STABILITY OF THE SUCKER ROD'S PERIODIC SOLUTION

Aloisio Freiria Neves

Janeiro RP 01/94

Relatório de Pesquisa

Instituto de Matemática
Estatística e Ciência da Computação

UNIVERSIDADE ESTADUAL DE CAMPINAS
Campinas - São Paulo - Brasil

ABSTRACT – We study the stability properties of the sucker rod pumping systems. We show the existence of a global attractor. Using results of nonautonomous differential equations, dynamical systems and α-contractions, we prove under very natural assumptions on the external σ-periodic force, that the attractor is exactly a periodic orbit.

IMECC – UNICAMP
Universidade Estadual de Campinas
CP 6065
13081-970 Campinas SP
Brasil

Janeiro – 1994

STABILITY OF THE SUCKER ROD'S PERIODIC SOLUTION

by
Aloisio Freiria Neves
IMECC - UNICAMP
CP 6065
13081–970, Campinas, SP, Brazil

Abstract

We study the stability properties of the sucker rod pumping systems. We show the existence of a global attractor. Using results of nonautonomous differential equations, dynamical systems and α-contractions, we prove under very natural assumptions on the external σ-periodic force, that the attractor is exactly a periodic orbit.

Key words: Periodic Solution, Bounded Dissipative, α-contraction, Attractor.

1 Introduction

In [2] we proved the existence of a strong periodic solution for a mathematical model of sucker rod pumping systems given by:

$$\begin{aligned} &u_{tt} - u_{xx} + cu_t = \hat{f}(t,x); \quad 0 < x < \ell, \ t > 0, \\ &u(t,0) = \mu(t), \\ &u_x(t,\ell) = m\,[1 - H(u_t(t.\ell))]. \end{aligned} \tag{1.1}$$

Here $u(t,x)$ denotes the displacement at time $t > 0$ of a material point at position $x \in [0,\ell]$ at reference configuration; ℓ is the length of the rods; c is a positive constant that embodies the dissipation mechanisms; $\hat{f}(t,x)$ are external σ–periodic forces, like gravity forces, acting on the body of the rods; $\mu(t)$ are the σ–periodic motion imparted by the engine on the top of the rods; m is a positive constant associated to the weight of the column of fluid that acts on the rods when the walking valve closes; $H(.)$ is the Heaviside function, that is $H(x) = 1$ if $x > 0$ and $H(x) = 0$ if $x < 0$.

The systems can be made homogeneous at the end $x = 0$, through a convenient changing of variable, the action of $\mu(t)$ can be incorporated to the external force $\hat{f}(t, x)$, and system the will became

$$\begin{aligned} &u_{tt} - u_{xx} + cu_t = f(t, x); \quad 0 < x < \ell, \ t > 0, \\ &u(t, 0) = 0, \\ &u_x(t, \ell) = m\,[1 - H(u_t(t.\ell))]. \end{aligned} \tag{1.2}$$

The mechanism will work imparted by the σ-periodic force $f(t, x)$. We observe that, if the amplitude and frequency of f are small, the system can not work, that is, the down end, $x = \ell$ can be stopped, $u_t(t, \ell) \equiv 0$. It will be assumed, that is not the case, we will suppose

H) *$f(t, x)$ is such that, for strong solutions, $u_t(t, \ell)$ assume positive and negative values.*

therefore, in this way, at least for the strong solutions, we have $u_x(t, \ell)$ assuming the maximum value m, and the minimum value 0.

In this paper we will analyze the stability properties of the periodic solution of the system (1.2). The point of view adopted here is the same of Hale [6], Sell [3] and Ceron and Lopes [1] for nonautonomous equations and dissipative processes.

We will work with the same abstract formulation used in [2], that is:

$$\dot{w} + Aw = (0, f) \tag{1.3}$$

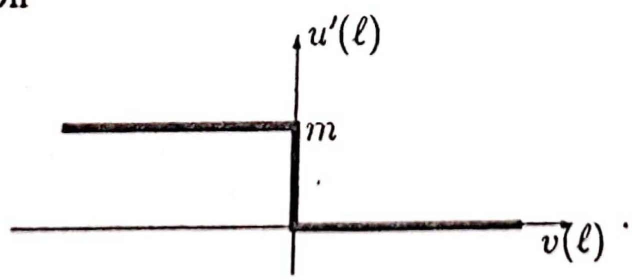
where $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is the operator given by

$$A(u, v) = (-v, -u'' + cv), \tag{1.4}$$

on the domain:

$$\mathcal{D}(A) = \{(u, v) \in H^2(0, \ell) \cap H_{1,0} \times H_{1,0} : (v(\ell), u'(\ell)) \in \Gamma\}; \tag{1.5}$$

where Γ is the graph



$w = (u, v) \in \mathcal{H} = H_{1,0} \times L^2(0, \ell)$, $H_{1,0} = \{u \in H^1(0, \ell) : u(0) = 0\}$, and

$$\|w\|^2 = \|(u, v)\|^2 = |u'|^2_{L^2} + |v|^2_{L^2}$$

2 The Abstract Theory

Let X be a complete metric space, $\mathbb{R}^+ = [0, \infty)$. A family of mappings $S(t) : X \to X$, $t \geq 0$, is said to be a dynamical systems (or a C^0-semigroup) if:

1. $S(0) = I$,
2. $S(t+s) = S(t)S(s)$,
3. $S(t)x$ is continuous in t, x.

Definition 2.1 *For any $x \in X$, the orbit $\gamma(x)$ through x is defined as $\gamma(x) = \{S(t)x,\ t \geq 0\}$, the ω-limit set $\omega(x)$ of x is defined as*

$$\omega(x) = \bigcap_{s\geq 0}\left(\overline{\bigcup_{t\geq s} S(t)x}\right),$$

and, a set $B \subset X$ is said to attract a set C (under $S(t)$) if $d(S(t)C, B) \to 0$ as $t \to \infty$.

Definition 2.2 *Let α be the Kuratowski measure of noncompactness, that is,*

$$\alpha(B) = inf\{d : \ B \text{ has a finite cover by sets of diameter } \leq d\}$$

The semigroup $S(t)$, $t \geq 0$, is said to be a α-contraction if there is a continuous function $k : \mathbb{R}^+ \to \mathbb{R}^+$ such that $k(t) \to 0$ as $t \to \infty$, and, for each $t > 0$ and each bounded set $B \subset X$ we have $\{S(s)B,\ 0 \leq s \leq t\}$ bounded and $\alpha(S(t)B) \leq k(t)\alpha(B)$

Definition 2.3 *A pseudo-metric ρ is said to be precompact if any bounded sequence (with respect to the distance of X) has a subsequence which is Cauchy with respect to ρ.*

Theorem 2.1 *If for each $t > 0$, $S(t)$ satisfies*

$$d(S(t)x, S(t)y) \leq k(t)d(x,y) + \rho_t(x,y),$$

where $k(t) \geq 0$ and ρ_t is a precompact pseudo-metric, then

$$\alpha(S(t)B) \leq k(t)\alpha(B)$$

for any bounded set B.

The proof of the theorem can be found in [1].

Theorem 2.2 *If $S(t)$, is a α-contraction and B is a nonempty set in X such that $\gamma(B)$ is bounded, then $\omega(B)$ is nonempty, compact, invariant, and $\omega(B)$ attracts B. If, in addition, B is connected, then $\omega(B)$ is connected. In particular, if, for some $x \in X$, $\gamma(x)$ is bounded, then $\gamma(x)$is compact and $\omega(x)$ is nonempety, compact, connected, and invariant.*

The proof can be found in [6].

3 Application to Sucker Rod Pumping Systems

We proved in [2] that the operator given in (1.4), (1.5) is maximal monotone, so, if $f \in BV_{loc}(0,\infty;L^2(0,\ell))$ we can apply the results of this theory (see, [4]), to obtain, for every initial condition $w_0 = (u_0, v_0) \in \mathcal{H}$, a unique (mild) solution $w(t) = (u(t), v(t))$, $t \geq 0$, of (1.3) such that $w(0) = w_0$ and, moreover, if $w_0 \in \mathcal{D}(A)$ then, $w(t) \in \mathcal{D}(A)$, $t \geq 0$, $w(t)$ has a right derivative that satisfies $\frac{d^+w}{dt} = f(t+0) - Aw(t)$.

Defining $T_f(t)w_0 = w(t) = (u(t), v(t))$, $t \geq 0$, $w_0 \in \mathcal{H}$ where $w(.)$ is the solution of (1.3) such that $w(0) = w_0$, we have that $T_f(t)w_0$ is continuous in t and w_0 [4, Theorem 3.16, pg 102], or more specifically, if $w_n \to w_0$ then $T_f(t)w_n \to T_f(t)w_0$ uniformly for t in compact interval of $\mathbb{R}^+$. For simplicity we will denote $T_f(t)$ by $T(t)$.

Theorem 3.1 *If $f \in BV_{loc}(0,\infty;L^2(0,\ell)) \cap L^\infty(0,\infty;L^2(0,\ell))$, then the solutions of (1.3), corresponding to initial values in a bounded set go exponentially to a bounded set, in particular, the problem (1.3) is bounded dissipative and orbits of bounded sets are bounded.*

The proof is given in [2, remark on pg. 10].

Theorem 3.2 *If $W(w)$ is given by*

$$W(w) = W(u,v) = \int_0^\ell [\frac{1}{2}(u')^2 + \frac{1}{2}v^2 + 2\beta uv]dx, \qquad \beta = \frac{2c}{8 + c^2\ell^2},$$

then,

$$i) \quad \frac{1}{4}(|u'|^2 + |v|^2) \leq W(u,v) \leq \frac{3}{4}(|u'|^2 + |v|^2),$$

and, moreover, there exists a constant $\lambda > 0$, such that

$$ii) \quad W(T(t)w_1 - T(t)w_2) \leq e^{-\lambda t} W(w_1 - w_2) + \rho_t(w_1, w_2)$$

where ρ_t is a precompact pseudo-metric, for every $t > 0$.

Proof: The proof of first item $i)$ can be found in [2]. For the the second one, if $w_1, w_2 \in \mathcal{D}(A)$, we can use the same kind of computation of [2, theorem 3.1] to obtain

$$W(T(t)w_1 - T(t)w_2) \leq e^{-\lambda t} W(w_1 - w_2) + 4\beta m \int_0^t |u_1(s,\ell) - u_2(s,\ell)| ds.$$

where $u_i(s,.)$ is the first component of $T(s)w_i$.
Since $\mathcal{D}(A)$ is dense in $\mathcal{H}$, we can conclude that this inequality remains true for every $w_1, w_2 \in \mathcal{H}$.

Setting, for $t > 0$.

$$\rho_t(w_1, w_2) = 4\beta m \int_0^t |u_1(s,\ell)) - u_2(s,\ell)| ds.$$

we have that ρ_t is a precompact pseudo-metric. Indeed, if $\{w_{0,n}\}$ is a bounded sequence in H, then from theorem 3.1 we have for the corresponding solutions that $\|u_n\|_{L^2(0,t;H^1(0,\ell))}$ and $\|\dot{u}_n\|_{L^2(0,t;L^2(0,\ell))}$ are bounded, and since the inclusion $H^1(0,\ell) \hookrightarrow C[0,\ell]$ is compact the Aubin–Lions lemma (Lions [7], theorem 5.1, p. 58) implies that $\{u_n\}$ is relatively compact in $L^2(0,t;C[0.\ell])$. Therefore, there is a Cauchy subsequence, that we keep denoting by $\{u_n\}$, in $L^2(0,t;C[0.\ell])$. Therefore, since

$$\rho_t(w_n, w_m) = 4\beta m \int_0^t |u_n(s,\ell)) - u_m(s,\ell)| ds.$$

$$\leq 4\beta m \int_0^t |u_n(s)) - u_m(s)|_{L^\infty} ds \leq 4\beta m t^{1/2} \left(\int_0^t |u_n(s)) - u_m(s)|^2_{L^\infty} ds \right)^{1/2}$$

$$= 4\beta m t^{1/2} \|u_n - u_m\|_{L^2(0,t;C[0.\ell])}.$$

we have ρ_t is a compact pseudo-metric, and the theorem is proved.

Using the results of the two previous theorems and the results of [5] and [6] we can state the following result:

Theorem 3.3 *If $f \in BV_{loc}(0,\infty; L^2(0,\ell))$ is σ-periodic in t, the Poincaré map $w \to T(\sigma)w$ has a fixed point (and therefore (1.3) a σ-periodic solution) and a global, compact, invariant attractor.*

Given $f \in BV(0,\sigma; L^2(0,\ell))$, defined in the interval $[0,\sigma]$ we will extend f periodically to $\mathbb{R}$, and consider the function space $\mathcal{F}$, of all translates of f,

$$\mathcal{F} = \{f_\tau : f_\tau(t) = f(\tau + t)\} \subset L^1(0,\sigma; L^2(0,\ell)).$$

Also, we will denote by X the following metric space:

$$X = \mathcal{H} \times \mathcal{H} \times \mathcal{F}.$$

where the metric considered is the maximum one. Under this conditions we have:

Lemma 3.1 *The function space $\mathcal{F} \subset L^1(0,\sigma; L^2(0,\ell))$ is compact*

Proof: Given a sequence f_{τ_n}, we can suppose, using the fact of f is σ-periodic, that $\tau_n \in [0,\sigma]$, and moreover, that $\tau_n \to \tau$ in $[0,\sigma]$ taking a subsequence if necessary. Setting $h_n = |\tau_n - \tau|$ and $V_f(t) = Var(f : [-\sigma, t])$, we can change the coordinates and use the σ-periodicity of f to obtain

$$\int_0^\sigma |f_{\tau_n}(t) - f_\tau(t)|dt = \int_0^\sigma |f(\tau_n + t) - f(\tau + t)|dt$$

$$= \int_{-h_n}^{\sigma - h_n} |f(t + h_n) - f(t)|dt \leq \int_{-h_n}^{\sigma - h_n} V_f(t + h_n) - V_f(t)dt$$

$$\leq \int_{\sigma - h_n}^{\sigma} V_f(t)dt \leq h_n V_f(\sigma).$$

therefore $f_{\tau_n} \to f_\tau \in \mathcal{F}$.

The next lemma is standard.

Lemma 3.2 *If X and Y are Banach spaces, then considering $X \times Y$ with the maximum norm, we have the following:*

1. $diam(B) = diam(P_X(B) \times P_Y(B))$
   $= max\{diam(P_X(B)), diam(P_Y(B))\}$

2. $\alpha(B) = \alpha(P_X(B) \times P_Y(B)) = max\{\alpha(P_X(B), \alpha(P_Y(B))\}$

where $B \subset X \times Y$ is a bounded set and P_X and P_Y are the respective projections. Moreover, if $T_1 : X \to X$ and $T_2 : Y \to Y$ satisfies

$$\alpha(T_1(B_1)) \leq q_1\alpha(B_1); \qquad \alpha(T_2(B_2)) \leq q_2\alpha(B_2)$$

then, $T : X \times Y \to X \times Y$; $T(x,y) = (T_1x, T_2y)$, satisfy

$$\alpha(T(B)) \leq max\{q_1, q_2\}\alpha(B)$$

where $B \subset X \times Y$, $B_1 \subset X$ and $B_2 \subset Y$ are bounded sets.

Theorem 3.4 *The family of mappings $S(t) : X \to X$, $t \geq 0$, given by*

$$S(t)(w_1, w_2, f) = (T_f(t)w_1, T_f(t)w_2, f_t)$$

is a dynamical system

Proof: X is a complete metric space. The Semi-group propriety $S(t+s) = S(t)S(s)$ follows using the same arguments of Sell [3], and the continuity is a consequence of Brezis [4, theorem 3.16, pg 102].

Theorem 3.5 *If $f \in BV(0,\sigma; L^2(0,\ell))$ satisfies H), then the periodic solution of (1.3) is globally asymptotically stable.*

Proof: Theorems 3.1, 3.2 and lemmas 3.1, 3.2 imply $\{S(t) : t \geq 0\}$ is a α-contraction and orbits of bounded sets are bounded, therefore, from theorem 2.2 we have that $\gamma(w_1, w_2, f)$ is pre-compact and, the ω-limit set, $\omega(w_1, w_2, f)$, is nonempty, compact connected and, invariant.

Now we will consider the function $V : X \to I\!R$ given by $V(w_1, w_2, f) = \|w_1 - w_2\|^2$. If $w_1,\ w_2 \in \mathcal{D}(A)$, and $t \geq 0$, we have, since the graph Γ is non increasing, that,

$$\frac{d^+}{dt}V(S(t)(w_1, w_2, f)) = \frac{d^+}{dt}\|T(t)w_1 - T(t)w_2\|^2$$

$$= - < T(t)w_1 - T(t)w_2, AT(t)w_1 - AT(t)w_2 >$$

$$\leq -c\int_0^\ell (v_1 - v_2)^2 dx \ \leq 0$$

where v_i is the second component of $T(t)w_i$, $i = 1, 2$.
Then, $V(S(t)(w_1, w_2, f))$ is non increasing, and a simple argument of density imply that this is also true for $w_1, w_2 \in \mathcal{H}$. Therefore,

$$V(S(t)(w_1, w_2, f)) \ \to \ d \geq 0, \quad \text{as } t \to \infty,$$

and

$$V \equiv d \quad \text{on } \omega(w_1, w_2, f).$$

If $(\bar{w}_1, \bar{w}_2, \bar{f}) \in \omega(w_1, w_2, f)$ and $(\bar{u}_i, \bar{v}_i) = T_{\bar{f}}(t)\bar{w}_i$, $i = 1, 2$, we can approximate $\bar{w}_1,\ \bar{w}_2$ by sequences $w_{1,n},\ w_{2,n} \in \mathcal{D}(A)$ and therefore, since $\omega(w_1, w_2, f)$ is invariant,

$$V_n(t) := V(S(t)(w_{1,n}, w_{2,n}, \bar{f})) \quad \to \quad V(t) := V(S(t)(\bar{w}_1, \bar{w}_2, \bar{f})) = d,$$

as $n \to \infty$, uniformly for t in compact intervals.
If $v_{i,n}$ is the second component of $T_{\bar{f}}w_{i,n}$ $(i = 1, 2)$, we have

$$\frac{d^+}{dt}V_n(t) \leq -c\int_0^\ell (v_{1,n} - v_{2,n})^2 dx$$

that imply

$$V_n(t) - V_n(0) \leq -c\int_0^t \int_0^\ell (v_{1,n} - v_{2,n})^2 dx ds$$

then, as $n \to \infty$, we have

$$0 \leq -c\int_0^t \int_0^\ell (\bar{v}_1 - \bar{v}_2)^2 dx ds \Longrightarrow \bar{v}_1 = \bar{v}_2.$$

Therefore, setting $\varphi = \bar{u}_1 - \bar{u}_2$, we have $\varphi_t = \bar{v}_1 - \bar{v}_2 = 0$ and then φ is an equilibrium solution of the linear problem:

$$\begin{aligned} u_{tt} - u_{xx} &= 0 \\ u(t,0) &= 0 \\ u_t(t,cll) &= 0 \end{aligned}$$

that is $\varphi(t,x) = kx$, k constant, and

$$\bar{u}_1(t,x) = \bar{u}_2(t,x) + kx$$

therefore

$$(\bar{u}_1)_x(t,\ell) = (\bar{u}_2)_x(t,\ell) + k$$

and then k must be zero, since the maximum and minimum of $(\bar{u}_i)_x(t,\ell)$, $i = 1,2$, are respectively m and 0. Therefore $\bar{w}_1 = \bar{w}_2$ and $d = 0$.
In particular for w_2 the initial condition of the strong periodic solution γ of (1.3) and w_1 arbitrary, we have $V(S(t)(w_1, w_2, f)) \to 0$, as $t \to \infty$, that is $T(t)w_1 \to \gamma$, and the theorem is proved. .

References

[1] S.S. Ceron and O. Lopes, *α-contractions and attractors for dissipative semilinear hyperbolic equations and systems*, Annali di Matematica pura ed applicata, IV, CLX(1991), 193-206.

[2] A.F. Neves and J.L. Boldrini, *Periodic motions of sucker rod pumping systems*, Nonlinear Analysis TMA, to appear.

[3] G.R. Sell, *Nonautonomous differential equations and topologial dynamics I, II*, Trans. Amer. Math. Soc., 127, (1967), 241-283.

[4] H. Brézis, *Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert*, North-Holland, Amsterdam, 1973.

[5] J.K. Hale, O.F. Lopes, *Fixed point theorem and dissipative processes*, J. Diff. Eq. 13, (2), (1973), pp 391–402.

[6] J.K. Hale, *Asymptotic behavior of dissipative systems*, Mathematical survey and monographs, 25, AMS, Providence, 1988.

[7] J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthier–Villars, Paris, 1969.