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SHOCK FORMATION FOR A SYSTEM OF CONSERVATION LAWS IN TWO SPACE DIMENSIONS

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ABSTRACT – We study shock formation for the 2×2 system $u_t + (u^2)_x + (uv)_y = 0$ and $v_t + (uv)_x + (v^2)_y = 0$. Our result is based on the argument, due to J.Keller and L.Ting [?], about the evolution along a characteristic of the compression rate of nearby characteristics. This system is one of a class of systems, called partially aligned, which exhibit a degenerate characteristic structure where a pair of directions substitutes the usual cone at every point. Our analysis can be extended to a set of partially aligned systems satisfying a sharp algebraic constraint.

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I. M. E. C. C.
B I B L I O T E C A

Shock formation for a system of conservation laws in two space dimensions

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Abstract

We study shock formation for the 2×2 system $u_t+(u^2)_x+(uv)_y = 0$ and $v_t+(uv)_x+(v^2)_y = 0$. Our result is based on the argument, due to J.Keller and L.Ting [1], about the evolution along a characteristic of the compression rate of nearby characteristics. This system is one of a class of systems, called partially aligned, which exhibit a degenerate characteristic structure where a pair of directions substitutes the usual cone at every point. Our analysis can be extended to a set of partially aligned systems satisfying a sharp algebraic constraint.

The system we will study,

$$\begin{cases} u_t + (u^2)_x + (uv)_y = 0 \\ v_t + (uv)_x + (v^2)_y = 0, \end{cases} \tag{1}$$

consists of the balance of momentum equations for an incompressible homogeneous 2-dimensional fluid at constant pressure. One should not expect these equations to model fluid behavior, but this fact motivates the consideration of this particular system.

Tan and Zhang [7] studied the Riemann problem for this system and introduced δ-shock waves as components of their solution. In contrast, our work is basically concerned with the smooth solutions of (1). The only general result on the formation of singularities for multi-D systems is the work of Sideris [5]. His theorem applies, however, to symmetrizable strictly hyperbolic systems, which excludes those under present consideration.

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System (1) exhibits a degeneracy in characteristic structure which puts it into a class we call *partially aligned* (see Definition 1). Our analysis applies to other partially aligned systems, but a sharp algebraic constraint on the fluxes, which system (1) happens to satisfy, is also required. We will discuss some of the basic properties of partially aligned systems in a forthcoming paper [3].

Let $U = (u, v)$ denote a point in a state space domain $\Omega \subseteq \mathbb{R}^2$ and let A and B be two smooth functions defined on Ω, with values in the set of 2×2 matrices with real eigenvalues. We consider the quasilinear hyperbolic systems

$$U_t + A(U)U_x + B(U)U_y = 0. \tag{2}$$

Definition 1 *The system above is partially aligned at $U_0 \in \Omega$ if $A(U_0)$ and $B(U_0)$ have an eigenvector in common. We say it is partially aligned in $\Omega_0 \subseteq \Omega$ if it is partially aligned at every state in Ω_0. We call a common eigenspace of A and B a direction of alignment.*

An important property of partially aligned systems is that they admit a Riemann invariant, associated to the direction of alignment. It is easy to see that system (1) is partially aligned. First we write it in quasilinear form (2), with

$$A = \begin{bmatrix} 2u & 0 \\ v & u \end{bmatrix} \text{ and } B = \begin{bmatrix} v & u \\ 0 & 2v \end{bmatrix}. \tag{3}$$

The eigenvalues $2u$ of A and $2v$ of B have left eigenvectors respectively $(1, 0)$ and $(0, 1)$. The remaining eigenvalues, u of A and v of B have a common left eigenvector $(v, -u)$.

We will put system (1) in upper triangular form. Consider functions $h(u, v)$ with gradient spanning the direction of alignment at every point away from the origin. The function h is called a Riemann invariant for system (1) and its local existence for arbitrary partially aligned systems follows from state space being two-dimensional. The argument is identical to the one used in 2×2 systems in one space dimension; see [6] for details. In our case, we will exhibit Riemann invariant functions explicitly.

Since the direction of alignment $(v, -u)$ is linearly dependent with the gradient of the Riemann invariants, they must be constant on rays departing the origin. There are no smooth nonsingular functions defined everywhere in

$\mathbb{R}^2 - \{(0,0)\}$ constant on rays. Thus, there are no globally smooth Riemann invariants. We pick the angle function as Riemann invariant,

$$\Theta(u,v) = \arctan\left(\frac{u}{v}\right), \tag{4}$$

choosing the branch of the arctan(·) with values in the interval $(-\pi, \pi)$. This corresponds to fixing a state space domain that does not contain the negative u-axis. We also remove a closed neighborhood of the state $(0,0)$ from the domain. For $\varepsilon > 0$ set

$$\Omega = \mathbb{R}^2 - \left(\{(u,v) : v = 0, u \leq 0\} \cup \{(u,v) : u^2 + v^2 \leq \varepsilon^2\}\right). \tag{5}$$

Other domains could be considered, by choosing different branches for the arctan(·), but our argument would not change in substance.

We choose another smooth function, which together with the Riemann invariant Θ forms a new pair of dependent variables, in which system (1) becomes upper triangular. The only requirement the other coordinate must satisfy is that its gradient be linearly independent from $\nabla\Theta$ everywhere in Ω. We use the radius variable $R = \sqrt{u^2 + v^2}$, as it is the familiar complement to Θ, but calling attention that it is an arbitrary choice. In the new variables,

$$W = \begin{bmatrix} R \\ \Theta \end{bmatrix},$$

system (1) becomes

$$W_t + \begin{bmatrix} 2R\sin\Theta & R^2\cos\Theta \\ 0 & R\sin\Theta \end{bmatrix} W_x + \begin{bmatrix} 2R\cos\Theta & -R^2\sin\Theta \\ 0 & R\cos\Theta \end{bmatrix} W_y = 0. \tag{6}$$

Assume $R(x,y,t)$ and $\Theta(x,y,t)$ are smooth solutions of (6), defined on $\mathcal{D} \equiv \mathbb{R}^2 \times (0,T)$, continuous on the closure $\overline{\mathcal{D}}$. The characteristics for system (1) become evident when the system is written as above. The second equation in (6) expresses the fact that the Riemann invariant Θ is constant along characteristics determined by

$$\begin{cases} dx/dt = (R\sin\Theta)(x,y,t) \\ dy/dt = (R\cos\Theta)(x,y,t) \\ (x(0), y(0)) = (\alpha_1, \alpha_2). \end{cases} \tag{7}$$

Here $\alpha \equiv (\alpha_1, \alpha_2)$ is a Lagrangian marker, labelling the starting position of the characteristic in physical space: The other characteristic wavefield is given by the eigenvalues of A and B associated to the non-aligned eigenspaces,

$$\begin{cases} dx/dt = (2R\sin\Theta)(x,y,t) \\ dy/dt = (2R\cos\Theta)(x,y,t) \\ (x(0),y(0)) = (\alpha_1,\alpha_2). \end{cases}$$

This wavefield does not carry a conserved quantity.

System (1) is hyperbolic, but not strictly so. At each state $(u_0, v_0) \in \Omega$, there is a direction where the two speeds of plane wave propagation coincide, given by $(-v_0, u_0)$. This failure of strict hyperbolicity is a feature of partially aligned systems in general, and it will be discussed in detail in [3]. The structure of the characteristic fields in this particular example exhibits invariance under rotational symmetries of state space, which is a reflection of the covariance of these equations with respect to simultaneous rotation of dependent and independent variables. For a discussion of symmetry and characteristics for multi-D systems see [2].

We will now begin to discuss shock formation per se. For 2×2 systems in one space dimension there is a well-known argument, due to Klainerman and Majda, which can be used to prove shock formation (we refer the reader to [4] rather than to the original source). Their idea relies on finite-time decoupling of information transported along the two characteristic wavefields, for sufficiently small compactly supported perturbations of a constant state. Unless one requires global separation of the speeds of the two wavefields, an argument of this kind fails in partially aligned systems because it is not possible to propagate in time the initial separation of wavefields with only one Riemann invariant. We will now show that the other classical shock formation analysis for 2×2 systems in one space dimension can actually be adapted to our system. We refer the reader once more to [4] for details on the one-dimensional case.

We are going to define a compression rate matrix, measuring the relative displacement of neighboring characteristics for the Riemann invariant wavefield. An ordinary differential equation for the compression rate along the characteristic will be derived (Lemma 2). We will show that as long as the solution remains confined to Ω, smoothness will be lost through infinite compression (shock formation) in finite time.

Definition 2 *Let $(x(\alpha,t),y(\alpha,t))$ be the characteristic curve, solution of system (7). We define the compression rate matrix M by:*

$$M(\alpha,t) = \begin{bmatrix} \partial x/\partial\alpha_1 & \partial x/\partial\alpha_2 \\ \partial y/\partial\alpha_1 & \partial y/\partial\alpha_2 \end{bmatrix}.$$

Note that $M(\alpha,0)$ is the identity, and hence invertible for t small. Our first result consists of the observation that when this matrix becomes singular shocks will form.

Lemma 1 *If $M(\alpha^*,T)$ is singular, with n a nonzero vector in its kernel; if the directional derivative $<\nabla\Theta_0,\mathrm{n}>$ is nonzero at α^* and if the solution $W=(R,\Theta)$ is smooth in a tubular neighborhood of the characteristic emanating from α^* for times $t<T$ then $|\nabla\Theta(x(\alpha^*,t),y(\alpha^*,t),t)| \rightarrow \infty$ as $t \rightarrow T$.*

Proof: This is a consequence of the transport equation

$$\left(\nabla_{(x,y)}\Theta\right) M = \nabla_\alpha\Theta_0, \tag{8}$$

which comes from Θ being constant along characteristics and is valid as long as the solution W is C^1, i.e. as long as $t<T$ by hypothesis. Multiply this identity by n on the right, and pass to the limit $t\rightarrow T$.

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Hereafter, we will denote the transpose of any matrix by $(\cdot)^T$. Vectors will be represented by row matrices. Consequently, matrices operate on vectors through multiplication on the right. If v_1 and v_2 are two vectors then $v_1(v_2)^T$ denotes their euclidian inner product and $(v_1)^T v_2$ denotes the matrix $v_1 \otimes v_2$. Set $\Lambda(R,\Theta) = (R\sin\Theta, R\cos\Theta)$, the vector of eigenvalues of A and B (the matrices defined in (3)), determining the Riemann invariant characteristic speed. Our next Lemma is the derivation of the differential equation satisfied by the compression rate matrix along the characteristic.

Lemma 2 *Fix a Lagrangian marker α. As in Lemma 1, assume that the solution W is smooth in a tubular neighborhood of the characteristic given by (7) with initial position α. Then the compression rate matrix $M(\alpha, \cdot)$ satisfies the following ordinary differential equation:*

$$\frac{dM}{dt} - (\frac{\partial \Lambda}{\partial R})^T (\nabla R) M = (\frac{\partial \Lambda}{\partial \Theta})^T \nabla_\alpha \Theta_0.$$

Proof: The equation obtained by differentiating M along the characteristic is:

$$\frac{dM}{dt} = (\frac{\partial \Lambda}{\partial R})^T (\nabla R) M + (\frac{\partial \Lambda}{\partial \Theta})^T (\nabla \Theta) M.$$

The Lemma is proved using the transport identity (8) on the last term of the right-hand side.

∎

We have not yet used the particular form of system (1). In what follows we restrict our attention to this system. Below we state the main result of this paper.

Theorem Let $U_0(x, y) = (u_0(x, y), v_0(x, y))$ be compactly supported initial data; let $W_0(x, y) = (R_0(x, y), \Theta_0(x, y))$ be this initial data expressed in W coordinates. Suppose the data satisfies the properties below:

1. The vector field $\nabla \Theta_0$ has but a finite number of singular points in the interior of the support of the data.
2. There exists a smooth solution $U(x, y, t)$ of system (1), with values in the domain Ω, defined up to time $T(U_0)$. This is the maximal time of smoothness and will be determined precisely in the proof.

Let $m(x, y, t) = \max\{|\nabla u|, |\nabla v|\}$. Then there exists a point $\alpha^* \in \mathrm{supp}(U_0)$ such that

$$m(x(\alpha^*, t), y(\alpha^*, t), t) \longrightarrow \infty \qquad \text{as} \qquad t \rightarrow T(U_0).$$

We will see from the proof that the shock formation time $T(U_0)$ depends on the derivatives of the initial data, and on precisely how far the solution

$U(x, y, t)$ is from blow-up (by blow-up we mean the solution leaves Ω in finite time).

Proof: Put system (1) in the upper triangular form (6) and recall the definition of Λ, the Riemann invariant characteristic wavespeed vector. Consider $\Lambda_R^\perp \equiv (-\cos\Theta, \sin\Theta)$, a nonzero vector perpendicular to $\partial\Lambda/\partial R$ everywhere. The crucial property of this vector, for our purposes is that it does not depend on R, in other words,

$$\frac{\partial}{\partial R}\left(\Lambda_R^\perp\right) = 0. \tag{9}$$

Take any simple, closed level curve of Θ_0, inverse image of a regular value. The vector field $\Lambda_R^\perp$ is constant along this level curve and the outward unit normal to a C^1 Jordan curve such as this one is a surjective map onto the unit circle. Hence there exists α^* such that $\Lambda_R^\perp(\alpha^*, 0)$ is parallel to the vector $\nabla\Theta_0(\alpha^*)$ and has the same orientation. Therefore we can choose a real constant $C > 0$ such that

$$\Lambda_R^\perp(\alpha^*, 0) = C\,(\nabla\Theta_0(\alpha^*))\,. \tag{10}$$

Define $\mathbf{n}(t) \equiv \Lambda_R^\perp(\alpha^*, t)$. Along the direction spanned by $\mathbf{n}$ the equation for the compression rate matrix M in Lemma 2 becomes a closed 2×2 system which can be integrated explicitly. The important facts here are:

1. $\mathbf{n}$ is constant along the Riemann invariant characteristic,
2. $\mathbf{n}$ belongs to the left kernel of the matrix $(\partial\Lambda/\partial R)^T(\nabla R)$.

Multiplying the equation for M by $\mathbf{n}$ one gets:

$$\mathbf{n}\frac{dM}{dt} - \mathbf{n}(\frac{\partial\Lambda}{\partial R})^T(\nabla R)M = \mathbf{n}(\frac{\partial\Lambda}{\partial\Theta})^T\nabla_\alpha\Theta_0.$$

Since $\mathbf{n}$ was chosen perpendicular to $\partial\Lambda/\partial R$, the second term on the left hand side vanishes, leaving us with

$$\mathbf{n}\frac{dM}{dt} = \mathbf{n}(\frac{\partial\Lambda}{\partial\Theta})^T\nabla_\alpha\Theta_0. \tag{11}$$

We recall that the time derivative here is in fact a derivative along the characteristic. Calculating this derivative for $\mathbf{n}$ gives

$$\frac{d\mathbf{n}}{dt} = \frac{\partial\mathbf{n}}{\partial R}\frac{dR}{dt} + \frac{\partial\mathbf{n}}{\partial\Theta}\frac{d\Theta}{dt}.$$

The term $d\Theta/dt$ vanishes because the Riemann invariant is constant along the characteristic, whereas $\partial\mathbf{n}/\partial R$ vanishes in virtue of the algebraic relation (9). Therefore we conclude that $\mathbf{n}$ is also constant along the characteristic, or in other words,

$$\frac{d\mathbf{n}}{dt} = 0. \tag{12}$$

Rewrite equation (11), using (12), obtaining an exact equation with the form:

$$\frac{d}{dt}(\mathbf{n}M) = \mathbf{n}(\frac{\partial\Lambda}{\partial\Theta})^T\nabla_\alpha\Theta_0.$$

We integrate this equation and get an explicit formula for $\mathbf{n}M$ along the characteristic;

$$(\mathbf{n}M)(t) = \left(C + \int_0^t \Lambda_R^{\perp}(\frac{\partial\Lambda}{\partial\Theta})^T ds\right)\nabla\Theta_0(\alpha^*), \tag{13}$$

which then becomes:

$$(\mathbf{n}M)(t) = \left(C - \int_0^t R ds\right)\nabla\Theta_0(\alpha^*).$$

This expression vanishes after some time $T(U_0)$ as long as $R > \varepsilon$. This is the case if the solution is to remain inside Ω. The conclusion follows from Lemma 1.

∎

Consider now general partially aligned systems cast in upper triangular form as in (6). The Riemann invariant wavefield is determined by a vector, which we still call Λ, whose components are the eigenvalues associated to the common eigenvector. We call the Riemann invariant variable V and its complementary variable $\tilde{V}$. An additional condition must be imposed in order to make an argument similar to the one above hold true. We must assume that there exists a smooth scalar function $\phi(\tilde{V}, V)$ and a vector $\Xi(V)$ such that

$$\frac{\partial\Lambda}{\partial\tilde{V}} = \phi(\tilde{V}, V)\Xi(V), \tag{14}$$

i.e. that the vector $\partial\Lambda/\partial\tilde{V}$ has direction independent of $\tilde{V}$. Granted that, we can choose the vector $\mathbf{n} \equiv \Lambda_{\tilde{V}}^{\perp}$ independent of $\tilde{V}$ so that (12) holds in the general case.

Another necessary requirement is that the integrand in (13) have a distinguished sign, and be kept far from vanishing. This condition plays the role of genuine nonlinearity in our problem. For partially aligned systems satisfying (14) and this genuine nonlinearity, a similar theorem holds. The singularity at the state $(0,0)$ in system (1) can also be interpreted as *coincidence* in the sense defined in [3], and hence as failure of a condition structurally similar to strict hyperbolicity for 1-D systems. Near the state $(0,0)$, system (1) has characteristics of variable multiplicity – a situation known to be difficult to deal with even in the linear case.

What we have proved is that either shocks form or the solution leaves the region Ω, which is not enough to guarantee shock formation. To prove shock formation, an additional weak L^∞ estimate would have to be added; we intend to address this in future investigation. We have shown evidence that for system (1) and a few others like it, shock formation is to be expected, and quite frequently. It is a well-accepted heuristic principle that the multi-D nonlinear waves such as those studied here tend to be less compressive than their 1-D counterparts. We have shown that partially aligned system such as ours are constrained to have 1-D type compressive behavior along a certain direction, without losing their essential multi-D character.

Two relevant questions are left unanswered. The first one is how sharp the present prediction of the direction of shock formation is. This could be addressed by suitable numerical investigation, running, however, into the difficulties of placing precisely the position of shocks in a numerical solution for a multi-D system of conservation laws. The second set of questions is concerned with extending this argument for general partially aligned systems. Condition (14) seems at this time more convenient than essential.

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