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MULTIDIMENSIONAL HYPERBOLIC SYSTEMS WITH DEGENERATE CHARACTERISTIC STRUCTURE

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ABSTRACT – In this work we study 2 × 2 hyperbolic systems of the form $U_t + A(U)U_x + B(U)U_y = 0$ with degenerate characteristic structure. We define *partially aligned* systems as those for which A and B have a common eigenvector, and we show that the characteristic structure degenerates into a pair of curves if and only if the system is partially aligned. We describe examples and some basic properties of such systems.

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I. M. E. C. C.
B I B L I O T E C A

Multidimensional hyperbolic systems with degenerate characteristic structure

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Abstract

In this work we study 2×2 hyperbolic systems of the form $U_t + A(U)U_x + B(U)U_y = 0$ with degenerate characteristic structure. We define *partially aligned* systems as those for which A and B have a common eigenvector, and we show that the characteristic structure degenerates into a pair of curves if and only if the system is partially aligned. We describe examples and some basic properties of such systems.

When studying multidimensional systems of conservation laws, the most obvious difficulty one faces is the complexity of the wave propagation structure that even the simplest of these systems present. Most of what is known applies to systems in one space dimension and scalar multi-D equations, situations where information propagates along characteristic curves, rather than the cones or more complicated geometric structures of the common multi-D systems. This paper is dedicated to a third group of problems with this property, two-dimensional 2×2 systems whose characteristic structures degenerate into curves. These give a simplified wave propagation picture, analogous to the known cases but complicated by their essential multi-D character. We hope to convince the reader that these offer a natural starting point for multidimensional theory for systems.

Examples of systems such as these have appeared in the literature, for instance in the work by Tan and Zhang on Riemann problems for a system related to the two-dimensional Euler equations for incompressible, ideal fluids, [9]. The literature of multidimensional systems of conservation laws is

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not yet extensive. We mention the work of Lax [2,3] on hyperbolicity and multi-D systems and the landmark monography by Majda [5] as basic literature leading to our set of concerns. Further related work includes the study on symmetry in multi-D systems [1].

This paper is organized as follows. We begin with the definition of hyperbolicity and characteristics in two space dimensions, to fix notation. We show that the characteristics being a pair of lines is equivalent to an algebraic condition (Lemma 1). We then introduce the class of *partially aligned* systems. Our main result states that a system is partially aligned if and only if the characteristics are a pair of curves. The characteristic structure of partially aligned systems is a generalization of that of a pair of decoupled equations. In that case they consist of straight lines emanating from every point in physical space. We also describe specific examples of partially aligned systems and some of their properties.

We start from the notion of hyperbolicity. Let A and B be a pair of constant 2×2 matrices, and let $\xi = (\xi_1, \xi_2)$ be a nonzero vector in $\mathbb{R}^2$. Define $C(\xi) = \xi_1 A + \xi_2 B$. Consider the system of differential equations

$$U_t + AU_x + BU_y = 0. \tag{1}$$

Definition 1 *System (1) is hyperbolic in the direction ξ if $C(\xi)$ has real eigenvalues. It is strictly hyperbolic if $C(\xi)$ has distinct real eigenvalues. We will say that the system is (strictly) hyperbolic if it is (strictly) hyperbolic in every direction.*

Assume the system (1) to be hyperbolic. We define its symbol S to be the matrix-valued function $S(\tau, \xi) \equiv \tau I + C(\xi)$. We will also consider the homogeneous quadratic polynomial $p(\tau, \xi) \equiv \det S(\tau, \xi)$.

Definition 2 *Define the co-characteristic variety*

$$\Gamma = \{(\tau, \xi) \in (\mathbb{R} \times \mathbb{R}^2)^* \mid p(\tau, \xi) = 0\}.$$

The characteristic variety is defined as

$$\Lambda = \{(t, x, y) \in \mathbb{R} \times \mathbb{R}^2 \mid (t, x, y) = \nabla p(\tau, \xi), \ \textit{for some } (\tau, \xi) \in \Gamma\}.$$

Both Λ and Γ are *conic* subsets of $\mathbb{R}^3$ and $(\mathbb{R}^3)^*$. This means that if a vector v belongs to one of them, then any real multiple of v also belongs to it. Γ is conic because it is the zero set of a homogeneous function. Λ is conic because p is quadratic, and hence its gradient is linear. There is a natural duality between Λ and Γ.

The function $p(\tau, \xi)$ is a second degree polynomial in τ, for every ξ fixed. Looking at it this way we identify it with the characteristic polynomial of the matrix $C(-\xi)$. We consider the even function of ξ

$$\Delta(\xi) = (\mathrm{Tr} C(\xi))^2 - 4 \det C(\xi), \tag{2}$$

which is the discriminant of $p(\tau, \xi) = 0$. Clearly (strict) hyperbolicity in the direction ξ is equivalent to (strict) nonnegativity of $\Delta(-\xi)$. Our first result describes the kind of degenerate characteristic structure in which we are interested, in terms of Δ.

Lemma 1 *The characteristic variety Λ of system (1) consists of a pair of straight lines through the origin if and only if the discriminant $\Delta(\xi)$ is the square of a linear homogeneous function of ξ. The characteristic variety consists of a single line if and only if $\Delta(\xi)$ is identically zero.*

Proof: We begin by observing that Λ consists of a pair of lines if and only if Γ consists of a pair of planes. Suppose that Γ consists of a pair of planes and let $\mathbf{n}_1$ and $\mathbf{n}_2$ be their normal vectors. Since Γ is the zero level set of $p(\tau, \xi)$ then ∇p is normal to Γ. Thus $\nabla p(\tau, \xi)$ must be linearly dependent with one of $\mathbf{n}_1$ or $\mathbf{n}_2$. By definition of Λ and by homogeneity of p we see that Λ must contain the spaces spanned by $\mathbf{n}_1$ and $\mathbf{n}_2$, a pair of straight lines. Conversely, assume Λ consists of a pair of lines and let $\mathbf{n}_1$ and $\mathbf{n}_2$ be their generators. The set Γ is hence normal to either $\mathbf{n}_1$ or $\mathbf{n}_2$ everywhere. Thus it is contained in the planes normal to $\mathbf{n}_1$ and $\mathbf{n}_2$. Consider the intersection of Γ with the plane normal to $\mathbf{n}_1$. That must contain a point q where $(\nabla p)(q)$ is not zero, since otherwise $\mathbf{n}_1$ would not be in Λ. By the implicit function theorem, Γ is a two-dimensional surface near q. So, p restricted to this plane is a quadratic function vanishing on a nonempty open set. This implies p is identically zero on the whole plane. The same argument applies to the plane normal to $\mathbf{n}_2$, which proves our assertion.

Clearly, the argument above also proves that Γ reduces to a single plane if and only if Λ reduces to a single line. It is thus enough to show that Γ consists of a pair of planes if and only if Δ is the square of a linear homogeneous function of ξ. If Δ is the square of such a linear function, one may by inspection conclude what we want immediately. The converse is a little more delicate.

Suppose that Γ consists of a pair of planes, given by $a_i\tau + b_i\xi_1 + c_i\xi_2 = 0$, $i = 1, 2$. Observe that both a_1 and a_2 must not be zero, because a vertical plane cannot be contained in a level set of p since the coefficient of τ^2 is nonzero. Now we have expressions, $\tau_i = -(1/a_i)(b_i\xi_1 + c_i\xi_2)$. We compute Δ explicitly, and obtain $\Delta = ((b_1/a_1 - b_2/a_2)\xi_1 + (c_1/a_1 - c_2/a_2)\xi_2)^2$.

If $\Delta \equiv 0$, clearly Γ is a single plane. The converse hypothesis will mean that the roots τ_i above are identical, which implies the vanishing of Δ. ∎

We now turn to the definition of partial alignment. Let A and B be smooth functions, defined on a domain $\Omega \subseteq \mathbb{R}^2$ with values in the set of 2×2 real matrices. We will consider the quasilinear system

$$U_t + A(U)U_x + B(U)U_y = 0. \tag{3}$$

We assume that this system is hyperbolic, i.e. for any $U_0 \in \Omega$, the linearized system $U_t + A(U_0)U_x + B(U_0)U_y = 0$ is hyperbolic.

Definition 3 *System (3) is partially aligned at $U_0 \in \Omega$ if $A(U_0)$ and $B(U_0)$ have an eigenvector in common. We say it is partially aligned in Ω if it is partially aligned at every state in Ω. We call a common eigenspace of A and B a direction of alignment. If, for each U in Ω, $A(U)$ and $B(U)$ have two common linearly independent eigenvectors then the system is said to be totally aligned.*

We will discuss several properties of partially aligned systems. Assume the choice of common eigenvector can be made smoothly in state space. The most important property is the existence of a Riemann invariant associated with the direction of alignment. The construction of Riemann invariants for hyperbolic 2×2 systems in one space dimension is possible due to the fact that all smooth vector fields in a two-dimensional space are locally conformally equivalent to a gradient vector field (see [8]). Since state space is two-dimensional, this fact can be applied to the smoothly varying family of common eigenvectors of the partially aligned matrices A and B. In Proposition 1 we give a sufficient condition for existence of a smoothly varying family of eigenvectors.

Let $r = r(U)$ be a smooth function, defined on a domain $\Omega_0 \subseteq \Omega$, such that $\nabla r(U)$ is a common left eigenvector of $A(U)$ and $B(U)$ for each U in Ω_0. In particular this vector field does not vanish.

Let $w(U)$ be a smooth function, defined on Ω_0 above, and such that, together, w and r form a new coordinate system for the neighborhood Ω_0 in state space. What is required for w is that the map $V(U) = (w(U), r(U))$ is a diffeomorphism. In particular, ∇w and ∇r have to be linearly independent. In these new coordinates, system (3) becomes upper triangular. We will write it as:

$$\begin{cases} w_t + < f(V), V_x > + < g(V), V_y > & = 0 \\ r_t + \lambda_A(V) r_x + \lambda_B(V) r_y & = 0. \end{cases} \tag{4}$$

Denote by $f = (f^1, f^2)$ and $g = (g^1, g^2)$ the vectors that appear in the first equation. The scalar functions λ_A and λ_B are the eigenvalues of A and B respectively, associated with the common eigenvector.

The main result in this paper is the following description of partial alignment in terms of the characteristic structure.

Theorem 1 *System (3) is partially aligned if and only if its characteristic structure consists of a pair of curves emanating from every point in physical space.*

Proof: Fix $U_0 \in \Omega$ and call $A = A(U_0)$ and $B = B(U_0)$. By Lemma 1, the proof is reduced to showing the equivalence of partial alignment and the property that the discriminant $\Delta(\xi)$ be the square of a homogeneous linear function of ξ.

First assume A and B have a common eigenvector $\mathbf{n}$, associated to the eigenvalues λ_A and λ_B. Set $L = (\lambda_A, \lambda_B)$. Then $l_\xi \equiv < L, \xi >$ is an eigenvalue of $C(\xi)$, and $\mathbf{n}$ is its eigenvector. The other eigenvalue is $\mathrm{Tr}(C(\xi)) - l_\xi$, also a linear function of ξ. The characteristic polynomial of $C(\xi)$ can be written as

$$p(\tau, -\xi) = \det(\tau I - C(\xi)) = \tau^2 - \mathrm{Tr}(C(\xi))\tau + \det(C(\xi)),$$

with discriminant $\Delta = (\mathrm{Tr} C)^2 - 4 \det C$. This expression is exactly the square of the difference of the eigenvalues. In this case, $\Delta = (2l_\xi - \mathrm{Tr} C(\xi))^2$, as we wanted.

Conversely, assume that the discriminant Δ is the square of a homogeneous linear function of ξ,

$$\Delta = (m\xi_1 + n\xi_2)^2. \tag{5}$$

First assume A is diagonalizable. Since the trace and determinant are invariant under conjugation, we will rewrite the problem on a basis of eigenvectors of A. We denote the new matrices A and B by:

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Now we write the discriminant of the characteristic polynomial of $C(\xi) = \xi_1 A + \xi_2 B$, with A and B above. We obtain the expression

$$\Delta = \xi_1^2(a_1 - a_2)^2 + 2\xi_1\xi_2(a_1 - a_2)(b_{11} - b_{22}) + \xi_2^2\left[(b_{11} - b_{22})^2 + 4b_{12}b_{21}\right]. \tag{6}$$

Matching the corresponding coefficients of the two expressions (5) and (6) for the quadratic polynomial $\Delta(\xi)$, we see that either $a_1 = a_2$ or $b_{12}b_{21} = 0$. In the case $a_1 = a_2$, the matrix A was originally a scalar multiple of the identity, and therefore, any eigenvector of B is a common eigenvector. In the second case, if $b_{21} = 0$, then the first basis element of the chosen basis of eigenvectors of A is also an eigenvector of B and if b_{12} vanished, the second basis element would then be the common eigenvector.

Next suppose A is not diagonalizable. The matrix A must have repeated eigenvalues and a one-dimensional eigenspace. Thus there is a basis on which the problem can be rewritten with new matrices:

$$A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

The discriminant of the characteristic polynomial of $C(\xi)$ in this case is

$$\Delta = \xi_1\xi_2 4b_{21} + \xi_2^2\left[(b_{11} - b_{22})^2 + 4b_{12}b_{21}\right].$$

By matching coefficients again we see that b_{21} has to be 0, hence the vector $(1, 0)$ in the new basis is also an eigenvector of B. This completes the proof.

∎

The degenerate case where the characteristics reduce to a single line is the case where both A and B have repeated eigenvalues. We call a state with this kind of degeneracy *coincident*. A system that is totally aligned and coincident has both dependent variables functioning as Riemann invariants, propagating along the same characteristic. This implies that the characteristic is a straight line. Thus, totally aligned, coincident systems behave very much like scalar equations.

In studying partially aligned systems, a hypothesis of non-coincidence may play the role that strict hyperbolicity plays in 1-D theory. Systems that are everywhere coincident are nonlinear examples of constant multiplicity multiple characteristic systems. Systems that possess both coincident and non-coincident states are examples of systems with characteristics of variable multiplicity, and are notoriously difficult to deal with (see [6] for some of the linear theory of hyperbolic equations with multiple characteristics). Another way of stressing the role of coincidence for the class of partially aligned systems is the following Proposition.

Proposition 1 *Let $A(\tau)$ and $B(\tau)$ be smooth one-parameter families of partially aligned 2×2 matrices. Let τ_0 be such that $A(\tau_0)$ and $B(\tau_0)$ are not coincident. Then there exists a neighborhood I of τ_0 such that $A(\tau)$ and $B(\tau)$ are not coincident in I. Furthermore, there exists a smooth vector-valued function $\mathrm{n}(\tau)$ which is a common eigenvector of $A(\tau)$ and $B(\tau)$ for all τ near τ_0.*

Proof: The first statement is a trivial consequence of the characterization of coincidence in terms of the vanishing of Δ.

Let us proceed to the second conclusion. Assume, without loss of generality, that $A(\tau)$ has distinct eigenvalues near τ_0. Therefore there are two smoothly varying families of eigenvalues of A. The respective eigenvectors can hence be chosen smoothly. One of these must necessarily be a common eigenvector. ∎

Another important observation on partially aligned systems is that they are always nonstrictly hyperbolic, in the sense that at every state U_0 there exists at least one direction ξ where $C(\xi)$ has coincident eigenvalues. We determine this direction by looking at the discriminant $\Delta(\xi) = (m_1\xi_1 + m_2\xi_2)^2$.

We observe that it always vanishes on a straight line. Nonstrict hyperbolicity in more than one direction implies Δ vanishing and hence coincidence.

Consider a linearization of a partially aligned system at a constant state. We can rotate physical space, to make the direction of non-strict hyperbolicity the x-axis. With that rotation, the matrix A will have repeated eigenvalues, which after a Galilean transformation of physical variables, can be assumed to be zero. In upper triangular form, this linearized system has the form:

$$U_t + \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} U_x + \begin{bmatrix} \mu_1 & b \\ 0 & \mu_2 \end{bmatrix} U_y = 0.$$

This is a particularly simple form for the linearized systems, that calls attention to the non-strict hyperbolic nature of these.

Finally, we describe some examples of partially aligned systems of interest.

1. The first example is the particular class of totally aligned systems. They have two linearly independent common eigenvectors and can be put in diagonal form, with a pair of Riemann invariants constant along each one of the wavefields. This provides local L^∞ estimates for smooth solutions, and behavior similar to 1-D 2×2 systems. The nonlinear sharp Huygens principle argument for shock formation, due to Klainerman and Majda (see [5]) works in this case, exactly as in the original 1-D case. Although they are nonstrictly hyperbolic they have better analytic behavior than general strictly hyperbolic systems, as was observed by Rauch in [7].

2. We describe a family of examples of partially aligned systems with simple form. Consider the systems:

$$\begin{cases} u_t + (u^2/2)_x + (f(u,v))_y = 0 \\ v_t + (g(u,v))_x + (v^2/2)_y = 0. \end{cases} \tag{7}$$

If this system is partially aligned, and the alignment direction is neither horizontal nor vertical, then the eigenvalues for the common eigenvector are u and v, and the following relation holds:

$$(g_v - u)(f_u - v) = g_u f_v.$$

Where the direction of alignment is either vertical or horizontal, the eigenvalues change and the corresponding relation becomes $f_v = 0$ or

$g_u = 0$ respectively. On the other hand, if either $(g_v - u)(f_u - v) = g_u f_v$ or $f_v = 0$ or $g_u = 0$ then the system is partially aligned. For instance, $f(u,v) = \varepsilon u^2 + (uv)/2 + \varepsilon v^2$ and $g(u,v) = -\varepsilon u^2 + (uv)/2 - \varepsilon v^2$ are a specific case of these examples.

3. A system of conservation laws can be obtained from the incompressible 2-D Euler equations, setting pressure and density constant in the momentum balance equations. This system was first studied by Tan and Zhang [9] who discussed Riemann problems. The system has the form

$$\begin{cases} u_t + (u^2)_x + (uv)_y = 0 \\ v_t + (uv)_x + (v^2)_y = 0. \end{cases} \tag{8}$$

This system is partially aligned, with a rather singular coincident state at $(0,0)$, and non-coincident elsewhere. Its structure was used in [4] to study shock formation, via a compression rate argument. In this shock formation study, this system was used as a template for a small class of partially aligned systems for which that analysis holds.

We add a few concluding remarks. What we have done can be generalized to 2×2 systems in many space dimensions. The notion of partial alignment is, however, far more singular in that context. The introduction of the class of partially aligned systems poses a wealth of interesting problems, in terms of generalizing known theory to them. There is enough structure to make some interesting generalizations possible, as the authors have shown in [4]. If these problems will be interesting or not will depend on whether partially aligned systems can be used as physically meaningful models, even if only in special circumstances. We think this may well turn out to be the case. In our view, the most important open problem is deciding if a-priori estimates for weak solutions are available. We mentioned the work of Rauch [7], which shows one cannot expect BV or L^p estimates, $p \neq 2$, for multi-D systems, but the only partially aligned systems that satisfy his hypothesis are totally aligned ones, which are exactly the systems that turn out to admit BV estimates after all.

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