$

and the Fourier transform of the source for the localized particle is given by

$$J(p) = e \int d\tau \; \exp(i(E_p/a)\sinh a\tau - i(p_z/a)\cosh a\tau). \tag{C3}$$

We are interested in the number of photons emitted per unit transverse momentum and per unit proper-time $dN/d\tau d^2p_\perp$, which can be obtained from the evaluation of the differential in 3-momentum

$$\frac{dN}{d^3p} = \frac{1}{(2\pi)^3 2E_p} \left|J(p)\right|^2 . \tag{C4}$$

Changing into relative coordinates $\bar{\tau} \equiv \frac{1}{2}(\tau + \tau')$ and $\delta\tau \equiv \frac{1}{2}(\tau - \tau')$, we can write the expression for the square of the current's Fourier transform as

$$\left|J(p)\right|^2 = e^2 \int d\tau d\tau' \exp\left[(2i/a)\sinh a\delta\tau (E_p \cosh a\bar{\tau} - p_z \sinh a\bar{\tau})\right]. \tag{C5}$$

Now we parametrize the particle's momentum by new hyperbolic variables $E_p = p_\perp \cosh\eta$, $p_z = p_\perp \sinh\eta$, and obtain the alternative representation

$$\left|J(p)\right|^2 = 2e^2 \int d\delta\tau d\bar{\tau} \exp\left[(2ip_\perp/a)\sinh(a\delta\tau)\cosh(\eta - a\bar{\tau})\right], \tag{C6}$$

which makes clear that the integral is independent of the rapidity η. We can remove η directly from the integral, which would yield an exact expression [50, Eq. 8.432-5] for $\left|J(p)\right|^2$ as

$$\left|J(p)\right|^2 = e^2 \left| \int d\tau \exp\left(i(p_\perp/a)\sinh a\tau\right) \right|^2 = \frac{4e^2}{a^2} K_0^2(p_\perp/a). \tag{C7}$$

where K_0 is a modified Bessel functions of the second kind. In terms of the η coordinate, the dN differential takes the form

$$\frac{dN}{d\eta\, d^2p_\perp} = \frac{1}{2(2\pi)^3} \left|J(p_\perp)\right|^2 , \tag{C8}$$

and the final expression is given by

$$\frac{dN}{d\eta\, d^2p_\perp} = \frac{e^2}{4\pi^3 a^2} K_0^2(p_\perp/a). \tag{C9}$$

We can also obtain the same espression in terms of a differential on the mean proper-time $\bar{\tau}$. First we integrate over all longitudinal momenta

$$\int dp_z \frac{dN}{d^3p} = \frac{1}{(2\pi)^3} \int \frac{dp_z}{2E_p} \left|J(p)\right|^2 . \tag{C10}$$

In terms of the momentum rapidity η, we get

$$\frac{dN}{d^2p_\perp} = \frac{e^2}{(2\pi)^3} \int d\eta\, d\delta\tau d\bar{\tau} \exp\left[(2ip_\perp/a)\sinh(a\delta\tau)\cosh(\eta - a\bar{\tau})\right]. \tag{C11}$$

Changing variables for the η-integral and extracting the linearly divergent total proper-time $\int d\bar{\tau}$, we get

$$\frac{dN}{d\bar{\tau}d^2p_\perp} = \frac{e^2}{(2\pi)^3}\int d\eta\, d\delta\tau \exp\left[(2ip_\perp/a)\sinh(a\delta\tau)\cosh(\eta)\right]. \tag{C12}$$

Again from Eq. 8.432-5 of Ref. [50], we get an exact expression for the integral in terms of another modified Bessel function of the second kind

$$\int d\delta\tau \exp\left[(2ip_\perp/a)\sinh(a\delta\tau)\cosh(\eta)\right] = \frac{2}{a}K_0(2(p_\perp/a)\cosh\eta), \tag{C13}$$

and the remaining η integral can be evaluated to (cf. Eq. 6.663-1 of [50])

$$\int d\eta\, K_0(2(p_\perp/a)\cosh\eta) = K_0^2(p_\perp/a). \tag{C14}$$

The final result for the distribution of scalar particles created per transverse momentum and proper time is thus

$$\frac{dN}{d\bar{\tau}d^2p_\perp} = \frac{e^2}{4\pi^3 a}K_0^2(p_\perp/a), \tag{C15}$$

which coincides with the previous direct calculation from the "momentum rapidity" η by the direct substitution $\eta \leftrightarrow a\bar{\tau}$.

We are interested in the mean squared transverse momentum transfer for the theory, so we calculate

$$2\kappa_{\rm scalar} = \int d^2p_\perp p_\perp^2 \frac{dN}{d\bar{\tau}d^2p_\perp} = \frac{e^2}{4\pi^3 a}\int d^2p_\perp p_\perp^2 K_0^2(p_\perp/a). \tag{C16}$$

From Eq. 6.576-4 of Ref. [50], we get

$$\int d^2p_\perp p_\perp^2 K_0^2(p_\perp/a) = 2\pi\int dp_\perp p_\perp^3 K_0^2(p_\perp/a) = \frac{2\pi a^4}{3}, \tag{C17}$$

which yields the final expression

$$\kappa = \frac{e^2a^3}{12\pi^2}, \tag{C18}$$

in agreement with the previous results when taking into account the spin degrees of freedom of the underlying field.

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Chapter 5

The strong CP problem, general covariance, and horizons

In this chapter, we present our article discussing the vacuum structure of topologically non-trivial gauge theories in the context of the causal structure of spacetime. In it, we discuss how the vacuum structure of quantum fields are sensitive to spacetime topology, and connect this discussion with the well-known strong CP problem in quantum chromodynamics. We argue that since observers bounded by horizons have limited access to spacetime, one must be very careful in how you treat the vacuum of a quantum field, since it is, in a sense, a "global" object. We discuss and speculate on the implications of treating the horizon as a source of decoherence for the topologically non-trivial sectors of the gauge theory, with applications to the CP-breaking phase θ of QCD_4. Finally, we propose an analogy between this horizon decoherence mechanism and a dissipative toy model in condensed matter physics—the quantum particle in a periodic potential interacting with a dissipative environment—and discuss possible experimental scenarios where these ideas can be explored quantitatively.

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The strong CP problem, general covariance, and horizons

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We discuss the strong CP problem in the context of quantum field theory in the presence of horizons. We argue that general covariance places constraints on the topological structure of the theory. In particular, it means that coherence between different topological sectors must have no observable consequence, because the degrees of freedom beyond a causal horizon must be traced over for general covariance to apply. Since the only way for this to occur in QCD is for $\theta = 0$, this might lead to a solution of the strong CP problem without extra observable dynamics.

I. INTRODUCTION TO THE STRONG CP PROBLEM

The strong CP problem [1–9] is the only fundamental naturalness problem connected to QCD. It arises generically in Yang-Mills theories as a consequence of the existence of an internal direction in the "color-symmetry" gauge group G, where the vector potential $A_\mu \to A_\mu^a$ is allowed to twist in topologically non-trivial ways. The non-trivial twisting of A_μ means that finite-action field configurations are separated into topologically separated equivalence classes, labeled by how they twist in gauge space over the asymptotic region $A_\mu(r \to \infty)$ in Euclidean space [7, 8]. The label for these equivalence classes is an integer $n[A] \in \mathbb{Z}$, called the winding number of A_μ, given by

$$n[A] = \frac{1}{32\pi^2} \int d^4x F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a, \quad \tilde{F}_{\mu\nu}^a \equiv \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma} F_{\lambda\sigma}, \tag{1}$$

and all classical solutions of the Euclidean equations of motion of the theory with finite action fall within one of these winding sectors.

Classically, the winding number cannot be changed due to energy conservation. Quantum mechanically, however, that can happen due to the existence of tunneling solutions of the classical Euclidean equations of motion (called instantons) that interpolate between two given equivalence classes $n \to n+1$, having the form (up to a parameter ρ)

$$A_\mu^a = 2\frac{x_\nu}{x^2}\left(\frac{\eta_{\mu\nu}^a \rho^2}{x^2+\rho^2}\right), \tag{2}$$

where a is the gauge index and $\epsilon_{\mu\nu a}$ mixes internal and spacetime degrees of freedom.

This means the winding number is only conserved perturbatively. Field strength fluctuations

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{3}$$

corresponding to field configurations of the type of Eq. (2) occur locally, suppressed non-perturbatively by the field's action content Γ

$$\Gamma \sim \exp\left[-\frac{1}{4g^2}\int d^4x \, F_{\mu\nu}^a F_{\mu\nu}^a\right], \tag{4}$$

and change the field's winding number by one unit.

Note that winding numbers are global properties of given field configurations defined over all space. Since the topological charge density $F_{\mu\nu}\tilde{F}_{\mu\nu}$ can be written in terms of a total derivative $\partial_\mu K_\mu$ (the so-called Chern-Simons current), they are only "observable" (not in practice but with QFT sources, see footnote 3) by probing the celestial sphere at at infinity

$$n \sim \int d^4x F_{\mu\nu}\tilde{F}^{\mu\nu} \sim \lim_{r\to\infty} \oint d^3S\, \epsilon_{\mu\nu\rho\sigma} n_\mu \mathrm{Tr}\left(A_\nu \partial_\rho A_\sigma + \frac{2}{3} A_\nu A_\rho A_\sigma\right). \tag{5}$$

Instantons, however, are localized, and dominate at a scale in momentum space (parametrized by ρ in Eq. (2)) where the theory becomes non-perturbative.

Instantons break the degeneracy of the classical vacuum solutions with different winding numbers, and their presence means that the Yang-Mills vacuum will be in general a superposition of the different topological sectors weighted by expansion coefficients c_n. This coefficients are fixed to a phase $c_n = \exp(in\theta)$ by the homogeneity of Minkowski space [2], where the arbitrary constant θ parametrizes the coherent superposition of states with topological winding number n

$$Z[\theta] = \sum_n e^{in\theta} Z_n \iff |\theta\rangle = \mathcal{N} \sum_n e^{in\theta} |n\rangle . \tag{6}$$

This superposition of winding sectors in the vacuum is equivalent, via imposing the first relation in Eq. (5) as a constraint in the partition function, to an additional gauge-invariant [9] term in the Euclidean effective Lagrangian

$$\mathcal{L}_{\text{eff}} \equiv \ln Z = \mathcal{L}_{\text{YM}} + \frac{i\theta}{16\pi^2} \mathrm{Tr}(F_{\mu\nu}\tilde{F}_{\mu\nu}), \tag{7}$$

which breaks the CP symmetry of the original Yang-Mills Lagrangian. In QCD, for example, this CP symmetry breaking gives rise to observable effects, such as θ-dependent electric dipole moments for neutral particles. Experiments, however, have constrained this CP-violating parameter to the small bound of $|\theta| \leq 10^{-9}$ (the experimental limit of the neutron's dipole moment is 10^{-18} e.m.), beyond naturalness, and the explanation for such a small value is called the strong CP problem.

A variety of solutions [1] were proposed for this problem, most involving beyond the standard model dynamics coupling to θ ("axions") or extra symmetries that fix θ to zero. Phenomenologically, no signature suggestive of such models has so far been detected.

In this work, we will try to "go back to the basics" of quantum field theory and reflect on the nature of the quantum vacuum. In the following three sections, we will make a series related arguments that link the strong CP problem to the interplay between quantum field theory and general coordinate transformations. Section III argues that a theory with a non-zero θ will likely lose general covariance of any observables depending on θ. We shall also show that this constraint on θ will not affect "local" topology fluctuations necessary for phenomenology (such as [3, 5, 10–12]). Section IV will show that the "toy model" often used to describe θ vacua together with the tracing over of trans-horizon degrees of freedom leads to $\theta = 0$ asymptotic states independently of the "real" θ value. While our work is new, approaches incorporating some ideas discussed below, such as

- Constraints from general covariance on local physics [13–20]
- The coherence of instantons [21]
- Infrared fixed points in the running of θ [22]
- The effect of analytical continuation in curved spacetime [23–25]
- The stability of periodic states against decoherence [26–28]
- The non-invariance of semiclassical states under non-inertial transformations [29, 30]

have been investigated before. The appendices will also discuss the relationship of our idea to exactly solvable toy models [31, 32].

II. QUANTUM FIELD THEORY AT DIFFERENT SCALES: BACKGROUND INDEPENDENCE AS AN INFRARED SYMMETRY

At first sight the claim at the end of the last section appears far-fetched. We are expected to believe that something as "global" as the topology of the universe plays an important role in local physics, the effective Lagrangian of QCD[1]. To justify and also sharpen this claim, we need to discuss a little bit more the sensitivity of quantum field theory observables to different scales.

Analogously to quantum mechanics, in quantum field theory all information about the vacuum state is encoded in the generating functional, and perturbations generated by sources

[1] There is a lot of discussion about weather General relativity is "Machian", although Mach's principle played a big role in constructing it. However the Gibbons-Hawking boundary term looks conceptually like a quantum field version of Mach's principle [50, 51]. The fact Machian-type reasoning can affect local observables was realized in [13] with regard to the gravitomagnetic moment form factor.

can be used to calculate correlations around this state. In terms of fields ϕ and source functions $J(x)$, the generating functional is given by the expression

$$Z = \int [\mathcal{D}\phi] \exp\left[\left[\int \sqrt{-g} d^4x \left[\mathcal{L}(\phi) + J(x)\phi\right]\right]\right]. \quad (8)$$

For fields of any given spin, both ϕ and J acquire the Lorentz and internal symmetry properties necessary for $J(x)\phi$ to be a Lorentz scalar.

Unlike quantum mechanics, quantum field theory [2] has a continuous infinity of degrees of freedom. As a result, key theorems behind quantum mechanics, such as the Stone-Von Neumann theorem, stop being valid, and we are left with the possibility of unitarily inequivalent representations of the canonical commutation relations (symmetry breaking in field theory is a particularly important example of this). Mathematically, the construction of observables can be beset by ambiguities, such as divergences.

The main way physicists deal with these issues is in the language of the renormalization group. Observables in quantum mechanics are not "states" but time ordered field correlators of operators $\hat{O}$, $\langle \hat{O}(x_1)\hat{O}(x_2)\rangle$, measured at a scale $Q \sim |x_1 - x_2|^{-1}$. Thus, unlike in quantum mechanics, even fundamental parameters of the Lagrangian of the theory become ambiguous, related to an observable and sensitive to Q. Provided a stable vacuum is well-defined, if one calls the scale of the detector interaction $Q \sim \mu$ and chooses a scale $\Lambda \gg \mu$, one can evolve correlators from Λ to μ and separate the Lagrangian into a finite number of renormalizable terms for which Λ can be absorbed into unobservable divergences, and a series in Q/Λ. (For correlators, equations of this form are called Callan-Symanzik equations [2], while for Lagrangians they are called the Wetterich equations [33–35]).

This construction has, however, some limitations. For one, it is only well-defined in Euclidean rather than Minkowski space [36, 37], which is obviously an issue when analytical continuation is non-trivial. As a related point, it has yet not been univocally and generally extended to fully cover *infrared physics*, for which Q is much smaller than μ, the inverse of the "detector size". Within a locally Minkowski spacetime, this infrared limit could exhibit a non-trivial horizon and causal structure, and evolving from a scale $k \ll \mu$ to a scale $k \sim \mu$ could depend on this.

For instance, it has long been known that the energy of a vacuum with the boundary is very sensitive to the shape of that boundary, to the effect that it even changes sign [38] in ways different from naive dimensional analysis [39, 42]. Within the context of the cosmological constant and inflation these topics are subject of active study [39–41]. While the cosmological constant is a super-renormalizable operator, and dimensionally θ is superficially a marginal coupling, topological terms are also known to depend on infrared wavelengths [43]; the relationship with vacuum energy made explicit in color counting [6] and the dependence on the metric topology of Eq. (5).

In the case of the θ vacuum, this mixing between IR and measured scales can be seen in the instanton liquid model by calculating the energy expectation value, as was done in [8]

$$< 0|H|0 >_{\theta} \sim \int_0 \frac{d\rho}{\rho^5} F\left(\rho\Lambda\right), \tag{9}$$

where $F(...)$ can qualitatively depend on both the measured and IR limit non-trivially. Physically, this mixing of scales is generated by the fact that these configurations are dominated by high occupation numbers of ground state quanta.

Since effective field theory techniques rely on a $Q \ll \mu$ expansion, these subtleties are generally incorporated in the $\mathcal{O}\left(1\right)$ terms of the effective Lagrangian. For instance, the calculation of the θ-dependence of the vacuum energy in [44] has no trace of the integral in Eq. (9) and appears to only be set by dimensionful parameters of the order of Λ_{QCD}.

The arguments above show that, while one must be careful with infrared scales, effective field theories can be a useful guide because they can indicate which terms are compatible with symmetries at the infrared. For instance, in QED and linearized gravity the fact that emission of quanta of the order $Q \ll \mu$ does not effect observables at $Q \sim \mu$ is related to the symmetries of the theory [45]. The full consequences of this have been the subject of a lot of theoretical development [37, 45, 46], and the topology of the space in question (the BMS group in asymptotically flat space) is crucial to this.

We therefore turn to the other ingredient necessary to construct effective field theories: Unbroken symmetries in the infrared and in the semiclassical approximation. The obvious symmetry mentioned previously is the equivalence principle, built out of general covariance of observables. Extending general covariance to the non-linear gravitational and quantum regime is of course a central, unsolved problem of theoretical physics. It has long been known that at one loop in the effective theory correlators do not transform covariantly (for a particular example, see the light-bending calculations at [47]). It has also been known that quantum vacua with different topologies fall into unitarily inequivalent representations (the "black hole information paradox" is a direct consequence of this, as can be seen from

[49–51]).

However, from a semiclassical effective theory point of view, the problem does appear more tractable. It can be argued [17, 18] that if general covariance is to be fundamental one must give up unitarity and treat the degrees of freedom behind the horizon in the language of open quantum systems. In this case, the vacuum's "state purity" is ill-defined, since it is frame dependent, but observables could acquire general covariance via the boundary term. Instead of unitarity, a fluctuation-dissipation theorem would constrain the partition function [48]. Several lines of evidence point to the fact that such a construction might be achievable. The soft graviton theorems mentioned earlier [45] relate the independence of infrared physics to Lorentz invariance. To first order, it has been known for a while that correlators along Rindler paths [15, 49] transform covariantly once boundary terms are added. For a correlator $\langle\psi(x_1^\mu(\tau_1)\psi(x_2^\mu(\tau_2)X\rangle$,where $x_{1,2}^\mu, \tau_{1,2}$ are along a Rindler trajectory and X is either a photon in QED [14] or a neutrino in Fermi theory [16] one can calculate the Minkowski correlator in terms of interactions with a classical field and the Rindler correlator in terms of interactions with the Unruh bath. The "interpretation" of the calculation will be different but the calculated matrix elements will be the same.

It is therefore worth thinking about the form that a "generally covariant" interacting quantum field theory would have, and in particular if, in analogy with renormalization requirements, some terms in the Lagrangian would be forbidden by imposing background independence as an infrared symmetry. In [13], it was shown, for example, that a non-zero gravitomagnetic moment would violate local covariance of observables under rotations.

In this regard, the θ term looks suspicious. The vacuum energy density depends on the boundary and on θ separately. The effective θ, $\sim \frac{\delta \ln \mathcal{Z}}{\delta F\tilde{F}}$ will also depend on the boundary and on the temperature. Hence, there is no a priori reason for the effective θ calculated from a Rindler boundary to be the same as the effective θ at finite temperature, something already clear from group theory arguments [29] in a generic scenario with degenerate vacua constructed from conformal zero modes (semiclassical instantons are similar in this respect). But local Lorentz invariance forces this equivalence [52, 53], leading to non-zero θ being forbidden.

This suspicious is is complemented, in effective theory language, by the scale separation necessary for both the Unruh effect and instantons to be properly defined. Topological configurations are dominated by high occupation numbers of ground state quanta. In the moving mirror picture mentioned earlier [30], if one were to construct topologically non-trivial configurations in such a setup, the relation between Minkowski and mirror boundary conditions would map "soft quanta" carrying topological information into harder ones which, because of asymptotic freedom, are insensitive to the presence of instantons in the vacuum.

Acceleration and force are by their nature semiclassical concepts, since they consider momentum a differentiable number rather than an operator. This means that acceleration's lifetime T needs to be long-lived compared to its scale a, and also smaller than the fundamental scale of the detector, for example its mass M

$$1/T \ll a \ll M \underbrace{\Rightarrow}_{instanton} \mu \ll a \ll \rho \tag{10}$$

in this context, the instanton's bulk of the action is in the peak, of size $\sim \rho$ (Eq. (2)), yet the topology information will be in the tail, dominated by a diverging occupation number of quanta of characteristic frequency $\ll 1/\rho$. One can therefore have an acceleration small enough w.r.t. the instanton size, long enough to maintain the semiclassical approximation, where the semiclassical expansion will break down at the tail. The first hierarchy in Eq. (10) is therefore far more dubious in its applicability than the second[2].

This insight can be sharpened in lower-dimensional theories, where the non-trivial interplay of asymptotic global symmetry properties of the theory and topological terms is well-known [58]. Appendix A 1 describes in detail a quantum particle on a ring with a magnetic field in thermal equilibrium. The different geometric and thermodynamic temperature in this system can be mocked up by a device that changes the magnetic field based on the system's heat capacity. While such a setup can be engineered to add no entropy (no microstates are measured, and the adiabatic limit is maintained) in this case, the thermodynamic temperature is different from the geometric temperature (given by the time periodicity) by a generally non-perturbative factor, Eq. (A42). Note that the difference is not necessarily connected to a scale separation, since the two dimensionful scales combine non-perturbatively.

Section A 2 summarizes an equivalent 1+1 dimensional topological theory. As shown there, while in such a theory the θ term is set as a boundary condition, a source "communi-

[2] One can choose a gauge, such as the "singular gauge" [4], where the Winding number is not asymptotic, at the price of having a singularity in the center, $A_\mu \sim \partial_\mu \ln\left(1 + \rho^2 (x - x_0)^{-2}\right)$. However, the continuation of this Gauge in Minkowski and Rindler space, discussed in the next section as well as [23], is probably impossible for obvious reasons

cating" with a boundary could modify the temperature from the Unruh value by arbitrary amounts. Dynamics spontaneously creating such topological terms in 3+1 theory, therefore, could potentially mean that the geometric temperature of an accelerating observer would be different from the temperature the observer would measure from the background. A dynamical symmetry breaking would be equivalent to assuming an effective $\langle J_L \rangle$ in Eq. (A55), coinciding with that of Eq. (5). As the next section III will argue, unless $\theta = 0$ it is impossible to make such a term truly background independent. Section IV will further show, using the often-used periodic potential analogy, how the IR tracing out could wash out an arbitrary θ to an effective $\theta = 0$.

Summarizing, a "background-independent quantum field theory"'s requirement would be the separation of local physics (at scale μ) from scales sensitive to the topology of space. This is very different from saying the latter are relevant to observable physics (indeed, as [44] shows explicitly, they are not). Analogously to counter-terms in Wilsonian renormalization enforcing independence of detectable physics from its UV completion, infrared constraints enforce independence of local physics from the topological features of the chosen coordinate system. One such constraint would be $\theta = 0$.

III. THE TOPOLOGICAL TERM FROM THE PARTITION FUNCTION

As can be seen in the previous section, the question of the general covariance of quantum field theories is a subtle one. It is straightforward to write an action invariant under general coordinate transformations, since $\sqrt{-g}d^4x$ is generally covariant and $\mathcal{L}$ can be made so by the use of covariant derivatives. Indeed, a θ term can also arise within an effective theory extension of general relativity [59].

The issue is the role of the integration measure $\mathcal{D}\phi$. What configurations are counted if the coordinate system contains singularities, or the spacetime is divided into causally disconnected regions? We note that, as shown in [60], Hawking radiation can be thought of as an anomaly, i.e. a tension between the fundamental symmetry of the theory and the integration measure – in this case provoked by the boundary structure of the Schwarzschild spacetime at the horizon.

At the partition function level, it has long been known [50, 51] that the partition function for general coordinate systems with causal horizons will necessitate of a surface term, which for a timelike Killing horizon can be modeled by a thermal bath. This is the essence of the Unruh effect, and, in 1+1D, can be rigorously connected to topological terms (see appendix and [31]).

In [17], it was proposed that perhaps promoting the partition function with a source to a dynamical object would generate a generally covariant quantum theory. Since bulk general relativity is *always* holographic [18] (something that can be seen as a consequence of Lovelock's theorem [19]), a general non-inertial transformation will alter both bulk and boundary, but there is a possibility that, with Lagrangians describing both, the total partition function will be invariant under general coordinate transformations. In [17] we argued that imposing this must lead to treating all quantum states as open, since such general coordinate transformations necessarily break unitarity.

This extension, however, is unobservable in experiments done so far as it concerns situations with strong classical accelerations. Such an approach, as we argued, could be used to write down an effective quantum field theory covariant under general coordinate transformations. Not quantum gravity, of course, but a field theory respecting the symmetries of gravity at quantum level.

In this spirit, let us try to define θ as a running parameter. We immediately see that Eq. (7) and Eq. (5) are only valid in flat Euclidean space, but any infrared subtlety, like the presence of a horizon, would change the asymptotic shape of the instanton and hence the $k \ll \mu$ dynamics. Put it differently, while of course $F_{\mu\nu}\tilde{F}^{\mu\nu}$ is a generally covariant scalar the equation 5 does not transform covariantly and hence the constraint leading to Eq. (7) is not generally covariant.

We can however circumvent this problem remembering the effective action can be defined also in terms of a source in Eq. (8). The winding number will be related to the infrared limit of a probe such as a "loop" of "color sources" J_a^μ placed at infinity[3], while θ will be related to the entanglement of such winding numbers

$$n \propto \lim_{r_0 \to \infty} \mathrm{Tr} \oint dS_{d-1}.J^a \delta(r - r_0) A_a \quad , \quad \theta \propto -i \langle n \rangle \tag{11}$$

[3] A technical note: of course it is impossible even in principle to put a probe at infinity. However, one can define such a measurement in terms of Bayesian limits. Here, by "a probe at infinity" we mean a probe placed at a sequence of larger and larger distances. The chance of the winding number not being inferred correctly by the measurement goes down with distance in a calculable way. Such a sequence can be used to define something like Eq. (5)

however, in Minkowski space S_{d-1} can be oriented in either a time-like or a space-like direction. In the first case, we will be projecting on θ, as it commutes with the Hamiltonian and relation Eq. (5) holds. As a particular case, if instead of an isotropic dS_{d-1} we consider a "long thin sausage", our observable coincides with the dipole moment of a neutral particle antiparticle combination. This measurement is in fact "asymptotic", since observing the neutron for any amount of finite time there is a finite probability that topological fluctuations will give us an effective non-zero dipole moment (of course this is irrelevant for the macroscopic scales where the neutron dipole moment is measured).

In the second, spacelike loop, we will be sensitive to the topological number $|n>$ of our system, rather than θ. This ambiguity is unique for "topological terms", and suggests that general topological terms are *not* generally covariant. Indeed, it is obvious that the boundary integral has the same structure of equation (5); A function of A_μ^a integrated over the surface dS_{d-1} which in turn is the edge of a particular geometry. A non-zero θ term means that different n's are coherently entangled, and the degree of this coherence would be modified by the boundary term. Hence, the entanglement entropy, *and all of its derivatives* (θ is proportional to the first derivative, the topological susceptibility to the second), would be modified by the boundary term in a way that is very sensitive to the geometry in question. In fact, calculating Eq. (5) using the WKB approximation [61] and a path crossing the horizon can easily be seen to be boundary-dependent.

This suggestion has been put on a firmer footing within [23], following an unsuccessful attempt by the same author [24] to argue, in a manner similar to [25], that Yang-Mills instantons are incompatible with a Schwarzschild geometry. As it turns out [23] this is not quite correct. However, unlike in Minkowski the instanton in Euclidean Schwarzschild space cannot be gauge transformed into a smooth temporal gauge; By looking at a Rindler patch of this space, it is clear that a Minkowski and a Rindler observer, or a Schwarzschild vs a Lemaitre observer, will see a different instanton content and a different n. As n is a scalar, the only way to preserve an undetermined n (required by local quantum mechanics) that transforms covariantly is to make sure the summation over $|n\rangle$ was incoherent even in the freely falling frame, where geometry is closest to Minkowski. This corresponds to the case of $\theta = 0$. Note that this argument only applies to the *average* topological value $\left\langle F\tilde{F}\right\rangle$. Phenomenologically useful fluctuations $\sim \left\langle F\tilde{F}F\tilde{F}\right\rangle$ [10] are not affected because of the time-ordered nature of the product in the fluctuation [6], which isolates the *local* (instanton peak, always in causal contact with the observer) over the global asymptotic state (possibly affected by the horizon). In a Euclidean spacetime, without horizons, the infrared limit of $\left\langle F\tilde{F}F\tilde{F}\right\rangle$ is related to $\left\langle F\tilde{F}\right\rangle$

$$\lim_{k\to 0}\int d^4x e^{ik(x-x')}\left\langle F\tilde{F}(x')F\tilde{F}(x)\right\rangle \sim \left\langle F\tilde{F}\right\rangle \tag{12}$$

but in locally Minkowski spacetime, whose global causality structure (set of points where $k_\mu(x-x')^\mu = 0$ of Eq. (12)) is non-trivial, the two could be different (See the discussion about time orderings in [6]). The analogy here is the cosmological constant (of which $F\tilde{F}$ is the "topological" QCD part, [6]) and local gravitational physics. Background independence requires that infrared physics could get corrections from quantum fluctuations and decoherence [39], but the same symmetry requires that locally physics is unaffected. This can happen if all local physics is sensitive to the second derivative and the first derivative is zero, as indeed seems to be the case.

Let us explore this argument in more detail, but concentrating on Rindler patches. For static and quasi-static spacetimes this is equivalent to filling the manifold with a bath of quanta with the temperature $T_h \sim 1/R_h$, where R_h is the horizon scale (the Schwarzschild radius for black holes, the Hubble radius for dS space, the acceleration for Rindler space and so on). Thermality appears as a consequence of the symmetries of the quasi-static accelerated spacetime [29, 62], and hence can be thought of as as embedded in the *effective action*

$$S_{eff} \sim \ln\left[\int \mathcal{D}\phi \exp \underbrace{\int \sqrt{-g}d^4x}_{\to\oint dt \int d^3x \sim T\sum_n \delta\left(t-\frac{n}{T}\right)} \left[\mathcal{L}(\phi)+J(x)\phi\right]\right]. \tag{13}$$

As shown in [52, 53], for axiomatic field theory (defined in terms of correlators), Lorentz invariance implies this relation is exact.

As further shown in [15, 49] and references therein, one can derive the Unruh effect by tracing over degrees of freedom beyond the horizon. The two pictures are complementary since, at least for Rindler patches [63, 64], tracing over in Minkowski space can be achieved by complexifying the action and appropriately choosing contours constructed to respect the periodicity of the time coordinate. This way we can write, up to a normalization factor, the

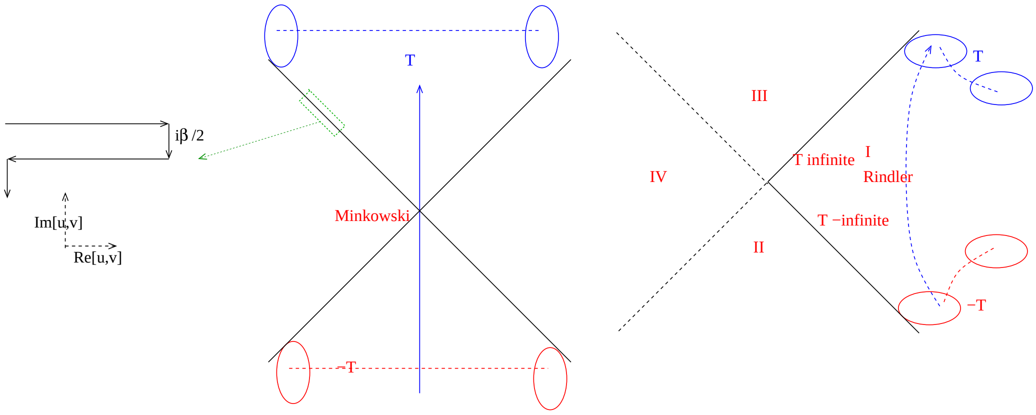

FIG. 1. The asymptotic horizons of Minkowski and Rindler observers, on which winding numbers are measured (middle and right panels). As shown in [49], a rigorous dictionary between the two partition functions can be established by complexifying coordinates and a choice of contours, shown on the left

generating functional for a field as

$$Z[J] \sim \int \mathcal{D}_{\mathcal{C}}\phi \exp\left\{ i \int_{\mathcal{C}} \sqrt{-g} d^4x \left(\mathcal{L}[\phi, \partial\phi] + J\phi \right) \right\}, \tag{14}$$

where $\mathcal{C}$ corresponds to a contour choice in complex coordinate space – analogous to the Schwinger-Keldysh and thermofield-dynamical formalisms of usual finite-temperature quantum field theory. An extension of Rindler to Minkowski spacetime can be realized, in Rindler null-coordinates $l_\pm = \tau \pm \xi$, by the horizontal patches $\mathcal{C}_1, \mathcal{C}_2$ of the contour shown in figure (1), defined by

$$\mathcal{C}_1 \equiv l_\pm, \quad \mathcal{C}_2 \equiv l_\pm - \frac{i\beta}{2}, \quad l_\pm \in \mathbb{R}. \tag{15}$$

In this way, all four wedges $W_{\pm,F,P}$ of Minkowski space correspond to different combinations of horizontal sections of the Rindler null-coordinate contours. The fields associated with the horizontal sections $l_\pm \in \mathcal{C}_2$ (with non-zero imaginary part $\Im\mathcal{C} \neq 0$) correspond to the causally inaccessible regions of spacetime $W_{-,F,P}$, and act as invisible fields from the point of view of an accelerated observer. In other words: The extension of the fields from Rindler to Minkowski spacetimes acts as a purification of the Hawking-Unruh thermal state[4].

[4] Contour choice has also been contentious in the question of whether de-Sitter space evaporates or not. The instability of de-Sitter space was argued for in [40], while [41] argues de Sitter space is stable. At the heart of the disagreement is the contour definition, with [39, 40] relying on a Feynman type locally causal quantization, and the definition of the “stable” Bunch-Davies vacuum [41] relying on a time-symmetric contour constructed not to evaporate. This work takes the first of these two approaches.

Extending such a calculation to a topologically non-trivial Yang-Mills theory is a formidable project. However, the arguments in [23] make it clear that the fact that winding numbers can be both in the observed and the hidden patches will generally change the topological structure of the resulting partition function. A physical reason is that for real positive quark masses, the partition function in equation (6) admits a quasi-probabilistic interpretation

$$Z = \sum_n Z_n e^{in\theta}, \quad P_n \sim \frac{Z_n}{Z(\theta=0)}, \tag{16}$$

with P_n is the "probability" to "measure" a winding number n [44]. Note that [44] this is a Wigner Quasi-probability [65] rather than a probability, since for generic θ it might not be real-valued. However, it expresses the quantum uncertainty of winding numbers when θ is fixed, and it obeys Wigner's quasi-probability axioms. The winding number, in this picture, is measurable via the probe in equation (5).

In the quasi-probabilistic interpretation motivated in equation (6), Z_n and the topological susceptibility $\partial^2 \ln Z/\partial^2\theta$ will transform between Minkowski and Rindler space with a factor representing the ratio of visible to invisible fields. By Bayes's theorem

$$Z_n = Z(\theta=0) \sum_\mu \frac{Z_\mu}{Z(\theta=0)} \times P\left(m-n \in \mathcal{C}_2\right), \tag{17}$$

and the $P(...)$ term will be directly proportional to the proportion of the given Rindler time-slice covering each section of Rindler space. This is not unity, and will depend on the proper time of the Rindler observer. It will also be an observable, measurable in a Gedankenexperiment by repeated applications of the operator defined in equation (5).

This lack of covariance has a root in two issues: the winding number is not a conserved quantum number, and hence is expected to change with the Hamiltonian. However, "time" is not just a coordinate but also defines the order in which the co-moving observer makes observations ("collapses the wavefunction", or, rather, samples correlators). If one sequentially "observes it", over time intervals of the order of an instanton size, one expects it to change by one unit. However, this change in Hamiltonians will also affect the partition function according to

$$Z = \langle 0| \exp\left[i \int dt \hat{H} \right] |0\rangle, \tag{18}$$

and so the procedure

$$\int dt \to \frac{1}{N} \sum_n \int_{nT/N}^{(n+1)T/N} dt$$

would break down. If the winding number is observed along a Minkowski vs. a Rindler trajectory, the degree of quantum coherence, and $P(...)$ will vary. The only way to make equation (17) generally covariant appears to be for $P\left(m-n \in \mathcal{C}_2\right)$ to be a unity operator in the winding numbers basis, which is equivalent to assuming $\theta = 0$.

Note that, since we are describing the vacuum rather than correlators, axiomatic field theory results regarding the equivalence of thermal and accelerated dynamics [52, 53] need not apply. In fact, symmetry arguments can be used to understand how non-perturbative vacuum degeneracies change the Unruh state w.r.t. a thermal state [29] (note that the θ state there is not a QCD one but a generic zero mode condensate). The result of [29] and the qualitative discussion here would imply the accelerated effective θ is not the same of the effective θ at finite temperature.

To get a physical feeling of what is going on here, consider the case of accelerated photon "Bremsstrahlung" emission, or the famous $p \to n e^+ \nu$ decay examined in [15]. As [15] makes the case, the "interpretation" in inertial and comoving frames is different (in one case the decay is a quantum reaction to a semiclassical field, in the other it is interaction with the Unruh bath) the decay matrix elements calculated in both cases will be the same. Neutrino oscillations somewhat complicate this last point, and there is no consensus to resolve this[5].

The equivalent Gedankenexperiment would have us compare the EDM measured on an accelerated neutron, interpreted by a Minkowski observer (who sees the effect of acceleration) and the comoving observer (who sees a finite temperature θ). Because the causal structure of the two observers are different, so the EDM operator sampled by Eq. (11) will contain a different combination of winding numbers and θ. In fact, the violation of the equivalence principle at finite temperature, long known and recently calculated in [20], makes it quite likely from thermal considerations alone. Hence, a Minkowski observer and a comoving observer will see different Z_n, and hence different effective θ's and topological susceptibilities. *The only way general covariance is to be a fundamental principle*, θ must be equal to zero, so the sum of different topological sectors appears incoherent in all frames.

This looks quite an abstract argument, but it has been known for some time in the context of lower dimensional Chern-Simons theories, known as the "finite temperature puzzle"

[5] In particular [54] argues that general covariance implies fundamental states mass be the usual mass-shell irreducible representations of the Lorentz group, while [55, 56] argues that on the contrary flavor states are fundamental, with the latter paper showing CP violation would generate an extra violation of general covariance in the mass basis. Finally [57] argues that a generally covariant effective action, taking condensates into account, is necessary to resolve the issue.

(see section 5.4 of [58] and references therein), where finite temperature breaking of Lorentz invariance introduces violations of large Gauge invariance order by order. In [58], the breaking of Lorentz invariance is physical, because the system is prepared at finite temperature. However, such a “finite temperature” state could never describe the ground state from the vantage point of a reference frame.

The above discussion can also be incorporated into the effective theory [66] because in such local models “local” and “large” Gauge transformations are separated, and the former handled perturbatively by adding a Wilson line U_∞ at infinity to each color charge. This way, in the perturbative limit, topological transformations completely decouple from local transformations and only the latter are relevant for Feynman diagram expansion. The problem is that once a horizon exists, U_∞ will span both “visible” and “invisible” fields. Hence, no effective theory, perturbative or otherwise, can be made from visible fields alone and topological and local transformations are inseparable. Naively we can speculate that since (as argued in [66]) local color-charged objects are forbidden by confinement, this means the θ angle should also go to zero. Note that the association between confinement and θ can be made using renormalization group arguments [22]. Of course, as argued in section 4.1 of [34, 35], there is a case for relating these arguments, with renormalization scheme independence taking the role of general covariance. If one wants, the symmetry of general covariance and the arguments in section II provide a theoretical justification for the infrared fixed point to be of the form of [22].

To summarize this section, we made a heuristic and speculative argument that the only way to restore general covariance is for θ to be equal to zero. The crux is that only for $\theta = 0$ the coefficient $c_m = e^{im\theta}$ does not change between a coherent and an incoherent sum. In the next section we shall show, using a toy model from condensed matter physics solved in the 80’s [27, 28], that explicitly tracing over degrees of freedom beyond the horizon will generally confirm this conclusion.

IV. THE PERIODIC POTENTIAL WELLS PICTURE OF θ

Since it is fashionable to illustrate the θ problem using the periodic potential well [3, 7, 8], let us try to get some additional understanding using such a toy model. The models are physically completely different, in the sense that the periodic potential well is “engineered”

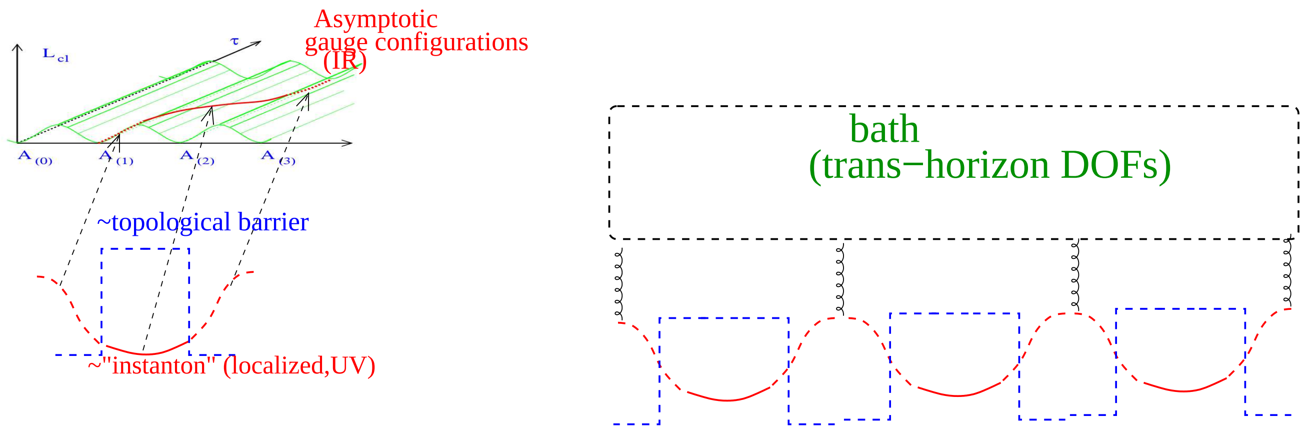

FIG. 2. A schematic illustration of the periodic well analogy of instantons, and the role that horizon radiation plays in them. The QCD solution is represented by a periodic potential, whose wavefunction is represented by a tunneling process (corresponding to a localized field configuration, the instanton) and a dominant "peak" (corresponding to the asymptotic field configuration, delocalized). Horizon radiation decoheres the IR peak without altering the localized tunneling. Top image from [7]

to have a θ-like parameter characterizing the eigenstates of the Hamiltonian and topological terms have no kinetic modes. Nevertheless, we think that a dictionary between these problems is useful enough to extend it to an open quantum system. Let us consider a system consisting of a quantum mechanical particle moving through potential wells satisfying the periodicity condition $V(x \pm a) = V(x)$, shown by the black lines in Figure (2). The wells, in this example, represent topological configurations (winding numbers) of a non-abelian gauge theory at the horizon. The Hamiltonian for this system is given by the kinetic and potential terms

$$\hat{H}_S = \frac{\hat{p}^2}{2M} + V(\hat{x}). \tag{19}$$

The solution for these kinds of systems is well known [67], and given in terms of the energy eigenstates

$$|\theta\rangle = \sum_{n\in\mathbb{Z}} \exp(in\theta) |n\rangle \,, \tag{20}$$

where $|n\rangle$ is state of the particle localized at the $n$the site (or winding number, in the Yang-Mills correspondence), and θ is a parameter that labels the simultaneous eigenstates of the Hamiltonian and the a-translation operator $T^\dagger(a)V(x)T(a) = V(x+a)$. The associated density matrix for such θ state is given by

$$\hat{\rho} = \sum_{m,n\in\mathbb{Z}} \exp(i(m-n)\theta) |m\rangle \langle n| . \tag{21}$$

If the potential barriers are tall with respect to the energy of the system, we can use the tight-binding approximation and obtain an approximation to the θ-state energy

$$E(\theta) = E_0 - 2\Delta \cos\theta \tag{22}$$

from the energy of the localized states $E_0 = \langle n| \hat{H} |n\rangle$ and the splitting energy between adjacent sites $\Delta = |\langle n \pm 1| \hat{H} |n\rangle |$. The ground-state wavefunction $\psi_\theta(x) = \langle x|\theta\rangle$ takes the Bloch form

$$\psi_\theta(x) = e^{ikx} u_k(x), \quad k = \theta/a, \tag{23}$$

where u_k a periodic function with period a and $k = \theta/a$ is the "Bloch momentum" associated with the periodic potential.

We now need to add to the toy model of the θ vacuum a toy model of the thermal bath due to the horizon, and the interactions of the system with the bath. Following [27, 28], let us add a "bath" consisting of an infinite number harmonic oscillators interacting with the particle in the periodic potential as

$$\hat{H} = \hat{H}_S + \hat{H}_B \tag{24}$$

where $\hat{H}_B$ is the Hamiltonian for the simple harmonic oscillators of the thermal bath together with a linear interaction term between the bath and the particle, given by

$$\hat{H}_B = \sum_{n\in\mathbb{Z}} \left(\frac{1}{2m_n}\hat{p}_n^2 + \frac{1}{2}m_n\omega_n^2 \left(\hat{q}_n + \frac{C_n\hat{x}}{m_n\omega_n^2} \right)^2 \right) . \tag{25}$$

For convenience, we define the pure interacting Hamiltonian by $\hat{H}_I = \sum_{n\in\mathbb{Z}} C_n\hat{q}_n\hat{x}$.

In the interaction picture, the equation of motion for the reduced density matrix of the particle is given by

$$\frac{d\hat{\rho}_S}{dt} = -i \,\mathrm{Tr}_B \left[\hat{H}_I(t), \hat{\rho}_{S+B}(t) \right] . \tag{26}$$

When the couplings C_n are equal to zero one recovers equation (20) for the ground state of the particle, since $\hat{\rho}_S$ and $\hat{\rho}_B$ decouple and evolve, respectively, under $\hat{H}_S$ and $\hat{H}_B$ separately. If C_n are not equal to zero, the toy model can only be solved in very particular cases and under certain approximations. Analytical solutions can be found for simple potential profiles, such as the double well potential [27], and the (biased) periodic potential [28]. These solutions hinges on the fact that the interaction of the Brownian particle couples to the thermal bath frequencies via the spectral density function

$$J(\omega) = \frac{\pi}{2} \sum_n \frac{C_n^2}{m_n \omega_n} \delta(\omega - \omega_n), \quad \omega > 0. \tag{27}$$

Exact solutions can be found by limiting the particle's response to the particular case of a thermal bath with ohmic profile and a high-frequency cutoff $\Lambda > 0$, given by

$$J(\omega) = \eta\omega, \quad 0 < \omega < \Lambda. \tag{28}$$

This means that for these kinds of toy models, the dissipative dynamics induced by environment interactions mostly involves the low-frequency modes of the environment. In the Yang-Mills correspondence, that means topological information (associated with infrared degrees of freedom) decoheres, while localized field configurations, such as instantons, would remain intact.

In the particular case of a double-well potential [27], for example, we can map the high-frequency dynamics of the model to an effective two-level system. The degrees of freedom in this case are the symmetric and anti-symmetric combinations of the damped harmonic oscillator states centered at the bottom of each potential well

$$\Psi_\pm(x, \{q_n\}) = \frac{1}{\sqrt{2}}[\Psi_R(x, \{q_n\}) \pm \Psi_L(x, \{q_n\})]. \tag{29}$$

Generally one can separate the associated density matrix into a coherent and an incoherent part, where the coherent part can be rotated as the projection part within a certain direction $\hat{S}_z$ for the two-level system above [27]

$$\hat{\rho}_S = \sum_n A_n |n\rangle \langle n| + P(t) \sum_{m \neq n} B_{mn} |m\rangle \langle n| . \tag{30}$$

Since H_I does not commute with H_S, the equation of motion for the reduced density matrix of the system becomes an initial value problem, with all time dependence can put into $P(t)$ (which could be a matrix in the "winding number" basis). It can be shown then that $P(t)$ obeys the damped harmonic oscillator equation

$$\ddot{P} + T^{-1}\dot{P} + \Delta^2 P = 0, \tag{31}$$

with T^{-1} and Δ functions of the original parameters.

The exact form of the parameters can be an involved calculation, but their dependence on the length scales of the problem is universal. As shown in [27], while Δ is dominated by short-range physics (in our context this means instantons equation (2), hence $\Delta^{-1} \sim \rho$), the damping time-scale T depends on the softest scale connected to the size of the reservoir (in our context, this is the horizon radius). The latter can be removed from the system by "adiabatic renormalization", where all dimensionful parameters are presented as a ratio of the cutoff frequency. In the limit we want to reach, where the cutoff frequency dependence is very small, one needs $\alpha \to 0$ (defined in chapter 9 of [27]). In fact, if one compares the scales in Eq. (10) to the running of Eq. (31) one sees that the applicability of the Unruh EFT in an instanton context is equivalent to the theory being near the $\alpha \to 1$ infrared fixed point. This means in a theory having a generally covariant infrared limit, the effective P ($\sim \theta$) will decay to zero on a time-scale set and inversely proportional to the lowest frequency.

To compare our models to a real quantum field theory vacuum, either Yang-Mills or its effective theory implementation, we need to be a bit more rigorous. The path integral formalism can be connected to density matrix language via [69], where the similarity between QCD and the periodic potential setup is more clear.

The density matrix is related to the partition function via

$$\langle x|\,\rho\,|x'\rangle = \frac{1}{Z}\int \mathcal{D}\phi(t=t_0,x) < \phi|\Psi >< \Psi|\phi > \tag{32}$$

$$= \frac{1}{Z}\int_{\tau=-\infty}^{\tau=\infty}\int \left[\mathcal{D}\phi, \mathcal{D}y(\tau)\,\mathcal{D}y'(\tau)\right] e^{-iS(\phi y,y')}\cdot\delta\left[y(0^+)-x'\right]\,\delta\left[y'(0^-)-x\right]\ ,$$

where $|\Psi>$ is some eigenstate basis and ψ are field configurations to be integrated over.

Let us now consider expand Ψ in terms of Ψ_n, the wave functional corresponding to equation (16). In both the QCD case and the periodic potential case,

$$\hat{\rho} \quad \underbrace{\to \int \mathcal{D}\phi}_{\lim_{x\to 0^+-O^-}\mathcal{D}\phi(t=t_0,x)} \quad < \phi|\Psi_n >< \Psi_m|\phi > \exp\left[\theta(n-m)\right] \tag{33}$$

where Ψ_n are Eigenstates of the Hamiltonian with, additionally, boundary condition set by the winding number (For the periodic potential, it is a fixed number of turns around N wells, for the QCD case it is a given winding number).

Now, the previous section has argued that background independence implies independence from horizon terms. In this section, the tracing over the horizon terms was argued to be equivalent to tracing over the "infrared" degrees of freedom $k \sim (0^+ - 0^-)^{-1}$ in equation (32).

Our mechanism adds to the system (S) a thermal bath (B) and in both cases

$$\hat{\rho}_{QCD} = \mathrm{Tr}_B \hat{\rho}_{S+B}, \quad \hat{\rho}_{S+B} \sim c_{kk'} \left|k\right\rangle \left\langle k'\right|, \quad k_B \sim (0^+ - 0^-)^{-1} \ll k_S, \tag{34}$$

where k refers to momentum and $c_{kk'}$ encode all structure of the vacuum. The arguments of the previous section make it clear that for background independence to be achieved, the tracing of the bath degrees of freedom must make no difference to the effective Lagrangian.

Let us, as above, refer to Ψ_m as physical states in flat space of winding number m and Φ_m as Ψ_m with the infrared limit decohered (all dependence of k_B removed, and $k_S \sim \rho^{-1}$ of equation (2) unchanged). While Ψ_m and Ψ_n are not Eigenstates of the Hamiltonian, they do form an orthogonal set. In contrast, the decoherence of k_{bath} means that

$$\left\langle \Phi_m | \Phi_n \right\rangle \sim Tr_k c_{k \sim k_{Bath}} \neq 0, \tag{35}$$

with the fact that instanton states have diverging infrared Fourier coefficients and infinite occupation numbers ensuring a finite overlap even if the ultraviolet part of the instanton is unchanged.

Thus, the decohered density matrix $\hat{\rho}_D$, will not be diagonal in the $|n\rangle$ basis. Shifting them to a diagonal basis $|\Psi'\rangle$ will involve a generally complex rotation in phase space, whose jth eigenvalue can be represented as complex numbers $r_j e^{i\alpha_j}$. Putting these together we get, from equation (33), (34) and (35),

$$\hat{\rho} \propto \sum_{n,m,j} e^{i(\theta n + \sum_j \alpha_j)} \mathcal{D}\phi \left\langle \phi | \Psi'_n \right\rangle \left\langle \Psi'_n | \phi \right\rangle \tag{36}$$

$|\Psi'_n\rangle$ and $|\Psi_n\rangle$ are distinguishable only via the IR part of the partition function and an unobservable renormalization parameter given by r_j. The phase, however, was rotated by an "infinite" number of angles α_j, and hence can be safely assumed to decohere.

Considering the density matrix to evolve dynamically, we would conclude the infrared form of the dynamics of this decoherence will be controlled by a damped equation of the type equation (31), with the damping time of the order of the horizon parameter (the cosmological constant in a de-Sitter space, acceleration for Rindler space and so on). This "coincidence" can actually be seen from the form of equation (8) and the argument, made in [48], that general covariance of the time parameter requires correlated fluctuation ("many outcomes for an initial condition") and dissipation ("one outcome for many initial conditions") that transform covariantly. Topological quantum fluctuations that fix the θ term are "the softest scale", set at the horizon scale. They must be matched by an equally soft dissipation, making such "slow" dynamics is unobservable. This implies θ damping is irrelevant for any measurement $k \sim k_{Bath}$.

Finally, the interpretation of decoherence as the effect of unmeasured degrees of freedom can connect this section to the previous section III. Because the horizon hides the information allowing the winding number to be instantaneously measured, the density matrix in the basis of $|n\rangle$ becomes

$$e^{in\theta} |n\rangle \langle m| \delta_{mn} \to \sum_{m',n'} e^{in\theta} P(n',n) P(m',m) |n'\rangle \langle m'| \delta_{m'n'}, \tag{37}$$

where $P(n', n)$ represent *semi-classical* probabilities of winding numbers disappearing behind a horizon. Rotating bases will add complex coefficients, parametrized by the $r_j e^{i\alpha_j}$ of the previous equation equation (36). The r_j's reflect the normalization of the field strength in response to the horizon changing the configuration space. It should be unobservable for frequencies higher than the horizon radius. The phases α_j however remains and rotates θ chaotically for each momentum mode, equivalent to decoherence (see a good discussion of "coherent" and "chaotic" sources here [70]).

In conclusion, we note that a similar mechanism to what we suggest was proposed in the 80's as an origin of *spontaneously broken* symmetries [26]. The success of the Higgs model and chiral symmetry breaking invalidated further development of this idea, but for *anomalously broken* symmetries it has potential.

V. DISCUSSION

The previous section IV makes it clear that any tracing over of degrees of freedom outside a causal horizon generally results in the asymptotic relaxation of a density matrix of a theory with connected topological sectors to a density matrix where they are connected incoherently. This means that equation (6) is therefore updated to

$$\sum_{m,n} e^{i(m-n)\theta} |m\rangle \langle n| \to \sum_{n} c_n |n\rangle \langle n| \tag{38}$$

where c_n doesn't depend on θ and is put to unity via field strength renormalization. $\theta = 0$ in this regard is "special" because coherent and incoherent summation is indistinguishable.

Note that instantons, as any "UV" degrees of freedom, are preserved (as they should be if there is any hope of the idea presented in this paper to be correct, since they are necessary for phenomenology (the η' mass issue) [3, 5, 6], seen on the lattice [10] and hinted at in experiment [11, 12]).

This can be seen as a consequence of local Lorentz invariance, since, unlike the local Euclidean signature (where Eq. (12) holds), the infrared limit is determined by the spacetime global horizon structure as well as the local quantum field theory correlations. Thus, Eq. (38) only modifies the infrared sector of the theory, where the infrared scale could be affected by the curvature as per Eq. (10), while local instanton fluctuations and topological susceptibility should therefore not be different from analytical continuations from Euclidean space used in [3, 10–12]. This also means that theories where CP violation is due to a condensate, such as the electroweak sector of the standard model, do not suffer from the ambiguities discussed in this paper, since there the hyerarchy defined in Eq. (10) is well defined, with the equivalent of μ in Eq. (10) being absent and the condensate gap energy taking the place of ρ^{-1}. In that situation, to check general covariance one just has to take interactions with the condensate into account properly in both the inertial and the co-moving frame, a non-trivial procedure discussed in [56, 57].

If a universe is "prepared" with a finite θ and a horizon, the timescale for the θ to relax to its effectively zero value can be given by the methods of section IV. The asymptotic density matrix picture is also covariant under general coordinate transformations, at least for non-inertial transformations having an acceleration smaller than the horizon size, according to section III.

The mechanism described here could be valid, however, even if general covariance is broken by quantum effects. For instance, it could arise dynamically in a cosmological scenario. In a "long" high cosmological constant phase in the early universe, natural in the slow-roll inflation scenario [68], the topological sector of the QCD vacuum would have time to decohere if this phase is also deconfined (either due to temperature of a dS constant above T_c). Afterwards, the decohered $\theta = 0$ state would be "locked", since long wavelength colored perturbations which cause tunneling would be below the confinement mass gap. Such a dynamical non-perturbative QFT problem is of course well outside this work's scope, but qualitatively this scenario might be implementable.

An obvious drawback of the explanation presented here is its lack of falsifiability. Unlike with more traditional mechanisms of the resolution of the θ problem, we do not predict new particles, and is founded on a fundamental modification of the quantum field theory vacuum state [17] that in turn generally does not produce verifiable predictions. However, one might be able to experimentally test our model with analogue systems. Provided a fluid with internal symmetries exhibiting topological properties similar to Yang-Mills theory is found (for example a fluid with polarization [71]), one might be able to put this fluid in Minkowski and de-Sitter configurations [72]. An effective θ term could then appear in the former and disappear in the latter, decohered by Hawking sound. The violation of unitarity that we believe is implicit in the structure of spacetime would here arise out of the fluid dynamics limit. In this regard,the impossibility of relativistic local equilibrium in "polymeric fluids" where spin and vorticity are not parallel at thermal equilibrium, pointed out in [71] is a useful analogy, as in this system local equilibrium is impossible because of topologically constrained local Goldstone modes. If the reasoning in this paper is broadly correct, we would predict that no topological insulators are possible which are also perfect liquids. As far as we know this is indeed the case to date.

Since the effect suggested here involves integrating out infrared degrees of freedom, there might be a parallel to non-perturbative renormalization group approaches, investigated elsewhere [22] and motivated by the analogy between renormalization group scheme independence and general covariance [34, 35]. In both cases, the approach does not rely on new physics, but rather on making sure the underlying theory's local physics is insensitive to unobserved infrared perturbations.

In conclusion, we have heuristically discussed the topology of a quantum field theory in curved space. We have argued that the presence of causal horizons, or, equivalently, the expectation of general covariance of the theory at the quantum level, generally require the asymptotic wave functional of the theory to have different topological sectors incoherently summed. This is just what one needs to require the θ angle of the theory to vanish, in accordance with observations. While we are very far from proving that this is the correct explanation of the strong CP problem, it is a suggestion that merits further development.

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Appendix A: Toy models

In this section we shall describe toy models which include a boundary term and the possibility to calculate the density matrix and the various forms of entropy explicitly. They are limited in that they do not allow us to fully understand how topological terms are generated by the dynamics, since for such 1+1 theories topology arises out of boundary conditions rather than dynamics, and is decoupled from the bulk.

However, dynamical coupling between bulk and boundary can be put in by hand in an adiabatic way which generates no entropy. In this case, it becomes clear that the "geometric" and "statistical" definitions of entropy do not match, thereby giving an example of the tension between general covariance and topological terms.

1. Quantum particle on a ring with a magnetic field

We will base ourselves on [49] and Gregory Moore's lectures on Chen-Simons theory [73].

We consider the quantum mechanical system of a particle in a ring S^1, characterized by the Euclidean action

$$iS = \int dt \left(\frac{1}{2}I\dot{\phi}^2 + \mathcal{B}\dot{\phi}\right) \longrightarrow -S_E = -\int d\tau \left(\frac{1}{2}I\dot{\phi}^2 - i\mathcal{B}\dot{\phi}\right) \tag{A1}$$

for the angular variable ϕ. The periodic-boundary conditions characterizing the system are given by $\phi(\tau+\beta) = \phi(\tau)$ and $\phi \sim \phi + 2\pi n$. This theory then represents a $(0+1)$ dimensional field theory $\phi : S^1 \to S^1$, which is a topological theory since $\pi_1(S^1) = \mathbb{Z}$. The topological charge is represented by how many times around the circle the particle runs on the course of one period β – counted by the $i\mathcal{B}\dot{\phi}$ term on the action.

We use this model as a toy example for the Laflamme procedure [49] of obtaining a description for the density matrix of a (free) field theory in an accelerated frame. This procedure is described as follows: The comoving-time of an accelerated observer is periodic in Euclidean signature, due to the natural invariance of Minkowski spacetime under boosts with imaginary-boost parameter. That means the accelerated observer's proper-time is represented in Euclidean signature by the angular variable in a polar-coordinate representation

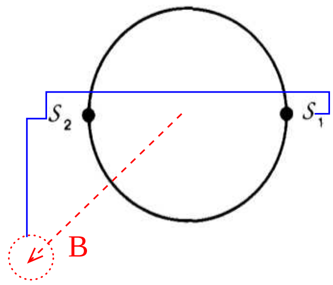

FIG. 3. Figure from [49], representing the Euclidean section of spacetime representing an accelerated observer. $\mathcal{M}_\pm$ are the pieces representing the right and left wedges respectively, and $\mathcal{S}_k$ the boundaries associated with the presence of causal horizons. The circumference of the circle is β, and $\tau_{\mathcal{S}_2} = \tau_{\mathcal{S}_1} + \beta/2$.

We also include the magnetic field term $\mathcal{B}$ and the boundary regulator described at the end of this section, leading to Eq. (A42). These are represented by the circuit connecting $\mathcal{B}$ to $S_{1,2}$

of spacetime. The presence of causal horizons for the accelerated observer is reflected on the fact that you can't cover the circle with a single coordinate chart: That can be seen from the transformation $\tau \to \tau + \pi$ on the circle, which maps the points $(\tau + \pi, r) \to (\tau, -r)$. Under reverse Wick rotation $\tau \to i\tau$, that transformation maps a Rindler wedge into its causal complement (that means mapping the right wedge into left wedge and vice versa).

Now we can describe the causally disconnected regions of Minkowski space by considering a division of the circle into two sections, with the common boundary (shown in figure (3)) representing the causal horizon structure of the accelerated frame. To obtain a description of physics from the point of view of the accelerated observer, we integrate out the contributions from the inaccessible region.

In mathematical terms, that corresponds to describing the theory on $\mathcal{M}_+$ by the path-integral summing over the complementary region $\mathcal{M}_-$: Let $\Psi_\pm(\tilde{\phi}_1, \tilde{\phi}_2)$ be the partition function for the fields on $\mathcal{M}_\pm$ satisfying the boundary conditions $\phi(\mathcal{S}_1) = \tilde{\phi}_1$ and $\phi(\mathcal{S}_2) = \tilde{\phi}_2$. The path integral we are interested in evaluating is, then, given by

$$\rho(\tilde{\phi}_1, \tilde{\phi}_1') = \int \mathcal{D}\tilde{\phi}_2 \Psi_+(\tilde{\phi}_1, \tilde{\phi}_2)\Psi_-(\tilde{\phi}_1', \tilde{\phi}_2), \tag{A2}$$

where ρ is the density matrix associated with the states $\tilde{\phi}_1, \tilde{\phi}_1' \in \mathcal{M}_+$ accessible to the observer.

In the context of quantum field theory, figure (3) is only a representation of the true

Euclidean section of spacetime, whose spatial dimensions compose the correct picture in higher dimensions. However, for free theories such as the scalar field

$$S = \frac{1}{2} \int d^{D+1}x \sqrt{g} \left(g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi + m^2 \Phi^2 \right), \tag{A3}$$

we can make use of the Fourier transform and study a single mode of the field [49]

$$S_E^\lambda = \frac{1}{2} \int d\tau \left(\dot{\phi}_\lambda^2 + \lambda \phi_\lambda^2 \right), \tag{A4}$$

which reduces to a quantum harmonic oscillator in a ring. Given the close connection with Euclidean-signature quantum mechanics on a ring with the physics of accelerated observers, we use this analogy to study the topological system from equation (A1) in an accelerated frame.

Our target is therefore calculating the density matrix in equation (A2) for the model described by the action (A1) for the particle in the ring. First, then, we calculate the partition functions

$$\Psi_\pm(\tilde{\phi}_1, \tilde{\phi}_2) = \int_{\mathcal{C}[\mathcal{M}_\pm]} \mathcal{D}\phi \, e^{-S_E} \tag{A5}$$

for the $(0+1)$-dimensional field ϕ with corresponding boundary conditions

$$\mathcal{C}[\mathcal{M}_\pm] \equiv \{\phi(\mathcal{S}_1) = \tilde{\phi}_1 \text{ and } \phi(\mathcal{S}_2) = \tilde{\phi}_2\}. \tag{A6}$$

Being a free theory, the path-integral can be solved by means of the solutions to the Euclidean equations of motion

$$\frac{d^2}{d\tau^2} \phi_{cl}(\tau) = 0 \tag{A7}$$

that extremizes the action. By virtue of the topological nature of the theory, the solutions of this classical equation of motion are parametrized by an integer $n \in \mathbb{Z}$ that counts how many times the particle winds around the circle for given boundary conditions. Let $\phi_{cl}^\pm$ be the classical solutions to the Euclidean equations of motion (A7)

$$\begin{aligned} \phi_{cl}^+(\tau) &= \tilde{\phi}_1 + \frac{2}{\beta} \left(\tilde{\phi}_2 - \tilde{\phi}_1 + 2\pi n_+ \right) \tau \\ \phi_{cl}^-(\tau) &= 2\tilde{\phi}_2 - \tilde{\phi}_1 - 2\pi n_- + \frac{2}{\beta} \left(\tilde{\phi}_1 - \tilde{\phi}_2 + 2\pi n_- \right) \tau \end{aligned} \tag{A8}$$

satisfying the boundary conditions (A6). The integers $n_\pm \in \mathbb{Z}$ are the winding numbers of the field configurations, which appear in the combination $2\pi n_\pm$ since $\phi(\tau)$ lives on the circle S^1. The field configurations

$$\exp\left\{ i\phi_{cl}^\pm(\tau) \right\} \in S^1 \tag{A9}$$

are called instantons, and they interpolate between the classical, τ-independent, zero-action "vacuum" solutions $\tilde{\phi}^{vac}_{cl}(\tau) = \tilde{\phi}_{1/2}$, on each $\mathcal{M}_\pm$ manifold.

The angular velocity of the classical solutions take the form

$$\dot{\phi}^{\pm}_{cl}(\tau) = \frac{4\pi}{\beta}\left(n_\pm \pm \frac{\tilde{\phi}_2 - \tilde{\phi}_1}{2\pi}\right), \tag{A10}$$

and the classical action for each instanton solution is given by the expression

$$\begin{aligned} -S^{\pm}_E &= -\int_{\mathcal{C}[\mathcal{M}_\pm]} d\tau \left\{\frac{1}{2}I\left(\frac{4\pi}{\beta}\right)^2\left(n_\pm \pm \frac{\tilde{\phi}_2 - \tilde{\phi}_1}{2\pi}\right)^2 - \frac{4\pi i\mathcal{B}}{\beta}\left(n_\pm \pm \frac{\tilde{\phi}_2 - \tilde{\phi}_1}{2\pi}\right)\right\} \\ &= -\frac{4\pi^2 I}{\beta}\left(n_\pm \pm \frac{\tilde{\phi}_2 - \tilde{\phi}_1}{2\pi}\right)^2 + 2\pi i\mathcal{B}\left(n_\pm \pm \frac{\tilde{\phi}_2 - \tilde{\phi}_1}{2\pi}\right). \end{aligned} \tag{A11}$$

Now, the partition functions $\Psi_\pm(\tilde{\phi}_1, \tilde{\phi}_2)$ are given by the sum over all possible field configurations $\phi^\pm$, and all such field configurations falls under a particular equivalence class determined by its winding number $n[\phi^\pm] \in \mathbb{Z}$. That means we must sum over all classical instanton solutions, taken to be the representatives of each winding-sector, as well as all possible fluctuations

$$\delta\phi^\pm = \phi - \phi^\pm_{cl} \tag{A12}$$

around these classical backgrounds. The fluctuating fields $\delta\phi^\pm$ are topologically trivial, meaning $n[\delta\phi^\pm] = 0$, and satisfy the boundary conditions

$$\delta\phi^\pm(\mathcal{S}_i) = 0. \tag{A13}$$

Since we are expanding around the solutions to the classical equations of motion, the total action is merely the sum

$$S[\phi] = S[\phi^\pm_{cl}] + S[\delta\phi^\pm], \tag{A14}$$

and therefore we obtain the factorization

$$\Psi_\pm(\tilde{\phi}_1, \tilde{\phi}_2) = \int \mathcal{D}\phi\, e^{-S_E} \sim Z_{\delta\phi^\pm} \sum_{n_\pm \in \mathbb{Z}} e^{-S_E[\phi^\pm_{cl}]}, \tag{A15}$$

with $Z_{\delta\phi^\pm}$ the (topologically trivial) partition function for the fluctuating field $\delta\phi^\pm$

$$Z_{\delta\phi^\pm} = \int_{\delta\phi^\pm(\mathcal{S}_1)=0}^{\delta\phi^\pm(\mathcal{S}_2)=0} \mathcal{D}\delta\phi^\pm e^{-S[\delta\phi^\pm]}. \tag{A16}$$

Since we are interested in the topological properties of the reduced density matrix, and $Z_{\delta\phi^{\pm}}$ is only part of the normalization of $\Psi_{\pm}$, we only deal with the instanton sum of equation (A15). Substituting the expression for the classical action of each instanton solution (A11), we get

$$\Psi_{\pm}(\tilde{\phi}_1, \tilde{\phi}_2) \sim \sum_{n_{\pm}} \exp\left\{ -\frac{4\pi^2 I}{\beta}\left(n_{\pm} \pm \frac{\tilde{\phi}_2 - \tilde{\phi}_1}{2\pi}\right)^2 + 2\pi i \mathcal{B}\left(n_{\pm} \pm \frac{\tilde{\phi}_2 - \tilde{\phi}_1}{2\pi}\right)\right\}. \qquad \text{(A17)}$$

Now we are in a position to evaluate the reduced density matrix (A2) obtained by integrating out all $\mathcal{M}_{-}$ fields. An elegant approach is to make use of the representation of the partition functions $\Psi_{\pm}$ in terms of the Riemann's theta function

$$\mathcal{R}\begin{bmatrix} n_0 \\ z_0 \end{bmatrix}(z, \tau) = \sum_{n \in \mathbb{Z}} \exp\left(i\pi\tau(n+n_0)^2 + 2\pi i(n+n_0)(z+z_0)\right). \qquad \text{(A18)}$$

Writing $\Delta\tilde{\omega} = (\phi_2 - \phi_1)/2\pi$, we get the compact expression for the wedge partition functions

$$\Psi_{\pm}(\tilde{\phi}_1, \tilde{\phi}_2) \sim \mathcal{R}\begin{bmatrix} \pm\Delta\tilde{\omega} \\ \mathcal{B} \end{bmatrix}\left(0, \frac{4\pi i I}{\beta}\right). \qquad \text{(A19)}$$

Using the transformation property of the $\mathcal{R}$ function under modular transformations

$$\mathcal{R}\begin{bmatrix} n_0 \\ z_0 \end{bmatrix}(z, \tau) = (-i\tau)^{-1/2} e^{2\pi i z_0 n_0} e^{-i\pi z^2/\tau} \mathcal{R}\begin{bmatrix} z_0 \\ -n_0 \end{bmatrix}\left(\frac{-z}{\tau}, \frac{-1}{\tau}\right), \qquad \text{(A20)}$$

we obtain an alternative and equivalent representation

$$\mathcal{R}\begin{bmatrix} \pm\Delta\tilde{\omega} \\ \mathcal{B} \end{bmatrix}\left(0, \frac{4\pi i I}{\beta}\right) = \left(\frac{4\pi I}{\beta}\right)^{-1/2} e^{\mp 2\pi i \mathcal{B}\Delta\tilde{\omega}} \mathcal{R}\begin{bmatrix} \mathcal{B} \\ \mp\Delta\tilde{\omega} \end{bmatrix}\left(0, -\frac{\beta}{4\pi i I}\right). \qquad \text{(A21)}$$

Expanding this expression in terms of the sum representation of $\mathcal{R}$ we end up with the relationship

$$\Psi_{\pm}(\tilde{\phi}_1, \tilde{\phi}_2) \sim \sum_{n_{\pm}} \exp\left\{ -\frac{4\pi^2 I}{\beta}(n_{\pm} \pm \Delta\tilde{\omega})^2 + 2\pi i \mathcal{B}(n_{\pm} \pm \Delta\tilde{\omega})\right\} =$$

$$= \left(\frac{4\pi I}{\beta}\right)^{-1/2} \sum_{n_{\pm}} \exp\left\{ -\frac{\beta}{4I}(n_{\pm} - \mathcal{B})^2 \pm 2\pi i n_{\pm}\Delta\tilde{\omega}\right\}. \qquad \text{(A22)}$$

Now we can proceed to integrate out $\tilde{\phi}_2$ using the new representation of $\Psi_\pm$. In this representation, the reduced density matrix is given by the expression

$$\rho(\tilde{\phi}_1, \tilde{\phi}_1') \sim \int_0^{2\pi} d\tilde{\phi}_2 \Psi_+(\tilde{\phi}_1, \tilde{\phi}_2)\Psi_-(\tilde{\phi}_1', \tilde{\phi}_2) \sim$$
$$\sim \left(\frac{\beta}{4\pi I}\right) \int_0^{2\pi} d\tilde{\phi}_2 \sum_{n_\pm \in \mathbb{Z}} \exp\left\{-\frac{\beta}{4I}(n_+ - \mathcal{B})^2 + i n_+ \left(\tilde{\phi}_2 - \tilde{\phi}_1\right)\right\} \times$$
$$\times \exp\left\{-\frac{\beta}{4I}(n_- - \mathcal{B})^2 - i n_- \left(\tilde{\phi}_2 - \tilde{\phi}_1'\right)\right\}, \quad \text{(A23)}$$

and by using the integral representation of the Kronecker delta

$$\delta_{mn} = \frac{1}{2\pi} \int_0^{2\pi} d\tilde{\phi}_2 \, e^{i(m-n)\tilde{\phi}_2} \quad \text{(A24)}$$

we obtain

$$\rho(\tilde{\phi}_1, \tilde{\phi}_1') \sim \left(\frac{\beta}{2I}\right) \sum_{n \in \mathbb{Z}} \exp\left\{-\frac{\beta}{2I}(n - \mathcal{B})^2 - i n \left(\tilde{\phi}_1 - \tilde{\phi}_1'\right)\right\}. \quad \text{(A25)}$$

Once again we normalizing the density matrix by demanding unity trace

$$\mathrm{Tr}\, \rho = \int_0^{2\pi} d\tilde{\phi}\, \rho(\tilde{\phi}, \tilde{\phi}) = 1, \quad \text{(A26)}$$

and that yields the expression for the matrix elements of the density matrix

$$\rho(\tilde{\phi}_1, \tilde{\phi}_1') = \frac{1}{2\pi Z} \sum_{n \in \mathbb{Z}} \exp\left\{-\frac{\beta}{2I}(n - \mathcal{B})^2 - i n \left(\tilde{\phi}_1 - \tilde{\phi}_1'\right)\right\}, \quad \text{(A27)}$$

$$Z = \sum_{n \in \mathbb{Z}} \exp\left\{-\frac{\beta}{2I}(n - \mathcal{B})^2\right\}. \quad \text{(A28)}$$

We could have obtained this result more directly from the expression of the Euclidean propagator

$$\langle \tilde{\phi}_1' | e^{-\beta H} | \tilde{\phi}_1 \rangle = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \exp\left\{-\frac{\beta}{2I}(n - \mathcal{B})^2 - i n (\tilde{\phi}_1 - \tilde{\phi}_1')\right\}. \quad \text{(A29)}$$

This propagator can be obtained from the system Hamiltonian

$$H = \frac{1}{2I}(-i\partial_\phi - \mathcal{B})^2 \quad \text{(A30)}$$

whose spectrum consists in the eigenvectors $\{|m\rangle, m \in \mathbb{Z}\}$, given by

$$\langle \phi | m \rangle = \Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}. \quad \text{(A31)}$$

The procedure of integrating out half of the spacetime circle corresponds to the process of separating $\beta = \frac{\beta}{2} + \frac{\beta}{2}$ and introducing a complete set of states in between

$$\langle \tilde{\phi}_1' | e^{-\beta H} | \tilde{\phi}_1 \rangle = \int_0^{2\pi} d\tilde{\phi}_2 \, \langle \tilde{\phi}_1' | e^{-\frac{1}{2}\beta H} | \tilde{\phi}_2 \rangle \langle \tilde{\phi}_2 | e^{-\frac{1}{2}\beta H} | \tilde{\phi}_1 \rangle$$
$$= \int_0^{2\pi} d\tilde{\phi}_2 \, \Psi_+(\tilde{\phi}_1, \tilde{\phi}_2) \Psi_-(\tilde{\phi}_1', \tilde{\phi}_2) \sim \rho(\tilde{\phi}_1, \tilde{\phi}_1'). \quad \text{(A32)}$$

Following this, we proceed to calculate thermodynamical quantities from the model. First, we note that we can write the identity in two equivalent ways

$$\int_0^{2\pi} d\phi |\phi\rangle\langle\phi| = \sum_{m\in\mathbb{Z}} |m\rangle\langle m| = \mathbb{I}. \quad \text{(A33)}$$

For convenience, we note that (A27) can be rewritten in terms of the eigenfunctions $\Phi_m(\phi)$ of the Hamiltonian

$$\langle \phi' | \rho | \phi \rangle = \frac{1}{2\pi Z} \sum_{n\in\mathbb{Z}} \exp\left\{ -\frac{\beta}{2I} (n - \mathcal{B})^2 - in(\phi - \phi') \right\} = \frac{1}{Z} \sum_{n\in\mathbb{Z}} e^{-\beta E_n} \Phi_n(\phi) \Phi_n^*(\phi'). \quad \text{(A34)}$$

We can also find the matrix elements of the density matrix in the Hamiltonian eigenstate basis

$$\langle m | \rho | n \rangle = \frac{1}{Z} e^{-\beta E_m} \delta_{mn}. \quad \text{(A35)}$$

Note that in the basis of Hamiltonian eigenstates ρ is diagonal, and we find that

$$\rho = \frac{1}{Z} e^{-\beta H}, \quad \text{(A36)}$$

which indeed confirms that ρ represents a finite temperature system of excitations of H. The energy and entropy of the system are obtained by using the diagonal representation of ρ in the basis of eigenstates of H, and are given by

$$E \equiv \langle H \rangle \equiv \operatorname{Tr} \rho H = \frac{1}{Z} \sum_{m\in\mathbb{Z}} E_m e^{-\beta E_m} \quad \text{(A37)}$$

in this picture we can calculate the Von Neumann entropy and recover the usual thermodynamic relation

$$S \equiv -\langle \ln \rho \rangle \equiv -\operatorname{Tr} \rho \ln \rho = \beta \langle H \rangle + \ln Z. \quad \text{(A38)}$$

Differentiating the entropy with respect to the internal energy we get the expected relation

$$\frac{\partial S}{\partial E} = \beta, \quad \text{(A39)}$$

Confirming that indeed $\beta^{-1} = T$ represents the temperature of the system. Taking the entropy to be an explicit function of both β and $\mathcal{B}$ we obtain the usual thermodynamical expression

$$dS = \left(\frac{\partial S}{\partial \beta}\right)_{\mathcal{B}} d\beta + \left(\frac{dS}{d\mathcal{B}}\right)_{\beta} d\mathcal{B} = \beta\, dE + \left(\frac{\partial S}{\partial \mathcal{B}}\right)_{E} d\mathcal{B}. \tag{A40}$$

Observing equations (A27), it would appear that that the $\mathcal{B}$-term on the density matrix doesn't depend on the temperature of the field $\beta \sim T^{-1}$. That means the topological sector, in this particular case, does not seem to be sensitive to the Hawking-Unruh temperature associated with the Euclidean-periodicity in proper-time. In this toy-model, the magnetic field $\mathcal{B}$ takes the role of the theta-parameter in other scenarios, such as Chern-Simons theory and the CP-violating θQCD_4 scenario.

On the other hand, in real 3+1 field theory $\mathcal{B}$ is determined dynamically from the theory. While we cannot reproduce this effect explicitly, we can "mock it up" and study its consequences in the following way: let us hook up a system prepared as in Fig. 3, a quantum particle on a circle in equilibrium, to an adiabatic device measuring heat capacity. This device in turn regulates a magnetic solenoid generating the field $\mathcal{B}$. This device is adiabatic and does not measure any microstates, so, unlike "Maxwell's demon type setups", its added entropy content is negligible.

What it does, however, is bring the system away from the thermodynamic limit, so the thermodynamic entropy is not automatically the Legendre transform of the energy. Instead, it is defined by

$$\beta' = \left.\frac{dS}{dE}\right|_{\hat{n}} \tag{A41}$$

where $\hat{n}$ is the direction in $\mathcal{B}, E$ space where two systems hooked up in parallel stop.

Considering the tangent vector $\hat{n}$ as the derivative of a curve $\gamma(\lambda) = (E(\lambda), \mathcal{B}(\lambda))$ at $\lambda = 0$, we get $\beta' = \gamma'(0) \cdot \nabla S$. Parameterizing the curve with the internal energy we get that the equilibrium configuration of such a system is related to the usual equilibrium as

$$\beta'(E, \mathcal{B}) \equiv \hat{n} \cdot \nabla S = \beta + \left(\frac{\partial S}{\partial \mathcal{B}}\right)_{E} \frac{d\mathcal{B}}{dE}. \tag{A42}$$

The moral of the story here is that if there is coupling between bulk and topological degrees of freedom, one could expect that geometric and thermodynamic temperatures are different. While this was a quantum problem without a "real" Unruh effect, in the next section we shall generalize this reasoning to a one dimensional quantum field theory.

2. Maxwell theory in (1+1) dimensions

Let us now consider the 1+1D Maxwell theory examined in [9, 31, 32]. This is a quantum field theory, where the Unruh effect is possible. At first site, the equivalence of Eq. 2.28 of [31] for the topological configuration with Eq. 49 of [29], together with the discussion following Eq. 49, would seal our case that a non-trivial topological charge will spoil the Unruh effect. However, this is not the case, since the theory examined in [31] has no propagating degrees of freedom.

In fact, this theory can be mapped on the problem examined in the previous section, since pure electrodynamics in $(1+1)$ dimensions is a topological theory, since $\pi_1(U(1)) = \mathbb{Z}$. This means pure electrodynamics admits a topological θ term, similar in nature to the QCD case, given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{e\theta}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu} \equiv \frac{1}{2}\mathcal{E}^2 + \frac{e\theta}{2\pi}\mathcal{E}, \tag{A43}$$

where $\mathcal{E} \equiv F_{01}$. Since the Lagrangian only depends on $\mathcal{E}$, we can treat it like the canonical field and find its equation of motion

$$\mathcal{E} = -\frac{e\theta}{2\pi}. \tag{A44}$$

Since the field is just a constant $\mathbb{C}$-number, the theory is trivial. That means there is only one physical state, the vacuum state $|\theta\rangle$, satisfying

$$\mathcal{E}\,|\theta\rangle = -\frac{e\theta}{2\pi}\,|\theta\rangle\,, \quad \mathcal{H}\,|\theta\rangle = \frac{1}{2}\left(\frac{e\theta}{2\pi}\right)^2 |\theta\rangle \quad , \quad \dot{F} = 0 \tag{A45}$$

In other words, θ is the simultaneous eigenstate of the Hamiltonian density and the electric field, and θ can be interpreted as a constant background electric field $\mathcal{E} = \frac{e\theta}{2\pi}$. In terms of $\mathcal{E}$, that means the Hamiltonian density takes the form

$$\mathcal{H}\,|\mathcal{E}\rangle = \frac{\mathcal{E}^2}{2}\,|\mathcal{E}\rangle\,. \tag{A46}$$

Furthermore, as shown in [32], this theory on a disk can be mapped exactly to the particle on a ring problem of the previous section. From the electromagnetic duality in $(1+1)$ dimensions, the Euclidean partition function of the theory in Eq. (A43) is related to the particle in a ring by

$$
\begin{aligned}
Z &= \int \mathcal{D}A \exp\left\{-\int d^2x \left(\frac{1}{4}F_{\mu\nu}F_{\mu\nu} - i\frac{e\theta}{2\pi}\epsilon_{\mu\nu}F_{\mu\nu}\right)\right\} \\
&= \int \mathcal{D}\mathcal{E}\delta(\dot{\mathcal{E}}) \exp\left\{-\int_0^\beta d\tau \left(\frac{a}{2}\mathcal{E}^2 - i\frac{e\theta}{2\pi}\mathcal{E}\right)\right\} \\
&\equiv \int \mathcal{D}\phi \exp\left\{-\int_0^\beta d\tau \left(\frac{1}{2a}\dot{\phi}^2 - i\frac{e\theta}{2\pi}\dot{\phi}\right)\right\},
\end{aligned}
\tag{A47}
$$

where a is defined in relation to the area enclosed by the ring $a = \beta^{-1}\int\sqrt{g}$. Thus, the correspondence becomes

$$
\mathcal{E} \to \dot{\phi}, \quad a \to I^{-1}, \quad \text{and} \quad \frac{e\theta}{2\pi} \to \mathcal{B}. \tag{A48}
$$

As seen in [69] the partition function requires asymptotic states as well as the action. However, we can use this equivalence as a starting point to calculate exactly the vacua in inequivalent frames.

Let us consider such a field prepared to be in a Minkowski vacuum state. By means of equation (4.24) of [31], defining the action of the Wilson line on the wedge-decomposition of a state as

$$
\mathcal{W}_e \left|\Psi\right\rangle = \mathcal{W}_e \frac{1}{Z}\sum_{\mathcal{E}} e^{-A\frac{\mathcal{E}^2}{2}} \left|\mathcal{E}\right\rangle_L \otimes \left|\mathcal{E}\right\rangle_R = \frac{1}{Z}\sum_{\mathcal{E}} e^{-A\frac{\mathcal{E}^2}{2}} \left|\mathcal{E}+e\right\rangle_L \otimes \left|\mathcal{E}+e\right\rangle_R, \tag{A49}
$$

we can reconstruct the Minkowskian wavefunctional for $\mathcal{E} = 0$, given in equation (4.30) of [31], using the Wilson line for an electric field $\mathcal{E} = \mathcal{E}_0$ as

$$
\begin{aligned}
\mathcal{W}_{\mathcal{E}_0} \left|\Psi\right\rangle = \mathcal{W}_{\mathcal{E}_0} \frac{1}{Z}\sum_{\mathcal{E}} e^{-\frac{\pi R^2}{2}\frac{\mathcal{E}^2}{2}} \left|\mathcal{E}\right\rangle_L \otimes \left|\mathcal{E}\right\rangle_R &= \frac{1}{Z}\sum_{\mathcal{E}} e^{-\frac{\pi R^2}{2}\frac{\mathcal{E}^2}{2}} \left|\mathcal{E}+\mathcal{E}_0\right\rangle_L \otimes \left|\mathcal{E}+\mathcal{E}_0\right\rangle_R \\
&= \frac{1}{Z}\sum_{\mathcal{E}} e^{-\frac{\pi R^2}{4}(\mathcal{E}-\mathcal{E}_0)^2} \left|\mathcal{E}\right\rangle_L \otimes \left|\mathcal{E}\right\rangle_R.
\end{aligned}
\tag{A50}
$$

On the limit $R \to \infty$, we get a delta function $e^{-\frac{\pi R^2}{4}(\mathcal{E}-\mathcal{E}_0)^2} \to \delta(\mathcal{E}-\mathcal{E}_0)$ around $\mathcal{E}_0$ on the exponential and thus obtain

$$
\left|\mathcal{E}_0\right\rangle = \left|\mathcal{E}_0\right\rangle_L \otimes \left|\mathcal{E}_0\right\rangle_R. \tag{A51}
$$

Remembering that such constant electric fields are corresponding to θ originally, we get

$$
\left|\theta\right\rangle_M = \left|\theta\right\rangle_L \otimes \left|\theta\right\rangle_R. \tag{A52}
$$

The separation of spacetime into right and left wedges correspond to a separation of the state into two sectors, both corresponding to the same parameter θ. That result can also be obtained directly from equation (4.31) on [31] by applying the Wilson line operator $\mathcal{W}_{\mathcal{E}}$

$$|\mathcal{E}\rangle_M = \mathcal{W}_{\mathcal{E}} |0\rangle_M = \mathcal{W}_{\mathcal{E}} |0\rangle_L \otimes |0\rangle_R \equiv |\mathcal{E}\rangle_L \otimes |\mathcal{E}\rangle_R \,. \tag{A53}$$

These results were calculated using a vacuum prepared as an Eigenstate of the field theory in Minkowski space. Hence, in this particular example (of $(1+1)$-dimensional pure photodynamics) the θ's seem to just factorize into right and left sectors, without having to be constrained to the $\theta = 0$ case.

As a consequence, take a localized observable $\mathcal{O}$ with support on the right Rindler wedge. Then $\mathcal{O}$ is non-zero only on $x \in \mathcal{M}_R$, and therefore completely accessible to both an inertial and a right-accelerated observer that has $\mathcal{M}_R$ as its causal region. Then

$$\langle \mathcal{O} \rangle_\Psi = \int_{\mathcal{B}(\Psi)} [\mathcal{D}A]\, \mathcal{O}\, e^{-S_E} \to \mathrm{Tr}[\mathcal{O}\, e^{-\beta H_R}], \tag{A54}$$

because the functional integral reduces to a sum over field configurations supported on $\mathcal{M}_R$. With H_R the Rindler Hamiltonian associated with the θ-dependent action above.

The discussion above is specific to the 1+1 theory because of the relation [9] between the topological current $\epsilon_{\mu\nu} A^\mu$ and the charge at the horizon $\int dx A_1$. The charge at the horizon introduces a local Gauss-law constraint which can be defined and changed locally. As the Reissner-Nordstrom black holes (where the charge is also "local" at the singularity) show, this constraint is valid across horizons. In 3+1 dimensions there is no Gauss constraint on θ and K_μ is controlled by Eq. (5), which has no equivalent local gauge constraint.

We can make the 1+1 theory a bit more "3D like" if, in analogy to the previous section A 1 we add the following source terms to the Lagrangian in Eq. (A43)

$$\mathcal{L} \to \mathcal{L} + J_\theta \times (\theta + J_L(\theta)) \tag{A55}$$

corresponding to a Rindler-type detector that measures θ and a device positioned asymptotically at the lest wedge that tunes θ according to some function (this of course breaks the Gauge constraint). In this case, we reproduce the same situation as in Eq. (A42). Equation A54 still holds, but the entropy density calculated via Eq. (32) from

$$s(\beta') \equiv \mathrm{Tr}_R \left[[\hat{\rho} \ln \hat{\rho}]_{\beta'} - [\hat{\rho} \ln \hat{\rho}]_{\beta' \to \infty} \right] \propto (\beta')^{-2} \qquad , \qquad \hat{\rho} = \hat{\rho}_L \times \hat{\rho}_R$$

will not scale with the same $\beta'^{-1} = a/(2\pi)$ as expected from the Unruh effect but will instead obey an equation such as A42. As a consequence, the ratio of β' and β is not necessarily set by the microscopic scale,which in this case is the acceleration

Of course, axiomatic quantum field theory [52, 53] is not violated here because the source terms J_θ, J_L introduce a preferred coordinate system which breaks local Lorentz invariance. However, the effect of topological dynamics in 3D can be argued to be equivalent to a spontaneous generation of J_θ, J_L. While the scale of J_θ is local (instanton peak) and of J_L is global (instanton tail), as shown in Eq. (A42) scale separation in such a case is not automatically perturbative. This illustrates that, as argued in section II topological terms can provide a mixing between "local" detector-scale physics and infrared scales.

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Chapter 6

Discussion

6.1 On the electron response to radiation under linear acceleration

In the article presented in Chapter 4, we talked about how to construct electron observables in order to probe the thermality of acceleration. The main idea is to start from a general, effective description of how a classical relativistic electron responds to a a field thermalized into a KMS state, such as in the Unruh effect. Based on general thermodynamical grounds, we expect that any degree of freedom of the electron that is sensitive to the radiating field's vacuum fluctuations should thermalize at the Unruh temperature if the electron is uniformly accelerating. Despite the subtleties involved in what it means to be a "finite temperature" state in an accelerated frame, we can still make these considerations due to the fact that the vacuum noise spectrum of a massless field seen by an uniformly accelerated observer can be directly related to the noise spectrum of an inertial observer immersed within a usual thermal bath of field excitations.

This connection is expressed by Eq. (1.6), where in four spacetime dimensions $n = 4$ the power noise spectra F_{accel} and F_{mink} of the electromagnetic field $s = 1$ are related by [15]

$$F_{\text{accel}}(\omega) \simeq \left(1 + \frac{a^2}{\omega^2}\right) F_{\text{mink}}(\omega). \tag{6.1}$$

Deviation from equality only happens in the low-frequency range of the spectrum, $\omega < a$. Low frequencies in the spectrum are sensitive to the presence of the Rindler horizon, whose characteristic length scale is $L \sim a^{-1}$. This means that the low frequency part of the spectrum is what differentiates between electrons that accelerate for a finite amount of time and electrons that accelerate eternally. Since we are interested in electrons that are accelerated for a long-enough time for thermalization to occur, differences between the power spectrum at low-frequency does not affect the qualitative arguments we make. Therefore, for all intents and purposes, the uniformly accelerated electron behaves, at first approximation, just like a stationary charge interacting with a thermal distribution of photons viewed from the point of view of an inertial observer.

Under these conditions, we've proposed the study of fluctuations around uniform acceleration to act as a thermometer for acceleration temperature, following similar ideas found in [51, 52, 53]. In particular, we focus on the known signal of such thermal contribution coming from the thermalization of the transverse-momentum fluctuations of the classical electron's trajectory [53],

$$\frac{1}{2m}\langle \delta p_\perp^i \delta p_\perp^j \rangle = \frac{1}{2} T_U \delta^{ij} + \mathcal{O}\left(\frac{a^2}{m^2}\right), \tag{6.2}$$

which is the equipartition relation between the fluctuations of the transverse momentum at the Unruh temperature $T_U \propto a$. This equipartition relation is defined on the model where the electron is represented by a classical, point-like relativistic charge coupled to the electromagnetic field, as in Eq. (3.11).

From general thermodynamical considerations we expect that, in this model, the dynamics of the probe on the transverse plane to be described by Brownian motion, which is described by a Langevin equation of the form

$$\frac{d\delta p_\perp^i}{d\tau} = -\frac{1}{\tau_D}\delta p_\perp^i + \xi^i, \quad \langle \xi^i(\tau)\xi^j(\tau')\rangle = \kappa\delta(\tau - \tau')\delta^{ij}. \tag{6.3}$$

Associated with the Langevin equation (6.3), we established the observable quantities relevant to the fluctuation and dissipation dynamics of accelerated electrons: the mean-squared momentum transfer per unit proper-time, represented (up to a factor) by the amplitude of the ξ-noise,

$$\kappa = \frac{1}{2}\frac{d}{d\tau}\langle \delta \mathbf{p}_\perp^2 \rangle, \tag{6.4}$$

the diffusion constant D associated with the stochastic dynamics of the electron on the transverse plane,

$$D = \frac{1}{4\tau}\langle \delta \mathbf{x}_\perp^2(\tau)\rangle, \tag{6.5}$$

and the dissipation time scale τ_D, corresponding to the scale where initial correlations of transverse-momentum fluctuations washout,

$$\tau_D = \frac{2mT_U}{\kappa} = \frac{mD}{T_U}. \tag{6.6}$$

From the thermodynamical relations in Eq. (6.6), we only need to calculate one of the quantities κ, D, τ_D in order to completely specify all of them. We calculate these thermodynamical quantities from three major microscopic theories that model accelerated electrons: classical electromagnetism, quantized photodynamics with classical electrons, and full quantum electrodynamics. We've shown that both the classical electromagnetic calculations and the quantized photodynamical calculations with a classical source agree on their predictions for the observables, while the fully-quantized QED calculation, where we consider the internal structure of the electron as a fermionic particle, shows deviation from the classical and semiclassical predictions as the acceleration gets close to the scale of the mass of the electron, $a/m \sim 1$. The deviation between the fully quantum electrodynamical treatment and the classical and semiclassical predictions are summarized in Figure 4.

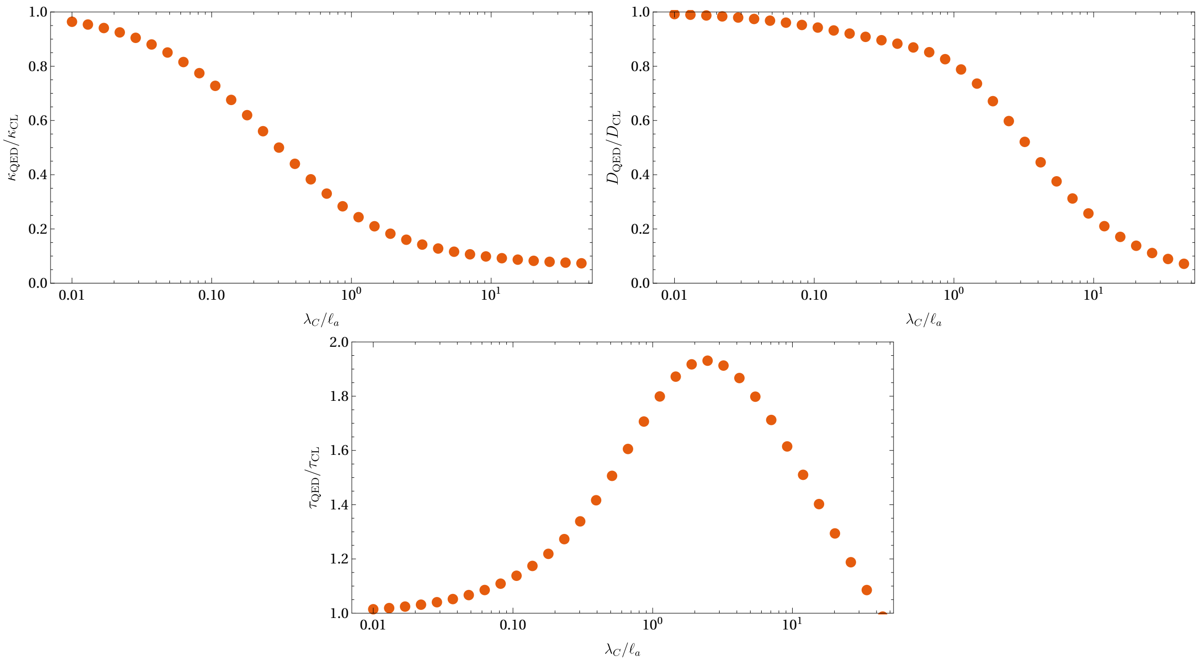

Figure 4 – Summary of the quantum-to-classical diffusion observables. On the top right is the mean transverse-momentum transfer κ; on the top right is the diffusion coefficient for the transverse plane D; on the bottom is the dissipation timescale τ_D. All plots are with respect to the adimensional scale λ_C/ℓ_a, where $\lambda_C = \hbar/mc$ is the Compton wavelength of the electron and $\ell_a = c^2/a$ is the length parameter for the acceleration.

Quantitative deviations from the classical and semiclassical predictions are expected around the critical scale $a/m \sim 1$. Since the acceleration being impressed on the electron by an external electric field $\mathbf{E} = E_z\hat{z}$, that means the electric field is close to the Schwinger limit $a/m \sim E_z/E_{\text{crit}} \sim 1$, where $E_{crit} = m^2c^3/e\hbar$. This corresponds to the critical acceleration $a_{\text{crit}} = eE_{crit}/m$ where the electric field begins probing the quantum nature of the electron—in particular, it gets close to the onset of electron-positron pair production by the external field. In this regime, we cannot justify the applicability of a point-particle description of the electron by a semiclassical current, such as in (3.3), but we can still make sense of the quantities defined in the QED approach. Since the quantitative diffferences between the classical and quantum approaches don't change the qualitative physics of the phenomenon, we take the QED results as an extension of the Brownian motion interpretation of the electron dynamics to "quantum" scales. This means the accelerated electron's transverse momentum fluctuations is well approximated by a Langevin-type equation associated with an Unruh temperature of the vacuum, even in the QED approach to the problem.

What is really interesting about the quantitative predictions for the fluctuation-dissipation observables is that we can give well-defined estimations for the "thermalization time" τ_D as a function of the acceleration-to-mass ratio a/m. Using this relation, we can compare the magnitude of the electric fields necessary to impose the acceleration, $|\mathbf{E}|$, with the time needed to maintain the field approximately constant for the uniform-acceleration approximation to be valid. As we've discussed on the paper, acceleration magnitudes $a/m \simeq 0.01$, associated with electric fields of the order $|\mathbf{E}| \simeq 10^{-16}\,\mathrm{V/m}$, correspond to a dissipation time τ_D of 1 femtosecond. This means any experimental setup designed to measure this effects should be able to generate electric fields of this order for at least 1 femtosecond in order for the Langevin-dynamics on the transverse plane of the acceleration axis to thermalize and pick up the signal of the vacuum acceleration temperature.

At the moment, the most promising methods for reaching high electron acceleration profiles come from high-intensity laser pulses, but the current laser technology [54, 55, 56] is still a few orders of magnitude too weak to probe the effect with enough sensitivity: either the electric field magnitudes and linear acceleration is too low, which means low Unruh temperature and a weak signal, or the duration of the laser pulses are too short compared to the dissipative timescale, not leaving enough time for the electron to thermalize with the acceleration temperature of the vacuum. Thus, experimental efforts to observe these effects must focus on improving accelerator precision and control. Additionally, future work should aim to refine the observables that can distinguish between classical and quantum predictions more clearly.

6.2 On the strong CP problem, general covariance, and horizons

In the article presented in Chapter 5, we talked about how the topology of spacetime plays a major role in the structure of the vacuum of non-abelian Yang-Mills theories. The main motivation for our study was the so-called strong CP problem: In the usual treatment of quantum chromodynamics, it is well known that the total-derivative term

$$\mathcal{L}_\theta = \frac{\theta g^2}{16\pi^2}\,\mathrm{tr}\left(F_{a\mu\nu}\tilde{F}_a^{\mu\nu}\right) = \frac{\theta g^2}{32\pi^2}\partial^\mu J^5_\mu \tag{6.7}$$

has non-perturbative effects on the observables of the theory. The θ-term $\mathcal{L}_\theta$ breaks the CP-symmetry in the strong sector of the standard model, which has as its most famous consequence the generation of a small electric dipole moment (EDM) for the neutron [57], $D_n = 3.6 \times 10^{-26}\, e \cdot \mathrm{cm}$. The EDM of the neutron is very well-determined by experiments, which imposes the bound $|\theta| < 10^{-10}$ on the CP-breaking phase in the strong sector. Trying to understand why this phase is so small is the so-called *strong CP problem*.

Many proposals have been put forth to try to explain why the θ-term in the QCD_4 Lagrangian is so close to zero. The most popular solution is perhaps the Peccei-

Quinn symmetry [58], which introduces a new field that couples to the quarks and relaxes the value of θ to zero by means of spontaneous symmetry breaking. This mechanism generates a new particle the Nambu-Goldstone boson corresponding to the breaking of the Peccei-Quinn symmetry, which is called the *axion*. So far, no particle resembling an axion has been found, which lead us to search for a first-principles explanation for the CP-symmetry of the strong sector.

The main interest of the article was to think about how the presence of horizons would interfere with the usual description of how non-abelian Yang-Mills theories pick up the CP-breaking θ-term $\mathcal{L}_\theta$. In the usual approach, the mechanism that introduces the θ-term in the QCD_4 Lagrangian comes from the study of the Euclidean partition function,

$$Z = \int [\mathcal{D}A] \exp\left(- \int d^4x\, \mathcal{L}_E[A] \right), \tag{6.8}$$

where A is the gauge-potential for the gluon field and $\mathcal{L}_E$ is the associated Euclidean Lagrangian. This integral receives its dominant contributions from the saddle points of the Euclidean action $S_E[A] = \int d^4x\, \mathcal{L}_E[A]$, which are field configurations of finite action. The field configurations of finite action are described by the gauge-fields that fall-off sufficiently fast at infinity, $F^a_{\mu\nu}(|x|) = \mathcal{O}(|x|^{-3})$, which are given by the gauge-potentials that fall off at infinity to pure-gauge configurations $A_\mu(|x| \to \infty) = g\partial_\mu g^{-1} + \mathcal{O}(|x|^{-2})$. These finite-action gauge-field configurations are classified by a topological charge called the winding number of the configuration,

$$n[A] = -\frac{1}{16\pi^2} \int d^4x\, F^a_{a\mu\nu} \tilde{F}^{\mu\nu}_a \in \mathbb{Z}, \tag{6.9}$$

and they can be used to construct the representation of eigenstates of the Hamiltonian of the theory by summing over the winding sectors

$$Z[\theta] = \sum_{n\in\mathbb{Z}} \int [\mathcal{D}A]\, e^{in\theta} \exp\left(- \int d^4x\, \mathcal{L}_E[A] \right). \tag{6.10}$$

This expression for the new partition function $Z[\theta]$ is interpreted as the vacuum expectation value $\langle\theta|e^{-Ht}|\theta\rangle$ in the energy eigenstate

$$|\theta\rangle = \sum_{n\in\mathbb{Z}} e^{in\theta}|n\rangle, \tag{6.11}$$

which is the coherent sum of the pure-gauge field configurations with winding number $n \in \mathbb{Z}$.

The main idea is to notice how dependent of the topology of spacetime all this procedure is. It is built on the Wick-rotated Euclidean section of Minkowski space, and it relies heavily on the topology of flat spacetime in global, inertial coordinates to make the arguments, such as the vanishing of the fields in the asymptotic regions $|x| \to \infty$. Thus, there is reason to believe that things might change if the background spacetime is different from flat Minkowski spacetime, or if we are dealing with non-inertial observers which experience causal horizons on their rest-frame.

Thoughts about whether the strong CP problem survives in more general spacetimes are not new. The idea that maybe the presence of the cosmological horizon might decohere the state (6.11), inspired by Hawking's idea of averaging over the states of the system over the horizon, had already been proposed by Linde in the 80s [59]. In the article, we especulate that this might be relevant even when considering causal horizons, which are not caused by the curvature of spacetime but by the rest-frame of a given observer. We argue that because an observer bounded by a causal horizon loses information of large sections of spacetime that get hidden behind the horizon. That means that topological information about both spacetime and the gauge field might be hidden behind his own horizon, which in turn forces the observer to sum over the winding sectors in an incoherent manner. That would be tantamount to decohering the θ-vacuum into a density matrix representing the incoherent mix of winding sectors,

$$|\theta\rangle\langle\theta| = \sum_{m,n\in\mathbb{Z}} e^{i(m-n)\theta}|n\rangle\langle m| \to \sum_{n\in\mathbb{Z}} |c_n|^2\, |n\rangle\langle n|. \tag{6.12}$$

In fact, the incoherent sum over the states at the horizon is one of the ways we can represent the Unruh effect in Euclidean signature [60]: if the Euclidean manifold $\mathcal{M}^E$ is divided into two disjoint regions $\mathcal{M}^E = W_R^E \cup W_L^E$, then the reduced density matrix for the field configurations $\varphi_R \in W_R^E$, obtained by integrating out the fields in the complementary region W_L^E is given by

$$\rho(\tilde{A}_R, \tilde{A}'_R) = \int [\mathcal{D}\tilde{A}_L]\, \Psi_R[\tilde{A}_R, \tilde{A}_L]\Psi_L[\tilde{A}'_R, \tilde{A}_L], \tag{6.13}$$

where $\Psi_{R/L}[\tilde{A}_R, \tilde{A}_L]$ is the "partition function" of the $W_{R/L}^E$ sector for field configurations that take the values $\tilde{A}_R$ and $\tilde{A}_L$ at the boundary $\partial(W_R^E \cap W_L^E)$,

$$\Psi_\pm[\tilde{A}_R, \tilde{A}_L] = \int_{\mathcal{C}_\pm[\tilde{A}_R,\tilde{A}_L]} [\mathcal{D}A]\, e^{-S_E[A]}. \tag{6.14}$$

This procedure, when applied to the Euclidean Rindler space, yields the formal Gibbs density matrix for the reduced density matrix ρ at the Unruh temperature. This suggests that the same decoherence mechanism that can represent the Unruh (and Hawking) effects may be capable to transform the coherent sum of the Yang-Mills θ-vacua into an incoherent superposition of pure-winding sectors, and thus washing out any θ-dependent observable to zero.

Since non-abelian Yang-Mills theories are generally self-interacting, it is very hard to carry out these kind of calculations exactly except for very contrived toy models that don't present all the physics of the original QCD_4 problem. To give a picture of how this mechanism would occur, we proposed an analogy with the famous toy model of a one-dimensional quantum mechanical particle $\hat{q}$ in a periodic potential $V(\hat{q})$. This analogy is very well known: the ground-states of the quantum mechanical particle in a periodic potential are given by the Bloch-waves, which have the same coherent structure as the QCD_4 θ-vacua (6.11). The Bloch-states are represented in Figure 5.

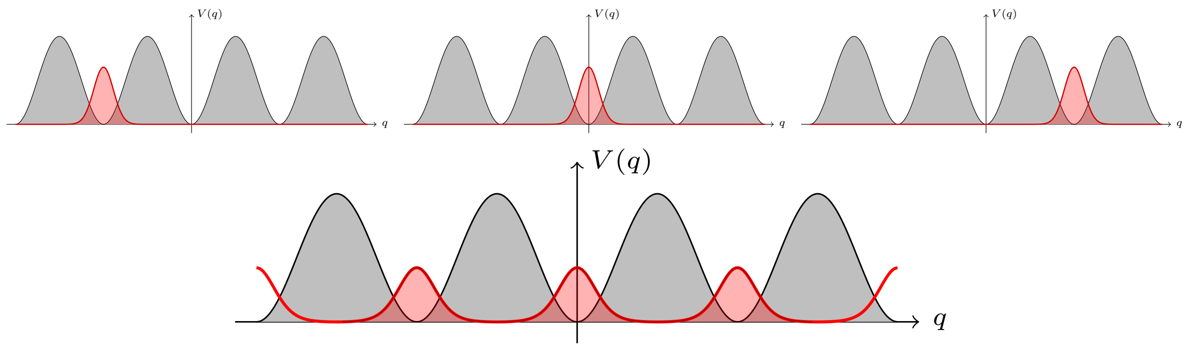

Figure 5 – A representation of the states of a particle $\hat{q}$ in a periodic potential $V(q)$. Above are the localized states $|n\rangle$ that correspond to the local sites at the minima of the potential, while below is the representation of the Bloch-wave $|\theta\rangle$ that is composed of a coherent superposition of $|n\rangle$.

In the analog model, the particle $\hat{q}$ represents the topological configurations of a non-abelian gauge theory A with winding number n, while the potential represent the barrier for tunneling between winding sectors, which in Yang-Mills is enacted by instantons. We model the dissipation due to the horizon by coupling the particle to $N \to \infty$ harmonic oscillators $\hat{Q}_k$ at finite Hawking-Unruh temperature, which yield a Caldeira-Legget type of Lagrangian

$$L = \frac{1}{2}\dot{q}^2 - V(q) + \sum_k \frac{1}{2}[\dot{Q}_k^2 - \omega_k^2 Q_k^2]. \tag{6.15}$$

The dissipative behavior of these kind of theories is well known [61, 62, 63, 64]. Even though the quantitative details vary depending on the model of choice, qualitatively we observe that the interaction of the thermal bath $\{Q_k\}$ can break the coherence of Bloch-type states such as $|\theta\rangle$, leaving the system in a totally incoherent mixture of states localized at the minima of the potential and emulating the horizon decoherence we speculated on Eq. (6.12).

Admittedly, since we don't have direct access to exact calculations for a non-trivial Yang-Mills theory, the arguments we make on the paper are merely heuristic. Despite that, we still argue that the same mechanism can be investigated in analogue models, provided they exhibit similar topological properties to Yang-Mills theories. One possible candidate is the theory of fluids with polarization, where the search for θ-decoherence by the horizon would be similar to the experiments on acoustic black holes, as first proposed by Unruh [65]. Even though the operational viability of such an experiment seems very far-fetched at the present, it still remains as valuable way of verifying experimentally the soundness of the ideas put forth.

Summarizing, the failure to detect axions that solve the strong CP problem [66] forces us to keep an open mind regarding the solution of this issue. In particular, perhaps the answer is in the infrared (IR) limit with respect to scales usually relevant to QCD_4. Our solution is of this kind, but others of a similar kind are possible. Two recent examples focus on IR fixed points with instanton interactions [67], as well as the correct limit of the QFT wave functional [68, 69]. Perhaps these solutions are all related, and say something general about the global structure of the wave functional of QCD_4.

Chapter 7

Conclusion

In this thesis, we have discussed several aspects of quantum field theory in the context of accelerated systems, such as the consequences of the thermality of the vacuum from an accelerated observer's point of view and the consequences of causal horizons to topologically non-trivial gauge theories.

In the first part, dealing with the study of the thermodynamics of the vacuum under acceleration, we discussed the Unruh temperature under general thermodynamical considerations, connecting the dynamics of accelerated charges with the Brownian-motion dynamics typical of particles interacting with external systems. Since Brownian-motion type of dynamics is a natural place to the discussion of fluctuation and dissipation, we proposed observables connected to the fluctuation-dissipation theorem that are sensitive to the thermality of the quantum vacuum at the Unruh temperature.

We calculated the dissipative time-scale, the diffusion constant, and the mean-squared momentum transfer for the electron's momentum fluctuations in the plane transverse to the acceleration axis. These observables were calculated in the classical electromagnetic regime, the semiclassical regime where only photons are quantized, and the fully quantum electrodynamical regime. We found that for accelerations lower than the scale set by the mass of the electron, $a/m \ll 1$, all these approaches coincide and are well-described by a dissipative Langevin-type of dynamics typical of Brownian motion. For higher accelerations, $a/m \gtrsim 1$, we have found that the quantum electrodynamical prediction deviates from the classical and semiclassical counterparts, even without considering non-perturbative effects that happen at that scale, such as electron-positron pair production due to the Schwinger mechanism. Still, we found that QED gives corrections to the classical predictions even as soon as $a/m \sim 0.1$, where we can still argue for strong-field dynamical corrections while having suppressed contributions to the electron dynamics from backreaction due to pair production.

Since one of the observables obtained was the dissipative time-scale for the effective Langevin dynamics of the electron's transverse momentum, we have quantitative predictions for the minimum acceleration duration necessary for the sufficient thermalization of the electron's transverse momentum distribution at the Unruh temperature $T_U \propto a$. This imposes constraints to the external background field that accelerated the electron. In order for the Langevin approximation to be valid, we need an electric field profile that can imprint linear acceleration for sufficiently long acceleration duration $\Delta\tau \gtrsim \tau_D$, which provides a lower bound on the magnitude of the background electric field used to accelerate the charge. We have found that for the particular observables we studied, the best fit for the magnitude and duration scales—given by laser wakefield technology—is still a few orders of magnitude outside the range where sufficient thermalization of the observables is expected.

In the second part, dealing with the study of the vacuum structure of gauge theories bounded by horizons, we discussed how the structure of the vacuum can be impacted by the spacetime topology and by the presence of said causal/event horizons. In particular, we focused on the possible consequences that the causal structure of spacetime can have on the strong CP problem: Since the strong CP problem is connected with a θ-term in the effective QCD_4 action that is given by the topological charge of the field, the presence of this term crucially depends on the specific topological structure of the spacetime under consideration.

In particular we argue that, since the inaccessibility of degrees of freedom beyond horizons forces us to trace-out their contributions on the partition function for the observer bounded by it, then we expect that the vacuum structure of a topologically non-trivial theory only sums its topological sectors incoherently. Since the strong CP problem crucially depends on the vacuum structure of the theory to coherently sum over the topological sectors in order to introduce the famous CP breaking θ-term, this idea has the potential to explain why the strong sector of the standard model conserves CP while still maintaining non-trivial topological structures—such as instantons—that can be observed in lattice simulations.

Since non-abelian gauge theories are generally self-interacting, we drew from the well-known analogy between the Yang-Mills vacuum structure and the condensed-matter model of a particle in a periodic potential to investigate how the decoherence process might look like. We introduce the horizon-induced dissipation by means of a Caldeira-Leggett type model, coupling the particle linearly to an infinite number of harmonic oscillators that induce the decoherence between superpositions of localized states. In this model, we find that the dissipation time-scale T depends on the softest scale within the problem, which, refering back to the original problem, should be given by the horizon radious. We view this as a way to understand how a θ-vacuum state would dissipate towards a mixed state with respect to its winding sectors, and thus washing out any θ dependency in physical observables.

Since these ideas are not based on new symmetries and particles, but on the behaviors of quantum fields in non-trivial background spacetimes, we also propose that we may view the effects discussed on the paper by means of analogue models with similar topological properties as the Yang-Mills fields. Despite being a speculative idea, the recent successes on acoustic black hole experiments and simulations [70, 71] allow us to hope that, in the future, this kinds of questions about the vacuum structure of gauge fields can be studied in the same context.

In conclusion, we have studied the effects of the effects of the "acceleration temperature" of the vacuum of quantum field theory in the contexts of both strong-field electrodynamics and in non-abelian Yang-Mills theories, and found that non-trivial consequences can be studied from the point of view of accelerated observer's thermodynamical considerations. Despite the technical difficulties of probing linear acceleration in the laboratory, we've still found ways of studying its effects in experimental contexts, both in currently available scenarios such as electron-laser experiments and in speculative but feasible scenarios such as acoustic black hole evaporation experiments. The avenues to explore the thermodynamical effects of acceleration and horizons are now reaching the point where they are accessible to laboratory, which gives us hope that soon we will be able to test some of the ideas presented in this in a concrete experimental setting. Other possible paths to expand the ideas presented here are manyfold: we can extend the analysis of linear acceleration to more general acceleration profiles, such as circular [27] and non-uniform [72] acceleration profiles; we can extend the analysis to other initial inertial states that are not the Poincaré vacuum [73]; we can study and compare the horizon dissipation picture with other current proposals to solve the strong CP problem [69] that appeal to boundary-condition arguments. The roads are open to exploration, and we hope that the present work may have helped to pave at least a bit of the path forward to the understanding of acceleration and horizon thermodynamical effects.

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APPENDIX A

Citations to previously published work

The work in this thesis is in collaboration with Donato Giorgio Torrieri, Lance Labun, Ou Z. Labun, and Manuel Hegelich. Chapter 4 appeared in [33], and Chapter 5 appeared in [35].