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THE EQUATIONS OF A VISCOUS INCOMPRESSIBLE CHEMICAL ACTIVE FLUID I: UNIQUENESS AND EXISTENCE OF THE LOCAL SOLUTIONS

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ABSTRACT - By using the spectral Galerkin method, we prove the existence and uniqueness of strong local solutions for the motion of a chemical active fluid. We also derive estimates of the solution that are useful for obtaining error bounds for the approximate solutions.

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Abstract

By using the spectral Galerkin method, we prove the existence and uniqueness of strong local solutions for the motion of a chemical active fluid. We also derive estimates of the solution that are useful for obtaining error bounds for the approximate solutions.

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Key Words: Chemical active fluid, local strong solutions, Galerkin method, Navier-Stokes equations.

1 Introduction

In this work we study the initial value problem for the equations that describe the motion of a viscous-chemically-active fluid in a bounded domain $\Omega \subseteq \mathbb{R}^n$, $n = 2$ or 3, in the time interval $[0, T]$, $0 < T < +\infty$.

In the Oberbeck- Boussinesq approximation, the state of such a system is described by the equations (see Joseph [14]).

*Short title: chemical active fluid.

$$\left.\begin{aligned}&\frac{\partial u}{\partial t}+u\cdot\nabla u-\Delta u+\nabla p=j+(\tilde\theta+\tilde\psi)g\ ,\\&\frac{\partial\tilde\theta}{\partial t}+u\cdot\nabla\tilde\theta-\Delta\tilde\theta=f\\&\frac{\partial\tilde\psi}{\partial t}+u\cdot\nabla\tilde\psi-\Delta\tilde\psi=h\\&\text{div }u=0\ .\end{aligned}\right\}\tag{1.1}$$

Here $u(t,x)\in\mathbb{R}^n$, $\tilde\theta(t,x)\in\mathbb{R}$, $\tilde\psi(t,x)\in\mathbb{R}$ and $p(t,x)\in\mathbb{R}$ denote respectively the unknowns velocity, temperature, concentration of material in the liquid and the pressure at a point $x\in\Omega$ at time $t\in[0,T]$; $g(t,x)$, $j(t,x)$, $f(t,x)$ and $h(t,x)$ are given source functions.

On the boundary Γ, we assume that

$$u(t,x)=0\ ;\quad \tilde\theta(t,x)=\theta_1\ ;\quad \tilde\psi(t,x)=\psi_1\tag{1.2}$$

where θ_1 and ψ_1 are known functions, and the initial conditions are expressed by

$$u(0,x)=u_0(x)\ ;\quad \tilde\theta(0,x)=\tilde\theta_0(x)\ ;\quad \tilde\psi(0,x)=\tilde\psi_0(x)\tag{1.3}$$

where u_0, $\tilde\theta_0$ and $\tilde\psi_0$ are given functions on the variable $x\in\Omega$.

The expression ∇,Δ and div, as usual, denote the gradient, Laplacian and divergence operators, respectively; the i$^{\underline{\text{th}}}$ component of $u.\nabla u$ is given by $[u.\nabla u]_i=\sum_{j=1}^{n}u_j\frac{\partial u_i}{\partial x_j}$; $(u.\nabla)\phi=\sum_{j=1}^{n}u_j\frac{\partial\phi}{\partial x_j}$, for $\phi=\tilde\theta$ or $\tilde\psi$.

The main goal in this paper is to show existence and uniqueness of strong solutions. Our strategy for setting this question consists of transforming problem (1.1)-(1.3) into another initial-value problem with homogeneous boundary condition; next, this new initial value problem is treated by using spectral Galerkin approximation (spectral in the sense that the eigenfunctions of the Stokes and Laplacian operators are used as basis, functions).

It is now appropriate to mention some earlier works on the initial boundary-value problem (1.1)-(1.3), which are related to ours.

When chemical reactions are absent ($\tilde\psi\equiv0$), the problem (1.1)-(1.3) is equivalent to the classical Boussinesq's problem, which has been investigated by several authors; see for instance Hishida [12], Korenev [15], Morimoto [19], Shinbrot and Kotorynski [24] and the references therein. Concerning the system (1.1)-(1.3), Gil's [10] studied the stationary model, Belov and Kapitonov [3], the stability of the solutions of the system (1.1)-(1.3) with different boundary conditions. They used linearization and fixed point arguments. The more constructive Galerkin method was used by Morimoto [19], in the case of Boussinesq's problem, to obtain global in time weak solution for $2 \leq n \leq 4$ and by Korenev [15], again in the Boussinesq's problem, to obtain local and global in time strong solutions for $2 \leq n \leq 3$, both with different boundary conditions, and by Rojas-Medar and Lorca [21] by using the Spectral Galerkin method to show the global existence in time of the weak solutions for any $n \geq 2$.

In the case of the Classical Navier-Stokes equations ($\tilde{\theta} \equiv \tilde{\psi} \equiv 0$), Prodi [20], by using the eigenfunctions of the Stokes operator as basis for the Galerkin method, obtained more regular solution, under weaker assumption on the data. Also, by using this basis Heywood [11] showed the classical regularity of the solution in a way that was easier and independent of potential theory (for this last technique see for example Ito [13], Fujita and Kato [8], Ladyzhenskaya [16], Giga and Miyakawa [9]).

In this paper we extend the ideas of Prodi [20] and Heywood [11] to the system (1.1)-(1.3). We prove the local existence of strong solution of (1.1)-(1.3). Our results also are valid in the case $\tilde{\psi} \equiv 0$ (Boussinesq's problem), and they extend the results of Korenev [15] and Hishida [12], since the initial data can be more irregular than theirs. Also, differently from Hishida [12], we will use the more constructive Spectral Galerkin method of approximation. Thus, the results in this paper form the theoretical basis for future numerical analysis of the problem: here we will obtain estimates for the approximate solutions that will be fundamental in a paper in which we will obtain optimal error estimates for such approximations (see [22]). These estimates will also play a rol in the proof of global existence of solutions of (1.1)-(1.3). (See [23]). In another publication we will study the regularity for $t > 0$ of solution obtained in this paper.

Finally, the paper is organized as follows: In Section 2 we state the basic assumptions and results that will be used later in the paper, we state the transform problem and we also rewrite the transform problem in a more suitable weak form; we describe the approximation method. In Section 3 we prove our first result of the existence of strong solution. In Section 4 we prove ours second result of the existence of strong solution; in Section 5 we state the results on the hydrostatic pression.

2 Preliminaires

Let $\Omega \subseteq I\!R^n$, $n = 2$ or 3, be a bounded domain with boundary Γ of class $C^{1,1}$. Let $H^s(\Omega)$ be the usual Sobolev spaces on Ω with norm $\|\cdot\|_s$ (s real), $(\cdot, \cdot)$ denote the usual inner product in $L^2(\Omega)$ and $|\cdot|$ denote the L^2-norm on Ω. By $H_0^1(\Omega)$ we denote the completion of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|_1$, the L^p-norm on Ω is denoted by $|\cdot|_p$, $1 \leq p \leq \infty$. If B is a Banach space, we denote by $L^q(0,T;B)$ the Banach space of the B-valued functions defined is the interval $(0,T)$ that are L^q-integrable in the sense of Bochner. Let $H^{s-\frac{1}{2}}(\Gamma)$, $s = 1, 2, \ldots$ be the usual trace space obtained as the image of $H^s(\Omega)$ by the boundary value mapping on Γ, equipped with the norm

$$\|\gamma\|_{H^{s-\frac{1}{2}}(\Gamma)} = \inf\{\|v\| \ , \ v \in H^s(\Omega) \ , \ v = \gamma \ \text{ on } \ \Gamma\}$$

(see, Adams [1] for their properties of the above spaces).

$H^{-1/2}(\Gamma)$ and $H^{-3/2}(\Gamma)$ denote the dual space of $H^{1/2}(\Gamma)$ and $H^{3/2}(\Gamma)$ respectively.

The functions in this paper are either $I\!R$ or $I\!R^n$-valued and we will not distinguish them in our notations.

We shall consider the following spaces of divergence-free functions.

$$C_{0,\sigma}^\infty(\Omega) = \{v \in C_0^\infty(\Omega) \ / \ \text{div } v = 0 \text{ in } \Omega\}$$

$$H = \text{closure of } C_{0,\sigma}^\infty(\Omega) \text{ in } L^2(\Omega)$$

$$V = \text{closure of } C_{0,\sigma}^\infty(\Omega) \text{ in } H^1(\Omega) \ .$$

We observe that the space V is characterized by

$$V = \{u \in H_0^1(\Omega \ / \ \text{div } u = 0 \text{ in } \Omega\} \ .$$

The space $L^2(\Omega)$ has the decomposition $L^2(\Omega) = H \oplus H^\perp$, where $H^\perp = \{\phi \in L^2(\Omega)/ \text{ exists } p \in H^1(\Omega) \text{ with } \phi = \nabla p\}$ (Helmholtz descomposition).

Throughout the paper P will denote the orthogonal projection from $L^2(\Omega)$ onto H. Then the operator $A : H \to H$ given by $A = -P\Delta$ with domain $D(A) = H^2(\Omega) \cap V$ is called the Stokes operator. It is well known that the operator A is positive definite, self-adjoint operator and is characterized by the relation

$$(Aw, v) = (\nabla w, \nabla v) \quad \text{for all} \quad w \in D(A), v \in V.$$

The operator A^{-1} is linear continuous from H into $D(A)$, and since the injection of $D(A)$ is H is compact, A^{-1} can be considered as a compact operator in H. As an operator in H it is also self-adjoint. By a well know theorem of Hilbert spaces, there exists a sequence of positive numbers $\mu_j > 0, \mu_{j+1} \leq \mu_j$ and an orthonormal basis of H, $\{w_j\}_{j=1}^{\infty}$ such that $A^{-1}w_j = \mu_j w_j$. We denote $\lambda_j = \mu_j^{-1}$. Since A^{-1} has range in $D(A)$ we obtain that

$$Aw_j = \lambda_j w_j \quad , \quad w_j \in D(A)$$

$0 < \lambda_1 < \cdots \leq \lambda_j \leq \lambda_{j+1} \leq \cdots \lim_{j\to\infty} \lambda_j = +\infty$ and $\{w_j\}_{j=1}^{\infty}$ are an orthonormal basis of H.

Therefore, $\{w_j|\sqrt{\lambda_j}\}_{j=1}^{\infty}$ and $\{w_j|\lambda_j\}_{j=1}^{\infty}$ form an orthonormal basis in V (with the inner product $((\nabla u, \nabla v), u, v \in V)$ and $H^2(\Omega) \cap V$ (with inner product $(Au, Av), u, v \in D(A))$, respectively. We denote by $V_k = \text{span}[w^1, \ldots, w^k]$.

We observe that for the regularity properties of the Stokes operator, it is usually assumed that Ω is of class C^2; this being in order to use Cattabriga's results [5]. We use instead the stronger results of Amrouche and Girault [2] which implies, in particular, that when $Au \in L^2(\Omega)$ then $u \in H^2(\Omega)$ and $||u||_{H^2}$ and $||Au||$ are equivalent norms when Ω is of class $C^{1,1}$.

Similar considerations are true for the Laplacian operator $B = -\Delta : D(B) \to L^2(\Omega)$ with the Dirichelet boundary conditions with domain $D(B) = H^2(\Omega) \cap H_0^1(\Omega)$ and we will denote $\varphi^k(x), \gamma_k$ by the eigenfunctions and eigenvalues of B, respectively. We denote by $H_k = \text{span}\ [\varphi^1, \ldots, \varphi^k]$.

Before we define strong solution, we will transform problem (1.1) - (1.3) into another one with homogeneous boundary value. In order to do it, we consider the following problem:

$$\left.\begin{array}{rl} \varphi_t - \Delta\varphi = 0 & \text{in } (0,T)\times\Omega \\ \varphi = \eta & \text{in } (0,T)\times\Gamma \\ \varphi(0) = \varphi_0. & \end{array}\right\} \tag{2.1}$$

By using spectral Galerkin method we can obtain the following results

Lemma 2.1. Let Ω be a bounded domain of class $C^{1,1}$. Assume that $\eta \in L^2(0,T;H^{1/2}(\Gamma))$, $\eta_t \in L^2(0,T;H^{-3/2}(\Gamma))$ and $\varphi_0 \in L^2(\Omega)$. Then, there exists an unique solution φ of (2.1) such that for any $t \in [0,T]$

$$|\varphi(t)|^2 + \int_0^t |\nabla\varphi(s)|^2 ds \le C\int_0^t (\|\eta\|^2_{H^{1/2}(\Gamma)} + \|\eta_t\|^2_{H^{-3/2}(\Gamma)})ds + |\varphi_0|^2$$

Moreover, if $\eta \in L^2(0,T;H^{3/2}(\Gamma))$, $\eta_t \in L^2(0,T;H^{-1/2}(\Gamma))$ and $\varphi_0 \in H^1(\Omega)$, $\eta(0) = \varphi_0$ on Γ, then φ satisfy

$$|\nabla\varphi(t)|^2 + \int_0^t |\Delta\varphi(s)|^2 ds \le C\int_0^t (\|\eta\|^2_{H^{3/2}(\Gamma)} + \|\eta_t\|^2_{H^{-1/2}(\Gamma)})ds + \|\varphi_0\|^2_1 .$$

for any $t \in [0,T]$.

Also, if $\eta \in L^\infty((0,T)\times\Gamma)$ and $\varphi_0 \in L^\infty(\Omega)$, then the following Maximum Principle holds

$$|\varphi|_{L^\infty((0,T)\times\Omega)} \le |\eta|_{L^\infty((0,T)\times\Gamma)} + |\varphi_0|_{L^\infty(\Omega)} .$$

Applying the above Lemma for $\eta = \theta_1$ and φ_0 any function such that $\varphi_0 = \theta_1(0)$ on Γ (we observe that such function exists by hypothesis done to $\eta = \theta_1$), we obtain the existence of $\varphi = \theta_2$ such that θ_2 is an unique solution of the problem (2.1), moreover θ_2 satisfies the conclusions of Lemma 2.1.

Analogously, we can obtain the existence of ψ_2 such that ψ_2 is a unique solution of the problem (2.1) and ψ_2 satisfying the conditions of Lemma 2.1.

Now, we can transform the equations (1.1)-(1.3) by introduction the new variables $\theta = \tilde{\theta} - \theta_2$ and $\psi = \tilde{\psi} - \psi_2$ we obtain

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + u.\nabla u - \Delta u + \nabla p = (\theta + \psi)g + g_1 \\ &\frac{\partial \theta}{\partial t} + u.\nabla \theta - \Delta \theta = f - u.\nabla \theta_2 \\ &\frac{\partial \psi}{\partial t} + u.\nabla \psi - \Delta \psi = h - u.\nabla \psi_2 \\ &\text{div } u = 0 \qquad \text{in} \quad (0,T) \times \Omega \,, \end{aligned}\right\} \tag{2.2}$$

$$u = 0 \;\; ; \;\; \theta = 0 \;\; ; \;\; \psi = 0 \quad \text{on} \quad (0,T) \times \Gamma \,, \tag{2.3}$$

$$u|_{t=0} = u_0 \;\; ; \;\; \theta|_{t=0} = \theta_0 \;\; ; \;\; \psi|_{t=0} = \psi_0 \,, \tag{2.4}$$

where $g_1 = (\theta_2 + \psi_2)g + j$; $\theta_0 = \tilde{\theta}_0 - \theta_2(0)$ and $\psi_0 = \tilde{\psi}_0 - \psi_2(0)$.

Now, by using the properties of P, we can reformulate problem (2.2)–(2.4) as follows: find

$$(u, \theta, \psi) \in C([0,T];\; V \times (H_0^1(\Omega))^2) \cap L^2(0,T; D(A) \times (D(B))^2)$$

such that

$$\left.\begin{aligned} &(u_t, v) + (u.\nabla u, v) + (Au, v) = ((\theta + \psi)g, v) + (g_1, v), \quad \forall v \in V \\ &(\theta_t, \xi) + (u.\nabla \theta, \xi) + (B\theta, \xi) = (f, \xi) - (u.\nabla \theta_2, \xi) \,, \quad \forall \xi \in H_0^1(\Omega) \\ &(\psi_t, \phi) + (u.\nabla \psi, \phi) + (B\psi, \phi) = (h, \phi) - (u.\nabla \psi_2, \phi) \,, \quad \forall \phi \in H_0^1(\Omega) \end{aligned}\right\} \tag{2.5}$$

$$(u(0),\; \theta(0),\; \psi(0)) = (u_0, \theta_0, \psi_0) \,. \tag{2.6}$$

The spectral Galerkin approximations for (u, θ, ψ) are defined for each $k \in N$ as the solution $(u^k, \theta^k, \psi^k) \in C^2([0,T];\; V_k \times (H_k)^2) \cap C^1([0,T] \times \overline{\Omega})$ of

$$(u_t^k, v) + (u^k.\nabla u^k, v) + (Au^k, v) = ((\theta^k + \psi^k)g, v) + (g_1, v), \quad \forall v \in V \tag{2.7}$$

$$(\theta_t^k, \xi) + (u^k.\nabla \theta^k, \xi) + (B\theta^k, \xi) = (f, \xi) - (u^k.\nabla \theta_2, \xi) \,, \quad \forall \xi \in H_k \tag{2.8}$$

$$(\psi_t^k, \phi) + (u^k.\nabla \psi^k, \phi) + (B\psi^k, \phi) = (h, \phi) - (u^k.\nabla \psi_2, \phi) \,, \quad \forall \phi \in H_k \tag{2.9}$$

$$u^k(0,x) = u_0^k(x) \,, \quad \theta^k(0,x) = \theta_0^k(x) \,, \psi^k(0,x) = \psi_0^k(x) \tag{2.10}$$

Here, u_0^k are the projections of u_0 on V_k, analogously, θ_0^k and ψ_0^k are the projections of θ_0 and ψ_0 on H_k, respectively.

Equations (2.7)–(2.10) are equivalent to a system of ordinary differential equations, which define u^k, θ^k and ψ^k in an interval $[0, t_k)$. We will show some a priori estimates independent on k and t, in order to take $t_k = T$. Also, we will prove that the sequences by u^k, θ^k and ψ^k converge in appropriate sense to a solution (u, θ, ψ) of (2.5)–(2.6). Our first result concern the local existence of solutions (2.5)–(2.6) is the following.

Theorem 2.2. Let Ω be a bounded domain in $\mathbb{R}^n (n = 2$ or $3)$ with boundary Γ of class $C^{1,1}$. Suppose that

$$(\theta_2, \psi_2) \in L^\infty(0, T; (H^1(\Omega))^2); (u_0, \theta_0, \psi_0) \in V \times (H_0^1(\Omega))^2;$$
$$j \in L^2(0, T; L^2(\Omega)); g \in L^2(0, T; L^3(\Omega)); f, h \in L^2(0, T; L^2(\Omega))$$

Then, there exists $T_1 > 0$ with $T_1 \leq T$ such that the problem (2.5)–(2.6) (or (2.2)–(2.4)) has a unique solution in the interval $[0, T_1)$. Moreover the approximations u^k, θ^k and ψ^k satisfy the estimates

$$|\nabla u^k|^2 + |\nabla \theta^k|^2 + |\nabla \psi^k|^2 + \int_0^t (|Au^k|^2 + |B\theta^k|^2 + |B\psi^k|^2) ds \leq F(t);$$
$$\int_0^t (|u_t^k|^2 + |\theta_t^k|^2 + |\psi_t^k|^2) ds \leq G(t) .$$

The functions on the right hand sides depend on their argument t, and in addition on T, Γ and the norms, $||u_0||_V$, $||\theta_0||_{H_0^1}$, $||\psi_0||_{H_0^1}$,

$$\int_0^T (|f|^2 + |j|^2 + |h|^2 + |g|_3^2) ds \quad \text{and} \quad \sup_{[0,T]} \{||\theta_2||_1 + ||\psi_2||_1\}$$

on the interval in question the functions are continuonsly differentiable with respect to t.

With stronger assumptions on the initial values and the external fields, we are able to prove the following.

Theorem 2.3. Under the hypothesis of Theorem 2.2 and suppose the forces satisfy

$$\mathcal{F}_1(t) = \int_0^T (|\partial_t g|^2 + |\partial_t j|^2 + |\partial_t f|^2 + |\partial_t h|^2) ds < +\infty ,$$

and

$$\mathcal{F}_2(t) = \sup_{t\in[0,T]} \{|\partial_t\theta_2(t)| + |\partial_t\psi_2(t)|\} + \int_0^T (||\partial_t\theta_2(t)||_1^2 + ||\partial_t\psi_2(t)||_1^2)\, ds < +\infty$$

and the initial data $u_0 \in D(A), \theta_0, \psi_0 \in D(B)$. Then the solution (u, θ, ψ) obtained in Theorem 2.2 belongs to $C([0,T_1]; D(A) \times D(B)^2) \cap C^1([0,T_1];\ H \times (L^2(\Omega))^2)$. Furthermore, the approximations u^k, θ^k and ψ^k satisfy

$$\begin{aligned}
&|u_t^k|^2 + |\theta_t^k|^2 + |\psi_t^k|^2 + \int_0^t (|\nabla u_t^k|^2 + |\nabla\theta_t^k|^2 + |\nabla\psi_t^k|^2) ds \leq H(t);\\
&|Au^k|^2 + |B\theta^k|^2 + |B\psi^k|^2 \leq L(t).
\end{aligned}$$

The functions on the right hands sides depend as their argument t, and in addition on T, Γ and the norms $||u_0||_{H^2}$, $||\theta_0||_{H^2}$, $||\psi_0||_{H^2}$, $\mathcal{F}_1(t)$ and $\mathcal{F}_2(t)$. On the interval in questions these functions are continuonsly differentiable with respect to t.

3 Proof of Theorem 2.2

Setting $v = u^k$ in (2.7), we obtain

$$\frac{1}{2}\frac{d}{dt}|u^k|^2 + |\nabla u^k|^2 = ((\theta^k + \psi^k)g, u^k) + (g_1, u^k) \tag{3.1}$$

since $(u^k.\nabla u^k, u^k) = 0$. We observe that

$$\begin{aligned}
|((\theta^k + \psi^k)g, u^k)| &\leq C_\delta|g|_3^2(|\theta^k|^2 + |\psi^k|^2) + \frac{\delta}{4}|\nabla u^k|^2\\
|(g_1, u^k)| &= |(j + (\theta_2 + \psi_2)g, u^k)|\\
&\leq C_\delta|j|^2 + C_\delta|g|_3^2(|\theta_2|^2 + |\psi_2|^2) + \frac{\delta}{4}|\nabla u^k|^2.
\end{aligned}$$

So, we deduce of (3.1)

$$\frac{d}{dt}|u^k|^2 + |\nabla u^k|^2 \leq C_\delta|g|_3^2(|\theta^k|^2 + |\psi^k|^2 + |\theta_2|^2 + |\psi_2|^2) + C|j|^2 + \frac{\delta}{2}|\nabla u^k|^2\ . \tag{3.2}$$

Moreover, setting $\xi = \theta^k$ in (2.8), we obtain

$$\frac{1}{2}\frac{d}{dt}|\theta^k|^2 + |\nabla\theta^k|^2 = (f,\theta^k) + (u^k.\nabla\theta_2,\theta^k) \tag{3.3}$$

since $(u^k.\nabla\theta^k,\theta^k) = 0$. Now, we observe that

$$\begin{aligned}
|(u^k.\nabla\theta_2,\theta^k)| &= |(u^k.\nabla\theta^k,\theta_2)| \\
&\leq C|u^k|_3|\theta_2|_6|\nabla\theta^k| \\
&\leq C_\varepsilon|u^k|\,|\nabla u^k|\,|\theta_2|_6^2 + \frac{\varepsilon}{2}|\nabla\theta^k|^2 \\
&\leq C_{\varepsilon,\delta}|u^k|^2\,|\theta_2|_6^4 + \delta|\nabla u^k|^2 + \frac{\varepsilon}{2}|\nabla\theta^k|^2
\end{aligned}$$

thanks to Hölder's, Sobolev's and Young's inequalities. By using the above estimate in (3.3), we get

$$\frac{d}{dt}|\theta^k|^2 + |\nabla\theta^k|^2 \leq C_\varepsilon|f|^2 + C_{\varepsilon,\delta}|u^k|^2|\theta_2|_6^4 + \varepsilon|\nabla\theta^k|^2 + \frac{\delta}{4}|\nabla u^k|^2. \tag{3.4}$$

Similarly, we obtain

$$\frac{d}{dt}|\psi^k|^2 + |\nabla\psi^k|^2 \leq C_\varepsilon|h|^2 + C_{\varepsilon,\delta}|u^k|^2|\psi_2|_6^4 + \varepsilon|\nabla\psi^k|^2 + \frac{\delta}{4}|\nabla u^k|^2. \tag{3.5}$$

By taking $\varepsilon > 0$ and $\delta > 0$ small enough, by adding (3.2), (3.4) and (3.5), we obtain

$$\begin{aligned}
&\frac{d}{dt}(|u^k|^2 + |\theta^k|^2 + |\psi^k|^2) + |\nabla u^k|^2 + |\nabla\theta^k|^2 + |\nabla\psi^k|^2 \\
\leq\ & C(|g|_3^2 + |\theta_2|_6^4 + |\psi_2|_6^4)(|u^k|^2 + |\theta^k|^2 + |\psi^k|^2) + C|g|_3^2(|\theta_2|^2 + |\psi_2|^2) + C|f|^2 + C|h|^2 + C|j|^2.
\end{aligned}$$

Integrating this last inequality, we get for any $t \in [0,T]$

$$\begin{aligned}
&|u^k(t)|^2 + |\theta^k(t)|^2 + |\psi^k(t)|^2 + \int_0^t(|\nabla u^k|^2 + |\nabla\theta^k|^2 + |\nabla\psi|^2)ds \\
\leq\ & |u^k(0)|^2 + |\theta^k(0)|^2 + |\psi^k(0)|^2 + C\int_0^t(|g|_3^2 + |\theta_2|_6^4 + |\psi_2|_6^4)(|u^k|^2 + |\theta^k|^2 + |\psi^k|^2)ds \\
&+ C\int_0^t |g|_3^2(|\theta_2|^2 + |\psi_2|^2)ds + C\int_0^t(|f|^2 + |h|^2 + |j|^2)ds \\
\leq\ & |u_0|^2 + |\theta_0|^2 + |\psi_0|^2 + C\int_0^t(|g|_3^2 + |\theta_2|_6^4 + |\psi_2|_6^4)(|u^k|^2 + |\theta^k|^2 + |\psi^k|^2)ds \\
&+ C\int_0^t |g|_3^2(|\theta_2|^2 + |\psi_2|^2)ds \ + \ C(\int_0^t(|f|^2 + |h|^2 + |j|^2)ds\ ,
\end{aligned}$$

since $|u^k(0)| = |P_k u_0| \leq |u_0|$, $|\theta^k(0)| = |R_k\theta_0| \leq |\theta_0|$, $|\psi^k(0)| = |R_k\psi_0| \leq |\psi_0|$.

Consequently, by using the Gronwall's inequality, we have

$$|u^k(t)|^2 + |\theta^k(t)|^2 + |\psi^k(t)|^2 + \int_0^t (|\nabla u^k(s)|^2 + |\nabla \theta^k(s)|^2 + |\nabla \psi^k(s)|^2 ds \leq C,$$

thanks our hypothesis and where C is a positive constant that only depends on the regularity of Γ and the initial datas, the above inequality implies that (u^k), (θ^k) and (ψ^k) exist globally in t and are uniformly bounded sequence in $L^\infty(0,T;H) \cap L^2(0,T;V)$ and $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, respectively.

The next step of the proof consists of proving that there exist $T_1 > 0, T_1 \leq T$ such that (u^k, θ^k, ψ^k) is a sequence uniformly bounded in $L^\infty(0,T_1;V) \times (L^\infty(0,T;H_0^1(\Omega))^2$.

To this end, we put $v = Au^k$ in (2.7); we obtain

$$\frac{1}{2}\frac{d}{dt}|\nabla u^k|^2 + |Au^k|^2 = ((\theta^k + \psi^k)g, Au^k) + (g_1, Au^k) - (u^k.\nabla u^k, Au^k). \tag{3.6}$$

Hölder's and Young's inequalities, together with the Sobolev embedding $H^1 \hookrightarrow L^6$ imply

$$|((\theta^k + \psi^k)g, Au^k)| \leq c_\varepsilon(|\nabla\theta^k|^2 + |\nabla\psi^k|^2)|g|_3^2 + \varepsilon|Au^k|^2$$

$$|(g_1, Au^k)| = |(j + (\theta_2 + \psi_2)g, Au^k)| \leq C_\varepsilon|j|^2 + C_3(||\theta_2||_1^2 + ||\psi_2||_1^2)|g|_3^2 + \varepsilon|Au^k|^2$$

where $\varepsilon > 0$.

Also, by using the estimate given in Duff [7, p. 154], we have

$$|(u^k.\nabla u^k, Au^k)| \leq c_3|\nabla u^k|^6 + \varepsilon|Au^k|^2.$$

Consequently, by setting $\varepsilon = 1/6$ in (3.6), we have

$$\frac{d}{dt}|\nabla u^k|^2 + |Au^k|^2 \leq C(|\nabla\theta^k|^2 + |\nabla\psi^k|^2)|g|_3^2 + C|\nabla u^k|^6 + + C(|\theta_2||_1^2 + ||\psi_2||_1^2)|g|_{L^3}^2. \tag{3.7}$$

Now, we take $\xi = B\theta^k$ in (2.8) to get

$$\frac{1}{2}\frac{d}{dt}|\theta^k|^2 + |B\theta^k|^2 = (f, B\theta^k) - (u^k.\nabla\theta^k, B\theta^k) - (u^k.\nabla\theta_2, B\theta^k). \tag{3.8}$$

Also, we observe that for all $\delta > 0$

$$|(f, B\theta^k)| \leq C_\delta|f|^2 + \delta|B\theta^k|^2.$$

The second term in the right-hand side of (3.8) will be estimate by means of the inequalities of Hölder, Sobolev and Young as follows

$$\begin{aligned}
|(u^k.\nabla\theta^k, B\theta^k) &\leq |u^k|_6|\nabla\theta^k|_3|B\theta^k| \\
&\leq |\nabla u^k|\ |\nabla\theta^k|_6^{1/2}|\nabla\theta^k|^{1/2}|B\theta^k| \\
&\leq C|B\theta^k|^{3/2}|\nabla u^k|\ |\nabla\theta^k|^{1/2} \\
&\leq C_\delta|\nabla u^k|^4|\nabla\theta^k|^2 + \delta|B\theta^k|^2
\end{aligned}$$

where $\delta > 0$.

The third term in the right-hand side of (3.8) will be estimate by means of the inequalities of Hölder, Sobolev, Young and the following inequality of Nirenberg [7, p. 149]:

$$|u|_{L^\infty} \leq C(|u|_6^{1/2}|\nabla u|_6^{1/2} + |u|_6)$$

as follows

$$\begin{aligned}
|(u^k.\nabla\theta_2, B\theta^k)| &\leq |u^k|_{L^\infty}|\nabla\theta_2|\ |B\theta^k| \\
&\leq C(|\nabla u^k|^{1/2}|Au^k|^{1/2} + |\nabla u^k|)|\nabla\theta_2|\ |B\theta^k| \\
&\leq C(|\nabla u^k|^{1/2}|Au^k|^{1/2}|\nabla\theta_2|\ |B\theta^k| \\
&\quad + C|\nabla u^k|\ |\nabla\theta_2|\ |B\theta^k| \\
&\leq C_\delta|\nabla u^k|\ |\nabla\theta_2|^2\ |Au^k| + \delta|B\theta^k|^2 \\
&\quad + C_\delta|\nabla u^k|^2|\nabla\theta_2|^2 + \delta|B\theta^k|^2 \\
&\leq C_{\delta,\varepsilon}|\nabla u^k|^2|\nabla\theta_2|^4 + C_\delta|\nabla u^k|^2|\nabla\theta_2|^2 \\
&\quad + 2\delta|\nabla\theta^k|^2 + \varepsilon|Au^k|^2,
\end{aligned}$$

where $\varepsilon, \delta > 0$.

Analogously the terms that involving ψ^k can be estimate as before. Consequently, for appropriate ε and δ, we obtain

$$\begin{aligned}
&\frac{d}{dt}(|\nabla u^k|^2 + |\nabla\theta^k|^2 + |\nabla\psi^k|^2) + |Au^k|^2 + |B\theta^k|^2 + |B\psi^k|^2 \\
&\leq C(|\nabla\theta^k|^2 + |\nabla\psi^k|^2)|g|_3^2 + C|\nabla u^k|^6 + C|j|^2 + C(||\theta_2||_1^2 + ||\psi_2||_1^2)|g|_3^2 \\
&\quad +C|\nabla u^k|^4|\nabla\theta^k|^2 + C|f|^2 + C|\nabla u^k|^2|\nabla\theta_2|^2 + C|\nabla u^k|^2|\nabla\theta^k|^4 \\
&\quad +C|\nabla u^k|^4|\nabla\psi^k|^2 + C|h|^2 + C|\nabla u^k|^2|\nabla\psi_2|^2 + C|\nabla u^k|^2|\nabla\psi^k|^4. \qquad (3.9)
\end{aligned}$$

Setting $\eta(t) = |\nabla u^k(t)|^2 + |\nabla \theta^k(t)|^2 + |\nabla \psi^k(t)|^2$, the above differential inequality imply

$$\frac{d}{dt}\eta \leq C\eta^3 + C\eta + C(|j|^2 + |f|^2 + |h|^2 + (||\theta_2||_1^2 + ||\psi_2||_1^2)|g|_3^2)$$

thanks to our hypothesis.

By applying Lemma 3 in Heywood [11, p. 656], we conclude that there exists $T_1 \in (0, T]$ such that

$$\eta(t) \leq F_0(t, \eta(0)) \qquad \forall t \in [0, T_1]$$

where $\eta(0) = |\nabla u_0|^2 + |\nabla \theta_0|^2 + |\nabla \psi_0|^2$, and F_0 is the solution of the initial value problem

$$\begin{aligned} F_0' &= CF_0^3 + CF_0 + C(|j|^2 + |f|^2 + |h|^2 + (||\theta_2||_1^2 + ||\psi_2||_1^2)|g|_3^2) \\ F_0(0) &= \eta(0). \end{aligned}$$

By returning to (3.9), we are left with

$$\begin{aligned} & |\nabla u^k(t)|^2 + |\nabla \theta^k(t)|^2 + |\nabla \psi^k(t)|^2 + \int_0^t (|Au^k s)|^2 + |B\theta^k(s)|^2 + |B\psi^k(s)|^2) ds \\ \leq\ & |\nabla u_0|^2 + |\nabla \theta_0|^2 + |\nabla \psi_0|^2 + CF_0^3(t, \eta(0)) + CF_0(t, \eta(0)) \\ & + C\int_0^t (|j(s)|^2 + |f(s)|^2 + |h(s)|^2) ds + C\int_0^t (||\theta_2(s)||_1^2 + ||\psi_2(s)||_1^2)|g(s)|_3^2 ds \\ \equiv\ & F(t). \end{aligned} \tag{3.10}$$

Thus,

u^k is uniformly bounded in $L^\infty(0, T_1; V) \cap L^2(0, T; D(A)), \theta^k, \psi^k$ are uniformly bounded in $L^\infty(0, T_1; H_0^1(\Omega)) \cap L^2(0, T; D(B))$.

Now, by taking $v = u_t^k, \xi = \theta_t^k$ and $\phi = \psi_t^k$ in (2.7), (2.8) and (2.9), respectively, we get

$$\begin{aligned} |u_t^k|^2 &= ((\theta^k + \psi^k)g, u_t^k) + (g_1, u_t^k) - (u^k.\nabla u^k, u_t^k) - (Au^k, u_t^k), \\ |\theta_t^k|^2 &= (f, \theta_t^k) - (u^k.\nabla \theta_2, \theta_t^k) - (u^k.\nabla \theta^k, \theta_t^k) - (B\theta^k, \theta_t^k), \\ |\psi_t^k|^2 &= (h, \psi_t^k) - (u^k.\nabla \psi_2, \psi_t^k) - (u^k.\nabla \psi^k, \psi_t^k) - (B\psi^k, \psi_t^k). \end{aligned}$$

From this, we have

$$\begin{aligned}
\int_0^t |u_t^k(s)|^2 ds &\leq C\int_0^t [(|\nabla\theta^k|^2+|\nabla\psi^k|^2)|g|_{L^3}^2+|g_1|^2+|u.\nabla u^k|^2+|Au^k|^2]ds,\\
\int_0^t |\theta_t^k(s)|^2 ds &\leq C\int_0^t [|f|^2+|u^k.\nabla\theta_2|^2+|u^k.\nabla\theta^k|^2+|B\theta^k|^2]ds, \qquad (3.11)\\
\int_0^t |\psi_t^k(s)|^2 ds &\leq C\int_0^t [|h|^2+|u^k.\nabla\psi_2|^2+|u^k.\nabla\psi^k|^2+|B\psi^k|^2]ds.
\end{aligned}$$

Now, bearing in mind (3.10) and the Sobolev embedding $H^2 \hookrightarrow L^\infty$, we obtain the following estimate:

$$\begin{aligned}
|u^k.\nabla u^k|^2 &\leq |u^k|_{L^\infty}^2|\nabla u^k|^2 \leq C|Au^k|^2|\nabla u^k|^2\\
&\leq C \sup_{0\leq t\leq T_1} F(t)\ |Au^k|^2,
\end{aligned}$$

Analogously, we prove

$$\begin{aligned}
|u^k.\nabla\theta_2|^2 &\leq C|\nabla\theta_2|^2\ |Au^k|^2 \leq C||\theta_2||_1|Au^k|^2,\\
|u^k.\nabla\psi_2|^2 &\leq C|\nabla\psi_2|^2|Au^k|^2 \leq C||\psi_2||_1|Au^k|^2,\\
|u^k.\nabla\theta^k|^2 &\leq C\sup_{0\leq t\leq T_1} F(t)|Au^k|^2\\
|u^k.\nabla\psi^k|^2 &\leq C\sup_{0\leq t\leq T_1} F(t)|Au^k|^2.
\end{aligned}$$

By using this estimates in the inequality (3.11) together our hypothesis, we obtain for all $t\in[0,T_1]$

$$\begin{aligned}
\int_0^t |u_t^k(s)|^2 ds &\leq C|g|_{L^\infty(0,T;L^3(\Omega))}\int_0^t(|\nabla\theta^k|^2+|\nabla\psi^k|^2)ds + C\int_0^t |g_1(s)|^2ds\\
&\quad + C(\sup_{0\leq t\leq T_1} F(t)+1)\int_0^t |Au^k(s)|^2 ds\\
&\equiv G_1(t)
\end{aligned}$$

Moreover, (u_t^k) is an uniformly bounded sequence in $L^2(0,T_1;H)$.
Also, by using the above estimates together with the hypothesis we have

$$\begin{aligned}
\int_0^t |\theta_t^k(s)|^2 ds &\leq C\int_0^t |f(s)|^2ds + C(|\theta_2|_{L^\infty(0,T;H^1(\Omega))}+\sup_{0\leq t\leq T_1} F(t))\int_0^t |Au^k(s)|^2ds\\
&\quad + C\int_0^t |B\theta^k(s)|^2ds\\
&\equiv G_2(t)
\end{aligned}$$

for all $t \in [0, T_1]$, so, (θ_t^k) is an uniformly bounded sequence in $L^2(0, T_1; L^2(\Omega))$. Analogously, we prove that (ψ_t^k) is an uniformly bounded sequence is $L^2(0, T_1; L^2(\Omega))$.

Now, by standard methods (see for instance [17], [11], [20]), these estimates enable us to take the limit as $k \to +\infty$ in (2.7)-(2.9). We conclude that a solution for (2.5)-(2.6) exists in stated class. We have also that $u_t \in L^2(0, T; H)$, (resp. $\theta_t, \psi_t \in L^2(0, T; L^2(\Omega))$). This condition, together with $u \in L^2(0, T; D(A))$, (resp. $\theta, \psi \in L^2(0, T; D(B))$), implies by interpolation (See, Temam [25], p. 260), that u (resp. θ, ψ) is almost everywhere equal to a continuous function from $[0, T_1]$ into V (resp. $[0, T]$ into $H_0^1(\Omega)$), consequently the initial conditions $u(0) = u_0$ (resp. $\theta(0) = \theta_0, \psi(0) = \psi_0$) are meaningful. ∎

4 Proof of Theorem 2.3

We will need further estimates for the approximations u^k, θ^k, ψ^k. To this end, we differentiable (2.7)-(2.9) with respect to t and set $v = u_t^k, \xi = \theta_t^k$ and $\phi = \psi_t^k$. We are left with

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u_t^k|^2 + |\nabla u_t^k|^2 &= ((\theta^k + \psi^k)g_t, u_t^k) + ((\theta_t^k + \psi_t^k)g, u_t^k) \\ &+ ((g_1)_t, u_t^k) - (u_t^k.\nabla u^k, u_t^k) - (u^k.\nabla u_t^k, u_t^k), \quad (4.2)\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\theta_t^k|^2 + |\nabla \theta_t^k|^2 &= (f_t, \theta_t^k) - (u_t^k.\nabla\theta_2, \theta_t^k) \\ &- (u^k.\nabla(\theta_2)_t, \theta_t^k) - (u_t^k.\nabla\theta^k, \theta_t^k) - (u^k.\nabla\theta_t^k, \theta_t^k), (4.3)\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\psi_t^k|^2 + |\nabla \psi_t^k|^2 &= (h_t, \psi_t^k) - (u_t^k.\nabla\psi_2, \psi_t^k) \\ &+ (u^k.\nabla(\psi_2)_t, \psi_t^k) - (u_t^k.\nabla\psi^k, \psi_t^k) - (u^k.\nabla\psi_t^k, \psi_t^k). (4.4)\end{aligned}$$

We observe that

$$(u^k.\nabla u_t^k, u_t^k) = (u^k.\nabla\theta_t^k, \theta_t^k) = (u^k.\nabla\psi_t^k, \psi_t^k) = 0$$

Also, by using the Cauchy - Schwarz and Young inequalities, we obtain

$$|(f_t, \theta_t^k)| \le \frac{1}{2}|f_t|^2 + \frac{1}{2}|\theta_t^k|^2$$
$$|(h_t, \psi_t^k)| \le \frac{1}{2}|h_t|^2 + \frac{1}{2}|\psi_t^k|^2 .$$

By using the Hölder and Young inequalities, we get

$$|((\theta^k + \psi^k)g_t, u_t^k)| \le C_\varepsilon(|\nabla\theta^k|^2 + |\nabla\psi^k|^2)|g_t|^2 + \varepsilon|\nabla u_t^k|^2$$
$$|((\theta_t^k + \psi_t^k)g, u_t^k)| \le C_\varepsilon(|\nabla\theta_t^k|^2 + |\psi_t^k|^2)|g|_{L^3}^2 + \varepsilon|\nabla u_t^k|^2.$$

Similarly, we have

$$\begin{aligned} |((g_1)_t, u_t^k)| &= |(j_t + (\theta_2 + \psi_2)g_t + (\theta_2 + \psi_2)_t g, u_t^k)| \\ &\le C_\varepsilon|j_t|^2 + C_\varepsilon(||\theta_2||_1^2 + ||\psi_2||_1^2)|g_t|^2 + C_\varepsilon(|(\theta_2)_t|^2 + |(\psi_2)_t|^2)|g|_{L^3}^2 \\ &\quad +\varepsilon|\nabla u_t^k|^2. \end{aligned}$$

To estimate the fourth term in (4.1), we use the Hölder's and Young's inequalities together with the Sobolev embedding $H^1 \hookrightarrow L^4$; we obtain for any $\varepsilon > 0$ and suitable $C_\varepsilon > 0$,

$$\begin{aligned} |(u_t^k.\nabla u^k, u_t^k) &\le |u_t^k|\ |\nabla u^k|_4\ |u_t^k|_4 \\ &\le C_\varepsilon|u_t^k|^2|Au^k|^2 + \varepsilon|\nabla u_t^k|^2 . \end{aligned}$$

Similarly, we have for any $\delta > 0$ and suitable $C_\delta > 0$

$$|(u_t^k.\nabla\theta^k, \theta_t^k)| \le C_\delta|u_t^k|^2\ |B\theta^k|^2 + \delta|\nabla\theta_t^k|^2$$

$$\begin{aligned} |(u_t^k.\nabla\theta_2, \theta_t^k)| &= |(u_t^k \cdot \nabla\theta_t^k, \theta_2)| \\ &\le |u_t^k|_3|\theta_2|_6|\nabla\theta_t^k| \\ &\le C_\delta|u_t^k|\ |\nabla u_t^k|\ |\theta_2|_6^2 + \delta|\nabla\theta_t^k|^2 \\ &\le C_{\delta,\varepsilon}|u_t^k|^2\ |\theta_2|_6^4 + \varepsilon|\nabla u_t^k|^2 + \delta|\nabla\theta_t^k|^2 \\ |(u^k.(\nabla\theta_2)_t, \theta_t^k)| &= |(u^k.\nabla\theta_t^k, (\theta_2)_t)| \\ &\le C_\delta|Au^k|^2|(\theta_2)_t|^2 + \delta|\nabla\theta_t^k|^2 . \end{aligned}$$

Analogous estimates are valid for the terms that involve ψ^k.

By taking $\varepsilon > 0$ and $\delta > 0$ small enough, by adding (4.1), (4.2) and (4.3) and by using the above estimates, we are left with the following differential inequality

$$\begin{aligned}
&\frac{d}{dt}(|u_t^k|^2 + |\theta_t^k|^2 + |\psi_t^k|^2) + (|\nabla u_t^k|^2 + |\nabla\theta_t^k|^2 + |\nabla\psi_t^k|^2) \\
\leq\ & C(|j_t|^2 + f_t|^2 + |h_t|^2) + C(|\nabla\theta^k|^2 + |\nabla\psi^k|^2 + |\nabla\theta_2|^2 + |\nabla\psi_2|^2)|g_t|^2 \\
& + C(|\theta_t^k|^2 + |\psi_t^k|^2 + |(\theta_2)_t|^2 + |(\psi_2)_t|^2)|g|_{L^3}^2 + C(|u_t^k|^2(|Au^k|^2 + |B\theta^k|^2 + |B\psi^k|^2 \\
& + |\theta_2|_6^4 + |\psi_2|_6^4) + |Au^k|^2(|(\theta_2)_t|^2 + |(\psi_2)_t|^2) \\
\leq\ & C\varphi_1(t) + C|g_t|^2(|\nabla\theta^k|^2 + |\nabla\psi^k|^2) + C|g|_3^2(|\theta_t^k|^2 + |\psi_t^k|^2) + C|u_t^k|^2\varphi_2(t) + (Au^k|^2\varphi_3(t),
\end{aligned}$$

where
$$\begin{aligned}
\varphi_1(t) &= |j_t|^2 + |f_t|^2 + |h_t|^2 + |g_t|^2(|\nabla\theta_2|^2 + |\nabla\psi_2|^2) + |g|_3^2(|(\theta_2)_t|^2 + |(\psi_2)_t|^2), \\
\varphi_2(t) &= |Au^k|^2 + |B\theta^k|^2 + |\theta_2|_6^4 + |\psi_2|_6^4, \\
\varphi_3(t) &= |(\theta_2)_t|^2 + |(\psi_2)_t|^2 .
\end{aligned}$$

Consequently, for $0 \leq t \leq T_1$; we obtain

$$\begin{aligned}
&|u_t^k(t)|^2 + |\theta_t^k(t)|^2 + |\psi_t^k(t)|^2 + \int_0^t (|\nabla u_t^k(s)|^2 + |\nabla\theta_t^k(s)|^2 + |\nabla\psi_t^k(s)|^2)ds \\
&\leq |u_t^k(0)|^2 + |\theta_t^k(0)|^2 + |\psi_t^k(0)|^2 + C\int_0^t \varphi_1(s)ds \\
&\quad + C\int_0^t |g_t(s)|^2(|\nabla\theta^k(s)|^2 + |\nabla\psi^k(s)|^2)ds + C\int_0^t |u_t^k(s)|^2\varphi_2(s)ds \\
&\quad + C\int_0^t |Au^k(s)|^2\varphi_3(s)ds.
\end{aligned} \tag{4.5}$$

We observe that by hypothesis $\varphi_1 \in L^1(0,T), \varphi_3 \in L^\infty(0,T)$, consequently

$$\begin{aligned}
\int_0^t |Au^k(s)|^2\varphi_3(s)ds &\leq |\varphi_3|_{L^\infty(0,T)} \int_0^t |Au^k(s)|^2 ds \\
&\leq |\varphi_3|_{L^\infty(0,T)} F(t)
\end{aligned}$$

thanks to the estimate (3.10).

Moreover, by using the estimate (3.10), we conclude that

$$|\nabla\psi^k(t)|^2 + |\nabla\theta^k(t)|^2 \leq \sup_{0\leq t\leq T_1} F(t) \leq C < +\infty.$$

so, $\int_0^t |g_t(s)|^2(|\nabla\theta^k(s)|^2 + |\nabla\psi^k(s)|^2)ds \leq C\int_0^t |g_t(s)|^2 ds \leq C < +\infty$, since $g_t \in L^2(\Omega \times (0,T))$.

Now, analogously as in Heywood [11, p. 665], we can prove that

$$|u_t^k(0)|^2 + |\theta_t^k(0)|^2 + |\psi_t^k(0)|^2 \le L,$$

where $L > 0$ is a constant independent of k.

Thus, by using the above estimates in (4.5), we get

$$\begin{aligned}
&|u_t^k(t)|^2 + |\theta_t^k(t)|^2 + |\psi_t^k(t)|^2 + \int_0^t (|\nabla u_t^k(s)|^2 + |\nabla \theta_t^k(s)|^2 + |\nabla \psi_t^k(s)|^2) ds \\
&\le L + C|\varphi_1|_{L^2(0,T)} + C|\varphi_3|_{L^\infty(0,T)} F(t) + C + C\int_0^t |u_t^k(s)|^2 \varphi_2(s) ds \\
&\le C + C\int_0^t |u_t^k(s)|^2 \varphi_2(s) ds.
\end{aligned}$$

Therefore, applying Gronwall's inequality to the above integral inequality, we obtain

$$\begin{aligned}
&|u_t^k(t)|^2 + |\theta_t^k|^2 + |\psi_t^k(t)|^2 + \int_0^t (|\nabla u_t^k(s)|^2 + |\nabla \theta_t^k(s)|^2 |\nabla \psi_t^k(s)|^2) ds \\
&\le Ce^{\int_0^t \varphi_2(s)}ds \\
&\equiv H(t)
\end{aligned}$$

for all $t \in [0, T_1]$. By the estimate (3.10) and hypothesis imply

$$\int_0^t \varphi_2(s) ds \le F_1(t)$$

for all $t \in [0, T_1]$, where $F_1(t)$ is a continuous function independent of k. Consequently, we conclude that u_t^k is uniformly bounded in $L^\infty(0, T_1; H) \cap L^2(0, T_1; V)$ and θ_t^k, ψ_t^k are uniformly bounded in $L^\infty(0, T_1; L^2(\Omega)) \cap L^2(0, T_1; H_0^1(\Omega))$.

Now, by taking $v = Au^k, \xi = B\theta^k$ and $\phi = B\psi^k$ in (2.7), (2.8) and (2.9), respectively, we obtain

$$\begin{aligned}
|Au^k|^2 &= ((\theta^k + \psi^k)g, Au^k) + (g_1, Au^k) - (u^k.\nabla u^k, Au^k) - (u_t^k, Au^k), \\
|B\theta^k|^2 &= (f, B\theta^k) - (u^k.\nabla\theta_2, B\theta^k) - (u^k.\nabla\theta^k, B\theta^k) - (\theta_t^k, B\theta^k), \\
|B\psi^k|^2 &= (h, B\psi^k) - (u^k.\nabla\psi_2, B\psi^k) - (u^k.\nabla\psi^k, B\psi^k) - (\psi_t^k, B\psi^k).
\end{aligned}$$

In what follows, we observe that if $\varphi \in L^P(0, T; X)$ and $\varphi_t \in L^P(0, T; X)$, where X is a Banach space and $1 \le p \le \infty$, then $\varphi \in C([0, T]; X)$ (see, Lions [17], p.7). Thus, we have $g_1 \in C([0, T]; L^2(\Omega))$. This together with the estimates (3.10) and (4.5) imply

$$|Au^k|^2 \le H_1(t)$$

for any $t \in [0, T_1]$.

Similarly, and by using this last estimate we get

$$|B\theta^k|^2 \leq H_2(t), \quad |B\psi^k|^2 \leq H_3(t)$$

for any $t \in [0, T_1]$.

Thus, u^k is uniformly bounded in $L^\infty(0, T; D(A))$; θ^k, ψ^k are uniformly bounded in $L^\infty(0, T; D(B))$. To prove the continuity of $u_t(t)$ in the L^2-norm, we only need to show that u_{tt} is in $L^2(0, T_1; V^*)$. In fact, if $u_{tt} \in L^2(0, T_1; V^*)$ then the fact that u_t is in $L^2(0, T_1; V)$, implies that $u \in C^1([0, T]; H)$ (Lemma 1.2, p. 260 in Temam [25]).

To prove that $u_{tt} \in L^2(0, T_1; V^*)$, it is enough to show the existence of $C > 0$ independent of k such that

$$\int_0^{T_1} |u_{tt}^k(s)|_{V^*}^2 ds \leq C.$$

To this end, we differentiable equation (2.7) with respect to t; we obtain

$$\begin{aligned} u_{tt}^k &= P_k((g_1)_t + (\theta_t^k + \psi_t^k)g + (\theta^k + \psi^k)g_t - u_t^k.\nabla u^k - u^k.\nabla u_t^k) - Au_t^k. \\ &\equiv G^k. \end{aligned}$$

The above estimates for u^k, θ^k and ψ^k imply that G^k is uniformly bounded in $L^2(0, T_1; V^*)$. In fact, we have

$$\begin{aligned} |P_k(u_t^k.\nabla u^*)|_{V^*} &= \sup_{|v|_V \leq 1} |(P_k u_t^k.\nabla u^k, v)| \\ &\leq \sup_{|v|_V \leq 1} |(u_t^k.\nabla u^k, P_k v)| \\ &\leq C \sup_{|v|_V \leq 1} |u_t^k|_4 |\nabla u^k| \, |v|_4 \\ &= C|\nabla u_t^k|. \end{aligned}$$

Here we have used the Sobolev embedding $H^1 \hookrightarrow L^4$, the estimates (4.5) and the continuity of P_k in L^4 (Von Wahl [26, p. XXIII]); C denotes a general constant depending only the previous estimates. Consequently, due to estimate (4.5) we obtain

$$\int_0^{T_1} |P_k(u_t^k.\nabla u^k)|_{V^*}^2 ds \leq C \int_0^{T_1} |\nabla u_t^k|^2 ds \leq C,$$

where $C > 0$ is independent of $k \in N$.

Also, we have

$$|P_k(u^k.\nabla u_t^k)|_{V^*} = \sup_{|v|_v \leq 1} |(u^k.\nabla u_t^k, P_k v)| \leq |u^k|_{L^\infty}|\nabla u_t^k| \leq C|\nabla u_t^k|.$$

Thus, $\int_0^{T_1} |P_k(u^k.\nabla u_t^k)|_{V^*}^2 ds \leq C$ thanks to the estimate (4.5).

Also, from

$$|Au_t^k|_{V^*} = \sup_{|v|_V \leq 1} |(Au_t^k, v)| = \sup_{|v|_V \leq 1} |(\nabla u_t^k, \nabla v) \leq |\nabla u_t^k|,$$

we conclude that $\int_0^{T_1} |Au_t^k|_{V^*}^2 ds \leq C$. The other terms in the G^k are analogously estimate.

To prove the continuity of $\theta_t(t)$ and $\psi_t(t)$ in the L^2-norm, we work exactly as before.

To finish the proof we have to show the continuity of $u(t), \theta(t)$ and $\psi(t)$ in the $H^2(\Omega)-$norm. We will only prove the continuity of $\theta(t)$ and $u(t)$; the proof of $\psi(t)$ is quite similar.

Also, we will prove this continuity only at $t_0 = 0$; for other $t_0 > 0$ the argument is analogous.

We observe that $\theta \in L^\infty(0, T_1; D(B))$ (resp. $u \in L^\infty(0, T_1; D(A))$). Thus, given any sequence $\{t_k\}_{k=0}^\infty \subseteq \mathbb{R}_+$, with $t_k \to 0^+$ we can extract a subsequence such that $\theta(t_{k_n}) \to \overline{\theta}$ weakly in H^2 for some $\overline{\theta} \in H^2$ (resp. $u(t_{k_n}) \to \overline{u}$ weakly in H^2 for some $\overline{u} \in H^2$). Since we know (Theorem 2.2) that $\theta(t_{k_n}) \to \theta_0$ strongly in H^1, (resp. $u(t_{k_n}) \to u_0$ strongly in H^1), the above implies that $\overline{\theta} = \theta_0$ (resp. $\overline{u} = u_0$). Moreover, since this holds for any sequence $\{t_k\}_{k=0}^\infty$ with $t_k \to 0^+$, we conclude that $\theta(t) \to \theta_0$ weakly in H^2 as $t \to 0^+$ (resp. $u(t) \to u_0$ weakly in H^2 as $t \to 0^+$). Consequently, due to the lower semicontinuity with respect to the weak topology of the norm, we have $|B\theta_0| \leq \liminf_{t \to 0^+} |B\theta(t)|$ (resp. $|Au_0| \leq \liminf_{t\to 0^+} |Au(t)|$).

Now, if we are able to prove that

$$\limsup_{t \to 0^+} |B\theta(t)| \leq |B\theta_0|, \tag{4.6}$$

(resp. $\limsup_{t\to 0^+} |Au(t)| \leq |Au_0|$) then we will have $\lim_{t\to 0^+} |B\theta(t)| = |B\theta_0|$ (resp. $\lim_{t \to 0^+} |Au(t)| = |Au_0|$), which together with the fact that $B\theta(t) \to B\theta_0$ weakly in

L^2 (resp. $Au(t) \to Au_0$ weakly in L^2) will imply that $B\theta(t) \to B\theta_0$ strongly in L^2 (resp. $Au(t) \to Au_0$ strongly in L^2) (see Brezis [4], p. 52).

In order to prove (4.6) we proceed as follows: put $\xi = B\theta_t^k$ in (2.8) to obtain

$$\begin{aligned} |\nabla\theta_t^k|^2 + \frac{1}{2}\frac{d}{dt}|B\theta^k|^2 &= (f, B\theta_t^k) - (u^k.\nabla\theta_2, B\theta_t^k) - (u^k.\nabla\theta^k, B\theta_t^k) \\ &= \frac{d}{dt}(f - u^k.\nabla\theta_2 - u^k.\nabla\theta^k, B\theta^k) \\ &\quad - (f_t - u_t^k.\nabla\theta_2 - u^k.\nabla(\theta_2)_t - u_t^k.\nabla\theta^k - u^k.\nabla\theta_t^k, B\theta^k). \end{aligned}$$

By integration with respect to time, and by using our previous estimates for u^k and θ^k, we obtain

$$\begin{aligned} |B\theta^k(t)|^2 &\leq |B\theta_0|^2 + 2\{(f - u^k.\nabla\theta_2 - u^k.\nabla\theta^k, B\theta^k) \\ &\quad - (f(0) - u_0^k.\nabla(\theta_2)(0) - u_0^k\nabla\theta_0^k, B\theta_0^k)\} + Mt^{1/2} \end{aligned}$$

where M is a positive constant depending on the previous estimates. From this, we conclude

$$\begin{aligned} |B\theta(t)|^2 &\leq |B\theta_0|^2 + 2\{(f - u.\nabla\theta_2 - u.\nabla\theta, B\theta) \\ &\quad - (f(0) - u_0.\nabla(\theta_2)(0) - u_0.\nabla\theta_0, B\theta_0)\} + Mt^{1/2}\ . \end{aligned}$$

Now, since $u.\nabla\theta_2 \to u_0.\nabla(\theta_2)(0), u.\nabla\theta \to u_0\nabla\theta_0, f \to f(0)$ in L^2 and $B\theta \to B\theta_0$ weakly in L^2 as $t \to 0^+$. We obtain (4.6).

For the velocity u, working exactly as before, we have

$$\begin{aligned} |Au(t)|^2 &\leq |Au_0|^2 + 2\{(\theta + \psi)g + g_1 - u.\nabla u, Au) \\ &\quad - ((\theta_0 + \psi_0)g(0) + g_1(0) - u_0.\nabla u_0, Au_0) + Mt^{1/2} \end{aligned}$$

observe that by the continuity of $\theta(t)$ and $\psi(t)$ in the $H^2(\Omega)-$norm, we have

$$((\theta + \psi)g, Au) \to ((\theta + \psi)(0)g(0), Au_0).\ \text{as}\ t \to 0^+ \tag{4.7}$$

The other terms are worked as before. Thus we obtain that $\lim\sup_{t\to 0+} |Au(t)| \leq |Au_0|$.

This completes the proof of the Theorem. ∎

Remark. We observe that we cannot obtain firstly the continuity of $u(t)$ in the H^2-norm, since $g \in L^2(0,T;L^3(\Omega)), g_t \in L^2(0,T;L^2(\Omega))$ implies only the continuity of g in L^2; and it is not sufficient to obtain the convergence (4.7). ∎

5 Results on the pressure.

In a standar way we can obtain information on the pressure, in fact, we have

Proposition 5.1 Under the hypothesis of Theorem 2.2, there exist a unique function $p \in L^2(0,T_1;H^1(\Omega)/\mathbb{R})$ such that (u,θ,ψ,p) is solution of (2.2) - (2.4). Under the hypothesis of Theorem 2.3, $p \in L^\infty(0,T_1;H^1(\Omega)/\mathbb{R}) \cap C([0,T_1];L^2(\Omega)/\mathbb{R})$.

Proof. We observe that (2.2)(i) is equivalent to $Au = P(F)$, where $F = g_1 + (\theta + \psi)g - u_t - u.\nabla u$.

Now, we observe that under the hypothesis of the Theorem 2.2 (resp. Theorem 2.3), we have $F \in L^2(0,T_1;L^2(\Omega))$ (resp. $F \in L^\infty(0,T_1;L^2(\Omega))$.

Therefore, Amrouche and Girault's results [2] imply that there is unique $p \in L^2(0,T_1;H^1(\Omega)/\mathbb{R})$ (resp. $p \in L^\infty(0,T_1;H^1(\Omega)/\mathbb{R}) \cap C([0,T_1];L^2(\Omega)/\mathbb{R})$) such that

$$\begin{aligned} &-\Delta u + \nabla p = F \\ &\text{div } u = 0 \quad \text{in} \quad \Omega, \\ &u|_\Gamma = 0 \ , \end{aligned}$$

thus, the Proposition is proved. ∎

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