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ON THE TOPOLOGY OF COMPLETE RIEMANNIAN MANIFOLDS WITH NONNEGATIVE CURVATURE OPERATOR

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ABSTRACT – We will describe a result on the topology of complete manifolds with non negative curvature operator and apply it to several situations where the positivity of the curvature operator is equivalent to the positivity of the sectional curvatures.

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INTRODUCTION.

One of the most interesting problems in riemannian geometry is the study of the topology of riemannian manifolds with non negative curvature. The classical Gauss-Bonnet theorem gives a complete answer in the case of compact surfaces. For higher dimensions several concept of curvature are available generalizing the Gaussian curvature of a surface and probably the most interesting one is the sectional curvature. Many results are known for manifolds with non negative sectional curvature, pinching theorems, soul theorem, finiteness theorems etc., but in general the problem is quite open.

The Hodge theory approach to the problem leads naturally, as we will see, to consider a stronger positivity assumption namely the positivity of the curvature operator We want to describe results due to several authors which lead to a topological classification of manifolds with non negative curvature operator. Such classification may be stated as follow:

THEOREM: *Let M be a complete simply connected riemannian manifold with non negative curvature operator. Then M is the riemannian product of manifolds of the following types:*

i) Manifolds homeomorphic to spheres:

ii) Manifolds diffeomorphic to Euclidean spaces

iii) Manifolds byholomorphic to complex projective spaces

iv) Symmetric spaces of compact type.

We will describe, in the last section, some particular situations in which the positivity of the curvature operator is equivalent to the positivity of the sectional curvatures, with particular emphasis on the case of low codimensional submanifolds of Euclidean space.

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1. A THEOREM OF GALLOT AND MEYER.

Let M^n be an n-dimensional riemannian manifold. Using the riemannian metric we will identify, when convenient, the tangent and cotangent bundle and their exterior algebras. We just notice that we have a naturally induced metric on the space of p-forms or p-vectors, modulo the above identification, requiring that, if $\{X_1,\dots,X_n\}$ is an orthonormal basis for the tangent space at some point, then $\{X_{i_1}\wedge\dots\wedge X_{i_p}: i_1<\dots<i_p\}$ is an orthonormal bases for the space $\Lambda^p(M)$ of p-vectors at that point.

Let $\Omega^p(M)$ be the space of differentiable p-forms on M. If M is compact we define a scalar product on $\Omega^p(M)$ by:

1.1. $$(\omega,\tau) = \int_M \langle\omega(x),\tau(x)\rangle dx \qquad \omega,\tau \in \Omega^p(M)$$

where dx is the riemannian volume density on M and $\langle.,.\rangle$ the naturally induced scalar product on the p-vectors at x.

The exterior differential d: $\Omega^p(M) \longrightarrow \Omega^{p+1}(M)$ has a formal adjoint δ: $\Omega^{p+1}(M) \longrightarrow \Omega^p(M)$ with respect to the scalar product defined above, i.e. an operator such that:

1.2. $$(d\phi,\rho) = (\phi,\delta\rho) \qquad \phi \in \Omega^{p-1}(M),\ \rho \in \Omega^p(M)$$

and the Laplace-Beltrami operator Δ: $\Omega^p(M) \longrightarrow \Omega^p(M)$ is defined by:

1.3. $$\Delta\omega = \delta d\omega + d\delta\omega.$$

The forms in the kernel of Δ are called *harmonic.*

The basic results of Hodge de-Rham theory may be stated as follows:

1.4. THEOREM: The cohomology of the de-Rham complex:

$$\ldots \longrightarrow \Omega^{p-1}(M) \longrightarrow \Omega^{p}(M) \longrightarrow \Omega^{p+1}(M) \longrightarrow \ldots$$

is isomorphic, as graded algebra, to the singular cohomology of M with real coefficients.

1.5. THEOREM: If M is compact the real cohomology of M is isomorphic, as graded vector space, to the kernel of the Laplace-Beltrami operator. More precisely in any de-Rham cohomology class there is a unique harmonic representative.

We observe that exterior product of harmonic forms is not, in general, an harmonic form so the kernel of Δ is not, in general a graded algebra.

In order to compute $\ker\Delta$ it is convenient to have an expression of Δ in terms of the riemannian invariants of M. We will denote, as usual, by ∇ the Levi Civita connection of M and by R it's curvature tensor. The connection ∇

acts on p-forms by:

$$1.6.\qquad (\nabla_X\omega)(X_1,\dots,X_p) = X\cdot\omega(X_1,\dots,X_p) + - \sum_{i=1}^{p} \omega(X_1,\dots X_{i-1},\nabla_X X_i,X_{i+1},\dots,X_p)$$

and therefore R acts on p-forms by:

$$1.7.\qquad R(X,Y)\omega = \nabla_X\nabla_Y\omega - \nabla_Y\nabla_X\omega - \nabla_{[X,Y]}\omega$$

Let $\{X_1,\dots,X_n\}$ be an orthonormal basis for the tangent space of M at x, T_xM. The Ricci operator $Q : T_xM \longrightarrow T_xM$ is defined by:

$$1.8.\qquad Q(X) = \sum_i R(X,X_i)X_i$$

and the Ricci curvature is the associated quadratic form, $Ricc(X) = \langle Q(X),X\rangle$. The Ricci operator extends to an operator on p-forms defined by:

$$1.9.\quad (Q_p\omega)(X_{i_1},\dots X_{i_p}) = \sum_{k=1}^{n}\sum_{j=1}^{p} [R(X_k,X_{i_j})\omega](X_{i_1},\dots,\hat{X}_{i_j},X_k,\dots X_{i_p})$$

where $\hat{X}$ means that we are omitting X.

Clearly $Q = Q_1$ modulo the identification of the tangent and cotangent spaces given by the metric. Using the expression of d and δ in terms of the Levi Civita connection:

$$1.10.\quad (d\omega)(X_{i_1},\dots,X_{i_{p+1}}) = \sum_{k=1}^{p+1} (-1)^{k+1}(\nabla_{X_{i_k}}\omega)(X_{i_1},\dots,\hat{X}_{i_k},\dots X_{i_{p+1}})$$

$$1.11.\quad (\delta\omega)(X_{i_1},\dots,X_{i_{p-1}}) = -\sum_{i=1}^{n} (\nabla_{X_j}\omega)(X_j,X_{i_1},\dots X_{i_{p-1}})$$

it is not difficult to deduce the following *Weitzenböck formula:*

$$1.12.\quad \langle\Delta\omega\ ,\ \omega\rangle = \frac{1}{2}\,\Delta(\|\omega\|^2) + \sum_{k=1}^{n} \|\nabla_{X_k}\omega\|^2 + \langle Q_p\omega\ ,\ \omega\rangle$$

and integration over M held :

$$1.13.\quad (\Delta\omega\ ,\ \omega) = \sum_{k=1}^{n} (\nabla_{X_k}\omega,\nabla_{X_k}\omega) + \int_M \langle Q_p\omega\ ,\ \omega\rangle\, dx.$$

As a preview of what we want to discuss we examine the case $p = 1$. Formula 1.13. (and theorems 1.4. and 1.5.) gives immediately that if the Ricci curvature is non negative and positive at some point then the first cohomology group of M with real coefficients vanishes. Even assuming only that the Ricci curvature is non negative, we get quite a bit of information: Formula 1.13. tells us that harmonic 1-forms are parallel and therefore if k is the dimension of the first cohomology group (as a real vector space), there are k linearly independent parallel vector fields. In particular $k \leq n = \dim M$ and M admits a riemannian submersion over a k dimensional torus (see [BM]).

In order to generalize the above arguments we are naturally led to look for conditions on the geometric invariants of M, that guarantee the positivity of the quadratic form Q_p. The positivity of the sectional curvature is not enough even for $p = 2$. For example the complex projective space, with the Fubini Study metric, has positive sectional curvature but non vanishing second cohomology. A geometric invariant which is well adapted to this type of argument is the curvature operator.

The curvature operator is defined as the linear map ρ_x on the space of bi-vectors at a point $x \in M$ uniquely defined by the condition:

1.14. $$\langle \rho_x(X \wedge Y), Z \wedge W \rangle = \langle R(X,Y)Z, W \rangle$$

The well known symmetries of R imply that 1.14. defines indeed a linear map and this map is symmetric. The sectional curvature of the plane spanned by X and Y is then given by $K(X,Y) = \langle \rho_x(X \wedge Y), X \wedge Y \rangle / \|X \wedge Y\|^2$ and therefore the positivity of ρ implies the positivity of the sectional curvatures. But, in general, the latter does not imply the positivity of ρ essencially since the eigenvectors of ρ may not be decomposable, i.e. may not be of the form $X \wedge Y$.

There have been various tentatives of relating the positivity of the curvature operator to the one of Q_p. Notably the ones of Bochner and Yano who proved the positivity of Q_p under the condition that the biggest eigenvalue of ρ is at most twice the smallest; Berger proved that Q_2 is positive if ρ is positive and finally Meyer proved in 1974 the following result (see [GM]):

1.15.THEOREM: If the curvature operator is non negative (resp. positive) then Q_p is non negative (resp. positive) for $0<p<n$.

From 1.15. (and the preceding discussion) we get immediately:

1.16.THEOREM: If M is a compact connected orientable n-dimensional riemannian manifold with curvature operator which is non negative and positive at some point then M has the real cohomology of an n-dimensional sphere.

Subsequently, in 1975, Gallot and Meyer studied the case where the curvature operator is non negative (see [GM]). In this case, by the formula of Weitzenböck we conclude that the harmonic forms are parallel. The general philosophy is that parallel forms are rather rare and the existence of such forms gives strong costrains on the geometry of the manifold. We will recall now some basic facts on holonomy which are essential for understanding those ideas.

For a riemannian manifold M and a point $x \in M$, the holonomy group at x, $\Phi(M,x)$, is defined as the subgroup of the group of orthogonal transformations of the tangent space at x, whose elements are parallel translations along picewise differentiable loops based at x. This group, viewed as a subgroup of $O(n)$, does not depend on x in the sense that if M is connected for all pair of points $x, y \in M$, $\Phi(M,x)$ and $\Phi(M,y)$ are conjugate in $O(n)$.
It is reasonably clear that *parallel differential forms on M correspond to exterior forms on* T_xM *which are invariant under* $\Phi(M,x)$. Now, for a generic metric, $\Phi(M,x)$ is trivial, i.e. is isomorphic to $O(n)$ or, if M is orientable, to $SO(n)$. Therefore the existence of non-zero parallel forms gives quite a bit of information on the geometry of M. Important examples of how the holonomy group influences the geometry of a riemannian manifold are the following:

1.17.THEOREM *(de-Rham decomposition,* see[KN] vol. 2*)*: Let M be an n-dimensional simply connected riemannian manifold whose holonomy splits as a product of subgroups G_i of $SO(n)$. Then M is the riemannian product of manifolds M_i with the holonomy of M_i isomorphic to G_i. The decomposition of M into product of irreducible manifolds is unique up to order.

1.18.THEOREM *(Berger,* see [Be_1]): If $\Phi(M,x)$ is irreducible and not transitive on the unit sphere of T_xM then the metric of M is symmetric.

For further use we will recall also the following:

1.19.THEOREM (*Ambrose-Singer,* see [KN] vol. 1): The Lie algebra of $\Phi(M,x)$ is generated by parallel translation of $\rho_x(\Lambda^2(T_xM))$.

We will give now an outline of the proof of the following result of Gallot and Meyer (see [GM]):

1.20.THEOREM: Let M be a compact irreducible (in the sense of 1.16.) simply connected riemannian manifold with non negative curvature operator. Then one of the following holds

i) M has the real cohomology of a sphere and holonomy the full special orthogonal group

ii) M has the real cohomology of a complex projective space and holonomy the full unitary group,

iii) M is a symmetric space of compact type.

Proof: If the holonomy group is not transitive on the unit sphere, then M is a symmetric space of compact type by the theorem of Berger. So we can suppose that the holonomy group is transitive. Such groups were classified by Berger and they are:

i) $SU(d)$ $(n = 2d)$, $Sp(d)$ $(n = 4d)$, $Spin(7)$ $(n = 8)$, G_2 $(n = 7)$

ii) $Spin(9)$ $(n = 16)$,

iii) $Sp(d)\cdot Sp(1)$ $(n = 4d)$,

iv) $SO(n)$, $U(d)$ $(n = 2d)$.

It is known that:

a) Manifolds with holonomy like in i) are Ricci flat (see [S]) and therefore flat, in our case, since they have non negative sectional curvature. But flat simply connected complete manifolds are diffeomorphic to R^n by the Cartan-Hadamard theorem, which contradicts compactness.

b) A compact manifold with holonomy $Spin(9)$ is isometric to the Cayley plane by a theorem of Brown and Gray (see [BG]) and therefore is a symmetric space.

c) A manifold with holonomy $Sp(d)\cdot SP(1)$ is an Einstein manifold, by a theorem of Berger (see [Be$_2$]), and a compact simply connected Einstein manifold with non negative curvature operator is a symmetric space, by a theorem of Tachibana (see [GM]).

Finally if the holonomy is the full special orthogonal or the full unitary group, then the invariant exterior forms (and therefore the real cohomology) are, clearly, an algebra isomorphic to the real cohomology algebra of a sphere or a complex projective space.

The above result, several results on the pinching problem as well as results that we will discuss later on, lead naturally to the following:

CONJECTURE : A compact simply connected manifold with positive curvature operator is diffeomorphic to a sphere.

2. A THEOREM OF MICALLEF AND MOORE

Working on the last question of the preceding section M.Micallef and J.D.Moore proved, in 1988, the following result (see[MM]):

2.1.THEOREM: An n-dimensional compact simply connected riemannian manifold with positive curvature operator is homeomorphic to a sphere.

In this section we will outline a proof of the above result.

2.2.REMARK: As we will comment later on theorem 2.1. is true under weaker hypothesis, a fact that may be useful in other contexts. Also the proof requires $n \geq 4$ but the case $n = 2$ is a consequence of Gauss Bonnet Theorem and the case $n = 3$ is a consequence of a result of Hamilton that we will comment in the next section.

The basic idea of the proof of 2.1. came from the classical proof of the Synge's theorem which states that an even-dimensional, compact, orientable riemannian manifold with positive sectional curvature is simply connected.

The proof of Synge's theorem is based on the following two facts:

a) Let M be a compact manifold and $\alpha: S^1 \longrightarrow M$ a continuous map. Then there exists a periodic geodesic $\gamma: S^1 \longrightarrow M$, homotopic to α, which has minimum energy between all picewise smooth maps of S^1 into M, homotopic to α.

b) If γ is a periodic geodesic and V a periodic vector field along γ, the hessian of the energy is given by:

$$E_{**}(V,V) = \int_{S^1} [\|\nabla_{\dot{\gamma}} V\|^2 - K(\dot{\gamma},V)]\, dt$$

where K(X,Y) is the sectional curvature of the plane spanned by X and Y.

The proof of Synge's theorem then goes as follows:

Let us assume that there is a closed curve not homotopic to a constant. Then, in its homotopy class there exists a periodic geodesic γ like in a). Parallel translation along γ induces an orientation preserving orthogonal transformation of the tangent space of M at $\gamma(0)$. This space is even-dimensional and parallel translation has $\dot{\gamma}(0)$ as fixed point so it has at least an other fixed point orthogonal to $\dot{\gamma}(0)$. So there exists a parallel periodic vector field V, along γ, orthogonal to γ. Now, on one side γ is a local minimum of the energy so the hessian is positive semidefinite; On the other we get, from b), $E_{**}(V,V) < 0$.

The idea in the proof of 2.1 is to consider harmonic non constant maps from the unit 2-sphere into M, $f: S^2 \longrightarrow M$, and try to use arguments similar to the ones used above applied to the energy integral:

2.3. $$E(f) = (1/2)\int_{S^2} \|df\|^2 \, dA$$

The harmonic maps are the critical points of E. At a critical point f the hessian of E on a section V of the pull back f^*TM, i.e. on a vector field along f, is given by:

2.4. $$E_{**}(V,V) = \int_{S^2} (\|\nabla V\|^2 - \langle K(V),V\rangle) dA$$

In terms of isothermal coordinates x_i, $i = 1,2$, on S^2, if $ds^2 = \lambda^2(dx_1^2 + dx_2^2)$, then $dA = \lambda^2 dx_1 \wedge dx_2$ and:

$$\|\nabla V\|^2 = \lambda^{-2} \{ \|\nabla_{\partial/\partial x_1} V\|^2 + \|\nabla_{\partial/\partial x_2} V\|^2 \}$$

$$K(V) = \lambda^{-2} \{ R(V,\partial f/dx_1)\partial f/dx_1 + R(V,\partial f/dx_2)\partial f/\partial x_2 \}.$$

A direct extension of Synge's theorem argument is not possible because of two facts:

a) The existence of minimum energy harmonic maps in a given homotopy class is not at all clear since the energy functional does not verify a reasonable compactness condition like, for example, the condition (C) of Palais Smale, which guarantees the existence of minima of a bounded functional.

b) The existence of parallel fields along such an f is extremely rare being related to the vanishing of certain curvatures and so we can not avoid easily the positive term $\|\nabla V\|^2$ in 2.4..

It is somehow possible to go around the difficulty explained in a) considering the α-energy functional:

2.5. $$E_\alpha(f) = \int_{S^2} (1 + \|df\|^2)^\alpha \, dA$$

which, for $\alpha > 1$ satisfies the compactness condition of Palais and Smale. Developing the Morse theory for E_α and "letting α go to 1" Sacks and Uhlenbeck were able to prove the following result (see [SU]):

2.6.THEOREM: Let M be a compact riemannian manifold and m the smallest integer such that $\pi_m(M) \neq 0$. If $m \geq 2$ then there exists a harmonic map $f: S^2 \longrightarrow M$ of index $\leq m-2$.

2.7.REMARK: First of all let us recall that the index of f is the maximum of the dimensions of subspaces of the sections of f^*TM on which E_{**} is negative definite. In the general philosophy of Morse Theory, if we have a function defined on a manifold (possibly infinite dimensional) which is a Morse function (i.e. has only non degenerate critical points is bounded below and verifies a compactness condition), and the i^{th} Betti number of the manifold in non-zero, then the function has critical points of index i. In our case the manifold is a suitable manifold of maps of S^2 into M so, in the hypothesis of the theorem, it's first non-vanishing homology group, with suitable coefficients, occurs in dimension m - 2, and it is possible to apply the above ideas after perturbing slightly E_α in order to have a Morse function.

The main problem is to control the critical points of E_α when α goes to 1. Just to have an idea of the difficulties we remark that there exist sequences of harmonic maps, homotopically not trivial, whose energy goes to zero.

We will discuss now on the index of E_{**} trying to give an idea of the proof of the following result (see [MM]):

2.8.THEOREM: If the curvature operator is positive, then a non constant harmonic map $f : S^2 \longrightarrow M$ has index at least n - 2.

Before proceeding on discussing 2.8. let us observe how 2.6. and 2.8. prove theorem 2.1. In fact if m is the smallest integer for which $\pi_m(M)$ is not trivial, we obtain $(m - 2) \geq (n - 2)$. Therefore $\pi_i(M) = \{0\}$ for $i < n$ and by the theorem of Hurewicz, M has the integral homology of a sphere. Since M is simply connected, M is a homotopy sphere by Whitehead theorem and therefore, if $n \geq 4$, M is homeomorphic to a sphere by the positive solution of the Poincaré conjecture in dimension bigger than 3.

We go back now to discuss 2.8.. A harmonic map $f : S^2 \longrightarrow M$ is a conformal branched minimal immersion and induces on f^*TM a metric and a connection. The main idea is to *complexify the situation* in order to replace the search of parallel fields along f (which do not exist, in general, by. b) above) with the search of holomorphic fields.

The induced metric extends to a bilinear complex form and to an hermitian product on $\mathbb{E} = f^*TM \otimes \mathbb{C}$:

$$(.,.) : \mathbb{E} \times \mathbb{E} \longrightarrow \mathbb{C}\ , \qquad \langle\langle .,. \rangle\rangle : \mathbb{E} \times \mathbb{E} \longrightarrow \mathbb{C}$$

and the connection ∇ to a connection on $\mathbb{E}$ which is hermitian with respect to $\langle\langle .,. \rangle\rangle$. There exist, therefore, a unique structure of holomorphic vector bundle on $\mathbb{E}$ such that a section $W: S^2 \longrightarrow \mathbb{E}$ is holomorphic if and only if $\nabla_{\partial/\partial\bar{z}} W = 0$, where $\partial/\partial\bar{z} = 1/2(\partial/\partial x_1 + i\partial/\partial x_2)$. The fact that f is harmonic is equivalent to $\nabla_{\partial/\partial\bar{z}}(\partial f/\partial z) = 0$, i.e. to the section $\partial f/\partial z$ being holomorphic.

The bilinear form 2.4. extends to a bilinear hermitian form on the sections of **E**:

2.9. $$E^{\mathbf{C}}_{**}(W,W) = \int_{S^2} \{\|\nabla W\|^2 - \ll K(W),W\gg\} dA$$

Let us extend the curvature operator ρ to a linear operator $\rho^{\mathbf{C}}$ on the complexification $\Lambda^2 M \otimes \mathbf{C}$. Then we have:

2.10. $$E^{\mathbf{C}}_{**}(W,W) = 4\int_{S^2} \{\|\nabla_{\partial/\partial\bar{z}} W\|^2 - \ll \rho^{\mathbf{C}}(W\wedge\partial f/\partial z), W\wedge\partial f/\partial z \gg$$

We observe that since $E^{\mathbf{C}}_{**}$ is the hermitian extension of E_{**}, if the first is negative defined on a subspace of complex dimension l then the latter is negative defined on a subspace of real dimension l. So, in order to prove 2.8., is enough to find (n-2) holomorphic sections of **E** which are **C**-linearly independent and not colinear, at some point, with $\partial f/\partial z$ (so $W\wedge\partial f/\partial z \neq 0$).

By a theorem of Grothendick, an holomorphic vector bundle on S^2 decomposes as direct sum of holomorphic line bundles:

2.11. $$\mathbf{E} = \mathbf{L}_1 \oplus \ldots\ldots \oplus \mathbf{L}_n \qquad \dim_{\mathbf{C}} \mathbf{L} = 1$$

The isomorphism classes of the $\mathbf{L}_i$'s are determined uniquely up to the order and we will suppose

$$c_1(L_1) \geq c_1(L_2) \geq \ldots\ldots \geq c_1(L_n)$$

where $c_1(L_i)$ is the first Chern class of L_i computed on the fundamental class of S^2.

We observe now that the bilinear form $(\cdot,\cdot)$ is parallel and defines an isomorphism between E and it's dual E^*. It follows that

2.12. $$c_1(L_i) = -c_1(L_{n-i+1})$$

The dimension d_i of the space of holomorphic sections of L_i may be computed using the Riemann-Roch theorem. This gives:

2.13. $$d_i = \begin{cases} c_1(L_i) + 1 & \text{if } c_1(L_i) \geq 0 \\ 0 & \text{if } c_1(L_i) < 0 \end{cases}$$

Let l be the biggest integer such that $c_1(L_l) > 0$. From 2.12. we get $c_1(L_j) = 0$ for $l+1 \leq j \leq n-l$. Therefore by 2.13. E has $\sum_{i \leq n-l} [c_1(L_i) + 1]$ linearly independent holomorphic sections. All those sections are not colinear, somewhere, with $\partial f/\partial z$ if the latest is not a section of some of the L_i's. If

$\partial f/\partial z$ is a section of some of the L_i's then, at worst is a section of L_1 and so colinear with at most with $[\ c_1(L_1)+1\ ]$ of the above sections. Therefore for the index of $E^{\mathbf{C}}_{**}$ we get:

$$2.14. \qquad \mathrm{index}(E^{\mathbf{C}}_{**}) \geq \sum_{2\leq i\leq n-1} [\ c_1(L_i) + 1\ ] \geq n-2$$

which prove 2.8..

2.15.REMARK: Theorem 2.1. still holds with the less restrictive hypothesis that $\rho^{\mathbf{C}}$ is positive on totally isotropic 2-planes, i.e. on bi-vectors $W \wedge Z$ in $\Lambda^2 M \otimes \mathbf{C}$ such that $(W,W) = (Z,Z) = 0$. In fact, refining the above arguments is possible to prove, in this case, the existence of $(n-3)/2$ linearly independent isotropic sections not colinear, somewhere, with $\partial f/\partial z$ (which is isotropic!). From 2.6. it follows that, in this case, the first non vanishing homotopy group occurs in dimension $m > (n-1)/2$. This is sufficient to guarantee that M is a homotopy sphere. This weaker condition holds in several interesting cases: For example if the sectional curvature takes values in an interval of the type $(A,4A]$, $A > 0$, then $\rho^{\mathbf{C}}$ is positive on totally isotropic 2-planes. As a corollary we get the well known sphere theorem of Berger and Klingenberg.

3. MODIFICATION OF THE METRIC FOLLOWING HAMILTON.

The theorem of Micallef and Moore require the curvature operator to be positive at all points: In fact, even if the basic argument in an integral argument, and so in principle we need the integrand to be non negative and positive only at one point, we can not guarantee, a priori, that this point of positive curvature is in the image of the harmonic map we are working with. On the other hand the theorem of Meyer requires the curvature operator to be positive only at one point, since we are integrating on the all manifold; The conclusion is weaker essentially since does not say anything on the torsion part of the integral homology.

In this section we will discuss a link between the two results through a method introduced, essentially, by Richard Hamilton. R. Hamilton was interested in solving a "Riemannian version of the Poincare' conjecture". More specifically he was trying to prove that every compact 1-connected riemannian manifold with positive Ricci curvature is diffeomorphic to a sphere. The starting point was that for 3-manifolds an Einstein metric, i.e. a metric of constant Ricci curvature, is of constant sectional curvature. If we look at the space of smooth riemannian metric on a manifold M, which is a cone in the Frechet space of smooth section of the bundle of symmetric 2-forms on M, the Einstein metric are the critical points of the total scalar curvature functional, $S(g) = \int_M s_g dx_g$ where s_g is the scalar curvature of the metric g and dx_g is the volume density associated to g (critical points with respect to variations which keep the volume fixed). So, in order to find those critical points, it is natural to follow the integral lines of the gradient of S. Naturally the above is a very rough idea since we are not working on a Hilbert manifold so there is not a well defined gradient field. If we take the L^2 gradient we find that the integral curve should be solution of a differential equation that, in general does not have solutions (see [Bes]). Anyhow after some natural modifications of the above mentioned equation Hamilton ended up considering the evolution equation:

$$
3.1 \qquad \begin{cases} \partial g/\partial t(g(t)) = -2\mathrm{Ricc}(g(t)) \\ \\ g(0) = g \end{cases}
$$

where g is the original metric.

If $n = \dim M = 3$ and the Ricci curvature of g is positive, then Hamilton proves that 3.1. has a solution for all $t \geq 0$ and this solution tends, as $t \longrightarrow \infty$, to a metric of constant curvature. So he obtains (see [H_1]) :

3.2.THEOREM: A compact 1-connected riemannian 3-manifold with positive Ricci curvature is diffeomorphic to a sphere.

In a subsequent paper, (see [H_2]), Hamilton studies the 4-dimensional case. He proves that if M is a 1-connected riemannian 4-manifold with positive curvature operator then 3.1. has still solutions tending to a metric of constant sectional curvature. In this paper moreover, he proves a lemma which holds for any dimension and furnish the link between the theorem of Micallef and Moore and the one of Gallot and Meyer:

3.3.LEMMA: Let M be an n-dimensional compact Riemannian manifold with a metric g with non negative curvature operator ρ. Then 3.1. has a solution g(t) defined in a small interval and the curvature operator ρ_t of g_t is non negative. Moreover, for t > 0, $\rho_t(\Lambda^2_x M)$ is parallel.

From the above lemma and the results described in the preceding sections we have:

3.4.THEOREM: If M is a compact simply connected riemannian manifold with non negative curvature operator and holonomy SO(n), then M is homeomorphic to a sphere.

Proof: We consider the solution of 3.1. at a positive time t. Since $\rho_t \geq 0$ and the topology of M does not depend on t, the holonomy of g_t is still SO(n), by the theorem of Gallot and Meyer. But, by 1.19, the Lie algebra of the holonomy group is generated by parallel translation of $\rho_t(\Lambda^2_x M)$, which is parallel, and therefore $\rho_t(\Lambda^2_x M) = \Lambda^2_x M$, the Lie algebra of $SO(n)$. Therefore ρ_t is non negative and non singular, hence positive, and the conclusion follow from the theorem of Micallef and Moore.

Using similar argument it is shown in [CY] that if the holonomy group is $U(n)$, then the modified metric has positive holomorphic bisectional curvatures and therefore, by the (positive) solution of the Frankel conjecture (see [SY]), we obtain:

3.5.THEOREM: Let M be a simply connected compact riemannian manifold with non negative curvature operator and holonomy $U(n)$. Then M is biholomorphically equivalent to CP^n.

Combining those results with the theorem of Gallot and Meyer we get the result quoted in the introduction for the compact case:

3.6.THEOREM: Let M be a simply connected compact riemannian manifold with non negative curvature. Then M is the riemannian product of symmetric spaces of compact type, manifolds biholomorphically equivalent to complex projective spaces and manifolds homeomorphic to spheres.

4. OPEN MANIFOLDS WITH NON NEGATIVE CURVATURE OPERATOR.

In this section we will discuss the case of open manifolds, i.e. complete non compact riemannian manifolds.

A well known theorem of Cheeger and Gromoll (see [CG]), guarantees that for an open n-manifold M with non negative sectional curvature there exists a totally convex embedding of a compact k-manifold S^k, such that M is diffeomorphic to the total space of the normal bundle of S^k. We just recall that totally convex is a stronger condition than totally geodesic: If a geodesic of M intersects a totally convex set in two distinct points, then it lies entirely in S. The submanifold S^k is called a *soul* of M (in general is not unique) and has several interesting properties between which we recall the following:

a) For all $x \in S^k$, $X \in T_xS^k$, $Z \in (T_xS^k)^\perp$, the sectional curvature $K(X,Z)$ of the plane spanned by X and Z vanishes (see [CG]).

b) A soul has minimal volume in its homology class (see [Y]).

From the first property we see that if the sectional curvature is strictly positive then the soul is a point and M^n is diffeomorphic to R^n. This, among other considerations, leads to the following question (posed by Cheeger and Gromoll):

CONJECTURE: If M^n is an open manifold with non negative sectional curvature and there exists a point $x \in M^n$ such that all the sectional curvatures at x are positive, then M^n is diffeomorphic to R^n.

Just to better justify the conjecture let us observe that the assertion is obviously true if the point x belongs to a soul. Now the construction of a soul has, as starting point, any point of M. From this point we construct a totally convex set which, in general has to be reduced to give a soul so that we can not guarantee that this point will still belong to the soul after the refined construction. In this section, beside other results, *we will indicate a proof of the above conjecture for the case of non negative curvature operator.*

Another interesting question is when the normal bundle of the soul is trivial. As an easy example when this is not the case we can consider the tangent bundle to the 2-dimensional sphere S^2. On TS^2 we have a metric of non negative sectional curvature obtained by considering TS^2 as the quotient of the product of the associated frame bundle of S^2, which is SO(3), with R^2 modulo the obvious diagonal action of SO(2). The two spaces admit (obvious) non negative curved metrics such that the action of SO(2) is by isometry, so the quotient has an induced riemannian metric which, by O'Neil formula, is still of non negative curvature. However TS^2 is not diffeomorphic to a trivial bundle over S^2. As we will see, and this is really the starting point of our discussion, if the curvature operator is non negative then, at least if the manifold is simply connected, the bundle is trivial.

The main result that we will present in this section is due, independently, in one form or an other, to Noronha (see [N_1]) Strake (see [St]) and Yim (see [Y]):

4.1.THEOREM: Let M^n be an open simply connected riemannian manifold with non negative curvature operator. Then M^n is isometric to the riemannian product of a soul S^k and a non negatively curved manifold of dimension $(n-k)$, diffeomorphic to $\mathbb{R}^{n-k}$.

We will outline a proof of 4.1.. M^n will be an n-dimensional manifold satisfying the hypothesis of 4.1. and S^k a soul of M^n. Since M^n is simply connected and S^k carries all the homotopy of M, then $k > 1$. The starting point of the proof is the following:

4.2.LEMMA: The normal connection of S^k in M^n is flat.

Proof: Let $x \in S^k$ and ω_i, $i=1,\dots,n$, be an orthonormal basis of $\Lambda^2_x M$ of eigenvectors of ρ, with eigenvalues λ_i. Let X, Y be orthonormal vectors tangent to S^k at x and Z a unit vector normal to S^k at x. Let $X \wedge Z = \sum a_i \omega_i$, $i = 1,\dots,\binom{n}{k}$. Then:

$$0 = K(X,Y) = \langle\rho(X\wedge Z), X\wedge Z\rangle = \sum a_i^2 \lambda_i$$

Since $\lambda_i \geq 0$, $a_i\lambda_i = 0$ for all i, and therefore $\rho(X\wedge Z) = 0$. For the same reason $\rho(Y\wedge Z) = 0$ and, by the Bianchi identity, $R(X,Y)Z = 0$. The conclusion follows from the Ricci equation of the embedding $S^k \longrightarrow M^n$, which is totally geodesic.

Let ξ be a vector normal to S^k at some point x. Since S^k is simply connected and the normal connection is flat we can extend ξ to a vector field along S^k, still denoted by ξ, which is normal to S^k and parallel in the normal connection. This allows to define a map:

$$f_\xi : S^k \times \mathbb{R} \longrightarrow M^n \quad , \quad f_\xi(x,t) = \exp_x t\xi(x).$$

Then, at least for small t's, we have a 1-parameter family of submanifolds $S_{t,\xi} = f_\xi(S^k,t)$, called *pseudo-souls* of M^n.

4.3.PROPOSITION: f_ξ is an isometric totally geodesic immersion.

Proof: For fixed t, $f_\xi(\cdot,t)$ is distance non incrising by Rauch comparison theorem. On the other hand, by b) above, $f_\xi(S^k,0)$ is volume minimizing in it's homology class and so $f_\xi(\cdot,t)$ is an isometric immersion. From the construction it follows easily that f_ξ is isometric. The fact that f_ξ is totally geodesic comes from the rigidity part of the Rauch comparison theorem.

Let $x \in M^n$. Take a minimal geodesic $\gamma : [0,1] \longrightarrow M^n$ between S^k and $x = \gamma(1)$ and extend $\dot{\gamma}(0)$ to a normal parallel vector field ξ along S^k. So there is a pseudo-soul S_x passing through x. Those pseudo-souls have still flat normal connection (need some calculations) and are volume minimizing in their homology classes. Some geometric arguments (see [N_1] for details), allow to conclude that for each point there is a unique pseudo-soul of the above form containing the point. Therefore we have two differentiable distributions defined on M: The first one, D_1, given by the tangent spaces to the pseudo-souls, and the second one $D_2 = D_1^{\perp}$. If we prove that the D_i's are parallel, the conclusion will follows from the de-Rham decomposition theorem. Since the pseudo-souls are totally geodesic, D_1 is parallel. So we are left to prove that D_2 is parallel, i.e.:

(*) $\langle \nabla_X Y, V \rangle = 0$ for X, Y vector fields in D_2 and V in D_1.

For this we first note that the leaves of the foliation F determined by D_1, the pseudo-souls, are totally geodesic, simply connected and equidistant, and this allows us to put in the quotient space M/F a smooth manifold structure and a riemannian metric such that the projection map is a riemannian submersion (see [He]). For this submersion, horizontal vectors are the ones in D_2 and vertical vectors the ones in D_1. To prove (*) we will use the O'Neill formulas for submersions (see [O]):

Let H and V be the projectors on the horizontal and vertical subspaces respectively. The O'Neill tensors T and A are given by:

$$T_V X = H(\nabla_{V(V)} V(X))+V(\nabla_{V(V)} H(X); \quad A_X V = V(\nabla_{H(X)} H(V))+H(\nabla_{X(X)} V(V))$$

If X is horizontal and V vertical, $T_V X = V(\nabla_V X) = 0$ (since the pseudo souls are totally geodesic) and $A_X V = H(\nabla_X V)$. So, *in order to prove (*) is enough to show that* $A_X V = 0$. The sectional curvature of the plane spanned by X and V is given by: $K(X,V) = <(\nabla_X T)_V V,X> + \|A_X V\|^2 - \|T_V X\|^2$, that, in our situation, becomes:

$$<(\nabla_X T)_V V,X> = -\|A_X V\|^2.$$

Expanding the left hand side we get:

$$-\|A_X V\|^2 = <\nabla_X T_V V - T_{\nabla_X V} V - T_V \nabla_X V \, , X> =$$
$$<\nabla_X \, H(\nabla_V V) - H(\nabla_{V(\nabla_X V)} V) - H(\nabla_V V(\nabla_X V) - V(\nabla_V H(\nabla_X V)) \, , X> = 0.$$

and therefore the second distribution is parallel.

As a corollary of 4.1. we will prove now a result which implies in a positive answer to the conjecture of Cheeger and Gromoll, quoted at the beginning of this section, for the case of manifolds with non negative curvature operator:

4.4.COROLLARY: Let M^n be an open manifold with non negative curvature operator. Then M is locally isometric to a product over the soul. In particular, if the sectional curvature is positive at some point, then M^n is diffeomorphic to $\mathbb{R}^n$.

Proof: Let S be a soul of M and $\tilde{M}$ and $\tilde{S}$ their universal coverings. By theorem 9.1. of [CG], $\tilde{S}$ is isometrically diffeomorphic to a product $S_0\times\mathbb{R}^m$, with S_0 compact and the splitting is the Topogonov lines splitting (see [To]). Then the lines in $\tilde{S}$ must split off in $\tilde{M}$ too and hence $\tilde{M}$ is isometrically diffeomorphic to $M_0\times\mathbb{R}^m$. Now M_0 is simply connected and so, by theorem 4.1., M_0 is the product of a soul S_1 and a manifold P^r diffeomorphic to $\mathbb{R}^r$. It is not difficult to see that $S_0 \subseteq S_1$ and since they are closed and homotopy equivalent, $S_0 = S_1$ (see [N1] for details). So, we have the following diagram:

$$\begin{array}{ccc} S_0\times P^r\times\mathbb{R}^m & \xrightarrow{\pi} & M \\ p_1 \downarrow & & \downarrow p_2 \\ S_0\times\mathbb{R}^m & \xrightarrow[\pi]{} & S \end{array}$$

Where the π's are the covering maps and p_1 is the projection. Since the π's are local isometries and the fundamental group preserve the splitting, p_2 is a submersion which is a local product.

5. SOME APPLICATIONS TO THE GEOMETRY OF SUBMANIFOLDS.

In this section we will consider some situation in which the positivity of the sectional curvature is equivalent to the positivity of the curvature operator and apply the results of the previous sections to those cases.

Let M and $\bar{M}$ be riemannian manifolds of dimension n and $(n+p)$ respectively and $f : M \longrightarrow \bar{M}$ an isometric immersion. We will denote by ∇ and $\bar{\nabla}$ the riemannian connections of M and $\bar{M}$, by ρ and $\bar{\rho}$ the curvature operators, by K and $\bar{K}$ the sectional curvatures. $\nu(f)$ will denote the normal bundle of the immersion so, for $x \in M$, we have the orthogonal decomposition:

5.1. $$T_{f(x)}\bar{M} \cong T_xM \oplus \nu_x(f)$$

As usual, for all local considerations we will identify M with its image $f(M) \subseteq \bar{M}$. If X,Y are vector fields tangent to M and ξ is a vector field normal to M, 5.1. gives orthogonal decompositions:

$$5.2.\quad \begin{cases} \bar{\nabla}_X Y = \nabla_X Y + \alpha(X,Y) & \text{(Gauss formula)} \\ \bar{\nabla}_X \xi = -A_\xi X + \nabla^{\perp}{}_X \xi & \text{(Weingarten formula)} \end{cases}$$

$\alpha : TM \otimes TM \longrightarrow \nu(f)$ is a symmetric tensor called the *second fundamental form*, A_ξ: $TM \longrightarrow TM$ is a symmetric tensor called the *Weingarten operator in the* ξ *direction*, $\nabla^{\perp}$ is the *normal connection* . A_ξ and α are related by $\langle A_\xi X,Y\rangle = \langle \alpha(X,Y),\xi\rangle$.

The curvature tensors of M and $\bar{M}$ and the second fundamental form are related by the Gauss equation:

5.3. $\langle(\rho-\bar{\rho})(X\wedge Y),Z\wedge W\rangle = \langle\alpha(X,Z),\alpha(Y,W)\rangle - \langle\alpha(X,W),\alpha(Y,Z)\rangle$

If $\{\xi_1,\ldots,\xi_p\}$ is an orthonormal basis for $\nu_x(f)$, 5.3. can be written:

5.4. $$(\rho-\bar{\rho})(X\wedge Y) = \sum_{i=1}^{p} A_{\xi_i}(X)\wedge A_{\xi_i}(Y)$$

The starting point of our discussion is the following result of Weinstein (see [W]):

5.5.PROPOSITION: With the above notations we have:

a) If the A_{ξ_i}'s are positive then $(\rho-\bar{\rho})$ is positive;

b) If $(\rho-\bar{\rho})$ is positive then $(K-\bar{K})$ is positive;

c) If $p = 2$ and $(K-\bar{K})$ is positive, then $(\rho-\bar{\rho})$ is positive.

(In the above we can substitute positive with non negative everywhere).

Proof: a) Set $\rho_i(X\wedge Y) = A_{\xi_i}(X)\wedge A_{\xi_i}(Y)$. Since $(\rho-\bar{\rho}) = \sum_{i=1}^{p} \rho_i$, it is enough to prove that the ρ_i's are positive. Let $\{e_1,\ldots e_n\}$ be an orthonormal basis of eigenvectors of A_{ξ_i} with eigenvalues $\lambda_k = \langle A_{\xi_i}(e_k),e_k\rangle$. Now $\rho_i(e_k\wedge e_m) = \lambda_k\lambda_m e_k\wedge e_m$ and therefore ρ_i is positive.

b) Obvious

c) If $(K-\bar{K}) > 0$ then, for all $X,Y \in T_xM$ with $X\wedge Y \neq 0$, the Gauss equation gives $\langle\alpha(X,X),\alpha(Y,Y)\rangle - \|\alpha(X,Y)\|^2 > 0$. Therefore, the vectors $\alpha(X,X)$ span a cone in $\nu_x(f)$ such that any two vectors of this cone form an angle less then $\pi/2$. Therefore there exist two orthonormal vectors ξ_1 and ξ_2 in $\nu_x(f)$ such that, for all $X \in T_xM$, $\langle\alpha(X,X),\xi_i\rangle > 0$. The conclusion follows from a).

5.6.COROLLARY: If $p = 2$, M is simply connected and complete, $\bar{\rho} \geq 0$ and $(K-\bar{K})$ is non negative and positive at some point, then M is either homeomorphic to a sphere or diffeomorphic to a Euclidean space.

Let us now specialize to the case $\bar{M} = \mathbb{R}^{n+2}$. If M is compact and orientable, a theorem of Bishop (see [Bi]) guarantees that the holonomy of M is of the form $SO(m)\times SO(n-m)$ (possibly $m = 0$) or $U(2)$ ($n = 4$).

Now, as we have seen in the previous sections, if the holonomy is $U(2)$ the manifold is biholomorphic to complex projective plane and such a manifold does not admit immersions in $\mathbb{R}^6$, by purely topological reasons. If the holonomy is $SO(m)\times SO(n-m)$, $m \neq 0, n$, then the manifold is a riemannian product, by de Rham decomposition theorem. In this case results of J.D.Moore (see [M_1]), Alexander and Maltz (see [AM]), imply that the immersion is a product of two hypersurfaces immersions. In conclusion we have:

5.7.THEOREM: If $f: M^n \longrightarrow \mathbb{R}^{n+2}$ is an isometric immersion of a compact, simply connected riemannian manifold with non negative sectional curvature, then either f is a product of two convex embeddings or M is is homeomorphic to a sphere.

5.8.REMARK: The above result was first stated in [BM_1] but the proof contains an error if n is odd. In the above mentioned paper is also studied the non simply connected case and the non orientable case was studied in [BM_2]. It is interesting to point out that the above mentioned results where obtained before some of the results discussed in this paper, using an inequality involving the total absolute curvature which imply, via Morse theory that, for any field of coefficients, the sum of the Betti numbers with those coefficients is less or equal to 4 (the inequality being strict if the sectional curvature is positive at some point). This is still the only method we know to handle the non simply connected case (see also [M_2]).

We want to observe explicitly, for further use, that all the above mentioned results hold true in codimension bigger that two if the first normal space (i.e. the subspace of the normal space spanned by the image of the second fundamental form) is of dimension at most two (see [B]). In particular we will need the following result:

5.9.THEOREM: Let $f: M^n \longrightarrow \mathbb{R}^{n+m}$ be an isometric immersion, $n \geq 3$, M^n compact with non negative sectional curvatures, and suppose that the dimension of the first normal space is at most two and there exists a point on M where all the sectional curvatures are positive. Then M is simply connected (and homeomorphic to a sphere).

We will now to discuss briefly the open case. We notice, first of all, that that proposition 5.5. and corollary 4.4., imply the conjecture of Cheeger and Gromoll stated in the preceding section for the case of codimension two submanifolds of Euclidean spaces, with non negative sectional curvature (the result was known for hypersurfaces).

Before stating the results we recall that an isometric immersion $f: M^n \longrightarrow \mathbb{R}^N$ is called s-cylindrical if there exist isometric splitting $M^n = M_1^{n-s} \times \mathbb{R}^s$ and $f = f_1 \times 1_s$ where $1_s: \mathbb{R}^s \longrightarrow \mathbb{R}^s$ is the identity map. Therefore in order to classify isometric immersions $f: M^n \longrightarrow \mathbb{R}^{n+2}$ with non negative sectional curvatures we can suppose that f is not cylindrical which, by [Ha], is equivalent to the existence of a point $x \in M^n$ such that the index of relative nullity, $\nu(x) = 0$, where

$$\nu(x) = \dim.\{X \in T_xM: \alpha(X,Y) = 0 \ \forall \ Y \in T_xM\}.$$

5.10.THEOREM: Let $f: M^n \longrightarrow R^{n+2}$, $n \geq 3$, be a non-cylindrical isometric immersion of an open manifold with non negative sectional curvature. Then either M^n is simply connected or the soul is homeomorphic to a circle or to a real projective plane, RP^2. In the latter case, $n = 3$.

Proof: Let $x \in M$ be a point such that $\nu(x) = 0$. Let $r(x)$ be the Lie algebra generated by the range of the curvature operator. By the theorem of Bishop mentioned above, such algebra is either the orthogonal algebra $o(n)$, the product of two orthogonal algebras $o(k)\times o(n-k)$ or the unitary algebra $u(2)$ and, in the latter case, $n = 4$. By corollary 4.4., M splits locally and so $r(x) \cong o(k)\times o(n-k)$ where k is the dimension of the soul. If M is not simply connected, $k > 0$. If $k = 1$ the soul is an S^1, so suppose $k \geq 2$. In this case it can be shown that there exist normal fields to the soul, ξ_1 and ξ_2 such that the curvature operator restricted to $o(k) = o(T_xS^k)$, is given by $A_{\xi_1}\wedge A_{\xi_2}$ and $Ker(A_{\xi_1}) = \{0\}$ (see [N_2] for details). It follows that there is a point in the soul where all sectional curvatures (of the soul) are positive. Let us restrict the immersion to the soul. Since it is totally geodesic in M, the first normal space to this restriction is at most two dimensional and therefore, by 5.10., if $k \geq 3$, the soul is simply connected (and so is M). If $k = 2$ then either the soul is an S^2 or a projective plane RP^2. We will sketch a proof that the latter case can not occur if $n \geq 4$. In fact suppose that the soul is a real projective plane and let $\tilde{M}$ be the universal cover of M and $\tilde{f}$ the induced immersion of $\tilde{M}$ into R^{n+2}. By theorem 4.1. $\tilde{M}$ is isometric to the riemannian product of the soul with another manifold of dimension $n-2$. If $n-2 \geq 2$, by a factorization theorem (see $[M_1]$), $\tilde{f}$ is product of hypersurfaces immersions and in particular, $\tilde{\alpha}(\tilde{X},\tilde{Y}) = 0$ where $\tilde{\alpha}$ is the second fundamental form of $\tilde{f}$, $\tilde{X}$ is tangent to the soul and $\tilde{Y}$ is normal. This implies that $\alpha(X,Y) = 0$ if X is tangent to the soul of M and Y is normal. This implies again that M is a product and f factors as product of hypersurfaces immersions. But this would give an isometric immersion of a projective plane in R^3 with non negative curvature, which is impossible.

It follows from the above that if M is not simply connected, M is topologically classified (see [CG] for the case $k = 1$ or $n = 3$). For the simply connected case we can prove:

5.11.THEOREM: Let $f\colon M^n \longrightarrow R^{n+2}$ be an isometric immersion of an open simply connected n-dimensional manifold with non negative sectional curvature, $n \geq 3$. Then:

i) either f is cylindrical over the soul and then either M is a product of a manifold homeomorphic to a sphere with a flat R^{n-k}, or f restricted to the soul is the product of two convex embeddings,

ii) or M is the riemannian product of a manifold homeomorphic to a sphere with a manifold diffeomorphic to $\mathbb{R}^{n-k}$, and, if $n-k > 1$, f is a product of hypersurfaces immersions.

Proof: If, after Hartman factorization, M_1 is compact, then M_1 is the soul of M and $f|M_1$ is described by theorem 5.8.. If M_1 is non compact and $f|M_1$ is non cylindrical, the proof of 5.10. implies that there is a point on the soul where the holonomy algebra is $o(k)$, and therefore by 3.4. the soul is homeomorphic to a sphere. Now theorem 4.1. and the factorization theorem of Moore conclude the proof.

Still in submanifolds theory, another case where the positivity of the curvature operator and of the sectional curvatures are equivalent, is the case of immersion with flat normal connection. Suppose $f: M^n \longrightarrow \mathbb{R}^N$ is an isometric immersion and the curvature $R^\perp$ of the normal connection vanishes. By the Ricci equation:

5.12. $\langle R^\perp(X,Y)\xi,\eta\rangle = \langle [A_\xi,A_\eta](X),Y\rangle$

it follows that all the Weingarten operators commute and therefore there exists an orthonormal bases $\{e_1,\dots,e_n\}$ in the tangent space which diagonalizes all those operators simultaneously. From the Gauss equation

5.3., $\rho(e_i \wedge e_j) = K(e_i, e_j) e_i \wedge e_j$. Therefore if the sectional curvatures are non negative, ρ is non negative. In general if M is a riemannian manifold such that there exists a tangent basis $\{e_1, \ldots, e_n\}$ such that the bi-vectors $e_i \wedge e_j$ are eigenvectors of ρ, we say that M has *pure curvature operator.* Beside the above example another interesting class of riemannian manifolds with pure curvature operator are the conformally flat manifolds. It is clear that the curvature operator of such manifold is non negative if the sectional curvatures are non negative. For this class of manifolds we have the following result (see [DMN]):

5.13.THEOREM: Let M be a complete simply connected riemannian manifold with pure non negative curvature operator. Then M is the riemannian product of a manifold diffeomorphic to $\mathbb{R}^k$ and manifolds homeomorphic to spheres.

Proof (an idea): It is clear that a product of manifolds with pure curvature operator has pure curvature operator. The converse is also true and this allows to work on the irreducible factors of M. For manifolds with pure curvature operator all the Pontrjagin forms vanish and therefore such a manifold can not be homeomorphic to complex projective space. Finally, if an irreducible symmetric space has pure curvature operator it has constant curvature and the conclusion follow from our main result.

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