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REMOVABLE SINGULARITIES THEOREMS
FOR THE YANG–MILLS FUNCTIONAL

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ABSTRACT - These notes are given as a reference for the minicourse on gauge theories and analysis presented by the author in UNICAMP, in August 1992. In this course we define gauge theories and the very basic geometric ideas involved. We focus mostly on the analytical aspects of these theories and introduce concepts such as conformal invariance, subharmonicity and apriori estimates for the curvature F, gauge invariance (optimal choice of gauge), weak-compactness of Sobolev spaces of connections. We use these tools to outline proofs of removable singularities theorems for Yang-Mills fields, also on manifolds with boundary.
The minicourse is divided in four lectures, the first three corresponding to the three chapters of these notes. In the fourth lecture, we explain Dirichlet and Neumann boundary value problems for Yang-Mills connection. We denote particular attention to the removable singularities theorem for boundary points. For the fourth lecture we refer to [10] and the last section in [11]. These notes are mostly self-contained, but not completely detailed. For more details we give further reference to the reader.

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Introduction.
In the first chapter we introduce gauge theories and the Yang-Mills equations as they arose as an extension of the relativistically invariant formulation of Maxwell's

equations. We describe the main geometrical ingredients of gauge theories, introduce the concept of gauge invariance and construct different gauges that are known as "good gauges", in which the analysis can be carried out to prove smoothness properties of Y. M. potentials.
We describe how gauge theories can be interpreted as a non-linear extension of Hodge theory and how the Y. M. equations arise as the Euler-Lagrange equations for a suitable action functional.
We also introduce the coupling with electrons, i.e. the Yang-Mills Higgs functional.
In the second chapter, we state and outline the proof of the removable singularities theorem in dimension four [3] and the good gauge theorem [8], due to K. Uhlenbeck.
In the third chapter we give a direct method for minimizing the Yang-Mills functional over a compact manifold with no boundary.
The method is due to Sedlacek [9].

Chapter 1. Gauge theories and the Yang-Mills equations.

The Yang-Mills equations were introduced by physicits in the 1950's as a nonlinear extension of Maxwell's equations.

Only in the 1970's they came to the attention of mathematicians, starting with Michael Atyah [1].

Nowadays, gauge theories are considered very interesting from the mathematical point of view, even independently of applications in physics.

§1. The Yang-Mills equations as a generalization of Maxwell's equations

Maxwell equations form a first order system of p.d.e. that describes the electric field $E \equiv (E_1, E_2, E_3)$ and the magnetic field $B \equiv (B_1, B_2, B_3)$.

Upon setting all physical constants equal to one, they are written as follows

$$(1) \qquad \begin{cases} \text{div } E = 4\pi\rho & (\rho \equiv \text{ charge density}) \\ \text{curl } B - \dfrac{\partial}{\partial t} E = 4\pi j & (j \equiv (j_1, j_2, j_3) \equiv \text{ vector current}) \end{cases}$$

$$(2) \qquad \begin{cases} \text{div } B = 0 \\ \text{curl } E + \dfrac{\partial}{\partial t} B = 0 . \end{cases}$$

These equations can be written in a more compact way, which is also relativistically invariant, by introducing an electromagnetic field $F \equiv \sum F_{jk} dx^j dx^k$.

The $(x^i)'s$ are coordinates on the four dimensional spacetime, and

$$(F_{jk}) \equiv \begin{pmatrix} 0 & -E_1 & E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \quad ; \quad j, k = 0, 1, 2, 3$$

(Note that $F_{jk} = -F_{kj}$; i.e. F is a two-form on spacetime).

Let now J be a differential form that encodes the charge density and the vector current,

$$J \equiv \rho dx^0 + j_1 dx^1 + j_2 dx^2 + j_3 dx^3 ,$$

and $*$ be the Hodge operator in the Minkowski spacetime.

Then, Maxwell's equations become

(1) $$d^*F \equiv \left(-\frac{\partial}{\partial t}F_{0k} + \sum_1^3 \frac{\partial}{\partial x^j} F_{jk}\right) dx^k = 4\pi j$$

(2) $$dF \equiv \left(\sum \frac{\partial}{\partial x^i}F_{jk} + \frac{\partial}{\partial x^j}F_{ki} + \frac{\partial}{\partial x^k}F_{ij}\right) dx^i dx^j dx^k = 0\,.$$

(Some details: Let $k=0$ in (1). Then (1) gives

$$-\frac{\partial}{\partial t}F_{00} + \frac{\partial}{\partial x^1}F_{10} + \frac{\partial}{\partial x^2}F_{20} + \frac{\partial}{\partial x^3}F_{30} = \frac{\partial}{\partial x^1}E_1 + \frac{\partial}{\partial x^2}E_2 + \frac{\partial}{\partial x^3}E_3 \equiv 4\pi\rho\,.$$

Let $k=1$ in (1). One obtains

$$-\frac{\partial}{\partial t}E_1 + \frac{\partial}{\partial x^2}B_3 + \frac{\partial}{\partial x^3}(-B_2) = 4\pi j\,, \quad \text{etc...}$$

Note that d^* is the divergence operator $\nabla\cdot \equiv \left(-\frac{\partial}{\partial t}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)\cdot$ in Minkowski space).

Equation (1) depends on the metric and can be generalized to a Riemannian metric and to k-forms (also $k \neq 2$).

If the $(k-1)$-form $J=0$, then F is called "harmonic".

The study of solutions of the equations

$$\begin{aligned} d^*w &= 0 \qquad (w \text{ a } k\text{-form}) \\ dw &= 0 \end{aligned}$$

is known as "linear Hodge-theory" – [2].

We will talk more about this later in the course. From Maxwell's equations, more precisely equation (2), we see that F is a closed 2-form. By Poincaré lemma, F can be locally written as $F = dA$, the differential of a one-form A. If we add an exact differential dU (U is a function), the curvature $F = d(A+dU) = dA$ does not change. Hence A is not uniquely determined.

The one-form A is called potential (or connection) and U represents the "choice of gauge".

F is also called "curvature" of the connection A.

To have a better understanding of the meaning of "gauge", we look at a more complete physical description where the electromagnetic field F interacts with matter. A beam of electrons is decribed as a complex function $\phi : M \to \mathbb{C}$. ($M \equiv$ spacetime). The coupling is expressed by the term $|D_A\phi|^2$ in the so-called Yang-Mills-Higgs functional; here $D_A\phi_j \equiv (d + iA)\phi_j$.

Only the intensity of the beam, $|\phi|$ and relative angles between beams $\phi_1 \cdot \overline{\phi}_2$ have physical meaning.

If we choose a different angle $\widehat{\phi}_j = u^{-1}\phi_j$, where $u : M \to S^1$, the new potential $\widehat{A} = A - iu^{-1}du = A - id(lnu)$, and the field $\widehat{F} = F$ is left unchanged.

A simple computation shows that $\widehat{D_A\phi} = D_{\widehat{A}}\widehat{\phi}$. The theory described so far is the abelian theory, since the arbitrariness of the potential A is the freedom of choice of the function u, valued in the abelian group S^1.

Now, we generalize to the non-linear gauge theory, in which the structure group is $G \equiv SU(N)$, a non abelian group.

The potential $A = \sum A_i dx^i$, where the A_i's are matrices in $su(N)$, the Lie Algebra of $SU(N)$. The field $F \equiv \sum F_{ij} dx^i dx^j$, where F_{ij} are also valued in $su(N)$, is the curvature of the connection A, i.e. $F = dA + A \wedge A$.

Maxwell's equations extend to the non-linear equations

$$(1) \qquad D_A^* F = \sum_{jk} \Big(\frac{\partial}{\partial x^j} F_{jk} + [A_j, F_{jk}] \Big) dx^k = 0$$

$$(2) \qquad D_A F = \sum_{ijk} \Big(\frac{\partial}{\partial x^i} F_{jk} + [A_i, F_{jk}] \Big) dx^i dx^j dx^k = 0$$

Here, we wrote (1) explicitely in the flat metric.

If we choose a different gauge $u : M \to SU(N)$, the potential A and the field F transform as follows

$$A \longmapsto u^{-1}du + u^{-1}Au$$
$$F \longmapsto u^{-1}Fu\,.$$

Equations (1) and (2) are gauge-invariant.

Equations (1) are known as Yang-Mills equations and equations (2) are the Bianchi identity, always satisfied by the curvature of a connection.
Again, (1) depend on the metric considered.

§2. The Yang-Mills functional

The pure Yang-Mills functional is given by

$$\text{Y.M.}(A) \equiv \frac{1}{2}\int_M tr(F_A \wedge *F_A) \equiv \frac{1}{2}\int_M |F_A|^2 d\,Vol =$$

(in the flat metric)

$$= \frac{1}{2}\int_M \Big(\sum_{ij} tr F_{ij} F_{ij}^{\dagger}\Big) d\,Vol.$$

Here $tr \equiv$ trace, F_{ij} are matrices in some Lie Algebra (for example $su(N)$), and $F_{ij}^{\dagger}$ is the matrix adjoint of F_{ij}.
The variation of the Yang-Mills functional is

$$\delta_{\varphi}(\text{Y.M.}(A)) = \frac{1}{t} \quad (\text{first}-\text{order terms in } t \text{ of Y.M.}(A + t\varphi))$$

$$= \int_M \langle d\varphi + [A, \varphi], F\rangle = \int_M \langle \varphi, D^* F\rangle\,.$$

Hence, the Yang-Mills equations with $J = 0$ are the Euler-Lagrange equations for the pure Yang-Mills functional.
Likewise, the coupled Yang-Mills equations are obtained as the Euler-Lagrange equations for the Yang-Mills-Higgs functional

$$\text{Y.M.}(A) = \frac{1}{2}\int_M \Big(|F_A|^2 + |D_A\phi|^2 + \lambda(1 - |\phi|^2)^2\Big) d\,Vol.$$

In the following section, as well as in the rest of the course, we consider the pure Yang-Mills functional.

§3. Construction of gauges

Choosing the right gauge is very important from the analytical point of view to obtain smooth solutions.

In the Abelian case, the Y.M. equations are written in the form

$$d^*dA \equiv \sum_{ij} \frac{\partial}{\partial x^i}\left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^i}\right) dx^j = 0\,.$$

(This is in coordinates and in the flat metric).

Hence, if A is expressed in a gauge in which $d^*A \equiv \sum_i \frac{\partial A_i}{\partial x^i} = 0$, the two equations combined give the elliptic equation

$$\Delta A \equiv [d^*d + dd^*]A \equiv \sum_j \Big(\sum_i \frac{\partial^2}{(\partial x^i)^2} A_j\Big) dx^j = 0\,.$$

Such gauge is called "Hodge" gauge, or "Coulomb" (by Physicists).

Coulomb gauges can be found under certain hypotheses. To prove existence of Coulomb gauges one uses the implicit function theorem between Banach spaces.

Hence, the first step is finding a gauge in which A is small.

More precisely, one needs an estimate of $||A||_\infty$ in terms of $||F||_\infty$.

Everything is done locally (there is no global method for finding good gauges).

We are going to construct exponential gauges on S^{n-1}, matched by a rotation along the equator and outline a similar construction on

$$U^n \equiv \{x \in D^n \quad \text{s.t.} \quad \rho_1 < |x| < \rho_2\}\,.$$

This material is needed in the next lecture to construcxt Coulomb gauges and to prove the removable singularities theorem in dimension four.

Exponential gauges [3].

Let us consider a geodesic ball centered at the origin, partametrized by coordinates (r, ψ).

It is possible to choose a gauge in which $A_r = 0$. [0]

In a Euclidian ball, $A_r(x) = \sum x^i A_i(x) = 0$.

From this equation, one finds that

$$\sum_i x^i F_{ij} = \sum_i x^i \Big(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^i} + [A_i, A_j] \Big) =$$

$$= \sum_i \Big(r \frac{\partial A_j}{\partial r} - \frac{\partial}{\partial x^j}(x^i A^i) + \delta_{ij} A_i \Big) = r \frac{\partial A_j}{\partial r} + A_j = r \frac{\partial}{\partial r}(r A_j)\,,$$

and by integrating and estimating in the sup-norms one obtains

$$|A_j(x)| \le \frac{1}{2}|x| \sup_{|y|\le|x|} |F(y)|\,.$$

The exponential gauges for S^{n-1} centered at the north (south) pole are such that in these gauges $A_\varphi = 0$, where $\varphi \in (0, \pi)$ is the polar angle.

In S^{n-1} one has similar estimates that take into account the geometry of S^{n-1} (here we use coordinates (φ, θ), $\varphi \in (0, \pi)$, and $\theta \in S^{n-2}$):

$$||A(\varphi, \theta)||_\infty \le \tan \frac{\varphi}{2} ||F||_\infty\,.$$

The following lemma holds:

Lemma 3.1. There exist constants $\alpha_0, K < \infty$ depending on G such that if $||F||_{L^\infty(S^{n-1})} < \alpha_0$, then there exists a gauge in which $||A||_{L^\infty(S^{n-1})} < K||F||_{L^\infty(S^{n-1})}$.

Proof. One chooses the exponential gauge based at the north pole $d^0 + A^0$, and at the south pole $d^\pi + A^\pi$.

In these gauges

$$||A^0|| \le \tan \frac{\varphi}{2} ||F||_\infty\,,$$

$$||A^\pi|| \le \tan \Big(\frac{\pi - \varphi}{2} \Big) ||F||_\infty\,.$$

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[0] these gauges are obtained by choosing frames at a point x_0 and parallel translating along geodesics departing from x_0.

On the overlap $0 \doteq sA^{\pi}_{\varphi} - A^{0}_{\varphi}s = \dfrac{\partial s}{\partial \varphi}$; i.e. s depends only on θ, i.e. $s(\varphi,\theta) = s_0 \exp u(\theta)$, where $u : S^{n-2} \to g$.
Assuming $\int_{S^{n-2}} u = 0$, the following estimate holds

$$||du||_\infty \leq \mathrm{const}(G)||ds(\pi/2,\theta)||_\infty \leq \mathrm{const}||F||_\infty .$$

One can obtain a gauge globally defined on S^{n-1} by rotating A^0 by

$$h(\varphi,\theta) = s_0 \exp(\sin^2(\varphi/2)u(\theta))$$

and A^{π} by

$$q(\varphi,\theta) = \exp(-\cos^2(\varphi/2)u(\theta)).$$

the new globally defined gauge is now

$$A = h^{-1}A^0 h + h^{-1}dh = q^{-1}A^{\pi}q + q^{-1}dq,$$

and it satisfies

$$|A_\varphi(\varphi,\theta)| = \frac{1}{2}\sin\varphi|u(\theta)| \leq \ \mathrm{const}\ ||F||_\infty,$$

and, for $\varphi \leq \pi/2$

$$|A_\theta(\varphi,\theta)| \leq |A^0(\varphi,\theta)| + |d_\theta h|$$

or, for $\varphi \geq \pi/2$

$$|A_\theta(\varphi,\theta)| \leq |A^{\pi}(\varphi,\theta)| + |d_\theta q|.$$

Hence, the final result

$$||A(\varphi,\theta)||_\infty \leq K||F||_\infty,$$

where K is a constant depending on the geometry of G.
This completes the proof of lemma 3.1.
To construct a gauge in U that satisfies the same kind of estimates, the idea is to match transverse gauges constructed on the spheres $S_1 \equiv \{x\ s.t.\ |x| = 1\}$, and $S_2 = \{x\ s.t.\ |x| = 2\}$. Similarly, one constructs a gauge in D^n, by matching the exponential gauge at zero, with the transverse gauge at S_1 [3].

Chapter 2. Smoothness properties of Yang-Mills connections.

In this chapter we introduce a removable singularities theorem that uses the Yang-Mills equations [3].
The most interesting case deals with a four-dimensional base manifold and with point singularities (these are the singularities that arise when finding Yang-Mills solutions via a directly minimizing procedure, in the hypothesis of finite action).
This is historically the first removable singularities theorem in gauge-theories. It is conceptually easier, since it uses the equations, but somehow quite technical.
(There are some more sophisticated theorems, which do not use Yang-Mills equations) [4].
The proof of this theorem is inspired by the theory of harmonic maps by Sacks and Uhlenbeck [5].
The ingredients used in this theorem are: a local theory, working in suitable gauges (exponential gauge, broken harmonic gauge, harmonic gauge) Morrey's theorem [6] (for the purpose of bounding $||F||_\infty$ in terms of the bound on the action functional) standard elliptic theory for the final regularity of A.

§1 Statement of the result

The following is a removable singularities theorem for Yang-Mills fields in dimension 4. [3]

Theorem 1. Let A be a Y.M. connection in a bundle P over $B^4\setminus\{0\}$, such that

$$\int_{B^4} |F|^2 < \infty .$$

Then, there exists a gauge in which (P, A) extends to a smooth bundle $(\overline{P}, \overline{A})$ over B^4.

The following corollary is a theorem of extension of a Y.M. field defined outside a ball, to the one-point compactification.

It is a direct consequence of the removable singularities theorem, since the Y.M. functional is conformally invariant.

Corollary 1. Let A be a Y.M. connection in a bundle P over

$$\begin{array}{rcl} \mathrm{Ext}(N) \equiv \{x \in I\!\!R^4 \text{ s.t. } |x| \geq N\}, \text{ and let} \\ f : B^4 \backslash \{0\} & \rightarrow & \mathrm{Ext}(N) \\ x & \longmapsto & N\dfrac{x}{|x|^2}, \end{array}$$

such that $\int_{\mathrm{Ext}(N)} |F|^2 < \infty$.

Then (f^*P, f^*A) extend to $(\overline{f^*P}, \overline{f^*A})$ over B^4 in a suitable gauge.

Corollary 2. There is a natural bijection between Y.M. connections on bundles over $I\!\!R^4$ with finite action and Y.M. connections on bundles over S^4.

Proof of corollary 2. Let A be a connection over $I\!\!R^4$, $f : S^4 \backslash \{p_0\} \rightarrow I\!\!R^4$ the stereographic projection.
Then, f^*A is a connection over $S^4 \backslash \{p_0\}$.
Let us consider a geodesic ball centered at p_0.
By dilation we may assume that the metric is close to the flat metric. We can use conformal invariance of the Y.M. functional and theorem 1 to extend f^*A across p_0, in a suitable gauge.

Example. [7].
Every bundle over $I\!\!R^4$ is topologically trivial. Bundles over M^4, a compact Riemanian manifold are classified by the second Stiefel Whitney class and the first Pontryagin class p_1 (second Chern class c_2). If $M \equiv S^4$, $H^2(M) = 0$ with any coefficients. Hence, bundles over S^4 are classified by $p_1(P) = c_2(P) = i(P)$ also called the instanton number.

Take a self-dual connection on $I\!R^4$, the extension obtained is classified by the instanton number

$$i(P) \equiv c_2(P) = \frac{1}{8\pi^2}\int_{I\!R^4} |F|^2 = \frac{1}{4\pi^2}\ \text{Y.M.}(A)$$

Although any bundle on $I\!R^4$ is trivial, the extensions to the compactification S^4 are not necessarily so.

On can consider a quaternion line line bundle $P = I\!R^4 \times I\!H,\ D = d + A^\lambda$, where $A^\lambda = Im\left(\frac{x d\bar{x}}{\lambda^2 + |x|^2}\right)$.

The curvature of A^λ is given by $F^\lambda \equiv \frac{\lambda^2 dx \wedge d\bar{x}}{(\lambda^2 + |x|^2)^2}$.

One can check that $*F^\lambda = F^\lambda$, and Y.M.$(D) = 4\pi^2$.

(Note that the metric on the sphere is given by $ds^2 = \frac{4|dx|^2}{(1+|x|^2)^2}$, where x is the stereographic projection).

Hence, this bundle over $I\!R^4$ extends to a non trivial bundle, with instanton number 1 over S^4.

For general connections, $i(P) \equiv c_2(P) \equiv \frac{1}{8\pi^2}\int F \wedge F$, and Y.M.$(A) = \frac{1}{2}\int F \wedge *F = \frac{1}{2}\int |F|^2$.

The second chern class is a topological lower bound for the Yang-Mills functional. In fact, the Hodge operator $* : \Lambda^2 M \to \Lambda^2 M$ is such that $*^2 = 1$ and is an isometry. So, the space of two-forms splits into the ± 1 orthogonal eigenspaces of $*$, i.e. $F_A = F_A^+ + F_A^-$, where

$$*F_A^+ = F_A^+ \text{ (instanton), and } *F_A^- = -F_A^- \text{ (anti-instanton).}$$

(Note that $*F = \pm F$ are Y.M. fields. In fact $D(*F) = \pm DF = 0$, by the Bianchi identity).

One obtains

$$\frac{1}{8\pi^2}\int F \wedge F = \frac{1}{8\pi^2}\int (F^+ + F^-) \wedge (F^+ + F^-) = \frac{1}{8\pi^2}\int |F^+|^2 - |F^-|^2,$$

thus

$$\text{Y.M.}(A) \geq 4\pi^2 |c_2| \,,$$

and the minimum is achieved when F is either an instanton, or an anti-instanton.

Proof of theorem 1 (outline):

We may assume without loss of generality that Y.M.$(A) < \varepsilon$ (since Y.M. is conformally invariant).

The proof divides in two parts:

(1) Estimate $||F||_\infty$ over $B^4\backslash\{0\}$ in terms of action,

(2) Use "harmonic gauges" to extend A smoothly across the origin.

Part (1) is divided in three steps.

1st step. Finding an inequality for $\Delta|F|$, by means of Bochner-Weitzenbock formulas. Estimating by means of Morrey's theorem.

The Bochner-Weitzenbock formula for a differential form ψ gives

$|\psi|\Delta|\psi| = \langle\psi, \nabla^2\psi\rangle + \langle\nabla\psi, \nabla\psi\rangle - |d|\psi||^2 \geq \langle\psi, \nabla^2\psi\rangle = \langle\psi, \Delta\psi\rangle + \langle\psi, [F, \psi]\rangle$, (by definition).

Here $\Delta = DD^* + D^*D$, and ∇^2 is the crude Laplacian, and

$$[F, \psi] \equiv \sum_j \Big([F_{i,j}, \psi_{jk}] - [\psi_{kj}, F_{ij}] \Big) dx^i dx^k \,.$$

Substituting $\psi = F$ in this formula, one finds

$$|F|\Delta|F| \geq \langle F, [F, F]\rangle \,,$$

(since F satisfies the Yang Mills equations and the Bianchi identity).

Hence,

$$\Delta|F| \geq -4|F|^2 \,,$$

and we are in the hypotheses of the following theorem by Morrey, for $f = |F| = b$.

Morrey's theorem [3], [6]. Let $b \in L^q(B_{x_0}, a_0), q > n/2$, f a non negative function such that $f^\gamma \in L^2_1(B_{x_0}; a_0)$ for $\frac{1}{2} \leq \gamma < 1$, and $-\Delta f \leq b\, f$ in a weak sense. Then for $B_{x;a} \subset B_{x_0}; a_0$,

$$f^{2\gamma}(x) \leq K a^{-n} \int_{B_{x_0};a_0} f^{2\gamma} ,$$

where K depends uniformly on $n, q, \gamma, a_0^{q-n/2} \int_{B_{x_0};a_0} |b|^q$.

Uhlenbeck in [3] gives some technical lemmas and some minor extensions of this theorem.
She shows that if $\int f^p < \infty$, for some $p > 2$, then f is uniformly bounded in terms of the Y.M. action.
If $p = 2$, there exist constants c, K such that

$$\text{if} \quad \int_{B^4} f^2 < K \ , \ \text{then} \quad f^2 \leq \frac{c}{r^4} \int_{B_{x_0;r}} f^2 \ ,$$

for all $B_{x_0;r} \subset B_{x_0;1/2}$.

2^{rd} step. Choosing a suitable gauge, in which $||A||_\infty$ estimates in terms of $||F||_\infty$.

To find approximation to a good gauge over $B^4 \backslash \{0\}$, one works on annuli

$$U_m \equiv \{x \text{ s.t. } \frac{1}{2^{m+1}} \leq |x| \leq \frac{1}{2^m}\} \ ,$$

and constructs a broken gauge, as follows.
By dilation, we can assume $\int_{B^4} |F|^2 < \varepsilon$.

Hence $|F| \leq \varepsilon c |x|^{-2}$, i.e., on U_m $|F| \leq \varepsilon c 2^{m+1}$. In particular, if $m = 0$ than $|F| \leq 2c\varepsilon$.

We construct the "exponential gauge" on $S_1 \equiv \{x \in \mathbb{R}^4 \quad \text{s.t.} \quad |x| = 1\}$.
(Recall: two exponential gauges are constructed, one based at the north pole, the other based at the south pole and matched by a rotation at the equator).
This gauge satisfies

$$||A^1||_\infty \leq K_1 ||F^1||_\infty .$$

At this point, one extends trivially along radial lines to the annulus U_0.

3rd step. Constructing a "broken harmonic gauge" on $B^4 \backslash \{0\}$.
On U_0, we construct a gauge in which $d^*A = 0$.
This is done by means of the implicit function theorem between Banach spaces.
The construction of this gauge is somehow technical. For this construction we refer to [3].
This gauge (called Hodge, or Coulomb) is nearly harmonic (in the Abelian case, or in the linearized problem $(d^*d + dd^*)A = 0$, since $d^*dA = d^*F = 0$ and $d^*A = 0$).
At this point, one dilates by a factor 2 and repeats the process above, for the new annulus U_0. Working in the broken harmonic gauge, one shows that $|F|$ is uniformly bounded in the punctured ball $B^4 \backslash \{0\}$, and the new connection satisfies $|A(x)| \leq |x||F(x)|$.
Hence A is defined (continuous) over B^4.
Part (2) consists of finding a Hodge gauge in B^4, yielding the final smoothness of A.
This completes the proof of theorem 1.

§2. Good gauge theorem

We state now the good gauge theorem (Hodge gauge) for a connection A.
For more details, we refer to [9].

Theorem 2. Let B^n be a geodesic ball, of radius one, G a compact Lie group, $\frac{n}{2} \leq p < n$, D a connection that trivializes as $d + A$ with $A \in L_1^p$. There exists positive constants $k(n), c(n)$ such that if $\int |F_A|^{n/2} < K$, then A is gauge equivalent to a connection $\hat{A} \in L_1^p$ satisfying:

(a) $d^*\hat{A} = 0$.
(b) $A_\nu = 0$ on $S^{n-1} \equiv \partial B^n$ ($A_\nu \equiv$ normal component of A).
(c) $||A||_{L_1^{n/2}} \leq c(n)||F||_{L^{n/2}}$.
(d) $||A||_{L_1^p} \leq c(n)||F||_{L^p}$.

Outline of the proof. One shows that the set of connections that satisfy the hypotheses of theorem 2 is connected.
Then shows that the subset of those connections for which the theorem holds is open and closed.
The openness argument is the harder one, and is the one that uses the implicit function theorem between Banach spaces.
Let A be a connection for which the theorem is satisfied, and let's also assume that $||A||_{L^n} \leq K(n)$ (with no restriction).
Then, there exists ε such that if $||\lambda||_{L_1^p} < \varepsilon$, and $\lambda_\nu = 0$ on ∂B^n, the non-linear equation

$$d^*(g^{-1}dg + g^{-1}(A+\lambda)g) = 0$$

admits a solution $g \in L_2^p$.
One considers the operator

$$\Lambda : (u, \lambda) \longmapsto d^*(e^{-u}de^u + e^{-u}(A+\lambda)e^u)\,.$$

By linearization, one obtains

$$\begin{array}{rcl} \Lambda'_A : L^{p\perp}_{2,\nu} & \rightarrow & L^{p\perp} \\ \psi & \longmapsto & d^*(d\psi + [A,\psi]) = \Delta\psi + [A, d\psi]\,. \end{array}$$

(Here $\perp$ means $\int \psi = 0$, and ν means $\psi_\nu = 0$ at ∂B^n).
This operator is an isomorphism between these two spaces, if $||A||_n$ is small enough.
In fact

$$||\Lambda'_A(\psi)||_p \geq ||d^*d\psi||_p - ||A||_n d\psi||_q \geq ||d\psi||_{L_1^p}(K' - ||A_n||K'')\,,$$

with $q = \dfrac{np}{n-p}$

Now, the implicit function theorem yields existence of a solution $g = e^u \in L_2^p$.
The solution g depends smoothly on λ.

§3. Chern classes of Sobolev connections - [4]

There exist removable singularities theorems that do not use the Yang-Mills equations.

In §1, we show that Yang-Mills fields over $\mathbb{R}^4$, with finite action, arise from connections over S^4. Hence, the Chern numbers are integers.

There are reasons (from Physics) to inquire whether this is true for arbitrary connections.

It has been proven, first with some growth hypothesis, then in general in [4], that the same is true for connections with finite action, over a bundle with fiber isomorphic to a compact Lie group G.

The theorem is the following:

Theorem 3. Let $A \in L^{n/2}_{1,loc}(\mathbb{R}^n, g)$. If n is even, $n \neq 2$, and $\int_{\mathbb{R}^n} |F|^{n/2} < \infty$, then $c_n \equiv n^{th}$ Chern number that arises from a representation $\rho : G \to SU(N)$ is integral.

Corollary. A is the pull-back of a connection over S^n, via stereographic projection.

For a proof of theorem 3, we refer the reader to [4].

In the fourth lecture, [10], [11] we give a removable singularities theorem for boundary points, the proof of which does not use the Yang-Mills equations, for the first part, but uses similar arguments.

Chapter 3. A direct method for minizing the Yang-Mills functional over a four-dimensional manifold [8], [9].

In this chapter we see how to apply the good gauge theorem and the removable singularities theorem to find a smooth solution of the Yang-Mills equations over a four-dimensional manifold.

The method is due to Sedlacek [9] and it is strongly based on Uhlenbeck's compactness theorem (theorem 2.1 in [8]).

It involves working locally and then patching together the local solutions to find a global solution that lives on a smooth bundle.

§1. Brief geometric background

We work on a four-dimensional, Riemannian, compact manifold with no boundary, M. We consider a principal fiber bundle P over M, with fiber isomorphic to the Lie group G; i.e. locally $P|_{U_\alpha} \simeq U_\alpha \times G$, where the $\{U_\alpha\}$'s form a covering of M. The bundle P can be described (up to isomorphisms) by a collection of functions $\{g_{\alpha\beta}\}, g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$.

They are the transition functions from the trivialization over U_α, to the one over U_β and satisfy the cocycle conditions

$$g_{\alpha\beta}g_{\beta\alpha} = id \,,$$
$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\delta} = id \,.$$

We consider connections A, with curvature $F = dA + A \wedge A$, and the pure Yang-Mills functional

$$\text{Y.M.}(A) = \frac{1}{2}\int_M |F_A|^2 \,.$$

Recall that the Yang-Mills equations are obtained from Y.M.(A) via a variational principle and they are $D^*F = 0$ (in a weak sense; i.e. $\langle D_A w, F\rangle = 0$, for any smooth one-form w).

For reasons that will be explained later, we do not wish to fix the bundle we work on. We work with families of connections $\{A_i\}$ defined on bundles P_i over M. These

bundle have fibers isomorphic to the same Lie group G, which is assumed to be compact (although this last hypothesis can be removed).
Also, they have in common a topological obstructuion η, which is defined via Čech cohomology.
If the Lie group G is simply connected this obstruction is zero. If $G = SO(3)$, then $\eta \in H^2(M, Z_2)$ is known as the second Stiefel-Whitney class.

§2. A direct method of minimization – Analysis background.

We consider the family $\mathcal{F}$ of all connections on principal G-bundles over M with given obstruction η. Since the Y.M. functional is bounded below, we can define

$$m(\eta) \equiv \inf_{A \in \mathcal{F}} \text{Y.M.}(A).$$

We look for a connection A such that $Y.M.(A) = m(\eta)$. A direct method for finding A is the following classical method. One takes a minimizing sequence, i.e. a sequence $\{A_i\}$ such that

$$\text{Y.M.}(A_i) \longrightarrow m(\eta)$$

and hopes to be able to pass to a converging subsequence $\{A_i'\}$, $A_i' \to A_\infty$, such that the Y.M.$(A_\infty) = m(\eta)$. The good gauge theorem enables us to consider connections in some Sobolev space. Hence, in order to be able to pass to a converging subsequence, we need a compactness theorem for such Sobolev spaces of connections. For our purpose, it is enough have weak-convergence.

Definition. Let X be a Banach space and X^* its topological dual. We say $x_n \rightharpoonup x$ (converges weakly to x) *iff* $\lambda(x_n) \to \lambda(x)$ for every $\lambda \in X^*$.
A basic theorem of functional analysis says that a ball of radius r in a Banach space is weakly-compact (i.e. compact in the weak-topology) *iff* $X^{**} \approx X$.
The Sobolev space $L_k^p(U)$ of vector functions over neighborhoods $U \subset M$ are reflexive Banach spaces.

Hence, the following theorem holds:

Theorem. Any bounded, closed subset of $L^p_k(U)$ is weakly compact.

To prove the main theorem, stated and outlined in the next section, we use Sobolev embedding theorems and multiplication theorems.
In particular we use the embedding $L^2_1 \subset L^4$ and continuity of the multiplication maps $L^2_2 \oplus L^2_1 \to L^2_1$ and $L^4 \oplus L^4 \to L^2$, in dimension four.

§3. The main Theorem.

The following theorem states existence of a limiting connection for a sequence that minimizes the Y.M. action, over a G-bundle over M except at most a finite number of points.

Theorem 1 Let $\{A_i\}$ be a sequence of smooth connections on G-bundles P_i, with obstruction $\eta \in \check{H}^2(M, \pi, (G))$. Suppose Y.M. $(A_i) \to m(\eta)$. Then there exists a subsequence $\{A_i'\}$ and a countable cover $\{U_\alpha\}$ of M minus at most a finite number of points, $q_1, \ldots, q_k$, sections $\sigma_\alpha(i) : U_\alpha \to P_i$, connections $A_\alpha \in L^2_1(\Lambda' U_\alpha \otimes g)$, and functions $g_{\alpha\beta} \in L^2_2(U_\alpha \cap U_\beta, G)$ such that

$$
\begin{aligned}
&\sigma_\alpha(i)^* A_i\Big|_{U_\alpha} \equiv A^{(i)}_\alpha \rightharpoonup A_\alpha \quad (L^2_1)\\
&F_\alpha(i) \rightharpoonup F_\alpha \equiv dA_\alpha + A_\alpha \wedge A_\alpha \quad (L^2)\\
&g_{\alpha\beta}(i) \rightharpoonup g_{\alpha\beta} \quad (L^2_2)\\
&A_\alpha = g^{-1}_{\alpha\beta} A_\beta g_{\alpha\beta} + g^{-1}_{\alpha\beta} dg_{\alpha\beta}.
\end{aligned}
$$

Idea of the proof:
One uses weak-compactness of closed and bounded subsets of Sobolev spaces $L^2_1(U)$ and Uhlenbeck's good gauge theorem.

Since $\{A_i\}$ minimizes the action, if i is large enough Y.M. (A_i) are uniformly bounded.
One takes a countable family of covers C_j of radius δ_j, with $\delta_j \to 0$.
Each C_j is a finite cover of M and there are at most h balls in C_j that have non-empty intersection, with h independent of j.
The "bad" balls for A_i in a cover C_j are those $U \in C_j$ for which $\int_U |F_{A_i}|^2 > K(4)$.
Let $N_{ij} \equiv$ number of bad balls in C_j for the connection A_i.
Then $N_{ij}K(4) < \sum_{U \in C_j} \int_U |F_{A_i}|^2 < h\int_M |F_{A_i}|^2 < hL$, where L is a uniform bound for Y.M.(A_i).
Fixing C_j, one finds a subsequence $\{A_i'\}$ for which the centers of the bad balls are eventually fixed.
By diagonalization, one finds a new subsequence $\{A_i'\}$ such that the following holds: $\forall j$, $\exists J$ s.t. the centers of the bad balls are fixed for every A_i, with $i > J$.
By throwing away from every C_j the "eventually" bad balls and taking everything that is left, one ends up with a cover $C \equiv \{U_\alpha\}$ in which U_α is eventually good for the new sequence.
By means of conformal invariance, one can apply the good gauge theorem to the ball of radius one and find good gauges for the connections $A_i\big|_{U_\alpha}$.
In these gauges the $A_\alpha(i)$ satisfy uniform estimates $||A_\alpha(i)||_{L_1^2} \leq$ const $||F||_{L^2} <$ const.
At this point one uses weak-compactness to find local limiting connections.
To show (2), (3), (4) one needs to use multiplication theorems and Sobolev embeddings mentioned in the previous section.

Remarks

Why is the dimension four special? The L^2-norm of curvature, i.e. the Y.M. functional is conformally invariant in dimension four. The covering procedure adopted barely works in dimension four, leaving some isolated singularities. The multiplication theorems used also barely work in dimension four. In higher dimension, not only the minimizing procedure does not work locally, but also patching the local solutions to form a connection on a reasonably smooth bundle does not work. In the next section, we see more in details what can happen over isolated points $q_1, \dots, q_k$.

Theorem 2. Let $A_\infty \equiv \{A_\alpha\}$ be the connection on the bundle $P_\infty \equiv \{g_{\alpha\beta}\}$ over $M \backslash \{q_1, \dots, q_k\}$ found in theorem 1.
The following holds:
(1) A_∞ satisfies the Yang-Mills equations. Moreover A_∞ is in a Hodge gauge, i.e. $d * A_\infty = 0$.
(2) A_∞ can be extended to a connection $\widehat{A}_\infty$ on a G-bundle $\widehat{P}_\infty$ over M.
(3) $\eta(\widehat{P}_\infty) = \eta$ (the obstructions is preserved).
(4) Y.M.$(A_\infty) = m(\eta)$

Idea of the proof: One shows that A_∞ satisfies the Y.M. equations by contradiction.
The connection A_∞ must satisfy the Y.M. equations, otherwise one could construct a sequence of smooth connections that for indexes sufficiently large have energy less than $m(\eta)$.
In fact, let α be such that $\langle D_{A_\alpha} w, F_{A_\alpha} \rangle < 0$ for some smooth one-form w with compact support on U_α.
One can think of w as the trivialization over U_α of one-forms w_i on the bundles P_i.
The connections $A_i + tw_i$ differ from A_i only on U_α, and for i sufficiently large and t small Y.M. $(A_i + tw_i) =$ Y.M.$(A_i) + t\langle D_{A_\alpha(i)} w, F_\alpha(i) \rangle + \{$ higher order terms in $t\} <$ Y.M.(A_i).
This is a contradiction, since there is sufficient regularity to show that all terms in the above inequality converge to the obvious limits.
One obtains (2) as consequence of the removable singularities theorem for Yang-Mills connections, proven in the previous chapter.
The obstruction η does not see isolated points, hence (3) holds.

One obtains (4) as consequence of (3) and lower semicontinuity of the Y.M. functional.
In fact, $\text{Y.M.}(A_\infty) \leq \underline{\lim}\,\text{Y.M.}(A_i) \equiv m(\eta)$ implies $\text{Y.M.}(A_\infty) = m(\eta)$.

§4. What can happen over isolated points $q_1, \ldots, q_k$? (the bubble theorem).

We showed that for any given $\eta \in H^2(M, \pi_1 G)$, there exist a bundle and a connection (P_∞, A_∞) with obstruction η, such that A_∞ satisfies the Yang-Mills equations and is an absolute minimum for the Yang-Mills functional over all bundles with obstruction η.
What happens if in the minimizing procedure we fix the bundle P and consider a sequence of connections $\{A_i\}$ over P, that minimize action over P, i.e. $\text{Y.M.}(A_i) \to m(p)$?
The minimizing procedure could be carried out in a similar way, to find a critical point A_∞ (A_∞ satisfies Y.M. equations) on a bundle P_∞, also with obstruction $\eta = \eta(P)$.
Since the Y.M. functional is lower semicontinuous, $\text{Y.M.}(A_\infty) \leq m(P)$.
The bundle P_∞ is in general different from the bundle P we start with, since the second Chern class is not preserved in the procedure.
In general, if $\{A_i\}$ is a minimizing sequence over a fixed bundle P, one has

$$\int_M |F^+_{A_\infty}|^2 \leq \underline{\lim} \int_M |F^+_{A_i}|^2 ,$$

and

$$\int_M |F^-_{A_\infty}|^2 \leq \underline{\lim} \int_M |F^-_{A_i}|^2 .$$

Thus,

$$4\pi(p_1(P) - p_1(P_\infty)) = \int |F^+_{A_i}|^2 - |F^-_{A_i}|^2 - |F^+_{A_\infty}|^2 + |F^-_{A_\infty}|^2 =$$

$$= \int |F^+_{A_i}|^2 + |F^-_{A_\infty}|^2 - \int |F^-_{A_i}|^2 + |F^+_{A_\infty}|^2 \leq$$

$$\leq \int |F^+_{A_i}|^2 + \underline{\lim} \int |F^-_{A_i}|^2 - \underline{\lim}|F^-_{A_i}|^2 + |F^+_{A_\infty}|^2 \leq$$

$$\leq \int |F^+_{A_i}|^2 + \int |F_{A_i}|^2 - \int |F^-_{A_\infty}|^2 + |F^+_{A_\infty}|^2 =$$

$$= \text{Y.M.}(A_i) - \text{Y.M.}(A_\infty) \leq m(P) - \text{Y.M.}(A_\infty).$$

Changing orientation on M, one obtains

$$4\pi^2 |p_1(P) - p_1(P_\infty)| \leq m(P) - \text{Y.M.}(A_\infty).$$

In other words, it is possible when minimizing on a fixed bundle that the limiting connection correspond to a lower energy. This happens because energy is concentrating on the fibers over the points $q_1, \ldots, q_k$, that are removed.

This phenomenon is expressed in the following theorem (see pg. 64 in [7]):

The bubble theorem: Let $\{D_i\}$ be a sequence of self-dual connections on a $SU(2)$-bundle with instanton number 1, over a 4-dimensional, compact, Riemannian manifold M, with no boundary, then $\{D_i\}$ admits a subsequence $\{D'_i\}$ such that one of the following holds:

(1) $\{D'_i\}$ is gauge-equivalent to $\{\widehat{D'_i}\}$, such that $\widehat{D'_i} \to \widehat{D}$ in the moduli space $\mathcal{M}$.

(2) there exists a point $q \in M$, and, $\forall \varepsilon > 0$, there exists a trivialization of the bundle over $M \backslash B_{q,\varepsilon} \equiv K$ such that $\{D'_i\}$ is gauge-equivalent to $\{\widehat{D'_i}\}$, with $\widehat{D'_i} = d + \hat{A}'_i$, $\hat{A}'_i \to 0$ on K.

On $B_{q,\varepsilon}$, there is a sequence of dilations

$$\rho_i(x) \equiv \frac{x}{\varepsilon_i} \quad \text{with} \quad \varepsilon_i \to 0,$$

and a sequence of connections $\{\widehat{D'_i}\}$ gauge equivalent to $\{D'_i\}$ over B_ε, such that for some $x_0 \in \mathbb{R}^4$,

$$\rho_i^* \widehat{A}_i' \to \mathrm{Im}\left(\frac{(x - x_0)d\overline{x}}{1 + |x - x_0|^2}\right).$$

The phenomenon described in (2) is known as bubbling. What happens is that, by taking smaller and smaller neighborhoods and a sequence of blow-ups, the sequence of connections is not converging to something with zero energy, but to the t'Hooft instanton. In the limit, the bundle we start with becomes flat and a four-sphere breaks off at q, carrying the energy of an instanton.

One can think of doing the inverse process, i.e. gluing instantons to Y.M. connections, to obtain new solutions with higher energy.

In the gluing process the new connection won't be exaxtly Yang-Mills, one needs to apply implicit function theorem arguments to find exact solutions.

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