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DIRAC-HESTENES SPINOR FIELDS, THEIR COVARIANT DERIVATIVES, AND THEIR FORMULATION ON RIEMANN-CARTAN MANIFOLDS

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ABSTRACT - In this paper we study Dirac-Hestenes spinor fields (DHSF) on a four-dimensional Riemann-Cartan spacetime (RCST). We prove that these fields must be defined as certain equivalence classes of even sections of the Clifford bundle (over the RCST), thereby being certain particular sections of a new bundle named Spin-Clifford bundle (SCB). The conditions for the existence of the SCB are studied and are shown to be equivalent to the famous Geroch's theorem concerning to the existence of spinor structures in a Lorentzian spacetime. We introduce also the covariant and algebraic Dirac spinor fields and compare these with DHSF, showing that all the three kinds of spinor fields contain the same mathematical and physical information. We clarify also the notion of (Crumeyrolle's) amorphous spinors (Dirac-Kähler spinor fields are of these type), showing that they cannot be used to describe fermionic fields. We develop a rigorous theory for the covariant derivatives of Clifford fields (sections of the Clifford bundle (CB)) and of Dirac-Hestenes spinor fields. We show how to generalize the original Dirac-Hestenes equation in Minkowski spacetime for the case of a RCST. Our results are obtained from a variational principle formulated through the multiform derivative approach to Lagrangian field theory in the Clifford bundle.

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Abstract

In this paper we study Dirac-Hestenes spinor fields (DHSF) on a four-dimensional Riemann-Cartan spacetime (RCST). We prove that these fields must be defined as certain equivalence classes of even sections of the Clifford bundle (over the RCST), thereby being certain particular sections of a new bundle named Spin-Clifford bundle (SCB). The conditions for the existence of the SCB are studied and are shown to be equivalent to the famous Geroch's theorem concerning to the existence of spinor structures in a Lorentzian spacetime. We introduce also the covariant and algebraic Dirac spinor fields and compare these with DHSF, showing that all the three kinds of spinor fields contain the same mathematical and physical information. We clarify also the notion of (Crumeyrolle's) amorphous spinors (Dirac-Kähler spinor fields are of these type), showing that they cannot be used to describe fermionic fields. We develop a rigorous theory for the covariant derivatives of Clifford fields (sections of the Clifford bundle (CB)) and of Dirac-Hestenes spinor fields. We show how to generalize the original Dirac-Hestenes equation in Minkowski spacetime for the case of a RCST. Our results are obtained from a variational principle formulated through the multiform derivative approach to Lagrangian field theory in the Clifford bundle.

1. Introduction

In the following we study the theory of Dirac-Hestenes spinor fields (DHSF) and the theory of their covariant derivatives on a Riemann-Cartan spacetime (RCST) using the formalism developed in [1]. We also show how to generalize the so-called Dirac-Hestenes equation (originally introduced in [2, 3] for the formulation of Dirac theory of the electron using the spacetime algebra $C\ell_{1,3}$ in Minkowski spacetime) for an arbitrary Riemann-Cartan spacetime. We use a novel approach based on the multiform derivative formulation of Lagrangian field theory to obtain the above results. They are important for the study of spinor fields in gravitational theory and are essential for an understanding of the relationship between Maxwell and Dirac theories and quantum mechanics.[4]

In order to achieve our goals we start clarifying many misconceptions concerning the usual presentation of the theory of covariant, algebraic and Dirac-Hestenes spinors.

Section 2 is dedicated to this subject and we must say that it improves over other presentations, e.g., [4]–[12] introducing a new and important fact, namely that all kind of spinors refered above must be defined as special equivalence classes in appropriate Clifford algebras. The hidden geometrical meaning of the covariant Dirac Spinor is disclosed and the physical and geometrical meaning of the famous Fierz identities[8, 9, 13, 14] becomes obvious.

In Section 3 we study the Clifford bundle of a Riemann-Cartan spacetime and its irreducible module representations. This permit us to define Dirac-Hestenes spinor fields (DHSF) as certain equivalence classes of even sections of the Clifford bundle. DHSF are then naturally identified with sections of a new bundle which we call the Spin-Clifford bundle.

We discuss also the concept of amorphous spinor fields (ASF) (a name introduced by Crumeyrolle[15]). The so-called Dirac-Kähler spinors[16] discussed by Graf[17] and used in presentations of field theories in the lattice[18, 19] are examples of ASF. We prove that they cannot be used to describe fermion fields because they cannot be used to properly formulate the Fierz identities.

In Section 4 we show how the Clifford and Spin-Clifford bundle techniques permit us to give a simple presentation of the concept of covariant derivative for Clifford fields, algebraic Dirac Spinor Fields and for the DHSF.[20] We show that our elegant theory agrees with the standard one developed for the so-called covariant Dirac spinor fields as developed, e.g., in [21, 22]

In Section 5 we introduce the concepts of Dirac and Spin-Dirac operators acting respectively on sections of the Clifford and Spin-Clifford bundles. We show how to use the Spin-Dirac operator on the representatives of DHSF on the Clifford bundle.

In Section 6 we present the multiform derivative approach to Lagrangian field theory and derive the Dirac-Hestenes equation on a Riemann-Cartan Spacetime.[23] We compare our results with some others that appear in the literature for the covariant Dirac Spinor field[24, 25] and also for Dirac-Kähler fields.[16, 17, 26]

Finally in Section 7 we present our conclusion.

2. Covariant, Algebraic and Dirac-Hestenes Spinors

2.1. *Some General Features about Clifford Algebras*

In this section we fix the notations to be used in this paper and introduce the main ideas concerning the theory of Clifford algebras necessary for the intelligibility of the paper. We follow with minor modifications the conventions used in [1, 8, 9].

Formal Definition of the Clifford Algebra $C\ell(V,Q)$

Let K be a field, char $K \neq 2$,[1] V a vector space of finite dimension n over K, and Q a nondegenerate quadratic form over V. Denote by

$$\mathbf{x}\cdot\mathbf{y} = \tfrac{1}{2}(Q(\mathbf{x}+\mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y})) \tag{2.1}$$

[1]In our applications in this paper, K will be $\mathbb{R}$ or $\mathbb{C}$, respectively the real or complex field. The quaternion ring will be denoted by $\mathbb{H}$.

the associated *symmetric* bilinear form on V and define the *left contraction* $\lrcorner : \bigwedge V \times \bigwedge V \to \bigwedge V$ and the *right contraction* $\llcorner : \bigwedge V \times \bigwedge V \to \bigwedge V$ by the rules

(i) $\mathbf{x} \lrcorner \mathbf{y} = \mathbf{x} \cdot \mathbf{y}$
$\quad \mathbf{x} \llcorner \mathbf{y} = \mathbf{x} \cdot \mathbf{y}$

(ii) $\mathbf{x} \lrcorner (u \wedge v) = (\mathbf{x} \lrcorner u) \wedge v + \hat{u} \wedge (\mathbf{x} \lrcorner v)$
$\quad (u \wedge v) \llcorner \mathbf{x} = u \wedge (v \llcorner \mathbf{x}) + (u \llcorner \mathbf{x}) \wedge \hat{v}$

(iii) $(u \wedge v) \lrcorner w = u \lrcorner (v \lrcorner w)$
$\quad u \llcorner (v \wedge w) = (u \llcorner v) \llcorner w$

where $\mathbf{x}, \mathbf{y} \in V$ and $u, v, w \in \bigwedge V$. The notation $\mathbf{a} \cdot \mathbf{b}$ will be used for contractions when it is clear from the context which factor is the contractor and which factor is being contracted. When just one of the factors is homogeneous, it is understood to be the contractor. When both factors are homogeneous, we agree that the one with lower degree is the contractor, so that for $\mathbf{a} \in \bigwedge^r V$ and $\mathbf{b} \in \bigwedge^s V$, we have $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \lrcorner \mathbf{b}$ if $r \leq s$ and $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \llcorner \mathbf{b}$ if $r \geq s$.

Define the (Clifford) product of $\mathbf{x} \in V$ and $u \in \bigwedge V$ by

$$\mathbf{x}u = \mathbf{x} \wedge u + \mathbf{x} \lrcorner u \tag{2.2}$$

and extend this product by linearity and associativity to all of $\bigwedge V$. This provides $\bigwedge V$ with a new product, and provided with this new product $\bigwedge V$ becomes isomorphic to the *Clifford algebra* $C\ell(V, Q)$.

We recall that $\bigwedge V = T(V)/I$ where $T(V)$ is the tensor algebra of V and $I \subset T(V)$ is the bilateral ideal generated by the elements of the form $\mathbf{x} \otimes \mathbf{x}$, $\mathbf{x} \in V$. It can also be shown that the Clifford algebra of (V, Q) is $C\ell(V, Q) = T(V)/I_Q$, where I_Q is the bilateral ideal generated by the elements of the form $\mathbf{x} \otimes \mathbf{x} - Q(\mathbf{x})$, $\mathbf{x} \in V$. The Clifford algebra so constructed is an associative algebra with unity. Since K is a field, the space V is naturally imbedded in $C\ell(V, Q)$

$$\begin{gathered} V \overset{i}{\hookrightarrow} T(V) \overset{j}{\to} T(V)/I_Q = C\ell(V, Q) \\ I_Q = j \circ i \quad \text{and} \quad V \equiv i_Q(V) \subset C\ell(V, Q) \end{gathered} \tag{2.3}$$

Let $C\ell^+(V, Q)$ [resp., $C\ell^-(V, Q)$] be the j-image of $\oplus_{i=0}^{\infty} T^{2i}(V)$ [resp., $\oplus_{i=0}^{\infty} T^{2i+1}(V)$] in $C\ell(V, Q)$. The elements of $C\ell^+(V, Q)$ form a subalgebra of $C\ell(V, Q)$ called the even subalgebra of $C\ell(V, Q)$.

$C\ell(V, Q)$ has the following property: If A is an associative K-algebra with unity then all linear mappings $\rho : V \to A$ such that $(\rho(x))^2 = Q(x)$, $x \in V$, can be extended in a unique way to an algebra homomorphism $\rho : C\ell(V, Q) \to A$.

In $C\ell(V, Q)$ there exist three linear mappings which are quite natural. They are extensions of the mappings

Main involution: an automorphism $\hat{}: C\ell(V, Q) \to C\ell(V, Q)$, extension of $\alpha : V \to T(V)/I_Q, \alpha(x) = -i_Q(x) = -x$, $\forall x \in V$.

Reversion: an antiautomorphism $\tilde{}: C\ell(V,Q) \to C\ell(V,Q)$, extension of $^t : T^r(V) \to T^r(V)$; $T^r(V) \ni x = x_{i_1} \otimes \ldots \otimes x_{i_r} \mapsto x^t = x_{i_r} \otimes \ldots \otimes x_{i_1}$.

Conjugation: $\bar{}: C\ell(V,Q) \to C\ell(V,Q)$, defined by the composition of the main involution $\hat{}$ with the reversion $\tilde{}$, i.e., if $x \in C\ell(V,Q)$ then $\bar{x} = (\hat{x})\tilde{} = (\tilde{x})\hat{}$.

$C\ell(V,Q)$ can be described through its generators, i.e., if $\Sigma = \{E_i\}$ $(i = 1,2,\ldots,n)$ is a Q-orthonormal basis of V, then $C\ell(V,Q)$ is generated by 1 and the E_i's are subjected to the conditions

$$\begin{aligned} &E_iE_i = Q(E_i) \\ &E_iE_j + E_jE_i = 0, \qquad i \neq j;\ i,j = 1,2,\ldots,n \\ &E_1E_2\cdots E_n \neq \pm 1. \end{aligned} \tag{2.4}$$

The Real Clifford Algebra $C\ell_{p,q}$

Let $\mathbb{R}^{p,q}$ be a real vector space of dimension $n = p+q$ endowed with a nondegenerate metric $g : \mathbb{R}^{p,q} \times \mathbb{R}^{p,q} \to \mathbb{R}$. Let $\Sigma = \{E_i\}$, $(i = 1,2,\ldots,n)$ be an orthonormal basis of $\mathbb{R}^{p,q}$,

$$g(E_i,E_j) = g_{ij} = g_{ji} = \begin{cases} +1, & i = j = 1,2,\ldots p \\ -1, & i = j = p+1,\ldots,p+q = n \\ 0, & i \neq j \end{cases} \tag{2.5}$$

The Clifford algebra $C\ell_{p,q} = C\ell(\mathbb{R}^{p,q},Q)$ is the Clifford algebra over $\mathbb{R}$, generated by 1 and the $\{E_i\}$, $(i = 1,2,\ldots,n)$ such that $E_i^2 = Q(E_i) = g(E_i,E_i)$, $E_iE_j = -E_jE_i$ $(i \neq j)$, and[27] $E_1E_2\ldots E_n \neq \pm 1$. $C\ell_{p,q}$ is obviously of dimension 2^n and as a vector space it is the direct sum of vector spaces $\bigwedge^k \mathbb{R}^{p,q}$ of dimensions $\binom{n}{k}$, $0 \leq k \leq n$. The canonical basis of $\bigwedge^k \mathbb{R}^{p,q}$ is given by the elements $e_A = E_{\alpha_1}\ldots E_{\alpha_k}$, $1 \leq \alpha_1 < \ldots < \alpha_k \leq n$. The element $c_J = E_1 \ldots E_n \in \bigwedge^n \mathbb{R}^{p,q}$ commutes (n odd) or anticommutes (n even) with all vectors $E_1,\ldots,E_n \in \bigwedge^1 \mathbb{R}^{p,q} \equiv \mathbb{R}^{p,q}$. The center of $C\ell_{p,q}$ is $\bigwedge^0 \mathbb{R}^{p,q} \equiv \mathbb{R}$ if n is even and its is the direct sum $\bigwedge^0 \mathbb{R}^{p,q} \oplus \bigwedge^n \mathbb{R}^{p,q}$ if n is odd.

All Clifford algebras are semi-simple. If $p+q = n$ is even, $C\ell_{p,q}$ is simple and if $p+q = n$ is odd we have the following possibilities:

(*i*) $C\ell_{p,q}$ is simple $\leftrightarrow c_J^2 = -1 \leftrightarrow p-q \neq 1 \pmod 4 \leftrightarrow$ center of $C\ell_{p,q}$ is isomorphic to $\mathbb{C}$

(*ii*) $C\ell_{p,q}$ is not simple (but is a direct sum of two simple algebras) $\leftrightarrow c_J^2 = +1 \leftrightarrow p-q = 1 \pmod 4 \leftrightarrow$ center of $C\ell_{p,q}$ is isomorphic to $\mathbb{R} \oplus \mathbb{R}$.

All these semi-simple algebras are direct sums of two simple algebras.

If A is an associative algebra on the field $K, K \subseteq A$, and if E is a vector space, a homomorphism ρ from A to $\mathrm{End}\,E$ ($\mathrm{End}\,E$ is the endomorphism algebra of E) which maps the unit element of A to Id_E is a called a *representation* of A in E. The dimension of E is called the degree of the representation. The addition in E together with the mapping $A \times E \to E$, $(a,x) \mapsto \rho(a)x$ turns E in an A-module, the *representation module*.

Conversely, A being an algebra over K and E being an A-module, E is a vector space over K and if $a \in A$, the mapping $\gamma : a \to \gamma_a$ with $\gamma_a(x) = ax$, $x \in E$, is a homomorphism $A \to \operatorname{End} E$, and so it is a representation of A in E. The study of A modules is then equivalent to the study of the representations of A. A representation ρ is *faithful* if its kernel is zero, i.e., $\rho(a)x = 0, \forall x \in E \Rightarrow a = 0$. The kernel of ρ is also known as the annihilator of its module. ρ is said to be *simple* or irreducible if the only invariant subspaces of $\rho(a)$, $\forall a \in A$, are E and $\{0\}$. Then the representation module is also simple, this meaning that it has no proper submodule. ρ is said to be *semi-simple*, if it is the direct sum of simple modules, and in this case E is the direct sum of subspaces which are globally invariant under $\rho(a), \forall a \in A$. When no confusion arises $\rho(a)x$ will be denoted by $a \bullet x$, $a * x$ or ax. Two A-modules E and E' (with the exterior multiplication being denoted respectively by $\bullet$ and $*$) are *isomorphic* if there exists a bijection $\varphi : E \to E'$ such that,

$$\begin{aligned} \varphi(x+y) &= \varphi(x) + \varphi(y), \quad \forall x, y \in E, \\ \varphi(a \bullet x) &= a * \varphi(x), \quad \forall a \in A, \end{aligned} \tag{2.6}$$

and we say that representations φ and φ' of A are equivalent if their modules are isomorphic. This implies the existence of a K-linear isomorphism $\varphi : E \to E'$ such that $\varphi \circ \rho(a) = \rho'(a) \circ \varphi$, $\forall a \in A$ or $\rho'(a) = \varphi \circ \rho(a) \circ \varphi^{-1}$. If $\dim E = n$ then $\dim E' = n$. We shall need:

Wedderburn Theorem:[28] If A is simple algebra then A is equivalent to $F(m)$, where $F(m)$ is a matrix algebra with entries in F, F is a division algebra and m and F are unique (modulo *isomorphisms*).

2.2. Minimal Left Ideals of $C\ell_{p,q}$

The minimal left (resp., right) ideals of a semi-simple algebra A are of the type Ae (resp., eA), where e is a primitive idempotent of A, i.e., $e^2 = e$ and e cannot be written as a sum of two non zero annihitating (or orthogonal) idempotents, i.e, $e \neq e_1 + e_2$, where $e_1 e_2 = e_2 e_1 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$.

Theorem: The maximum number of pairwise annihilating idempotents in $F(m)$ is m.

The decomposition of $C\ell_{p,q}$ into minimal ideals is then characterized by a spectral set $\{e_{pq,i}\}$ of idempotents of $C\ell_{p,q}$ satisfying (i) $\sum_i e_{pq,i} = 1$; (ii) $e_{pq,i} e_{pq,j} = \delta_{ij} e_{pq,i}$; (iii) rank of $e_{pq,i}$ is minimal $\neq 0$, i.e., $e_{pq,i}$ is primitive $(i = 1, 2, \ldots, m)$

By rank of $e_{pq,i}$ we mean the rank of the $\bigwedge \mathbb{R}^{p+q}$-morphism $e_{pq,i} : \psi \mapsto \psi e_{pq,i}$ and $\bigwedge \mathbb{R}^{p,q} = \oplus_{k=0}^{n} \bigwedge^k(\mathbb{R}^{p,q})$ is the exterior algebra of $\mathbb{R}^{p,q}$. Then $C\ell_{p,q} = \sum_i I^i_{p,q}$, $I^i_{p,q} = C\ell_{p,q} e_{pq,i}$ and $\psi \in I^i_{p,q}$ is such that $\psi e_{pq,i} = \psi$. Conversely any element $\psi \in I^i_{p,q}$ can be characterized by an idempotent $e_{pq,i}$ of minimal rank $\neq 0$ with $\psi e_{pq,i} = \psi$. We have the following

Theorem:[29] A minimal left ideal of $C\ell_{p,q}$ is of the type $I_{p,q} = C\ell_{p,q} e_{pq}$ where $e_{pq} = \frac{1}{2}(1 + e_{\alpha_1}) \ldots \frac{1}{2}(1 + e_{\alpha_k})$ is a primitive idempotent of $C\ell_{p,q}$ and are $e_{\alpha_1}, \ldots, e_{\alpha_k}$ commuting elements of the canonical basis of $C\ell_{p,q}$ such that $(e_{\alpha_i})^2 = 1$, $(i = 1, 2, \ldots, k)$ that generate a group of order 2^k, $k = q - r_{q-p}$ and r_i are the Radon-Hurwitz numbers, defined by the recurrence formula $r_{i+8} = r_i + 4$ and

i	0	1	2	3	4	5	6	7
r_i	0	1	2	2	3	3	3	3

If we have a linear mapping $L_a : C\ell_{p,q} \to C\ell_{p,q}$, $L_a(x) = ax$, $x \in C\ell_{p,q}$, $a \in C\ell_{p,q}$, then since $I_{p,q}$ is invariant under left multiplication with arbitrary elements of $C\ell_{p,q}$ we can consider $L_a|_{I_{p,q}} : I_{p,q} \to I_{p,q}$ and taking into account Wedderburn theorem we have

Theorem: If $p + q = n$ is even or odd with $p - q \neq 1 \pmod 4$ then

$$C\ell_{p,q} \simeq \mathrm{End}_F(I_{p,q}) \simeq F(m)$$

where $F = \mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$, $\mathrm{End}_F(I_{p,q})$ is the algebra of linear transformations in $I_{p,q}$ over the field $F, m = \dim_F(I_{p,q})$ and $F \simeq eF(m)e$, e being the representation of e_{pq} in $F(m)$.

If $p + q = n$ is odd, with $p - q = 1 \pmod 4$ then

$$C\ell_{p,q} = \mathrm{End}_F(I_{p,q}) \simeq F(m) \oplus F(m)$$

and $m = \dim_F(I_{p,q})$ and $e_{pq} C\ell_{p,q} e_{pq} \simeq \mathbb{R} \oplus \mathbb{R}$ or $\mathbb{H} \oplus \mathbb{H}$.

Observe that F is the set

$$F = \{T \in \mathrm{End}_F(I_{p,q}), TL_a = L_a T, \ \forall\, a \in C\ell_{p,q}\}$$

Periodicity Theorem:[28] For $n = p + q \geq 0$ there exist the following isomorphisms

$$\begin{aligned} C\ell_{n+8,0} &\simeq C\ell_{n,0} \otimes C\ell_{8,0} \qquad & C\ell_{0,n+8} &\simeq C\ell_{0,n} \otimes C\ell_{0,8} \\ C\ell_{p+8,q} &\simeq C\ell_{p,q} \otimes C\ell_{8,0} \qquad & C\ell_{p,q+8} &\simeq C\ell_{p,q} \otimes C\ell_{0,8} \end{aligned} \tag{2.7}$$

We can find, e.g., in [28, 5, 6] tables giving the representations of all algebras $C\ell_{p,q}$ as matrix algebras. For what follows we need

$$\begin{array}{rl} \text{complex numbers} & C\ell_{0,1} \simeq \mathbb{C} \\ \text{quarternions} & C\ell_{0,2} \simeq \mathbb{H} \\ \text{Pauli algebra} & C\ell_{3,0} \simeq M_2(\mathbb{C}) \\ \text{spacetime algebra} & C\ell_{1,3} \simeq M_2(\mathbb{H}) \\ \text{Majorana algebra} & C\ell_{3,1} \simeq M_4(\mathbb{R}) \\ \text{Dirac algebra} & C\ell_{4,1} \simeq M_4(\mathbb{C}) \end{array} \tag{2.8}$$

We also need the following

Proposition: $C\ell^+_{p,q} = C\ell_{q,p-1}$, for $p > 1$ and $C\ell^+_{p,q} = C\ell_{p,q-1}$ for $q > 1$.

From the above proposition we get the following particular results that we shall need later

$$C\ell^+_{1,3} \simeq C\ell^+_{3,1} = C\ell_{3,0} \qquad C\ell^+_{4,1} \simeq C\ell_{1,3}, \tag{2.9}$$

$$C\ell_{4,1} \simeq \mathbb{C} \otimes C\ell_{3,1} \qquad C\ell_{4,1} \simeq \mathbb{C} \otimes C\ell_{1,3}, \tag{2.10}$$

which means that the Dirac algebra is the complexification of both the spacetime or the Majorana algebras.

Right Linear Structure for $I_{p,q}$

We can give to the ideal $I_{p,q} = C\ell_{p,q}e$ (resp. $I_{pq} = eC\ell_{pq}$) a right (resp. left) linear structure over the field $F(C\ell_{p,q} \simeq F(m)$ or $C\ell_{p,q} \simeq F(m) \oplus F(m))$. A right linear structure, e.g, consists of an additive group (which is $I_{p,q}$) and the mapping

$$I \times F \to I; \quad (\psi, T) \mapsto \psi T$$

such that the usual axioms of a linear vector space structure are valid, e.g., we have[2] $(\psi T)T' = \psi(TT')$.

From the above discussion it is clear that the minimal (left or right) ideals of $C\ell_{p,q}$ are representation modules of $C\ell_{p,q}$. In order to investigate the equivalence of these representations we must introduce some groups that are subsets of $C\ell_{p,q}$. As we shall see, this is the key for the definition of algebraic and Dirac-Hestenes spinors.

2.3. *The Groups: $C\ell^{\star}_{p,q}$, Clifford, Pinor and Spinor*

The set of the invertible elements of $C\ell_{p,q}$ constitutes a non-abelian group which we denote by $C\ell^{\star}_{p,q}$. It acts naturally on $C\ell_{p,q}$ as an algebra homomorphism through its adjoint representation

$$\mathrm{Ad} : C\ell^{\star}_{p,q} \to \mathrm{Aut}(C\ell_{p,q}); \; u \mapsto \mathrm{Ad}_u, \text{ with } \mathrm{Ad}_u(x) = uxu^{-1}. \tag{2.11}$$

The Clifford-Lipschitz group is the set

$$\Gamma_{p,q} = \{u \in C\ell^{\star}_{p,q} \mid \forall x \in \mathbb{R}^{p,q}, ux\hat{u}^{-1} \in \mathbb{R}^{p,q}\}. \tag{2.12}$$

The set $\Gamma^{+}_{p,q} = \Gamma_{p,q} \cap C\ell^{+}_{p,q}$ is called special Clifford-Lipschitz group.

Let $N : C\ell_{p,q} \to C\ell_{p,q}$, $N(x) = \langle \bar{x}x \rangle_0$ ($\langle\ \rangle_0$ means the scalar part of the Clifford number). We define further:

The *Pinor group* $\mathbf{Pin}(p,q)$ is the subgroup of $\Gamma_{p,q}$ such that

$$\mathbf{Pin}(p,q) = \{u \in \Gamma_{p,q} | N(u) = \pm 1\}. \tag{2.13}$$

The *Spin group* $\mathbf{Spin}(p,q)$ is the set

$$\mathbf{Spin}(p,q) = \{u \in \Gamma^{+}_{p,q} | N(u) = \pm 1\}. \tag{2.14}$$

The $\mathbf{Spin}_{+}(p,q)$ group is the set

$$\mathbf{Spin}_{+}(p,q) = \{u \in \Gamma^{+}_{p,q} | N(u) = +1\}. \tag{2.15}$$

Theorem: $\mathrm{Ad}_{|\mathbf{Pin}(p,q)} : \mathbf{Pin}(p,q) \to \mathrm{O}(p,q)$ is onto with kernel $\mathbf{Z}_2$. $\mathrm{Ad}_{|\mathbf{Spin}(p,q)} : \mathbf{Spin}(p,q) \to \mathrm{SO}(p,q)$ is onto with kernel $\mathbf{Z}_2$.

[2] For $C\ell_{3,0}$, $I = C\ell_{3,0}\frac{1}{2}(1+\sigma_3)$ is a minimal left ideal. In this case it is also possible to give a left linear structure for this ideal. See [4, 5]

$O(p,q)$ is the pseudo-orthogonal group of the vector space $\mathbb{R}^{p,q}$, $SO(p,q)$ is the special pseudo-orthogonal group of $\mathbb{R}^{p,q}$. We also denote by $SO_+(p,q)$ the connected component of $SO(p,q)$. $\mathbf{Spin}_+(p,q)$ is connected for all pairs (p,q) with the exception of $\mathbf{Spin}_+(1,0) \simeq \mathbf{Spin}_+(0,1) \simeq \{\pm 1\}$ and $\mathbf{Spin}_+(1,1)$. We have,

$$O(p,q) = \frac{\mathbf{Pin}(p,q)}{\mathbf{Z}_2} \quad SO(p,q) = \frac{\mathbf{Spin}(p,q)}{\mathbf{Z}_2} \quad \underset{+}{SO}(p,q) = \frac{\mathbf{Spin}_+(p,q)}{\mathbf{Z}_2}.$$

In the following the group homomorphism between $\mathbf{Spin}_+(p,q)$ and $SO_+(p,q)$ will be denoted

$$\mathcal{H} : \mathbf{Spin}_+(p,q) \to \underset{+}{SO}(p,q). \tag{2.16}$$

We also need the important result:

Theorem:[29] For $p+q \leq 5$, $\mathbf{Spin}_+(p,q) = \{u \in \mathcal{C}\ell^+_{p,q} | u\tilde{u} = 1\}$.

Lie Algebra of $\mathbf{Spin}_+(1,3)$

It can be shown that for each $u \in \mathbf{Spin}_+(1,3)$ it holds

$$u = \pm e^F, \qquad F \in \bigwedge^2 \mathbb{R}^{1,3} \subset \mathcal{C}\ell_{1,3} \tag{2.17}$$

and F can be chosen in such a way to have a positive sign in Eq. 2.17, except in the particular case $F^2 = 0$ when $u = -e^F$. From Eq. 2.17 it follows immediately that the Lie algebra of $\mathbf{Spin}_+(1,3)$ is generated by the bivectors $F \in \bigwedge^2 \mathbb{R}^{1,3} \subset \mathcal{C}\ell_{1,3}$ through the commutator product.

2.4. *Geometrical and Algebraic Equivalence of the Representation Modules $I_{p,q}$ of Simple Clifford Algebras $\mathcal{C}\ell_{p,q}$*

Recall that $\mathcal{C}\ell_{p,q}$ is a ring. We already said that the minimal lateral ideals of $\mathcal{C}\ell_{p,q}$ are of the form $I_{p,q} = \mathcal{C}\ell_{p,q}e_{pq}$ (or $e_{pq}\mathcal{C}\ell_{p,q}$) where e_{pq} is a primitive idempotent. Obviously the minimal lateral ideals are modules over the ring $\mathcal{C}\ell_{p,q}$, they are representation modules. According to the discussion of Section 2.1, given two ideals $I_{p,q} = \mathcal{C}\ell_{p,q}e_{pq}$ and $I'_{p,q} = \mathcal{C}\ell_{p,q}e'_{pq}$ they are by definition isomorphic if there exists a bijection $\varphi : I_{p,q} \to I'_{p,q}$ such that,

$$\varphi(\psi_1 + \psi_2) = \varphi(\psi_1) + \varphi(\psi_2)\ ;\ \varphi(a\psi) = a\varphi(\psi)\ ,\ \forall a \in \mathcal{C}\ell_{p,q}, \forall \psi_1, \psi_2 \in I_{p,q} \tag{2.18}$$

Recalling the Noether-Skolem theorem, which says that all automorphisms of a simple algebra are inner automorphisms, we have:

Theorem: When $\mathcal{C}\ell_{p,q}$ is simple, its automorphisms are given by inner automorphisms $x \mapsto uxu^{-1}$, $x \in \mathcal{C}\ell_{p,q}$, $u \in \mathcal{C}\ell^{\star}_{p,q}$.

We also have:

Proposition: When $\mathcal{C}\ell_{p,q}$ is simple, all its finite-dimensional irreducible representations are equivalent (i.e., isomorphic) under inner automorphisms.

We quote also the

Theorem:[15] $I_{p,q}$ and $I'_{p,q}$ are isomorphic if and only if $I'_{p,q} = I_{p,q}X$ for non-zero $X \in I'_{p,q}$.

We are thus lead to the following definitions:

(*i*) The ideals $I_{p,q} = C\ell_{p,q}e_{pq}$ and $I'_{p,q} = C\ell_{p,q}e'_{pq}$ are said to be *geometrically equivalent* if, for some $u \in \Gamma_{p,q}$,

$$e'_{pq} = ue_{pq}u^{-1}. \tag{2.19}$$

(*ii*) $I_{p,q}$ and $I'_{p,q}$ are said to be *algebraically equivalent* if

$$e'_{pq} = ue_{pq}u^{-1}, \tag{2.20}$$

for some $u \in C\ell^{\star}_{p,q}$, but $u \notin \Gamma_{p,q}$.

It is now time to specialize the above results for $C\ell_{1,3} \simeq M_2(\mathbb{H})$ and to find a relationship between the Dirac algebra $C\ell_{4,1} \simeq M_4(\mathbb{C})$ and $C\ell_{1,3}$ and their respective minimal ideals.

Let $\Sigma_0 = \{E_0, E_1, E_2, E_3\}$ be an orthogonal basis of $\mathbb{R}^{1,3} \subset C\ell_{1,3}$, $E_\mu E_\nu + E_\nu E_\mu = 2\eta_{\mu\nu}$, $\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$. Then, the elements

$$e = \tfrac{1}{2}(1+E_0) \qquad e' = \tfrac{1}{2}(1+E_3E_0) \qquad e'' = \tfrac{1}{2}(1+E_1E_2E_3), \tag{2.21}$$

are easily verified to be primitive idempotents of $C\ell_{1,3}$. The minimal left ideals, $I = C\ell_{1,3}e$, $I' = C\ell_{1,3}e'$, $I'' = C\ell_{1,3}e''$ are right two dimensional linear spaces over the quaternion field (e.g., $\mathbb{H}e = e\mathbb{H} = eC\ell_{1,3}e$). According to the definition (ii) above these ideals are algebraically equivalent. For example, $e' = ueu^{-1}$, with $u = (1+E_3) \notin \Gamma_{1,3}$

The elements $\Phi \in C\ell_{1,3}\frac{1}{2}(1+E_0)$ will be called *mother* spinors.[9, 10] We can show[4, 5] that each Φ can be written

$$\Phi = \psi_1 e + \psi_2 E_3E_1 e + \psi_3 E_3E_0 e + \psi_4 E_1E_0 e = \sum_i \psi_i s_i, \tag{2.22}$$

$$s_1 = e, \quad s_2 = E_3E_1e, \quad s_3 = E_3E_0e, \quad s_4 = E_1E_0e \tag{2.23}$$

and where the ψ_i are formally complex numbers, i.e., each $\psi_i = (a_i + b_iE_2E_1)$ with $a_i, b_i \in \mathbb{R}$.

We recall that $\mathbf{Pin}(1,3)/\mathbf{Z}_2 \simeq \mathrm{O}(1,3)$, $\mathbf{Spin}(1,3)/\mathbf{Z}_2 \simeq \mathrm{SO}(1,3)$, $\mathbf{Spin}_+(1,3)/\mathbf{Z}_2 \simeq \mathrm{SO}_+(1,3)$, $\mathbf{Spin}_+(1,3) \simeq \mathrm{SL}(2,\mathbb{C})$ the universal covering group of $\mathcal{L}^\uparrow_+ \equiv \mathrm{SO}_+(1,3)$, the restrict Lorentz group.

In order to determine the relation between $C\ell_{4,1}$ and $C\ell_{1,3}$ we proceed as follows: let $\{F_0, F_1, F_2, F_3, F_4\}$ be an orthogonal basis of $C\ell_{4,1}$ with $-F_0^2 = F_1^2 = F_2^2 = F_3^2 = F_4^2 = 1$, $F_AF_B = -F_BF_A$ ($A \neq B$; $A, B = 0,1,2,3,4$). Define the pseudoscalar

$$\mathbf{i} = F_0F_1F_2F_3F_4 \qquad \mathbf{i}^2 = -1 \qquad \mathbf{i}F_A = F_A\mathbf{i} \qquad A = 0,1,2,3,4 \tag{2.24}$$

Define

$$\mathcal{E}_\mu = F_\mu F_4 \tag{2.25}$$

We can immediately verify that $\mathcal{E}_\mu\mathcal{E}_\nu+\mathcal{E}_\nu\mathcal{E}_\mu = 2\eta_{\mu\nu}$. Taking into account that $C\ell_{1,3} \simeq C\ell^+_{4,1}$ we can explicitly exhibit here this isomorphism by considering the map $g : C\ell_{1,3} \to C\ell^+_{4,1}$ generated by the linear extension of the map $g^\# : \mathbb{R}^{1,3} \to C\ell^+_{4,1}, g^\#(E_\mu) = \mathcal{E}_\mu = F_\mu F_4$, where E_μ, $(\mu = 0,1,2,3)$ is an orthogonal basis of $\mathbb{R}^{1,3}$. Also $g(1_{C\ell_{1,3}}) = 1_{C\ell^+_{4,1}}$, where $1_{C\ell_{1,3}}$ and $1_{C\ell^+_{4,1}}$ are the identity elements in $C\ell_{1,3}$ and $C\ell^+_{4,1}$. Now consider the primitive idempotent of $C\ell_{1,3} \simeq C\ell^+_{4,1}$,

$$e_{41} = g(e) = \tfrac{1}{2}(1 + \mathcal{E}_0) \tag{2.26}$$

and the minimal left ideal $I^+_{4,1} = C\ell^+_{4,1}e_{4,1}$. The elements $Z_{\Sigma_0} \in I^+_{4,1}$ can be written in an analogous way to $\Phi \in C\ell_{1,3}\frac{1}{2}(1 + E_0)$ (Eq. 2.22), i.e.,

$$Z_{\Sigma_0} = \Sigma\ z_i\bar{s}_i \tag{2.27}$$

where

$$\bar{s}_1 = e_{41},\quad \bar{s}_2 = -\mathcal{E}_1\mathcal{E}_3e_{41},\quad \bar{s}_3 = \mathcal{E}_3\mathcal{E}_0e_{41},\quad \bar{s}_4 = \mathcal{E}_1\mathcal{E}_0e_{41}, \tag{2.28}$$

and

$$z_i = a_i + \mathcal{E}_2\mathcal{E}_1 b_i,$$

are formally complex numbers, $a_i, b_i \in \mathbb{R}$.

Consider now the element $f_{\Sigma_0} \in C\ell_{4,1}$,

$$\begin{aligned} f_{\Sigma_0} &= e_{41}\tfrac{1}{2}(1 + \mathrm{i}\mathcal{E}_1\mathcal{E}_2) \\ &= \tfrac{1}{2}(1 + \mathcal{E}_0)\tfrac{1}{2}(1 + \mathrm{i}\mathcal{E}_1\mathcal{E}_2), \end{aligned} \tag{2.29}$$

with i given by Eq. 2.24.

Since $f_{\Sigma_0}C\ell_{4,1}f_{\Sigma_0} = \mathbb{C}f_{\Sigma_0} = f_{\Sigma_0}\mathbb{C}$ it follows that f_{Σ_0} is a primitive idempotent of $C\ell_{4,1}$. We can easily show that each $\Phi_{\Sigma_0} \in I_{\Sigma_0} = C\ell_{4,1}f_{\Sigma_0}$ can be written

$$\Psi_{\Sigma_0} = \sum_i \psi_i f_i,\quad \psi_i \in \mathbb{C}$$

$$f_1 = f_{\Sigma_0}\ ,\ f_2 = -\mathcal{E}_1\mathcal{E}_3 f_{\Sigma_0}\ ,\ f_3 = \mathcal{E}_3\mathcal{E}_0 f_{\Sigma_0}, f_4 = \mathcal{E}_1\mathcal{E}_0 f_{\Sigma_0} \tag{2.30}$$

with the methods described in [4, 5] we find the following representation in $M_4(\mathbb{C})$ for the generators $\mathcal{E}_\mu$ of $C\ell^+_{4,1} \simeq C\ell_{1,3}$

$$\mathcal{E}_0 \mapsto \gamma_0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} \leftrightarrow \mathcal{E}_i \mapsto \gamma_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{2.31}$$

where 1_2 is the unit 2×2 matrix and σ_i, $(i = 1,2,3)$ are the standard Pauli matrices. We immediately recognize the γ-matrices in Eq. 2.31 as the standard ones appearing, e.g., in [30]

The matrix representation of $\Psi_{\Sigma_0} \in I_{\Sigma_0}$ will be denoted by the same letter without the indice, i.e., $\Psi_{\Sigma_0} \mapsto \Psi \in M_4(\mathbb{C})f$, where

$$f = \tfrac{1}{2}(1 + \gamma_0)\tfrac{1}{2}(1 + i\gamma_1\gamma_2) \qquad i = \sqrt{-1}. \tag{2.32}$$

We have

$$\Psi = \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix} \qquad \psi_i \in \mathbb{C}. \tag{2.33}$$

Eqs. 2.22, 2.27 and (2.30) are enough to prove that there are bijections between the elements of the ideals $C\ell_{1,3}\frac{1}{2}(1+E_0)$, $C\ell^+_{4,1}\frac{1}{2}(1+\mathcal{E}_0)$ and $C\ell_{4,1}\frac{1}{2}(1+\mathcal{E}_0)\frac{1}{2}(1+\mathbf{i}\mathcal{E}_1\mathcal{E}_2)$.

We can easily find that the following relation exists between $\Psi_{\Sigma_0} \in C\ell_{4,1}f_{\Sigma_0}$ and $Z_{\Sigma_0} \in C\ell^+_{4,1}\frac{1}{2}(1+\mathcal{E}_0)$,

$$\Psi_{\Sigma_0} = Z_{\Sigma_0}\tfrac{1}{2}(1+\mathbf{i}\mathcal{E}_1\mathcal{E}_2). \tag{2.34}$$

Decomposing Z_{Σ_0} into even and odd parts relative to the $\mathbf{Z}_2$-graduation of $C\ell^+_{4,1} \simeq C\ell_{1,3}$, $Z_{\Sigma_0} = Z^+_{\Sigma_0} + Z^-_{\Sigma_0}$ we obtain $Z^+_{\Sigma_0} = Z^-_{\Sigma_0}\mathcal{E}_0$ which clearly shows that all information of Z_{Σ_0} is contained in $Z^+_{\Sigma_0}$. Then,

$$\Psi_{\Sigma_0} = Z^+_{\Sigma_0}\tfrac{1}{2}(1+\mathcal{E}_0)\tfrac{1}{2}(1+\mathbf{i}\mathcal{E}_1\mathcal{E}_2). \tag{2.35}$$

Now, if we take into account[4, 5] that $C\ell^{++}_{4,1}\frac{1}{2}(1+\mathcal{E}_0) = C\ell^+_{4,1}\frac{1}{2}(1+\mathcal{E}_0)$ where the symbol $C\ell^{++}_{4,1}$ means $C\ell^{++}_{4,1} \simeq C\ell^+_{1,3} \simeq C\ell_{3,0}$ we see that each $Z_{\Sigma_0} \in C\ell^+_{4,1}\frac{1}{2}(1+\mathcal{E}_0)$ can be written

$$Z^-_{\Sigma_0} = \psi_{\Sigma_0}\tfrac{1}{2}(1+\mathcal{E}_0) \qquad \psi_{\Sigma_0} \in (C\ell^+_{4,1})^+ \simeq C\ell^+_{1,3}. \tag{2.36}$$

Then putting $Z^+_{\Sigma_0} = \psi_{\Sigma_0}/2$, Eq. 2.35 can be written

$$\begin{aligned} \Psi_{\Sigma_0} &= \psi\tfrac{1}{2}(1+\mathcal{E}_0)\tfrac{1}{2}(1+\mathbf{i}\mathcal{E}_1\mathcal{E}_2) \\ &= Z_{\Sigma_0}\tfrac{1}{2}(1+\mathbf{i}\mathcal{E}_1\mathcal{E}_2). \end{aligned} \tag{2.37}$$

The matrix representations of Z_{Σ_0} and ψ_{Σ_0} in $M_4(\mathbb{C})$ (denoted by the same letter without index) in the spinorial basis given by Eq. 2.30 are

$$\Psi = \begin{pmatrix} \psi_1 & -\psi_2^* & \psi_3 & \psi_4^* \\ \psi_2 & \psi_1^* & \psi_4 & -\psi_3^* \\ \psi_3 & \psi_4^* & \psi_1 & -\psi_2^* \\ \psi_4 & -\psi_3^* & \psi_2 & \psi_1^* \end{pmatrix}, \quad Z = \begin{pmatrix} \psi_1 & -\psi_2^* & 0 & 0 \\ \psi_2 & \psi_1^* & 0 & 0 \\ \psi_3 & \psi_4^* & 0 & 0 \\ \psi_4 & -\psi_3^* & 0 & 0 \end{pmatrix}. \tag{2.38}$$

2.5. *Algebraic Spinors for $\mathbb{R}^{p,q}$*

Let $\mathcal{B}_\Sigma = \{\Sigma_0, \dot{\Sigma}, \ddot{\Sigma}, \ldots\}$ be the set of all ordered orthonormal basis for $\mathbb{R}^{p,q}$, i.e., each $\Sigma \in \mathcal{B}_\Sigma$ is the set $\Sigma = \{E_1, \ldots, E_p, E_{p+1}, \ldots, E_{p+q}\}$, $E_1^2 = \ldots = E_p^2 = 1$, $E_{p+1}^2 = \ldots = E_{p+q}^2 = -1$, $E_rE_s = -E_sE_r$, $(r \neq s;\ r, s = 1, 2, \ldots, p+q = n)$. Any two basis, say, $\Sigma_0, \dot{\Sigma} \in \mathcal{B}_\Sigma$ are related by an element of the group $\mathbf{Spin}_+(p,q) \subset \Gamma_{pq}$. We write,

$$\dot{\Sigma} = u\Sigma_0 u^{-1}, \quad u \in \mathbf{Spin}_+(p,q). \tag{2.39}$$

A primitive idempotent determined in a given basis $\Sigma \in \mathcal{B}_\Sigma$ will be denoted e_Σ. Then, the idempotents $e_{\Sigma_0}, e_{\dot{\Sigma}}, e_{\ddot{\Sigma}}$, etc., such that, e.g.,

$$e_{\dot{\Sigma}} = u e_{\Sigma_0} u^{-1}, \quad u \in \mathbf{Spin}_+(p,q), \tag{2.40}$$

define ideals $I_{\Sigma_0}, I_{\dot{\Sigma}}, I_{\bar{\Sigma}}$, etc., that are geometrically equivalent according to the definition given by Eq. 2.19. We have,

$$I_{\dot{\Sigma}} = uI_{\Sigma_0}u^{-1} \qquad u \in \mathrm{Spin}_+(p,q) \tag{2.41}$$

but since $uI_{\Sigma_0} \equiv I_{\Sigma_0}$, Eq. 2.41 can also be written

$$I_{\dot{\Sigma}} = I_{\Sigma_0}u^{-1}. \tag{2.42}$$

Eq. 2.42 defines a new correspondence for the elements of the ideals, $I_{\Sigma_0}, I_{\dot{\Sigma}}, I_{\bar{\Sigma}}$, etc. This suggests the

Definition: An algebraic spinor for $\mathbb{R}^{p,q}$ is an equivalence class of the quotient set $\{I_\Sigma\}/R$, where $\{I_\Sigma\}$ is the set of all geometrically equivalent ideals, and $\Psi_{\Sigma_0} \in I_{\Sigma_0}$ and $\Psi_{\dot{\Sigma}} \in I_{\dot{\Sigma}}$ are equivalent, $\Psi_{\dot{\Sigma}} \simeq \Psi_{\Sigma_0}$ (mod R) if and only if

$$\Psi_{\dot{\Sigma}} = \Psi_{\Sigma_0}u^{-1}. \tag{2.43}$$

Ψ_Σ will be called the representative of the algebraic spinor in the basis $\Sigma \in \mathcal{B}_\Sigma$. Recall that $\dot{\Sigma} = u\Sigma u^{-1} = L\Sigma$, $u \in \mathrm{Spin}_+(1,3)$, $L \in \mathcal{L}_+^\uparrow$.

2.6. *What is a Covariant Dirac Spinor (CDS)*

As we already know $f_{\Sigma_0} = \frac{1}{2}(1+\mathcal{E}_0)(1+i\mathcal{E}_1\mathcal{E}_2)$ (Eq. 2.29) is a primitive idempotent of $C\ell_{4,1} \simeq M_4(\mathbb{C})$. If $u \in \mathrm{Spin}_+(1,3) \subset \mathrm{Spin}_+(4,1)$ then all ideals $I_{\dot{\Sigma}} = I_{\Sigma_0}u^{-1}$ are geometrically equivalent to I_{Σ_0}. Since $\Sigma_0 = \{\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$ is a basis for $\mathbb{R}^{1,3} \subset C\ell_{4,1}^+$, the meaning of $\dot{\Sigma} = u\Sigma_0 u^{-1}$ is clear. From Eq. 2.30 we can write

$$I_{\Sigma_0} \ni \Psi_{\Sigma_0} = \sum \psi_i f_i, \qquad \text{and} \qquad I_{\dot{\Sigma}} \ni \Psi_{\dot{\Sigma}} = \sum \dot{\psi}_i \dot{f}_i, \tag{2.44}$$

where

$$f_1 = f_{\Sigma_0}, \quad f_2 = -\mathcal{E}_1\mathcal{E}_3 f_{\Sigma_0}, \quad f_3 = \mathcal{E}_3\mathcal{E}_0 f_{\Sigma_0} \quad f_4 = \mathcal{E}_1\mathcal{E}_0 f_{\Sigma_0}$$

and

$$\dot{f}_1 = f_{\dot{\Sigma}}, \quad \dot{f}_2 = -\dot{\bar{\mathcal{E}}}_1\dot{\bar{\mathcal{E}}}_3 f_{\dot{\Sigma}}, \quad \dot{f}_3 = \dot{\bar{\mathcal{E}}}_3\dot{\bar{\mathcal{E}}}_0 f_{\dot{\Sigma}}, \quad \dot{f}_4 = \dot{\bar{\mathcal{E}}}_1\dot{\bar{\mathcal{E}}}_0 f_{\dot{\Sigma}}$$

Since $\Psi_{\dot{\Sigma}} = \Psi_{\Sigma_0}u^{-1}$, we get

$$\Psi_{\dot{\Sigma}} = \sum_i \psi_i u^{-1} f_i = \sum_{i,k} S_{ik}(u^{-1})\psi_i \dot{f}_k = \sum_k \dot{\psi}_k \dot{f}_k.$$

Then

$$\dot{\psi}_k = \sum_i S_{ik}(u^{-1})\psi_i, \tag{2.45}$$

where $S_{ik}(u^{-1})$ are the matrix components of the representation in $M_4(\mathbb{C})$ of $u^{-1} \in \mathrm{Spin}_+(1,3)$. As proved in [4, 5] the matrices $S(u)$ correspond to the representation $D^{(1/2,0)} \oplus D^{(0,1/2)}$ of $SL(2,\mathbb{C}) \simeq \mathrm{Spin}_+(1,3)$.

We remark that all the elements of the set $\{I_\Sigma\}$ of the ideals geometrically equivalent to I_{Σ_0} under the action of $u \in \mathbf{Spin}_+(1,3) \subset \mathbf{Spin}_+(4,1)$ have the same image $I = M_4(\mathbb{C})f$ where f is given by Eq. 2.32, i.e.,

$$f = \tfrac{1}{2}(1+\gamma_0)(1+i\gamma_1\gamma_2) \qquad i = \sqrt{-1},$$

where γ_μ, $\mu = 0,1,2,3$ are the Dirac matrices given by Eq. 2.31.

Then, if

$$\begin{array}{rcl} \gamma : C\ell_{4,1} & \to & M_4(\mathbb{C}) \equiv \mathrm{End}(M_4(\mathbb{C})f) \\ x & \mapsto & \gamma(x) : M_4(\mathbb{C})f \to M_4(\mathbb{C})f \end{array} \tag{2.46}$$

it follows that $\gamma(\mathcal{E}_\mu) = \gamma(\dot{\mathcal{E}}_\mu) = \gamma_\mu$, $\gamma(f_{\Sigma_0}) = \gamma(f_{\dot\Sigma}) = f$ for all $\mathcal{E}_\mu, \dot{\mathcal{E}}_\mu$ such that $\dot{\mathcal{E}}_\mu = u\mathcal{E}_\mu u^{-1}$ for some $u \in \mathbf{Spin}_+(1,3)$. Observe that all the information concerning the orthonormal frames $\Sigma_0, \dot\Sigma$, etc., disappear in the matrix representation of the ideals $I_{\Sigma_0}, I_{\dot\Sigma}, \ldots$ in $M_4(\mathbb{C})$ since all these ideals are mapped in the same ideal $I = M_4(\mathbb{C})f$.

With the above remark and taking into account Eq. 2.45 we are then lead to the following

Definition: A Covariant Dirac Spinor (CDS) for $\mathbb{R}^{1,3}$ is an equivalent class of triplets $(\Sigma, S(u), \Psi)$, Σ being an orthonormal basis of $\mathbb{R}^{1,3}$, $S(u) \in D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $\mathbf{Spin}_+(1,3)$, $u \in \mathbf{Spin}_+(1,3)$ and $\Psi \in M_4(\mathbb{C})f$ and

$$(\Sigma, S(u), \Psi) \sim (\Sigma_0, S(u_0), \Psi_0)$$

if and only if

$$\Psi = S(u)S^{-1}(u_0)\Psi_0, \quad \mathcal{H}(uu_0^{-1}) = L\Sigma_0, \quad L \in \mathcal{L}_+^\uparrow, \quad u \in \mathbf{Spin}_+(1,3). \tag{2.47}$$

The pair $(\Sigma, S(u))$ is called a spinorial frame. Observe that the CDS just defined depends on the choice of the original spinorial frame (Σ_0, u_0) and obviously, to different possible choices there correspond isomorphic ideals in $M_4(\mathbb{C})$. For simplicity we can fix $u_0 = 1$, $S(u_0) = 1$.

The definition of CDS just given agrees with that given by Choquet-Bruhat[31] except for the irrelevant fact that Choquet-Bruhat uses as the space of representatives of a CDS the complex four-dimensional vector space $\mathbb{C}^4$ instead of $I = M_4(\mathbb{C})f$. We see that Choquet-Bruhat's definition is well justified from the point of view of the theory of algebraic spinors presented above.

2.7. *Algebraic Dirac Spinors (ADS) and Dirac-Hestenes Spinors (DHS)*

We saw in Section 2.4 that there is bijection between $\psi_{\Sigma_0} \in C\ell_{4,1}^{++} \simeq C\ell_{1,3}^{+}$ and $\Psi_{\Sigma_0} \in I_{\Sigma_0} = C\ell_{4,1}^{+} f_{\Sigma_0}$, namely (Eq. 2.37),

$$\Psi_{\Sigma_0} = \psi_{\Sigma_0}\tfrac{1}{2}(1+\mathcal{E}_0)\tfrac{1}{2}(1+i\mathcal{E}_1\mathcal{E}_2)$$

Then, as we already said, all information contained in Ψ_{Σ_0} (that is the representative in the basis Σ_0 of an algebraic spinor for $\mathbb{R}^{1,3}$) is also contained in $\psi_{\Sigma_0} \in C\ell_{4,1}^{++} \simeq C\ell_{1,3}^{+}$. We are then lead to the following

Definition: Consider the quotient set $\{I_\Sigma\}/\mathcal{R}$ where $\{I_\Sigma\}$ is the set of all geometrically equivalent minimal left ideals of $C\ell_{1,3}$ generated by $e_{\Sigma_0} = \frac{1}{2}(1+E_0), \Sigma_0 = (E_0, E_1, E_2, E_3)$ [i.e., $I_{\bar{\Sigma}}, I_{\dot{\Sigma}} \in \{I_\Sigma\}$ then $I_{\bar{E}} = uI_{\dot{\Sigma}}u^{-1} \equiv I_{\dot{\Sigma}}u^{-1}$ for some $u \in \mathbf{Spin}_+(1,3)$]. An algebraic Dirac Spinor (ADS) is an element of $\{I_\Sigma\}/\mathcal{R}$. Then if $\Phi_{\dot{\Sigma}} \in I_{\dot{\Sigma}}, \Phi_{\bar{\Sigma}} \in I_{\bar{\Sigma}}$, then $\Phi_{\bar{\Sigma}} \simeq \Phi_{\dot{\Sigma}}(\mathrm{mod}\ \mathcal{R})$ if and only if $\Phi_{\bar{\Sigma}} = \Phi_{\dot{\Sigma}}u^{-1}$, for some $u \in \mathbf{Spin}_+(1,3)$.

We remark that (see Eq. 2.36)

$$\Phi_{\dot{\Sigma}} = \psi_{\dot{\Sigma}} e_{\dot{\Sigma}}, \quad \Phi_{\bar{\Sigma}} = \psi_{\dot{\Sigma}} e_{\bar{\Sigma}} \qquad \psi_{\dot{\Sigma}}, \psi_{\bar{\Sigma}} \in C\ell^+_{1,3}$$

and since $e_{\bar{\Sigma}} = ue_\Sigma u^{-1}$ for some $u \in \mathbf{Spin}_+(1,3)$ we get[3]

$$\psi_{\bar{\Sigma}} = \psi_{\dot{\Sigma}}u^{-1}. \tag{2.48}$$

Now, we quoted in Section 2.3 that for $p+q \leq 5$, $\mathbf{Spin}_+(p,q) = \{u \in C\ell^+_{p,q} | u\tilde{u} = 1\}$. Then for all $\psi_\Sigma \in C\ell^+_{1,3}$ such that $\psi_\Sigma \tilde{\psi}_\Sigma \neq 0$ we obtain immediately the polar form

$$\psi_\Sigma = \rho^{1/2} e^{\beta E_5/2} R_\Sigma, \tag{2.49}$$

where $\rho \in \mathbb{R}^+, \beta \in \mathbb{R}, R_\Sigma \in \mathbf{Spin}_+(1,3), E_5 = E_0E_1E_2E_3$. With the above remark in mind we present the

Definition: A Dirac-Hestenes spinor (DHS) is an equivalence class of triplets (Σ, u, ψ_Σ), where Σ is an oriented orthonormal basis of $\mathbb{R}^{1,3} \subset C\ell_{1,3}$, $u \in \mathbf{Spin}_+(1,3)$, and $\psi_\Sigma \in C\ell^+_{1,3}$). We say that $(\Sigma, u, \psi_\Sigma) \sim (\Sigma_0, u_0, \psi_{\Sigma_0})$ if and only if $\psi_\Sigma = \psi_{\Sigma_0} u_0^{-1} u$, $\mathcal{H}(uu_0^{-1}) = L$, $\Sigma = L\Sigma_0 (\equiv u^{-1}u_0\Sigma_0 u_0^{-1} u)$, $u, u_0 \in \mathbf{Spin}_+(1,3)$, $L \in \mathcal{L}^\uparrow_+$. u_0 is arbitrary but fixed. A DHS determines a set of vectors $X_\mu \in \mathbb{R}^{1,3}$, $(\mu = 0,1,2,3)$ by a given representative $\psi_{\dot{\Sigma}}$ of the DHS in the basis Σ by

$$\psi : \dot{\Sigma} \to \mathbb{R}^{1,3}, \quad \psi_{\dot{\Sigma}} \dot{E}_\mu \tilde{\psi}_{\dot{\Sigma}} = X_\mu \quad (\dot{\Sigma} = (\dot{E}_0, \dot{E}_1, \dot{E}_2, \dot{E}_3)). \tag{2.50}$$

We give yet another equivalent definition of a DHS

Definition: A Dirac-Hestenes spinor is an element of the quotient set $C\ell^+_{1,3}/\mathcal{R}$ such that given the basis $\Sigma, \dot{\Sigma}$ of $\mathbb{R}^{1,3} \subset C\ell_{1,3}$, $\psi_\Sigma \in C\ell^+_{1,3}$, $\psi_{\dot{\Sigma}} \in C\ell^+_{1,3}$ then $\psi_{\dot{\Sigma}} \sim \psi_\Sigma(\mathrm{mod}\mathcal{R})$ if and only if $\psi_{\dot{\Sigma}} = \psi_\Sigma u^{-1}$, $\dot{\Sigma} = L\Sigma = u\Sigma u^{-1}$, $\mathcal{H}(u) = L$, $u \in \mathbf{Spin}_+(1,3)$, $L \in \mathcal{L}^\uparrow_+$.

With the canonical form of a DHS given by Eq. 2.49 some features of the hidden geometrical nature of the Dirac spinors defined above comes to light: Eq. 2.49 says that when $\psi_\Sigma \tilde{\psi}_\Sigma \neq 0$ the Dirac-Hestenes spinor ψ_Σ is equivalent to a Lorentz rotation followed by a dilation and a duality mixing given by the term $e^{\beta E_5/2}$, where β is the so-called Yvon-Takabayasi angle[32, 33] and the justification for the name duality rotation can be found in [4]. We emphasize that the definition of the Dirac-Hestenes spinors gives above is new. In the past objects $\psi \in C\ell^+_{1,3}$ satisfying $\psi X \tilde{\psi} = Y$, for $X, Y \in \mathbb{R}^{1,3} \subset C\ell_{1,3}$ have been called operator spinors (see, e.g., in [34, 9, 10]). DHS have been used as the departure point of many interesting results as, e.g., in [4, 35]–[38].

[3] In [9, 10] Lounesto calls 2Φ the mother of all the real spinors.

We remark that all the elements of the set $\{I_\Sigma\}$ of the ideals geometrically equivalent to I_{Σ_0} under the action of $u \in \mathbf{Spin}_+(1,3) \subset \mathbf{Spin}_+(4,1)$ have the same image $I = M_4(\mathbb{C})f$ where f is given by Eq. 2.32, i.e.,

$$f = \tfrac{1}{2}(1+\gamma_0)(1+i\gamma_1\gamma_2) \qquad i = \sqrt{-1},$$

where γ_μ, $\mu = 0,1,2,3$ are the Dirac matrices given by Eq. 2.31.

Then, if

$$\begin{array}{rcl} \gamma : C\ell_{4,1} & \to & M_4(\mathbb{C}) \equiv \mathrm{End}(M_4(\mathbb{C})f) \\ x & \mapsto & \gamma(x) : M_4(\mathbb{C})f \to M_4(\mathbb{C})f \end{array} \tag{2.46}$$

it follows that $\gamma(\mathcal{E}_\mu) = \gamma(\dot{\mathcal{E}}_\mu) = \gamma_\mu$, $\gamma(f_{\Sigma_0}) = \gamma(f_{\dot{\Sigma}}) = f$ for all $\mathcal{E}_\mu, \dot{\mathcal{E}}_\mu$ such that $\dot{\mathcal{E}}_\mu = u\mathcal{E}_\mu u^{-1}$ for some $u \in \mathbf{Spin}_+(1,3)$. Observe that all the information concerning the orthonormal frames $\Sigma_0, \dot{\Sigma}$, etc., disappear in the matrix representation of the ideals $I_{\Sigma_0}, I_{\dot{\Sigma}}, \ldots$ in $M_4(\mathbb{C})$ since all these ideals are mapped in the same ideal $I = M_4(\mathbb{C})f$.

With the above remark and taking into account Eq. 2.45 we are then lead to the following

Definition: A Covariant Dirac Spinor (CDS) for $\mathbb{R}^{1,3}$ is an equivalent class of triplets $(\Sigma, S(u), \Psi)$, Σ being an orthonormal basis of $\mathbb{R}^{1,3}$, $S(u) \in D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $\mathbf{Spin}_+(1,3)$, $u \in \mathbf{Spin}_+(1,3)$ and $\Psi \in M_4(\mathbb{C})f$ and

$$(\Sigma, S(u), \Psi) \sim (\Sigma_0, S(u_0), \Psi_0)$$

if and only if

$$\Psi = S(u)S^{-1}(u_0)\Psi_0, \quad \mathcal{H}(uu_0^{-1}) = L\Sigma_0, \quad L \in \mathcal{L}_+^\uparrow, \quad u \in \mathbf{Spin}_+(1,3). \tag{2.47}$$

The pair $(\Sigma, S(u))$ is called a spinorial frame. Observe that the CDS just defined depends on the choice of the original spinorial frame (Σ_0, u_0) and obviously, to different possible choices there correspond isomorphic ideals in $M_4(\mathbb{C})$. For simplicity we can fix $u_0 = 1, S(u_0) = 1$.

The definition of CDS just given agrees with that given by Choquet-Bruhat[31] except for the irrelevant fact that Choquet-Bruhat uses as the space of representatives of a CDS the complex four-dimensional vector space $\mathbb{C}^4$ instead of $I = M_4(\mathbb{C})f$. We see that Choquet-Bruhat's definition is well justified from the point of view of the theory of algebraic spinors presented above.

2.7. *Algebraic Dirac Spinors (ADS) and Dirac-Hestenes Spinors (DHS)*

We saw in Section 2.4 that there is bijection between $\psi_{\Sigma_0} \in C\ell_{4,1}^{++} \simeq C\ell_{1,3}^{+}$ and $\Psi_{\Sigma_0} \in I_{\Sigma_0} = C\ell_{4,1}^{+} f_{\Sigma_0}$, namely (Eq. 2.37),

$$\Psi_{\Sigma_0} = \psi_{\Sigma_0}\tfrac{1}{2}(1+\mathcal{E}_0)\tfrac{1}{2}(1+\mathrm{i}\mathcal{E}_1\mathcal{E}_2)$$

Then, as we already said, all information contained in Ψ_{Σ_0} (that is the representative in the basis Σ_0 of an algebraic spinor for $\mathbb{R}^{1,3}$) is also contained in $\psi_{\Sigma_0} \in C\ell_{4,1}^{++} \simeq C\ell_{1,3}^{+}$. We are then lead to the following

2.8. *Fierz Identities*

The formulation of the Fierz identities [13] using the CDS $\Psi \in \mathbb{C}^4$ is well known.[14] Here we present the identities for $\Psi_{\Sigma_0} \in I_{\Sigma_0} \simeq (\mathbb{C} \otimes C\ell_{1,3})f_{\Sigma_0}$ and for the DHS $\psi_{\Sigma_0} \in C\ell^{+}_{1,3}$ [9, 10]. Let then $\Psi \in \mathbb{C}^4$ be a representative of a CDS for $\mathbb{R}^{1,3}$ associated to the basis $\Sigma_0 = \{E_0, E_1, E_2, E_3\}$ of $\mathbb{R}^{1,3} \subset C\ell_{1,3}$. Then Ψ, Ψ_{Σ_0} determines the following so-called bilinear covariants,

$$\begin{aligned}
\sigma &= \Psi^{\dagger}\gamma_0\Psi = 4\langle\tilde{\Psi}^{*}_{\Sigma_0}\Psi_{\Sigma_0}\rangle_0,\\
J_\mu &= \Psi^{\dagger}\gamma_0\gamma_\mu\Psi = 4\langle\tilde{\Psi}^{*}_{\Sigma_0}E_\mu\Psi_{\Sigma_0}\rangle_0,\\
S_{\mu\nu} &= \Psi^{\dagger}\gamma_0 i\gamma_{\mu\nu}\Psi = 4\langle\tilde{\Psi}^{*}_{\Sigma_0}iE_{\mu\nu}\Psi_{\Sigma_0}\rangle_0,\\
K_\mu &= \Psi^{\dagger}\gamma_0 i\gamma_{0123}\Psi = 4\langle\tilde{\Psi}^{*}_{\Sigma_0}iE_{0123}E_\mu\Psi_{\Sigma_0}\rangle_0,\\
\omega &= -\Psi^{\dagger}\gamma_0\gamma_{0123}\Psi = -4\langle\tilde{\Psi}^{*}_{\Sigma_0}E_{0123}\Psi_{\Sigma_0}\rangle_0,
\end{aligned} \tag{2.51}$$

where $\dagger$ means Hermitian conjugation and $*$ complex conjugation. We remark that the reversion in $C\ell_{4,1}$ corresponds to the reversion plus complex conjugation in $\mathbb{C} \otimes C\ell_{1,3}$.

All the bilinear covariants are real and have physical meaning in the Dirac theory of the electron, but its geometrical nature appears clearly when these bilinear covariants are formulated with the aid of the DHS.

Introducing the Hodge dual of a Clifford number $X \in C\ell_{1,3}$ by

$$\star X = \tilde{X}E_5, \qquad E_5 = E_0E_1E_2E_3 \tag{2.52}$$

the bilinear covariants given by Eq. 2.51 become in terms of ψ_{Σ_0}, the representative of a DHS in the orthonormal basis $\Sigma_0 = \{E_0, E_1, E_2, E_3\}$ of $\mathbb{R}^{1,3} \subset C\ell_{1,3}$

$$\begin{array}{ll}
\psi_{\Sigma_0}\tilde{\psi}_{\Sigma_0} = \sigma + \star\omega & J = J_\mu E^\mu\\
\psi_{\Sigma_0}E_0\tilde{\psi}_{\Sigma_0} = J & S = \frac{1}{2}S_{\mu\nu}E^\mu E^\nu\\
\psi_{\Sigma_0}E_1E_2\tilde{\psi}_{\Sigma} = S & K = K_\mu E^\mu\\
\psi_{\Sigma_0}E_3\tilde{\psi}_{\Sigma_0} = K & E^\mu = \eta^{\mu\nu}E_\mu\\
\psi_{\Sigma_0}E_0E_3\tilde{\psi}_{\Sigma_0} = \star S & \eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)\\
\psi_{\Sigma_0}E_0E_1E_2\tilde{\psi}_{\Sigma_0} = \star K &
\end{array} \tag{2.53}$$

The Fierz identities are

$$J^2 = \sigma^2 + \omega^2, \quad J\cdot K = 0, \quad J^2 = -K^2, \quad J \wedge K = -(\omega + \star\sigma)S \tag{2.54}$$

$$\left\{\begin{array}{ll}
S\cdot J = \omega K & S\cdot K = \omega J\\
(\star S)\cdot J = -\sigma K & (\star S)\cdot K = -\sigma J\\
S\cdot S = \omega^2 - \sigma^2 & (\star S)\cdot S = -2\sigma\omega
\end{array}\right. \tag{2.55}$$

$$\left\{\begin{array}{ll}
JS = -(\omega + \star\sigma)K & KS = -(\omega + \star\sigma)J\\
SJ = -(\omega - \star\sigma)K & SK = -(\omega - \star\sigma)J\\
S^2 = \omega^2 - \sigma^2 - 2\sigma(\star\omega) & \\
S^{-1} = -S(\sigma - \star\omega)^2/(\sigma^2 + \omega^2) = KSK/(\sigma^2 + \omega^2)^2 &
\end{array}\right. \tag{2.56}$$

The proof of these identities using the DHS is almost a triviality.

The importance of the bilinear covariants is due to the fact that we can recover from them the CDS $\Psi_{\Sigma_0} \in M_4(\mathbb{C})f$ or all other kinds of Dirac spinors defined above through an algorithm due to Crawford (see also [9, 10]). Indeed, representing the images of the bilinear covariants in $\mathcal{C}\ell_{1,3}$ and $\mathcal{C}\ell^+_{4,1} \subset \mathcal{C}\ell_{4,1}$ under the mapping g (Eq. 2.25) by the same letter we have that the following result holds true: let

$$Z_{\Sigma_0} = (\sigma + J + iS + i(\star K) + \star\omega) \in \mathbb{C} \otimes \mathcal{C}\ell_{1,3} \tag{2.57}$$

where σ, J, S, K, ω are the bilinear covariants of $\Psi_{\Sigma_0} \simeq (\mathbb{C} \otimes \mathcal{C}\ell_{1,3})f_{\Sigma_0}$. Take $\eta_{\Sigma_0} \in (\mathbb{C} \otimes \mathcal{C}\ell_{1,3})f_{\Sigma_0}$ such that $\tilde{\eta}^*_{\Sigma_0}\Psi_{\Sigma_0} \neq 0$. Then Ψ_{Σ_0} and $Z_{\Sigma_0}\eta_{\Sigma_0}$ differ by a complex factor. We have

$$\Psi_{\Sigma_0} = \frac{1}{4N_{\eta\Sigma_0}} e^{-i\alpha} Z_{\Sigma_0}\eta_{\Sigma_0} \tag{2.58}$$

$$N_{\eta_{\Sigma_0}} = \sqrt{\langle \tilde{\eta}^*_{\Sigma_0} Z_{\Sigma_0} \eta_{\Sigma_0}\rangle_0}\,, \quad e^{-i\alpha} = \frac{4}{N\eta_{\Sigma_0}} \langle \tilde{\eta}^* \Psi \rangle_0 \tag{2.59}$$

Choosing $\eta_{\Sigma_0} = f_{\Sigma_0}$, we obtain

$$N_{f_{\Sigma_0}} = \frac{1}{2}\sqrt{\sigma + J \cdot E_0 - S \cdot (E_1 E_2) - K \cdot E_3}\,, \quad e^{-i\alpha} = \psi_1/|\psi_1|, \tag{2.60}$$

where ψ_1 is the first component of Ψ_{Σ_0} in the spinorial basis $\{s_i\}$.

It is easier to recuperate the CDS from its bilinear covariants if we use the DHS $\psi_{\Sigma_0} \in \mathcal{C}\ell^+_{1,3} \simeq (\mathcal{C}\ell^+_{4,1})^+$ since putting

$$\begin{cases} \psi_{\Sigma_0}(1 + E_0)\tilde{\psi}_{\Sigma_0} = P \\ \psi_{\Sigma_0}(1 + E_0)E_1 E_2 \tilde{\psi}_{\Sigma_0} = Q \end{cases} \tag{2.61}$$

$$\psi_{\Sigma_0}(1 + E_0)(1 + iE_1 E_2)\tilde{\psi}_{\Sigma_0} = (P + iQ) \tag{2.62}$$

results

$$P = \sigma + J + \omega \qquad Q = S + \star K \tag{2.63}$$

and

$$Z_{\Sigma_0} = P\tfrac{1}{2}(1 + \tfrac{i}{2\sigma}Q)^2 \tag{2.64}$$

valid for $\sigma \neq 0, \omega \neq 0$ (for other cases see [10]). From the above results it follows that Ψ_{Σ_0} can be easily determined from its bilinear covariant except for a "complex" E_2E_1 phase factor.

3. The Clifford Bundle of Spacetime and their Irreducible Module Representations

3.1. *The Clifford Bundle of Spacetime*

Let M be a four dimensional, real, connected, paracompact manifold. Let TM $[T^*M]$ be the tangent [cotangent] bundle of M.

Definition: A Lorentzian manifold is a pair (M, g), where $g \in \sec T^*M \times T^*M$ is a Lorentzian metric of signature (1,3), i.e., for all $x \in M$, $T_xM \simeq T_x^*M \simeq \mathbb{R}^{1,3}$, where $\mathbb{R}^{1,3}$ is the vector Minkowski space.

Definition: A spacetime $\mathcal{M}$ is a triple (M, g, ∇) where (M, g) is a time oriented and spacetime oriented Lorentzian manifold and ∇ is a linear connection for M such that $\nabla g = 0$. If in addition $\mathbf{T}(\nabla) = 0$ and $\mathbf{R}(\nabla) \neq 0$, where $\mathbf{T}$ and $\mathbf{R}$ are respectively the torsion and curvature tensors, then $\mathcal{M}$ is said to be a Lorentzian spacetime. When $\nabla g = 0$, $\mathbf{T}(\nabla) = 0$, $\mathbf{R}(\nabla) = 0$, $\mathcal{M}$ is called Minkowski spacetime and will be denote by $\mathbb{M}$. When $\nabla g = 0$, $\mathbf{T}(\nabla) \neq 0$ and $\mathbf{R}(\nabla) = 0$ or $\mathbf{R}(\nabla) \neq 0$, $\mathcal{M}$ is said to be a Riemann-Cartan spacetime.

In what follows $P_{SO_+(1,3)}(\mathcal{M})$ denotes the principal bundles of oriented Lorentz tetrads.[8, 23] By g^{-1} we denote the "metric" of the cotangent bundle.

It is well known that the natural operations on metric vector spaces, such as, e.g., direct sum, tensor product, exterior power, etc., carry over canonically to vector bundles with metrics. Take, e.g., the cotangent bundle T^*M. If $\pi : T^*M \to M$ is the canonical projection, then in each fiber $\pi^{-1}(x) = T_x^*M \simeq \mathbb{R}^{1,3}$, the "metric" g^{-1} can be used to construct a Clifford algebra $\mathcal{C}\ell(T_x^*M) \simeq C\ell_{1,3}$. We have the

Definition: The Clifford bundle of spacetime $\mathcal{M}$ is the bundle of algebras

$$\mathcal{C}\ell(\mathcal{M}) = \bigcup_{x \in M} \mathcal{C}\ell(T_x^*M) \tag{3.1}$$

As is well known $\mathcal{C}\ell(\mathcal{M})$ is the quotient bundle

$$\mathcal{C}\ell(\mathcal{M}) = \frac{\tau M}{\mathbf{J}(\mathcal{M})} \tag{3.2}$$

where $\tau M = \oplus_{r=0}^{\infty} T^{0,r}(M)$ and $T^{(0,r)}(M)$ is the space of r-covariant tensor fields, and $\mathbf{J}(\mathcal{M})$ is the bundle of ideals whose fibers at $x \in M$ are the two side ideals in τM generated by the elements of the form $a \otimes b + b \otimes a - 2g^{-1}(a, b)$ for $a, b \in T^*\mathcal{M}$.

Let $\pi_c : \mathcal{C}\ell(\mathcal{M}) \to M$ be the canonical projection of $\mathcal{C}\ell(\mathcal{M})$ and let $\{U_\alpha\}$ be an open covering of M. From the definition of a fibre bundle[22] we know that there is a trivializing mapping $\varphi\alpha : \pi_c^{-1}(U_\alpha) \to U_\alpha \times C\ell_{1,3}$ of the form $\varphi\alpha(p) = (\pi_c(p), \overset{\triangle}{\varphi}_\alpha(p))$. If $U_{\alpha\beta} = U_\alpha \cap U_\beta$ and $x \in U_{\alpha\beta}$, $p \in \pi_c^{-1}(x)$, then

$$\overset{\triangle}{\varphi}_\alpha(p) = f_{\alpha\beta}(x) \overset{\triangle}{\varphi}_\beta(p) \tag{3.3}$$

for $f_{\alpha\beta}(x) \in \mathrm{Aut}(C\ell_{1,3})$, where $f_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Aut}(C\ell_{1,3})$ are the transition mappings of $\mathcal{C}\ell(\mathcal{M})$. We know that every automorphism of $C\ell_{1,3}$ is inner and it follows that,

$$f_{\alpha\beta}(x) \overset{\triangle}{\varphi}_\beta(p) = g_{\alpha\beta}(x) \overset{\triangle}{\varphi}_\beta(p) g_{\alpha\beta}(x)^{-1} \tag{3.4}$$

for some $g_{\alpha\beta}(x) \in C\ell_{1,3}^*$, the group of invertible elements of $C\ell_{1,3}$. We can write equivalently instead of Eq. 3.4,

$$f_{\alpha\beta}(x) \overset{\triangle}{\varphi}_\beta(p) = \overset{\triangle}{\varphi}_\beta(a_{\alpha\beta} p a_{\alpha\beta}^{-1}) \tag{3.5}$$

for some invertible element $a_{\alpha\beta} \in C\ell(T_x^* M)$.

Now, the group $SO_+(1,3)$ has, as we know (Section 2), a natural extension in the Clifford algebra $C\ell_{1,3}$. Indeed we know that $C\ell_{1,3}^*$ acts naturally on $C\ell_{1,3}$ as an algebra automorphism through its adjoint representation $\mathrm{Ad} : u \mapsto \mathrm{Ad}_u$, $\mathrm{Ad}_u(a) = uau^{-1}$. Also $\mathrm{Ad}\,|_{\mathbf{Spin}_+(1,3)} = \sigma$ defines a group homeomorphism $\sigma : \mathbf{Spin}_+(1,3) \to SO_+(1,3)$ which is onto with kernel $\mathbf{Z}_2$. It is clear, since $\mathrm{Ad}_{-1} =$ identity, that $\mathrm{Ad} : \mathbf{Spin}_+(1,3) \to \mathrm{Aut}(C\ell_{1,3})$ descends to a representation of $SO_+(1,3)$. Let us call Ad' this representation, i.e., $\mathrm{Ad}' : SO_+(1,3) \to \mathrm{Aut}(C\ell_{1,3})$. Then we can write $\mathrm{Ad}'_{\sigma(u)}\, a = \mathrm{Ad}_u a = uau^{-1}$.

From this it is clear that the structure group of the Clifford bundle $C\ell(\mathcal{M})$ is reducible from $\mathrm{Aut}(C\ell_{1,3})$ to $SO_+(1,3)$. This follows immediately from the existence of the Lorentzian structure (M, g) and the fact that $C\ell(\mathcal{M})$ is the exterior bundle where the fibres are equipped with the Clifford product. Thus the transition maps of the principal bundle of oriented Lorentz tetrads $P_{SO_+(1,3)}(\mathcal{M})$ can be (through Ad') taken as transition maps for the Clifford bundle. We then have the result[39]

$$C\ell(\mathcal{M}) = P_{SO_+(1,3)}(\mathcal{M}) \times_{\mathrm{Ad}'} C\ell_{1,3} \tag{3.6}$$

3.2. Spinor Bundles

Definition:[24] A spinor structure for $\mathcal{M}$ consists of a principal fibre bundle $\pi_s : P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \to M$ with group $SL(2,\mathbb{C}) \simeq \mathbf{Spin}_+(1,3)$ and a map

$$s : P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \to P_{SO_+(1,3)}(\mathcal{M})$$

satisfying the following conditions

(*i*) $\pi(s(p)) = \pi_s(p)\ \forall p \in P_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$

(*ii*) $s(pu) = s(p)\mathcal{H}(u)\ \forall p \in P_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ and $\mathcal{H} : SL(2,\mathbb{C}) \to SO_+(1,3)$.

Now, in Section 2 we learned that the minimal left (right) ideals of $C\ell_{p,q}$ are irreducible left (right) module representations of $C\ell_{p,q}$ and we define a covariant and algebraic Dirac spinors as elements of quotient sets of the type $\{I_\Sigma\}/\mathbb{R}$ (sections 2.6 and 2.7) in appropriate Clifford algebras. We defined also in Section 2 the DHS. We are now interested in defining algebraic Dirac spinor fields (ADSF) and also Dirac-Hestenes spinor fields (DHSF).

So, in the spirit of Section 2 the following question naturally arises: Is it possible to find a vector bundle $\pi_s : S(\mathcal{M}) \to M$ with the property that each fiber over $x \in M$ is an irreducible module over $C\ell(T_x^* M)$?

The answer to the above question is in general no. Indeed it is now well known[40] that the necessary and sufficient conditions for $S(\mathcal{M})$ to exist is that the Spinor Structure bundle $P_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ exists, which implies the vanishing of the second Stiefel-Whitney class of M, i.e., $\omega_2(M) = 0$. For a spacetime $\mathcal{M}$ this is equivalent, as shown originally by Geroch[41, 42] that $P_{SO_+(1,3)}(\mathcal{M})$ is a trivial bundle, i.e., that it admits a global section. When $P_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ exists we said that $\mathcal{M}$ is a spin manifold.

Definition: A real spinor bundle for $\mathcal{M}$ is the vector bundle

$$S(\mathcal{M}) = P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \times_\mu \mathbf{M} \tag{3.7}$$

where $\mathbf{M}$ is a left (right) module for $C\ell_{1,3}$ and where $\mu : P_{\mathbf{Spin}_+(1,3)} \to \mathrm{SO}_+(1,3)$ is a representation given by left (right) multiplication by elements of $\mathbf{Spin}_+(1,3)$.

Definition: A complex spinor bundle for $\mathcal{M}$ is the vector bundle

$$S_c(\mathcal{M}) = P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \times_{\mu_c} \mathbf{M}_c \tag{3.8}$$

where $\mathbf{M}$ is a complex left (right) module for $\mathbb{C} \otimes C\ell_{1,3} \simeq C\ell_{4,1} \simeq M_4(\mathbb{C})$, and where $\mu_c : P_{\mathbf{Spin}_+(1,3)} \to \mathrm{SO}_+(1,3)$ is a representation given by left (right) multiplication by elements of $\mathbf{Spin}_+(1,3)$.

Taking, e.g. $\mathbf{M}_c = \mathbb{C}^4$ and μ_c the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $\mathbf{Spin}_+(1,3)$ in $\mathrm{End}(\mathbb{C}^4)$, we recognize immediately the usual definition of the covariant spinor bundle of $\mathcal{M}$, as given, e.g., in [31].

Since, besides being right (left) linear spaces over $\mathbb{H}$, the left (right) ideals of $C\ell_{1,3}$ are representation modules of $C\ell_{1,3}$, we have the

Definition: $I(\mathcal{M})$ is a real spinor bundle for $\mathcal{M}$ such that $\mathbf{M}$ in Eq. 3.7 is I, a minimal left (right) ideal of $C\ell_{1,3}$.

In what follows we fix the ideal taking $I = C\ell_{1,3}\frac{1}{2}(1+E_0) = C\ell_{1,3}e$. If $\pi_I : I(\mathcal{M}) \to M$ is the canonical projection and $\{U_\alpha\}$ is an open covering of M we know from the definition of a fibre bundle that there is a trivializing mapping $\chi_\alpha(q) = (\pi_I(q), \overset{\Delta}{\chi}_\alpha(q))$. If $U_{\alpha\beta} = U_\alpha \cap U_\beta$ and $x \in U_{\alpha\beta}$, $q \in \pi_I^{-1}(U_\alpha)$, then

$$\overset{\Delta}{\chi}_\alpha(q) = g_{\alpha\beta}(x)\, \overset{\Delta}{\chi}_\beta(q) \tag{3.9}$$

for the transition maps in $\mathbf{Spin}_+(1,3)$.[4] Equivalently

$$\overset{\Delta}{\chi}_\alpha(q) = \overset{\Delta}{\chi}_\beta(a_{\alpha\beta} q) \tag{3.10}$$

for some $a_{\alpha\beta} \in C\ell(T^*_x\mathcal{M})$. Thus, for the transition maps to be in $\mathbf{Spin}_+(1,3)$ it is equivalent that the right action of $\mathbb{H}e = e\mathbb{H} = eC\ell_{1,3}e$ be the defined in the bundle, since for $q \in \pi_x^{-1}(x), x \in U_\alpha$ and $a \in \mathbb{H}$ we define qa as the unique element of $\pi_q^{-1}(x)$ such that

$$\overset{\Delta}{\chi}_\alpha(qa) = \overset{\Delta}{\chi}_\alpha(q)a \tag{3.11}$$

Naturally, for the validity of Eq. 3.11 to make sense it is necessary that

$$g_{\alpha\beta}(x)(\overset{\Delta}{\chi}_\alpha(q)a) = (g_{\alpha\beta}(x)\, \overset{\Delta}{\chi}_\alpha(q))a \tag{3.12}$$

---

[4] We start with transition maps in $C\ell^*_{p,q}$ and then by the bundle reduction process we end with $\mathbf{Spin}_+(1,3)$.

and Eq. 3.12 implies that the transition maps are $\mathbb{H}$-linear.[5]

Let $f_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Aut}(C\ell_{1,3})$ be the transition functions for $C\ell(\mathcal{M})$. On the intersection $U_\alpha \cap U_\beta \cap U_\alpha$ it must hold

$$f_{\alpha\beta} f_{\beta\gamma} = f_{\alpha\gamma} \tag{3.13}$$

We say that a set of *lifts* of the transition functions of $C\ell(\mathcal{M})$ is a set of elements in $C\ell^\star_{1,3}$, $\{g_{\alpha\beta}\}$ such that if

$$\begin{aligned} &\mathrm{Ad} : C\ell^\star_{1,3} \to \mathrm{Aut}(C\ell_{1,3}) \\ &\mathrm{Ad}(u)X = uXu^{-1} \ , \ \forall X \in C\ell_{1,3} \end{aligned}$$

then $\mathrm{Ad}_{g_{\alpha\beta}} = f_{\alpha\beta}$ in all intersections.

Using the theory of the Cěch cohomology[43] it can be shown that any set of lifts can be used to define a characteristic class $\omega(C\ell(\mathcal{M})) \in \check{H}^2(M, \mathbb{H}^*)$, the second Cěch cohomology group with values in $\mathbb{H}^*$, the space of all non zero $\mathbb{H}$-valued germs of functions in M.

We say that we can coherently lift the transition maps $C(\mathcal{M})$ to a set $\{g_{\alpha\beta}\} \in C\ell^\star_{1,3}$ if in the intersection $U_\alpha \cap U_\beta \cap U_\gamma$, $\forall \alpha, \beta, \gamma$, we have

$$g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma} \tag{3.14}$$

This implies that $\omega(C\ell(\mathcal{M})) = \mathrm{id}_{(2)}$, i.e., M is Cěch trivial and the coherent lifts can be classified by an element of the first Cěch cohomology group $\check{H}^1(M, \mathbb{H}^*)$. Benn and Turcker[43] proved the important result:

Theorem: There exists a bundle of irreducible representation modules for $C\ell(\mathcal{M})$ if and only if the transition maps of $C\ell(\mathcal{M})$ can be coherently lift from $\mathrm{Aut}(C\ell_{1,3})$ to $C\ell^\star_{1,3}$.

They showed also by defining the concept of equivalence classes of coherent lifts that such classes are in one to one correspondence with the equivalence classes of bundles of irreducible representation modules of $C\ell(\mathcal{M})$, $I(\mathcal{M})$ and $I'(\mathcal{M})$ being equivalent if there is a bundle isomorphsim $\rho : I(\mathcal{M}) \to I'(\mathcal{M})$ such that

$$\rho(a_x q) = a_x \rho(q), \quad \forall a_x \in C\ell(T^*_x M), \forall q \in \pi_I^{-1}(x)$$

By defining that a *spin structure* for M is an equivalence class of bundles of irreducible representation modules for $C\ell(\mathcal{M})$, represented by $I(\mathcal{M})$, Benn and Turcker showed that this agrees with the usual conditions for M to be a spin manifold.

Now, recalling the definition of a vector bundle we see that the prescription for the construction of $I(\mathcal{M})$ is the following. Let $\{U_\alpha\}$ be an open covering of M with $f_{\alpha\beta}$ being the transition functions for $C\ell(\mathcal{M})$ and let $\{g_{\alpha\beta}\}$ be a coherent lift which is then used to quotient the set $\cup_\alpha \ U_\alpha \times I$, where e.g., $I = C\ell_{1,3}\frac{1}{2}(1 + E_0)$ to form the bundle

[5] Without the $\mathbb{H}$-linear structure there exists more general bundles of irreducible modules for $C\ell(\mathcal{M})$.[43]

$\cup_\alpha U_\alpha \times I/\mathcal{R}$ where $\mathcal{R}$ is the equivalence relation defined as follows. For each $x \in U_\alpha$ we choose a minimal left ideal $I^\alpha_{\Sigma(x)}$ in $C\ell(T^*_x M)$ by requiring[6]

$$\overset{\Delta}{\varphi}_\alpha \left(I^\alpha_{\Sigma(x)}\right) = I \tag{3.15}$$

As before we introduce $a_{\alpha\beta} \in C\ell(T^*_x M)$ such that

$$\overset{\Delta}{\varphi}_\beta \left(a_{\alpha\beta}\right) = g_{\alpha\beta}(x) \tag{3.16}$$

Then for all $X \in C\ell(T^*_x\mathcal{M}), \overset{\Delta}{\varphi}_\alpha (X) = \overset{\Delta}{\varphi}_\beta (a_{\alpha\beta}Xa^{-1}_{\alpha\beta})$. So, if $X \in I^\alpha_{\Sigma(x)}$ then $a_{\alpha\beta}Xa^{-1}_{\alpha\beta}$ and also $Xa^{-1}_{\alpha\beta} \in I^\beta_{\dot{\Sigma}(x)}$. Putting $Y_\alpha = U_\alpha \times I^\alpha_{\Sigma(x)}$ $Y = \cup_\alpha Y_\alpha$, the equivalence relation $\mathcal{R}$ is defined on Y by $(U_\alpha, x, \psi_\Sigma) \simeq (U_\beta, x, \psi_{\dot{\Sigma}})$ if and only if

$$\psi_{\dot{\Sigma}} = \psi_\Sigma a^{-1}_{\alpha\beta} \tag{3.17}$$

Then, $I(\mathcal{M}) = Y/\mathcal{R}$ is a bundle which is an irreducible module representation of $\mathcal{C}(\mathcal{M})$. We see that Eq. 3.17 captures nicely for $a_{\alpha\beta} \in \mathbf{Spin}_+(1,3) \subset C\ell^+_{1,3}$ our discussion of ADS of Section 2. We then have

Definition: An algebraic Dirac Spinor Field (ADSF) is a section of $I(\mathcal{M})$ with $a_{\alpha\beta} \in \mathbf{Spin}_+(1,3) \subset C\ell^+_{1,3}$ in Eq. 3.17.

From the above results we see that ADSF are equivalence classes of sections of $C\ell(\mathcal{M})$ and it follows that ADSF can locally be represented by a sum of inhomogeneous differential forms that lie in a minimal left ideal of the Clifford algebra $C\ell_{1,3}$ at each spacetime point.

In Section 2 we saw that besides the ideal $I = C\ell_{1,3}\frac{1}{2}(1 + E_0)$, other ideals exist for $C\ell_{1,3}$ that are only algebraically equivalent to this one. In order to capture all possibilities we recall that $C\ell_{1,3}$ can be considered as a module over itself by left (or right) multiplication by itself. We are thus lead to the

Definition: The Real Spin-Clifford bundle of $\mathcal{M}$ is the vector bundle

$$C\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) = P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \times_\ell C\ell_{1,3} \tag{3.18}$$

It is a "principal $C\ell_{1,3}$ bundle", i.e., it admits a free action of $C\ell_{1,3}$ on the right.[7, 39] There is a natural embedding $P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \subset C\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ which comes from the embedding $\mathbf{Spin}_+(1,3) \subset C\ell^+_{1,3}$. Hence every real spinor bundle for $\mathcal{M}$ can be captured from $C\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$. $C\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ is different from $C\ell(\mathcal{M})$. Their relation can be discovered remembering that the representation

$$\mathrm{Ad} : \mathbf{Spin}_+(1,3) \to \mathrm{Aut}(C\ell_{1,3}) \quad \mathrm{Ad}_u X = uXu^{-1} \qquad u \in \mathbf{Spin}_+(1,3)$$

is such that Ad_{-1} = identity and so Ad descends to a representation Ad$'$ of $\mathrm{SO}_+(1,3)$ which we considered above. It follows that when $P_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ exists

$$C\ell(\mathcal{M}) = P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \times_{\mathrm{Ad}'} C\ell_{1,3} \tag{3.19}$$

[6] Recall the notation of Section 2 where Σ is an orthonormal frame, etc.

From this it is easy to prove that indeed $S(\mathcal{M})$ is a bundle of modules over the bundle of algebras $C\ell(\mathcal{M})$.[11]

We end this section defining the local Clifford product of $X \in \sec C\ell(\mathcal{M})$ by a section of $I(\mathcal{M})$ or $C\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$. If $\varphi \in I(\mathcal{M})$ we put $X\varphi = \phi \in \sec I(\mathcal{M})$ and the meaning of Eq. 3.19 is that

$$\phi(x) = X(x)\rho(x) \qquad \forall x \in M \tag{3.20}$$

where $X(x)\varphi(x)$ is the Clifford product of the Clifford numbers $X(x), \varphi(x) \in C\ell_{1,3}$.

Analogously if $\psi \in C\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$

$$X\psi = \xi \in C\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \tag{3.21}$$

and the meaning of Eq. 3.20 is the same as in Eq. 3.19.

With the above definition we can "identify" from the algebraically point of view sections of $C\ell(\mathcal{M})$ with sections of $I(\mathcal{M})$ or $C\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$.

3.3. Dirac-Hestenes Spinor Fields (DHSF)

The main conclusion of Section 3.2 is that a given ADSF which is a section of $I(\mathcal{M})$ can locally be represented by a sum of inhomogeneous differential forms in $C\ell(\mathcal{M})$ that lies in a minimal left ideal of the Clifford algebra $C\ell_{1,3}$ at each point $x \in M$. Our objective here is to define a DHSF on $\mathcal{M}$. In order to achieve our goal we need to find a vector bundle such that a DHSF is an appropriate section.

In Section 2.7 we defined a DHS as an element of the quotient set $C\ell^+_{1,3}/\mathcal{R}$ where $\mathcal{R}$ is the equivalence relation given by Eq. 2.50. We immediately realize that if it is possible to define globally on M the equivalence relation $\mathcal{R}$, then a DHSF can be defined as an even section of the quotient bundle $C\ell(\mathcal{M})/\mathcal{R}$.

More precisely, if $\Sigma = \{\gamma^a\}$, $(a = 0,1,2,3)$ and $\dot{\Sigma} = \{\dot{\gamma}^a\}$, $\gamma^a, \dot{\gamma}^a \in \sec \bigwedge^1(T^*M) \subset C\ell(M)$ are such that $\dot{\gamma}^a = R\gamma^a R^{-1}$, where $R \in \sec C\ell^+(\mathcal{M})$ is such that $R(x) \in \mathbf{Spin}_+(1,3)$ for all $x \in M$, we say that $\dot{\Sigma} \sim \Sigma$. Then a DHSF is an equivalence class of even sections of $C\ell(\mathcal{M})$ such that its representatives ψ_Σ and $\psi_{\dot{\Sigma}}$ in the basis Σ and $\dot{\Sigma}$ define a set of 1-form fields $X^a \in \sec \Lambda^1(T^*M) \subset \sec C\ell(\mathcal{M})$ by

$$X^a(x) = \psi_{\dot{\Sigma}}(x)\dot{\gamma}^a(x)\tilde{\psi}_{\dot{\Sigma}}(x) = \psi_\Sigma(x)\gamma^a(x)\tilde{\psi}_\Sigma(x) \tag{3.22}$$

i.e., ψ_Σ and $\psi_{\dot{\Sigma}}$ are equivalent if and only if

$$\psi_{\dot{\Sigma}} = \psi_\Sigma R^{-1}. \tag{3.23}$$

Observe that for $\dot{\Sigma} \sim \Sigma$ to be globally defined it is necessary that the 1-form fields $\{\gamma^a\}$ and $\{\dot{\gamma}^a\}$ are globally defined. It follows that $P_{\mathrm{SO}_+(1,3)}(\mathcal{M})$, the principal bundle of orthonormal frames must have a global section, i.e., it must be trivial. This conclusion follows directly from our definitions, and it is a necessary condition for the existence of a DHSF. It is obvious that the condition is also sufficient. This suggests the

Definition: A spacetime $\mathcal{M}$ admits a spinor structure if and only if it is possible to define a global DHSF on it.
Then, it follows the

Theorem: Let $\mathcal{M}$ be a spacetime ($\dim M = 4$). Then the necessary and sufficient condition for M to admit a spinor structure is that $P_{SO_+(1,3)}(\mathcal{M})$ admits a global section.

In Section 3.1 we defined the spinor structure as the principal bundle $P_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ and a theorem with the same statement as the above one is known in the literature as Geroch's Theorem.[41] Geroch's deals with the existence of covariant spinor fields on $\mathcal{M}$, but since we already proved, e.g., that covariant Dirac spinors are equivalent to DHS, our theorem and Geroch's one are equivalent. This can be seen more clearly once we verify that

$$\frac{C\ell(\mathcal{M})}{\mathcal{R}} \equiv C\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \tag{3.24}$$

where $C\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) = P_{\mathbf{Spin}_+(1,3)} \times_\ell C\ell_{1,3}$ is the Spin-Clifford bundle defined in Section 3.1. To see this, recall that a DHSF determines through Eq. 3.20 a set of 1-form fields $X^a \in \sec \bigwedge^1(T^*M) \subset \sec C\ell(\mathcal{M})$. Under an active transformation,

$$X^a \mapsto \dot{X}^a = RX^aR^{-1}, \quad R(x) \in \mathbf{Spin}_+(1,3), \qquad \forall x \in M \tag{3.25}$$

we obtain the active transformation of a DHSF which in the Σ-frame is given by[7]

$$\psi_\Sigma \mapsto \psi'_{\dot\Sigma} = R\psi_\Sigma \tag{3.26}$$

From Eq. 3.23 it follows that the action of $\mathbf{Spin}_+(1,3)$ on the typical fibre $C\ell_{1,3}$ of $C\ell(\mathcal{M})/\mathcal{R}$ must be through left multiplication, i.e. given $u \in \mathbf{Spin}_+(1,3)$ and $X \in C\ell_{1,3}$, and taking into account that $C\ell_{1,3}$ is a module over itself we can define $\ell_u \in \mathrm{End}(C\ell_{1,3})$ by $\ell_u(X) = ux, \forall X \in C\ell_{1,3}$. In this way we have a representation $\ell : \mathbf{Spin}_+(1,3) \to \mathrm{End}(C\ell_{1,3}), u \mapsto \ell_u$. Then we can write,

$$\frac{C\ell(\mathcal{M})}{\mathcal{R}} = P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \times_\ell C\ell_{1,3}$$

3.4. A Comment on Amorphous Spinor Fields

Crumeyrolle[15] gives the name of amorphous spinors fields to ideal sections of the Clifford bundle $C\ell(\mathcal{M})$. Thus an amorphous spinor field ϕ is a section of $C\ell(\mathcal{M})$ such that $\phi e = \phi$, with e being an idempotent section of $C\ell(\mathcal{M})$.

It is clear from our discussion of the Fierz identities that are fundamental for the physical interpretation of Dirac theory that these fields cannot be used in a physical theory. The same holds true for the so-called Dirac-Kähler fields[16]–[18, 25] which are sections of $C\ell(\mathcal{M})$. These fields do not have the appropriate transformation law under a Lorentz rotation of the local tetrad field. In particular the Dirac-Hestenes equation written for amorphous fields is not covariant (see Section 6). We think that with our definitions of algebraic and DH spinor fields physicists can safely use our formalism which is not only nice but extremely powerful.

[7] Observe also that in the $\dot\Sigma$ we have for the representative of the actively transformed DHSF the relation $\psi'_{\dot\Sigma} = R\psi_\Sigma R^{-1}$.

4. The Covariant Derivative of Clifford and Dirac-Hestenes Spinor Fields

In what follows, as in Section 3, $\mathcal{M} = \langle M, \nabla, g \rangle$ will denote a general Riemann-Cartan spacetime. Since $\mathcal{C}\ell(\mathcal{M}) = \tau M / J(\mathcal{M})$ it is clear that any linear connection defined in τM such that $\nabla g = 0$ passes to the quotient $\tau M / J(\mathcal{M})$ and thus define an algebra bundle connection.[15] In this way, the covariant derivative of a Clifford field $A \in \sec \mathcal{C}\ell(\mathcal{M})$ is completely determined.

Although the theory of connections in a principal fibre bundle and on its associate vector bundles is well described in many textbooks, we recall below the main definitions concerning to this theory. A full understanding of the various equivalent definitions of a connection is necessary in order to deduce a nice formula that permit us to calculate in a simple way the covariant derivative of Clifford fields and of Dirac-Hestenes spinor fields (Section 4.3). Our simple formula arises due to the fact that the Clifford algebra $C\ell_{1,3}$, the typical fibre of $\mathcal{C}\ell(\mathcal{M})$, is an associative algebra.

4.1. Parallel Transport and Connections in Principal and Associate Bundles

To define the concept of a connection on a PFB $(\mathbf{P}, M, \pi, G)$ over a four-dimensional manifold M ($\dim G = n$), we first recall that the total space $\mathbf{P}$ of that PFB is itself a $(n+4)$-dimensional manifold and each one of its fibres $\pi^{-1}(x)$, $x \in M$, is a n-dimensional submanifold of $\mathbf{P}$. The tangent space $T_p\mathbf{P}$, $p \in \pi^{-1}(x)$ is a $(n+4)$-dimensional linear space and the tangent space $T_p\pi^{-1}(x)$ of the fibre over x, at the same point $p \in \pi^{-1}(x)$, is a n-dimensional linear subspace of $T_p\mathbf{P}$. It is called *vertical subspace* of $T_p\mathbf{P}$ and denoted by $V_p\mathbf{P}$.

A connection is a mathematical object that governs the parallel transport of frames along smooth paths in the base manifold M. Such a transport takes place in $\mathbf{P}$, along directions specified by vectors in $T_p\mathbf{P}$, which *does not* lie within the vertical space $V_p\mathbf{P}$. Since the tangent vectors to the paths on on the base manifold, passing through a given point $x \in M$, span the entire tangent space T_xM, the corresponding vectors $\mathbf{X} \in T_p\mathbf{P}$ (in whose direction parallel transport can generally take place in $\mathbf{P}$) span a four-dimensional linear subspace of $T_p\mathbf{P}$, called *horizontal space* of $T_p\mathbf{P}$ and denoted by $H_p\mathbf{P}$. The mathematical concept of a connection is given formally by

Definition: A connection on a PFB $(\mathbf{P}, M, \pi, G)$ is a field of vector spaces $H_p\mathbf{P} \subset T_p\mathbf{P}$ such that

(*i*) $\pi' : H_p\mathbf{P} \to T_xM$, $x = \pi(p)$, is an isomorphism

(*ii*) $H_p\mathbf{P}$ depends diferentially on p

(*iii*) $H_{\hat{R}_g p} = \hat{R}'_g(H_p)$

The elements of $H_p\mathbf{P}$ are called *horizontal vectors* and the elements of $T_p\pi^{-1}(x) = V_p\mathbf{P}$ are called *vertical vectors*. In view of the fact that $\pi : \mathbf{P} \to M$ is a smooth map of the entire manifold $\mathbf{P}$ onto the base manifold M, we have that $\pi' \equiv \pi_* : T\mathbf{P} \to TM$ is a globally defined map from the entire tangent bundle $T\mathbf{P}$ (over the bundle space $\mathbf{P}$) onto the tangent bundle TM.

If $x = \pi(p)$, then due to the fact that $x = \pi(p(t))$ for any curve in $\mathbf{P}$ such that $p(t) \in \pi^{-1}(x)$ and $p(0) = 0$, we conclude that π' maps all vertical vectors into the zero vector in T_xM, that is $\pi'(V_p\mathbf{P}) = 0$, and we have

$$T_p\mathbf{P} = H_p\mathbf{P} \oplus V_p\mathbf{P}, \qquad p \in \mathbf{P}$$

so that every $\mathbf{X} \in T_p\mathbf{P}$ can be written

$$\mathbf{X} = \mathbf{X}_\mathrm{h} + \mathbf{X}_\mathrm{v}, \qquad \mathbf{X}_\mathrm{h} \in H_p\mathbf{P}, \quad \mathbf{X}_\mathrm{v} \in V_p\mathbf{P}.$$

Therefore, if $\mathbf{X} \in T_p\mathbf{P}$ we get $\pi'(\mathbf{X}) = \pi'(\mathbf{X}_\mathrm{h}) = X \in T_xM$. $\mathbf{X}_\mathrm{h}$ is then called *horizontal lift* of $X \in T_xM$. An equivalent definition for a connection on $\mathbf{P}$ is given by

Definition: A connection on the principal fibre bundle $(\mathbf{P}, M, \pi, G)$ is a mapping $\Gamma_p : T_xM \to T_pP$, $x = \pi(p)$ such that

(*i*) Γ_p is linear

(*ii*) $\pi' \circ \Gamma_p = \mathrm{Id}_{T_xM}$, where Id_{T_xM} is the identity mapping in T_xM, and π' is the differential of the canonical projection mapping $\pi : \mathbf{P} \to M$

(*iii*) the mapping $p \mapsto \Gamma_p$ is differentiable

(*iv*) $\Gamma_{R_gp} = R'_g\Gamma_p$, $g \in G$ and R_g being the right translation in $(\mathbf{P}, \pi, M, G)$.

Definition: Let $C : \mathbb{R} \supset I \to M$, $t \mapsto C(t)$, with $x_0 = C(0) \in M$ be a curve in M and let $p_0 \in \mathbf{P}$ be such that $\pi(p_0) = x_0$. The *parallel transport* of p_0 along C is given by the curve $\mathbf{C} : \mathbb{R} \supset I \to \mathbf{P}$, $t \mapsto \mathbf{C}(t)$ defined by

$$\frac{d}{dt}\mathbf{C}(t) = \Gamma_p \frac{d}{dt}C(t)$$

with $\mathbf{C}(0) = p_0$, $\mathbf{C}(t) = p_\parallel$, $\pi(p_\parallel) = x = C(t)$.

We need now to know more about the nature of the vertical space $V_p\mathbf{P}$. For this, let $\hat{X} \in T_eG = \mathfrak{G}$ be an element of the Lie algebra of G and let $f : G \supset U_e \to \mathbb{R}$, where U_e is some neighborhood of the identity element of $\mathfrak{G}$. The vector $\hat{\mathbf{X}}$ can be viewed as the tangent to the curve produced by the exponential map

$$\hat{\mathbf{X}}(f) = \frac{d}{dt} f(\exp(\hat{\mathbf{X}}t))|_{t=0}$$

Then to every $u \in \mathbf{P}$ we can attach to each $\hat{\mathbf{X}} \in T_eG$ a unique element of $V_p\mathbf{P}$ as follows: Let $\mathcal{F} : \mathbf{P} \to \mathbb{R}$ be given by

$$\hat{\mathbf{X}}_\mathrm{v}(p)(\mathcal{F}) = \frac{d}{dt}\mathcal{F}(p\exp(\hat{\mathbf{X}}t))|_{t=0}$$

By this construction we have attached to each $\hat{\mathbf{X}} \in T_eG$ a unique global section of TP, called fundamental field corresponding to this element. We then have the canonical isomorphism

$$\hat{\mathbf{X}}_\mathrm{v}(p) \leftrightarrow \hat{\mathbf{X}}, \qquad \hat{\mathbf{X}}_\mathrm{v}(p) \in V_p\mathbf{P}, \quad \hat{\mathbf{X}} \in T_eG$$

and we have

$$V_p\dot{\mathbf{P}} \simeq \mathfrak{G}$$

It follows that another equivalent definition for a connection is:

Definition: A connection on $(\mathbf{P}, M, \pi, G)$ is a 1-form field ω on $\mathbf{P}$ with values in the Lie algebra $\mathfrak{G}$ such that, for each $p \in \mathbf{P}$,

(i) $\omega_p(\mathbf{X}_v) = \hat{\mathbf{X}}$, $\mathbf{X}_v \in V_p\mathbf{P}$ and $\hat{\mathbf{X}} \in \mathfrak{G}$ are related by the canonical isomorphism

(ii) ω_p depends diferentially on p

(iii) $\omega_{\tilde{R}_g p}(\tilde{R}'_g \mathbf{X}) = (\mathrm{Ad}_{g^{-1}}\, \omega_p)(\mathbf{X})$

It follows that if $\{\mathcal{G}_a\}$ is a basis of $\mathfrak{G}$ and $\{\theta^i\}$ is a basis of $T_p^*\mathbf{P}$, we can write ω as

$$\omega_p = \omega^a \otimes \mathcal{G}_a = \omega_i^a \theta^i \otimes \mathcal{G}_a \tag{4.1}$$

where ω^a are 1-forms on $\mathbf{P}$.

The horizontal spaces $H_p\mathbf{P}$ can then be defined by

$$H_p\mathbf{P} = \ker(\omega_p)$$

and we can verify that this is equivalent to the definition of $H_p\mathbf{P}$ given in the first definition of a connection.

Now, for a given connection ω, we can associate with each differentiable local section of $\pi^{-1}(U) \subset \mathbf{P}$, $U \subset M$, a 1-form with values in $\mathfrak{G}$. Indeed, let

$$f : M \supset U \to \pi^{-1}(U) \subset \mathbf{P} \qquad \pi \circ f = \mathrm{Id}_M$$

be a local section of $\mathbf{P}$. We define the 1-form $f^*\omega$ on U with values in $\mathfrak{G}$ by the pull-back of ω by f. If $X \in T_x M$, $x \in U$,

$$(f^*\omega)_x(X) = \omega_{f(x)}(f'X)$$

Conversely, we have:

Theorem: Given $\omega \in TM \otimes \mathfrak{G}$ and a differentiable section of $\pi^{-1}(U)$, $U \subset M$, there exists one and only one connection ω on $\pi^{-1}(U)$ such that $f^*\omega = \omega$.

It is important to keep in mind also the following result:

Theorem: On each principal fibre bundle with paracompact base manifold there exists infinitely many connections.

As it is well known, each local section f determines a local trivialization

$$\Phi : \pi^{-1}(U) \to U \times G$$

of $\pi : \mathbf{P} \to M$ by setting $\Phi^{-1}(x, g) = f(x)g$. Conversely, Φ determines f, since $f(x) = \Phi^{-1}(x, e)$, where e is the identity of G. We shall also need the following

Proposition: Let be given a local trivialization (U, Φ), $\Phi : \pi^{-1}(U) \to U \times G$, and let $f : M \supset U \to \mathbf{P}$ be the local section associated to it. Then the connection form can be written:

$$(\Phi^{-1*}\omega)_{x,g} = g^{-1}dg + g^{-1}\omega g \tag{4.2}$$

where $\omega = f^*\omega \in TU \otimes \mathfrak{G}$. We usually write, for abuse of notation, $\Phi^{-1*}\omega \equiv \omega$. (The proof of this proposition is trivial.)

We can now determine the nature of span$(H_p\mathbf{P})$. Using local coordinates $\langle x^i \rangle$ for $U \subset M$ and g_{ij} for $U_e \in G$,[8] we can write

$$\omega = g_{ij}^{-1}dg_{ij} + g^{-1}\omega g$$

$$\omega = \omega_\mu^A \mathcal{G}_A dx^\mu = \omega^A \otimes \mathcal{G}_A \in T_xU \otimes \mathfrak{G}$$

and

$$[\mathcal{G}_A, \mathcal{G}_B] = f_{ABC}\mathcal{G}_C$$

with f_{ABC} being the structure constants of the Lie algebra $\mathfrak{G}$ of the group G.

Recall now that $\dim H_p\mathbf{P} = 4$. Let its basis be

$$\frac{\partial}{\partial x^\mu} + d_{\mu ij}\frac{\partial}{\partial g_{ij}}$$

$\mu = 0, 1, 2, 3$ and $i, j = 1, \ldots, n = \dim G$. Since $H_p\mathbf{P} = \ker(\omega_p)$, we obtain, by writing

$$\mathbf{X_h} = \beta^\mu \left(\frac{\partial}{\partial x^\mu} + d_{\mu ij}\frac{\partial}{\partial g_{ij}} \right)$$

that

$$d_{\mu ij} = -\omega_\mu^A \mathcal{G}_{Aik} g_{kl}$$

where $\mathcal{G}_{Aik}$ are the matrix elements of $\mathcal{G}_A$.

Consider now the vector bundle $E = \mathbf{P} \times_{\rho(G)} F$ associated to the PFB $(\mathbf{P}, M, \pi, G)$ through the linear representation ρ of G in the vector space F. Consider the local trivialization $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ of $(\mathbf{P}, M, \pi, G)$, $\varphi_\alpha(p) = (\pi(p), \overset{\Delta}{\varphi}_\alpha(p))$ with $\overset{\Delta}{\varphi}_{\alpha,x}(p) : \pi^{-1}(x) \to G$, $x \in U_\alpha \in M$. Also, consider the local trivialization $\chi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ of E where $\pi : E \to M$ is the canonical projection. We have $\chi_\alpha(y) = (\pi(Y), \overset{\Delta}{\chi}_\alpha(y))$ with $\overset{\Delta}{\chi}_{\alpha,x}(y) : \pi^{-1}(x) \to F$. Then, for each $x \in U_{\alpha\beta} = U_\alpha \cap U_\beta$ we must have,

$$\overset{\Delta}{\chi}_{\beta,x} \circ \overset{\Delta}{\chi}{}^{-1}_{\beta,x} = \rho(\overset{\Delta}{\varphi}_{\beta,x} \circ \overset{\Delta}{\varphi}{}^{-1}_{\alpha,x})$$

We then have

Definition: The *parallel transport* of $v_0 \in E$, $\pi(v_0) = x_0$ along the curve $C : \mathbb{R} \supset I \to M$, $x_0 = C(0)$ from x_0 to $x = C(t)$ is the element $v_\parallel \in E$ such that

[8] For simplicity, G is supposed here to be a matrix group. The g_{ij} are then the elements of the matrix representing the element $g \in G$.

(i) $\pi(v_\parallel) = x$

(ii) $\overset{\Delta}{\chi}_{\alpha,x}(v_\parallel) = \rho(\overset{\Delta}{\varphi}_{\alpha,x}(p_\parallel)\circ \overset{\Delta}{\varphi}{}^{-1}_{\alpha,x_0}(p_0))\,\overset{\Delta}{\varphi}_{\beta,x_0}(v_0)$

Definition: Let X be a vector at $x_0 \in M$ tangent to the curve $C : t \mapsto C(t)$ on M, $x_0 = C(0)$. The *covariant derivative* of $X \in \sec E$ in the direction of V at x_0 is $(\nabla_V X)_{X_0} \in \sec E$ such that

$$(\nabla_V X)(x_0) \equiv (\nabla_V X)_{x_0} = \lim_{t\to 0} \frac{1}{t}(X^0_{\parallel,t} - X_0) \tag{4.3}$$

where $X^0_{\parallel,t}$ is the "vector" $X_t \equiv X(x(t))$ of a section $X \in \sec E$ parallel transported along C from $x(t)$ to x_0, the unique requirement on C being $\left.\frac{d}{dt}C(t)\right|_{t=0} = V$.

In the local trivialization (U_α, χ_α) of E we have,

$$\overset{\Delta}{\chi}_\alpha(X^0_{\parallel,t}) = \rho(g_0 g_t^{-1})\,\overset{\Delta}{\chi}_{\alpha,x(t)}(X_t) \tag{4.4}$$

From this last definition it is trivial to calculate the covariant derivative of $A \in \sec \mathcal{C}\ell(\mathcal{M})$ in the direction of V. Indeed, since a *spin manifold* for M is (Section 3) $\mathcal{C}\ell(\mathcal{M}) = P_{SO_+(1,3)} \times_{Ad'} C\ell_{1,3} = P_{\mathbf{Spin}_+(1,3)} \times_{Ad} C\ell_{1,3}$, $g_0, g_t^{-1} \in \mathbf{Spin}_+(1,3)$ and ρ is the adjoint representation of $\mathbf{Spin}_+(1,3)$ in $C\ell_{1,3}$, we can verify (just take into account that our bundle is trivial and put $g_0 = 1$ for simplicity) that that we can write

$$A^0_{\parallel,t} = g_t^{-1} A_t\, g_t \qquad g_t = g(x(t)) \in \mathbf{Spin}_+(1,3) \tag{4.5}$$

Then,

$$(\nabla_V A)(x_0) = \lim_{t\to 0} \frac{1}{t}(g_t^{-1} A_t g_t - A_0) \tag{4.6}$$

Now, as we observed in Section 2, each $g \in \mathbf{Spin}_+(1,3)$ is of the form $\pm e^{F(x)}$, where $F \in \sec \bigwedge^2(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M})$, and F can be chosen in such a way to have a positive sign in this expression, except in the particular case where $F^2 = 0$ and $R = -e^F$. We then write,[9]

$$g_t = e^{-1/2\omega t} \tag{4.7}$$

and

$$\omega = -2 g_t' g_t^{-1}|_{t=0} \tag{4.8}$$

Using Eq. 4.8 in Eq. 4.7 gives

$$(\nabla_V A)(x_0) = \left\{\frac{d}{dt}A_t + \frac{1}{2}[\omega, A_t]\right\}|_{t=0} \tag{4.9}$$

Now let $<x^\mu>$ be a coordinate chart for $U \subset M$, $e_a = h_a^\mu \partial_\mu$, $a = 0,1,2,3$ an orthonormal basis for $TU \subset TM$.[10] Let $\gamma^a \in \sec(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M})$ be the dual basis of

[9]The negative sign in the definition of ω is only for convenience, in order to obtain formulas in agreement with known results.

[10]Since M is a spin manifold, $P_{SO_+(1,3)}(\mathcal{M})$ is trivial and $\{e_a\}$, $a = 0,1,2,3$ can be taken as a global tetrad field for the tangent bundle.

$\{e_a\} \equiv \mathcal{B}$. Let $\Sigma = \{\gamma^a\}$ and $\{\gamma_a, a = 0,1,2,3\}$ the reciprocal basis of $\{\gamma^a\}$, i.e., $\gamma^a \cdot \gamma_b = \delta^a_b$ where $\cdot$ is the internal product in $\mathcal{C}\ell_{1,3}$. We have $\gamma^a = h^a_\mu dx^\mu$, $\gamma_a = h^\mu_a \eta_{\mu\alpha} dx^\alpha$.

$$\nabla_{\partial_\mu} \partial_\nu = \Gamma^\alpha_{\mu\nu} \partial_\alpha, \qquad \nabla_{\partial_\mu}(dx^\alpha) = -\Gamma^\alpha_{\mu\beta}(dx^\beta) \tag{4.10}$$

$$\nabla_{e_a} e_b = \omega^c_{ab} e_c, \qquad \nabla_{e_a} \gamma^b = -\omega^b_{ac} \gamma^c, \qquad \nabla_{e_a} \gamma_b = \omega^c_{ab} \gamma_c \tag{4.11}$$

$$\nabla_{e_\mu} e_b = \omega^c_{\mu b} e_c, \qquad \nabla_\mu \gamma^b = -\omega^b_{\mu c} \gamma^c, \qquad \nabla_\mu \gamma_b = \omega^c_{\mu b} \gamma_c \tag{4.12}$$

From Eq. 4.10 we easily obtain ($\nabla_{\partial_\mu} \equiv \nabla_\mu$)

$$(\nabla_\mu A) = \partial_\mu A + \tfrac{1}{2}[\omega_\mu, A] \tag{4.13}$$

with

$$\omega_\mu = -2(\partial_\mu g) g^{-1} \in \sec \bigwedge^2 (T^* M) \subset \sec \mathcal{C}\ell(\mathcal{M}) \tag{4.14}$$

where $g \in \sec \mathcal{C}\ell^+(\mathcal{M})$ is such that $g|_{c(t)} \equiv g_t \in \mathbf{Spin}_+(1,3)$.

We observe that our formulas, Eq. 4.10 and Eq. 4.11 for the covariant derivative of an homogeneous Clifford field preserves (as it must be), its graduation, i.e., if $A_p \in \sec \bigwedge^p(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M}), p = 0,1,2,3,4,$ then $[\omega_\mu, A_p] \in \sec \bigwedge^p(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M})$ as can be easily verified.

Since

$$\tfrac{1}{2}[\omega_\mu, \gamma^\alpha] = \omega_\mu \cdot \gamma^\alpha = -\gamma^\alpha \cdot \omega_\mu \tag{4.15}$$

we have

$$\omega_\mu = \tfrac{1}{2} \omega^{ab}_\mu (\gamma_a \wedge \gamma_b) \tag{4.16}$$

and we observe that

$$\omega^{ab}_\mu = -\omega^{ba}_\mu \tag{4.17}$$

For $A = A_a \gamma^a$ we immediately obtain

$$\nabla_{e_a} A_b = e_a(A_b) = \omega^c_{ab} A_c \tag{4.18}$$

which agrees with the well known formula for the derivative of a covariant vector field.

Also we have

$$\begin{aligned} \nabla_\mu A_a &= \partial_\mu(A_a) - \omega^{\ b}_{\mu a} A_b \\ \nabla_\mu A_\alpha &= \partial_\mu(A_\alpha) - \Gamma^\beta_{\mu\alpha} A_\beta \end{aligned} \tag{4.19}$$

From the general formula 4.9 it follows immediately the

Proposition: The covariant derivative ∇_X on $\mathcal{C}\ell(\mathcal{M})$ acts as a derivation on the algebra of sections, i.e., for $A, B \in \sec \mathcal{C}\ell(\mathcal{M})$ it holds

$$\nabla_X(AB) = (\nabla_X A)B + A(\nabla_X B) \tag{4.20}$$

The proof is trivial.

4.2. *The Lie Derivative of Clifford Fields*

Let $V \in \sec TM$ be a vector field on M which induces a local one-parameter transformation group $t \mapsto \varphi_t$. It φ_{*t} stands as usual to the natural extension of the tangent map $d\varphi_t$ to tensor fields, the Lie derivative $\mathcal{L}_V$ of a given tensor field $X \in \sec TM$ is defined by

$$(\mathcal{L}_V X)(x) = \lim_{t \to 0} \frac{1}{t}(X_x - (\varphi_{*t}(x))_x) \tag{4.21}$$

$\mathcal{L}_V$ is a derivation in the tensor algebra $\tau\mathcal{M}$. Then, we have for $a, b \in \sec \bigwedge^1(T^*\mathcal{M}) \subset \mathcal{C}\ell(\mathcal{M})$.

$$\mathcal{L}_V(a \otimes b + b \otimes a - 2g^{-1}(a,b)) = (\mathcal{L}_V a) \otimes b + b \otimes (\mathcal{L}_V a) - 2\mathcal{L}_V(g^{-1}(a,b)) \tag{4.22}$$

Since $a \otimes b + b \otimes a - 2g^{-1}(a,b)$ belongs to $J(\mathcal{M})$, the bilateral ideal generating the Clifford bundle $\mathcal{C}\ell(\mathcal{M})$ we see from Eq. 4.21 that $\mathcal{L}_V$ preservers $J(\mathcal{M})$ if and only if $\mathcal{L}_V g = 0$, i.e., V induces a local isometry group and then V is a Killing vector.[23]

4.3. *The Covariant Derivative of Algebraic Dirac Spinor Fields*

As discussed in Section 3 ADSF are sections of the Real Spinor Bundle $I(\mathcal{M}) = P_{\mathbf{Spin}_+(1,3)}(M) \times_\ell I$ where $I = \mathcal{C}\ell_{1,3}\frac{1}{2}(1 + E_0)$. $I(\mathcal{M})$ is a subbundle of the Spin-Clifford bundle $\mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$. Since both $I(\mathcal{M})$ and $\mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ are vector bundles, the covariant derivatives of a ADSF or a DHSF can be immediately calculated using the general method discussed in Section 4.1.

Before we calculate the covariant spinor derivative ∇^s_V of a section of $I(\mathcal{M})$ [or $\mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$] where $V \in \sec TM$ is a vector field we must recall that ∇^s_V is a module derivation,[39] i.e., if $X \in \sec \mathcal{C}\ell(\mathcal{M})$ and $\varphi \in \sec I(\mathcal{M})$ [or $\sec \mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$] then it holds:

Proposition: Let ∇ be the connection in $\mathcal{C}\ell(\mathcal{M})$ to which ∇^s is related. Then,

$$\nabla^s_V(X\varphi) = (\nabla_V X)\varphi + X(\nabla^s_V \varphi) \tag{4.23}$$

The proof of this proposition is trivial once we derive an explicit formula to compute $\nabla^s_V(\varphi)$, $\varphi \in \sec I(\mathcal{M}) \subset \sec \mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$.

Let us now calculate the covariant derivative ∇^s_v in the direction of v, a vector at $x_0 \in M$ of $\phi \in \sec I(\mathcal{M}) \subset \sec \mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$.

Putting $g_0 = 1 \in \mathbf{Spin}_+(1,3)$ we have using the general procedure

$$\phi^0_{\parallel,t} = g_t^{-1}\phi_t \tag{4.24}$$

where $\phi^0_{\parallel,t}$ is the "vector" $\phi_t = \phi(x(t))$ of a section $\phi \in \sec I(\mathcal{M}) \subset \sec \mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ parallel transported along $C : \mathbb{R} \supset I \to M$, $t \mapsto C(t)$ from $x(t) \equiv C(t)$ to $x_0 = C(0)$, $\left.\frac{d}{dt}C(t)\right|_{t=0} = v$

Putting as in Eq. 4.8 $g_t = e^{-1/2wt}$, we get by using Eq. 4.4

$$(\nabla^s_v \phi)(x_0) = \left.\left(\frac{d}{dt}\phi_t + \frac{1}{2}\omega\phi_t\right)\right|_{t=0} \tag{4.25}$$

If $\{\gamma^a\}$ is an orthogonal field of 1-forms, $\gamma^a \in \sec \bigwedge^1(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M})$ dual to the orthogonal frame field $\{e_a\}$, $e_a \in \sec TM$, $g(e_a, e_b) = \eta_{ab}$ and if $\{\gamma_a\}$ is the reciprocal frame of $\{\gamma^a\}$, i.e., $\gamma^a \cdot \gamma_b = \delta^a_b$ $(a, b = 0, 1, 2, 3)$ then for Eq. 4.25 we get

$$\nabla^s_{e_a}\phi = e_a(\phi) + \tfrac{1}{2}\omega_a\phi \tag{4.26}$$

with

$$\omega_a = \tfrac{1}{2}\omega_a^{bc}\gamma_b \wedge \gamma_c \tag{4.27}$$

and we recognize the 1-forms ω_a as being $\omega_a = \omega(e_a)$ where $\omega = f^*\omega$, $f : M \to U \times G$ is the global section used to write Eq. 4.24. The Lie algebra of $\mathbf{Spin}_+(1,3)$ is, of course, generated by the "vectors" $\{\gamma_a \wedge \gamma_b\}$.

$$\nabla_{e_a}\gamma^b = -\omega_a^{bc}\gamma_c \tag{4.28}$$

If $\langle x^\mu \rangle$ is a coordinate chart for $U \subset M$ and $\gamma^a = h^a_\mu dx^\mu$, $a, \mu = 0, 1, 2, 3$, we also obtain

$$\nabla^s_\mu\phi = \partial_\mu(\phi) + \tfrac{1}{2}\omega_\mu\phi, \qquad \omega_\mu = \tfrac{1}{2}\omega_\mu^{bc}\gamma_b \wedge \gamma_c \tag{4.29}$$

Now, since $\phi \in \sec I(\mathcal{M}) \subset \sec \mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ is such that $\phi e_\Sigma = \phi$ with $e_\Sigma = \frac{1}{2}(1 + \gamma^0)$ it follows from $\nabla^s_{e_a}\phi = \nabla^s_{e_a}(\phi e_\Sigma)$ that

$$e_\Sigma \nabla^s_{e_a} e_\Sigma = 0 \tag{4.30}$$

Now, recalling Eq. 2.30 we have a spinorial basis for $I(\mathcal{M})$ given by $\beta^s = \{s^A\}$, $A = 1, 2, 3, 4$, $s^A \in \sec I(\mathcal{M})$ with

$$s^1 = e_\Sigma = \tfrac{1}{2}(1 + \gamma^0), \quad s^2 = -\gamma^1\gamma^3 e_\Sigma, \quad s^3 = \gamma^3\gamma^0 e_\Sigma, \quad s^4 = \gamma^1\gamma^0 e_\Sigma. \tag{4.31}$$

Then as we learn in Section 2, $\phi = \phi_A s^A$ where ϕ_A are formally complex numbers. Then

$$\begin{aligned}
\nabla^s_{e_a}\phi &= e_a(\phi) + \tfrac{1}{2}\omega_a\phi \\
&= \left[e_a(\phi_A) + \tfrac{1}{2}\omega_a\phi_A\right] s^A \\
&= \left(e_a(\phi_A) + \tfrac{1}{2}[\omega_a]^B_A\phi_B\right) s^A
\end{aligned} \tag{4.32}$$

with

$$\omega_a s^A = [\omega_a]^A_B s^B \tag{4.33}$$

$$\begin{aligned}
\nabla^s_{e_a}\phi &= \nabla^s_{e_a}(\phi_A s^A) \\
&= e_a(\phi_A)s^A + \phi_A \nabla^s_{e_a} s^A
\end{aligned} \tag{4.34}$$

From Eq. 4.32 and Eq. 4.34 it follows that

$$\nabla^s_{e_a} s^A = \tfrac{1}{2}[\omega_a]^A_B s^B \tag{4.35}$$

We introduce the dual space $I^*(\mathcal{M})$ of $I(M)$ where $I^*(M) = P_{\mathrm{Spin}_+(1,3)}(\mathcal{M}) \times_r I$ where here the action of $\mathrm{Spin}_+(1,3)$ on the typical fiber is on the right. A basis for $I^*(\mathcal{M})$ is then $\rho_s = \{s_A\}$, $A = 1,2,3,4$, $s_A \in \sec I^*(\mathcal{M})$ such that

$$s_A(s^B) = \delta_A^B \tag{4.36}$$

A simple calculation shows that

$$\nabla^s_{e_a} s_A = -\frac{1}{2}[\omega_a]^B_A s_B \tag{4.37}$$

Since $\mathcal{C}\ell(\mathcal{M}) = I^*(\mathcal{M}) \otimes I(\mathcal{M})$ (the "tensor-spinor space") is spanned by the basis $\{s^A \otimes s_B\}$ we can write

$$\gamma_a s_A = [\gamma_a]^B_A s_B \tag{4.38}$$

with

$$[\gamma_a]^B_A = \gamma^B_{aA} \equiv \gamma_a(s^B, s_A) \tag{4.39}$$

being the matricial representation of γ_a. It follows that

$$\nabla^s_{e_b} \gamma_a(s^B, s_A) = e_b([\gamma_a]^B_A) - \omega^c_{ba}\gamma^B_{CA} + \frac{1}{2}\omega^B_{bC}\gamma^C_{bA} - \frac{1}{2}\omega^C_{bA}\gamma^B_{aC} \tag{4.40}$$

Now,

$$(\frac{1}{2}\omega^B_{bc}\gamma^C_{aA} - \frac{1}{2}\omega^C_{bA}\gamma^B_{aC})s^A = (\gamma_a \cdot \omega_b)s^B \tag{4.41}$$

and from $\omega_b = \frac{1}{2}\omega_b^{cd}\gamma_c \wedge \gamma_d$, we get

$$(\gamma_a \cdot \omega_b)s^B = (-\omega^c_{ba}\gamma^B_{cA})s^A \tag{4.42}$$

From Eq. 4.41 and Eq. 4.42 we obtain

$$\tfrac{1}{2}\omega^B_{bC}\gamma^C_{aA} - \tfrac{1}{2}\omega^C_{bA}\gamma^B_{aC} = -\omega^d_{ba}\gamma^B_{dA} \tag{4.43}$$

and then

$$\nabla^s_{e_b}[\gamma_a]^B_A = e_b([\gamma_a]^B_A) = 0 \tag{4.44}$$

since according to a result obtained in Section 2.6 $[\gamma_a]^B_C$ are constant matrices. Eq. 4.43 agrees with the result presented, e.g., in [23]. Also from $\omega_a = \frac{1}{2}\omega_a^{bc}\gamma_b \wedge \gamma_c$ it follows

$$\omega^A_{aB} = \tfrac{1}{2}\omega_a^{bc}[\gamma_b, \gamma_c]^A_B. \tag{4.45}$$

We can also easily obtain the following results. Writing

$$\nabla^s_{e_a}\phi \equiv (\nabla^s_{e_a}\phi_A)s^A \tag{4.46}$$

it follows that

$$\nabla^s_{e_a}\phi_A = e_a(\phi_A) + \frac{1}{8}\omega^b_{ac}[\gamma_b, \gamma^c]^B_A\phi_B \tag{4.47}$$

and

$$\nabla^s_{e_a}\phi^A = e_a(\phi^A) - \frac{1}{8}\omega^b_{ac}[\gamma_b, \gamma^c]^A_B\phi^B \tag{4.48}$$

Eq. 4.48 agrees exactly with the result presented, e.g., by Choquet-Bruhat et al[23] for the components of the covariant derivative of a CDSF $\psi \in \sec P_{\mathrm{Spin}_+(1,3)}(\mathcal{M}) \times_\rho \mathbb{C}^4$. It is important to emphasize here that the condition given by Eq. 4.43, namely $\nabla^s_{e_b}[\gamma_a]^B_A = 0$ holds true but this does not imply that $\nabla_{e_b}\gamma^a = 0$, i.e., ∇ need not be the so called connection of parallelization of the $\mathcal{M} = \langle M, g, \nabla\rangle$, which as well known has zero curvature but non zero torsion.[44]

The main difference between ∇^s acting on sections of $I(\mathcal{M})$ or of $\mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}(\mathcal{M})$ and ∇ acting on sections of $\mathcal{C}\ell(\mathcal{M})$ is that, for $\phi \in \sec I(\mathcal{M})or\sec \mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}(\mathcal{M})$ and $A \in \sec \mathcal{C}\ell(\mathcal{M})$, we must have

$$\nabla^s_{e_a}(A\phi) = (\nabla_{e_a}A)\phi + A(\nabla^s_{e_a}\phi), \tag{4.49}$$

and of course ∇ cannot be applied to sections of $I(\mathcal{M})$ or of $\mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}(\mathcal{M})$.

4.4. *The Representative of the Covariant Derivative of a Dirac-Hestenes Spinor Field in $\mathcal{C}\ell(\mathcal{M})$*

In Section 3.2 we defined a DHSF ψ as an even section of $\mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}(\mathcal{M})$. Then, by the same procedure used in Section 4.3 we get[11]

$$\nabla^s_{e_a}\psi = e_a(\psi) + \tfrac{1}{2}\omega_a\psi \qquad \nabla^s_{e_a}\tilde{\psi} = e_a(\tilde{\psi}) - \tfrac{1}{2}\tilde{\psi}\omega_a \tag{4.50}$$

and as before

$$\omega_a = \tfrac{1}{2}\omega^{bc}_a\gamma_b \wedge \gamma_c \in \sec \mathcal{C}\ell(\mathcal{M}) \tag{4.51}$$

Now, let $\gamma^a \in \sec \mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}(\mathcal{M})$ such that $\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}$, $(a, b = 0, 1, 2, 3)$, and let us calculate $\nabla^s_{e_a}(\psi\gamma^b)$. Using Eq. 4.48 we have,

$$\nabla^s_{e_a}(\psi\gamma^b) = e_a(\psi\gamma^b) + \tfrac{1}{2}\omega_a\psi\gamma^b = (\nabla^s_{e_a}\psi)\gamma^b \tag{4.52}$$

On the other hand using Eq. 4.50 we get

$$\nabla^s_{e_a}(\psi\gamma^b) = (\nabla^s_{e_a}\psi)\gamma^b + \psi(\nabla^s_{e_a}\gamma^b) \tag{4.53}$$

Comparision of Eq. 4.52 and Eq. 4.53 implies that :

$$\nabla^s_{e_a}\gamma^b = 0 \tag{4.54}$$

We know that if $\psi, \tilde{\psi} \in \sec \mathcal{C}\ell^+_{\mathrm{Spin}_+(1,3)}(\mathcal{M})$ then $\psi\gamma^a\tilde{\psi} = \mathbf{X}^a$ is such that $\mathbf{X}^a(x) \in \mathbb{R}^{1,3}\ \forall x \in M$. Then,

$$\nabla^s_{e_a}(\psi\gamma^b\tilde{\psi}) = (\nabla^s_{e_a}\psi)\gamma^b\tilde{\psi} + \psi\gamma^b(\nabla^s_{e_a}\tilde{\psi}) \tag{4.55}$$

[11] The meaning of e_a, γ^b, etc. is as before.

and $\nabla_{e_a}(\psi\gamma^b\tilde{\psi})(x) \in \mathbb{R}^{1,3}$, $\forall x \in M$.

We are now prepared to find the representative of the covariant derivative of a DHSF in $\mathcal{C}\ell(\mathcal{M})$. We recall that ψ is an equivalence class of even sections of $\mathcal{C}\ell(\mathcal{M})$ such that in the basis $\Sigma = \{\gamma^a\}, \gamma^a \in \sec \bigwedge^1(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M})$ the representative of ψ is $\psi_\Sigma \in \mathcal{C}\ell^+(\mathcal{M})$ and the representative of $\mathbf{X}^a$ is $X^a \in \sec \bigwedge^1(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M})$ such that

$$X^a = \psi_\Sigma \gamma^a \tilde{\psi}_\Sigma \tag{4.56}$$

Let ∇ be the connection acting on sections of $\mathcal{C}\ell(\mathcal{M})$. Then,

$$\begin{aligned}
\nabla_{e_a}(\psi_\Sigma\gamma^b\tilde{\psi}_\Sigma) &= \left\{e_a(\psi_\Sigma) + \frac{1}{2}[\omega_a, \psi_\Sigma]\right\}\gamma^b\tilde{\psi}_\Sigma \\
&+\psi_\Sigma(\nabla_{e_a}\gamma^b)\tilde{\psi}_\Sigma + \psi_\Sigma\gamma^b\left\{e_a(\tilde{\psi}_\Sigma) + \frac{1}{2}[\omega_a, \tilde{\psi}_\Sigma]\right\} = \\
&= \left[e_a(\psi_\Sigma) + \frac{1}{2}\omega_a\psi_\Sigma\right]\gamma^b\tilde{\psi}_\Sigma + \psi_\Sigma\gamma^b\left[e_a(\tilde{\psi}_\Sigma) - \frac{1}{2}\tilde{\psi}_\Sigma\omega_a\right].
\end{aligned} \tag{4.57}$$

Comparing Eq. 4.55 and Eq. 4.57 we see that the following definition suggests by itself

Definition:

$$\begin{aligned}
(\nabla^s_{e_a}\psi)_\Sigma &\equiv \nabla^s_{e_a}\psi_\Sigma = e_a(\psi_\Sigma) + \frac{1}{2}\omega_a\psi_\Sigma \\
(\nabla^s_{e_a}\tilde{\psi})_\Sigma &\equiv \nabla^s_{e_a}\tilde{\psi}_\Sigma = e_a(\tilde{\psi}_\Sigma) - \frac{1}{2}\tilde{\psi}_\Sigma\omega_a \\
(\nabla^s_{e_a}\gamma^b)_\Sigma &\equiv \nabla^s_{e_a}\gamma^b = 0
\end{aligned} \tag{4.58}$$

where $(\nabla^s_{e_a}\psi)_\Sigma, (\nabla^s_{e_a}\tilde{\psi})_\Sigma, (\nabla^s_{e_a}\gamma^b)_\Sigma \in \sec \mathcal{C}\ell(\mathcal{M})$ are representatives of $\nabla^s_{e_a}\psi$ (etc...) in the basis Σ in $\mathcal{C}\ell(\mathcal{M})$

Observe that the result $\nabla^s_{e_a}\gamma^b = 0$ is compatible with the result $\nabla^s_{e_a}[\gamma_a]^B_A = 0$ obtained in Eq. 4.43 and is an important result in order to write the Dirac-Hestenes equation (Section 6)

5. The Form Derivative of the Manifold and the Dirac and Spin-Dirac Operators

Let $\mathcal{M} = \langle M, g, \nabla\rangle$ be a Riemann-Cartan manifold (Section 4), and let $\mathcal{C}\ell(\mathcal{M}), I(\mathcal{M})$ and $\mathcal{C}\ell_{\text{Spin}_+(1,3)}(\mathcal{M})$ be respectively the Clifford, Real Spinor and Spin Clifford bundles. Let ∇^s be the spinorial connection acting on sections of $I(\mathcal{M})$ or $\mathcal{C}\ell_{\text{Spin}_+(1,3)}(\mathcal{M})$. Let also $\{e_a\}, \{\gamma^a\}$ with the same meaning as before and for convenience when useful we shall denote the Pfaff derivative by $\partial_a \equiv e_a$.

Definition: Let Γ be a section of $\mathcal{C}\ell(\mathcal{M}), I(\mathcal{M})$ or $\mathcal{C}\ell_{\text{Spin}_+(1,3)}(\mathcal{M})$. The form derivative of the manifold is a canonical first order differential operator $\partial : \Gamma \mapsto \Gamma$ such that

$$\begin{aligned}
\partial\Gamma &= (\gamma^a\partial_a)\Gamma \\
&= \gamma^a \cdot (\partial_a(\Gamma)) + \gamma^a \wedge (\partial_a(\Gamma))
\end{aligned} \tag{5.1}$$

for $\gamma^a \in \sec\ \mathcal{C}\ell(\mathcal{M})$.

Definition: The Dirac operator acting on sections of $\mathcal{C}\ell(\mathcal{M})$ is a canonical first order differential operator $\boldsymbol{\partial} : A \mapsto \boldsymbol{\partial} A$, $A \in \sec \mathcal{C}\ell(\mathcal{M})$, such that

$$\boldsymbol{\partial} A = (\gamma^a \nabla_{e_a}) A = \gamma^a \cdot (\nabla_{e_a} A) + \gamma^a \wedge (\nabla_{e_a} A) \tag{5.2}$$

Definition: The Spin-Dirac operator[12] acting on sections of $I(\mathcal{M})$ or $\mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}(\mathcal{M})$ is a canonical first order differential operator $\mathbf{D} : \Gamma \to \mathbf{D}\Gamma$ ($\Gamma \in \sec\ I(\mathcal{M})$) [or $\Gamma \in \sec \mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}(\mathcal{M})$] such that

$$\begin{aligned} \mathbf{D}\Gamma &= (\gamma^a \nabla^s_{e_a})\Gamma \\ &= \gamma^a \cdot (\nabla^s_{e_a}\Gamma) + \gamma^a \wedge (\nabla^s_{e_a}\Gamma) \end{aligned} \tag{5.3}$$

The operator $\boldsymbol{\partial}$ is sometimes called the Dirac-Kahler operator when $\mathcal{M}$ is a Lorentzian manifold,[17] i.e., $\mathbf{T}(\nabla) = 0$, $\mathbf{R}(\nabla) = 0$, where $\mathbf{T}$ and $\mathbf{R}$ are respectively the torsion and Riemann tensors. In this case we can show that[1]

$$\boldsymbol{\partial} = d - \delta \tag{5.4}$$

where d is the differential operator and δ the Hodge codifferential operator. In the spirit of section 4, we use the convention that the representative of $\mathbf{D}$ (acting on sections of $\mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}(\mathcal{M})$) in $\mathcal{C}\ell(\mathcal{M})$ will be also denote by

$$\mathbf{D} = \gamma^a \nabla^s_{e_a} \tag{5.5}$$

6. The Dirac-Hestenes Equation in Minkowski Spacetime

Let $\mathcal{M} = \langle M, g, \nabla\rangle$ be the Minkowski spacetime, $\mathcal{C}\ell(\mathcal{M})$ be the Clifford bundle of $\mathcal{M}$ with typical fiber $\mathcal{C}\ell_{1,3}$, and let $\Psi \in \sec P_{\mathrm{Spin}_+(1,3)}(\mathcal{M}) \times_\rho \mathbb{C}^4$ (with ρ the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $\mathrm{SL}(2,\mathbb{C}) \simeq \mathrm{Spin}_+(1,3)$. Then, the Dirac equation for the charged fermion field Ψ in interaction with the electromagnetic field A is[30] ($\hbar = c = 1$)

$$\underline{\gamma}^\mu(i\partial_\mu - eA_\mu)\Psi = m\Psi \text{ or } i\mathrm{D}\!\!\!/\,\psi - \underline{\gamma}^\mu A_\mu \Psi = m\Psi \tag{6.1}$$

where $\underline{\gamma}^\mu\underline{\gamma}^\nu + \underline{\gamma}^\nu\underline{\gamma}^\mu = 2\eta^{\mu\nu}$, $\underline{\gamma}^\mu$ being the Dirac matrices given by Eq. 2.31 and $A = A_\mu dx^\mu \in \sec \bigwedge^1(T^*M)$

As showed, e.g., in [7] this equation is equivalent to the following equation satisfied by $\phi \in \sec I(\mathcal{M})$ [$\phi e_\Sigma = \phi, e_\Sigma = \frac{1}{2}(1+\gamma^0), \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}, \gamma^\mu \in \sec \mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}(\mathcal{M})$],

$$\mathbf{D}\phi\gamma^2\gamma^1 - eA\phi = m\phi, \tag{6.2}$$

where $\mathbf{D}$ is the Dirac operator on $I(\mathcal{M})$ and $A \in \sec \bigwedge^1(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M})$.

Since, as discussed in Section 3, each ϕ is an equivalence class of sections of $\mathcal{C}\ell(\mathcal{M})$ we can also write an equation equivalent to Eq. 6.2 for $\phi_\Sigma = \phi_\Sigma e_\Sigma$, $\phi_\Sigma, e_\Sigma \in \sec \mathcal{C}\ell(\mathcal{M})$,

---

12 In [39] this operator (acting on sections of $I(\mathcal{M})$) is called simply Dirac operator, being the generalization of the operator originally introduced by Dirac. See also [11] for comments on the use of this terminology.

$e_\Sigma = \frac{1}{2}(1+\gamma^0)$, $\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$, $\gamma^\mu \in \sec \mathcal{C}\ell(\mathcal{M})$, and $\gamma^\mu = dx^\mu$ for the global coordinate functions $\langle x^\mu \rangle$. In this case the Dirac operator $\partial = \gamma^\mu \nabla_\mu$ is equal to the form derivative $\partial = \gamma^\mu \partial_\mu$ and we have

$$\partial \phi_\Sigma \gamma^2 \gamma^1 - eA\phi_\Sigma = m\phi_\Sigma \gamma^0 \tag{6.3}$$

Since each ϕ_Σ can be written $\phi_\Sigma = \psi_\Sigma e_\Sigma$, ($\psi_\Sigma \in \sec \mathcal{C}\ell^+(\mathcal{M})$ being the representative of a DHSF) and $\gamma^0 e_\Sigma = e_\Sigma$, we can write the following equation for ψ_Σ that is equivalent to Dirac equation[7, 9, 10]

$$\partial \psi_\Sigma \gamma^2 \gamma^1 - eA\psi_\Sigma = m\psi_\Sigma \gamma^0 \tag{6.4}$$

which is the so called Dirac-Hestenes equation.[2, 3]

Eq.6.4 is covariant under passive (and active) Lorentz transformations, in the following sense: consider the change from the Lorentz frame $\Sigma = \{\gamma^\mu = dx^\mu\}$ to the frame $\dot{\Sigma} = \{\dot{\gamma}^\mu = d\dot{x}^\mu\}$ with $\dot{\gamma}^\mu = R^{-1}\gamma^\mu R$ and $R \in \mathbf{Spin}_+(1,3)$ being constant. Then the representative of the Dirac-Hestenes spinor changes as already discussed in Section 3 from ψ_Σ to $\psi_{\dot{\Sigma}} = \psi_\Sigma R^{-1}$. Then we have $\partial = \gamma^\mu \partial_\mu = \dot{\gamma}^\mu \partial / \partial \dot{x}^\mu$ where $\langle x^\mu \rangle$ and $\langle \dot{x}^\mu \rangle$ are related by a Lorentz transformation and

$$\partial \psi_\Sigma R^{-1} R \gamma^2 R^{-1} R \gamma^1 R^{-1} - eA\psi_\Sigma R^{-1} = m\psi_\Sigma R^{-1} R \gamma^0 R^{-1}, \tag{6.5}$$

i.e.,

$$\partial \psi_{\dot{\Sigma}} \dot{\gamma}^2 \dot{\gamma}^1 - eA\psi_{\dot{\Sigma}} = m\psi_{\dot{\Sigma}} \dot{\gamma}^0 \tag{6.6}$$

Thus our definition of the Dirac-Hestenes spinor fields as an equivalence class of even sections of $\mathcal{C}\ell(\mathcal{M})$ solves directly the question raised by Parra[45] concerning the covariance of the Dirac-Hestenes equation.

Observe that if ∇^s is the spinor covariant derivative acting on ψ_Σ (defined in Section 4.4) we can write Eq. 6.4 in intrinsic form, i.e., without the need of introducing a chart for $\mathcal{M}$ as follows

$$\gamma^a \nabla^s_a \psi_\Sigma \gamma^2 \gamma^1 - eA\psi_\Sigma = m\psi_\Sigma \gamma^0 \tag{6.7}$$

where γ^a is now an orthogonal basis of T^*M, and not necessarily it is $\gamma^a = dx^a$ for some coordinate functions x^a.

It is well-known that Eq. 6.1 can be derived from the principle of stationary action through variation of the following action

$$S(\Psi) = \int d^4x \mathcal{L} \tag{6.8}$$

$$\mathcal{L} = -\frac{i}{2}(\gamma^\mu \partial_\mu \Psi^+)\Psi + \frac{i}{2}\Psi^+(\gamma^\mu \partial_\mu \bar{\Psi}) - m\Psi^+\bar{\Psi} - eA_\mu \Psi^+ \underline{\gamma}_\mu \bar{\Psi} \tag{6.9}$$

with $\Psi^+ = \Psi^* \gamma^0$.

In the next section we shall present the rudiments of the multiform derivative approach to Lagrangian field theory (MDALFT) developed in [23] (see also [46]) and we apply this formalism to obtain the Dirac-Hestenes equation on a Riemann-Cartan spacetime.

7. Lagrangian Formalism for the Dirac-Hestenes Spinor Field on a Riemann-Cartan Spacetime

In this section we apply the concept of multiform (or multivector) derivatives first introduced by Hestenes and Sobczyk[34] (HS) to present a Lagrangian formalism for the Dirac-Hestenes spinor field DHSF on a Riemann-Cartan spacetime. In Section 7.1 we briefly present our version of the multiform derivative approach to Lagrangian field theory for a Clifford field $\phi \in \sec \mathcal{C}\ell(\mathcal{M})$ where $\mathcal{M}$ is Minkowski spacetime. In Section 7.2 we present the theory for the DHSF on Riemann-Cartan spacetime.

7.1. *Multiform Derivative Approach to Lagrangian Field Theory*

We define a Lagrangian density for $\phi \in \sec \mathcal{C}\ell(\mathcal{M})$ as a mapping

$$\mathcal{L} : (x, \phi(x), \boldsymbol{\partial} \wedge \phi(x), \boldsymbol{\partial} \cdot \phi(x)) \mapsto \mathcal{L}(x, \phi(x), \boldsymbol{\partial} \wedge \phi(x), \boldsymbol{\partial} \cdot \phi(x)) \in \bigwedge^4 (T^*M) \subset \mathcal{C}\ell(\mathcal{M}) \quad (7.1)$$

where $\boldsymbol{\partial}$ is the Dirac operator acting on sections of[13] $\mathcal{C}\ell(\mathcal{M})$, and by the above notation we mean an arbitrary multiform function of ϕ, $\boldsymbol{\partial} \wedge \phi$ and $\boldsymbol{\partial} \cdot \phi$.

In this section we shall perform our calculations using an orthonormal and coordinate basis for the tangent (and cotangent) bundle. If $\langle x^\mu \rangle$ is a global Lorentz chart, then $\gamma^\mu = dx^\mu$ and $\boldsymbol{\partial} = \gamma^\mu \nabla_\mu = \gamma^\mu \partial_\mu = \partial$, so that the Dirac operator ($\boldsymbol{\partial}$) coincides with the form derivative (∂) of the manifold.

We introduce also for ϕ a Lagrangian $L(x, \phi(x), \partial \wedge \phi(x), \partial \cdot \phi(x)) \in \bigwedge^0(T^*M) \subset \mathcal{C}\ell(\mathcal{M})$ by

$$\mathcal{L}(x, \phi(x), \partial \wedge \phi(x), \partial \cdot \phi(x)) = L(x, \phi(x), \partial \wedge \phi(x), \partial \cdot \phi(x))\tau_g \quad (7.2)$$

where $\tau_g \subset \sec \bigwedge^4(T^*M)$ is the volume form, $\tau_g = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ for $\langle x^\mu \rangle$ a global Lorentz chart.

In what follows we suppose that $\mathcal{L}[L]$ does not depend explicity of x and we write $L(\phi, \partial \wedge \phi, \partial \cdot \phi)$ for the Lagrangian. Observe that

$$L(\phi, \partial \wedge \phi, \partial \cdot \phi) = \langle L(\phi, \partial \wedge \phi, \partial \cdot \phi) \rangle_0 \quad (7.3)$$

As usual, we define the action for ϕ as

$$S(\phi) = \int_U L(\phi, \partial \wedge \phi, \partial \cdot \phi)\tau_g \qquad U \subseteq M \quad (7.4)$$

The field equations for ϕ is obtained from the principle of stationary action for $S(\phi)$. Let $\eta \in \sec \mathcal{C}\ell(\mathcal{M})$ containing the same grades as $\phi \in \sec \mathcal{C}\ell(\mathcal{M})$. We say that ϕ is stationary with respect to L if

$$\left. \frac{d}{dt} S(\phi + t\eta) \right|_{t=0} = 0 \quad (7.5)$$

[13] An example of a Lagrangian of the form given by Eq. 7.1 appears, e.g., in the theory of the gravitational field in Minkowski spacetime[47]

But, recalling HS[34] we see that Eq. 7.5 is just the definition of the multiform derivative of $S(\phi)$ in the direction of η, i.e., we have using the notation of HS

$$\eta * \partial_\phi S(\phi) = \frac{d}{dt} S(\phi + t\eta)\Big|_{t=0} \tag{7.6}$$

Then,

$$\frac{d}{dt} S(\phi + t\eta)\Big|_{t=0} = \int \tau_g \frac{d}{dt}\{L[(\phi + t\eta), \partial \wedge (\phi + t\eta), \partial \cdot (\phi + t\eta)]\}\Big|_{t=0} \tag{7.7}$$

Now

$$\begin{aligned} &\frac{d}{dt}\{[L(\phi + t\eta), \partial \wedge (\phi + t\eta), \partial \cdot (\phi + t\eta)]\}_{t=0} \\ &\quad = \eta * \partial_\phi L + (\partial \wedge \eta) * \partial_{\partial\wedge\phi} L + (\partial \cdot \eta) * \partial_{\partial\cdot\phi} L \end{aligned} \tag{7.8}$$

Before we calculate (7.8) for a general $\phi \in \sec \mathcal{C}\ell(\mathcal{M})$, let us suppose that $\phi = \langle\phi\rangle_r$, i.e., it is homogeneous. Using the properties of the multiform derivative[34] we obtain after some algebra the following fundamental formulas, $(\eta = \langle\eta\rangle_r)$

$$\eta * \partial_{\phi_r} L = \eta \cdot \partial_{\phi_r} L \tag{9a}$$
$$(\partial \wedge \eta) * \partial_{\partial\wedge\phi_r} L = \partial \cdot [\eta \cdot (\partial_{\partial\wedge\phi_r} L)] - (-1)^r \eta \cdot [\partial \cdot (\partial_{\partial\wedge\phi_r} L)] \tag{9b}$$
$$(\partial \cdot \eta) * \partial_{\partial\cdot\phi_r} L = \partial \cdot [\eta \cdot (\partial_{\partial\cdot\phi_r} L)] + (-1)^r \eta \cdot [\partial \wedge (\partial_{\partial\cdot\phi_r} L)] \tag{9c}$$

Inserting Eq. 7.9 into Eq. 7.8 and then in Eq. 7.7 we obtain, imposing $\frac{d}{dt} S(\phi_r + t\eta) = 0$,

$$\begin{aligned} &\int_U \{\eta \cdot [\partial_{\phi_r} L - (-1)^r \partial \cdot (\partial_{\partial\wedge\phi_r} L) + (-1)^r \partial \wedge (\partial_{\partial\cdot\phi_r} L)]\}\tau_g \\ &\quad + \int_U \partial \cdot [\eta \cdot (\partial_{\partial\wedge\phi_r} L + \partial_{\partial\cdot\phi_r} L)]\tau_g = 0 \end{aligned} \tag{7.10}$$

The last integral in Eq. 7.10 is null by Stokes theorem if we suppose as usual that η vanishes on the boundary of U.

Then Eq. 7.10 reduces to

$$\int_U \{\eta \cdot [\partial_{\phi_r} L - (-1)^r \partial \cdot (\partial_{\partial\wedge\phi_r} L) + (-1)^r \partial \wedge (\partial_{\partial\cdot\phi_r} L)]\}\tau_g = 0 \tag{7.11}$$

Now since $\eta = \langle\eta\rangle_r$ is arbitrary and $\partial_{\phi_r} L$, $\partial \cdot (\partial_{\partial\wedge\phi_r} L)$, $\partial \wedge (\partial_{\partial\cdot\phi_r} L)$ are of grade r we get

$$\langle \partial_{\phi_r} L - (-1)^r \partial \cdot (\partial_{\partial\wedge\phi_r} L) + (-1)^r (\partial_{\partial\cdot\phi_r} L) \rangle_r = 0 \tag{7.12}$$

But since $\partial_{\phi_r} \langle L\rangle_0 = \langle \partial_{\phi_r} L\rangle_r = \partial_{\phi_r} L, \partial_{\partial\wedge\phi_r} L = \langle \partial_{\partial\wedge\phi_r} L\rangle_{r+1}$, etc Eq. 7.12 reduces to

$$\partial_{\phi_r} L - (-1)^r \partial \cdot (\partial_{\partial\wedge\phi_r} L) + (-1)^r \partial \wedge (\partial_{\partial\cdot\phi_r} L) = 0 \tag{7.13}$$

Eq. 7.13 is a *multiform Euler-Lagrange equation*. Observe that as $L = \langle L\rangle_0$ the equation has the graduation of $\phi_r \in \sec \bigwedge^r(T^* M) \subset \sec \mathcal{C}\ell(\mathcal{M})$.

Now, let $X \in \sec \mathcal{C}\ell(\mathcal{M})$ be such that $X = \sum_{s=0}^{4}\langle X\rangle_r$ and $F(x) = \langle F(x)\rangle_0$. From the properties of the multivectorial derivative we can easily obtain

$$
\begin{aligned}
\partial_X F(x) &= \partial_X \langle F(x)\rangle_0 \\
&= \sum_{s=0}^{4} \partial_{\langle X\rangle_s}\langle F(x)\rangle_0 = \sum_{s=0}^{4}\langle \partial_{\langle X\rangle_s} F(X)\rangle_0
\end{aligned}
\tag{7.14}
$$

In view of this result if $\phi = \sum_{r=0}^{4}\langle\phi\rangle_r \in \sec \mathcal{C}\ell(\mathcal{M})$ we get as *Euler-Lagrange equation* for ϕ the following equation

$$
\sum_r [\partial_{\langle\phi\rangle_r} L - (-1)^r \partial \cdot (\partial_{\partial\wedge\langle\phi\rangle_r} L) + (-1)^r \partial \wedge (\partial_{\partial\cdot\langle\phi\rangle_r} L)] = 0 \tag{7.15}
$$

We can write Eq. 7.13 and Eq. 7.15 in a more convenient form if we take into account that $A_r \cdot B_s = (-1)^{r(s-1)} B_s \cdot A_r (r \leq s)$ and $A_r \wedge B_s = (-1)^{rs} B_s \wedge A_r$. Indeed, we now have for ϕ_r that

$$
\partial \cdot (\partial_{\partial\wedge\phi_r} L) \equiv \partial \cdot (\partial_{\partial\wedge\phi_r} L)_{r+1} = (-1)^r (\partial_{\partial\wedge\phi_r} L)_{r+1} \cdot \overleftarrow{\partial} \tag{7.16}
$$

$$
\partial \wedge (\partial_{\partial\cdot\phi_r} L) \equiv \partial \wedge (\partial_{\partial\cdot\phi_r} L)_{r-1} = (-1)^r (\partial_{\partial\cdot\phi_r} L)_{r+1} \wedge \overleftarrow{\partial} \tag{7.17}
$$

where $\overleftarrow{\partial}$ means that the internal and exterior products are to be done on the right. Then, Eq. 7.15 can be written as

$$
\partial_\phi L - (\partial_{\partial\wedge\phi} L) \cdot \overleftarrow{\partial} - (\partial_{\partial\cdot\phi} L) \wedge \overleftarrow{\partial} = 0 \tag{7.18}
$$

We now analyse the particular and important case where

$$
L(\phi, \partial\wedge\phi, \partial\cdot\phi) = L(\phi, \partial\wedge\phi + \partial\cdot\phi) = L(\phi, \partial\phi) \tag{7.19}
$$

We can easily verify that

$$
\partial_{\partial\cdot\phi} L(\partial\phi) = \langle \partial_{\partial\phi} L(\partial\phi)\rangle_{r-1} \tag{7.20}
$$

$$
\partial_{\partial\wedge\phi} L(\partial\phi) = \langle \partial_{\partial\phi} L(\partial\phi)\rangle_{r+1} \tag{7.21}
$$

Then, Eq. 7.18 can be written

$$
\begin{aligned}
&\partial_\phi L - \langle \partial_{\partial\phi} L\rangle_{r+1} \cdot \overleftarrow{\partial} - \langle \partial_{\partial\phi} L\rangle_{r-1} 1 \overleftarrow{\partial} \\
&= \partial_\phi L - \langle (\partial_{\partial_\phi} L) \cdot \overleftarrow{\partial}\rangle_r - \langle (\partial_{\partial\phi} L) \wedge \overleftarrow{\partial}\rangle_r \\
&= \langle \partial_\phi L - (\partial_\phi L) \cdot \overleftarrow{\partial} - (\partial_{\partial\phi} L) \wedge \overleftarrow{\partial}\rangle_r = 0 \\
&= \langle \partial_\phi L - (\partial_\phi L) \overleftarrow{\partial}\rangle_r = 0
\end{aligned}
\tag{7.22}
$$

from where it follows the very elegant equation

$$
\partial_\phi L - (\partial_{\partial\phi} L) \overleftarrow{\partial} = 0, \tag{7.23}
$$

also obtained in [46].

As an example of the use of Eq. 7.23 we write the Lagrangian in Minkowski space for a Dirac-Hestenes spinor field represented in the frame $\Sigma = \{\gamma^\mu\}, [\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}, \gamma^\mu \in \sec \bigwedge^1(T^*M)(\subset \sec \mathcal{C}\ell(\mathcal{M})]$ by $\psi \in \sec \mathcal{C}\ell(\mathcal{M})^+$ in interaction with the electromagnetic field $A \in \sec \bigwedge^1(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M})$. We have[14],

$$L = L_{DH} = \langle(\partial\psi\gamma^2\gamma^1 - m\psi\gamma^0)\gamma^0\tilde{\psi} - eA\psi\gamma^0\tilde{\psi}\rangle_0 \tag{7.24}$$

Then

$$\partial_{\tilde{\psi}} L = (\partial\psi\gamma^2\gamma^1 - m\psi\gamma^0)\gamma_0 - eA\psi\gamma^0 \qquad \text{and} \qquad \partial_{\partial\tilde{\psi}} L = 0 \tag{7.25}$$

and we get the Dirac-Hestenes equation

$$\partial\psi\gamma^2\gamma^1 - eA\psi = m\psi\gamma^0 \tag{7.26}$$

Also since $\langle A\psi\gamma^0\tilde{\psi}\rangle_0 = \langle\psi\gamma^0\tilde{\psi}A\rangle_0$ we have

$$\partial_\psi L = -m\tilde{\psi} - e\gamma^0\tilde{\psi}A \tag{7.27}$$

$$\partial_{\partial\psi} L = \gamma^{210}\tilde{\psi}, \quad (\gamma^{210} = \gamma^2\gamma^1\gamma^0). \tag{7.28}$$

Now,

$$(\partial_{\partial\psi} L)\overleftarrow{\partial} = (\gamma^{210}\tilde{\psi})\overleftarrow{\partial}$$

and from the above equations we get

$$-m\tilde{\psi} - e\gamma^0\tilde{\psi}A - (\gamma^{210}\tilde{\psi})\overleftarrow{\partial} = 0$$

and this gives again,

$$\partial\psi\gamma^2\gamma^1 - eA\psi = m\psi\gamma^0$$

Another Lagragian that also gives the DH equation is, as can be easily verified,

$$L'_{DH} = \langle\tfrac{1}{2}\partial\psi\gamma^{210}\tilde{\psi} - \tfrac{1}{2}\psi\gamma^{210}\tilde{\psi}\overleftarrow{\partial} - m\psi\tilde{\psi} - eA\psi\gamma^0\psi\rangle_0 \tag{7.29}$$

7.2. *The Dirac-Hestenes Equation on a Riemann-Cartan Spacetime*

Let $\mathcal{M} = \langle M, g, \nabla\rangle$ be a Riemann-Cartan spacetime (RCST), i.e., $\nabla g = 0, \mathbf{T}(\nabla) \neq 0, \mathbf{R}(\nabla) \neq 0$. Let $\mathcal{C}\ell(\mathcal{M})$ be the Clifford bundle of spacetime with typical fibre $\mathcal{C}\ell_{1,3}$ and let $\psi \in \sec \mathcal{C}\ell^+(\mathcal{M})$ be the representative of a Dirac-Hestenes spinor field in the basis $\Sigma = \{\gamma^a\}, [\gamma^a \in \sec \bigwedge^1(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M}), \gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}]$ dual to the basis $\mathcal{B} = \{e_a\}, e_a \in \sec TM, a, b = 0, 1, 2, 3$.

To describe the "interaction" of the DHSF ψ with the Riemann-Cartan spacetime we invoke the principle of minimal coupling. This consists in changing $\partial = \gamma^a\partial_a$ in the Lagrangian given by Eq. 7.29 by

$$\gamma^a\partial_a\psi \longmapsto \gamma^a\nabla^s_{e_a}\psi \tag{7.30}$$

[14] Note that we are omitting, for sake of simplicity, the reference to the basis Σ in the notation for ψ.

where $\nabla^{s}_{e_a}$ is the spinor covariant derivative of the DHSF introduced in Section 4.4. i.e.,

$$\nabla^{s}_{e_a}\psi = e_a(\psi) + \frac{1}{2}\omega_a\psi. \tag{7.31}$$

Let $\langle x^\mu \rangle$ be a chart for $U \subset M$ and let be $\partial_a \equiv e_a = h^\mu_a \partial_u$ and $\gamma^a = h^a_\mu dx^\mu$, with $h^a_\mu h^v_a = \delta^\mu_v, h^a_\mu h^\mu_b = \delta^a_b$.

We take as the action for the DHSF ψ on a RCST,

$$S(\psi) = \int_U \langle \frac{1}{2}\mathbf{D}\psi\gamma^{210}\tilde{\psi} - \frac{1}{2}\psi\gamma^{210}\tilde{\psi}\overleftarrow{\mathbf{D}} - m\psi\tilde{\psi}\rangle_0 h^{-1} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \tag{7.32}$$

where $\mathbf{D} = \gamma^a \nabla^{s}_{e_a}$ is the operator Dirac operator made with the spinor connection acting on sections of $\mathcal{C\ell}(\mathcal{M})$ and $h^{-1} = [\det(h^\mu_a)]^{-1}$. The Lagrangian $L = \langle L \rangle_0$ is then

$$\begin{aligned} L &= h^{-1}\langle \frac{1}{2}\mathbf{D}\psi\gamma^{210}\tilde{\psi} - \frac{1}{2}\psi\gamma^{210}\tilde{\psi}\overleftarrow{\mathbf{D}} - m\psi\tilde{\psi}\rangle_0 = \\ &= h^{-1}\langle \frac{1}{2}[\gamma^a(\partial_a + \frac{1}{2}\omega_a\psi)\gamma^{210}\tilde{\psi} - \psi\gamma^{210}(\partial_a\tilde{\psi} - \frac{1}{2}\tilde{\psi}\omega_a)\gamma^a] - m\psi\tilde{\psi}\rangle_0 \end{aligned} \tag{7.33}$$

As in Section 7.2 the principle of stationary action gives

$$\begin{aligned} \partial_{\tilde{\psi}} L - (\partial_{\partial_{\tilde{\psi}}} L)\overleftarrow{\partial} &= 0 \\ \partial_{\psi} L - (\partial_{\partial_{\psi}} L)\overleftarrow{\partial} &= 0. \end{aligned} \tag{7.34}$$

To obtain the equations of motion we must recall that

$$(\partial_{\partial_\psi} L)\overleftarrow{\partial} = \partial_\mu(\partial_{\partial_\mu\psi} L) \tag{7.35}$$

and

$$\partial_{\partial_\mu\psi} L = h^\mu_a \partial_{\partial_a\psi} L. \tag{7.36}$$

Then Eqs. 7.34 become

$$\begin{aligned} \partial_\psi L - \partial_\mu(h^\mu_a)\partial_{\partial_a\psi} L - \partial_a(\partial_{\partial_a\psi} L) &= 0, \\ \partial_{\tilde{\psi}} L - \partial_\mu(h^\mu_a)\partial_{\partial_a\tilde{\psi}} L - \partial_a(\partial_{\partial_a\tilde{\psi}} L) &= 0. \end{aligned} \tag{7.37}$$

Now, taking into account that $[e_a, e_b] = c^d_{ab} e_d$ and that $\partial_a h/h = h^a_\mu \partial_a h^\mu_a$ we get

$$\partial_\mu h^\mu_a = -c^b_{ab} + \partial_a \ln h \tag{7.38}$$

and Eqs. 7.37 become

$$\begin{aligned} \partial_\psi L - [\partial_a + \partial_a \ln h - c^b_{ab}]\partial_{\partial_a\psi} L &= 0, \\ \partial_{\tilde{\psi}} L - [\partial_a + \partial_a \ln h - c^b_{ab}]\partial_{\partial_a\tilde{\psi}} L &= 0. \end{aligned} \tag{7.39}$$

Let us calculate explicitly the second of Eqs. 7.37. We have,

$$\partial_{\tilde{\psi}} = h^{-1}[\frac{1}{2}\gamma^a(\nabla_{e_a}\psi)\gamma^{210} + \frac{1}{4}\omega_a\gamma^a\psi\gamma^{210} - m\psi], \tag{7.40}$$

$$\partial_{\partial_a\tilde{\psi}} L = h^{-1}(-\frac{1}{2}\gamma^a\psi\gamma^{210}). \tag{7.41}$$

Then,

$$\partial_a(\partial_{\partial_a\tilde{\psi}}L) = (\partial_a \ln h^{-1})h^{-1}(-\frac{1}{2}\gamma^a\psi\gamma^{210}) - h^{-1}\frac{1}{2}\gamma^a\partial_a\psi\gamma^{210} =$$
$$= -(\partial_a \ln h)\partial_{\partial_a\tilde{\psi}}L - h^{-1}\frac{1}{2}\gamma^a\partial_a\psi\gamma^{210}. \tag{7.42}$$

Using Eq. 7.38 and Eq. 7.40 in the second of Eqs. 7.37 we obtain

$$\frac{1}{2}(\mathbf{D}\psi)\gamma^{210} + \frac{1}{4}\omega_a\gamma^a\psi\gamma^{210} - m\psi + \frac{1}{2}\gamma^a\partial_a\gamma^{210} - \frac{1}{2}c^b_{ab}\gamma^a\psi\gamma^{210} = 0$$

or

$$\mathbf{D}\psi\gamma^{210} - \frac{1}{4}(\gamma^a\omega_a - \omega_a\gamma^a)\psi\gamma^{210} - m\psi - \frac{1}{2}c^b_{ab}\gamma^a\psi\gamma^{210} = 0.$$

Then

$$\mathbf{D}\psi\gamma^{210} - \frac{1}{2}(\gamma^a\cdot\omega_a)\psi\gamma^{210} - \frac{1}{2}c^b_{ab}\gamma^a\psi\gamma^{210} - m\psi = 0. \tag{7.43}$$

But

$$\gamma^a\cdot\omega_a = \omega^b_{ba}\gamma^a \tag{7.44}$$

and since $\omega^b_{ab} = 0$ because it is $\omega^{bc}_a = -\omega^{cb}_a$ we have

$$\gamma^a\cdot\omega_a = (\omega^b_{ba} - \omega^b_{ab})\gamma^a. \tag{7.45}$$

Using Eq. 7.45 in Eq. 7.43 we obtain

$$\mathbf{D}\psi\gamma^{210} - \frac{1}{2}[\omega^b_{ba} - \omega^b_{ab} + c^b_{ab}]\gamma^a\psi\gamma^{210} - m\psi = 0.$$

Recalling the definition of the torsion tensor, $T^c_{ab} = \omega^c_{ba} - \omega^c_{ba} + c^c_{ab}$, we get

$$(\mathbf{D} + \tfrac{1}{2}T)\psi\gamma^1\gamma^2 + m\psi\gamma^0 = 0, \tag{7.46}$$

where $T = T^b_{ab}\gamma^a$.

Eq. 7.46 is the Dirac-Hestenes equation on Riemann-Cartan spacetime. Observe that if $\mathcal{M}$ is a Lorentzian spacetime ($\nabla g = 0$, $\mathbf{T}(\nabla) = 0$, $\mathbf{R}(\nabla) \neq 0$) then Eq. 7.46 reduces to

$$\gamma^a(\partial_a + \frac{1}{2}\omega_a)\psi\gamma^1\gamma^2 + m\psi\gamma^0 = 0, \tag{7.47}$$

that is exactly the equation proposed by Hestenes[48] as the equation for a spinor field in a gravitational field modelled as a Lorentzian spacetime $\mathcal{M}$. Also, Eq. 7.46 is the representation in $\mathcal{C}\ell(\mathcal{M})$ of the spinor equation proposed by Hehl et al[25] for a covariant Dirac spinor field $\Psi \in P_{\mathrm{Spin}_+(1,3)} \times_\rho \mathbb{C}^4$ on a Riemann-Cartan spacetime. The proof of this last statement is trivial. Indeed, first we multiply ψ in Eq. 7.46 by the idempotent field $\frac{1}{2}(1+\gamma^0)$ thereby obtaining an equation for the representative of the Dirac algebraic spinor field in $\mathcal{C}\ell(\mathcal{M})$. Then we translate the equation in $I(\mathcal{M}) = P_{\mathrm{Spin}_+(1,3)} \times_\ell I$, from where taking a matrix representation with the techniques already discussed in Section 2 we obtain as equation for $\Psi \in P_{\mathrm{Spin}_+(1,3)} \times_\rho \mathbb{C}^4$,

$$i\left(\underline{\gamma}_a\nabla^s_a\Psi - \tfrac{1}{2}T\Psi\right) - m\Psi = 0 \qquad i = \sqrt{-1} \tag{7.48}$$

with $T = T^b_{ab}\gamma^a$, γ^a being the Dirac matrices (Eq. 2.31).

We must comment here that Eq. 7.46 looks like, but it is indeed very different from an equation proposed by Ivanenko and Obukhov[26] as a generalization of the so called Dirac-Kähler (-Ivanenko) equation for a Riemann-Cartan spacetime. The main differences in the equation given in [26] and our eq(7.46) is that in [26] $\Psi \in \sec C\ell(\mathcal{M})$ whereas in our approach $\psi_a \in C\ell^+(\mathcal{M})$ is only the representative of the Dirac-Hestenes spinor field in the basis $\Sigma = \{\gamma^a\}$ and also[26] use ∇_{e_a} instead of $\nabla^s_{e_a}$.

Finally we must comment that Eq. 7.46 have played an important role in our recent approach to a geometrical equivalence of Dirac and Maxwell equations[4, 38] and also to the double solution interpretation of Quantum Mechanics.[4, 49, 50]

8. Conclusions

We presented in this paper a thoughtful and rigorous study of the Dirac-Hestenes Spinor Fields (DHSF), their Covariant Derivatives and the Dirac-Hestenes Equations on a Riemann-Cartan manifold $\mathcal{M}$.

Our study shows in a definitive way that Covariant Spinor Fields (CDSF) can be represented by DHSF that are equivalence classes of even sections of the Clifford Bundle $C\ell(\mathcal{M})$, i.e., spinors are equivalence classes of a sum of even differential forms. We clarified many misconceptions and misanderstanding appearing on the ealier literature concerned with the representation of spinor fields by differential forms. In particular we proved that the so-called Dirac-Kähler spinor fields that are sections of $C\ell(\mathcal{M})$ and are examples of amorphous spinor fields (Section 4.3.4) cannot be used for representation of the field of fermionic matter. With amorphous spinor fields the Dirac-Hestenes equation is not covariant.

We presented also an elegant and consise formulation of Lagrangian theory in the Clifford bundle and use this powerfull method to derive the Dirac-Hestenes equation on a Riemann-Cartan spacetime.

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