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JUAN DAVID LONDOÑO ACEVEDO

An Euler-Lagrangian approach to rough incompressible Euler equations

Uma Abordagem Euler-Lagrangiana para as equações de Euler incompressíveis rough

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Juan David Londoño Acevedo

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Prof. Dr. CHRISTIAN HORACIO OLIVERA

Profa. Dra. GABRIELA DEL VALLE PLANAS

Prof. Dr. PEDRO JOSE CATUOGNO

Prof. Dr. DAVID ALEXANDER CHIPANA MOLLINEDO

Prof. Dr. WLADIMIR AUGUSTO DAS NEVES

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Resumo

Nesta tese, estudamos a relação entre formulações Lagrangianas e clássicas para as equações de Euler incompressíveis rough. Primeiramente, baseando-nos na fórmula de Itô-Kunita-Wentzell e em técnicas de análise estocástica, estabelecemos uma formulação Lagrangiana para as equações de Euler incompressíveis estocásticas. Em segundo lugar, estabelecemos uma formulação Lagrangiana para as equações de Euler incompressíveis conduzidas por um caminho Hölder. A prova é baseada na fórmula de Itô-Kunita-Wentzell para a integral de Young. Além disso, em ambos os casos, demonstramos um resultado de existência local para a formulação Lagrangiana em espaços de Sobolev adequados.
Finalmente, demonstramos que a equação de Euler incompressível conduzida por um rough path verifica a formulação Lagrangiana, e novamente a prova é baseada na fórmula de Itô-Wentzell para rough paths.

Palavras-chave: Equação de Euler, formulação Lagrangiana, formula de Itô-Kunita-Wentzell, rough paths, integral de Young, movimento Browniano, espaços de Sobolev.

Abstract

In this thesis, we study the relationship between Lagrangian and classical formulations for the rough incompressible Euler equations. Firstly, based on the Itô-Kunita-Wentzell formula and stochastic analysis techniques, we establish a Lagrangian formulation for stochastic incompressible Euler equations. Secondly, we establish a Lagrangian formulation for incompressible Euler equations driven by a Hölder path. The proof is based on Itô–Kunita-Wentzell's formula for the Young integral. Furthermore, in both cases, we show a local existence result for the Lagrangian formulation in suitable Sobolev spaces.
Finally, we prove that the incompressible Euler equation driven by a rough path satisfies the Lagrangian formulation, and again, the proof is based on the Itô-Wentzell formula for rough paths.

Keywords: Euler equation, Lagrangian formulation, Itô-Kunita-Wentzell's formula, rough paths, Young integral, Brownian motion, Sobolev spaces.

Contents

INTRODUCTION

This thesis deals with the Euler-Lagrangian formulation, called also Constantin-Iyer representation after [Con01], [CI08], (following [Con01], [FL19a] and [PR16]) of the rough incompressible Euler equations on the torus $\mathbb{T}^d$.

First, we consider the incompressible flows of homogeneous fluids on the torus $\mathbb{T}^d$ in the absence of external forcing. Consider the system of equations

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla p = 0 \\ \nabla \cdot u = 0, \;\; u|_{t=0} = u_0, \end{cases} \tag{1}$$

where u is the fluid velocity, p is the scalar pressure, and $\nu \geqslant 0$ is the kinematic constant viscocity. For $\nu > 0$, system (1) is called the Navier-Stokes equations; for $\nu = 0$ it reduces to the Euler equations. The difference between the closely related Euler equations and the Navier–Stokes equations are that the latter take viscosity into account while the former only model the inviscid flow. Such equations always attract the attention of many researchers, with enormous quantity of publications in the literature. There are books ([AK21], [Che98], [CM93], [MB02], [MP94]) and expository articles ([BT07], [Con06], [ES06]) on the subject, too numerous to all be listed here.

The Lagrangian formulation is a way of describing the dynamics of a fluid or a physical system by following individual fluid particles as they move through space and time. More precisely, the Lagrangian formulation for the incompressible Euler equations is stated as follows:

$$\begin{cases} \dfrac{dX}{dt}(t,x) = u(t, X(t,x)), \;\; X(0,x) = x, \\ A(t,\cdot) = X^{-1}(t,\cdot), \\ u_t(x) = \mathbb{P}\left[(\nabla A_t)^* u_0(A_t)\right](x) \end{cases} \tag{2}$$

for each $x \in \mathbb{T}^d$. Here $\mathbb{P}$ is the Leray-Hodge projector.

For the Euler-Lagrangian form in the deterministic setting, in 2001 Constantin [Con01] showed the equivalence between the incompressible Euler equations and the Lagrangian form (2), involving the back-to-labels map (the inverse of the trajectory map for each fixed time) and proved a local existence result in certain Hölder spaces on $\mathbb{R}^3$ satisfying suitable decay conditions, or for solutions that are periodic. Pooley and Robinson [PR16] in 2016 showed that the Lagrangian formulation is equivalent to the usual formulation of the Euler equations and prove an existence and uniqueness result for the Lagrangian formulation in $C^0([0,T]; H^s(\mathbb{T}^d))$ with $s > \frac{d}{2} + 1$.
In 2008, Constantin [CI08] established a probabilistic Lagrangian representation formula for the deterministic three-dimensional Navier-Stokes equations using stochastic flows. They show that u is a classical solution to the Navier-Stokes equations (1) if and only if u satisfies the stochastic system.

$$X_t(x) = x + \int_0^t u_r(X_r(x))dr + W_t, \tag{3}$$

$$u_t(x) = \mathbb{E}\mathbb{P}\left[(\nabla X_t^{-1})^* u_0(X_t^{-1})\right], \tag{4}$$

where W_t is a standard Brownian motion, $\mathbb{E}$ is the expectation and $*$ denotes the transposition of matrix. We mention that in 2018, Fang and Luo [FL18] established the formula (4) on compact Riemannian manifolds. Rezakhanlou [Rez16], in 2016, proved the representation (4) in the context of symplectic geometry, and in 2021, Olivera [Oli21] obtained the formula (4) for mild solutions of the Navier-Stokes equations on $\mathbb{R}^d$.

The Euler equations serve as the traditional model for describing the motion of an inviscid, incompressible fluid. By incorporating stochastic terms into these governing equations, we can better account for numerical, empirical, and physical uncertainties. This approach is particularly useful in various applications, including climatology and turbulence theory, where the stochastic flows, are essential for capturing the statistical properties and long-term behavior of turbulent systems, which deterministic models often fail to do. We refer to [AOBdL20], [Bes23], [BCF92], [BFM16], [CM23], [CFH19], [CT15], [FGP11], [FL19b], [LC23] for works considering stochastic Euler equations.

In this thesis, we study the rough incompressible Euler equations on a torus $\mathbb{T}^d$ driven by the rough signal B_t^j, which can be written as

$$\begin{cases} du_t + ((u_t \cdot \nabla)u_t + \nabla p_t)dt + \sum_k \mathcal{L}^*_{\sigma^j} u_t dB_t^j = 0 \\ \nabla \cdot u_t = 0, \;\; u|_{t=0} = u_0, \end{cases} \tag{5}$$

which describe the evolution of the velocity u of an incompressible inviscid fluid, as well as the internal pressure p, $\mathcal{L}^*_{\sigma^j} u := (\sigma^j \cdot \nabla)u + \left(\nabla \sigma^j\right)^* u$ is the dual operator of the Lie derivative $\mathcal{L}_{\sigma^j} u = (\sigma^j \cdot \nabla)u - (u \cdot \nabla)\sigma^j$ and we assume, to avoid technical difficulties, that there are finitely many smooth and divergence free vector fields $\left\{\sigma^j\right\}_j$.

The study of fluid dynamics equations with a "Lie noise", as in equation (5), is relevant in the study of some stochastic energy functionals. A variational approach to the full theory of stochastic ideal fluid dynamics was derived by Darryl Holm in [Hol15], by using transformation theory from geometric mechanics based on the Lagrange-to-Euler map for stochastic Lagrangian particle trajectories. Equations related to fluid dynamics with Lie noise appeared in several other works, see for instance [Bes23], [DHL20], [FGV01], [Fla11], [LC23] and many others.

And finally, a Lagrangian representation of the rough incompressible Euler equations using noisy flow paths is written as follows:

$$\begin{cases} dX_t = \sum_j \sigma^j(X_t)dB_t^j + u_t(X_t)dt, \\ A(t,\cdot) = X^{-1}(t,\cdot), \\ u_t(x) = \mathbb{P}\left[(\nabla A_t)^* u_0(A_t)\right](x). \end{cases}$$

The Lagrangian approach often simplifies the analysis of equations by transforming complex partial differential equations (PDEs) into stochastic ordinary differential equations (SDEs). This can make it easier to study properties like existence, uniqueness, and regularity of solutions. Finally, we refer to the work of Flandoli and Luo in 2019 [FL19a] for the Lagrangian representation formula of the three-dimensional Euler equation with Lie noise, using the vorticity equation, as in system (5), and the work of Drivas and Holm in 2020 [DH20] for a discussion of Kelvin circulation theorems for stochastic Euler equations.

The main purpose of this thesis is to establish the relation between Lagrangian and classical (Eulerian) formulations for rough incompressible Euler equations (5) in any dimension, considering three cases: the first where B_t^j is a Brownian motion, the second where B_t^j is an α-Hölder path with $\alpha \in (1/2, 1]$, and finally, the case where B_t^j is a weakly geometric rough path. Furthermore, using the Lagrangian formulation, we demonstrate a local-in-time existence result for solutions in $C^0([0,T]\,;(H^s(\mathbb{T}^d))^d)$ with $s > \frac{d}{2}+1$, which is novel for the
system (5) in both the stochastic case and the Young case.
The order in which the results of this work will be presented can be read below.

Structure of the thesis

I have tried, as much as possible, to present the topics included in this thesis in their most natural logical order, with each chapter being presented chronologically with respect to when the work was done. Each chapter has its own brief introduction, explaining the main motivations, as well as the notations and conventions adopted in it. For this reason, here I will only give a very short overview of the contents of the chapters.

In the first chapter, we recall definitions, notations, and basic properties of spaces of functions, Leray projector, stochastic processes, stochastic integrals, Itô's formula, Itô-Kunita-Wentzell's formula, and stochastic differential equations that are fundamental for the development of the work.

In chapter 2, considering B_t^j as a Wiener process, and integration in the Stratonovich sense, we study the Euler-Lagrangian formulation and discuss how it is formally equivalent to the usual stohastic Euler equations for any dimension. The Lagrangian formulation is subsequently employed to establish a local-in-time existence result for solutions in appropriate Sobolev spaces, which is a novel result for the system (5).

In chapter 3, we extends the results from chapter 2, considering B_t^j an α-Hölder path in $\mathbb{R}^d$ with $\alpha \in (1/2, 1]$, and integration in the Young sense. First we show the Euler-Lagrangian formulation is equivalent to the incompressible Euler equations (5). The Lagrangian formulation is then used to prove a local in time existence result for solutions in suitable Sobolev spaces, new for the system (5).

Finally, in Chapter 4, our aim is to demonstrate that the Lagrangian formulation is satisfied by the solution of the incompressible Euler equations (5), considering $B_t^j = \mathbf{Z}_t$ an α-Hölder weakly geometric rough path with $\alpha \in (\frac{1}{3}, \frac{1}{2}]$.

CHAPTER 1
PRELIMINARIES

In this chapter, we briefly introduce the mathematics needed in this thesis, which contains Hölder spaces, Sobolev spaces, L^p-spaces, stochastic processes, Brownian motion and Itô-Kunita-Wentzell's formula. These are widely used in mathematics. Based on them, we will develop our work (a more detailed introduction to such aspects of the theory can be found in [BCD11], [Bau14], [Cho15], and [Kun84]).

1.1 Spaces of functions

1.1.1 $C^{m,\alpha}$-functions

The spatial dimension will sometimes be denoted by d and, when it is, we will always assume that $d \geqslant 2$. We remark that we restrict ourselves to the (flat) d-torus by $\mathbb{T}^d := \mathbb{R}^d/2\pi\mathbb{Z}^d$ for simplicity.
Let $U \subseteq \mathbb{R}^d$, $\mathbb{T}^d$ or the whole space, we denote by $C^0(U)$ the set of continuous functions on U and by $C_c(U)$ the subspace consisting of those continuous functions with compact support. In certain situations we may simplify notation by omitting the set U.
More generally, for $m \in \mathbb{N} \cup \{0\}$, we will denote by $C^m(U)$ the space of functions on U with continuous derivatives up to order m, and by $C_c^m(U)$ the space of those with compact support. Of course C^∞ and C_c^∞ will denote the spaces of functions for which all derivatives exist and, in the latter case, that also have compact support.

Now the set of bounded continuous functions on U forms a normed vector space with the supremum norm, which we denote by

$$\|f\|_\infty := \sup_{x \in U} |f(x)|.$$

In addition, we say that $f : U \to \mathbb{R}$ satisfies the *α-Hölder condition* (or "f is *α-Hölder*") for some $\alpha \in (0,1]$ if there exists $C > 0$ such that

$$|f(x) - f(y)| \leqslant C|x-y|^\alpha \tag{1.1}$$

for all $x, y \in U$. We then define spaces of α-Hölder continuous functions by:

$$C^{0,\alpha}(U) := \{f : U \to \mathbb{R} : \ f \text{ satisfies } a \ \alpha\text{-Hölder condition for some } C > 0\}.$$

In particular $C^{0,1}$ is the space of *Lipschitz functions*.
We define a seminorn on $C^{0,\alpha}$ to encapsulate the α-Hölder property (1.1):

$$|f|_{C^{0,\alpha}} := \inf\{C > 0 : \ f \text{ is } \alpha\text{-Hölder, with coefficient } C\}$$

Now bounded Hölder-continuous functions form a normed vector space with the norm

$$\|f\|_\alpha = \|f\|_{C^{0,\alpha}} := \|f\|_\infty + |f|_{C^{0,\alpha}}.$$

More generally, if $f \in C^{0,\alpha}$ has α-Hölder derivatives up to order m, i.e. $D^\beta f \in C^{0,\alpha}$ if $|\beta| \leqslant m$, then we say $f \in C^{m,\alpha}(U)$. In the case that f is bounded we also define a norm

$$\|f\|_{m,\alpha} := \|f\|_\infty + \sum_{0 \leqslant |\beta| \leqslant m} |D^\beta f|_{C^{0,\alpha}}$$

Since any Hölder-continuous function is continuous we have, of course, that $C^{m,\alpha}(U) \subset C^m(U)$.
Note that when we discuss vector-valued functions $f : U \to \mathbb{R}^d$, statements like "$f \in V$", for a normed space V, should be understood in a componentwise sense. In this case, the norm $\|\cdot\|_V$ should be understood as a norm on $|f|$:

$$\|f\|_V := \||f|\|_V.$$

We will consider $U = \mathbb{T}^d$ or $U = \mathbb{R}^d$ throughout this work.

1.1.2 L^p-spaces

Much of the analysis in this work will concern functions in certain Lebesgue spaces or Sobolev spaces based upon them. In this subsection we will set out the notation we will use when working with these spaces and recall a few standard facts. More detailed discussion can be found in countless textbooks, for example [AF03] or [BCD11].
Unless otherwise specified, all integrals over subsets of $\mathbb{R}^d$ will be written with respect to the Lebesgue measure μ in the corresponding dimension.

For $1 \leqslant p < \infty$ and any μ-measurable set $U \subset \mathbb{R}^d$ the space $L^p(U)$ (which will usually be denoted by L^p, when the choice of domain is clear) denotes the set of all measurable functions $f : U \to \mathbb{R}$ such that

$$\int_U |f(x)|^p dx < \infty,$$

For the endpoint $p = \infty$, we define L^∞ to be the set of all measurable functions that are essentially bounded:

$$\operatorname*{ess\,sup}_U |f| := \inf \{\sup \{|f(x)| : x \in U \backslash E\} : \mu(E) = 0\} < \infty$$

That is, there exists a μ-null set E such that $\sup\limits_{U \backslash E} |f| < \infty$.
The set L^p forms a linear space under the pointwise addition of functions and moreover is a Banach space with the norm

$$\|f\|_{L^p(U)} := \left(\int_U |f|^p \right)^{1/p}$$

if $1 \leqslant p < \infty$, or

$$\|f\|_{L^\infty(U)} := \operatorname*{ess\,sup}_U |f|$$

for $p = \infty$.
In the following we shall use $\|f\|_p$ and $\|f\|_\infty$ for $\|f\|_{L^p(U)}$ and $\|f\|_{L^\infty(U)}$, respectively, when there is no ambiguity.
We shall need the following inequalities in dealing with integral estimates:

- *Young's inequality*

  $$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q};$$

  this holds for positive real numbers a, b, p, q satisfying $p^{-1} + q^{-1} = 1$.

- *Hölder's inequality.*

  $$\int_U fg dx \leqslant \|f\|_p \|g\|_q;$$

  this holds for function $f \in L^p(U)$, $g \in L^q(U)$, $p^{-1} + q^{-1} = 1$ and is a consequence of Young's inequality.

Finally $L^2(U)$ is a *Hilbert space* under the inner product

$$(f, g)_{L^2} := \int_U fg dx,$$

or

$$(u, v)_{L^2} := \sum_i (u_i, v_i)_{L^2},$$

in the case of vector valued functions u and v.

1.1.3 Sobolev spaces

A significant amount of the analysis in this work will be carried out in Sobolev spaces. Loosely speaking these are spaces of L^p functions with weak derivatives in L^p. More precisely, for $m \in \mathbb{N}_0$ and $1 \leqslant p < \infty$, the space $W^{m,p}$ consists of functions $u \in L^p$ such that the weak derivatives $D^\beta u$ exists and are in L^p for all multi-indices β such that $|\beta| \leqslant m$. On this space we define the Sobolev norm

$$\|u\|_{W^{m,p}} := \left(\sum_{|\beta| \leqslant m} \|D^\beta u\|_p \right)^{1/p}.$$

In the case that $p = 2$, we use the notation $H^m := W^{m,2}$ since this is a Hilbert space with the inner product

$$(f, g)_{H^m} := \sum_{|\beta| \leqslant m} \left(D^\beta f, D^\beta g \right)_{L^2}.$$

For (non-integer) $s \geqslant 0$, we use the following definition of the *(inhomogeneous) Sobolev space* $H^s(\mathbb{T}^d)$. The space H^s coincides with the Sobolev space $W^{s,2}$, see Section 7.62 of [AF03]. For $f \in L^2(\mathbb{T}^d)$, we say $f \in H^s(\mathbb{T}^d)$ if the Fourier coefficients satisfy

$$\sum_{k \in \mathbb{Z}^d} |k|^{2s} |\hat{f}(k)|^2 < \infty.$$

($\hat{f}$ will be defined in the next section).

For $f \in H^s(\mathbb{T}^d)$, we define "modulus of s derivatives" Λ^s by

$$\Lambda^s f(x) := (2\pi)^{-d/2} \sum_{k \in \mathbb{Z}^d} |k|^s \hat{f}(k) \exp(ik \cdot x) \in L^2(\mathbb{T}^d).$$

In particular $\Lambda^2 f(x) = (-\Delta) f$ for any $f \in H^2$. Moreover, the norm in H^s is given by

$$\| \cdot \|_s := \left(\| \cdot \|_{L^2}^2 + \|\Lambda^s \cdot \|_{L^2}^2 \right)^{1/2}.$$

Note that we will sometimes use the fact that this is equivalent to the norm $\| \cdot \|_{L^2} + \|\Lambda^s \cdot \|_{L^2}$. We will also make use of the fact that for a function $f \in H^r(\mathbb{T}^d)$, $\|\Lambda^s f\|_{L^2} \leqslant \|\Lambda^r f\|_{L^2}$ if $0 < s \leqslant r$. Almost analogously, one can define $H^s(\mathbb{R}^d)$, using Fourier transforms (see for example [BCD11]).

1.2 Fourier Transforms

The Fourier basis for $L^2(\mathbb{T}^d)$ consists of periodic functions of the form

$$x \mapsto \frac{1}{(2\pi)^{d/2}} \exp(ix \cdot k)$$

where $k \in \mathbb{Z}^d$, and the Fourier coefficients of a function $f \in L^2(\mathbb{T}^d)$ will be denoted by $\hat{f}(k) \in \mathbb{C}$ (or $\hat{f}(k) \in \mathbb{C}^d$ if f is vector valued). The formula that defines $\hat{f}(k)$ is

$$\hat{f}(k) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} f(x) \exp(-ik \cdot x) dx,$$

and the corresponding decomposition of f is

$$f(x) = \frac{1}{(2\pi)^{d/2}} \sum_{k \in \mathbb{Z}^d} \hat{f}(k) \exp(ix \cdot k).$$

Note that if all components of f are real valued then $\hat{f}(k) = \overline{\hat{f}(-k)}$ for all $k \in \mathbb{Z}^d$ where $\overline{x}$ denotes the complex conjugate.

1.3 The Helmholtz-Weyl decomposition and the Leray projector

A well-known family of results, most commonly attributed to [Hel70], show that a smooth vector field on $\mathbb{R}^3$ with sufficiently fast decay (or compact support) can be decomposed into a divergence-free part, and a curl-free (gradient) part:

$$u = \nabla \times h + \nabla g.$$

For our purposes it will suffice to consider the cases of $L^2(\mathbb{T}^d)$, and $L^2(\mathbb{R}^d)$, for $d \geqslant 2$. In either domain we have

$$L^2 = H \oplus G,$$

where H is the closure of the set of smooth divergence-free functions in L^2, and G is the space of gradients of H^1 functions. By considering Fourier series, this decomposition can be written explicitly (see, for example Chapter 2 of [RRS16]). Indeed for $u \in L^2(\mathbb{T}^d)$,

$$u(x) = (2\pi)^{-d/2} \sum_{k \in \mathbb{Z}^d} \hat{u}(k) \exp(ix \cdot k) = (2\pi)^{-d/2} \sum_{k \in \mathbb{Z}^d} \left(\hat{g}(k) + \hat{h}(k) \right) \exp(ix \cdot k)$$

where

$$\hat{g}(k) := \frac{\hat{u}(k) \cdot k}{|k|^2} k, \text{ for } k \neq 0; \;\; \hat{g}(0) := 0, \;\; \text{and } \hat{h}(k) := \hat{u}(k) - \hat{g}(k).$$

It is straihgtforward to check that $\hat{g}$ and $\hat{h}$ are the coefficients of convergent Fourier series, let us call the corresponding limits g and h, respectively. It is also not difficult to see that g is te weak derivative of the scalar-valued H^1 function f, with Fourier coefficients

$$\hat{f}(k) = -i \frac{\hat{u}(k) \cdot k}{|k|^2}, \;\; \hat{f}(0) = 0.$$

Moreover, it can be seen that $h \in H$ since $\hat{h}(k) \cdot k = 0$ for all $k \in \mathbb{Z}^d$.

To see that the decomposition of a given function u is unique, it suffices to consider $u = 0$. In that case $h = -g = \nabla f$, in a weak sense for some $f \in H^1$. Formal

consideration of the Fourier series of f, assuming that $\nabla \cdot h = 0$, implies that $\hat{f}(k)|k|^2 = 0$ for all $k \in \mathbb{Z}^d$, hence $h = g = 0$. This can be justified by considering the Fourier series of a sequence of smooth divergence-free approximations to h. The projection of L^2 onto H will play an important role in the analysis herein; we will denote it by $\mathbb{P} : L^2 \to H$.
On $L^2(\mathbb{T}^d)$, $\mathbb{P}$ can be calculated explicitly in Fourier space, following the discussion above. For example, on $\mathbb{T}^d$ we have

$$\mathbb{P}\left(\sum_{k\in\mathbb{Z}^d} \hat{u}(k)\exp(ix\cdot k)\right)(x) = \hat{u}(0) + \sum_{k\in\mathbb{Z}^d\setminus\{0\}}\left(\hat{u}(k) - \frac{\hat{u}(k)\cdot k}{|k|^2}k\right)\exp(ix\cdot k)$$

This is usually called the *Leray projection* (or sometimes the Helmholtz projection). Clearly $\mathbb{P}$ is a bounded operator on L^2, moreover it follows easily from the Fourier-series definition that for any $s \geqslant 0$ and any $u \in H^s(\mathbb{T}^d)$

$$\|\Lambda^s \mathbb{P} u\|_{L^2} \leqslant \|\Lambda^s u\|_{L^2}.$$

Furthermore $\mathbb{P}$ and Λ^s commute on $H^s(\mathbb{T}^d)$ (this is discussed in [CM93], [MB02], and in the aforementioned references).

1.4 Stochastic analysis

In this section we recall the basic vocabulary and results of probability theory. A *probability space* associated with a random experiment is a triple $(\Omega, \mathcal{F}, P)$ where Ω is the set of all possible outcomes of the random experiment, $\mathcal{F}$ is a σ-algebra of subsets of Ω, and P is a probability measure on $\mathcal{F}$.
If $(\Omega, \mathcal{F}, P)$ is a given probability space, then a function $f : \Omega \to \mathbb{R}^d$ is called *$\mathcal{F}$-measurable* if

$$f^{-1}(U) := \{\omega \in \Omega : f(\omega) \in U\} \in \mathcal{F}$$

for all open sets $U \subset \mathbb{R}^d$ (or, equivalently, for all Borel sets $U \subset \mathbb{R}^d$). A *random variable* X is a $\mathcal{F}$-measurable function $X : \Omega \to \mathbb{R}^d$

1.4.1 Stochastic Processes

We fix a probability space $(\Omega, \mathcal{F}, P)$.

Definition 1.1. *On $(\Omega, \mathcal{F}, P)$, a (d-dimensional) stochastic process is a sequence $(X_t)_{t\geqslant 0}$ of $\mathbb{R}^d$-valued random variables that are $\mathcal{F}$-measurable*

For every fixed $\omega \in \Omega$, the applications $t \to X_t(\omega)$ are called the paths of the process

Definition 1.2. *A process* $(X_t)_{t\geqslant 0}$ *is said to be measurable if the application*

$$(t,\omega) \mapsto X_t(\omega)$$

is measurable with respect to the σ*-algebra* $\mathcal{B}([0,+\infty)) \otimes \mathcal{F}$*, that is, if*

$$\forall A \in \mathcal{B}(\mathbb{R}^d), \{(t,\omega), X_t(\omega) \in A\} \in \mathcal{B}([0,+\infty)) \otimes \mathcal{F}.$$

The paths of a measurable process are, of course, measurable functions $[0,+\infty) \to \mathbb{R}$*.*

The process X_t is called *continuous process* if $X_t(\omega)$ is a continuous function of t for almost all ω. Moreover, a continuous process is measurable in the sense of the Definition 1.2, see Proposition 1.8. of [Bau14].

A stochastic process $(X_t)_{t\geqslant 0}$ may also be seen as a random system evolving in time. This system carries some information. More precisely, if one observes the paths of a stochastic process up to a time $t > 0$, one is able to decide if an event

$$A \in \sigma(X_r, r \leqslant t)$$

has occured (here and in the sequel $\sigma(X_r, r \leqslant t)$ denotes the smallest σ-field that makes all the random variables $\{(X_{t_1}, \cdots, X_{t_n}), 0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant t\}$ measurable). This notion of information carried by a stochastic process is modeled by filtrations.

Definition 1.3. *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space. A filtration* $(\mathcal{F}_t)_{t\geqslant 0}$ *is a non-decreasing family of sub-*σ*-algebras of* $\mathcal{F}$*.*

As a basic example, if $(X_t)_{t\geqslant 0}$ is a stochastic process defined on $(\Omega, \mathcal{F}, P)$, then

$$\mathcal{F}_t = \sigma(X_r, r \leqslant t)$$

is a filtration. This filtration is called the natural filtration of the process X and often denoted by $(\mathcal{F}_t^X)_{t\geqslant 0}$.

A *filtered probability space* $(\Omega, (\mathcal{F}_t)_{\geqslant 0}, \mathcal{F}, P)$ consists of a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $(\mathcal{F}_t)_{t\geqslant 0}$ contained in $\mathcal{F}$. The filtered probability space is said to satisfy the *usual conditions* if the following conditions are met:

1. The probability space $(\Omega, \mathcal{F}, P)$ is complete,

2. The filtration $(\mathcal{F}_t)_{t\geqslant 0}$ is right continuous, that is, for every $t \geqslant 0$

$$\mathcal{F}_t = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}.$$

Definition 1.4. *A stochastic process* $(X_t)_{t\geqslant 0}$ *is said to be adapted to a filtration* $(\mathcal{F}_t)_{t\geqslant 0}$ *if for every* $t \geqslant 0$*, the random variable* X_t *is measurable with respect to* $\mathcal{F}_t$*.*

Of course, a stochastic process is always adapted with respect to its natural filtration. We may observe that if a stochastic process $(X_t)_{t\geqslant 0}$ is adapted to a filtration $(\mathcal{F}_t)_{t\geqslant 0}$ and that if $\mathcal{F}_0$ contains all the subsets of $\mathcal{F}$ that have a zero probability, then every process $(\tilde{X}_t)_{t\geqslant 0}$ that satisfies

$$P(\tilde{X}_t = X_t) = 1,\ t \geqslant 0,$$

is still adapted to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$.

Quite often, it is important to be able to evaluate the statistics of solutions to stochastic differential equations (SDEs) at appropriate random times, the so-called stopping times.

Definition 1.5. *Let $(\mathcal{F}_t)_{t\geqslant 0}$ be a filtration on a probability space $(\Omega, \mathcal{F}, P)$. Let $\tau : \Omega \to [0, \infty]$ be a random variable, measurable with respect to $\mathcal{F}$. We say that τ is a stopping time of the filtration $(\mathcal{F}_t)_{t\geqslant 0}$ if for $t \geqslant 0$,*

$$\{\omega : \tau(\omega) \leqslant t\} \in \mathcal{F}_t$$

Often, a stopping time will be the time during which a stochastic process adapted to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$ satisfies a given property. The above definition means that for any $t \geqslant 0$, at time t, one is able to decide if this property is satisfied or not.

1.4.2 Martingales and Semimartingales

We introduce and study in this section martingales in continuous time.

Definition 1.6. *Let $(\mathcal{F}_t)_{t\geqslant 0}$ be a filtration defined on a probability space $(\Omega, \mathcal{F}, P)$. A process $(M_t)_{t\geqslant 0}$ that is adapted to $(\mathcal{F}_t)_{t\geqslant 0}$ is called a submartingale with respect to this filtration if:*

1. *For every $t \geqslant 0$, $\mathbb{E}(|M_t|) < +\infty$;*
2. *For every $t \geqslant r \geqslant 0$*
$$\mathbb{E}(M_t|\mathcal{F}_r) \geqslant M_r.$$

A stochastic process $(M_t)_{t\geqslant 0}$ that is adapted to $(\mathcal{F}_t)_{t\geqslant 0}$ and such that $(-M_t)_{t\geqslant 0}$ is a submartingale, is called a *supermartingale*.
Finally, a stochastic process $(M_t)_{t\geqslant 0}$ that is adapted to $(\mathcal{F}_t)_{t\geqslant 0}$ and that is at the same time a submartingale and a supermartingale is called a *martingale*. If X_t is a $\mathbb{R}^d$-valued martingale with $\mathbb{E}|X_t|^p < \infty$, $t \in [0, +\infty)$, for some $p \geqslant 1$, then it is called a L^p-martingale.

Definition 1.7 (Local martingale)**.** *A stochastic process $(M_t)_{t\geqslant 0}$ is called a local martingale (with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$) if there is a sequence of stopping times $(\tau_n)_{n\geqslant 0}$ such that:*

1. *The sequence* $(\tau_n)_{n\geqslant 0}$ *is increasing and almost surely satisfies* $\lim_{n\to\infty} \tau_n = +\infty$;
2. *For* $n \geqslant 1$, *the process* $(M_{t\wedge\tau_n})_{t\geqslant 0}$ *is a uniformly integrable martingale with respect to the filtration* $(\mathcal{F}_t)_{t\geqslant 0}$

Of course, any martingale turns out to be a local martingale. But, in general the converse is not true.

Definition 1.8 (Semimartingale)**.** *Let* $(X_t)_{t\geqslant 0}$ *be an adapted continuous stochastic process on the filtered probability space* $(\Omega, (\mathcal{F}_t)_{t\geqslant 0}, \mathcal{F}, P)$. *We say that* $(X_t)_{t\geqslant 0}$ *is a semimartingale with respect to the filtration* $(\mathcal{F}_t)_{t\geqslant 0}$ *if* $(X_t)_{t\geqslant 0}$ *may be writtten as:*

$$X_t = X_0 + A_t + M_t,$$

where $(A_t)_{t\geqslant 0}$ *is a bounded variation process and* $(M_t)_{t\geqslant 0}$ *is a continuous local martingale such that* $M_0 = 0$. *If exists, the previous decomposition is unique.*

1.4.3 Brownian motion and stochastic integrals

The most well-known example of a continuous martingale is the Brownian motion $\{W_t, t \geqslant 0\}$. The mathematical definition of a Brownian motion is the following.

Definition 1.9. *A stochastic process* $\{W_t, t \geqslant 0\}$ *is called a Brownian motion if it satisfies the following conditions:*

i) $W_0 = 0$

ii) For all $0 \leqslant t_1 < \cdots < t_n$ *the increments* $W_{t_n} - W_{t_{n-1}}, ..., W_{t_2} - W_{t_1}$, *are independent random variables.*

iii) If $0 \leqslant r < t$, *the increment* $W_t - W_r$ *has the normal distribution* $N(0, t-r)$

iv) $\{W_t, t \geqslant 0\}$ *is a continuous process.*

A d-dimensional stochastic process $(W_t)_{t\geqslant 0}$ is called a Brownian motion if

$$(W_t)_{t\geqslant 0} = (W_t^1, \cdots, W_t^d)_{t\geqslant 0},$$

where the process $(W_t^i)_{t\geqslant 0}$ are independent Brownian motions.

The inception of the stochastic integral is credited to K. Itô, who originally formulated it in relation to a standard Brownian motion. Subsequently, it was extended to include local martingales and semimartingales.

Let X_t, $t \in [0,T]$, be a continuous real-valued $\mathcal{F}_t$-adapted stochastic process. Let $\pi_t = \{0 = t_0 < t_1 < \cdots < t_m = T\}$ be a partition of $[0,T]$ with $|\pi_T| = \max_{1 \leqslant k \leqslant m} (t_k - t_{k-1})$. Define $\xi_t^m(\pi_T)$ as

$$\xi_t^m(\pi_T) = \sum_{k=1}^{m} \left(X_{t_k \wedge t} - X_{t_{k-1} \wedge t}\right)^2 .$$

If, for any sequence of partitions π_T^n, $\xi_t(\pi_T^n)$ converges in probability to a limit $\langle X \rangle_t$, or $\langle X_t \rangle$, as $|\pi_T^n| \to 0$ (as $nn \to \infty$) for $t \in [0,T]$, then $\langle X \rangle_t$ is called the *quadratic variation* of X_t. For instance, if $X_t = W_t$ is a Brownian motion, then $\langle W \rangle_t = t$ a.s. If X_t is a process of bounded variation, then $\langle X \rangle_t = 0$ a.s. Similarly, let X_t, Y_t, $0 \leqslant t \leqslant T$ be two continuous real-valued $\mathcal{F}_t$-adapted processes. Define

$$\eta_t(\pi_T) = \sum_{k=1}^{m} \left(X_{t_k \wedge t} - X_{t_{k-1} \wedge t}\right) \left(Y_{t_k \wedge t} - Y_{t_{k-1} \wedge t}\right) .$$

Then the *covariation* of X_t and Y_t, denoted by $\langle X, Y \rangle_t$ or $\langle X_t, Y_t \rangle$ is defined as the limit of $\eta_t(\pi_T^n)$ in probability as $|\pi_T^n| \to 0$.

Let M_t be a continuous, real-valued L^2-martingale and let $f(t)$ be a continuous adapted process in $\mathbb{R}$ for $0 \leqslant t \leqslant T$. For any partition $\Delta_T^n = \{0 = t_0 < t_1 < \cdots < t_n = T\}$, define

$$I_t^n = \sum_{k=1}^{n} f_{t_{k-1} \wedge t} \left(M_{t_k \wedge t} - M_{t_{k-1} \wedge t}\right) . \tag{1.2}$$

Then I_t^n is a continuous martingale with the quadratic variation

$$\langle I^n \rangle_t = \int_0^t |f_r^n|^2 d\langle M \rangle_r ,$$

where $f_t^n = f_{t_{k-1}}$ for $t_{k-1} \leqslant t < t_k$. Suposse that $\int_0^T |f_r|^2 d\langle M \rangle_r < \infty$, a.s. Then the sequence I_t^n will converge uniformly in probability as $|\Delta_T^n| \to 0$ (as $n \to \infty$) to a limit

$$I_t = \int_0^t f_r dM_r ,$$

which is independent of the choice of the partition. The limit I_t is called the *Itô integral* of f_t with respect to the martingale M_t. Instead of I_t^n given by (1.2), define

$$J_t^n = \sum_{k=1}^{n} \frac{1}{2} \left(f_{t_{k-1} \wedge t} + f_{t_k \wedge t}\right) \left(M_{t_k \wedge t} - M_{t_{k-1} \wedge t}\right) . \tag{1.3}$$

The corresponding limit J_t of J_t^n as

$$J_t = \int_0^t f_r \circ dM_r ,$$

is known as the *Stratonovich integral* of f_t with respect to M_t. Similar to the Itô integral, the Stratonovich integral (1.3) is a generalization from the case when $M_t = W_t$ is a Brownian motion.

Theorem 1.1. *Let M_t and f_t be given as above. Then the Itô integral $I_t = \int_0^t f_r dM_r$ is a continuous local martingale satisfying $\mathbb{E}(I_t) = 0$ and*

$$\langle I \rangle_t = \int_0^t |f_r|^2 d\langle M \rangle_r, \quad a.s.$$

Moreover, the Stratonovich integral is related to the Itô integral as follows

$$\int_0^t f_r \circ dM_r = \int_0^t f_r dM_r + \frac{1}{2}\langle f, M \rangle_t \tag{1.4}$$

1.4.4 Itô's Formula and Itô-Kunita-Wentzell's Formula

The Itô's formula is certainly the most important and useful formula of stochastic calculus. It is the change of variable formula for stochastic integrals.
Our starting point is precisely the Itô's formula for continuous semimartingales:

Theorem 1.2 (Itô's Formula)**.** *Let $X_t = (X_t^1, ..., X_t^d)$ be a continuous semimartingale. Suposse that $\Phi : \mathbb{R}^d \to \mathbb{R}$ be a twice continuously differentiable function. Then the following formula holds:*

$$\Phi(X_t) = \Phi(X_0) + \sum_{i=1}^{d} \int_0^t \frac{\partial \Phi}{\partial x_i}(r, X_r) dX_r^i + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \frac{\partial^2 \Phi}{\partial x_i \partial x_j}(r, X_r) d\langle X^i, X^j \rangle_r.$$

If, in addition, Φ is three-time differentiable function, then the above formula can be written simply as

$$\Phi(t, X_t) = \Phi(0, X_0) + \sum_{i=1}^{d} \int_0^t \frac{\partial \Phi}{\partial x_i}(r, X_r) \circ dX_r^i.$$

An immediate consequence of this result is the well-known integration by parts formula for semimartingales:

Corollary 1.1 (Itô's formula for the product)**.** *Let $(X_t)_{t \geqslant 0}$ and $(Y_t)_{t \geqslant 0}$ be two continuous semimartingales, then the process $(X_t Y_t)_{t \geqslant 0}$ is a continuous semimartingale and we have:*

$$X_t Y_t = X_0 Y_0 + \int_0^t X_r dY_r + \int_0^t Y_r dX_r + \langle X, Y \rangle_t, \quad t \geqslant 0$$

Proof. See Theorem 5.39 of [Bau14]. □

One of the fundamental tools to develop our work is the so-called Itô-Kunita-Wentzell's Formula, which describes the differential rule for change of variables. We present here a differential rule for the composition of two semimartingales, which is a generalization of the well known Itô's Formula (see Theorem 8.3. in Chapter I of [Kun84]).

Theorem 1.3 (Itô-Kunita-Wentzell's formula)**.** *Let $\Phi_t(x)$, $t \in [0, T]$, $x \in \mathbb{R}^d$ be a random field continuous in (t, x) a.s., satisfying*

1. *For each t, $\Phi_t(\cdot)$ is a C^3-map from $\mathbb{R}^d$ into $\mathbb{R}$ a.s. ω*

2. *For each x, $\Phi_t(x)$ is a continuous semimartingale and it satisfies*

$$\Phi_t(x) = \Phi_0(x) + \sum_{j=1}^{m} \int_0^t f_r^j(x) \circ dY_r^j, \ \text{for all } x \in \mathbb{R}^d \ \text{a.s.}, \tag{1.5}$$

where $Y_r^1, ..., Y_r^m$ are continuous semimartingales, $f_r^j(x)$ with $r \in [0,T]$, $x \in \mathbb{R}^d$ are random fields which are continuous in (r,x) and satisfy

i) $f_r^j(x)$ are twice continuously differentiable in x,

ii) for each x, $f_r^j(x)$ are adapted processes.

Let now $X_t = (X_t^1, ..., X_t^d)$ be a continuous semimartingales. Then we have

$$\Phi_t(X_t) = \Phi_0(X_0) + \sum_{j=1}^{m} \int_0^t f_r^j(X_r) \circ dY_r^j + \sum_{i=1}^{d} \int_0^t \frac{\partial \Phi_r}{\partial x_i}(X_r) \circ dX_r^i. \tag{1.6}$$

1.5 Stochastic Differential Equations

We will concerned with the SDEs

$$dX_t(x) = b(t, X_t(x))dt + \sigma(t, X_t(x))dW_t, \ X_0 = x_0 \in \mathbb{R}^d, \tag{1.7}$$

where $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$ are measurable vector-valued and matrix-valued functions, respectively, and $(W_t)_{t \in [0,T]}$ is a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ endowed with the filtration $(\mathcal{F}_t^W)_{t \in [0,T]}$. We assume that the initial condition is a random variable that is independent of the Brownian motion W_t.

We will say that (1.7) has *strong solution* if there exists continuous adapted process X_t to the filtration $(\mathcal{F}_t^W)_{t \in [0,T]}$, such that verifies the stochastic integral equation

$$X_t(x) = x_0 + \int_0^t b(r, X_r(x))dr + \int_0^t \sigma(r, X_r(x))dW_r,$$

with $b(\cdot, X_\cdot) \in L^1([0,T]; \mathbb{R}^d)$ and $\sigma(\cdot, X_\cdot) \in L^2([0,T]; \mathbb{R}^{d \times d})$ almost surely.
We will state the existence and uniqueness when the coefficients b and σ satisfy the following two assumptions: there exists a constant $C > 0$ such that for all $x \in \mathbb{R}^d$ and $t \in [0,T]$,

$$|b(t,x)| + |\sigma(t,x)| \leqslant C(1 + |x|), \tag{1.8}$$

and for all $x, y \in \mathbb{R}^d$ and $t \in [0,T]$,

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leqslant C|x - y|. \tag{1.9}$$

Under these assumptions, a global, unique solution exists for the SDE (1.7).

Theorem 1.4. *Let b and σ satisfy assumptions (1.8) and (1.9). Then, for every initial condition $x_0 \in \mathbb{R}^d$, the SDE (1.7) has a unique solution X_t with*

$$\mathbb{E}\left[\sup_{0\leqslant r\leqslant t} |X_r|^2 dr\right] < \infty$$

for all $t > 0$.

Proof. See Theorem 5.2.1. of [Øks03]. □

1.5.1 Flow Properties

We recall the relevant definition from [Kun84]

Definition 1.10. *A stochastic flow of diffeomorphisms (resp. the $C^{m,\alpha}$), associated to equation (1.7) is a map $(s, t, x, \omega) \mapsto \phi_{s,t}(x)(\omega)$ defined for $0 \leqslant s \leqslant t$, $x \in \mathbb{R}^d$ $\omega \in \Omega$ with values in $\mathbb{R}^d$ such that*

- *given any $s \geqslant 0$, $x \in \mathbb{R}^d$ the process $X_t^{s,x} = \phi_{s,t}(x)$ is continuous $\mathcal{F}_{s,t}$ measurable solution of the equation (1.7),*
- *P-a.s, for $0 \leqslant s \leqslant t$ the function, $\phi_{s,t}$ is a diffeomorphisms, and the functions $\phi_{s,t}, \phi_{s,t}^{-1}, D^m\phi_{s,t}, D^m\phi_{s,t}^{-1}$, are continuous in (s, t, x) (resp. the $C^{m,\alpha}$ class in x uniformly in $0 \leqslant s \leqslant t \leqslant T$),*
- *P-a.s, $\phi_{s,t} = \phi_{u,t}(\phi_{s,u})$ for all $0 \leqslant s \leqslant u \leqslant t$, $x \in \mathbb{R}^d$ and $\phi_{s,s} = x$*

We present the following relevant theorem on stochastic flows without proof. Unfortunately the rigorous proof contains a lot technical difficulties and is very long to be demonstrated here, as it falls outside the scope of this work. A proof can be read at [Kun84].

Theorem 1.5. *if $b, \sigma \in L^\infty([0, T], C_b^{m,\alpha}(\mathbb{R}^d))$. Then the map $x \mapsto X_t(x)$ is a stochastic flow of $C^{m,\alpha'}$-diffeomorphisms with $\alpha' < \alpha$.*

This concludes the mathematical background for this work. In the remaining chapters we discuss the main content of this thesis.

CHAPTER 2 THE BROWNIAN CASE

In this chapter, we consider the stochastic Euler equations

$$\begin{cases} du + ((u \cdot \nabla u) + \nabla p)dt + \sum_j \mathcal{L}^*_{\sigma^j} u \circ dW^j_t = 0 \\ \nabla \cdot u = 0, \;\; u|_{t=0} = u_0, \end{cases} \tag{2.1}$$

where W^j_t is a family of independent real-valued Brownian motions, $\mathcal{L}^*_{\sigma^j} u := (\sigma^j \cdot \nabla)u + \left(\nabla \sigma^j\right)^* u$ is the dual operator of the Lie derivative $\mathcal{L}_{\sigma^j} u = (\sigma^j \cdot \nabla)u - (u \cdot \nabla)\sigma^j$ and the integration is in the Stratonovich sense.

First, we show the Euler-Lagrangian formulation is equivalent to the stochastic Euler equations (2.1), see Proposition 2.1. The proof is based on Itô-Kunita-Wentzell's formula and stochastic analysis techniques. Furthermore, we use the stochastic flow decomposition of the Lagrangian formulation to obtain a deterministic fixed-point problem, and apply this result to demonstrate a time-local existence result for solutions in $C^0([0,T]\,;(H^s(\mathbb{T}^d))^d)$ with $s > \dfrac{d}{2} + 1$.

2.1 Equivalent formulations.

Let $\mathcal{L}^*_{\sigma^j}$ be the adjoint operator of the Lie derivative $\mathcal{L}_{\sigma^j}$ with respect to the inner product in $L^2(\mathbb{T}^d, \mathbb{R}^d)$:

$$\langle \mathcal{L}^*_{\sigma^j} v, w \rangle_{L^2} = -\langle v, \mathcal{L}_{\sigma^j} w \rangle_{L^2},$$

for all smooth vector fields v, w, σ^j. When σ^j is divergence free, the adjoint Lie operator is given by

$$\mathcal{L}^*_{\sigma^j} v = \left(\sigma^j \cdot \nabla\right) v + \left(\nabla \sigma^j\right)^* v,$$

or in vector components by

$$(\mathcal{L}^*_{\sigma^j} v)_i = \sum_k \left(\sigma^j_k \partial_k v_i + v_k \partial_i \sigma^j_k\right).$$

In fact, by integration by parts and using $\nabla \cdot \sigma^j = 0$

$$\begin{aligned}
\langle u, \mathcal{L}_{\sigma^j} w\rangle_{L^2} &= \sum_i \int_{\mathbb{T}^d} u_i(x)(\mathcal{L}_{\sigma^j} w)_i(x)dx \\
&= \sum_i \int_{\mathbb{T}^d} u_i(x) \left((\sigma^j \cdot \nabla)w - (w \cdot \nabla)\sigma^j\right)_i dx \\
&= \sum_i \sum_k \int_{\mathbb{T}^d} u_i(x) \left(\sigma^j_k \partial_k w_i - w_k \partial_k \sigma^j_i\right)(x)dx \\
&= \sum_i \sum_k \int_{\mathbb{T}^d} \left(-(\sigma^j_k \partial_k u_i)(x) w_i(x) - w_k(x)\partial_k \sigma^j_i(x) u_i(x)\right) dx \\
&= \sum_i \sum_k \int_{\mathbb{T}^d} \Big(-\left(\sigma^j_k \partial_k u_i + u_k \partial_i \sigma^j_k\right)(x) w_i(x) \\
&\qquad + u_k(x)\partial_i \sigma^j_k(x) w_i(x) - u_i(x)\partial_k \sigma^j_i(x) w_k(x)\Big) dx \\
&= -\sum_i \int_{\mathbb{T}^d} \left((\sigma^j \cdot \nabla)u_i(x) + \left(\left(\nabla \sigma^j\right)^* u\right)_i\right) w_i(x)dx \\
&= -\sum_i \int_{\mathbb{T}^d} \left(\mathcal{L}^*_{\sigma^j}\right)_i(x) w_i(x)dx \\
&= -\langle \mathcal{L}^*_{\sigma^j} u, w\rangle_{L^2}.
\end{aligned}$$

We assume that the vector fields $\sigma^j \in H^r(\mathbb{R}^d, \mathbb{R}^d)$ and satisfy the condition

$$\sum_j \|\sigma^j\|^2_{\lfloor r\rfloor,\theta} < \infty \tag{2.2}$$

for some $r \in \mathbb{R}$ with $r > \dfrac{d}{2} + 3$. Here $\lfloor r\rfloor$ is the integer part of r and $\theta = r - \lfloor r\rfloor$.

Definition 2.1. *Given a (divergence free) velocity u and a semimartingale $\gamma \in C^0([0,T]; (C^1(\mathbb{T}^d))^d)$, we define the material differential $\mathcal{D}$ by*

$$\mathcal{D}\gamma := d\gamma + (u \cdot \nabla)\gamma dt + \sum_j \left(\sigma^j \cdot \nabla\right)\gamma \circ dW^j_t$$

Note that if $\gamma, \beta \in C^1$ (scalar valued) then

$$\partial_i(\gamma\beta) = (\partial_i \gamma)\beta + \gamma(\partial_i \beta),$$

for $i = 1, 2, ..., d$. Therefore, for $\gamma, \beta \in C^0([0,T]; C^1)$ we have

$$\mathcal{D}(\gamma\beta) = (\mathcal{D}\gamma)\beta + \gamma(\mathcal{D}\beta). \tag{2.3}$$

Moreover, if $\gamma \in C^2$,

$$\begin{aligned}[\nabla((u\cdot\nabla)\gamma)]_i &= \partial_i\left(\sum_j u_j\partial_j\gamma\right)\\ &= \sum_j \partial_j\gamma\partial_i u_j + \sum_j u_j\partial_j\partial_i\gamma\\ &= ((\nabla u)^*\nabla\gamma)_i + ((u\cdot\nabla)\nabla\gamma)_i,\end{aligned}$$

for $i = 1, 2, ..., d$.
Hence the commutation relation

$$\mathcal{D}\nabla\gamma = \nabla\mathcal{D}\gamma - (\nabla u)^*\nabla\gamma dt - \sum_j \left(\nabla\sigma^j\right)^*\nabla\gamma\circ dW_t^j. \tag{2.4}$$

holds when $\gamma \in C^1([0,T];C^2)$. We use the notation $\mathbb{P}$ for the Leray-Hodge projection onto the space of divergence-free functions.

Let us now define what we mean by solution of the stochastic differential equation (2.1):

Definition 2.2. *Given $u \in C^0([0,T];C^1(\mathbb{T}^d,\mathbb{R}^d))$ a semimartingale and divergence-free vector field (i.e. $\nabla\cdot u = 0$), and $\left\{W_t^j, t \geqslant 0\right\}$ be a family of d-dimensional independent Brownian motions. We have that u is solution of (2.1) if verifies*

$$u_t(x) = u_0(x) - \int_0^t (u_r(x)\cdot\nabla)u_r(x)dr - \int_0^t \nabla p_r(x)dr - \sum_j\int_0^t \mathcal{L}^*_{\sigma^j}u_r(x)\circ dW_r^j, \tag{2.5}$$

where $p \in C^0([0,T];C^1(\mathbb{T}^d,\mathbb{R}))$ is a scalar potential representing internal pressure and the integration is in the Stratonovich sense.

Proposition 2.1. *Let $\alpha \in (0,1)$. Assume (2.2) and that u is $C^{3,\alpha}$-continuous semimartingale. Then u is solution of the equation (2.1) if and only if the pair (X,u) verifies the Lagrangian formulation*

$$dX_t = \sum_j \sigma^j(X_t)\circ dW_t^j + u_t(X_t)dt \tag{2.6}$$

$$u_t(x) = \mathbb{P}\left[(\nabla A_t)^*u_0(A_t)\right](x), \tag{2.7}$$

where $*$ means the transposition of matrices and the back-to-labels map A is denoted by $A(\cdot,t) = X^{-1}(\cdot,t)$.

Proof. ($\Rightarrow$) We have

$$u(t,x) = u_0(x) - \int_0^t (u\cdot\nabla)u(r,x)dr - \sum_j\int_0^t \mathcal{L}^*_{\sigma^j}u(r,x)\circ dW_r^j - \int_0^t \nabla p(r,x)dr \tag{2.8}$$

and

$$X_t = x + \int_0^t u(r, X_r)\, dr + \sum_j \int_0^t \sigma^j (X_r) \circ dW_r^j. \tag{2.9}$$

Then, from Itô-Kunita-Wentzell's formula, see Theorem 1.3, the k-th component of the process $u(t, X_t)$ is given by

$$\begin{aligned} u^k (t, X_t) =& u_0^k (x) - \int_0^t \left((u \cdot \nabla) u^k (r, X_r) + \left(\frac{\partial p}{\partial x_k} \right) (r, X_r) \right) dr \\ & - \int_0^t \sum_j \left((\sigma^j \cdot \nabla) u^k (r, X_r) + \frac{\partial \sigma^j}{\partial x_k} (X_r) \cdot u(r, X_r) \right) \circ dW_r^j \\ & + \sum_{i=1}^d \int_0^t \frac{\partial u^k}{\partial x_i} (r, X_r) u^i (r, X_r) dr + \sum_{i=1}^d \sum_j \int_0^t \frac{\partial u^k}{\partial x_i} (r, X_r) \sigma_i^j (X_r) \circ dW_r^j. \end{aligned} \tag{2.10}$$

We observe that

$$\begin{aligned} \int_0^t (u \cdot \nabla) u^k (r, X_r) ds &= \int_0^t \sum_{i=1}^d u^i \frac{\partial u^k}{\partial x_i} \Big|_{X_r} dr \\ &= \sum_{i=1}^d \int_0^t u^i (r, X_r) \frac{\partial u^k}{\partial x_i} (r, X_r) dr \\ &= \sum_{i=1}^d \int_0^t \frac{\partial u^k}{\partial x_i} (r, X_r) u^i (r, X_r) dr \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} \int_0^t \sum_j (\sigma^j \cdot \nabla) u^k (r, X_r) \circ dW_r^j &= \sum_j \int_0^t \sum_{i=1}^d \sigma_i^j \frac{\partial u^k}{\partial x_i} \Big|_{X_r} \circ dW_r^j \\ &= \sum_j \sum_{i=1}^d \int_0^t \frac{\partial u^k}{\partial x_i} (r, X_r) \sigma_i^j (X_r) \circ dW_r^j. \end{aligned} \tag{2.12}$$

Making obvious cancellation we obtain

$$u^k (t, X_t) = u_0^k (x) - \int_0^t \left(\frac{\partial p}{\partial x_k} \right) (r, X_r) dr - \sum_j \int_0^t \frac{\partial \sigma^j}{\partial x_k} (X_r) \cdot u(r, X_r) \circ dW_r^j,$$

i.e.,

$$u(t, X_t) = u_0(x) - \int_0^t (\nabla p)(r, X_r)\, dr - \sum_j \int_0^t \left(\nabla \sigma^j \right)^* u_r \big|_{X_r} \circ dW_r^j.$$

Now, we observe that $\sum_{i=1}^d \frac{\partial X^i}{\partial x_k} u^i$ is the k-th coordinate of $(\nabla X)^* u$, and

$$\frac{\partial X^i}{\partial x_k} = \delta_{ik} + \int_0^t \nabla u^i (r, X_r) \cdot \frac{\partial X_r}{\partial x_k} dr + \sum_j \int_0^t \nabla \sigma_i^j (X_r) \cdot \frac{\partial X_r}{\partial x_k} \circ dW_r^j.$$

Thus from Itô's formula for the product of two semimartingales we deduce

$$\begin{aligned}\frac{\partial X^i}{\partial x_k}u^i\left(t,X_t\right) &= \delta_{ik}u_0^i(x) + \int_0^t u^i(r,X_r)\nabla u^i(r,X_r)\cdot\frac{\partial X_r}{\partial x_k}dr \\ &\quad + \int_0^t u^i(r,X_r)\left(\sum_j \nabla\sigma_i^j\left(X_r\right)\cdot\frac{\partial X_r}{\partial x_k}\right)\circ dW_r^j - \int_0^t \frac{\partial X^i}{\partial x_k}\left(\frac{\partial p}{\partial x_i}\right)(r,X_r)dr \\ &\quad - \int_0^t \frac{\partial X^i}{\partial x_k}\left(\sum_j\left(\frac{\partial \sigma^j}{\partial x_i}\right)(X_r)\cdot u(r,X_r)\right)\circ dW_r^j.\end{aligned}$$

Note that

$$\begin{aligned}\sum_{i=1}^d \frac{\partial X^i}{\partial x_k}\left[\left(\frac{\partial\sigma^j}{\partial x_i}\right)\cdot u\right] &= \sum_{i=1}^d \frac{\partial X^i}{\partial x_k}\sum_{n=1}^d\frac{\partial\sigma_n^j}{\partial x_i}u^n \\ &= \sum_{i=1}^d\sum_{n=1}^d \frac{\partial X^i}{\partial x_k}\frac{\partial\sigma_n^j}{\partial x_i}u^n.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\sum_{i=1}^d u^i\left[\nabla\sigma_i^j\cdot\frac{\partial X_r}{\partial x_k}\right] &= \sum_{i=1}^d u^i\sum_{m=1}^d\frac{\partial\sigma_i^j}{\partial x_m}\frac{\partial X_r^m}{\partial x_k} \\ &= \sum_{i=1}^d\sum_{m=1}^d u^i\frac{\partial\sigma_i^j}{\partial x_m}\frac{\partial X_r^m}{\partial x_k}.\end{aligned}$$

Thus we obtain

$$\begin{aligned}&\sum_{i=1}^d\int_0^t u^i(r,X_r)\left(\sum_j\nabla\sigma_i^j\left(X_r\right)\cdot\frac{\partial X_r}{\partial x_k}\right)\circ dW_r^j \\ &\quad = \sum_{i=1}^d\int_0^t\frac{\partial X^i}{\partial x_k}\left(\sum_j\left(\frac{\partial\sigma^j}{\partial x_i}\right)(X_r)\cdot u(r,X_r)\right)\circ dW_r^j\end{aligned}$$

Then the $k-$th term of $\left(\nabla X_t\right)^* u\left(t,X_t\right)$ is

$$\begin{aligned}\sum_{i=1}^d\frac{\partial X^i}{\partial x_k}u^i\left(t,X_t\right) &= \sum_{i=1}^d\delta_{ik}u_0^i(x) - \sum_{i=1}^d\int_0^t\frac{\partial X^i}{\partial x_k}\left(\frac{\partial p}{\partial x_i}\right)(r,X_r)dr \\ &\quad + \sum_{i=1}^d\int_0^t u^i\left(r,X_r\right)\nabla u^i\left(r,X_r\right)\cdot\frac{\partial X_r}{\partial x_k}dr,\end{aligned}$$

i.e.,

$$\begin{aligned}\left(\nabla X_t\right)^* u\left(t,X_t\right) &= u_0(x) + \int_0^t\left(\nabla X_r\right)^*\left(\nabla u(r,X_r)\right)^* u\left(r,X_r\right)dr \\ &\quad - \int_0^t\left(\nabla X_r\right)^*\left(\nabla p\right)(r,X_r)dr \\ &= u_0(x) + \frac{1}{2}\int_0^t\nabla\left(|u_r|^2\circ X_r\right)dr - \int_0^t\nabla\left(p_r\circ X_r\right)dr \\ &= u_0 + \nabla\tilde{q},\end{aligned}$$

where $\tilde{q} := \int_0^t \left[\frac{1}{2}\left(|u_r|^2 \circ X_r\right) - p_r \circ X_r\right] dr.$

Then, if we denote $M_t := (\nabla X_t)^*$, it follows that

$$\begin{aligned} u_t \circ X_t = u\,(t, X_t) &= M_t^{-1} u_0 + M_t^{-1} \nabla \tilde{q} \\ &= (\nabla A_t\big|_{X_t})^* u_0 + (\nabla A_t\big|_{X_t})^* \nabla \tilde{q}. \end{aligned}$$

Finally we conclude

$$\begin{aligned} u_t &= (\nabla A_t)^* u_0 \circ A_t + (\nabla A_t)^* (\nabla \tilde{q})(A_t) \\ &= (\nabla A_t)^* u_0 \circ A_t + \nabla q, \end{aligned}$$

where $q := \tilde{q} \circ A$. Therefore, $u_t = \mathbb{P}\left[(\nabla A_t)^* u_0 \circ A_t\right]$.

($\Leftarrow$) We follow directly from Proposition 2 in [PR16], where the authors showed this implication for the deterministic case. We shall also show an alternative formal proof. We set $v = u_0 \circ A$, then by Theorem 2.3.2 of [Cho15] we have $\mathcal{D}A = 0$ and $\mathcal{D}v = 0$. Since u satisfies (2.7) there exists a function q such that

$$u = (\nabla A)^* \, v - \nabla q.$$

Then by (2.3) and (2.4) we have

$$\begin{aligned} \mathcal{D}u &= \mathcal{D}\left[(\nabla A)^* \, v - \nabla q\right] \\ &= \mathcal{D}\left[(\nabla A)^* \, v\right] - \mathcal{D}\nabla q \\ &= \left[\mathcal{D}\,(\nabla A)^*\right] v + (\nabla A)^* \, \mathcal{D}v - \mathcal{D}\nabla q \\ &= \left[\nabla \mathcal{D}A - (\nabla u)^* \, (\nabla A)\, dt - \left(\nabla \sigma^j\right)^* \nabla A \circ dW_t^j\right]^* v \\ &\quad - \nabla \mathcal{D}q + (\nabla u)^* \, \nabla q dt + \left(\nabla \sigma^j\right)^* \nabla q \circ dW_t^j. \end{aligned}$$

Hence, after a calculation and a rearrangement of the terms, we get

$$\begin{aligned} \mathcal{D}u &= -\,(\nabla u)^* \, (\nabla A)^* \, v dt + (\nabla u)^* \, \nabla q dt - \nabla \mathcal{D}q \\ &\quad - \left(\nabla \sigma^j\right)^* \left[(\nabla A)^* \, v - \nabla q\right] \circ dW_t^j \\ &= -\,(\nabla u)^* \, u dt - \nabla \mathcal{D}q - \left(\nabla \sigma^j\right)^* u \circ dW_t^j \\ &= -\nabla \frac{|u|^2}{2} dt - \nabla \mathcal{D}q - \left(\nabla \sigma^j\right)^* u \circ dW_t^j. \end{aligned}$$

Then,

$$\begin{aligned} u(t,x) &= u_0(x) - \int_0^t (u \cdot \nabla) u dr - \int_0^t (\sigma^j \cdot \nabla) u \circ dW_r^j \\ &\quad - \nabla \int_0^t \frac{|u|^2}{2} dr - \nabla \int_0^t \mathcal{D}q - \int_0^t \left(\nabla \sigma^j\right)^* u \circ dW_r^i \\ &= u_0(x) - \int_0^t (u \cdot \nabla) u dr - \int_0^t \mathcal{L}_{\sigma^j}^* u \circ dW_r^j - \int_0^t \nabla p dt, \end{aligned}$$

where formally we have

$$p = \frac{|u|^2}{2} + \partial_t q + (u \cdot \nabla)q + (\sigma^j \cdot \nabla)q \; \partial_t W_t^j.$$

Therefore we conclude that u is solution of the Euler equation (2.1).

□

Remark 2.1. *After submitting the paper [OL23] we was alerted that similar calculations were made in [DH20] to show circulation theorem. However, we prove the equivalence between both formulations and an existence result in Sobolev spaces.*

2.2 An Existence Theorem

2.2.1 Decomposition of the Flow

We use the idea of [FL19a] to decompose the stochastic flow in the system (2.6)-(2.7). More precisely, we consider the stochastic equation without drift:

$$d\varphi_t = \sum_j \sigma_j(\varphi_t) \circ dW_t^j, \quad \varphi_0 = I, \tag{2.13}$$

where I is the identity diffeomorphism of $\mathbb{T}^d$. Under the assumption that (2.2) and $r > \frac{d}{2}+3$, the above equation generates a stochastic flow $\{\varphi_t\}_{t\geqslant 0}$ of $C^{\lfloor r\rfloor,\beta}-$diffeomorphisms on $\mathbb{T}^d$, where $\beta \in (0,\theta)$.

We denote by ω a generic random element in a probability space Ω. For a given random vector field $u : \Omega \times [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$, we define

$$\tilde{u}_t(\omega,x) = \left[\left(\varphi_t(\omega,\cdot)^{-1}\right)_* u_t(\omega,\cdot)\right](x) \tag{2.14}$$

which is the pull-back of the field $u_t(\omega,\cdot)$ by the stochastic flow $\{\varphi_t(\omega,\cdot)\}_{t\geqslant 0}$. If we denote by $K_t(\omega,x) = (\nabla\varphi_t(\omega,x))^{-1}$, i.e., the inverse of the Jacobi matrix, then

$$\tilde{u}_t(\omega,x) = K_t(\omega,x)\, u_t(\omega,\varphi_t(\omega,x)). \tag{2.15}$$

From this expression we see that if $u \in C^0([0,T];H^r)$ a.s., then one also has a.s. $\tilde{u} \in C^0([0,T];H^r)$. Moreover, if the process u is adapted, then so is $\tilde{u}$. Now, we consider the random ODE

$$\dot{Y}_t = \tilde{u}_t(Y_t), \quad Y_0 = I. \tag{2.16}$$

Applying the Itô-Kunita-Wentzell's formula, we see that

$$X_t = \varphi_t \circ Y_t$$

is the flow of $C^{\lfloor r\rfloor,\beta}-$diffeomorphisms associated to the SDE in (2.6)-(2.7).

2.2.2 Sobolev Estimations

All notations and results in this subsection we follow from [PR16]. For $s \geqslant 0$, we will use the notation H^s variously for scalar or vector valued functions in $H^s\left(\mathbb{T}^d\right)$ (componentwise), where this does not cause ambiguity. We will often consider functions in spaces of the norm $C^0([0,T];(H^s(\mathbb{T}^d))^d)$.
To simplify notation we define $\Sigma_s(T)$ (usually denoted Σ_s) for $T \geqslant 0$ and $s \geqslant 0$ by

$$\Sigma_s(T) := C^0([0,T];(H^s(\mathbb{T}^d))^d).$$

We consider the natural norm on Σ_s:

$$\|u\|_{\Sigma_s} = \sup_{t\in[0,T]} \|u(t)\|_s$$

We begin by stating two inequalities concerning the advection term $(u \cdot \nabla)v$, using the notation $B(u,v) := (u \cdot \nabla)v$. The following two results are taken from Lemma 1 and Lemma 2 of [PR16].

Lemma 2.1. *For $s > \frac{d}{2}$ there exists $C_1 > 0$ such that if $u \in H^s$ and $v \in H^{s+1}$ then $B(u,v) \in H^s$ and*

$$\|B(u,v)\|_s \leqslant C_1 \|u\|_s \|v\|_{s+1}.$$

Lemma 2.2. *If $s > \frac{d}{2} + 1$ there exists $C_2 > 0$ such that for $u \in H^s$, $v \in H^{s+1}$ with divergence-free we have*

$$|(B(u,v),v)_s| \leqslant C_2 \|u\|_s \|v\|_s^2$$

We use the following shorthand for closed balls in Σ_s:

$$B_M = \overline{B_{\|\cdot\|_{\Sigma_s}}(0,M)},$$

i.e., B_M is the closed ball centred at the origin of radius $M > 0$ with respect to the norm $\|\cdot\|_{\Sigma_s}$. Where ambiguity could arise we write $B_M(T)$ for the closed ball in $\Sigma_s(T)$.

We need the following key technical result, see Lemma 3 of [PR16].

Lemma 2.3. *If $s > \frac{d}{2} + 1$ and η, $v \in \Sigma_s(T)$ then $\mathbb{P}\left[(\nabla\eta)^* v\right] \in \Sigma_s(T)$ and there exists a constant $C_3 > 0$ (independent of η, v, t and T) such that for fixed t,*

$$\|\mathbb{P}\left[(\nabla\eta)^* v\right]\|_r \leqslant C_3 \|\eta\|_s \|v\|_r, \tag{2.17}$$

where $r = s$ or $r = s - 1$. Furthermore, there exists $C_3' > 0$ such that for any $M > 0$ and $T > 0$, the following bounds hold uniformly with respect to $t \in [0,T]$ for any η_1, η_2, v_1, $v_2 \in B_M(T)$:

$$\|\mathbb{P}\left[(\nabla\eta_1)^* v_1 - (\nabla\eta_2)^* v_2\right]\|_X \leqslant C_3' \left(\|\eta_1 - \eta_2\|_X + \|v_1 - v_2\|_X\right), \tag{2.18}$$

where X is $L^2\left(\mathbb{T}^d\right)$ or H^{s-1}.

The next lemma gives uniform bounds on the H^s norms of solutions to the transport equations (2.23) and (2.24), see Lemma 4 of [FL19a]. We will consider the following system:

$$\begin{cases} \partial_t f + (u \cdot \nabla) f = g, \\ f(0) = f_0. \end{cases} \tag{2.19}$$

where $f,\, g : [0,T] \times \mathbb{T}^d \to \mathbb{T}^d$ and u is divergence-free.

Lemma 2.4. *Let $s > \frac{d}{2} + 1$ and fix $f_0 \in H^s$, $g \in \Sigma_s$. If $u \in \Sigma_s$ is non-zero and divergence free then there exists a unique solution f to (2.19). Furthermore, the solution $f \in \Sigma_s \cap C^1([0,T]; H^{s-1}) \cap C^1([0,T] \times \mathbb{T}^d)$ and there exists $C_4 > 0$ (from Lemma 2.2) such that if $r,\, t \in [0,T]$ we have:*

$$\|f(t)\|_s \leqslant \left(\|f(r)\|_s + \frac{\|g\|_{\Sigma_s}}{C_4\|u\|_{\Sigma_s}}\right) \exp\left(C_4|t-r|\|u\|_{\Sigma_s}\right) - \frac{\|g\|_{\Sigma_s}}{C_4\|u\|_{\Sigma_s}}. \tag{2.20}$$

The following result, see Lemma 5 of [FL19a], is key to demonstrate the main result of this chapter.

Lemma 2.5. *For $s > \frac{d}{2} + 1$ fix $u_1,\, u_1 \in \Sigma_s$ and $f_0 \in H^s$. Let $g_1 = g_2 = 0$ or $g_i = -u_i$ for $i = 1,\, 2$. If $f_1,\, f_2$ are the solutions of (2.19) corresponding to $u_1,\, u_2,\, g_1,\, g_2$ respectively, then in the case that $g_1 = g_2 = 0$, there exists $C_5 > 0$ depending only of s such that*

$$\|f_1(t) - f_2(t)\|_{L^2} \leqslant C_5\|f_1 + f_2\|_{\Sigma_s}\|u_1 - u_2\|_{\Sigma_0} t \tag{2.21}$$

for all $t \in [0,T]$. In the case that $g_i = -u_i$ for $i = 1,\, 2$ we instead have

$$\|f_1(t) - f_2(t)\|_{L^2} \leqslant \left(C_5\|f_1 + f_2\|_{\Sigma_s} + 1\right)\|u_1 - u_2\|_{\Sigma_0} t. \tag{2.22}$$

We consider the following transport equation:

$$\partial_t B + (\tilde{u} \cdot \nabla) B = 0, \tag{2.23}$$

$$\partial_t v + (\tilde{u} \cdot \nabla) v = 0, \tag{2.24}$$

where $\tilde{u}_t(x) = \left[\left(\varphi_t^{-1}\right)_* u_t\right](x)$.
Given an initial divergence-free velocity u_0 for the classic equations, we choose initial conditions for the above system as follows:

$$B(x,0) = x, \tag{2.25}$$

$$u(x,0) = v(x,0) = u_0(x). \tag{2.26}$$

Also $\eta(x,t) := B(x,t) - x$ and replace (2.23) and (2.25) with the equations

$$\partial_t \eta + (\tilde{u} \cdot \nabla)\eta + \tilde{u} = 0, \tag{2.27}$$

$$\eta(x,0) = 0, \tag{2.28}$$

respectively. We do this because the identity map (hence B) does not have sufficient Sobolev regularity when considered as a function on the torus with values in $\mathbb{R}^d$ (i.e. without accounting for the topology of the target torus).

2.2.3 Contraction

The objective of the remainder of this chapter is to demonstrate the following theorem. In this outcome, we shall prove that a map (defined by Su in (2.29) below) has a fixed point in a close ball in $\Sigma_s(T)$.

Theorem 2.1. *Assume that $r \geqslant s+1$. If $d \geqslant 2$, $s > \frac{d}{2} + 1$ and $u_0 \in H^s$ is divergence free, then there exists $T(\omega) > 0$ such that*

$$u = \mathbb{P}\left[(\nabla A)^* u_0(A)\right]$$

has solution $u \in \Sigma_s(T)$.

Proof. For $u \in \Sigma_s$, we consider the following system:

$$\begin{cases} \tilde{u}_t(x) = \left[\left(\varphi_t^{-1}\right)_* u_t\right](x), \\ \dot{Y}_t = \tilde{u}_t(Y_t), \quad Y_0 = I, \\ X_t = \varphi_t(Y_t), \\ Su_t(x) = \mathbb{P}\left[\left(\nabla X_t^{-1}\right)^* u_0\left(X_t^{-1}\right)\right](x). \end{cases} \tag{2.29}$$

Here, we assume φ_t is the solution of the equation (2.13), and that we are given a family of diffeomorphisms $\{\varphi_t\}_{t\in[0,T]}$ of $\mathbb{T}^d$ satisfying $\varphi \in C^0([0,T]; C^{\lfloor r\rfloor,\beta})$ and $\varphi_0 = I$. As mentioned above, $\beta \in (0,\theta)$ and $\theta = r - \lfloor r \rfloor$.

From Chapter 3 of [AF03] we have that there exist positive constants $C_0 := C_0(t,d,s,\varphi)$, $C_1 := C_1(t,d,s,\varphi)$, $C_2 := C_2(t,d,s,\varphi)$ such that

$$\|\varphi^{-1}\|_s \leqslant C_1; \tag{2.30}$$

$$\|\tilde{u}\|_s = \|\left(\nabla\varphi^{-1}\right)(\varphi)u(\varphi)\|_s \leqslant C_0\|u\|_s; \tag{2.31}$$

$$\|\eta \circ \varphi^{-1}\|_s \leqslant C_2\|\eta\|_s \text{ and } \|v \circ \varphi^{-1}\|_s \leqslant C_2\|v\|_s. \tag{2.32}$$

Fix $s > \dfrac{d}{2} + 1$ and let C_3, C_4 be the constants in (2.17), (2.20) (from Lemmas 2.3 and 2.4) respectively. Fix $\|u_0\|_s < M$ and $T > 0$ so that

$$C\|u_0\|_s \exp\left(C_4 C_0 T M\right)\left[\frac{C_2^2}{C_1 C_4}\left(\exp\left(C_4 C_0 T M\right) - 1\right) + 1\right] \leqslant M,$$

where C is a constant to be defined later.
Let $u \in B_M(T)$ be a divergence free vector field and let $\eta = Y_t^{-1} - x$ is unique solution of (2.27) with initial data $\eta_0 = 0$. Let $v = u_0(Y_t^{-1})$ be the unique solution of (2.24) for initial data $v_0 = u_0$.

From (2.31) it follows that

$$\|\tilde{u}\|_{\Sigma_s} = \sup_{t\in[0,T]} \|\tilde{u}\|_s \leqslant \sup_{t\in[0,T]} C_0\|u\|_s \leqslant C_0 M,$$

Then by Lemma 2.4,

$$\|v(t)\|_s \leqslant \|u_0\|_s \exp\left(C_4 t\|\tilde{u}\|_{\Sigma_s}\right) \leqslant \|u_0\|_s \exp\left(C_0 C_4 T M\right) \tag{2.33}$$

and

$$\|\eta(t)\|_s \leqslant \frac{1}{C_4}\left(\exp\left(C_0 C_4 T M\right) - 1\right). \tag{2.34}$$

Hence, with $C = C_1 C_2 C_3$, (the constants C_1, C_2, C_3 are given by (2.30), (2.32) and Lemma 2.3 respectively)

$$\begin{aligned}
\|Su(t)\|_s &\leqslant \|\mathbb{P}\left[(\nabla(\eta\circ\varphi^{-1}))^* v(\varphi^{-1})\right]\|_s + \|\mathbb{P}\left[(\nabla\varphi^{-1})^* v(\varphi^{-1})\right]\|_s \\
&\leqslant C_3\|\eta\circ\varphi^{-1}\|_s\|v(\varphi^{-1})\|_s + C_3\|\varphi^{-1}\|_s\|v(\varphi^{-1})\|_s \\
&\leqslant C_3 C_2^2\|\eta\|_s\|v\|_s + C_3 C_1 C_2\|v\|_s \\
&= C_3 C_2\|v\|_s\left(C_2\|\eta\|_s + C_1\right),
\end{aligned}$$

here the third inequality follows from (2.30) and (2.32). Then by (2.33) and (2.34)

$$\begin{aligned}
\|Su(t)\|_s &\leqslant \|u_0\|_s \exp\left(C_4 C_0 t M\right)\left[\frac{C_3 C_2^2}{C_4}\left(\exp\left(C_4 C_0 t M\right) - 1\right) + C_3 C_1 C_2\right] \\
&= C\|u_0\|_s \exp\left(C_4 C_0 t M\right)\left[\frac{C_2^2}{C_1 C_4}\left(\exp\left(C_4 C_0 t M\right) - 1\right) + 1\right] \leqslant M
\end{aligned} \tag{2.35}$$

for all $t \in [0, T]$. Hence $S : B_M(T) \to B_M(T)$.
Before proving that the map S is a contraction in a certain space, we need to prove a couple of inequalities. By Lemmas 2.3, 2.4 and 2.5 we have

$$
\begin{aligned}
&\left\|\mathbb{P}\left[\left(\nabla\left(\eta_1\circ\varphi^{-1}\right)\right)^* v_1\left(\varphi^{-1}\right)-\left(\nabla\left(\eta_2\circ\varphi^{-1}\right)\right)^* v_2\left(\varphi^{-1}\right)\right]\right\|_{L^2}\\
&\leqslant C_3M\left[\left\|\eta_1\circ\varphi^{-1}-\eta_2\circ\varphi^{-1}\right\|_{L^2}+\left\|v_1\circ\varphi^{-1}-v_2\circ\varphi^{-1}\right\|_{L^2}\right]\\
&= C_3M\left[\|\eta_1-\eta_2\|_{L^2}+\|v_1-v_2\|_{L^2}\right]\\
&\leqslant C_3M\left[\left(C_5\|\eta_1+\eta_2\|_{\Sigma_s}+1\right)\|\tilde u_1-\tilde u_2\|_{\Sigma_0} t+C_5\|v_1-v_2\|_{\Sigma_s}\|\tilde u_1-\tilde u_2\|_{\Sigma_0} t\right]\\
&\leqslant C_3M\left[\left(\frac{2C_5}{C_4}\left(\exp\left(C_4C_0tM\right)-1\right)+1\right)t\|\tilde u_1-\tilde u_2\|_{\Sigma_0}\right.\\
&\qquad\qquad \left.+2C_5Mt\|u_0\|_s\exp\left(C_4C_0tM\right)\|\tilde u_1-\tilde u_2\|_{\Sigma_0}\right]\\
&\leqslant C_3M\left[\left(\frac{2C_5}{C_4}\left(\exp\left(C_4C_0TM\right)-1\right)+1\right)t+2C_5t\|u_0\|_s\exp\left(C_4C_0TM\right)\right]\|\tilde u_1-\tilde u_2\|_{\Sigma_0},
\end{aligned}
$$

and

$$
\begin{aligned}
\left\|\mathbb{P}\left[\left(\nabla\varphi^{-1}\right)^*\left(v_1(\varphi^{-1})-v_2(\varphi^{-1})\right)\right]\right\|_{L^2}&\leqslant C_3M\left\|v_1(\varphi^{-1})-v_2(\varphi^{-1})\right\|_{L^2}\\
&= C_3M\|v_1-v_2\|_{L^2}\\
&\leqslant C_3C_5M\|v_1-v_2\|_{\Sigma_s}\|\tilde u_1-\tilde u_2\|_{\Sigma_0} t\\
&\leqslant 2C_3C_5Mt\|u_0\|_s\exp\left(C_4tC_0M\right)\|\tilde u_1-\tilde u_2\|_{\Sigma_0},
\end{aligned}
$$

where C_3, C_4, C_5 are the constants from Lemmas 2.3, 2.4 and 2.5, respectively.

Now we will show that the map S is a contraction on $B_M(T)$ in the L^2-norm if T is sufficiently small. For u_1, $u_2 \in B_M(T)$ we construct v_i and η_i from u_i as above for $i = 1, 2$ with $v_1(\cdot,0) = v_2(\cdot,0) = u_0$. Then by the inequalities above

$$
\begin{aligned}
\|Su_1-Su_2\|_{L^2}&\leqslant\left\|\mathbb{P}\left[\left(\nabla\left(\eta_1\circ\varphi^{-1}\right)\right)^* v_1\left(\varphi^{-1}\right)-\left(\nabla\left(\eta_2\circ\varphi^{-1}\right)\right)^* v_2\left(\varphi^{-1}\right)\right]\right\|_{L^2}\\
&\qquad+\left\|\mathbb{P}\left[\left(\nabla\varphi^{-1}\right)^*\left(v_1(\varphi^{-1})-v_2(\varphi^{-1})\right)\right]\right\|_{L^2}\\
&\leqslant C_3M\left(\frac{2C_5}{C_4}\left(\exp\left(C_4C_0TM\right)-1\right)+1\right)t\|\tilde u_1-\tilde u_2\|_{\Sigma_0}\\
&\qquad+4C_3C_5Mt\|u_0\|_s\exp\left(C_4C_0TM\right)\|\tilde u_1-\tilde u_2\|_{\Sigma_0}\\
&\leqslant 2TMC_3C_6\left(C_5\left(2\|u_0\|_s+\frac{1}{C_4}\right)\exp\left(C_4C_0TM\right)+\frac{1}{2}-\frac{C_5}{C_4}\right)\|u_1-u_2\|_{\Sigma_0}\\
&= C(u_0,\omega,M,T)\|u_1-u_2\|_{\Sigma_0},
\end{aligned}
\tag{2.36}
$$

where the third inequality follows from a change of variables and Hölder's inequality

$$
\begin{aligned}
\|\tilde u_1-\tilde u_2\|_{\Sigma_0}&=\sup_{t\in[0,T]}\|\tilde u_1-\tilde u_2\|_{L^2}=\sup_{t\in[0,T]}\left\|(\nabla\varphi)^{-1}u_1(\varphi)-(\nabla\varphi)^{-1}u_2(\varphi)\right\|_{L^2}\\
&=\sup_{t\in[0,T]}\left\|\left(\nabla\varphi^{-1}\right)(\varphi)\left(u_1-u_2\right)(\varphi)\right\|_{L^2}\\
&=\sup_{t\in[0,T]}\left\|\left(\nabla\varphi^{-1}\right)\left(u_1-u_2\right)\right\|_{L^2}\leqslant\sup_{t\in[0,T]}\left(\left\|\nabla\varphi^{-1}\right\|_{L^\infty}\|u_1-u_2\|_{L^2}\right)\\
&\leqslant C_6\|u_1-u_2\|_{\Sigma_0},
\end{aligned}
$$

where $C_6 := \sup_{t \in [0,T]} \left\| \nabla \varphi^{-1} \right\|_{L^\infty}$.
We observe that $C(u_0, \omega, M, T)$ is given by the formula

$$C(u_0, \omega, M, T) := 2T \left[C_5 M C_3 C_6 \left(2 \left\| u_0 \right\|_s + \frac{1}{C_4} \right) \exp\left(C_4 C_0 T M \right) + C_3 C_6 M \left(\frac{1}{2} - \frac{C_5}{C_4} \right) \right]. \tag{2.37}$$

Note that the smoothness of the diffeomorphisms φ implies that the constants C_0, C_1, C_2 and C_6 are limited uniformly in time. Then taking the supremum of (2.36) with respect to t and choosing $T > 0$ small enough, we see that S is a contraction in the required sense. With the above preparations, the proof is the same as that in [PR16]. The Banach fixed-point Theorem guarantees the existence and uniqueness of a fixed point u for S in the closure of $B_M(T)$ with respect to $\| \cdot \|_{\Sigma_0}$. We conclude that S has a unique accumulation point u in the closure of B_M with respect to $\| \cdot \|_{\Sigma_0}$. Now, since $B_M(T)$ is convex and closed in Σ_s then by the Mazur's Theorem it is weakly closed, hence $u \in B_M(T)$ is a fixed point of S with respect to $\| \cdot \|_{\Sigma_s}$. A fixed point of S, along with associated back-to-labels map and virtual velocity, clearly give a solution to the Eulerian-Lagrangian formulation of the Euler equations with the required regularity.

□

CHAPTER 3
THE YOUNG CASE

We will prove the equivalence of the Eulerian-Lagrangian formulation with the classical Euler equations

$$du_t + (u_t \cdot \nabla u_t + \nabla p_t)\, dt + \sum_j \mathcal{L}^*_{\sigma_j} u_t dY^j_t = 0, \tag{3.1}$$

where for each j, Y^j_t is an α-Hölder paths with $\alpha \in (1/2, 1]$, and the integration is in the Young sense. Again, as in the previous chapter, we consider $\mathcal{L}^*_{\sigma^j} u := (\sigma^j \cdot \nabla) u + \left(\nabla \sigma^j\right)^* u$ is the dual operator of the Lie derivative $\mathcal{L}_{\sigma^j} u = (\sigma^j \cdot \nabla) u - (u \cdot \nabla)\sigma^j$.

Using this formulation we prove a local in time existence result for solutions in $C^0([0,T]\,;(H^s(\mathbb{T}^d))^d)$ with $s > \dfrac{d}{2} + 1$, new for equation (3.1). Our work also includes a solution theory for fractional Brownian motion driven equations, which enables memory effects to be introduced through our formulation.

3.1 The Young's integral

We will introduce α-Hölder paths, which play a fundamental role in the theory of continuous stochastic processes, including the fractional Brownian motion, which we will mention later on. More precisely, we have the following definition. Let U, V and W be Banach spaces. Given a path $\phi : [0,T] \to V$ and $s, t \in [0,T]$ we write $\phi_{st} = \phi_t - \phi_s$.

Definition 3.1. *Let $0 < \alpha \leqslant 1$. $\mathcal{C}^\alpha([0,T]; V)$ is the space of functions on $[0,T]$ taking values in V such that the following α-Hölder seminorm*

$$|\phi|_\alpha := \sup_{0 \leqslant s < t \leqslant T} \frac{\|\phi_{st}\|_V}{|t-s|^\alpha}.$$

is finite.

The space $\mathcal{C}^\alpha([0,T];V)$ is a Banach space with the norm

$$\|\phi\|_\alpha = |\phi|_\alpha + \|\phi\|_\infty,$$

where, as usual, $\|\phi\|_\infty = \sup\limits_{t\in[0,T]} \|\phi(t)\|_V$.
We note here that the product of Hölder continuous functions is again Hölder continuous, in fact.

Theorem 3.1. *If $\phi, \gamma \in C^\alpha([0,T];V)$ then $\phi\gamma \in C^\alpha([0,T];V)$ and exist a constant $c > 0$ such that*

$$\|\phi\gamma\|_\alpha \leqslant c\|\phi\|_\alpha\|\gamma\|_\alpha$$

Proof. From the definition of α-Hölder seminorm and properties of supremum we have

$$\begin{aligned}
|\phi\gamma|_\alpha &= \sup_{0\leqslant s<t\leqslant T}\left(\frac{\|(\phi\gamma)_{st}\|_V}{|t-s|^\alpha}\right) = \sup_{0\leqslant s<t\leqslant T}\left(\frac{\|\phi(t)\gamma(t)-\phi(s)\gamma(s)\|_V}{|t-s|^\alpha}\right)\\
&\leqslant \sup_{0\leqslant s<t\leqslant T}\left(\|\phi(t)\|_V\frac{\|\gamma(t)-\gamma(s)\|_V}{|t-s|^\alpha} + \|\gamma(s)\|_V\frac{\|\phi(t)-\phi(s)\|_V}{|t-s|^\alpha}\right)\\
&\leqslant \|\phi\|_\infty|\gamma|_\alpha + \|\gamma\|_\infty|\phi|_\alpha.
\end{aligned}$$

For this inequality and the definition of α-Hölder norm, the relation indeed holds since

$$\begin{aligned}
\|\phi\gamma\|_\alpha &= |\phi\gamma|_\alpha + \|\phi\gamma\|_\infty\\
&\leqslant \|\phi\|_\infty|\gamma|_\alpha + \|\gamma\|_\infty|\phi|_\alpha + \|\phi\|_\infty\|\gamma\|_\infty\\
&\leqslant \|\phi\|_\infty\left(|\gamma|_\alpha + \|\gamma\|_\infty\right) + \|\gamma\|_\alpha|\phi|_\alpha\\
&\leqslant c\|\gamma\|_\alpha\left(\|\phi\|_\infty + |\phi|_\alpha\right)\\
&= c\|\phi\|_\alpha\|\gamma\|_\alpha.
\end{aligned}$$

□

We will denote by $L(U,V)$ the space of continuous linear operators from U to V equipped with the operator norm, and

$$\Delta_T := \left\{(s,t) \in [0,T]^2 : 0 \leqslant s \leqslant t \leqslant T\right\}.$$

We define the Young's integral $\int XdY$ when $Y \in C^\alpha([0,T];V)$ and $X \in C^\beta([0,T];L(V,W))$ with $\alpha+\beta > 1$. The cornerstone of this theory is the following Young-Loeve estimative, see Proposition 3 in [Gub04] and [You36].

Theorem 3.2. *Let $Y \in C^\alpha([0,T];V)$ and $X \in C^\beta([0,T];L(V,W))$ for some $\alpha,\beta \in (0,1]$ with $\alpha+\beta > 1$. Then the limit*

$$\int_0^t X_r dY_r := \lim_{|\pi|\to 0}\sum_{[r,s]\in\pi} X_r Y_{rs}$$

exists for every $t \in [0,T]$*, where the limit is taken over any* $\pi \in \mathcal{P}([0,t])$*, and* $\mathcal{P}([0,t])$ *the set of all partitions* π *of the interval* $[0,t]$*. This limit is called the* Young integral *of* X *against* Y*. Moreover it holds the following estimative*

$$\left|\int_s^t X_r dY_r - X_s Y_{st}\right|_W \leqslant C_{\alpha+\beta}|Y|_\alpha |X|_\beta |t-s|^{\alpha+\beta} \tag{3.2}$$

for all $(s,t) \in \Delta_T$*.*

We recall the independence of the Young integral with respect to the choice of the partitioning points. Let $r \in [s,\theta]$ denote an arbitrary point in the interval $[s,\theta]$. The Young integral of X against Y is equal to the limit

$$\int_0^t X_s dY_s := \lim_{|\pi|\to 0} \sum_{[s,\theta]\in\pi} X_r Y_{s\theta}$$

for any $t \in [0,T]$.

Example of Hölder noise: The fractional Brownian motion, which was introduced by Kolmogorov in [Kol40] and further developed by Mandelbrot and Van Ness in [MVN68], is a stochastic process that significantly differs from classical Brownian motion and semimartingales, which are commonly used in stochastic calculus. It is a centered Gaussian process distinguished by the stationarity of its increments and its medium- or long-memory property, contrasting sharply with the properties of martingales and Markov processes.

We have that $(Y_t)_{t\in[0,T]}$, a *fractional Brownian motion* (fBm in short) with Hurst (or self-similarity) parameter $H \in (0,1)$, is a centered continuous Gaussian process with covariance

$$\mathbb{E}[Y_t Y_s] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) \text{ for every } s,t \in [0,T].$$

According to Proposition 1.6 of [Nou12], fractional Brownian motion will exists and has Hölder continuous paths of order $\alpha \in (0,H)$.

The Hurst parameter of fractional Brownian motion determines the degree of roughness of the fBm path:

- When $H = 1/2$, the fBm reduces to standard Brownian motion, which exhibits no long-range dependence.

- When $H > 1/2$, the fBm path is smoother than Brownian motion, and the process exhibits positive long-range dependence. This means that large fluctuations are more likely to be followed by large fluctuations and vice versa.

- When $H < 1/2$, the fBm path becomes rougher, and the process exhibits negative long-range dependence. In this case, large fluctuations are more likely to be followed by small fluctuations and vice versa.

As the next lemma shows that Young integration satisfies the classical integration by parts formula.

Lemma 3.1. *Let $X \in C^{\alpha}$ and $Y \in C^{\beta}$ for some $\alpha, \beta \in (0,1]$ with $\alpha + \beta > 1$. Then*

$$X_T Y_T = X_0 Y_0 + \int_0^T X_u dY_u + \int_0^T Y_u dX_u.$$

Lemma 3.2. *Let $X \in C^{\alpha}([0,T], V)$ and $f \in C^{1+\beta}(V, W)$ for some $\alpha, \beta \in (0,1]$, such that $\alpha(1+\beta) > 1$. Then $\int_0^T Df(X_r)dX_r$ is well-defined Young integral, and*

$$f(X_T) = f(X_0) + \int_0^T Df(X_r)dX_r.$$

Remark 3.1. *Composition of a differentiable function with an α-Hölder path is also an α-Hölder path.*

Here is the main result of this section: the generalized Itô-Wentzell's formula for the Young integral, see Theorem 3.1 of [CC22].

Theorem 3.3. *Let $\alpha \in (\frac{1}{2}, 1]$, $Y \in C^{\alpha}([0,T]; V)$ and $h : [0,T] \times U \to L(V, W)$ continuous and differentiable in U such that*

1. *$(t, x) \to D_x h_t(x)$ is continuous,*
2. *$h \in C^0(U, C^{\beta}([0,T]; L(V, W)))$ for some $\beta \in (\frac{1}{2}, 1]$.*

Let

$$g_t(x) = g_0(x) + \int_0^t h_r(x)dY_r. \tag{3.3}$$

Asssume that $g : [0,T] \times U \to W$ is twice differentiable in $U$and the functions $(t, x) \mapsto D_x^2 g_t(x)$ are continuous. Then for any $X \in C^{\alpha}([0,T]; U)$,

$$g_t(X_t) = g_0(X_0) + \int_0^t h_r(X_r)dY_r + \int_0^t D_x g_r(X_r)dX_r, \tag{3.4}$$

where the integral $\int_0^t D_x g_r(X_r)dX_r$ is understood in the Riemann-Stieltjes sense. If $D_x g \in C^0(U, C^{\gamma}([0,T]; L(V, W)))$ for some $\gamma \in (\frac{1}{2}, 1]$ we have that the integral $\int_0^t D_x g_r(X_r)dX_r$ is a Young integral, and $t \mapsto g_t(X_t) \in C^{\alpha}([0,T]; W)$.

Now, we assume that $\sigma^j \in H^r(\mathbb{R}^d, \mathbb{R}^d)$ and

$$\sum_j \|\sigma^j\|^2_{\lfloor r\rfloor,\theta} < \infty \tag{3.5}$$

for some $r \in \mathbb{R}$ with $r > \frac{d}{2} + 3$. Here $\lfloor r\rfloor$ is the integer part of r and $\theta = r - \lfloor r\rfloor$.

Finally, let's see what we mean by a solution of the stochastic differential equation (3.1). But before that, let us consider the following remark:

Remark 3.2. *Let us consider $\sigma^j : \mathbb{R}^d \to \mathbb{R}^d$ a sequence of continuous, divergence free vector fields, and $\{Y^j_t\}_j$ a sequence of α-Hölder paths on $\mathbb{R}^d$. The following condition on the σ^j and Y^j_t should hold*

$$\sum_j \|\sigma^j\|_{\infty,x} |Y^j_t|_\alpha + \sum_j \|\nabla\sigma^j\|_{\infty,x} |Y^j_t|_\alpha < \infty.$$

This ensures that the integration in the following definition is well-defined.

Definition 3.2. *Given $u : [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$ divergence–free vector field (i.e. $\nabla \cdot u = 0$) with $u \in C^\alpha([0,T]; C^1(\mathbb{T}^d, \mathbb{R}^d))$ and $Y^j : [0,T] \to \mathbb{R}^d$ α-Hölder paths with $\alpha \in (1/2, 1]$, we say that u is solution of (3.1) if verifies*

$$u_t(x) = u_0(x) - \int_0^t (u_r(x) \cdot \nabla) u_r(x) dr - \int_0^t \nabla p_r(x) dr - \sum_j \int_0^t \mathcal{L}^*_{\sigma^j} u_r(x) dY^j_r, \tag{3.6}$$

where $p \in C^0([0,T]; C^1(\mathbb{R}^d, \mathbb{R}))$ is a scalar potential representing internal pressure and the last integration is in the Young sense.

3.2 Equivalent formulations

Let $\mathcal{L}^*_{\sigma^j}$, as in the previous chapter, be the adjoint operator of $\mathcal{L}_{\sigma^j}$ with respect to the inner product in $L^2(\mathbb{T}^d, \mathbb{R}^d)$.

Definition 3.3. *Given a (divergence free) velocity $u \in C^0([0,T]; (C^0(\mathbb{T}^d))^d)$ and $\gamma \in C^\alpha([0,T]; (C^1(\mathbb{T}^d))^d)$, we define the material differential $\mathcal{D}$ by (with summation over repeated indices)*

$$\mathcal{D}\gamma := d\gamma + (u \cdot \nabla)\gamma dt + \left(\sigma^j \cdot \nabla\right)\gamma dY^j_t$$

Analogously to the stochastic and deterministic cases discussed in the previous chapter and Proposition 2 in [PR16], respectively, through simple calculation, we obtain the identities.

$$\mathcal{D}(\gamma\beta) = (\mathcal{D}\gamma)\beta + \gamma(\mathcal{D}\beta), \tag{3.7}$$

$$\mathcal{D}\nabla\gamma = \nabla\mathcal{D}\gamma - (\nabla u)^* \nabla\gamma dt - \left(\nabla\sigma^j\right)^* \nabla\gamma dY_t^j, \tag{3.8}$$

for the last identity (3.8) we must consider $\gamma \in C^{\alpha}([0,T];(C^2(\mathbb{T}^d))^d)$, so that the Young integration is well defined.

The proof of the following result is based on Itô-Kunita-Wentzell's formula for Young's integral and the continuity properties of Young's integral.

Proposition 3.1. *Assume that $u \in C^{\alpha}\left([0,T];C^3\left(\mathbb{T}^d,\mathbb{R}^d\right)\right)$ and (3.5) holds. Then u is solution of equation (3.1) if and only if the pair (X,u) verifies the Lagrangian formulation*

$$dX_t = \sigma^j(X_t)dY_t^j + u_t(X_t)dt \tag{3.9}$$

$$u_t(x) = \mathbb{P}\left[(\nabla A_t)^* u_0(A_t)\right](x), \tag{3.10}$$

where $*$ means the transposition of matrices and the back-to-labels map A is given $A(\cdot,t) = X^{-1}(\cdot,t)$.

Proof. ($\Rightarrow$) We have

$$u(t,x) = u_0(x) - \int_0^t (u\cdot\nabla)u(r,x)dr - \sum_j \int_0^t \mathcal{L}^*_{\sigma^j} u(r,x)dY_r^j - \int_0^t \nabla p(r,x)dr \tag{3.11}$$

and

$$X_t = x + \int_0^t u\left(r,X_r\right)dr + \sum_j \int_0^t \sigma^j\left(X_r\right)dY_r^j. \tag{3.12}$$

Then, from Itô-Kunita-Wentzell's formula for the Young's integral, see Theorem 3.3, we have that $u(t,X_t)$ is given by

$$\begin{aligned} u\left(t,X_t\right) =& u_0\left(X_0\right) - \int_0^t \left[\left(u\cdot\nabla\right)u\left(r,X_r\right) + \nabla p\left(r,X_r\right)\right]dr - \sum_j \int_0^t \mathcal{L}^*_{\sigma^j} u\left(r,X_r\right)dY_r^j \\ &+ \int_0^t \left(\nabla u_r\right)\left(X_r\right)\left[\sum_j \sigma^j\left(X_r\right)dY_r^j + u_r\left(X_r\right)dr\right] \end{aligned}$$

Analogously to previously chapter, see equations (2.11) and (2.12), we have the identities

$$\int_0^t (u\cdot\nabla)u(r,X_r)dr = \int_0^t \left[\left(\nabla u_r\right)u_r\right]\big|_{X_r} dr$$

and

$$\sum_j \int_0^t (\sigma^j\cdot\nabla)u(r,X_r)dY_r^j = \sum_j \int_0^t \left[\left(\nabla u_r\right)\sigma^j\right]\big|_{X_r} dY_r^j.$$

Then we deduce

$$u(t, X_t) = u_0(x) - \int_0^t \nabla p(r, X_r)\, dr - \sum_j \int_0^t \left(\nabla \sigma^j\right)^* u_r\big|_{X_r} dY_r^j. \tag{3.13}$$

Now, from the Proposition 8 of [Lej10] we have

$$\nabla X_t = I + \int_0^t (\nabla u_r(X_r))\, \nabla X_r dr + \sum_j \int_0^t \left(\nabla \sigma^j\right)^* (X_r)\, \nabla X_r dY_r^j.$$

From Lemma 3.1 we have

$$\begin{aligned}(\nabla X_t)^* u_t(X_t) = u_0(x) &- \int_0^t (\nabla X_r)^* \left[(\nabla p)(r, X_r)\, dr + \sum_j \left(\nabla \sigma^j(X_r)\right)^* u_r(X_r)\, dY_r^j\right] \\ &+ \int_0^t u_r(X_r) \left[(\nabla X_r)^* (\nabla u_r(X_r))^* dr + \sum_j (\nabla X_r)^* \left(\nabla \sigma^j\right)(X_r)\, dY_r^j\right].\end{aligned}$$

We observe that

$$\sum_j \int_0^t (\nabla X_r)^* \left(\nabla \sigma^j\right)(X_r)\, u_r(X_r)\, dY_r^j = \sum_j \int_0^t (\nabla X_r)^* \left(\nabla \sigma^j(X_r)\right)^* u_r(X_r)\, dY_r^j.$$

Then we deduce

$$\begin{aligned}(\nabla X_t)^* u(t, X_t) &= u_0(x) + \int_0^t (\nabla X_r)^* (\nabla u_r(X_r))^* u(r, X_r)\, dr - \int_0^t (\nabla X_r)^* (\nabla p)(r, X_r) dr \\ &= u_0(x) + \frac{1}{2}\int_0^t \nabla\left(|u_r|^2 \circ X_r\right) dr - \int_0^t \nabla(p_r \circ X_r)\, dr \\ &= u_0 + \nabla \tilde{q},\end{aligned}$$

where $\tilde{q} := \int_0^t \left[\frac{1}{2}\left(|u_r|^2 \circ X_r\right) - p_r \circ X_r\right] dr.$

Then, if we denote $M_t := (\nabla X_t)^*$, it follows that

$$\begin{aligned}u_t \circ X_t = u(t, X_t) &= M_t^{-1} u_0 + M_t^{-1} \nabla \tilde{q} \\ &= (\nabla A_t\big|_{X_t})^* u_0 + (\nabla A_t\big|_{X_t})^* \nabla \tilde{q}.\end{aligned}$$

Finally we conclude

$$\begin{aligned}u_t &= (\nabla A_t)^* u_0 \circ A_t + (\nabla A_t)^* (\nabla \tilde{q})(A_t) \\ &= (\nabla A_t)^* u_0 \circ A_t + \nabla q,\end{aligned}$$

where $q := \tilde{q} \circ A$. Therefore, $u_t = \mathbb{P}\left[(\nabla A_t)^* u_0 \circ A_t\right]$.
($\Leftarrow$) For each j let $\left(Y^{j,n}\right)_{n\in\mathbb{N}}$ be a sequence on $C^1\left([0,T];\mathbb{R}^d\right)$ such that

$$Y^{j,n} \to Y^j, \text{ as } n \to \infty$$

with respect to α-Hölder norm. Then from Theorem 3 of [CF09]

$$X^n \to X$$

in α-Hölder norm.
Consider the transport equation

$$v_t^n = u_0 - \int_0^t (u_r \cdot \nabla) v_r^n dr - \sum_j \int_0^t \left(\sigma^j \cdot \nabla\right) v_r^n dY_r^{j,n}.$$

Now, from the Theorem 11.12 and Theorem 11.13 of [FV10] we have that

$$A^n \to A \quad \text{and} \quad \nabla A^n \to \nabla A, \tag{3.14}$$

in α-Hölder norm.
Set $v^n := u_0 \circ A^n$, where the initial function u_0 is assumed to be $C^2\left(\mathbb{T}^d;\mathbb{R}^d\right)$, then by (3.14) and Remark 3.1 we have

$$v^n \to u_0 \circ A =: v.$$

Furthermore, by the triangle inequality and Theorem 3.1 we have

$$\begin{aligned}
\|\nabla v^n - \nabla v\|_\alpha &= \|Du_0(A^n)\nabla A^n - Du_0(A)\nabla A\|_\alpha \\
&\leqslant \|Du_0(A^n)\nabla A^n - Du_0(A)\nabla A^n\|_\alpha + \|Du_0(A)\nabla A^n - Du_0(A)\nabla A\|_\alpha \\
&= \|\left(Du_0(A^n) - Du_0(A)\right)\nabla A^n\|_\alpha + \|Du_0(A)\left(\nabla A^n - \nabla A\right)\|_\alpha \\
&\leqslant \|Du_0(A^n) - Du_0(A)\|_\alpha \|\nabla A^n\|_\alpha + \|Du_0(A)\|_\alpha \|\nabla A^n - \nabla A\|_\alpha \\
&\leqslant \|\nabla A^n\|_\alpha \left(\|Du_0\|_\infty + \|D^2 u_0\|_\infty\right) \|A^n - A\|_\alpha + \|Du_0(A)\|_\alpha \|\nabla A^n - \nabla A\|_\alpha
\end{aligned}$$

If we let $n \to \infty$, we deduce from (3.14) and the continuity of Du_0, that ∇v^n converge in α-Hölder norm to ∇v. Hence we get that v is a solution of

$$v_t = u_0 - \int_0^t (u_r \cdot \nabla) v_r dr - \sum_j \int_0^t \left(\sigma^j \cdot \nabla\right) v_r dY_r^j. \tag{3.15}$$

Therefore, A and v satisfy the equation (3.15) with initial conditions x and u_0, respectively. Thus, we conclude that $\mathcal{D}A = 0$ and $\mathcal{D}v = 0$.

Since u satisfies (3.10) there exists a function q such that

$$u = (\nabla A)^* v - \nabla q.$$

Then by (3.7) and (3.8) we have

$$\begin{aligned}
\mathcal{D}u &= \mathcal{D}\left[(\nabla A)^* v - \nabla q\right] \\
&= \mathcal{D}\left[(\nabla A)^* v\right] - \mathcal{D}\nabla q \\
&= \left[\mathcal{D}(\nabla A)^*\right] v + (\nabla A)^* \mathcal{D}v - \mathcal{D}\nabla q \\
&= \left[\nabla \mathcal{D}A - (\nabla u)^* (\nabla A)\, dt - \left(\nabla \sigma^j\right)^* \nabla A dY_t^j\right]^* v \\
&\quad - \nabla \mathcal{D}q + (\nabla u)^* \nabla q dt + \left(\nabla \sigma^j\right)^* \nabla q dY_t^j.
\end{aligned}$$

Hence, after a calculation and a rearrangement of the terms, we get

$$\begin{aligned}
\mathcal{D}u &= -(\nabla u)^* (\nabla A)^* v dt + (\nabla u)^* \nabla q dt - \nabla \mathcal{D}q \\
&\quad - \left(\nabla \sigma^j\right)^* \left[(\nabla A)^* v - \nabla q\right] dY_t^j \\
&= -(\nabla u)^* u dt - \nabla \mathcal{D}q - \left(\nabla \sigma^j\right)^* u dY_t^j \\
&= -\nabla \frac{|u|^2}{2} dt - \nabla \mathcal{D}q - \left(\nabla \sigma^j\right)^* u dY_t^j.
\end{aligned}$$

Then,

$$\begin{aligned}
u(t,x) &= u_0(x) - \int_0^t (u\cdot\nabla)u(r,x)dr - \int_0^t (\sigma^j\cdot\nabla)u(r,x)dY_r^j - \nabla\int_0^t \frac{|u|^2}{2}(r,x)dr \\
&\quad - \nabla \int_0^t \mathcal{D}q_r - \int_0^t (\left(\nabla\sigma^j\right)^* u)(r,x)dY_r^j \\
&= u_0(x) - \int_0^t (u\cdot\nabla)u(r,x)dr - \int_0^t \mathcal{L}^*_{\sigma^j} u(r,x)dY_r^j - \int_0^t \nabla p(r,x)dt,
\end{aligned}$$

where formally we have

$$p = \frac{|u|^2}{2} + \partial_t q + u\cdot\nabla q + \sigma^j\cdot\nabla q\ \partial_t Y_t^j.$$

Therefore we conclude that u is solution of the Euler equation (3.1).

□

3.3 An Existence Theorem

3.3.1 Decomposition of the Flow

We use the idea of [FL19a] to decompose the Young flow in the system (3.9)-(3.10). More precisely, we consider the Young differential equation (YDE) without drift:

$$d\varphi_t = \sum_j \sigma^j(\varphi_t)\, dY_t^j, \quad \varphi_0 = I, \tag{3.16}$$

where I is the identity diffeomorphism of $\mathbb{T}^d$. Under the assumption that (3.5) and $r > \frac{d}{2}+3$, the above equation generates a Young flow $\{\varphi_t\}_{t\geqslant 0}$ of $C^{\lfloor r\rfloor,\beta}-$diffeomorphisms on $\mathbb{T}^d$, where $\beta \in (0,\theta)$.

For a given vector field $u : [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$, we define

$$\tilde{u}_t(x) = \left[\left(\varphi_t(\cdot)^{-1}\right)_* u_t(\cdot)\right](x) \tag{3.17}$$

which is the pull-back of the field $u_t(\cdot)$ by the Young flow $\{\varphi_t(\cdot)\}_{t\geqslant 0}$. If we denote by $K_t(x) = (\nabla\varphi_t(x))^{-1}$, i.e., the inverse of the Jacobi matrix, then

$$\tilde{u}_t(x) = K_t(x)\, u_t(\varphi_t(x)) . \tag{3.18}$$

From this expression we see that if $u \in C^0([0,T], H^r)$ a.s., then one also has a.s. $\tilde{u} \in C^0([0,T], H^r)$. Now we consider the ODE

$$\dot{Z}_t = \tilde{u}_t(Z_t), \quad Z_0 = I. \tag{3.19}$$

Applying Theorem 3.4 of [CLR23] we have

$$X_t = \varphi_t \circ Z_t \tag{3.20}$$

is the flow of $C^{\lfloor r\rfloor,\beta}-$diffeomorphisms associated to the YDE in (3.9)-(3.10).

From the above discussions, we can observe that an advantage of using this decomposition is the fact that it yields a deterministic fixed-point problem, without needing to resort to Young integration.

3.3.2 Contraction

The aim of the rest of this chapter is to show the existence of a fixed point within the closed ball in $\Sigma_s(T)$ for a mapping (denoted by Su as described in equation (3.21) below). The proof is based on the decomposition of flow (3.20) and Theorem 2.1.

Theorem 3.4. *Assume $r \geqslant s+1$. If $d \geqslant 2$, $s > \frac{d}{2}+1$ and $u_0 \in H^s$ is divergence free then there exists $T > 0$, such that (3.10) has solution $u \in \Sigma_s(T)$.*

Proof. For $u \in \Sigma_s$, we consider the following system:

$$\begin{cases} \tilde{u}_t(x) = \left[\left(\varphi_t^{-1}\right)_* u_t\right](x), \\ \dot{Z}_t = \tilde{u}_t(Z_t), \quad Z_0 = I, \\ X_t = \varphi_t(Z_t), \\ Su_t(x) = \mathbb{P}\left[\left(\nabla X_t^{-1}\right)^* u_0\left(X_t^{-1}\right)\right](x), \end{cases} \tag{3.21}$$

where φ_t is the solution of the equation (3.16).

Analogously to the previous chapter, we have that there exist positive constants $C_0 := C_0(t, d, s, \varphi)$, $C_1 := C_1(t, d, s, \varphi)$, $C_2 := C_2(t, d, s, \varphi)$ such that

$$\|\varphi^{-1}\|_s \leqslant C_1; \tag{3.22}$$

$$\|\tilde{u}\|_s = \| \left(\nabla\varphi^{-1}\right)(\varphi)u(\varphi)\|_s \leqslant C_0\|u\|_s; \tag{3.23}$$

$$\|\eta \circ \varphi^{-1}\|_s \leqslant C_2\|\eta\|_s \text{ and } \|v \circ \varphi^{-1}\|_s \leqslant C_2\|v\|_s. \tag{3.24}$$

Fix $s > \frac{d}{2} + 1$ and let C_3, C_4 be the constants in (2.17), (2.20) (from Lemmas 2.3 and 2.4), respectively. Fix $\|u_0\|_s < M$ and $T > 0$ so that

$$C\|u_0\|_s \exp\left(C_4C_0TM\right)\left[\frac{C_2^2}{C_1C_4}\left(\exp\left(C_4C_0TM\right) - 1\right) + 1\right] \leqslant M,$$

where C is a constant to be defined later.
Let $u \in B_M(T)$ be a divergence free vector field and let $\eta_t = Z_t^{-1} - x$ be unique solution of (2.27) with initial data $\eta_0 = 0$. Let $v_t = u_0(Z_t^{-1})$ be the unique solution of (2.24) for initial data $v_0 = u_0$.

From (3.23) we have

$$\|\tilde{u}\|_{\Sigma_s} = \sup_{t\in[0,T]} \|\tilde{u}\|_s \leqslant C_0M,$$

Then by Lemma 2.4,

$$\|v(t)\|_s \leqslant \|u_0\|_s \exp\left(C_4t\|\tilde{u}\|_{\Sigma_s}\right) \leqslant \|u_0\|_s \exp\left(C_0C_4TM\right) \tag{3.25}$$

and

$$\|\eta(t)\|_s \leqslant \frac{1}{C_4}\left(\exp\left(C_0C_4TM\right) - 1\right). \tag{3.26}$$

Hence, with $C = C_1C_2C_3$, (the constants C_1, C_2, C_3 are given by (3.22), (3.24) and Lemma 2.3, respectively)

$$\begin{aligned}
\|Su(t)\|_s &\leqslant \|\mathbb{P}\left[(\nabla(\eta \circ \varphi^{-1}))^* v(\varphi^{-1})\right]\|_s + \|\mathbb{P}\left[(\nabla\varphi^{-1})^* v(\varphi^{-1})\right]\|_s \\
&\leqslant C_3\|\eta \circ \varphi^{-1}\|_s\|v(\varphi^{-1})\|_s + C_3\|\varphi^{-1}\|_s\|v(\varphi^{-1})\|_s \\
&\leqslant C_3C_2^2\|\eta\|_s\|v\|_s + C_3C_1C_2\|v\|_s \\
&= C_3C_2\|v\|_s\left(C_2\|\eta\|_s + C_1\right),
\end{aligned}$$

here the third inequality follows from (3.22) and (3.24). Thus by (3.25) and (3.26)

$$\begin{aligned}
\|Su(t)\|_s &\leqslant \|u_0\|_s \exp\left(C_4C_0tM\right)\left[\frac{C_3C_2^2}{C_4}\left(\exp\left(C_4C_0tM\right) - 1\right) + C_3C_1C_2\right] \\
&= C\|u_0\|_s \exp\left(C_4C_0tM\right)\left[\frac{C_2^2}{C_1C_4}\left(\exp\left(C_4C_0tM\right) - 1\right) + 1\right] \leqslant M
\end{aligned} \tag{3.27}$$

for all $t \in [0,T]$. Hence $S : B_M(T) \to B_M(T)$.

By Lemmas 2.3, 2.4 and 2.5 we deduce

$$\begin{aligned}
&\left\|\mathbb{P}\left[\left(\nabla\left(\eta_1\circ\varphi^{-1}\right)\right)^* v_1\left(\varphi^{-1}\right)-\left(\nabla\left(\eta_2\circ\varphi^{-1}\right)\right)^* v_2\left(\varphi^{-1}\right)\right]\right\|_{L^2}\\
&\leqslant C_3M\left[\left\|\eta_1\circ\varphi^{-1}-\eta_2\circ\varphi^{-1}\right\|_{L^2}+\left\|v_1\circ\varphi^{-1}-v_2\circ\varphi^{-1}\right\|_{L^2}\right]\\
&= C_3M\left[\|\eta_1-\eta_2\|_{L^2}+\|v_1-v_2\|_{L^2}\right]\\
&\leqslant C_3M\left[\left(C_5\|\eta_1+\eta_2\|_{\Sigma_s}+1\right)\|\tilde u_1-\tilde u_2\|_{\Sigma_0}t+C_5\|v_1-v_2\|_{\Sigma_s}\|\tilde u_1-\tilde u_2\|_{\Sigma_0}t\right]\\
&\leqslant C_3M\left[\left(\frac{2C_5}{C_4}\left(\exp\left(C_4C_0tM\right)-1\right)+1\right)t\|\tilde u_1-\tilde u_2\|_{\Sigma_0}\right.\\
&\qquad\left.+2C_5Mt\|u_0\|_s\exp\left(C_4C_0tM\right)\|\tilde u_1-\tilde u_2\|_{\Sigma_0}\right]\\
&\leqslant C_3M\left[\left(\frac{2C_5}{C_4}\left(\exp\left(C_4C_0TM\right)-1\right)+1\right)t+2C_5t\|u_0\|_s\exp\left(C_4C_0TM\right)\right]\|\tilde u_1-\tilde u_2\|_{\Sigma_0},
\end{aligned}$$

and

$$\begin{aligned}
\left\|\mathbb{P}\left[\left(\nabla\varphi^{-1}\right)^*\left(v_1(\varphi^{-1})-v_2(\varphi^{-1})\right)\right]\right\|_{L^2} &\leqslant C_3M\left\|v_1(\varphi^{-1})-v_2(\varphi^{-1})\right\|_{L^2}\\
&= C_3M\|v_1-v_2\|_{L^2}\\
&\leqslant C_3C_5M\|v_1-v_2\|_{\Sigma_s}\|\tilde u_1-\tilde u_2\|_{\Sigma_0}t\\
&\leqslant 2C_3C_5Mt\|u_0\|_s\exp\left(C_4tC_0M\right)\|\tilde u_1-\tilde u_2\|_{\Sigma_0},
\end{aligned}$$

where C_3, C_4, C_5 are the constants from Lemmas 2.3, 2.4 and 2.5, respectively.

Now, we will show that the map S is a contraction on $B_M(T)$ in the L^2-norm if T is sufficiently small. For u_1, $u_2 \in B_M(T)$ we construct v_i and η_i from u_i as above for $i = 1, 2$ with $v_1(\cdot,0) = v_2(\cdot,0) = u_0$. Then by the inequalities above

$$\begin{aligned}
\|Su_1-Su_2\|_{L^2} &\leqslant \left\|\mathbb{P}\left[\left(\nabla\left(\eta_1\circ\varphi^{-1}\right)\right)^* v_1\left(\varphi^{-1}\right)-\left(\nabla\left(\eta_2\circ\varphi^{-1}\right)\right)^* v_2\left(\varphi^{-1}\right)\right]\right\|_{L^2}\\
&\qquad+\left\|\mathbb{P}\left[\left(\nabla\varphi^{-1}\right)^*\left(v_1(\varphi^{-1})-v_2(\varphi^{-1})\right)\right]\right\|_{L^2}\\
&\leqslant C_3M\left(\frac{2C_5}{C_4}\left(\exp\left(C_4C_0TM\right)-1\right)+1\right)t\|\tilde u_1-\tilde u_2\|_{\Sigma_0} \qquad (3.28)\\
&\qquad+4C_3C_5Mt\|u_0\|_s\exp\left(C_4C_0TM\right)\|\tilde u_1-\tilde u_2\|_{\Sigma_0}\\
&\leqslant 2TMC_3C_6\left(C_5\left(2\|u_0\|_s+\frac{1}{C_4}\right)\exp\left(C_4C_0TM\right)+\frac{1}{2}-\frac{C_5}{C_4}\right)\|u_1-u_2\|_{\Sigma_0}\\
&= C(u_0,M,T)\|u_1-u_2\|_{\Sigma_0},
\end{aligned}$$

where the third inequality follows from a change of variables and Hölder's inequality, and

$$\begin{aligned}\|\tilde{u}_1-\tilde{u}_2\|_{\Sigma_0} &= \sup_{t\in[0,T]}\|\tilde{u}_1-\tilde{u}_2\|_{L^2} = \sup_{t\in[0,T]}\left\|(\nabla\varphi)^{-1}u_1(\varphi)-(\nabla\varphi)^{-1}u_2(\varphi)\right\|_{L^2}\\ &= \sup_{t\in[0,T]}\left\|\left(\nabla\varphi^{-1}\right)(\varphi)\left(u_1-u_2\right)(\varphi)\right\|_{L^2}\\ &= \sup_{t\in[0,T]}\left\|\left(\nabla\varphi^{-1}\right)\left(u_1-u_2\right)\right\|_{L^2} \leqslant \sup_{t\in[0,T]}\left(\left\|\nabla\varphi^{-1}\right\|_{L^\infty}\|u_1-u_2\|_{L^2}\right)\\ &\leqslant C_6\|u_1-u_2\|_{\Sigma_0},\end{aligned}$$

where $C_6 := \sup_{t\in[0,T]}\left\|\nabla\varphi^{-1}\right\|_{L^\infty}$.

We observe that $C(u_0,M,T)$ is given by the formula

$$C(u_0,M,T) := 2T\left[C_5MC_3C_6\left(2\|u_0\|_s+\frac{1}{C_4}\right)\exp\left(C_4C_0TM\right)+C_3C_6M\left(\frac{1}{2}-\frac{C_5}{C_4}\right)\right]. \tag{3.29}$$

We observe that $\varphi\in C^0([0,T],C^{[r],\beta}(\mathbb{T}^d,\mathbb{T}^d))$ then this implies that C_0,C_1,C_2 and C_6 are limited uniformly in time. Then taking the supremum of (3.28) with respect to t and choosing $T>0$ small enough, we see that S is a contraction in the required sense. With the above preparations, the proof is the same as that in the Theorem 2.1. We conclude that S has a unique accumulation point u in the closure of B_M with respect to $\|\cdot\|_{\Sigma_0}$. Since $B_M(T)$ is convex and closed in Σ_s it is weakly closed, hence $u\in B_M$ is a fixed point of S. □

Finally, we will state and prove a α-Hölder regularity result for the fix point of the mapping (3.21).

Lemma 3.3. *Let u be the fix point of the contraction (3.21). Then $u\in C^\alpha([0,T];H^s)$. T here must be sufficiently small so that the constant C in (3.29) is less than 1.*

Proof. We use the notation $\varepsilon(t) := \eta_t\circ\varphi_t^{-1}$ and $w(t) := v_t\circ\varphi_t^{-1}$. Note that $\eta_t = Z_t^{-1}-x$ and $v_t = u_0\circ Z_t^{-1}$ are, from equation (3.20) and smoothness of u_0, at least twice differentiable. Now, by the equation (3.16) and the time reversal path defined in [CF09], we have that φ_t^{-1} is solution for

$$d\psi_t = \sum_j \sigma^j\left(\psi_t\right)dY^j_{t-r}, \tag{3.30}$$

where Y^j_{t-r} is a α-Hölder path with $r\in[0,t]$. Hence by the equation (3.30), $\varphi^{-1}\in C^\alpha([0,T];H^s)$. Then by Remark 3.1 we have that ε and w belong to $C^\alpha([0,T];H^s)$.

Now

$$\begin{aligned}
&\frac{\|u(t)-u(r)\|_s}{|t-r|^\alpha}\\
&=\frac{1}{|t-r|^\alpha}\left\|\mathbb{P}\left[(\nabla\varepsilon(t))^*w(t)+w(t)\right]-\mathbb{P}\left[(\nabla\varepsilon(r))^*w(r)+w(r)\right]\right\|_s\\
&=\frac{1}{|t-r|^\alpha}\left\|\mathbb{P}\left[(\nabla\varepsilon(t))^*(w(t)-w(r))+w(t)\right]+\mathbb{P}\left[(\nabla\varepsilon(t))^*w(r)\right]\right.\\
&\qquad\qquad\qquad\qquad\left.-\mathbb{P}\left[(\nabla\varepsilon(r))^*w(r)+w(r)\right]\right\|_s\\
&=\frac{1}{|t-r|^\alpha}\left\|\mathbb{P}\left[(\nabla\varepsilon(t))^*(w(t)-w(r))\right]+\mathbb{P}\left[(\nabla\varepsilon(t)-\nabla\varepsilon(r))^*w(r)\right]+\mathbb{P}\left[w(t)-w(r)\right]\right\|_s,
\end{aligned}$$

from Lemma 2.3 and Leray-Hodge projector properties, we have

$$\left\|\mathbb{P}\left[(\nabla\varepsilon(t))^*(w(t)-w(r))\right]\right\|_s\leqslant C_3\|\varepsilon(t)\|_s\|w(t)-w(r)\|_s,$$

$$\left\|\mathbb{P}\left[(\nabla\varepsilon(t)-\nabla\varepsilon(r))^*w(r)\right]\right\|_s\leqslant C_3\|w(r)\|_s\|\varepsilon(t)-\varepsilon(r)\|_s$$

and

$$\|\mathbb{P}[w(t)-w(r)]\|_s\leqslant\|w(t)-w(r)\|_s.$$

Hence, taking supremum over $0\leqslant s<t\leqslant T$ we have

$$\begin{aligned}
\sup_{0\leqslant r<t\leqslant T}\frac{\|u(t)-u(r)\|_s}{|t-r|^\alpha}&=|u|_{C^\alpha([0,T];H^s)}\\
&\leqslant C_3\sup_{0\leqslant r<t\leqslant T}\left[\|\varepsilon(t)\|_s\frac{\|w(t)-w(r)\|_s}{|t-r|^\alpha}+\|w(r)\|_s\frac{\|\varepsilon(t)-\varepsilon(r)\|_s}{|t-r|^\alpha}\right]\\
&\qquad+\sup_{0\leqslant r<t\leqslant T}\frac{\|w(t)-w(r)\|_s}{|t-r|^\alpha}\\
&=C_3|w|_{C^\alpha([0,T];H^s)}\sup_{0\leqslant r<t\leqslant T}\|\varepsilon(t)\|_s+C_3|\varepsilon|_{C^\alpha([0,T];H^s)}\sup_{0\leqslant r<t\leqslant T}\|w(r)\|_s\\
&\qquad+|w|_{C^\alpha([0,T];H^s)}\\
&\leqslant C\left[|w|_{C^\alpha([0,T];H^s)}\left(\|\varepsilon\|_{L^\infty([0,T];H^s)}+1\right)+|\varepsilon|_{C^\alpha([0,T];H^s)}\|w\|_{L^\infty([0,T];H^s)}\right]
\end{aligned}$$

Since $\varepsilon,w\in C^\alpha([0,T];H^s)$, the right-hand side is finite. Therefore $u\in C^\alpha([0,T];H^s)$. □

CHAPTER 4 THE ROUGH PATH CASE: LAGRANGIAN FORMULATION

In this chapter we consider the following rough incompressible Euler equation

$$du_t + (u_t \cdot \nabla u_t + \nabla p_t)\, dt + \mathcal{L}_\sigma^* u_t d\mathbf{Z}_t = 0, \tag{4.1}$$

where $\mathbf{Z}_t$ is a weakly geometric rough path. Here, again, we consider $\mathcal{L}_\sigma^* u := (\sigma \cdot \nabla)u + (\nabla\sigma)^* u$ ($\mathcal{L}$ is the Lie derivative defined as $\mathcal{L}_\sigma u = (\sigma \cdot \nabla)u - (u \cdot \nabla)\sigma$).

The main objective of this chapter is to show that the solution of the rough incompressible Euler equation (4.1) satisfies the Lagrangian formulation.

4.1 Introduction to rough paths

We will provide an overview of the theory of rough paths. We shall use freely concepts and notations of [FH20]. We invite the reader to consult [CCH22], [FV10] for more thorough expositions.
Rough path theory, originally developed by Terry Lyons in his seminal work [Lyo98] in 1998, is an analytic theory of differential equations driven by multidimensional irregular paths (e.g. Brownian motion). Its development is partly motivated by the pathwise integration of Hölder paths.

Indeed, let $X : [0, T] \to V$, $Z : [0, t] \to U$, $f : V \to L(U, V)$, and consider the equation

$$\begin{cases} dX_t = f(X_t)dZ_t, \quad t \in [0, T] \\ X_0 = x \in V, \end{cases}$$

where the path driving the equation is non-differentiable. Such a restriction is not imposed for the pleasure of generalization, but rather a reality. Since differential equations are often interpreted and solved in integral form, it is natural to first address the question of constructing the integral

$$I_t(X,Z) = \int_0^t f(X_r)dZ_r,$$

where f be a smooth function, and X, Z come from a suitable class of continuous paths that is at least rich enough to include generic Brownian sample paths. In the context of integration, one might suggest that for general continuous paths X, Z, one could simply take an approximating sequence of smooth paths with $X^n \to X$ and $Z^n \to Z$, and then define $I_t(X,Z)$ by the limit $\lim_{n\to\infty}\int_0^t f(X_r^n)dZ_r^n$, since each of the approximations $\int_0^t f(X_r^n)dZ_r^n$ is already well understood. Although this is in principle possible, the problem is knowing in which topology to take the limit. A natural candidate of path topology is the uniform topology. However, the following negative example shows that it fails to be continuous with respect to the uniform topology.

Example 4.1. *For each $n \geqslant 1$, define $X_t^n, Z_t^n : [0,2\pi] \to \mathbb{R}$ by*

$$X_t^n = \frac{1}{n}\text{sin} n^2 t \;\; \text{and} \;\; Z_t^n = \frac{1}{n}\cos n^2 t$$

It is clear that X^n, Z^n both converge to zero uniformly. However, from explicit calculation one finds that

$$I_t(X^n,Z^n) = \int_0^t X_r^n dZ_r^n = \frac{t}{2} + \frac{1}{4n^2}\text{sin} 2n^2 t,$$

which does not converge to the zero path.

As suggested by Young's integration theory, the α-Hölder topology with $\alpha \in (1/2,1]$ does work. However, the completion of smooth paths with respect to this topology is not rich enough to at least cover the Brownian motion case. Unfortunately, the following negative result (cf. Proposition 1.1. of [FH20]) indicates that there is not a clever choice of path topology which on the one hand ensures the continuity of I_t and on the other hand is weak enough to contain Brownian sample paths in the completion of smooth paths.

Proposition 4.1. *There exists no separable Banach space $E \subset C([0,T];\mathbb{R}^d)$ with the following properties:*

1. *Sample paths of Brownian motions lie in E almost surely.*
2. *The map $(X,Z) \mapsto I_\cdot(X,Z)$ defined on the smooth functions extends to a continuous map from $E \times E$ into the space of continuous functions on $[0,T]$.*

Therefore, if this strategy is to work we would need a topology considerably stronger than that of uniform convergence.

The following formal calculation reveals why paths need to be enhanced to include higher order structure that is not encoded in the original trajectory Z. Let us assume for now that X_r has the form $X_r = Z_r$ being an α-Hölder continuous path for some $\alpha \in (0,1]$. By a formal Taylor expansion of f, we have that

$$f(Z_r) = f(Z_s) + Df(Z_s)(Z_r - Z_s) + \cdots$$

Integrating with respect to Z, we obtain

$$\int_s^t f(Z_r)dZ_r = f(Z_s)(Z_t - Z_s) + Df(Z_s)\int_s^t (Z_r - Z_s) \otimes dZ_r + \cdots$$

It turns out that, provided $\alpha > 1/3$, the higher order terms we have omitted in the above expansion vanish upon applying $\lim\limits_{|\pi|\to 0} \sum\limits_{[s,t]\in\pi}$, where the limit is taken over any partition π of the interval $[0,T]$. In fact, if $\alpha > 1/2$, then one can show that

$$\lim_{|\pi|\to 0} \sum_{[s,t]\in\pi} \int_s^t (Z_r - Z_s) \otimes dZ_r = 0. \tag{4.2}$$

In this case we simply obtain

$$\int_0^T f(Z_r)dZ_r = \lim_{|\pi|\to 0} \sum_{[s,t]\in\pi} \int_s^t f(Z_r)dZ_r = \lim_{|\pi|\to 0} \sum_{[s,t]\in\pi} f(Z_s)(Z_t - Z_s),$$

which we recognise as the definition of the Young integral of $f(Z)$ against Z.

However, when $\alpha \leqslant 1/2$ the equality (4.2) does not necessarily hold, and this "second order" terms remains:

$$\begin{aligned}\int_0^T f(Z_r)dZ_r &= \lim_{|\pi|\to 0} \sum_{[s,t]\in\pi} \int_s^t f(Z_r)dZ_r \\ &= \lim_{|\pi|\to 0} \sum_{[s,t]\in\pi} \left(f(Z_s)(Z_t - Z_s) + Df(Z_s)\int_s^t (Z_r - Z_s) \otimes dZ_r \right).\end{aligned}$$

This suggest that, for $\alpha \in (1/3, 1/2]$, in order to compute the integral of $f(Z)$ against Z, we need as inputs both the path increments $Z_t - Z_s$ as well as the integrals $\int_s^t (Z_r - Z_s) \otimes dZ_r$ for each pair of times $s < t$. We therefore make the definition:

$$\int_s^t (Z_r - Z_s) \otimes dZ_r := \mathbb{Z}_{st}. \tag{4.3}$$

By a *rough path*, we mean the pair $(Z, \mathbb{Z})$. But to be clear, when we come later to the proper definition of a rough path we will abandon the equality in (4.3), and instead provide analytical and algebraic conditions which must be satisfied by $\mathbb{Z}$. One should therefore just think of (4.3) as motivation for the "information" encoded by $\mathbb{Z}$.

Fix a time interval $[0,T]$. Assume that $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. Let U and V be Banach spaces. We follow the construction of [FH20](Chapters 2, 4) to introduce the basic framework of the theory of rough paths. We will denote

$$\Delta_T := \left\{ (s,t) \in [0,T]^2 : 0 \leqslant s \leqslant t \leqslant T \right\}$$

and

$$\Delta_T^2 := \left\{(s,\theta,t) \in [0,T]^3 : 0 \leqslant s \leqslant \theta \leqslant t \leqslant T\right\}.$$

Given a path $X : [0,T] \to V$ and $s,t \in [0,T]$ we write $X_{st} = X_t - X_s$.

Definition 4.1. *$\mathcal{C}_2^\alpha(\Delta_T;U)$ is the space of functions on Δ_T taking values in U and such that the following α-Hölder seminorm is finite*

$$|\varphi|_\alpha := \sup_{s,t} \frac{|\varphi_{st}|}{|t-s|^\alpha}$$

A V-valued rough path, introduced below, is defined as a pair of a rough function and a double integral term.

Definition 4.2. *The space of rough paths $\mathscr{C}^\alpha([0,T];V)$ is the collection of pairs $\mathbf{Z} = (Z,\mathbb{Z})$ satisfying the following properties:*

(i) $Z \in \mathcal{C}^\alpha([0,T];V)$.

(ii) $\mathbb{Z} \in \mathcal{C}_2^{2\alpha}(\Delta_T;V\otimes V)$, *where $V\otimes V$ is the tensor product space equipped with the projective norm.*

(iii) $(Z,\mathbb{Z})$ *satisfies Chen's relation: for all* $(s,\theta,t) \in \Delta_T^2$,

$$\mathbb{Z}_{st} - \mathbb{Z}_{s\theta} - \mathbb{Z}_{\theta t} = Z_{s\theta} \otimes Z_{\theta t}. \tag{4.4}$$

Definition 4.3. *Let $\mathbf{Z} = (Z,\mathbb{Z})$ be an α-Hölder rough paths. The bracket of $\mathbf{Z}$ is defined by*

$$[\mathbf{Z}]_{st} = Z_{st} \otimes Z_{st} - 2Sym(\mathbb{Z}_{st}),$$

where $Sym(\mathbb{Z}_{st}) := \frac{1}{2}(\mathbb{Z}_{st} + \mathbb{Z}_{st}^*)$ denotes the symmetric part of $\mathbb{Z}_{st}$. If $[\mathbf{Z}] = 0$, we say that $\mathbf{Z}$ is a weak geometric rough path. We denote the set of all α-Hölder weakly geometric rough paths with respect to Z by $\mathscr{C}_g^\alpha([0,T];V)$.

Given a path $Z \in \mathcal{C}^\alpha([0,T];V)$. We define rough paths controlled by Z as follows:

Definition 4.4. *Let $Z \in \mathcal{C}^\alpha([0,T];V)$. We say that $X \in \mathcal{C}^\alpha([0,T];U)$ is controlled by Z if there exists $X' \in \mathcal{C}^\alpha([0,T];L(V;U))$ so that the remainder term $R^X \in \mathcal{C}_2^{2\alpha}(\Delta_T;U)$ given implicitly through the relation*

$$X_{st} = X'_s(Z_{st}) + R^X_{st}. \tag{4.5}$$

This defines the space of controlled rough paths $(X,X') \in \mathscr{D}_Z^{2\alpha}([0,T];U)$. The path X' is called a Gubinelli derivative of X with respect to Z.

Note that, given paths X and Z, the Gubinelli derivative X', when it exists, is not unique in general. For instance, if it happens that $Z \in C^{2\alpha}$ and $X \in C^{2\alpha}$, then any continuous path X' would satisfy (4.5) with $\|R^X\|_{2\alpha} < \infty$. On the other hand, as shown in the Chapter 6 of [FH20], if Z is far from smooth, i.e. genuinely rough in all directions, then X' is uniquely determined by X. More precisely, suppose that we are in the one dimensional case $d = 1$, and for $0 \leqslant s \leqslant T$

$$\overline{\lim_{t\to s^+}} \frac{|Z_{st}|}{|t-s|^{2\alpha}} = \infty. \tag{4.6}$$

The X' is uniquely determined. Indeed, it is given by the equality

$$X'_s = \lim_{t\to s^+} \frac{X_{st}}{Z_{st}}. \tag{4.7}$$

In fact, by (4.6)

$$\frac{|R^X_{st}|}{|Z_{st}|} \leqslant \|R^X\|_{2\alpha} \frac{|t-s|^{2\alpha}}{|Z_{st}|} \to 0,$$

so that, by using $R^X_{st} = X_{st} - X'_s Z_{st}$, the limit in (4.7) exists and is equal to X'_s.

In [FH20] the authors say that a rough path which verifies (4.6) is a "really rough" path. In fact, this condition is really necessary for getting uniqueness: suppose that the limit in (4.6) is finite uniformly. Then $Z \in C^{2\alpha}$ and consequently, for every $r \in \mathbb{R}$, we have the decomposition $X_{st} = (X'_s + r)Z_{st} + (R^X_{st} - rZ_{st})$ and $R^X_{st} - rZ_{st}$ is a "good remainder". This means that $X'_s + r$ may also be used as a Gubinelli derivative.

With an abuse of notations, we sometimes write $X \in \mathscr{D}^{2\alpha}_Z([0,T];U)$ instead of $(X, X') \in \mathscr{D}^{2\alpha}_Z([0,T];U)$.
Suppose that $Z \in \mathcal{C}^\alpha([0,T];V)$ and $(X, X') \in \mathscr{D}^{2\alpha}_Z(L(V;U))$. Then X' takes values in $L(V; L(V;U))$, which can be identified with $L(V \otimes V; U)$ via

$$\Phi(x)(y) = \Phi(x \otimes y),$$

where $\Phi \in L(V, L(V,U))$ and $x, y \in V$.
The next theorem defines the rough integral of a controlled path against the rough path $\mathbf{Z} = (Z, \mathbb{Z})$.

Theorem 4.1. *Let $\mathbf{Z} = (Z, \mathbb{Z}) \in \mathscr{C}^\alpha([0,T];V)$. Suppose that $(X, X') \in \mathscr{D}^{2\alpha}_Z([0,T]; L(V;U))$ and denote $\Xi_{t_i t_{i-1}} := X_{t_{i-1}}(Z_{t_{i-1}t_i}) + X'_{t_{i-1}}(\mathbb{Z}_{t_{i-1}t_i})$. Then the following "compensated Riemann-Stieltjes sum"*

$$\sum_{i=1}^n \Xi_{t_{i-1}t_i}, \tag{4.8}$$

converges as $|\pi| \to 0$, where $\pi = (s = t_1 < t_2 < \cdots < t_n)$ and $|\pi| = \max_{1\leqslant i\leqslant n} |t_i - t_{i-1}|$. Denote by $\mathcal{I}_{st}(\Xi)$ the limit of (4.8). Then, $\mathcal{I}_{st}(\Xi)$ is additive, that is $\mathcal{I}_{st} = \mathcal{I}_{s\theta} + \mathcal{I}_{\theta t}$ for any $(s, \theta, t) \in \Delta^2_T$. Moreover, the following estimate is satisfied for all $(s,t) \in \Delta_T$,

$$\|\mathcal{I}_{st}(\Xi) - \Xi_{st}\|_U \leqslant C \left(|Z|_\alpha |R^X|_{2\alpha} + |\mathbb{Z}|_{2\alpha} |X'|_\alpha \right) |t-s|^{3\alpha}, \tag{4.9}$$

where C is a constant depending only on α. By definition, the rough integral of X against $\mathbf{Z} = (Z, \mathbb{Z})$ is defined as follows, for all $(s,t) \in \Delta_T$,

$$\int_s^t X_r d\mathbf{Z}_r := \mathcal{I}_{st}(\Xi). \tag{4.10}$$

Similarly we can define the rough integral $\int_s^t X_r \otimes d\mathbf{Z}_r \in L(V;U)$, for any $\mathbf{Z} = (Z, \mathbb{Z}) \in \mathscr{C}^\alpha([0,T];V)$ and $(X, X') \in \mathscr{D}_Z^{2\alpha}([0,T]; L(V;U))$. Theorem 4.1 can be proved by using the Sewing Lemma.

Theorem 4.2 (Sewing Lemma)**.** *Let $\alpha \in (0,1]$, and let $\Xi \in \mathcal{C}_2^\alpha(\Delta_T; U)$. Suppose there exist $C > 0$ and $\gamma > 1$ such that the following inequality holds:*

$$\|\delta\Xi(s,\theta,t)\|_U := \|\Xi_{st} - \Xi_{s\theta} - \Xi_{\theta t}\|_U \leqslant C|t-s|^\gamma,$$

for any $(s,\theta,t) \in \Delta_T^2$. Then there exists a unique (up to additive constant) function $\mathcal{I}(\Xi) \in \mathcal{C}^\alpha([0,T];V)$, such that the following inequality holds

$$\|\mathcal{I}_{st}(\Xi) - \Xi_{st}\|_U = \|\mathcal{I}_t(\Xi) - \mathcal{I}_s(\Xi) - \Xi_{st}\|_U \leqslant (1-2^{1-\gamma})^{-1} C|t-s|^\gamma.$$

Moreover, $\mathcal{I}_{st}(\Xi)$ can be represented as follows,

$$\mathcal{I}_{st}(\Xi) = \lim_{|\pi|\to 0} \sum_{k=1}^n \Xi_{t_{k-1}t_k},$$

where $\pi = (s = t_0 < t_1 < \cdots < t_n = t)$ and the limit is independent of the choice of π.

Proof of Theorem 4.1. For $s \leqslant \theta \leqslant t$. We have

$$\begin{aligned}
\delta\Xi(s,\theta,t) &= \Xi_{st} - \Xi_{s\theta} - \Xi_{\theta t} \\
&= X_s Z_{st} - X_s Z_{s\theta} - X_\theta Z_{\theta t} + X'_s \mathbb{Z}_{st} - X'_s \mathbb{Z}_{s\theta} - X'_\theta \mathbb{Z}_{\theta t} \\
&= X_s Z_{\theta t} - X_\theta Z_{\theta t} + X'_s(\mathbb{Z}_{st} - \mathbb{Z}_{s\theta}) - X'_\theta \mathbb{Z}_{\theta t} \\
&= -X_{s\theta} Z_{\theta t} + X'_s(\mathbb{Z}_{\theta t} + Z_{s\theta} \otimes Z_{\theta t}) - X'_\theta \mathbb{Z}_{\theta t} \\
&= (-X_{s\theta} + X'_s Z_{s\theta}) Z_{\theta t} - X'_{s\theta} \mathbb{Z}_{\theta t} \\
&= R^X_{s\theta} Z_{\theta t} - X'_{s\theta} \mathbb{Z}_{\theta t},
\end{aligned}$$

and hence

$$\|\delta\Xi(s,\theta,t)\|_U = \|R^X_{s\theta} Z_{\theta t} - X'_{s\theta}\mathbb{Z}_{\theta t}\|_U \leqslant (|R^X|_{2\alpha}|Z|_\alpha + |X'|_\alpha |\mathbb{Z}|_{2\alpha})|t-s|^{3\alpha}.$$

Since $3\alpha > 1$, it follows from the Sewing lemma that there exists a α-Hölder continuous path $\mathcal{I}(\Xi) =: \int_0^\cdot X_r d\mathbf{Z}_r$ with the desired properties. ☐

The next proposition stablishes the integral of a controlled path with respect to another controlled path, see [FH20]. The proof is a consequence of the Sewing Lemma.

Proposition 4.2. *Suppose that* $\mathbf{Z} = (Z, \mathbb{Z}) \in \mathscr{C}^{\alpha}([0,T]; W)$ *and* $(Y, Y') \in \mathscr{D}_Z^{2\alpha}([0,T]; L(V;U))$. *Let* $(X, X') \in \mathscr{D}_Z^{2\alpha}([0,T]; V)$. *Then the limit*

$$\int_0^t Y_r dX_r = \lim_{|\pi|\to 0} \sum_{i=1}^{n} \left[Y_{t_{i-1}} X_{t_{i-1}t_i} + Y'_{t_{i-1}} \cdot X'_{t_{i-1}} \mathbb{Z}_{t_{i-1}t_i} \right] \tag{4.11}$$

there exists for all $t \in [0,T]$, *where* $\pi = (s = t_1 < t_2 < \cdots < t_n)$ *and* $|\pi| = \max_{1\leqslant i\leqslant n} |t_i - t_{i-1}|$. *We call this limit the integral of the controlled path* (Y, Y') *with respect to the controlled path* (X, X'). *Also the following estimative holds for all* $s \leqslant t$,

$$\left\| \int_s^t Y_r dX_r - Y_s X_{st} - Y'_s \cdot X'_s \mathbb{Z}_{st} \right\|_U$$
$$\leqslant C \left\{ \|R^Y\|_{2\alpha} \|X\|_\alpha + \|Y'\|_\infty \|Z\|_\alpha \left(\|R^X\|_{2\alpha} + \|X'\|_\alpha \|Z\|_\alpha \right) + \|Y'X'\|_\alpha \|\mathbb{X}\|_{2\alpha} \right\} |t-s|^{3\alpha},$$

where the constant C *depends only of* α.

We observe that the integrals (4.10) and (4.11) are controlled paths, in fact

$$\Big(\int_0^{\cdot} X_r d\mathbf{Z_r}, X_r\Big) \in \mathscr{D}_Z^{2\alpha}([0,T]; U) \text{ and } \Big(\int_0^{\cdot} Y_r dX_r, YX'\Big) \in \mathscr{D}_Z^{2\alpha}([0,T]; U).$$

Itô's formula is among the most valuable outcomes in stochastic calculus, serving as the stochastic counterpart to the chain rule and the fundamental theorem of calculus. In the setting of rough paths, we have the following analogous result.

Proposition 4.3. *Let* $\mathbf{Z} = (Z, \mathbb{Z}) \in \mathscr{C}_g^{\alpha}([0,T]; V)$ *be a weakly geometric rough path, and suppose that* $(Y, Y') \in \mathscr{D}_Z^{2\alpha}(L(V;U))$ *be a controlled path. Suppose further that*

$$Y_t = Y_0 + \int_0^t Y'_r d\mathbf{Z}_r$$

for all $t \in [0,T]$. *Let* $F \in C^3$, *then*

$$F(Y_t) = F(Y_0) + \int_0^t DF(Y_r) Y'_r d\mathbf{Z}_r$$

The following result is a particular case of Theorem 4.1. of [CCH22] applied to weakly geometric rough paths, and stablishes the Itô-Wentzell's formula for rough paths.

Theorem 4.3 (Itô-Wentzell's formula)**.** *Let* $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, $\mathbf{Z} = (Z, \mathbb{Z}) \in \mathscr{C}_g^{\alpha}([0,T]; V)$ *and* $(h, h') \in C\left(U, \mathscr{D}_Z^{2\alpha}([0,T], L(V;W))\right)$. *We assume that*

1. $h : [0,T] \times U \to L(V, W)$ *is continuous and twice differentiable in* U.

2. $\nabla h : [0,T] \times U \to L(U, L(V, W))$ *is continuous and differentiable in* U.

3. *For each* $t \in [0,T]$, $h'(t, \cdot) \in C^1(U, L(V, L(V, W)))$.

4. $\left(\nabla h, (\nabla h)'\right) \in C\left(U, \mathscr{D}_Z^{2\alpha}([0,T], L(U, L(V;W)))\right)$.

Let

$$g(t,x) = g(0,x) + \int_0^t h(r,x)d\mathbf{Z}_r.$$

Assume that $g : [0,T]\times U \to W$ *is continuous and twice differentiable in* U *and the functions* $(t,x) \mapsto \nabla g(t,x)$ *and* $(t,x) \mapsto \nabla^2 g(t,x)$ *are continuous. Then for any* $X \in \mathscr{D}_Z^{\alpha}([0,T];U)$,

$$g(t,X_t) = g(0,X_0) + \int_0^t h(r,X_r)d\mathbf{Z}_r + \int_0^t \nabla g(r,X_r)dX_r. \tag{4.12}$$

We recall that $\tilde{\mathbf{Z}} = (\tilde{Z}, \tilde{\mathbb{Z}})$ as (canonical) space-time rough path extension of $\mathbf{Z} \in \mathscr{C}_g^{\alpha}([0,T],\mathbb{R}^d)$ where $\tilde{Z} = (t, Z_t)$ and $\tilde{\mathbb{Z}}$ is given by $\mathbb{Z}$ and the "remaining cross integrals" of Z_t and t, given by usual Riemann-Stieltjes integration; see Chapter 8 of [FH20]. We know that $\tilde{\mathbf{Z}} \in \mathscr{C}_g^{\alpha}([0,T],\mathbb{R}^{d+1})$ as discussed in Section 9.4 of [FV10].

Definition 4.5. *Let* $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, $\mathbf{Z} = (Z,\mathbb{Z}) \in \mathscr{C}_g^{\alpha}([0,T];\mathbb{R}^d)$ *and* $\sigma \in C^{l,\beta}\left(\mathbb{R}^d,\mathbb{R}^d\right)$ *divergence–free vector field (i.e.* $\nabla \cdot u = 0$*), that is* $\|\sigma\|_{l,\beta}^2 < \infty$ *for* $\beta \in (0,1)$ *and some* $l \in \mathbb{N}$ *with* $l \geqslant \frac{d}{2} + 3$. *Given* $u \in C_b^0([0,T];C^1(\mathbb{R}^d,\mathbb{R}^d))$ *divergence–free vector field with* $(u,u') \in \mathscr{D}_Z^{2\alpha}([0,T], L(\mathbb{R}^d,\mathbb{R}^d))$, *and* $(\nabla u, (\nabla u)') \in \mathscr{D}_Z^{2\alpha}([0,T], L(\mathbb{R}^d, L(\mathbb{R}^d,\mathbb{R}^d)))$. u *is solution of* (4.1) *if verify*

$$u_t(x) = u_0(x) - \int_0^t (u_s(x)\cdot\nabla)u_s(x)ds - \int_0^t \nabla p_s(x)ds - \int_0^t \mathcal{L}_\sigma^* u_r(x)d\mathbf{Z}_r, \tag{4.13}$$

where p *is a scalar potential representing internal pressure with* $p \in C_b^0([0,T];C^1(\mathbb{R}^d,\mathbb{R}))$. *Here the last integral is understood in the rough path sense.*

4.2 Lagrangian formulation

Let's demonstrate the main result of this Chapter: the solution to the Euler equation (4.1) satisfies the Lagrangian formulation.

Theorem 4.4. *Let* $\mathbf{Z} = (Z,\mathbb{Z}) \in \mathscr{C}_g^{\alpha}([0,T],\mathbb{R}^d)$. *Let* $x \in \mathbb{R}^d \to (u(\cdot,x), u'(\cdot,x))$ *be a continuous family of controlled rough path respect to* Z *with values in* $L(\mathbb{R}^d,\mathbb{R}^d)$. *We assume that* $u \in C^0\left([0,T];C^3\left(\mathbb{R}^d,\mathbb{R}^d\right)\right)$ *and* $p \in C^0\left([0,T];C^3\left(\mathbb{R}^d,\mathbb{R}\right)\right)$.

If u *is solution of the equation* (4.1) *then the pair* (X,u) *verifies the Lagrangian formulation*

$$dX_t = \sigma(X_t)d\mathbf{Z}_t + u_t(X_t)dt \tag{4.14}$$

$$u_t(x) = \mathbb{P}\left[(\nabla A_t)^* u_0(A_t)\right](x), \tag{4.15}$$

where $*$ *means the transposition of matrices and denote the back-to-labels map* A *by setting* $A(\cdot,t) = X^{-1}(\cdot,t)$.

Proof. With the assumptions above, we have

$$u(t,x) = u_0(x) - \int_0^t (u_r(x)\cdot\nabla)u_r(x)dr - \int_0^t \nabla p_r(x)dr - \int_0^t \mathcal{L}_\sigma^* u_r(x)d\mathbf{Z}_r \tag{4.16}$$

$$= u_0(x) + \int_0^t h(r,x)d\tilde{\mathbf{Z}}_r, \tag{4.17}$$

where $h = (-(u\cdot\nabla)u - \nabla p, -\mathcal{L}_\sigma^* u)$. Furthermore, we consider

$$X_t(x) = X_0(x) + \int_0^t u_r(X_r(x))dr + \int_0^t \sigma(X_r(x))d\mathbf{Z}_r, \tag{4.18}$$

where $(X, X') \in \mathscr{D}_Z^{2\alpha}\left([0,T], L(\mathbb{R}^d, \mathbb{R}^d)\right)$.
From Itô-Wentzell's formula for the weakly geometric rough paths, see Theorem 4.3, we have that $u(t, X_t)$ is given by

$$\begin{aligned} u(t, X_t) =& u_0(x) + \int_0^t h_r(X_r)\, d\tilde{\mathbf{Z}}_r + \int_0^t \nabla u_r(X_r)\, dX_r \\ =& u_0(x) - \int_0^t ((u\cdot\nabla) u_r(X_r) + (\nabla p)(X_r))\, dr - \int_0^t \mathcal{L}_\sigma^* u_r(X_r)\, d\mathbf{Z}_r \\ &+ \int_0^t \nabla u_r(X_r)(u_r(X_r)\, dr + \sigma(X_r)\, d\mathbf{Z}_r). \end{aligned} \tag{4.19}$$

Analogously to chapter 2, see equations (2.11) and (2.12), we have the identities

$$\int_0^t (u\cdot\nabla)u(r, X_r)dr = \int_0^t ((\nabla u_r)\, u_r)\big|_{X_r} dr \quad \text{and} \quad \int_0^t (\sigma\cdot\nabla)u_r(X_r)d\mathbf{Z}_r = \int_0^t ((\nabla u_r)\,\sigma)\big|_{X_r} d\mathbf{Z}_r.$$

Then we deduce

$$u(t, X_t) = u_0(x) - \int_0^t \nabla p(X_r)\, dr - \int_0^t (\nabla\sigma)^* u_r\big|_{X_r} d\mathbf{Z}_r. \tag{4.20}$$

Now, from [FH20] we have

$$\nabla X_t = I + \int_0^t (\nabla X_r)^* \nabla u_r(X_r)\, dr + \int_0^t (\nabla X_r)^* (\nabla\sigma)(X_r)\, d\mathbf{Z}_r.$$

Then

$$(\nabla X_t)^* = I + \int_0^t (\nabla u_r(X_r))^* \nabla X_r dr + \int_0^t (\nabla\sigma(X_r))^* \nabla X_r d\mathbf{Z}_r.$$

From Proposition 4.3 applied to the product function, we conclude that

$$\begin{aligned} (\nabla X_t)^* u_t(X_t) = u_0(x) &+ \int_0^t u_r(X_r)\left((\nabla u_r(X_r))^* \nabla X_r dr + (\nabla\sigma(X_r))^* \nabla X_r d\mathbf{Z}_r\right) \\ &- \int_0^t (\nabla X_r)^* \left((\nabla p)(X_r)\, dr + (\nabla\sigma(X_r))^* u_r(X_r)\, d\mathbf{Z}_r\right). \end{aligned}$$

We observe that

$$\int_0^t u_r(X_r)(\nabla\sigma(X_r))^* \nabla X_r d\mathbf{Z}_r = \int_0^t (\nabla X_r)^* (\nabla\sigma(X_r))^* u_r(X_r)\, d\mathbf{Z}_r.$$

Then we deduce

$$\begin{aligned}
(\nabla X_t)^* u(t, X_t) &= u_0(x) + \int_0^t u(r, X_r)(\nabla u_r(X_r))^* \nabla X_r dr - \int_0^t (\nabla X_r)^* (\nabla p)(X_r) dr \\
&= u_0(x) + \int_0^t (\nabla X_r)^* (\nabla u_r(X_r))^* u(r, X_r)\, dr - \int_0^t (\nabla X_r)^* (\nabla p)(X_r) dr \\
&= u_0(x) + \frac{1}{2}\int_0^t \nabla\left(|u_r|^2 \circ X_r\right) dr - \int_0^t \nabla(p \circ X_r)\, dr \\
&= u_0 + \nabla\tilde{q},
\end{aligned}$$

where $\tilde{q} := \int_0^t \left[\frac{1}{2}\left(|u_s|^2 \circ X_r\right) - p \circ X\right] dr$. Then, if we denote $M_t := (\nabla X_t)^*$, it follows that

$$u_t \circ X_t = u(t, X_t) = M_t^{-1} u_0 + M_t^{-1}\nabla\tilde{q} = (\nabla A_t\big|_{X_t})^* u_0 + (\nabla A_t\big|_{X_t})^* \nabla\tilde{q}.$$

Finally we conclude

$$\begin{aligned}
u_t &= (\nabla A_t)^* u_0 \circ A_t + (\nabla A_t)^* (\nabla\tilde{q})(A_t) \\
&= (\nabla A_t)^* u_0 \circ A_t + \nabla q,
\end{aligned}$$

where $q := \tilde{q} \circ A$. Therefore, $u_t = \mathbb{P}\left[(\nabla A_t)^* u_0 \circ A_t\right]$. □

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