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Solvability of Complexes of Differential Operators

Resolubilidade de Complexos de Operadores Diferenciais

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Supervisor: Giuliano Angelo Zugliani

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Resumo

Discorremos sobre a teoria de complexos diferenciais de operadores e sistemas involutivos de campos vetoriais, estudando em maiores detalhes as condições suficientes de resolubilidade para os complexos diferenciais encontrados em duas situações distintas. Ambos os exemplos derivam de um modelo comum, o de sistemas do tipo tubo, sendo a resolubilidade determinada por condições no coeficiente do complexo diferencial. As hipóteses sobre os modelos levam a condições de resolubilidade qualitativamente distintas, a depender da homologia dos conjuntos de subnível do coeficiente, em um caso, e das integrais do coeficiente ao longo dos 1-ciclos geradores da homologia, no outro.

Palavras-chave: Sistemas involutivos. Sistemas sobredeterminados. Complexos de operadores diferenciais.

Abstract

We go through the theory of complexes of differential operators and involutive systems of vector fields, delving into sufficient conditions for solvability of differential complexes as they appear in two distinct situations. Both examples are derived from a common model – tube type systems – with solvability being determined by conditions over the coefficient of the differential complex. The hypothesis over the models lead to distinct solvability conditions, depending on the homology of the sublevel sets determined by the coefficient, in one case, and on the integrals of the coefficient along the generating 1-cycles, in the other.

Keywords: Involutive systems. Overdetermined systems. Complex of differential operators.

List of abbreviations and acronyms

a.e.	almost everywhere
iff	if and only if
ODE	ordinary differential equation
PDE	partial differential equation
TVS	topological vector space

List of symbols

$\mathbb{F}$	Scalar field of either $\mathbb{R}$ or $\mathbb{C}$
$\overline{\mathbb{R}}$	Set of extended real numbers $\mathbb{R} \cup \{\pm\infty\}$
$\mathbb{T}^n$	The n-torus $\mathbb{T} \times \cdots \times \mathbb{T}$
S^p	The permutation group of order p
$\mathcal{M}_{m\times n}(R)$	Set of $m \times n$ matrices with entries in a ring R
$\Gamma(E)$	Set of smooth sections of a vector bundle E
$\mathfrak{X}(\Omega)$	The $C^\infty(\Omega)$-module of vector fields over Ω
$[\cdot,\cdot]$	Lie bracket of vector fields
$\\|\cdot\\|_p$	The L^p norm over a measure space
$\doteq$	Equality defining the expression on the left side
$\cong$	Isomorphic objects in the appropriate category
$\oplus$	Direct sum
$\Subset$	Relatively compact subset
$\bigsqcup$	Disjoint union
Aut	Set of automorphisms in the appropriate category
span	Subset generated (in the appropriate sense) by a family of elements
$\bigwedge^\bullet V$	The exterior algebra of a vector space V
$\ker, \mathrm{ran}$	Kernel and range of a map
supp	Support of a scalar-valued function
Re, Im	Real and imaginary parts
$\mathcal{L}(V,W); \mathcal{L}(V)$	Set of linear maps from V to W; linear operators from V to V
$[v]_\gamma, [P]_\gamma$	Coordinate representation of a vector with respect to a basis γ, matrix representation of a linear map (or R-module homomorphism)
$C^\infty(\Omega)$	Set of scalar-valued smooth functions on a manifold Ω

$C_c^\infty(\Omega)$	Set of compactly supported smooth functions on a manifold Ω
$\mathcal{D}'(\Omega)$	Set of Schwartz distributions over Ω
$C^\infty(\Omega; E)$	Set smooth functions on a manifold Ω with values on E
$\mathrm{Diff}(\Omega)$	Set of differential operators $C^\infty(\Omega) \to C^\infty(\Omega)$
$\mathrm{Diff}(\Omega; V, W)$	Set of differential operators $C^\infty(\Omega; V) \to C^\infty(\Omega; W)$, where V and W are vector spaces
$\mathrm{ord}(P)$	Order of a differential operator P

Contents

Introduction

Among the better known systems in the theory of solvability for overdetermined linear PDEs are what we might call tube models. Its local description involves a family of complex vector fields $L_1, \ldots, L_\nu$ on a product smooth manifold $\mathbb{R}^\nu \times N$, with variables $t \in \mathbb{R}^\nu$ and $x \in N$. Supposing N has dimension m, the vector fields are locally given by

$$L_j = \partial_{t_j} + \sum_{k=1}^{m} c_{j,k}(t)\partial_{x_k}, \quad j = 1, \ldots, \nu \tag{1}$$

for some complex smooth coefficients $c_{j,k}$ which do not depend on x. Under a constraint of closedness over the 1-forms $c_k \doteq \sum_j c_{j,k} dt_j$ $(j = 1, \ldots, \nu)$, the family $L_1, \ldots, L_\nu$ can be associated to a complex of differential operators over p-forms on the variable t, denoted as

$$C^\infty(\mathbb{R}^\nu \times N; \mathbb{C}) \xrightarrow{\mathbb{L}^0} C^\infty(\mathbb{R}^\nu \times N; \textstyle\bigwedge^{1,0} \mathbb{C}^\nu) \xrightarrow{\mathbb{L}^1} \cdots \xrightarrow{\mathbb{L}^{\nu-1}} C^\infty(\mathbb{R}^\nu \times N; \textstyle\bigwedge^{\nu,0} \mathbb{C}^\nu) \tag{2}$$

with successive chain maps $\mathbb{L}^p$ given by

$$\mathbb{L}^p = d_t + \sum_k c_k(t) \wedge \partial_{x_k}, \quad p = 1, \ldots, \nu - 1 \tag{3}$$

Solving the overdetermined system $L_j u = f_j$ $(j = 0, 1, \ldots, \nu)$, in particular, is equivalent to solving an inhomogeneous problem with respect to $\mathbb{L}^0$.

Our subject matter is the solvability of complexes originated by differential operators closely related to (1). It is therefore in our interest to determine necessary and sufficient conditions to solve inhomogeneous linear equations, such as

$$\mathbb{L}^p u = f \tag{4}$$

for every $f \in C^\infty(\mathbb{R}^\nu \times N, \bigwedge^{p,0} \mathbb{C}^\nu)$ in a pre-established (*compatible*, as we usually call it) subset where solutions are reasonably expected. Weak solutions in spaces of distributions are to be regarded as well. The approach taken in the present work involves a general presentation of the theory of involutive systems followed by a detailed exposition of some of the techniques used in the proof of sufficient conditions for a couple different models.

Before further details, a bit of historical context is provided. In [Tre76], a number of innovative techniques are used to establish a necessary and sufficient condition for the existence of semi-global solutions at every level of a differential complex in the context of a pseudodifferential model over open Euclidean subsets. Soon after, [CH77] are able to adapt a few of those ideas to make a characterization of solvability for an analogous complex in a global setting. They consider operators on an arbitrary compact manifold in place of $\mathbb{R}^\nu$ and an abstract Hilbert space as the function space in the x variable.

In a different line, [BCM93] generalizes the Diophantine characterization of global hypoellipticity for constant coefficient vector fields in $\mathbb{T}^2$, found in [GW72], to two substantially wider classes of overdetermined systems of vector fields – they fit (3) when we take a compact manifold in place of $\mathbb{R}^\nu$ for the variable t and $N = \mathbb{T}^1$ for the variable x. Under suitable constraints, global hypoellipticity is fully characterized when the coefficient $c(t)$ is real-analytic.

Using the same Diophantine condition as [BCM93], [BP99] characterizes global solvability for real vector fields with variables $(t, x) \in \mathbb{T}^\nu \times \mathbb{T}$ at every level of the complex defined by (3) and for $c(t)$ closed, thus completing the picture sketched by [CH77] for such setup.

The contents here are ordered according to the following scheme.

In Chapter 1 we go through a handful of concepts and elementary results involved in the main material. We recall notions from differential topology, real analysis and differential equations the reader is most likely already familiar with. This is done for the sake of recalling concepts, providing references and introducing bits of terminology and notation we adopt in the remaining. The last section presents pseudo-differential operators in the very basic setting needed to handle the model of Chapter 3. If, at any rate, the concepts or terminology employed seem unclear, the reader is advised to consult the references cited in the heading of the statements.

The actual subject begins in Chapter 2, where we overview the theory of complexes of differential operators and involutive structures, in an attempt to establish a middle ground between the material in [Tre77], from a course given by Treves in UFPE, and the introductory chapter of the comprehensive modern treatise [BCH08]. We start by defining differential complexes of differential operators of a general kind and related notions, giving some of the most prominent cases as examples – de Rham's complex, Dolbeault's complex and the complex of currents. Next, we see how families of vector fields may be turned into complexes, in a situation that generalizes the construction of the de Rham complex. This line of reasoning naturally leads us to the so-called involutive structures, locally defined by vector fields attending to two specific requirements which propitiate the study of solutions.

Chapter 3 and Chapter 4 postulate the specific models and solvability problems investigated. The methods employed to obtain candidate solutions are the typical ones from Fourier analysis, but the different assumptions taken for each model lead to qualitatively distinct solvability conditions. The model given in Chapter 3 is the one featured in [Tre76]. We follow a proof of solvability for the first level of the complex which is not covered in the original paper, but was later made available by the author in the lecture notes [Tre77]. Obtaining the necessary estimates to the end of getting solutions for the family of ODEs parameterized by ξ (obtained through a partial Fourier transform) which ensures a corresponding solution to the original PDE relies on a good choice for the paths of integration $\gamma(t, \xi)$. This calls, on one hand, for a uniform bound on an exponential factor along the paths of integration and, on the other, for the growth on the length of $\gamma(t, \xi)$ to be controlled with respect to ξ. Those constraints lead to a solvability condition of a topological nature, fully determined by a criteria of connectivity on the sublevel sets induced by the coefficients. This is representative of the more general phenomena described in the paper, namely, the fact that solvability at the p-th level of the complex is fully determined by the p-th homology groups of the sublevel sets induced by those same coefficients.

Lastly, in Chapter 4, we follow the argument found in [BP99] to establish a sufficient condition for global solvability at every level of a differential complex on the $(n+1)$-torus. In this situation, the appropriate control on the asymptotic decay of the small coefficients, which may appear in the solutions for the Fourier coefficients, is done by means of an algebraic condition of Diophantine approximations on the integrals of the coefficient of the complex ($c(t)$ in (3)) over the generating 1-cycles of $\mathbb{T}^n$.

It is worth mentioning the subject studied here branches off into various lines of active research. To cite two recent examples, [HZ17] characterizes the global solvability at the first and last level of the complex, on the product of a compact manifold by a torus, when $m = 1$ and $c(t)$ is real-analytic and purely imaginary, while [Ara+24] studies the cohomology spaces on every degree of the differential complex and arbitrary m.

1 Preliminaries

1.1 Real analysis and smooth manifolds

A *topological manifold* Ω of dimension N is to be understood as a Hausdorff, second countable topological space Ω such that each point $p \in \Omega$ admits an open neighborhood $U \subset \Omega$ which is homeomorphic to $\mathbb{R}^N$. The pairs (U, x) where $x : U \to \mathbb{R}^N$ is an homeomorphism from an open set $U \subset \Omega$ are called *(local) charts* of Ω. They can be combined into a unique maximal collection $\mathcal{A}$, said an *atlas* on Ω, where the open sets cover Ω and the transition functions $y \circ x^{-1}$ between any two charts $(U, x), (V, y)$ are homeomorphisms of subsets $\mathbb{R}^N$, whenever the composition between y and x^{-1} makes sense. Conversely, given Ω a Hausdorff, second countable manifold, each atlas on Ω determines a unique topological manifold.

To turn a topological manifold Ω into a *smooth manifold*, we are required to choose a *differentiable structure* – a maximal, smoothly compatible subset of the atlas $\mathcal{A}$.

Definition 1.1. *A differentiable manifold (or smooth manifold) of dimension N is a topological manifold Ω of dimension N endowed with a differentiable structure $\mathcal{F}$, that is, a subcollection $\mathcal{F} = \{(U, x)\}$ of the atlas determined by Ω such that*

(i) The smooth chart domains $\{U : (U, x) \in \mathcal{F}\}$ cover Ω;

(ii) Given $(U, x), (V, y) \in \mathcal{F}$, the transition map $y \circ x^{-1} : x(U \cap V) \to y(U \cap V)$ is C^∞ whenever $U \cap V \neq \varnothing$;

(iii) It is maximal in the following sense: any chart (V, y) in the atlas determined by Ω, such that $\{(V, y)\} \cup \mathcal{F}$ satisfies (ii), necessarily belongs to $\mathcal{F}$.

This allows us to talk about smooth functions and maps as one would do in the Euclidean setting. Since all topological manifolds we shall ever consider are accompanied by a *smooth* structure, the adjective 'smooth' will often remain implicit when we talk about charts, coordinates, maps and so on. Moreover, if Ω is given as an open subset of the Euclidean space $\mathbb{R}^N$, the underlying differentiable structure is always to be regarded as the one induced by local diffeomorphisms from the Euclidean space $\mathbb{R}^N$ to itself. It is plain that the local coordinates of a smooth chart (U, x) may be offset by an arbitrary constant vector in $\mathbb{R}^N$ while still remaining within the differentiable structure. Therefore, in choosing a local chart (U, x) near $p \in U$, we may require $x(p) = 0$, which we indicate by saying (U, x) is centered at p.

1.2 Complex vector fields

Since our study concerns complex vector fields, we generally regard $C^\infty(\Omega)$ as the set of complex-valued smooth functions over Ω, which is to say, $f : \Omega \to \mathbb{C}$ such that, for every (U, x) in the differentiable structure of Ω we have $f \circ x^{-1}$ smooth (with the standard identification $\mathbb{C} \cong \mathbb{R}^2$). We make exceptions in a couple occasions: in Section 2.1, where $\mathbb{F}$ stands for either $\mathbb{R}$ or $\mathbb{C}$ (and the values are in $\mathbb{F}$) and in Chapter 4, where the vector fields are real (and the values are in $\mathbb{R}$). In the proceeding discussion, we shall always assume functions in $C^\infty(\Omega)$ to be complex-valued.

Notice $C^\infty(\Omega)$ constitutes a $\mathbb{C}$-algebra with a conjugate linear map $f \mapsto \overline{f}$ and which contains $C^\infty_{\mathbb{R}}(\Omega) \doteq \{f \in C^\infty(\Omega) : \operatorname{ran} f \subset \mathbb{R}\}$ as a $\mathbb{R}$-subalgebra. The subspace of compactly supported smooth functions $C^\infty_c(\Omega)$ is defined analogously.

A *germ of a C^∞ function at p* is an equivalence class on the set $\mathcal{B}_p$ of pairs (U, f), where U is open containing p and $f \in C^\infty(U)$, under the relation

$$(U, f) \sim (V, g) \iff f|_{U\cap V} = g|_{U\cap V} \tag{1.1}$$

The set $\mathcal{B}_p/\sim$ of germs at p and the germ defined by $f \in C^\infty(\Omega)$ at p are are denoted $C^\infty(p)$ and $\underline{f}_p$, respectively. It isn't hard to see how this defines a $\mathbb{C}$-algebra as well.

The $\mathbb{C}$-algebra of smooth functions is, in particular, a vector space over $\mathbb{C}$. One possible definition of a complex vector field is therefore the following.

Definition 1.2. *A smooth complex vector field over Ω is a $\mathbb{C}$-linear map $L : C^\infty(\Omega) \to \mathbb{C}^\infty(\Omega)$ which adheres to Leibniz's rule*

$$L(fg) = L(f)g + fL(g), \quad f, g \in C^\infty(\Omega) \tag{1.2}$$

The set of smooth complex vector fields over Ω is denoted $\mathfrak{X}(\Omega)$. We extend the obvious vector space multiplication $\mathbb{C} \times \mathfrak{X}(\Omega) \to \mathfrak{X}(\Omega)$ to multiplication by $C^\infty(\Omega)$ by defining for each $g \in C^\infty(\Omega)$

$$(gL)f = g \cdot Lf, \quad f \in C^\infty(\Omega) \tag{1.3}$$

It is easy to see gL as given by (1.3) still satisfies (1.2), so it makes $\mathfrak{X}(\Omega)$ a $C^\infty(\Omega)$-module. Furthermore, it has a conjugate operation

$$\overline{L}(f) = \overline{Lf}, \quad f \in C^\infty(\Omega) \tag{1.4}$$

and the so-called Lie bracket operation $[\cdot, \cdot] : \mathfrak{X}(\Omega) \times \mathfrak{X}(\Omega) \to \mathfrak{X}(\Omega)$

$$[L, M]f = L(Mf) - M(Lf), \quad \forall f \in C^\infty(\Omega) \tag{1.5}$$

making it a Lie algebra over $\mathbb{C}$.[1] To avoid cluttered expressions, given $L \in \mathfrak{X}(\Omega)$ and $f \in C^\infty(\Omega)$, we denote the value of $L(f)$ at a point $p \in \Omega$ by $(Lf)_p$.

[1] see footnote in [BCH08, p. 4] for the definition

Restrictions

The action of vector fields over smooth functions has a local character which is not very evident in Definition 1.2. The first thing to realize is that the support never increases – if f vanishes near a point p, the same is true of Lf.

Proposition 1.3 ([BCH08, Prop. I.1.3]). *If $L \in \mathfrak{X}(\Omega)$ and f is constant, then $Lf = 0$. We also have*

$$\operatorname{supp} Lf \subset \operatorname{supp} f, \quad \forall f \in C^\infty(\Omega), L \in \mathfrak{X}(\Omega) \tag{1.6}$$

□

Proof. The first claim follows at once replacing $f = g = 1$ in Definition 1.2 and using the fact L is $\mathbb{C}$-linear. To verify (1.6), we consider a set $V \subset \Omega$ where f vanishes and show the same must occur for Lf.

Fix a point $p \in \Omega$ and take (U, x) a local chart with $p \in U \subset V$. Also let $\phi \in C_c^\infty(\Omega)$ with $\operatorname{ran} \phi \subset [0, 1]$ be supported in U and such that $\phi(p) = 1$. Then $f = (1 - \phi)f$ and by Leibniz's rule

$$Lf = L(1-\phi)f + (1-\phi)Lf \implies (Lf)_p = (L(1-\phi))_p f(p) + (1-\phi(p))(Lf)_p = 0 \tag{1.7}$$

so we conclude $Lf|_U = 0$. □

Then, by linearity, it's easy to see the value of Lf at a point $p \in \Omega$ is determined solely by the *germ at p defined by f*, since, for any $g \in C^\infty(\Omega)$ which agrees with f in a neighborhood of p, we have $p \in (\Omega \backslash \operatorname{supp}(f - g))$ and thus, by Proposition 1.3, $(L(f-g))_p = (Lf)_p - (Lg)_p = 0$.

That allows us to restrict a vector field $L \in \mathfrak{X}(\Omega)$ to open subsets $U \subset \Omega$ by letting $L_U \in \mathfrak{X}(U)$ be the unique $\mathbb{C}$-linear map making

$$\begin{array}{ccc} C^\infty(\Omega) & \xrightarrow{L} & C^\infty(\Omega) \\ \downarrow & & \downarrow \\ C^\infty(U) & \xrightarrow{L_U} & C^\infty(U) \end{array} \tag{1.8}$$

a commutative diagram, with the vertical arrows denoting restrictions. More explicitly,

$$(L_U f)_p = (L\widetilde{f})_p, \quad p \in \Omega, f \in C^\infty(U) \tag{1.9}$$

where $\widetilde{f} \in C^\infty(\Omega)$ is any such that $\underline{\widetilde{f}}_p = \underline{f}_p$.

Local coordinate representations

If $U \subset \mathbb{R}^N$ is open, the partial derivatives

$$\frac{\partial}{\partial x_j} : f \mapsto \frac{\partial f}{\partial x_j}, \quad j = 1, \dots, N \tag{1.10}$$

define $\mathbb{C}$-linear operators over $C^\infty(U)$ which, furthermore, satisfy Definition 1.2 by virtue of the product rule.

Let $p \in U$, since the the values $(Lf)_p$ depend on f solely through $\underline{f}_p \in C^\infty(U)$, expressing the values of f near p with a first order approximation

$$f(x) - f(p) = \sum_{j=1}^{N} h_j(x)(x_j(x) - x_j(p)) \tag{1.11}$$

where $h = (h_1, \dots, h_N)$ is valued in $\mathbb{R}^N$ and such that $\lim_{x \to p} h(x) = f'(p)$, we may apply Leibniz's rule to the previous and evaluate at p to obtain $(Lf)_p$ as follows –

$$Lf = \sum_j Lh_j \, (x_j - x_j(p)) + h_j \, Lx_j \implies \tag{1.12}$$

$$(Lf)_p = \sum_j h_j(p)(Lx_j)_p = \sum_j (Lx_j)_p \left(\frac{\partial}{\partial x_j} f_j \right)_p \tag{1.13}$$

Therefore, L is given as a $C^\infty(\Omega)$-linear combination of the vector fields in (1.10)

$$L = \sum_j Lx_j \frac{\partial}{\partial x_j} \tag{1.14}$$

and $\left(\frac{\partial}{\partial x_j} \right)_{j=1,\dots,N}$ is a basis for the module $\mathfrak{X}(U)$.

Now suppose Ω is a manifold of dimension N. Each local diffeomorphism provided by a chart in the differentiable structure can be used to pullback the basic vector fields (1.10) in $\mathbb{R}^N$ to a basis of vector fields in the chart domain.

Proposition 1.4. *Each chart (U, x) in Ω induces a basis $\left(\frac{\partial}{\partial x_j} \right)_j$ of $\mathfrak{X}(U)$ with elements defined by the requirement of commutativity in the following diagram*

$$\begin{array}{ccc} C^\infty(U) & \xrightarrow{\frac{\partial}{\partial x_j}} & C^\infty(U) \\ \downarrow x & & \downarrow x \\ C^\infty(x(U)) & \xrightarrow{\frac{\partial}{\partial x_j}} & C^\infty(x(U)) \end{array} \tag{1.15}$$

If $(x_1, \dots, x_N) \subset C^\infty_{\mathbb{R}}(U)$ are the coordinates of a chart (U, x), the expression for a given $L \in \mathfrak{X}(U)$ is again provided by (1.14). In particular the Lie bracket of L with some $M \in \mathfrak{X}(U)$ may be written

$$[L, M] = \sum_j (L(Mx_j) - M(Lx_j)) \frac{\partial}{\partial x_j} \tag{1.16}$$

1.3 Vector bundles

The general definition of a smooth vector bundle is the following.

Definition 1.5 ([Lee13, Chap.10]). *Let Ω be a smooth manifold of dimension N, a smooth vector bundle of rank ν over Ω is a smooth manifold E with a smooth surjection $\rho : E \to M$ such that*

(i) for each $p \in \Omega$, the fiber $E_p \doteq \rho^{-1}(p)$ over p is endowed with the structure of a ν-dimensional vector space over $\mathbb{F}$;

(ii) for every $p \in \Omega$, there is an open neighborhood $U \subset \Omega$, which contains p, and a diffeomorphism

$$\Phi : \rho^{-1}(U) \to U \times \mathbb{F}^\nu \tag{1.17}$$

(said a local trivialization of E over U) such that the canonical projection π of $U \times \mathbb{F}^\nu$ onto $\mathbb{F}^\nu$ makes

$$\begin{tikzcd} \rho^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\rho"'] & & U \times \mathbb{F}^\nu \arrow[dl, "\pi"] \\ & U & \end{tikzcd} \tag{1.18}$$

commute. Moreover, the restriction of Φ to any of the fibers E_q defines a linear isomorphism $E_q \to \{q\} \times \mathbb{F}^\nu$.

It is a common abuse of language to refer to E, rather than (E, p), as the vector bundle, leaving ρ and Ω implicit. The bundle is said real if $\mathbb{F} = \mathbb{R}$ and complex if $\mathbb{F} = \mathbb{C}$. The smooth manifold Ω is often said to be the *base space*, while E is referred to as the *total space*.

Since ρ is surjective, the underlying set of E is a disjoint union of the ν-dimensional vector spaces

$$E = \bigsqcup_{p \in \Omega} E_p \tag{1.19}$$

Notice the differential structure of E can be fully recovered from condition (ii), which implies $\rho^{-1}(U)$ are chart domains on E for every chart domain U of Ω. In particular, if there exists a local trivialization whose domain is the entire base space E, then E is isomorphic to the product space $\Omega \times \mathbb{F}^\nu$ and $\rho : E \to \Omega$ is said a *trivial bundle*.

Smooth maps from $\sigma : \Omega \to E$ such that the composition $\rho \circ \sigma$ is the identity map on Ω are called *sections of E*. They constitute the space of sections of E, which is denoted $\Gamma(E)$.

For our purposes, the most essential instances of vector bundles will be the complex tangent and cotangent bundles of a smooth manifold Ω, as well as the lower rank bundle structures found within those.

Definition 1.6 ([BCH08, Definition I.3.1]). *A complex tangent vector v at a point $p \in \Omega$ is a $\mathbb{C}$-linear map $v \in \mathcal{L}(C^\infty(p), \mathbb{C})$ such that*

$$v(\underline{fg}) = f(p)v(\underline{g}) + g(p)v(\underline{f}), \quad \underline{f}, \underline{g} \in C^\infty(p) \tag{1.20}$$

We denote $\mathbb{C}T_p\Omega$ the vector space of complex tangent vectors at $p \in \Omega$.

As we have seen before, given a vector field L and a point $p \in \Omega$, the value of $(Lf)_p$ only depends on the germ of f at p. Since Leibniz's rule is built-in the definition of a vector field, we can define complex tangent vector with the following association

$$L_p : \underline{f}_p \in C^\infty(p) \mapsto (Lf)_p \in \mathbb{C}, \quad \underline{f}_p \in C^\infty(p) \tag{1.21}$$

The tangent vectors obtained in this manner from the basis $(\frac{\partial}{\partial x_j}) \in \mathfrak{X}(U)$ induced by a chart (U, x) and a point $p \in \Omega$ are denoted $\left.\frac{\partial}{\partial x_j}\right|_p$. It is easy to see they constitute a basis of $\mathbb{C}T_p\Omega$.

Definition 1.7. *The complexified tangent bundle $\mathbb{C}T\Omega$ of Ω is the disjoint union*

$$\mathbb{C}T\Omega = \bigsqcup_{p \in \Omega} \mathbb{C}T_p\Omega \tag{1.22}$$

If $\pi : \mathbb{C}T\Omega \to \Omega$ is the projection $v \in \mathbb{C}T_p\Omega \mapsto p$, a trivialization $\pi^{-1}(U) \to U \times \mathbb{C}^N$ of $\mathbb{C}T\Omega$ near p defined by choosing coordinates (U, x) for Ω with $p \in U$ and setting

$$v = \sum \alpha_j \left.\frac{\partial}{\partial x_j}\right|_p \in \pi^{-1}(U) \mapsto (p, (\alpha_1, \ldots, \alpha_N)) \tag{1.23}$$

This makes $(\mathbb{C}T\Omega, \pi)$ a smooth vector bundle of rank N.

Definition 1.8. *A complex vector subbundle $\mathcal{V}$ of $\mathbb{C}T\Omega$ of rank n and corank $N - n$ is a smooth vector bundle substructure of $\mathbb{C}T\Omega$, meaning it is given by a disjoint union*

$$\mathcal{V} = \bigsqcup_{p \in \Omega} \mathcal{V}_p \subset \mathbb{C}T\Omega \tag{1.24}$$

such that

1. *for each $p \in \Omega$, $\mathcal{V}_p$ is a n-dimensional (complex) vector subspace of $\mathbb{C}T\Omega$;*

2. *given $p \in \Omega$, there is $U \ni p$ open and vector fields $L_1, \ldots, L_n \in \mathfrak{X}(U)$ such that $\operatorname{span}_{\mathbb{C}}\{(L_1)_q, \ldots, (L_n)_q\} = \mathcal{V}_q$ for each $q \in U$.*

A family of vector fields $(L_1, \ldots, L_n)$ such that $(L_1)_p, \ldots, (L_n)_p$ spans $\mathcal{V}_p$ for each $p \in \Omega$ are called *generators* of $\mathcal{V}$. Condition *2.* in the definition can thus be rephrased as follows: for every $p \in \Omega$ there is $U \subset \Omega$ open and a family of n vector fields which generate $\mathcal{V} \cap \mathbb{C}TU$.

Notice a *section* of a subbundle $\mathcal{V} \subset \mathbb{C}T\Omega$ is nothing but a vector field $L \in \mathfrak{X}(\Omega)$ such that $L_p \in \mathcal{V}_p$ for every $p \in \Omega$.

Corresponding to vector fields and the tangent bundle, we have the dual notions of a 1-form and the cotangent bundle. They fulfill pretty much the same properties as the former.

Loosely speaking, we can say the cotangent bundle $\mathbb{C}T^*\Omega$ is the smooth vector bundle obtained from attaching continuous duals $\mathbb{C}T_p^*\Omega \doteq (\mathbb{C}T_p\Omega)^*$ as in Definition 1.8. A 1-form ω is a $C^\infty(\Omega)$ linear map

$$\omega : \mathfrak{X}(\Omega) \to C^\infty(\Omega) \tag{1.25}$$

They restrict to open sets as well. Given a chart (U, x), the canonical projections of $\mathbb{R}^N$ induce a basis of 1-forms in U, denoted $(dx_j)_{1,\ldots,N}$, characterized by the relations

$$dx_j \left(\frac{\partial}{\partial x_k} \right) = \delta_{jk}, \quad \forall j, k \tag{1.26}$$

More generally, a k-form is characterized as a smooth section of $\bigwedge^k T^*\Omega$, so we sometimes denote $\Gamma(\bigwedge^k T^*\Omega)$ the set of k-forms over Ω. See [BCH08, Section I.4] or [Lee13] for further details.

Proposition 1.9 (Exterior derivative axioms [Lee13, Prop. 14.24])**.**

Let M be a smooth manifold, then there are unique linear maps $d_p : \Gamma(\bigwedge^p T^*M) \to \Gamma(\bigwedge^{p+1} T^*M)$ with the following properties:

(i) If $\omega \in \Gamma(\bigwedge^k T^*M)$ and $\eta \in \Gamma(\bigwedge^l T^*M)$, then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta;$$

(ii) $d_{p+1} \circ d_p = 0$;

(iii) For all $f \in C^\infty(M)$, the dual relation

$$\langle df, L \rangle = L(f) \tag{1.27}$$

holds.

The validity of the following Lemma and its dual statement establish a one-to-one correspondence between subbundles of $\mathbb{C}T\Omega$ of rank n and subbundles of $\mathbb{C}T^*\Omega$ of rank $N-n$.[2] This relation will play a significant role in our classification of involutive structures at a later point in Chapter 2.

Lemma 1.10 ([BCH08, p. 9 Prop. 4.4])**.** *Let $\mathcal{V} \subset \mathbb{C}T\Omega$ be a complex vector subbundle, then the subset $\mathcal{V}^\perp \subset \mathbb{C}T^*\Omega$ with fiber elements*

$$\mathcal{V}_p^\perp = \{\lambda \in \mathbb{C}T_p^*(\Omega) : \lambda|_{\mathcal{V}_p} = 0\}, \quad p \in \Omega \tag{1.28}$$

is a vector subbundle of $\mathbb{C}T^*\Omega$.

Proof. Fix $p \in \Omega$ and a local chart (U, x) with $p \in U$. Take $L_1, \ldots, L_\nu \in \mathfrak{X}(U)$ generators for $\mathcal{V} \cap \mathbb{C}TU$ and write each of them as

$$L_j = \sum_{k=1}^{\nu} a_{jk} \frac{\partial}{\partial x_k}, \quad a_{jk} \in C^\infty(\Omega) \tag{1.29}$$

Since $(a_{jk}) \in \mathcal{M}_{N\times\nu}(C^\infty(\Omega))$ is smooth and has full rank in U, one of the $\binom{N}{\nu}$ square submatrices is non-singular in a shrunk neighborhood of p. Therefore, by taking U smaller and reindexing the matrix rows, we can assume the square matrix given by the first ν rows is invertible in U. Suppose its inverse is $(b_{jk}) \in \mathcal{M}_{\nu\times\nu}(C^\infty(U))$ and consider a new family of vector fields in U defined as

$$L_j' = \sum_k b_{jk} L_j, \quad j = 1, \ldots, \nu \tag{1.30}$$

Notice they also generate the subbundle $\mathcal{V} \cap \mathbb{C}TU$. Furthermore, the vector fields L_j' are written

$$L_j' = \frac{\partial}{\partial x_j} + \sum_{k=1}^{N-\nu} c_{jk} \frac{\partial}{\partial x_{\nu+k}}, \quad c_{jk} \in C^\infty(U) \tag{1.31}$$

so we let

$$\omega_j = dx_{\nu+j} - \sum_{k=1}^{\nu} c_{jl} dx_k, \quad l = 1, \ldots N-\nu \tag{1.32}$$

Then, $\omega_1, \ldots, \omega_{N-\nu}$ are linearly independent in the fibers of $\mathbb{C}T^*U$ while also satisfying

$$\omega_j(L_k') = dx_{n+l}(L_j') - c_{jl} = 0 \tag{1.33}$$

Thus they make generators for $\mathcal{V}^\perp \cap \mathbb{C}T^*U$. □

[2] To be sure, the dual statement is obtained by switching the roles of the tangent and cotangent bundles in the proposition.

1.4 Distribution theory and Fourier analysis

Our notation in the subject of real analysis follows the standard of classical books such as [Fol99] and [Gra14].

Given $x = (x_1, \ldots, x_n) \in \mathbb{R}^n, y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we set

$$x \cdot y = \sum_j x_j y_j, \quad |x| = \left(\sum_j x_j^2\right)^{1/2} \tag{1.34}$$

Let f be a function of x on $\mathbb{R}^n$, the partial derivative with respect to the j-th variable is written

$$\frac{\partial f}{\partial x_j} \quad \text{or} \quad \partial_{x_j} f \tag{1.35}$$

A *multi-index* (on $\mathbb{R}^n$) will be an ordered n-tuple of non-negative integers. Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we set

$$|\alpha| = \sum_j \alpha_j, \quad \alpha! = \sum_j \alpha_j!, \quad \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \tag{1.36}$$

and, given $x = (x_1, \ldots, x_n)$, $x^\alpha = \prod_j x_j^{\alpha_j}$. If x is the only variable of a function, the subscript x may be omitted in ∂_x^α. The order relations $<$ and $\leqslant$ on $\mathbb{N}$ induce partial orders denoted by the same symbols on $\mathbb{N}^n$. For instance, if α and β are multi-indices,

$$\alpha \leqslant \beta \iff \forall j,\ \alpha_j \leqslant \beta_j \tag{1.37}$$

Then, the general Leibniz's product rule for f, g functions on $\mathbb{R}^n$ reads

$$\partial^\alpha (fg) = \sum_{0 \leqslant \beta \leqslant \alpha} \binom{\alpha}{\beta} (\partial^\alpha f)(\partial^{\alpha - \beta} g), \quad \text{where} \quad \binom{\alpha}{\beta} = \prod_j \binom{\alpha_j}{\beta_j} \tag{1.38}$$

Topological vector spaces

A topological vector space (TVS) consists of a set X equipped with a topology τ and a vector space structure $(X, +, \cdot)$ over $\mathbb{F}$ such that

1. every element of X is closed in τ;
2. the vector spaces operations $+ : X \times X \to X$ and $\cdot : \mathbb{F} \times X \to X$ are continuous with respect to τ

It follows automatically from those requirements that X is Hausdorff and the topology τ is translation-invariant, the latter meaning given an open set $U \in \tau$, the translates

$v + U = \{v + u : u \in U\}$ are open as well, for whatever $v \in X$ we choose (see [Rud91]). The topology τ is then fully described by a *local base*, that is, a topological base at 0.

If there exists a choice of local base $\mathcal{B}$ such that every $U \in \mathcal{B}$ is convex, the TVS is said *locally convex*. This is relevant because many of the standard functional-analytic results for normed spaces have corresponding versions for locally convex TVS.

A seminorm on a vector space X is a function $p : X \to \mathbb{F}$ which is both additive and positive homogeneous. One convenient way to introduce a locally convex topology in a vector space X is by means of a family of seminorms $(\rho_\alpha)_{\alpha\in A}$ with the separating property –

$$\forall x \in X\backslash\{0\}, \quad \exists \alpha \in A \text{ such that } \rho_\alpha(x) \neq 0 \tag{1.39}$$

Theorem 1.11 ([Rud91, Th.1.37])**.** *Suppose $(\rho_\alpha)_{\alpha\in A}$ a separating family of seminorms on a vector space X. Associate to each $\alpha \in A$ and positive integer n the set*

$$V(\alpha, n) = \{x \in X : \rho_\alpha(x) < 1/n\} \tag{1.40}$$

then finite intersections of sets in the collection $(V(\alpha, n))_{\alpha\in A, n\in\mathbb{N}}$ constitute a local base for a topology τ on X, which turns X into a locally convex space such that all seminorms ρ_α continuous.

When the family of seminorms which induces the topology of a TVS is countable, say $(\rho_j)_{j\in\mathbb{N}}$, a compatible translation-invariant metric can be defined as

$$d(x, y) = \sum_j 2^{-j} \frac{\rho_j(x - y)}{1 + \rho_j(x - y)}, \quad x, y \in X \tag{1.41}$$

Spaces of functions

Definition 1.12 (Locally integrable functions)**.** *A Lebesgue measurable function $f : \mathbb{R}^n \to \Omega$ is locally integrable if for all $K \subset \mathbb{R}^n$ compact*

$$\int_K |f| d\lambda < \infty \tag{1.42}$$

The set of all locally integrable functions is denoted $L^1_{loc}(\mathbb{R}^n)$. It can be turned into a metrizable TVS by taking $(K_j) \subset \mathbb{R}^n$ an exhaustion of $\mathbb{R}^n$ by compact sets (i.e. $\bigcup K_j = \mathbb{R}^n$ with $K_j \Subset K_{j+1}$) and the topology described in Theorem 1.11 for the seminorms

$$\rho_j(f) = \int_{K_j} |f| d\lambda \tag{1.43}$$

Definition 1.13 ([Gra14, Def.2.2.1])**.** *Let $C^\infty(\mathbb{R}^n)$ be the space of smooth, complex-valued functions on $\mathbb{R}^n$. We introduce the following family of seminorms: given α, β multi-indices* *in* $\mathbb{R}^n$, *set*

$$\rho_{\alpha,\beta}(f) \doteq \sup |x^\alpha \partial^\beta f| \tag{1.44}$$

Then $f \in C^\infty(\mathbb{R}^n)$ *is in the space of Schwartz functions* $\mathcal{S}(\mathbb{R}^n)$ *if for all* α, β

$$\rho_{\alpha,\beta}(f) < \infty \tag{1.45}$$

The family $(\rho_{\alpha,\beta})$ defined in (1.44) is countable and each of its elements satisfy the axioms of a seminorm on the space of Schwartz functions due to (1.45). Furthermore, it is separating since $\rho_{0,0}(f) = 0$ implies $f = 0$; therefore, we can endow $\mathcal{S}(\mathbb{R}^n)$ with the topology of Theorem 1.11 to make it a locally convex metrizable TVS. The corresponding notion of convergence for a sequence $(f_k) \subset \mathcal{S}(\mathbb{R}^n)$ is the following –

$$f_k \xrightarrow{\mathcal{S}} 0 \iff \forall \alpha, \beta, \quad \rho_{\alpha,\beta}(f_k) \to 0 \tag{1.46}$$

It is not hard to show this makes $\mathcal{S}(\mathbb{R}^n)$ complete, since derivation is well-behaved with respect to uniform limits and $\rho_{0,0}(f_k) \to 0$ implies $f_k \to 0$ uniformly.

The setting of Schwartz functions is ideal for Fourier analysis because the Fourier transform defines an automorphism over $\mathcal{S}(\mathbb{R}^n)$. We opt the normalizations in the following definition.

Definition 1.14. *Given* $f \in \mathcal{S}(\mathbb{R}^n)$ *we define*

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx \tag{1.47}$$

the Fourier transform of f *and*

$$\check{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{2\pi i x \cdot \xi} dx \tag{1.48}$$

the inverse Fourier transform of f.

The standard formulas relating the Fourier transform and derivatives of a function are more concisely written if we introduce the normalized differential operators

$$D^\alpha \doteq \frac{1}{(2\pi i)^{|\alpha|}} \partial^\alpha \tag{1.49}$$

Proposition 1.15. *Given* $f \in \mathcal{S}(\mathbb{R}^n)$, $y \in \mathbb{R}^n$ *and* α *a multi-index, the Fourier transform given by Definition 1.14 is a homeomorphism from* $\mathcal{S}(\mathbb{R}^n)$ *to itself. Furthermore,*

1. $\|\widehat{f}\|_{L^\infty} \leqslant \|\widehat{f}\|_{L^1}$
2. $(D^\alpha f)^\wedge(\xi) = \xi^\alpha \widehat{f}(\xi)$
3. $\check{\widehat{f}} = f = \widehat{\check{f}}$

Definition 1.16 ([Gra14, Section 2.3.1]). *Convergence in the spaces C^∞ and C_c^∞ of smooth functions are regarded in the following sense:*

1. $f_k \to f \in C^\infty \iff f_k, f \in C^\infty$ *and* $\lim_{k\to\infty} \sup_{|x|\leqslant N} |\partial^\alpha (f - f_k)(x)| = 0$, $\forall \alpha$ *multi-indices and* $N > 0$
2. $f_k \to f \in C_c^\infty \iff f_k, f \in C^\infty$ *are commonly supported in a compact set* K *and* $\lim_{k\to\infty} \|\partial^\alpha (f - f_k)\| = 0$, $\forall \alpha$ *multi-indices*

1.4.1 Distributions

77 Distributions are the continuous linear functionals in the spaces of functions previously introduced. We denote

$$\mathcal{D}'(\mathbb{R}^n) \doteq (C_c^\infty(\mathbb{R}^n))^* \tag{1.50}$$

$$\mathcal{S}'(\mathbb{R}^n) \doteq (\mathcal{S}(\mathbb{R}^n))^* \tag{1.51}$$

The operator topology, in both cases, is given by the weak* topology, where sequential convergence amounts to

$$T_k \to T \text{ in } \mathscr{B}^* \iff T_k, T \in \mathscr{B} \text{ and } T_k(f) \to T(f), \quad \forall f \in \mathscr{B} \tag{1.52}$$

with $\mathscr{B}$ standing for either C_c^∞ or $\mathcal{S}$. It is usual to refer to an element of $\mathcal{S}'$ as a *tempered distribution.*

Those spaces allow us to define, among others, operations of multiplication by smooth function as differentiation. Consult [Fol99, Chapter 9] for the precise definitions.

For the subject of Hilbert Sobolev spaces, we refer to [Fol99, Section 9.3] and only discuss the basic facts we will use.

Let $s \in \mathbb{R}$, the function $\xi \mapsto (1 + |\xi|^2)^{s/2}$ is smooth and slowly increasing, therefore

$$\Lambda_s f = ((1 + |\xi|^2)^{s/2} \widehat{f})^\vee \tag{1.53}$$

is a well-defined, continuous linear operator on $\mathcal{S}'$. In fact, one can verify $\Lambda_s \circ \Lambda_{-s} = \Lambda_{-s} \circ \Lambda_s = id_{\mathcal{S}'}$ so that Λ_s is in fact an isomorphism. We define the Hilbert space H^s, for a given $s \in \mathbb{R}$, as the subset of tempered distributions such that $\Lambda_s f \in L^2$.

Definition 1.17. *The Sobolev space H^s, for $s \in \mathbb{R}$, is the set*

$$H^s = \{f \in \mathcal{S}' : \Lambda_s f \in L^2\} \tag{1.54}$$

endowed with the norm

$$\|f\|_{H^s} = \|\Lambda_s f\| = \left(\int |\widehat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi \right)^{1/2} \tag{1.55}$$

Notice $H^s \subset H^t$ whenever $s > t$.

Definition 1.18. *The Sobolev spaces H^∞ and $H^{-\infty}$ are defined as*

$$H^\infty = \bigcap_{s>0} H^s, \quad H^{-\infty} = \bigcup_{s<0} H^s \tag{1.56}$$

The former is endowed with the inductive limit topology obtained from the inclusions $H^\infty \hookrightarrow H^s$, while the latter receives the weak topology from its dual pairing with H^∞.*

1.4.2 Fourier analysis

Consider the Fourier transform $\mathcal{F}$ defined again by formula (1.47).

Theorem 1.19 ([Fol99, Th.8.22])**.** *Suppose $f \in L^1(\mathbb{R}^n)$.*

1. *if $x^\alpha f \in L^1$ for $|\alpha| \leqslant k$, then $\widehat{f} \in C^k$ and $D^\alpha \widehat{f} = (-1)^{|\alpha|}\widehat{x^\alpha f}$*
2. *$\mathcal{F}(L^1) \subset \{f \in C^0 : \lim_{x\to\infty} f(x) = 0\}$*

1.4.3 Measure theory

Definition 1.20 (Upper semi-continuity)**.** *A function $f : X \to \overline{\mathbb{R}}$ is upper semicontinuous if for every $p \in X$ and $r \in \mathbb{R}$ there is an open neighborhood U of p such that $f(U) \subset [-\infty, r)$*

Theorem 1.21 ([Fol99, Th.7.10] Lusin's theorem)**.** *Suppose that μ is a Radon measure on X and $f : X \to \mathbb{C}$ measurable and vanishing outside a set of finite measure. Then given $\epsilon > 0$, there exists $\phi \in C_c(X)$ such that $\phi = f$ except for a set of measure less than ϵ. If f is bounded, ϕ can be taken such that $\|\phi\|_\infty \leqslant \|f\|_\infty$.*

1.5 Pseudo-differential operators

Given a function $a(x,\xi)$, we may consider its corresponding action T_a as an integral operator

$$T_a f(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} a(x,\xi)\widehat{f}(\xi)\, d\xi \tag{1.57}$$

In particular, the action of a differential operator can be recovered in (1.57) by $a(x,\xi)$ its symbol. Thus, the class of pseudo-differential operators, which includes all integral operators represented as above, generalizes the usual differential operators.

Integrability on the right-side of (1.57) and continuity of the integral operator will naturally depend on the domain of definition, as well as the decay of $a(x,\xi)$ and its derivatives. The different conditions we may impose on $a(x, \xi) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ determine spaces of functions generally referred as symbol classes.

The rather simple case of symbol classes $S^m(\mathbb{R}^n \times \mathbb{R}^n)$ and its extension result for the Hilbert Sobolev spaces $H^{\pm\infty}(\mathbb{R}^n)$ are all we shall need for the purposes of Chapter 3.

Definition 1.22 ([RT10, Def 2.1.1] Symbol classes $S^m(\mathbb{R}^n \times \mathbb{R}^n)$)**.** *A smooth function $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ belongs to the class of symbols $S^m(\mathbb{R}^n \times \mathbb{R}^n)$ if for all multi-indices $\alpha, \beta \geqslant 0$ there are constants $A_\alpha > 0$ such that*

$$|\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \leqslant A_\alpha (1 + |\xi|)^{m-|\alpha|} \tag{1.58}$$

Theorem 1.23. *Let $a \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$, the pseudo-differential defined by the symbol a is*

$$a(X, D)f(x) = \int e^{2\pi x \cdot \xi} a(x, \xi) \widehat{f}(\xi) d\xi, \quad f \in \mathcal{S}(\mathbb{R}^n) \tag{1.59}$$

Then $a(X, D)f \in \mathcal{S}(\mathbb{R}^n)$.

Theorem 1.24 ([RT10, Th. 2.6.11])**.** *Let $T \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ be a pseudo-differential operator of order $m \in \mathbb{R}$ and $k \in \mathbb{R}$. Then T extends to a bounded linear operator from $H^k(\mathbb{R}^n)$ to $H^{k-m}(\mathbb{R}^n)$.*

Corollary 1.25. *If $T \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ is a pseudo-differential operator for some $m \in \mathbb{R}$, then T extends from a bounded linear operator in $\mathcal{S}(\mathbb{R}^n)$ to bounded linear operators in $H^\infty(\mathbb{R}^n)$ and $H^{-\infty}(\mathbb{R}^n)$.*

2 Differential complexes and involutive systems

We assume throughout the current chapter that Ω is a manifold of dimension ν. The symbol $\mathbb{F}$ will stand for the scalar fields of either $\mathbb{R}$ or $\mathbb{C}$ – we employ it whenever both options are suitable.

2.1 Complexes of differential operators

Let Ω be an open submanifold of $\mathbb{R}^\nu$.

Linear differential operators of scalar-valued functions

We denote $\mathrm{Diff}(\Omega)$ the collection of all linear differential operators $C^\infty(\Omega) \to C^\infty(\Omega)$. In multi-index notation, that means for each $P \in \mathrm{Diff}(\Omega)$ there are smooth coefficients $(a_\alpha)_{\alpha\in A} \subset C^\infty(\Omega)$, where A is a finite multi-index family, such that for any $f \in C^\infty(\Omega)$ and $x \in \Omega$,

$$Pf(x) = \sum_{\alpha\in A} a_\alpha(x) D^\alpha f(x) \tag{2.1}$$

This is often written more concisely as

$$P(x, D) = \sum_{\alpha\in A} a_\alpha(x) D^\alpha \tag{2.2}$$

The **support of a family** (a_α), denoted $\mathrm{supp}\,(a_\alpha)$ consists of those indexes $\alpha \in A$ for which a_α is non-zero.[1] Thus, each differential operator $P \in \mathrm{Diff}(\Omega)$ has an **order** given by $\mathrm{ord}(P) \doteq \max_{\alpha\in\mathrm{supp}(a_\alpha)} |\alpha|$. Perhaps including a few zero coefficients, the same operator P of (2.2) is written as

$$P(x, D) = \sum_{|\alpha|\leqslant \mathrm{ord}(P)} a_\alpha(x) D^\alpha \tag{2.3}$$

The subset of operators in $\mathrm{Diff}(\Omega)$ up to order k is denoted $\mathrm{Diff}^k(\Omega)$.

[1] if context allows it, we will often omit the indexing family

Linear differential operators of vector-valued functions

For operators mapping vector-valued functions, we make the following adjustment.

Definition 2.1. *Let V, W be finite dimensional $\mathbb{F}$-vector spaces with $\dim V = m$ and $\dim W = n$. Then, a* ***linear differential operator*** *$P \in \mathrm{Diff}(\Omega; V, W)$ is a linear mapping $P : C^\infty(\Omega; V) \to C^\infty(\Omega; W)$ which is represented by a matrix of coefficients in $\mathrm{Diff}(\Omega)$, that is, given γ_V, γ_W respective bases of V and W, there exists $[P] = (P_{jk}) \in \mathcal{M}_{n\times m}(\mathrm{Diff}(\Omega))$ such that for all $f \in C^\infty(\Omega; V)$*

$$[Pf]_{\gamma_W} = [P][f]_{\gamma_V}$$

Although the definition works with a specified basis of the vector spaces, all examples we shall present will come with a ‘natural’ choice at hand, ultimately coming from the standard basis of $\mathbb{R}^\nu$. Our initial examples should make this point clear. Identifying the vector values within a single vector space is a privilege of working with open submanifolds – if our description of Ω came from the point of view of intrinsic manifolds, the values would take place in vector bundles, as one often sees in the literature.

Definition 2.2. *Let $(E_j)_{j\in\mathbb{N}}$ be vector spaces of finite dimension and $(P_j)_{j\in\mathbb{N}} \subset \mathrm{Diff}(\Omega; E_j, E_{j+1})$. A* ***complex of differential operators*** *$P \doteq (C^\infty(\Omega; E_j), P_j)_j$ consists of a sequence*

$$0 \xrightarrow{P_0} C^\infty(\Omega; E_1) \xrightarrow{P_1} C^\infty(\Omega; E_2) \xrightarrow{P_2} \cdots \xrightarrow{P_{j-1}} C^\infty(\Omega; E_j) \xrightarrow{P_j} \cdots$$

such that $\forall j \in \mathbb{N},\ P_{j+1} \circ P_j = 0$.[2]

The j-th level of the complex, $C^\infty(\Omega; E_j)$, is attached to a cohomology space

$$H^j_P(\Omega) \doteq \frac{\ker P_{j+1}}{\mathrm{ran}\, P_j}$$

The sequence is ***exact at*** *$C^\infty(\Omega; E_j)$ if the corresponding cohomology $H^j_P(\Omega)$ is trivial, i.e., $\ker P_{j+1} = \mathrm{ran}\, P_j$. If P is exact at every level, we simply refer to P as an* ***exact sequence****.*

Examples

Example 2.3 (The de Rham complex)**.**

Let $E_p = \bigwedge^p \mathbb{F}^\nu$ be the $\binom{\nu}{p}$-dimensional vector space of alternating p-linear maps on $\mathbb{F}^\nu$, often referred simply as p-tensors. We first recall some basic facts about the

[2] notice each equality $P_{j+1} \circ P_j = 0$ is equivalent to an inclusion $\mathrm{ran}\, P_j \subset \ker P_{j+1}$

alternating vector algebra as they apply to the situation at hand. Let $\mathcal{I}_p^\nu = \{(i_1, \ldots, i_p) \in \{1, \ldots, \nu\}^p : i_1 < \ldots < i_p\}$, then given a basis $e_1, \ldots, e_\nu$ for E_1, dual to $e^1, \ldots, e^\nu \in E_0$, each $J = (j_1, \ldots, j_p) \in \{1, \ldots, \nu\}^p$ determines a p-tensor $e_J \doteq e_{j_1} \wedge \ldots \wedge e_{j_p}$ as follows: if any index of J is repeated, then e_J is zero; otherwise, e_J is characterized among p-linear maps in $\mathbb{F}^\nu$ by the fact $\forall I = (i_1, \ldots, i_p) \in \{1, \ldots, \nu\}^p$

$$e_J(e^{i_1}, \ldots, e^{i_p}) = \begin{cases} 0, & \text{if } I \neq J \text{ as sets} \\ \operatorname{sgn} \sigma, & \text{if } I = \sigma J \text{ for some } \sigma \in S^p \end{cases} \tag{2.4}$$

A permutation $\sigma \in S^p$ applied to the p-tuple J thus corresponds to multiplication by $\operatorname{sgn} \sigma \in \{\pm 1\}$ on $e_J \in E_p$, that is

$$e_{\sigma J} = (\operatorname{sgn} \sigma) e_J, \quad \forall \sigma \in S^p \tag{2.5}$$

In particular, all $e_{\sigma J}$ with $\sigma \in S^p$ are co-linear in E_p and $(e_J)_{J \in \mathcal{I}_p^\nu}$ is a basis for the same vector space. For practical reasons, whenever J is an index ranging over $\mathcal{I}_p^\nu$, most commonly in a summation, we shall abbreviate $[J] = p$, leaving ν understood from context.

Proceeding with our example, the function spaces $C^\infty(\Omega; E_p)$ with E_p can be understood as spaces of p-forms over Ω. The natural choice of basis for E_1 are the 1-tensors $(dx_j)_{j=1,\ldots,\nu}$, dual to $(\partial_{x_j})_{j=1,\ldots,\nu} \subset E_0$. As such, the algebraic structure of $C^\infty(\Omega; E_p)$ as a $C^\infty(\Omega)$-module over E_p means we are able to represent each $\omega \in C^\infty(\Omega; E_p)$ uniquely as a sum

$$\omega = \sum_{[J]=p} f_J dx_J, \quad f_J \in C^\infty(\Omega) \tag{2.6}$$

taking values $\omega(t) = \sum_{[J]=p} f_J(t) dx_J \in E_p$ for each $t \in \Omega$.

The differential operators $d_p \in \operatorname{Diff}(\Omega; E_p, E_{p+1})$ of the complex are the **exterior derivatives**. Given with respect to the standard coordinates $(x_j)_{j=1\ldots,\nu}$, those are defined as

$$d_p \left(\sum_{[J]=p} f_J e_J \right) = \sum_{[J]=p} \sum_{k=1}^{\nu} \partial_{x_k} f_J dx_k \wedge dx_I \tag{2.7}$$

This is equivalent to the the coordinate-free axiomatic approach of Proposition 1.9, which emphasizes the formal properties of the operator (notice the chain complex condition $d_{j+1} \circ d_j = 0$ is precisely axiom (ii)). In terms of the coordinate-wise definition, the complex condition can be seen as a consequence of the commutativity property of partial derivatives of smooth functions, as will become clear in the computation of Lemma 2.14.

The sequence $d \doteq (C^\infty(\Omega; E_j), d_j)$ is known as the **de Rham** complex. Taking the restriction of the exterior derivatives to the spaces of compactly supported smooth functions, another differential complex $(C_c^\infty(\Omega, E_j), d_j)$ is produced.

Example 2.4 (Complex of p-currents)**.**

We construct a differential complex similar to the previous one, where distributions take the role of smooth functions.[3] Here the p-th level $\bigwedge^p \mathcal{D}'(\Omega)$ of the complex will be a **p-current**, which essentially works as a p-form with distributional coefficients. Algebraically speaking, $\bigwedge^p \mathcal{D}'(\Omega)$ is still a $C^\infty(\Omega)$-module. The p-alternating tensors $(dx_{[J]})_{[J]=p}$ from the previous example allow us to write a p-current $u \in \bigwedge^p \mathcal{D}'(\Omega)$ uniquely as

$$u = \sum_{[J]=p} u_J dx_J, \quad u_J \in \mathcal{D}'(\Omega) \tag{2.8}$$

which adds and multiplies by scalar values as one would expect. Furthermore, there is a bilinear wedge product operation that turns $\bigwedge^\bullet \mathcal{D}'(\Omega)$ into an exterior algebra. The construction mirrors the wedge product of differential forms in almost every aspect. An important distinction, however, is that $\mathcal{D}'(\Omega)$ lacks the multiplicative structure of $C^\infty(\Omega)$, so we have to settle for multiplication by $C^\infty(\Omega)$ in $\bigwedge^p \mathcal{D}'(\Omega)$.

From an alternative, somewhat less artificial angle, we can understand p-currents through the dual relation they have with compactly supported $(\nu - p)$-forms. The bilinear pairing $\langle \cdot, \cdot \rangle_{\mathcal{D}'}$ in Ω which evaluates distributions on test functions induces a bilinear map.

$$\langle \cdot, \cdot \rangle : \bigwedge^p \mathcal{D}'(\Omega) \times \bigwedge^{\nu-p} C_c^\infty(\Omega) \to \mathbb{F}$$

which is uniquely determined (among bilinear maps) by the property

$$\forall J \in \mathcal{I}_p^\nu, I \in \mathcal{I}_{\nu-p}^\nu, \quad u_J \in \mathcal{D}'(\Omega), \varphi_I \in C_c^\infty(\Omega) \tag{2.9}$$

$$\langle u_J dx_J, \varphi_I dx_I \rangle = \operatorname{sgn}(I, J) \langle u_J, \varphi_I \rangle_{\mathcal{D}'} \tag{2.10}$$

where $sgn(I, J)$ is 0 if $I \cap J \neq \varnothing$ or equal to the ± 1 sign of the permutation $(I, J) \in S^\nu$ otherwise. Then, a p-current $u \in \bigwedge^p \mathcal{D}'(\Omega)$ corresponds to the continuous linear functional $\langle u, \cdot \rangle : \bigwedge^{\nu-p} C_c^\infty(\Omega) \to \mathbb{C}$. Finally, to match the topological space of distributions when $p = 0$, we endow $\bigwedge^p \mathcal{D}'(\Omega)$ with the weak-$*$ operator topology.

We remind that a function $f \in L^1_{\text{loc}}(\Omega)$ can be identified in the space of distributions by regarding it as a continuous linear functional which acts on test functions $\psi \in C_c^\infty(\Omega)$ by integration, that is, $\langle f, \psi \rangle = \int f\psi$. Now since $C_c^\infty(\Omega)$ and $C^\infty(\Omega)$ are included in L^1_{loc}, p-currents with distributional coefficients which correspond to smooth functions in Ω can be identified with p-forms over Ω, so that we have inclusions $\bigwedge^p C_c^\infty(\Omega) \subset \bigwedge^p C^\infty(\Omega) \subset \bigwedge^p \mathcal{D}'(\Omega)$.

[3] The present example is not, strictly speaking, a differential complex as we defined in Definition 2.2. Still, it is not difficult to define complexes a bit more generally so as to include it − the important fact is $C_c^\infty(\Omega)$ continuously embeds as a dense subspace of $\mathcal{D}'(\Omega)$. Currents will unavoidably appear when we consider distributional spaces of functions in the example models of the next sections, so we believe it is appropriate to introduce them sooner, in our general discussion, rather than later when our focus is on specific matters.

Once again, the chain maps d_p are given by formula (2.7). Of course, the symbols ∂^α in this case are to be understood in the sense of distributions, that is, each $\partial^\alpha f_I \in \mathcal{D}'(\Omega)$ in the formula is actually a continuous linear functional on $C^\infty(\Omega)$ whose value on a compactly supported function $\varphi \in C_c^\infty(\Omega)$ is

$$\langle \partial^\alpha f_I, \varphi \rangle = (-1)^{|\alpha|} \langle f_I, \partial^\alpha \varphi \rangle \tag{2.11}$$

The same computation we referred to in the previous example shows $(\mathcal{D}'(\Omega; \bigwedge^p E_p), d_p)$ is a differential complex. In fact, a theorem of de Rham even shows the cohomologies of this complex and the one from the former example are naturally isomorphic.

Example 2.5 (Dolbeault complex)**.**

Let $\nu = 2n$ and consider $\Omega = \mathbb{C}^n \cong \mathbb{R}^\nu$ identified through the natural isomorphism $(a_1 + ib_1, \dots, a_n + ib_n) \mapsto (a_1, \dots, a_n, b_1, \dots, b_n)$. Then, the standard $\mathbb{C}$-valued coordinates $(z_j)_{j=1,\dots,n} \in C^\infty(\Omega)$ lead to a set of smooth real coordinates $(x_j, y_j)_{j=1,\dots,n}$ with $z_j = x_j + iy_j$. Similarly, we may consider the coordinates $\bar{z}_j \doteq x_j - iy_j \in C^\infty(\Omega)$ coming from the isomorphism $(a_1 - ib_1, \dots, a_n - ib_n) \mapsto (a_1, \dots, a_n, b_1, \dots, b_n)$. Since $x_j = \frac{1}{2}(z_j + \bar{z}_j)$ and $y_j = \frac{1}{2i}(z_j - \bar{z}_j)$, the chain rule is suggestive of the following derivation rules with respect to the z_j and $\bar{z}_j$ variables: given $\phi \in C^\infty(\Omega)$, let

$$\partial_{z_j}\phi = \frac{\partial \phi}{\partial x_j}\frac{\partial x_j}{\partial z_j} + \frac{\partial \phi}{\partial y_j}\frac{\partial y_j}{\partial z_j} = \frac{1}{2}\left(\frac{\partial \phi}{\partial x^j} - i\frac{\partial \phi}{\partial y^j}\right) \tag{2.12}$$

$$\partial_{\bar{z}_j}\phi = \frac{\partial \phi}{\partial x_j}\frac{\partial x_j}{\partial \bar{z}_j} + \frac{\partial \phi}{\partial y_j}\frac{\partial y_j}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial \phi}{\partial x^j} + i\frac{\partial \phi}{\partial y^j}\right) \tag{2.13}$$

so we formally define $\partial_{z_j}, \partial_{\bar{z}_j} \in \mathbb{C}T\Omega$ using the RHS expressions above. Since $\mathrm{span}_{\mathbb{R}}\langle \partial_{x_j}, \partial_{y_j} \rangle = T\Omega$, we also have $\mathrm{span}_{\mathbb{C}}\langle \partial_{z_j}, \partial_{\bar{z}_j} \rangle = \mathbb{C}T\Omega$ and a direct splitting into the subbundles

$$\mathbb{C}T\Omega = \mathbb{C}T^{1,0}\Omega \oplus \mathbb{C}T^{0,1}(\Omega), \text{where} \tag{2.14}$$

$$\mathbb{C}T^{1,0}\Omega \doteq \operatorname*{span}_{\mathbb{C}}\langle \partial_{z_j} \rangle \quad \text{and} \quad \mathbb{C}T^{0,1}(\Omega) \doteq \operatorname*{span}_{\mathbb{C}}\langle \partial_{\bar{z}_j} \rangle \tag{2.15}$$

By duality, the corresponding tensors $(dz_j, d\bar{z}_j)_j$ are generators of $\mathbb{C}T^*\Omega$. Furthermore, in view of Lemma 1.10, the latter also splits into a direct sum of subbundles $\Lambda^{1,0}(\Omega) \doteq \mathbb{C}T^{0,1}(\Omega)^\perp, \Lambda^{0,1}(\Omega) \doteq \mathbb{C}T^{1,0}(\Omega)^\perp$, so 1-forms in Ω decompose directly into a section of $\Lambda^{0,1}(\Omega)$ plus a section of $\Lambda^{1,0}(\Omega)$. More generally, a complex k-form is a section $\alpha \in \Gamma(\bigwedge^k \mathbb{C}T^*\Omega)$ of the k-th exterior power of $\mathbb{C}T^*\Omega$, which is a $C^\infty(\Omega)$-linear combination of tensors $dz_I \wedge d\bar{z}_J$ with $|I| + |J| = k$. Let p, q such that $p + q = k$ be fixed, a k-form written as

$$\alpha = \sum_{\substack{|I|=p \\ |J|=q}} \alpha_{I,J} dz_I \wedge d\bar{z}_J, \quad \alpha_{I,J} \in C^\infty(\Omega) \tag{2.16}$$

is often referred to as a (p,q)-form on Ω. Such elements constitute a set we denote $\Lambda^{p,q}(\Omega)$.

For each p and q fixed we define complexes

$$\Lambda^{0,q}(\Omega) \xrightarrow{\partial} \Lambda^{1,q}(\Omega) \longrightarrow \dots \tag{2.17}$$

$$\Lambda^{p,0}(\Omega) \xrightarrow{\bar\partial} \Lambda^{p,1}(\Omega) \longrightarrow \dots \tag{2.18}$$

where $\partial, \bar\partial$ have the following formulas

$$\partial\Big(\sum_{\substack{|I|=p\\|J|=q}} \alpha_{I,J} dz_I \wedge d\bar z_J\Big) = \sum_{\substack{|I|=p\\|J|=q}} \sum_{j=1}^{n} \partial_{z_j}\alpha_{I,J}\, dz_j \wedge dz_I \wedge d\bar z_J \tag{2.19}$$

$$\bar\partial\Big(\sum_{\substack{|I|=p\\|J|=q}} \alpha_{I,J} dz_I \wedge d\bar z_J\Big) = \sum_{\substack{|I|=p\\|J|=q}} \sum_{j=1}^{n} \partial_{\bar z_j}\alpha_{I,J}\, d\bar z_j \wedge dz_I \wedge d\bar z_J \tag{2.20}$$

Again, the commutativity of $(\partial_{z_j})_j$ and $(\partial_{\bar z_j})_j$ ensures $\partial\circ\partial = \bar\partial\circ\bar\partial = 0$.

Remark 2.6 (Relation with de Rham complex)**.**

It should be noted for each j we have

$$\frac{\partial\alpha_{I,J}}{\partial z^j} dz_j + \frac{\partial\alpha_{I,J}}{\partial \bar z^j} d\bar z_j = \frac{1}{2}\left(\frac{\partial\alpha_{I,J}}{\partial x^j} - i\frac{\partial\alpha_{I,J}}{\partial y^j}\right)(dx_j + idy_j) \tag{2.21}$$

$$+\frac{1}{2}\left(\frac{\partial\alpha_{I,J}}{\partial x^j} + i\frac{\partial\alpha_{I,J}}{\partial y^j}\right)(dx_j - idy_j) = \frac{\partial\alpha_{I,J}}{\partial x^j} dx_j + \frac{\partial\alpha_{I,J}}{\partial y^j} dy_j \tag{2.22}$$

thus, from the defining expressions of $\partial, \overline{\partial}$, summing over j we have $d = \partial + \bar\partial$, where d is the exterior derivative of the de Rham complex. It then also follows $\partial\circ\overline{\partial} = -\overline{\partial}\circ\partial$, since $(\partial+\overline{\partial})\circ(\partial+\overline{\partial}) = 0$.

2.2 Principal symbol and ellipticity

Each $\mathbb{F}$-valued differential operator as in equation (2.3) induces a function we call the **symbol** (or total symbol) of P.[4] The part of order k of P is an homogeneous operator $P^{(k)} \in \mathrm{Diff}^k(M)$ where we only consider the terms in the representation which are associated to derivatives of order precisely k. If $k > \mathrm{ord}(P)$, then $P^{(k)}$ is zero, of course. Thus P is decomposed as $P^{(\mathrm{ord}(P))} + \dots + P^{(0)}$ where each of the symbols $P^{(k)}$ is a homogeneous polynomial of order k on ξ.

The **principal symbol** σ_P of P will be the total symbol of $P^{(\mathrm{ord}(P))}$, that is

$$\sigma_P(\xi) = \sum_{|\alpha|=\mathrm{ord}(P)} a_\alpha(x)\xi^\alpha, \quad \xi \in T_x^*\Omega \tag{2.23}$$

[4] $\Omega \subset \mathbb{R}^\nu$ being open, the cotangent bundle is trivially diffeomorphic to $\Omega \times \mathbb{R}^\nu$

The operator P is said of **elliptic** type if for each $x \in \Omega$, the only real root for $\sigma_P(\xi)$ with $\xi \in T_x^*\Omega$ is the trivial $\xi = 0$.

Let us see how those concepts are handled in the context of differential complexes.

Definition 2.7 (Principal symbol of $P \in \operatorname{Diff}(\Omega; V, W)$). *Consider as in Definition 2.1 that $P \in \operatorname{Diff}(\Omega; V, W)$ has a matrix representation $[P] = (P_{jl}) \in \mathcal{M}_{n\times m}(\operatorname{Diff}(\Omega))$ and let $k \geqslant \max_{j,l} \operatorname{ord} P_{jl}$ be fixed. Then the principal symbol $\sigma_P(x,\xi) \in \mathcal{L}(V, W)$ is the linear mapping where each entry (j,l) in matrix representation is given by $\sigma_{P_{jl}^{(k)}}$, that is, each entry is the principal symbol of the homogeneous part of order k of P_{jl}.*

Remark 2.8. *If Ω is simply a smooth manifold, the coordinates ξ on the fibers of $T^*\Omega$ depend on the choice of a local chart for Ω. As such, the coordinate representation of P for a fixed order k can vary, which makes the total symbol of P unsuitable for coordinate-invariant properties. In the case of the principal symbol, however, we can work around this issue as follows.*

Let $k = \operatorname{ord}(P)$ and $P \in \operatorname{Diff}(\Omega; V, W)$. For each choices of $u \in C^\infty(\Omega; V)$ and $f \in C^\infty(\Omega; \mathbb{R})$, we obtain a polynomial of degree k in W

$$\tau \longmapsto e^{-i\tau f(x_0)} P(e^{i\tau f} u)(x_0) \in W \tag{2.24}$$

as we will readily see by taking coordinates. The operator P is represented by a combination

$$[P] = \sum_{|\alpha| \leqslant k} \sum_{j,l} a_{jl}^\alpha [D_{jl}^\alpha], \quad a_{jl}^\alpha \in C^\infty(\Omega) \tag{2.25}$$

where $[D_{jl}^\alpha] \in \mathcal{M}_{n\times m}(\operatorname{Diff}(\Omega))$ is the elementary matrix with D^α at row j and column l as its only non-zero entry. Then, if $[v_l] \in \mathcal{M}_{n\times 1}(C^\infty(\Omega))$ is the coordinate representation of $v \in C^\infty(\Omega; V)$, we have

$$[D_{jl}^\alpha(v)]_p = \delta_{jp} D^\alpha v_l \tag{2.26}$$

thus, by linearity,

$$[P(e^{i\tau f} u)]_p = \sum_{|\alpha| \leqslant k} \sum_{j,l} \left[a_{jl}^\alpha D_{jl}^\alpha (e^{i\tau f} u)\right]_p = \sum_{|\alpha| \leqslant k} \sum_{l} a_{pl}^\alpha D^\alpha (e^{i\tau f} u_l) \tag{2.27}$$

Now Leibniz's product rule (1.38) allows us to write for each multi-index α

$$D^\alpha (e^{i\tau f} u_l) = \tau^{|\alpha|} e^{i\tau f} df^\alpha u_l + \sum_{\beta < \alpha} C_\beta \tau^{|\beta|} e^{i\tau f} df^\beta D^{\alpha-\beta} u_l \tag{2.28}$$

Then, multiplying (2.27) by $e^{i\tau f}$ and evaluating at x_0, we get (2.24) as sum of polynomial terms of τ on each coordinate. The maximal order coefficients are the ones associated to τ^k, which are obtained from indices α such that $|\alpha| = k$. If we single them out with (2.28) we get

$$e^{-i\tau f(x_0)} \left(\sum_{|\alpha|=k} \tau^{|\alpha|} e^{i\tau f} df^\alpha \sum_l (a^\alpha_{pl} u_l) \right)(x_0) = \tag{2.29}$$

$$\tau^k \sum_{|\alpha|=k} df(x_0)^\alpha \left(a^\alpha_{j\bullet}(x_0) \cdot u(x_0)\right) = \tau^k \left(\sum_{|\alpha|=k} df(x_0)^\alpha a_{j\bullet}(x_0), u(x_0) \right) \cdot u(x_0) \tag{2.30}$$

In particular, we see that the coefficient of τ^k in (2.24) depends on the choices of f and u only through the values of $\xi \doteq df(x_0)$ and $\eta \doteq u(x_0)$, depending linearly on the latter. It is therefore legitimate to define the principal symbol of P based on those values, as we do in the following.

Definition 2.9 ([Men23] Symbol)**.** *Let $P \in \mathrm{Diff}(\Omega; V, W)$, $\xi \in T^*_{x_0}\Omega$ and $k \geqslant \mathrm{ord}(P)$. The symbol $\sigma^k_P(\xi)$ is the linear map $V \to W$ given by*

$$\sigma^k_P(\xi)(\eta) = \lim_{\tau\to\infty} \frac{e^{-i\tau f(x_0)}}{\tau^k} P(e^{i\tau f} u)(x_0) \tag{2.31}$$

where $u \in C^\infty(\Omega; V)$, $f \in C^\infty(\Omega; \mathbb{R})$ are such that $df(x_0) = \xi$ and $u(x_0) = \eta$.

Remark 2.10. *The principal symbol of P refers exclusively to σ^k_P with $k = \mathrm{ord}(P)$. Still, in some contexts (such as Lemma 2.11), it is convenient to allow $k > \mathrm{ord}(P)$, in which case the definition above readily implies $\sigma^k_P = 0$.*

It suffices to compare (2.29) with the statement of Definition 2.7 to convince ourselves that both definitions of principal symbol agree.

Lemma 2.11. *Let $P \in \mathrm{Diff}^k(\Omega; V, W)$ and $Q \in \mathrm{Diff}^l(\Omega; W, X)$, then for any $\xi \in T^*\Omega$ the principal symbol of QP satisfies the relation $\sigma^{k+l}_{QP}(\xi) = \sigma^l_Q(\xi) \circ \sigma^k_P(\xi)$.*

Proof. Given $u \in C^\infty(\Omega; V)$ and $f \in C^\infty(\Omega; \mathbb{R})$, for each $\tau \in \mathbb{R}$ we can write

$$\frac{1}{\tau^{k+l}} e^{-i\tau f} QP(e^{i\tau f} u) = \left(\frac{1}{\tau^l} e^{-i\tau f} Q e^{i\tau f} \right) \left(\frac{1}{\tau^k} e^{-i\tau f} P e^{i\tau f} \right) u \tag{2.32}$$

By (2.31), we have the pointwise limit

$$\tau^{-k} e^{-i\tau f} P(e^{i\tau f}) u \overset{\tau\to\infty}{\to} \sigma^k_P(df)(u) \tag{2.33}$$

$$\text{where } \sigma^k_P(df)(u) : p \in \Omega \mapsto \sigma^k_P(df(p))(u(p)) \in W \tag{2.34}$$

and, likewise, given $v \in C^\infty(\Omega; W)$,

$$\tau^{-k} e^{-i\tau f} Q(e^{i\tau f}) v \overset{\tau\to\infty}{\to} \sigma^l_Q(df)(v) \tag{2.35}$$

Now let $\xi \in T^*_{x_0}\Omega$ and $\eta \in V$, take $u \in C^\infty(\Omega; V)$, $f \in C^\infty(\Omega; \mathbb{R})$ related to ξ and η as in Definition 2.9. Substituting $\sigma_P^k(df)u$ for v in (2.35), since the maps in (2.33) and (2.35) (from Ω to V and W, respectively) are all smooth, the limit for the composition at the right-hand side of (2.32) evaluated at x_0 simplifies to

$$\sigma_{QP}^{k+l}(\xi)(\eta) = \sigma_Q^l(df(p))(\sigma_P^k(\xi))(\eta) = (\sigma_Q^l(\xi) \circ \sigma_P^k(\xi))(\eta) \tag{2.36}$$

as we wanted. □

Definition 2.12. *Let $P = (C^\infty(\Omega; E_j), P_j)_j$ be a differential complex as in Definition 2.2. By virtue of Lemma 2.11, we know for each $\xi \in T^*\Omega$ the sequence*

$$0 \longrightarrow E_1 \xrightarrow{\sigma_{P_1}(\xi)} E_2 \xrightarrow{\sigma_{P_2}(\xi)} E_3 \xrightarrow{\sigma_{P_3}(\xi)} \cdots \tag{2.37}$$

*is in fact a chain complex of vector spaces (meaning for any j, $\sigma_{P_{j+1}}(\xi) \circ \sigma_{P_j}(\xi) = 0$). The operator P is then said **elliptic** if for all $\xi \in T^*\Omega$ the induced sequence (2.37) is exact.*

Example 2.13 (de Rham complex)**.** *We show the de Rham complex from Example 2.3 is elliptic.*

Let $u \in C^\infty(\Omega; \bigwedge^p \mathbb{C}^\nu)$ be a p-form and $f \in C^\infty(\Omega)$, the product rule of the exterior derivative (cf. (i) in Proposition 1.9) leads to

$$\frac{1}{\tau} e^{i\tau f} d(e^{-i\tau f} u) = idf \wedge u + \frac{1}{\tau} du \tag{2.38}$$

so taking the limit of Definition 2.9 obtain the principal symbol $\sigma_d(\xi) = i\xi \wedge \cdot$.

Let us check the induced chain complex

$$0 \xrightarrow{i\xi\wedge\cdot} \mathbb{C}^\nu \xrightarrow{i\xi\wedge\cdot} \bigwedge^1 \mathbb{C}^\nu \xrightarrow{i\xi\wedge\cdot} \bigwedge^2 \mathbb{C}^\nu \xrightarrow{i\xi\wedge\cdot} \cdots \tag{2.39}$$

is exact for any given ξ. Indeed suppose we have $\xi \in \bigwedge^1 \mathbb{C}^\nu \setminus \{0\}$ and $\alpha \in \bigwedge^p \mathbb{C}^\nu$ with $i\xi \wedge \alpha = 0$. Since $\xi \neq 0$, we can take a basis $(e_1, e_2, \ldots, e_\nu) \in \bigwedge^1 \mathbb{C}^\nu$ where $e_1 = \xi$. Then, we split $\bigwedge^{p+1} \mathbb{C}^\nu$ into a direct sum

$$\bigwedge^{p+1} \mathbb{C}^\nu = \overbrace{\underset{\substack{[J]=p+1 \\ 1\in J}}{\operatorname{span}} \langle e_J \rangle}^{E_j \doteq} \oplus \overbrace{\underset{\substack{[J]=p+1 \\ 1\notin J}}{\operatorname{span}} \langle e_J \rangle}^{F_j \doteq} \tag{2.40}$$

Writing the action of $i\xi \wedge \cdot$ in coordinates, it becomes clear $i\xi \wedge \cdot$ restricted to F_j is injetive. All summands in the coordinate expression of $i\xi \wedge \eta$ with $\eta \in E_j$ vanish identically due to $\xi \wedge \xi = 0$, so that $i\xi \wedge (E_j) = \{0\}$ as well. Thus, for each $\alpha = \alpha_{E_j} + \alpha_{F_j}$ with $(\alpha_{E_j}, \alpha_{F_j}) \in E_j \times F_j$

$$\xi \wedge \alpha = 0 \implies \xi \wedge \alpha_{F_j} = 0 \implies \alpha_{F_j} = 0 \tag{2.41}$$

Therefore $\alpha = \alpha_{E_j} \in E_j$. On the other hand, we easily see $i\xi \wedge (F_{j-1}) = E_j$, so $\alpha \in \operatorname{ran} i\xi \wedge \cdot$.

We then conclude $\ker i\xi \wedge \cdot = \operatorname{ran} i\xi \wedge \cdot$, which is the condition for exactness.

2.3 Complexes determined by vector fields and involutive systems

Now suppose $\mathbb{F} = \mathbb{C}$ and Ω is a smooth manifold of dimension N where a family of complex vector fields $\mathbb{L} = (L_1, \ldots, L_\nu) \in \mathfrak{X}(\Omega)$ is defined. The elements of $\mathbb{L}$ can be understood in agreement with Definition 2.1 as homogeneous differential operators $C^\infty(\Omega) \to C^\infty(\Omega)$ of order 1.

We set basic constraints on $\mathbb{L} = (L_1, \ldots, L_\nu) \subset \mathfrak{X}(\Omega)$ leading towards two distinct goals. The first is to be able to define a differential complex

$$0 \xrightarrow{\mathbb{L}^{(0)}} C^\infty(\Omega) \xrightarrow{\mathbb{L}^{(1)}} C^\infty(\Omega; \textstyle\bigwedge^1 \mathbb{C}^\nu) \xrightarrow{\mathbb{L}^{(2)}} C^\infty(\Omega; \textstyle\bigwedge^2 \mathbb{C}^\nu) \xrightarrow{\mathbb{L}^{(3)}} \cdots \tag{2.42}$$

in the sense of Definition 2.2. Once we set reasonable formula for the differential maps, a necessary and sufficient condition over $\mathbb{L}$, to the end of making the sequence a complex, will readily follow – it is the mutual commutativity of the vector fields.

The other goal has to do with the inhomogeneous problems posed by overdetermined systems of equations such as

$$L_j u = f_j, \quad j = 1, \ldots, \nu \tag{2.43}$$

We establish a couple of conditions over $\mathbb{L}$ that make an investigation of the basic matters concerning solutions feasible (existence, uniqueness, regularity, and so on). They are the following: the vector fields must be linearly independent pointwise, and the $C^\infty(\Omega)$-linear span of the vector fields span the Lie algebra generated by those same elements. Together, they give the minimal working assumptions for the theory of involutive systems to develop, while also pointing towards certain subbundles of $\mathbb{C}T\Omega$, known as formally integrable structures, as the intrinsic objects of study in regards to overdetermined systems.

From vector fields to a differential complex

Let $(U, \mathbf{x})$ be a local chart of Ω and denote $(\partial_{x_j}) \subset \mathfrak{X}(U)$, $(dx_j) \subset \Gamma(T^*U)$ the local generators obtained from such choice of coordinates. Elements of the family $\mathbb{L}$ are then locally given by

$$L_j = \sum_{k=1}^{N} \alpha_{jk} \partial_{x_k} \tag{2.44}$$

for some uniquely determined coefficients $\alpha_{jk} \in C^\infty(U)$. As laid out in the introduction, our intention is to define a differential complex (2.42) out of $\mathbb{L}$. A reasonable starting point is to take $\mathbb{L}^{(1)} \in \mathrm{Diff}(\Omega; \mathbb{C}^\nu, \bigwedge^1 \mathbb{C}^\nu)$ given by

$$\mathbb{L}^{(1)} u = \sum_{j=1}^{\nu} L_j u \; e_j \in C^\infty(\Omega; \textstyle\bigwedge^1 T^*\Omega), \quad \forall f \in C^\infty(\Omega) \tag{2.45}$$

with $(e_1, \dots, e_\nu)$ a basis for $\mathbb{C}^\nu$. Solutions to $\mathbb{L}^{(1)}u = f$ are then equivalent to solutions to the overdetermined system (2.43) by means of the isomorphism $C^\infty(\Omega; \mathbb{C}^\nu) \cong C^\infty(\Omega; \bigwedge^1 \mathbb{C}^\nu)$.

The operator just defined is remarkably similar to the differentials of the de Rham complex of Example 2.3; in fact, if Ω is open and $L_j = \partial_{x_j}$ with $\nu = N$, then $\mathbb{L}^{(1)}$ in equation (2.45) is precisely the first level of the exterior derivative. With that in mind, we consider extending the construction to the remaining levels of the complex in accordance with (2.7), that is, by making

$$\mathbb{L}^{(p)}\left(\sum_{[I]=p-1} \alpha_I e_I\right) = \sum_{[I]=p-1} \sum_{k=1}^{\nu} (L_k \alpha_I)\, e_k \wedge e_I, \quad \alpha_I \in C^\infty(U) \tag{2.46}$$

Doing so certainly makes (2.42) a sequence of first order differential maps. The question to be settled, then, is under which circumstances the definition above gives a differential complex.

Lemma 2.14. *Let $\mathbb{L} = (L_1, \dots, L_\nu) \subset \mathfrak{X}(\Omega)$ be a family of vector fields and $\mathbb{L}^{(p)} : C^\infty(\Omega; \bigwedge^{p-1} \mathbb{C}^\nu) \to C^\infty(\Omega; \bigwedge^{p} \mathbb{C}^\nu)$ the linear mappings given by (2.46), then (2.42) is a complex if and only if, $[L_k, L_l] = 0$ for each $k, l = 1, \dots, \nu$.*

Proof. Let $\alpha = \sum_{[I]=p-1} \alpha_I e_I \in C^\infty(\Omega; \bigwedge^{p-1} \mathbb{C}^\nu)$, then (2.46) leads to

$$\mathbb{L}^{(p+1)}\mathbb{L}^{(p)}\alpha = \sum_{k,l=1}^{\nu} \sum_{[I]=j-1} L_l(L_k \alpha_I)\, e_l \wedge e_k \wedge e_I \tag{2.47}$$

$$= \sum_{l<k}^{\nu} \sum_{[I]} (L_l L_k \alpha_I - L_k L_l \alpha_I)\, e_l \wedge e_k \wedge e_I \tag{2.48}$$

Hence, $[L_k, L_l] = 0$ suffices to ensure $\mathbb{L}^{(p+1)}\mathbb{L}^{(p)} = 0$.

Conversely, if the linear operators do form a complex, then in particular at $p = 1$ the expression in the last equation simplifies to

$$\mathbb{L}^{(2)}\mathbb{L}^{(1)} f = \sum_{l<k}^{\nu} [L_l, L_k] f\; e_l \wedge e_k = 0 \tag{2.49}$$

for each $f \in C^\infty(\Omega)$. Now given $(e_l \wedge e_k)_{l<k}$ is a basis $\bigwedge^2 \mathbb{C}^\nu$ and f varies freely, we conclude $[L_k, L_l] = 0$ for all $k \neq l$. $\square$

Involutive systems

We now present the assumptions made for overdetermined systems such as (2.43).

The special case of a single real vector field already provides some direction. Suppose L is defined in a neighborhood of the origin in $\mathbb{R}^N$ with local representation

$$L = \sum_{j=1}^{N} a_j \partial_{x_j} \tag{2.50}$$

If L is non-vanishing at 0, we can define coordinates $y_1, \dots, y_N$ near the origin so that $L = \partial_{y_1}$. Indeed, assuming we have $a_1(0) \neq 0$, that amounts to solving a Cauchy problem

$$\begin{cases} \partial_{y_1} x_j = a_j(x_1, \dots, x_N), & j \in \{1, \dots, N\} \\ x_1(0, y_2, \dots, y_N) = 0 \\ x_1(0, y_2, \dots, y_N) = y_j, & j \in \{2, \dots, N\} \end{cases} \tag{2.51}$$

Since $a_1(0) \neq 0$, $(y_1, \dots, y_N) \to (x_1, \dots, x_N)$ is a smooth local diffeomorphism, the (local) inverse $(x_1, \dots, x_N) \to (y_1, \dots, y_N)$ exists and attains to $L = \partial_{y_j}$ as we wanted. The problem of solving $Lu = f$ near the origin can be thus be solved by integration with respect to y_1. The assumption of L non-vanishing at the origin is crucial – nothing can be asserted in general otherwise.

Then, at least in the real setting, the basic requirement for the existence of local solutions to a vector field equation is that $L \in \mathfrak{X}(\Omega)$ is non-vanishing at the origin. More generally, for a number $n \geqslant 1$ of real vector fields, Frobenius' theorem states that for each point there are coordinates $y_1, \dots, y_n$ such that $\operatorname{span}(L_j) = \operatorname{span}(\partial_{y_j})$ locally, as long as $L_1, \dots, L_n$ are linearly independent pointwise (cf. [BCH08, Theorem 5.1]). Considering the case of complex vector fields is more general than the real case, we have motivated the following condition on $\mathbb{L}$ as a necessary one.

Definition 2.15. *The family $\mathbb{L} = (L_1, \dots, L_\nu) \subset \mathfrak{X}(\Omega)$ is said of* ***principal type*** *if for each $p \in \Omega$ the families of vectors $((L_1)_p, \dots, (L_\nu)_p) \subset \mathbb{C}T_p\Omega$ are linearly independent.*

Remark 2.16. *Clearly the maximum number of vector fields in a family of principal type is bounded by the dimension of the manifold, i.e. $\nu \leqslant N$ in the definition above. Also evident is the fact that $\nu = N$ can occur in the chart domains $U \subset \Omega$, since the standard Euclidean coordinate fields pull-back to a local basis in $\mathfrak{X}(U)$. In general, however, the maximum number of vector fields in a principal type family is particular to the manifold Ω under consideration. For example, in the real case, it is well known that the spheres S^N with N even forbid globally defined, non-vanishing vector fields, so that $\nu = 0$ is already the maximal constant. On the other hand, again in the real case, the class of parallelizable manifolds, which includes the tori $\mathbb{T}^N$ and the sphere S^3, can be characterized by the existence of a principal type family with ν equal to the dimension of the manifold (see [Lee13, p.179] for further details).*

For the second requirement, consider the system of equations (2.43) determined by $\mathbb{L}$ in the homogeneous case $f_j = 0, j = 1, \dots, \nu$. Let $\mathcal{L} \subset \mathfrak{X}(\Omega)$ be the set of vector fields L such that u solving the system implies $Lu = 0$. Then

$$L, M \in \mathcal{L} \implies \forall u,\ [L, M]u = L(Mu) - M(Lu) = 0 \implies [L, M] \in \mathcal{L} \tag{2.52}$$

so $\mathcal{L}$ is closed with respect to the bracket operation, and thus a Lie algebra. Furthermore, it easy to see $\mathcal{L}$ is $C^\infty(\Omega)$-linearly closed. It may well be the case that the Lie algebra generated by the family $\mathbb{L}$ is strictly larger than its $C^\infty(\Omega)$-linear span. Therefore, if we want make $\mathbb{L}$ a $C^\infty(\Omega)$-linear basis for $\mathcal{L}$, we might need to complete it by adjoining vector fields from $\mathcal{L}$. However, such process may prove incompatible with the principal type condition over $\mathcal{L}$, as the next example demonstrates. Therefore, working simultaneously with both conditions over $\mathbb{L}$ requires an assumption from the outset.

Definition 2.17. *The family of complex vector fields $\mathbb{L} = (L_1, \ldots, L_\nu) \in \mathfrak{X}(\Omega)$ is said* ***involutive*** *if it is of principal type and satisfies the Frobenius condition:*

$$\forall j, k,\ [L_j, L_k] \in \operatorname*{span}_{C^\infty(\Omega)} \mathbb{L} \tag{2.53}$$

Example 2.18.

Let $\Omega = \mathbb{R}^3$ and $\mathbb{L} = (L_1, L_2) = (\partial_{x_1}, \partial_{x_2} + x_1 x_3 \partial_{x_3})$, then $\mathbb{L}$ is of principal type, but cannot be extended to an involutive family while remaining within the same Lie sub-algebra. Indeed, (L_1, L_2) generates a proper Lie algebra of $\mathfrak{X}(\Omega)$, with linear basis (L_1, L_2, L_3) where $L_3 \doteq [L_1, L_2] = x_3 \partial_{x_3} \notin \operatorname{span}(L_1, L_2)$. However, to adjoin a vector field to (L_1, L_2) while maintaining pointwise linear independence we need the linear span of the fields at 0 to be $\mathbb{C}T_0\Omega \supsetneq \operatorname{span}(\partial_{x_1}|_0, \partial_{x_2}|_0)$.

2.4 Involutive structures

The properties pertaining to the solutions of locally defined overdetermined systems such as (2.43) do not vary either by a change of coordinates nor by diffeomorphic transformations. Therefore, if we want to regard those systems in an intrinsic manner, we must concern ourselves with the vector bundles generated by the principal type families, rather than the vector fields themselves. From now on, this is the perspective we adopt.

Definition 2.19. *A complex vector bundle $\mathcal{V} \subset \mathbb{C}T\Omega$ is said a* ***formally integrable structure*** *over Ω if for each $W \subset \Omega$ open we have*

$$[L, M] \in \Gamma(\mathbb{C}TU \cap \mathcal{V}) \textit{ whenever } L, M \in \Gamma(\mathbb{C}TU \cap \mathcal{V}) \tag{2.54}$$

In that case, $(\Omega, \mathcal{V})$ is said an ***involutive structure****.*

Let $\mathcal{V}$ be a formally integrable structure of rank n, then for each local chart $(U, \mathbf{x})$ we get generators $L_1, \ldots, L_n \in \mathfrak{X}(U)$ for $\mathcal{V} \cap \mathbb{C}TU$. Writing the vector fields as

$$L_j = \sum_{k=1}^{\nu} a_{jk} \frac{\partial}{\partial x_k} \tag{2.55}$$

the condition of linear independence implies the matrices $(a_{jk}(x)) \in \mathcal{M}_{n\times\nu}$ are of maximal rank for each $x \in \Omega$. Also notice that, since $L_1, \ldots, L_n$ are generators, there exists smooth coefficients $c^l_{jk} \in C^\infty(U)$ such that $\forall j, k$,

$$[L_j, L_k] = \sum_l c^l_{jk} L_l \tag{2.56}$$

Definition 2.20 (Characteristic set). *Let $\mathcal{V} \subset \mathbb{C}T\Omega$ be a formally integrable structure over Ω, the characteristic set $\mathcal{V}^0$ of $\mathcal{V}$ is the subset of the (real) cotangent bundle which is orthogonal to $\mathcal{V}$ through their natural duality, that is*

$$\mathcal{V}^0 \doteq \mathcal{V}^\perp \cap T^*\Omega \tag{2.57}$$

The symbol of a vector field $L \in \mathfrak{X}(\Omega)$ is the mapping

$$\sigma_L : \xi \in T^*_p\Omega \subset T^*\Omega \mapsto \xi(L_p) \tag{2.58}$$

Therefore, we have $\xi \in \mathcal{V}^0$ if and only if $\sigma_L(\xi)$ vanishes for every $L \in \Gamma(\mathcal{V})$.

Remark 2.21. *Take $(U, (x_1, \ldots, x_N))$ a local chart on Ω and $\xi \in T^*_p\Omega$ a covector at $p \in U$. Writing with respect to those coordinates $\xi = \sum_j \xi_j(dx_j)_p$ and $L = \sum_j \alpha_j \frac{\partial}{\partial x_j}$, the expression for the symbol becomes*

$$\sigma_L(\xi) = \xi(L_p) = \left\langle \sum_j \xi_j(dx_j)_p, \sum_k \alpha_k(p)\frac{\partial}{\partial x_j} \right\rangle = \sum_j \alpha_j(p)\xi_j \tag{2.59}$$

Then, if $(L_j)_{j=1,\ldots,\nu}$ given by $L_j = \sum_k a_{jk}\frac{\partial}{\partial x_k}$ are linearly independent local generators of $\mathcal{V}$, the characteristic set in the vicinity $\mathcal{V}^0 \cap T^*U$ is described by a system of equations

$$\sum_k a_{jk}(p)\xi_k = 0, \quad (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N, \ j = 1, \ldots, \nu \tag{2.60}$$

Notice that this definition of symbol for vector fields agrees with the one we gave for vector valued differential operators in Definition 2.7 and Definition 2.9. Furthermore, if $(L_1, \ldots, L_\nu)$ commute, and therefore define a differential complex $\mathbb{L}$ (as we described in Section 2.3), then a solution $(\xi_1, \ldots, \xi_N)$ to the system of equations (2.60) is equivalent to the equality

$$\ker \sigma_{\mathbb{L}^1}(\xi) = \operatorname{ran} \sigma_{\mathbb{L}^0}(\xi) = \{0\} \tag{2.61}$$

in the chain complex induced by $\xi = \sum_j \xi_j(dx_j)_p$ of (2.37). Said differently, a given $\xi \in T^*\Omega$ will be in $\mathcal{V}^0 \cap T^*\Omega$ iff the complex (2.37) induced by $\mathbb{L}$ and $\xi \in T^*\Omega$ in (2.37) is exact at $E_1 = \bigwedge^1 \mathbb{C}^N$.

Example 2.22 (Mizohata operator). *Denote (t, x) the standard coordinates in $\Omega = \mathbb{R}^2$. The Mizohata operator is the complex vector field*

$$M = \frac{\partial}{\partial x} - ix\frac{\partial}{\partial t} \in \mathfrak{X}(\mathbb{R}^2) \tag{2.62}$$

Since it doesn't vanish anywhere, M defines a locally integrable structure $\mathcal{V}$. The equation for its symbol $\sigma(M)_{(t,x)}(\xi,\tau) = \xi - ix\tau$ does not contain any real roots $(\xi,\tau) \neq 0$ unless $x = 0$. Therefore the characteristic set of M is given by

$$\mathcal{V}^0_{(t,x)} = \begin{cases} 0, & \text{if } x \neq 0 \\ \operatorname{span} dt_{(t,x)}, & \text{if } x = 0 \end{cases} \tag{2.63}$$

In particular, we see the characteristic set is not necessarily a submanifold.

We give names for structures $\mathcal{V}$ with special algebraic properties on the vector space structures of its fibers.

Definition 2.23 (Special structure types [BCH08, I.8, p 15]). *Let $\mathcal{V}$ be a formally integrable structure over Ω, we say $\mathcal{V}$ defines*

- *an **elliptic structure** if $\mathcal{V}^0_p = 0, \ \forall p \in \Omega$*
- *a **complex structure** if $\mathcal{V}_p \oplus \overline{\mathcal{V}}_p = \mathbb{C}T_p\Omega, \forall p \in \Omega$*
- *a **CR structure** if $\mathcal{V}_p \cap \overline{\mathcal{V}}_p = 0, \ \forall p \in \Omega$*
- *an **essentially real structure** if $\mathcal{V}_p = \overline{\mathcal{V}}_p, \ \forall p \in \Omega$*

A few remarks are due here. Notice that a complex structure is a special type of both elliptic and CR structures. Indeed, $\mathcal{V}_p \cap \overline{\mathcal{V}}_p = 0$ is the precondition to have a direct sum, while

$$\mathcal{V}_p \oplus \overline{\mathcal{V}}_p = \mathbb{C}T_p\Omega \implies \overline{\mathcal{V}}_p^{\perp} \oplus \mathcal{V}_p^{\perp} = 0 \implies \mathcal{V}_p^{\perp} = 0 \tag{2.64}$$

shows complex structures are elliptic. The name CR is short for Cauchy-Riemann, in reference to the quintessential example of the sub-bundles $\mathbb{C}T^{1,0}, \mathbb{C}T^{0,1}$ defined in the construction of the Dolbeault complex in Example 2.5. Finally, the defining property of essentially real structures makes it so that real vector fields can be taken as local generators. To be sure, let $L_1, \ldots, L_n$ be local generators of $\mathcal{V}$ near a point p, then $\mathcal{V}$ essentially real implies $(\operatorname{Re} L_j, \operatorname{Im} L_j)_{j=1,\ldots,n}$ are still sections of $\mathcal{V}$. Furthermore, n out of those span $\mathcal{V}_p$ at p. Since the rank of the subspaces generated pointwise by the vector fields must remain constant in a neighborhood, we conclude they define a set of local, real generators of $\mathcal{V}$.

Example 2.24 (The boundary operator on a hypersurface). *Let $\Omega = \{z \in \mathbb{C}^p : \sum_j |z_j|^2 = 1\}$ be the unit sphere regarded as an embedded manifold of codimension 1 in $\mathbb{C}^p \cong \mathbb{R}^{2p}$. With coordinates $(z_j, \bar{z}_j)_{j=1,\ldots,p}$ as in Example 2.5, we consider the vector fields in $\operatorname{span}_{C^\infty(\Omega)} (\partial_{\bar{z}_j})_{j=1,\ldots,p} \subset \mathfrak{X}(\mathbb{C}^p)$ which are also tangent to Ω and build a corresponding differential complex on $\Lambda^{0,\bullet}(\Omega)$.*

The sphere Ω is implicitly given by the equation $r(z) = 0$, where $r(z) = \sum_j z_j\bar{z}_j - 1$. The 1-form $dr = \sum_j \bar{z}_j dz_j + z_j d\bar{z}_j \in \Gamma(\mathbb{C}T^*\Omega)$ is therefore a generator of $\mathbb{C}T^*\Omega$ and, given $v \in T_w\mathbb{C}^p$ with $w \in \Omega$, tangency to Ω is equivalent to $\langle dr, v\rangle = 0$.

Define for $\nu = 1, \dots, p-1$ the vector fields

$$L_j \doteq z_p \frac{\partial}{\partial \bar{z}_j} - z_j \frac{\partial}{\partial \bar{z}_p} \in \mathfrak{X}(\mathbb{C}^p) \tag{2.65}$$

and notice for each $k = 1, \dots, p-1$,

$$\langle dr, L_k\rangle = \sum_j (z_j d\bar{z}_j + \bar{z}_j dz_j) \left(z_p \frac{\partial}{\partial \bar{z}_k} - z_k \frac{\partial}{\partial \bar{z}_p} \right) = z_k z_p - z_p z_k = 0 \tag{2.66}$$

Therefore, each L_j actually restricts to a vector field over Ω. Moreover, disregarding the set of points where $z_p = 0$, the family $(L_1, \dots, L_{p-1})$ is pointwise linearly independent and, for all j, k, its elements satisfy the relation

$$[L_j, L_k] = \left[z_p \frac{\partial}{\partial \bar{z}_j} - z_j \frac{\partial}{\partial \bar{z}_p}, z_p \frac{\partial}{\partial \bar{z}_k} - z_k \frac{\partial}{\partial \bar{z}_p} \right] = 0 \tag{2.67}$$

thus defining an involutive family over $\Omega \backslash \{z_p = 0\}$.

To generate a differential complex, we select 1-forms $\omega_1, \dots, \omega_{p-1} \in \operatorname{span}_{C^\infty(\Omega)}(d\bar{z}_j)$ (sections of $\Lambda^{0,1}(\mathbb{C}^p)$ cf. Example 2.5) in dual relation to the vector fields (2.66), that is,

$$\omega_k = \sum_{j=1}^{p} \alpha_{kj} d\bar{z}_j, \quad \alpha_{kj} \in C^\infty(\mathbb{C}^p) \tag{2.68}$$

$$\text{such that } \omega_k(L_j) = \delta_{kj}, \quad (k = 1, \dots, p-1,\ j = 1, \dots, p-1) \tag{2.69}$$

Substituting the expressions (2.68) and (2.65) for each equation given by (2.69), we obtain the following general solutions for the coefficients with $j \neq p$

$$\omega_k(L_j) = \alpha_{kj} z_p - z_j \alpha_{kp} = \delta_{jk} \implies \alpha_{kj} = \frac{\delta_{kj} + z_j \alpha_{kp}}{z_p} \tag{2.70}$$

which depend solely on the choices of $(\alpha_{kp})_{k=1,\dots,p-1} \subset C^\infty(\mathbb{C}^p)$.

Now let $(\omega_k), (\widetilde{\omega}_k)$ be 1-forms in $\Lambda^{0,1}(\mathbb{C}^p)$ corresponding different choices of $(\beta_k), (\widetilde{\beta}_k)$ for the free coefficients $(\alpha_{kp})_{k=1,\dots,p-1}$. Then, by (2.70) we may write

$$\omega_k - \widetilde{\omega}_k = \sum_j \frac{z_j}{z_p} (\beta_k - \widetilde{\beta}_k) d\bar{z}_j = \frac{\beta_k - \widetilde{\beta}_k}{z_p} \sum_j z_j d\bar{z}_j \tag{2.71}$$

Notice the 1-form $\sum_j z_j d\bar{z}_j$ vanishes on $\mathbb{C}T^{0,1}\Omega$. Indeed, if $v = \sum_k \gamma_k \frac{\partial}{\partial \bar{z}_k} \in \mathbb{C}T^{0,1}_w\Omega$ for some $w \in \Omega$, then

$$\langle dr, v\rangle = \sum_k \gamma_k(w) z_k = 0 \implies \sum_j z_j d\bar{z}_j(v) = \sum_j \gamma_j(w) z_j = 0 \tag{2.72}$$

Therefore, $\omega_k - \widetilde{\omega}_k$ restricted to $\mathbb{C}T^{0,1}\Omega$ is zero and we conclude the duality between (ω_k) and (L_j), as established by the system of equations (2.69), determines a unique 1-form $\omega_j^p \in \Lambda^{0,1}(\Omega)$. Thus we may choose, for instance, $\alpha_{kp} = 0$ $(k = 1, \ldots, p-1)$ for the free coefficients, so that $\omega_j^p = \frac{1}{z_p} d\bar{z}_j$.

Define the differential operator $\overline{\partial_b^p}$ as follows – given $f \in C^\infty(\Omega \backslash \{z_p = 0\}, \mathbb{C})$,

$$\overline{\partial_b^p} f = \sum_{j=1}^{p-1} (L_j f) \omega_j^p \tag{2.73}$$

Notice this defines $\overline{\partial_b^p}$ intrinsically by virtue of the fact (L_j) and (ω_j) were chosen as duals.

Of course, p had a privileged role in our construction and, in the same fashion, we can replace p with $q = 1, \ldots, p-1$, obtaining similar vector fields $L_j^q = z_q \frac{\partial}{\partial \bar{z}_j} - z_j \frac{\partial}{\partial \bar{z}_q}$, 1-forms $\omega_j^q = \frac{1}{z_q} d\bar{z}_j$ and differential operators

$$\overline{\partial_b^q} f = \sum_{j \neq q} (L_j^q f) \omega_j^q, \quad f \in C^\infty(\Omega \backslash \{z_q = 0\}, \mathbb{C}) \tag{2.74}$$

Let $r, s \in \{1, \ldots, p\}$ and $z_r, z_s \neq 0$, then

$$(\overline{\partial_b^r} - \overline{\partial_b^s}) f = \sum_j \frac{1}{z_r} \left(z_r \frac{\partial f}{\partial \bar{z}_j} - z_j \frac{\partial f}{\partial \bar{z}_r} \right) d\bar{z}_j - \frac{1}{z_s} \left(z_s \frac{\partial f}{\partial \bar{z}_j} - z_j \frac{\partial f}{\partial \bar{z}_s} \right) d\bar{z}_j \tag{2.75}$$

$$= \left(\frac{1}{z_q} \frac{\partial f}{\partial \bar{z}_q} - \frac{1}{z_p} \frac{\partial f}{\partial \bar{z}_p} \right) \sum_j z_j d\bar{z}_j \tag{2.76}$$

and once again the factor $\sum_j z_j d\bar{z}_j$ vanishes in Ω.[5] Since the domains $\Omega \backslash \{z_r = 0\}$ cover Ω and agree at their intersections, we arrive at a single, globally defined operator $\bar{\partial}_b$, which leads to the complex

$$C^\infty(\Omega) \xrightarrow{\bar{\partial}_b} \Lambda^{0,1}(\Omega) \xrightarrow{\bar{\partial}_b} \Lambda^{0,2}(\Omega) \xrightarrow{\bar{\partial}_b} \cdots$$

Locally integrable structures

Definition 2.25. *A complex subbundle $\mathcal{V} \subset \mathbb{C}T\Omega$ of rank n and corank $m = N - n$ is said a* ***locally integrable structure*** *if Ω has an open covering $(U_\alpha)_\alpha$ where each bundle $\mathcal{V}^\perp \cap \mathbb{C}T^*U_\alpha$ is generated by exact differentials of local smooth functions. More precisely, for all α there are $Z_1, \ldots, Z_m \in C^\infty(U_\alpha)$ such that*

$$\operatorname{span}\langle (dZ_1)_p, \ldots, (dZ_m)_p \rangle = \mathcal{V}_p^\perp, \forall p \in U_\alpha \tag{2.77}$$

If u is a local smooth function of Ω, the definition of $\mathcal{V}^\perp$ says du is a local section of $\mathcal{V}^\perp$ iff $du(L) = Lu = 0$ for all $L \in \Gamma(\mathcal{V})$. This implies $\mathcal{V}$ must be formally integrable as

[5] notice we can include the indexes $j = r$ and $j = s$ in the expression of (2.75) without affecting the sum

well: if L, M are local sections of $\mathcal{V}$, then $[L, M]Z_j = L(MZ_j) - ML(Z_j) = 0$; since those are local generators, it follows that any local sections of $\mathcal{V}^\perp$ vanish at $[L, M]$. Therefore, $[L, M]$ is a local section of $\mathcal{V}$ as well.

Local integrability of a formally integrable structure $\mathcal{V}$ can thus be reinterpreted as a statement of existence of a maximal set of solutions: for each homogeneous problem defined by a local set of generators of $\mathcal{V}$, a full set of m independent local solutions exists. From this viewpoint, we arrive at the following characterization.

Proposition 2.26. *A formally integrable structrure $\mathcal{V}$ is locally integrable iff for any $p \in \Omega$ and $L_1, \ldots, L_n$ local generators of $\mathcal{V}$ we have $U_\alpha \ni p$ such that $\exists Z_1, \ldots, Z_m \in C^\infty(U_\alpha)$ with*

$$dZ_1 \wedge \ldots \wedge dZ_m \neq 0 \text{ everywhere in } U_\alpha \tag{2.78}$$

$$\text{and } L_j Z_k = 0 \quad \forall j, k \tag{2.79}$$

Local generators

There are many possible choices of local coordinates on Ω and local generators for $\mathcal{V}$ and $\mathcal{V}^\perp$. By using a structural decomposition of complex vector subspaces in $\mathbb{C}^n$ ([BCH08, p. 17, Lemma 8.5]) one can choose local generators in a standard simplified form. Naturally, it is only in the case of locally integrable structures that we can induce coordinates on $\mathcal{V}$ and $\mathcal{V}^\perp$ simultaneously.

Remark 2.27. *With the exception of CR structures, all other types of formally integrable structures presented in Definition 2.23 are locally integrable as well. The fact that complex structures are locally integrable is a key result known as the Newlander-Nirenberg theorem. There are known explicit examples of CR structures that are not locally integrable, but the proofs are quite involved (see [BCH08, Section I.16]).*

We will avoid the technicalities and present the result of interest right away.

Theorem 2.28 ([BCH08, Corollary I.10.2])**.** *Let $\mathcal{V}$ be a locally integrable structure of rank n and corank m over Ω. Then given $p \in \Omega$ there is a coordinate system $(t_1, \ldots, t_n, x_1, \ldots, x_m)$ from a chart $(U, (t, x))$ centered at p such that $\mathcal{V}^\perp$ is spanned by the differentials of*

$$Z_k(x, t) = x_k + i\phi_k(t, x), \quad k = 1, \ldots, m \tag{2.80}$$

near the origin for some real-valued smooth functions $\phi_k \in C^\infty_{\mathbb{R}}(U)$ satisfying

$$\phi_k(0, 0) = 0,\ d_x\phi_k(0, 0) = 0, \quad k = 1, \ldots, m \tag{2.81}$$

Writing $Z(t, x) = (Z_1(t, x), \ldots, Z_m(t, x))$, notice (2.80) and (2.81) imply $D_x Z(0, 0)$ (the Jacobian matrix restricted to the x-variables) is the $m \times m$ identity. Then, we can choose smooth complex coefficients (μ_{kl}) near the origin so as to satisfy the relations

$$M_k Z_l = \delta_{kl} \tag{2.82}$$

with the vector fields

$$M_k = \sum_{l=1}^{m} \mu_{kl}(t,x) \frac{\partial}{\partial x_l}, \quad k = 1, \dots, m \tag{2.83}$$

Now setting

$$L_j \doteq \partial_{t_j} - i \sum_{k=1}^{m} (\partial_{t_j} \phi_k) M_k, \quad j = 1, \dots, n, \tag{2.84}$$

equation (2.82) implies $L_j Z_k = \partial_{t_j} Z_k - i \partial_{t_j} M_j = 0$. Furthermore, writing the coordinate matrix for the vector fields, it is clear that $L_1, \dots, L_n, M_1, \dots, M_m$ are linearly independent and thus span $\mathbb{C}T\mathbb{R}^{m+n}$ near the origin, with $L_1, \dots, L_n$ generators for $\mathcal{V}$.

A *tube structure* $\mathcal{V}$ corresponds to the case where the ϕ_k from (2.80) does not depend on x, so we can write $Z_k = x_k + i\phi_k(t)$. Then, $D_x Z(x,t)$ is the identity and we can take $M_k = \partial_{x_k}$ to solve for (2.82). The resulting expression for the vector fields L_j then becomes

$$L_j = \partial_{t_j} - i \sum_{k=1}^{m} (\partial_{t_j} \phi_k(t)) \partial_{x_k} \tag{2.85}$$

and one can easily verify, using the coordinate expression (1.16) for the Lie bracket operation, that $L_1, \dots, L_n, M_1, \dots, M_m$ are mutually commutative. In particular, $L_1, \dots, L_n$ are local generators for an involutive structure with a corresponding complex of differential operators, as determined in Section 2.3.

We shall investigate, in the next two chapters, how questions of solvability are handled for differential complexes related to this type of structure.

3 A model with complex coefficients

3.1 Definition

The system under study is obtained from a family (P_j) of ν operators acting over generalized functions of variables $(t, x) \in \Omega \times \mathbb{R}^n$, for some $\Omega \subset \mathbb{R}^\nu$ open. Each P_j corresponds to a smooth function from t to a symbol $(x, \xi) \mapsto b_j(t, \xi)$ in $S^1(\mathbb{R}^n \times \mathbb{R}^n)$ (Definition 1.22) which is constant with respect to x. Writing $b_j(t, D_x)$ for the operator associated to the symbol $b_j(t, \xi)$, the operators (P_j) are given by

$$P_j u = (\partial_{t_j} - b_j(t, D_x))u \tag{3.1}$$
$$= \int_{\xi \in \mathbb{R}^n} e^{2\pi i x \cdot \xi}(\partial_{t_j} - b_j(t, \xi))\widehat{u}\, d\xi, \quad j = 1, \ldots, \nu \tag{3.2}$$

where $\widehat{u}$ is the partial Fourier transform of u with respect to x (as will always be the case in the present chapter).

The operators act on distributions over Ω with values in a topological vector space E, the elements of which are generalized functions of x. A few requirements will be placed over the family of symbols. For now, it should be noted each P_j is a constant coefficients vector field with respect to t and a pseudo-differential operator with respect to x.

The function spaces

To give a suitable domain for the operators (P_j), we will need spaces of p-currents which take value in a topological vector space E. This is a generalization of the construction we encountered in Example 2.4, difference being the codomain E, which was assumed to be the complex field $\mathbb{C}$ previously.

Let E be a locally convex Hausdorff TVS over the complex field, the space of E-valued p-currents over Ω is denoted $\bigwedge^p \mathcal{D}'(\Omega; E)$, or simply $\mathcal{D}'(\Omega; E)$ when $p = 0$. Starting from the latter, a 0-current is a continuous linear mapping $C_c^\infty(\Omega) \to E$, where $C_c^\infty(\Omega)$ is to be equipped with the usual inductive limit topology which turns it into a complete locally convex TVS. Now let $p > 0$, analogously to (2.8), we represent $u \in \bigwedge^p \mathcal{D}'(\Omega; E)$ by combining coefficients in $\mathcal{D}'(\Omega; E)$ with alternating tensors in the variable t, that is,

$$u = \sum_{[J]=p} u_J dt_J, \quad u_J \in \mathcal{D}'(\Omega; E) \tag{3.3}$$

Again, an alternate description is possible. There is a bilinear mapping $\bigwedge^p \mathcal{D}'(\Omega; E) \times \bigwedge^{\nu-p} C_c^\infty(\Omega; E) \to E$, which comes from the dual pairing $\mathcal{D}'(\Omega; E) \times C_c^\infty(\Omega; E) \to E$ defined *mutatis mutandis* as in (2.9). The space of p-currents with values over E is then naturally identified with the space of continuous linear mappings $\bigwedge^{\nu-p} C_c^\infty(\Omega; E) \to E$.

The value spaces

Let us move on to the definition of the value spaces, referred to as E so far. The Fourier-transform on the variable x will be the basic starting point of all our investigations, but obtaining solutions requires not only the Fourier transform to be well-defined, but also a condition of local integrability allowing Fourier inversion. As usual, this is a problem of asymptotic decay. We can set those difficulties aside for now without losing any generality by considering the problem in a space of currents valued in a formal space of generalized functions, characterized by being mapped homeomorphically to $L^1_{\text{loc}}(\mathbb{R}^n)$ through the Fourier transform. Such value space will be suggestively denoted $\mathcal{F}^{-1}L^1_{\text{loc}}(\mathbb{R}^n)$, or $\mathcal{F}^{-1}L^1_{\text{loc}}$ for short. We qualify them as formal because they lack the properties of localization commonly known in the setting of distributions.

Let $(\mathcal{S}(\mathbb{R}^n), \tau)$ be the topological space of Schwartz functions in $\mathbb{R}^n$ with the subspace topology induced by $L^1_{\text{loc}}(\mathbb{R}^n)$ (see Definition 1.12). Since the Fourier transform is bijective from $\mathcal{S}$ to $\mathcal{S}$ (Proposition 1.15), the topology $\mathcal{F}^{-1}\tau = \{\mathcal{F}^{-1}(U) : U \in \tau\}$ makes $\mathcal{F} : (\mathcal{S}(\mathbb{R}^n), \mathcal{F}^{-1}\tau) \to (\mathcal{S}(\mathbb{R}^n), \tau)$ a homeomorphism. The topological space $E \doteq \mathcal{F}^{-1}L^1_{\text{loc}}(\mathbb{R}^n)$ is then defined as the completion of $(\mathcal{S}(\mathbb{R}^n), \mathcal{F}^{-1}\tau)$. Given $\mathcal{S}(\mathbb{R}^n) \subset L^1_{\text{loc}}(\mathbb{R}^n)$ is a dense subset and $L^1_{\text{loc}}(\mathbb{R}^n)$ complete ([Maz11, p.2]), the Fourier transform extends to a linear homeomorphism $\mathcal{F} : \mathcal{F}^{-1}L^1_{\text{loc}}(\mathbb{R}^n) \to L^1_{\text{loc}}(\mathbb{R}^n)$.[1]

Later on, we replace $E = L^1_{\text{loc}}$ with Sobolev spaces and define the type of solvability we intend to verify. Then, we must determine if the formal solutions obtained at first are appropriate.

3.2 Construction of the differential complex

Let $\mathbb{P} = \{P_1, \ldots, P_\nu\}$ be the family of pseudo-differential operators introduced in (3.1). We would like to construct a corresponding differential complex like the one we made for involutive systems in Section 2.3. Of course, the results obtained there are not readily available to us because the domain differs and our operators are no longer differential. Still, the same genera approach can be repeated. Referring to Example 2.4, we aim to construct a differential complex $(\mathbb{P}^{(p)}, \bigwedge^p \mathcal{D}'(\Omega; \mathcal{F}^{-1}L^1_{\text{loc}}))_p$ where $p = 0$ encodes the overdetermined system of equations $P_j u = f_j,\ j = 1, \ldots, \nu$.

[1] Since the spaces L^1_{loc} are non-metrizable, the completion invoked in our construction only makes sense in the context of topological vector spaces. This is no issue here: if a TVS is Hausdorff, the existence and uniqueness (up to an isomorphism) theorem of the completion says just as much as in the metric case. See [Tre67, p. 41 Theorem 5.2] for details.

Write $u \in \bigwedge^p \mathcal{D}'(\Omega; \mathcal{F}^{-1}L^1_{\text{loc}})$ with respect to its coefficients

$$u = \sum_{|J|=p} u_J dt_J, \quad u_J \in \mathcal{D}'(\Omega; \mathcal{F}^{-1}L^1_{\text{loc}}) \tag{3.4}$$

then the family of pseudo-differential maps b_j combines into chain maps $b(t, D_x)\wedge$: $\bigwedge^\bullet \mathcal{D}'(\Omega; \mathcal{F}^{-1}L^1_{\text{loc}}) \to \bigwedge^{\bullet+1} \mathcal{D}'(\Omega; \mathcal{F}^{-1}L^1_{\text{loc}})$ which act on u by integration

$$b(t,\xi) \wedge u = \sum_{j=1}^{\nu} \sum_{|J|=p} (b_j(t,\xi) \wedge u_J) dt_j \tag{3.5}$$

The evident choice of $\mathbb{P}^{(p)} : \bigwedge^p \mathcal{D}'(\Omega; \mathcal{F}^{-1}L^1_{\text{loc}}) \to \bigwedge^{p+1} \mathcal{D}'(\Omega; \mathcal{F}^{-1}L^1_{\text{loc}})$ fitting our requirements is

$$\mathbb{P} = d_t + b(t, D_x) \wedge \cdot \tag{3.6}$$

where d_t is the exterior derivative with respect to the variable t only. Parallel to that, it is useful to consider the action of $\mathbb{P}$ on the frequency domain of x. Set $\widehat{\mathbb{P}} : \bigwedge^p \mathcal{D}'(\Omega; L^1_{\text{loc}}) \to \bigwedge^{p+1} \mathcal{D}'(\Omega; L^1_{\text{loc}})$ with

$$\widehat{\mathbb{P}} = d_t + b(t,\xi) \wedge \cdot \tag{3.7}$$

then $\mathbb{P}$ and $\widehat{\mathbb{P}}$ are related by

$$\widehat{\mathbb{P}u} = \widehat{\mathbb{P}}\hat{u}, \quad \hat{u} \in \bigwedge^p \mathcal{D}'(\Omega; \mathcal{F}^{-1}L^1_{\text{loc}}) \tag{3.8}$$

We will also adhere to the notations $\mathbb{P}_b$ and $\widehat{\mathbb{P}}_b$ whenever we wish to make the dependence of b explicit.

It should be verified if the definition above satisfies the chain complex condition. The same computation done in Lemma 2.14 for complexes of vector fields shows $b(t, D_x) \wedge b(t, D_x) = 0$. Combining this fact with the axiomatic properties of Proposition 1.9 for d_t, the composition of successive operators in the sequence is [2]

$$\begin{aligned}
\mathbb{P}^{(p+1)} \circ \mathbb{P}^{(p)} &= (d_t + b(t, D_x) \wedge \cdot)(d_t + b(t, D_x) \wedge \cdot) && (3.9)\\
&= d_t \circ d_t + d_t(b \wedge \cdot) + b \wedge d_t + b \wedge b && (3.10)\\
&= d_t b \wedge \cdot - b \wedge d_t + b \wedge d_t = d_t b \wedge \cdot && (3.11)
\end{aligned}$$

Thus, $\mathbb{P}$ makes a complex iff $d_t b(t, D_x) = 0, \forall t \in \Omega$. In terms of $b(t,\xi) = \sum_j b_j(t,\xi) dt_j$, the same condition reads

$$(d_t b(t, D_x) \wedge u)(t,x) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} d_t b(t,\xi) \wedge \widehat{u}(t,\xi) d\xi = 0, \forall u \tag{3.12}$$

[2] the truncated formulas shown stand in accordance with the rather lengthy expressions one gets by evaluating $\mathbb{P}^{(p+1)} \circ \mathbb{P}^{(p)}(u)$ from the definitions

so we conclude (3.6) defines a complex iff for all $t \in \Omega$, $b(t,\xi)$ is a closed 1-form for almost every $\xi \in \mathbb{R}^n$.

The main achievement of [Tre76] (which came up with the model) was to establish a necessary and sufficient condition of solvability of a semi-global type (more on that later) for the equations

$$\mathbb{P}_b u = f, \quad f \in \bigwedge^p C^\infty(\Omega; H^\infty) \tag{3.13}$$

with f under suitable compatibility conditions and b with a few further requirements. We are going to state the problem in full generality, although our goal is only to show a proof of sufficiency specific to the case $p = 0$.

Requirements on the symbol $b(t,\xi)$

The most important assumption we make on b is exactness of $b(t,\xi)$. Beyond fulfilling the complex condition, that allows us to work with a primitive $B(t,\xi)$ in what follows.

If exactness is assumed, a primitive B can be obtained by integration component-wise: let Ω_0 be a connected component of Ω and take $p_0 \in \Omega_0$ fixed. For $t \in \Omega_0$ let

$$B(t,\xi) = \int_{\gamma(p_0,t)} b(s,\xi) \tag{3.14}$$

where $\gamma(p_0,t)$ is any path going from p_0 to t in Ω.

The remaining requirements are stated by means of B. It should be noted, however, that a primitive B fulfills them iff any other choice of primitive does, and in that sense the requirements are still set upon b. The full set of premises is the following:

(i) for a.e. $\xi \in \mathbb{R}^n, b(\cdot,\xi)$ is exact

(ii) Let B be a primitive of b, then $B(t,\xi) = B^0(t,\xi) + R(t,\xi)$ for some B^0, R such that, $\forall t \in \Omega$, $B(t,\cdot)$ is positive homogeneous of degree one and $R(t,\cdot) \in L^\infty(\mathbb{R}^n)$

(iii) Both B^0 and R are C^∞ with respect to $t \in \Omega$

(iv) The restriction of $B^0(t,\cdot)$ to $S^{n-1} = \{\xi : |\xi| = 1\}$ is a function of class C^1

The properties we impose on B (existence of B^0 and R in (ii) and regularity of (iii) and (iv)) imply analogous conditions on coefficients b_j of the model. Indeed, $d_t B = d_t B^0 + d_t R$ leads to a set of equations

$$b_j(t,\xi) = b_j^0(t,\xi) + r_j(t,\xi), \quad j = 1,\dots,n \tag{3.15}$$

where we take $d_t B^0 = \sum b_j^0 dt_j$ and $d_t R = \sum r_j dt_j$. In particular, the coefficients are still C^∞ on t and C^1 on ξ.

3.3 Reduction on the coefficient b and compatibility conditions

Write the homogeneous part of B as

$$B^0 = B_1^0 + iB_2^0 \doteq \operatorname{Re} B^0 + i \operatorname{Im} B^0 \tag{3.16}$$

We define a family $(U_t)_{t\in\Omega} \subset \mathcal{L}(E), E \in \{H^\infty(\mathbb{R}^n), H^{-\infty}(\mathbb{R}^n)\}$ of operators

$$\langle U_t, v\rangle(x) = \int e^{i(2\pi x\cdot\xi - B_2^0(t,\xi)) - R(t,\xi)} \widehat{v}(\xi) d\xi, \quad v \in E \tag{3.17}$$

which are continuous extensions of certain classes of transformations known as *Fourier integral operators*.

In a basic prototypical version, a Fourier integral operator T acts as a linear endomorphism on Schwartz functions $f \in \mathcal{S}(\mathbb{R}^N)$ by

$$(Tf)(x) = \int_{\mathbb{R}^N} e^{2\pi i \Phi(x,\xi)} a(x,\xi) \widehat{f}(\xi) d\xi \tag{3.18}$$

Here, the functions a and Φ are what we call the amplitude and phase of T, respectively. Well-definiteness depends upon the choice of a symbol class for $a(x,\xi)$ (it could be one of the symbol spaces $S^m(\mathbb{R}^N \times \mathbb{R}^N)$ we introduced in Definition 1.22, for example) and further conditions on the real-valued phase function Φ, which is usually homogeneous of degree 1. See [SM93, §IX.3] for an introductory treatise on the subject.[3]

Of course, familiar examples of such mappings are obtained from $\Phi(x,\xi) = x\cdot\xi$, when each choice of a corresponds to a pseudo-differential operator with the same symbol. Far more general classes of operators arise naturally as solutions in the study of hyperbolic equations, perhaps most famously in the context of the wave equation.

Going back to (3.17), we may regard each U_t as a Fourier integral operator with $\Phi_t(x,\xi) = x\cdot\xi - B_2^0(t,\xi)$ and $a_t(x,\xi) = e^{-R(t,\xi)}$. By doing so, we obtain Φ_t homogeneous of degree 1 and the boundedness of R with respect to ξ provides straightforward estimates

$$\left| \frac{\partial_\xi^\beta \partial_x^\alpha a_t(x,\xi)}{(1+|\xi|)^{|\alpha|}} \right| \leqslant |r(t,\xi)^\alpha e^{-R(t,\xi)}| \leqslant \sup |r(t,\cdot)|^\alpha e^{\sup |R(t,\cdot)|} \tag{3.20}$$

showing a_t is in the symbol class $S^0(\mathbb{R}^N \times \mathbb{R}^N)$. Taking $v \in H^s$, for each $t \in \Omega$ fixed,

$$\langle U_t, v\rangle \in H^s \iff \Lambda_s(U_t v) \in L^2 \tag{3.21}$$

[3] The cited author develops the theory for a class of Fourier integral operators such that the symbol $a(x,\xi)$ is compactly supported in x and such that, for each $(x,\xi) \in \operatorname{supp} a \cap \{(x,\xi) : \xi \neq 0\}$, Φ is a smooth function satisfying the nondegeneracy condition

$$\det\left(\frac{\partial^2 \Phi}{\partial x_i \partial \xi_j}\right) \neq 0 \tag{3.19}$$

(c.f. (1.53)). Considering the Fourier coefficients made evident in (3.17),

$$\|U_t v\|_{H^s} = \|\Lambda_s(U_t v)\|_2 = \left| \int e^{-2R_t}(1+|\xi|^2)^s |\widehat{v}|^2 d\xi \right|^{1/2} \tag{3.22}$$

$$\leqslant \sup \left| e^{-R(t,\cdot)} \right| \|v\|_{H^s} < \infty \tag{3.23}$$

so U_t acts boundedly and invariantly on each Sobolev space H^s, $s \in \mathbb{R}$. Since s is arbitrary, the same considerations hold for $v \in H^{\pm\infty}$.

Similarly, one can define $U_t^{-1} \in \mathcal{L}(H^s)$ by letting

$$\langle U_t^{-1}, v \rangle = \mathcal{F}\{\widehat{v}(\xi) e^{iB_2^0(t,\xi)+R(t,\xi)}\} \tag{3.24}$$

Rewriting (3.17) in the manner of (3.24), it is immediate that

$$U_t \circ U_t^{-1} = \mathcal{F} \circ \mathcal{F}^{-1}, \quad U_t^{-1} \circ U_t = \mathcal{F}^{-1} \circ \mathcal{F} \tag{3.25}$$

so the fact U_t and U_t^{-1} are inverses is a simple restatement of the Fourier inversion formula. Hence, U_t is an automorphism on each of the Sobolev spaces considered so far.

To bring it to use in our problem, the family of automorphisms $(U_t) \subset \mathrm{Aut}(E)$ of (3.17) must be reinterpreted as a single automorphism over $\bigwedge^p \mathcal{D}'(\Omega; E)$. Accordingly, we let

$$\langle U, v \rangle (t,x) \doteq \int e^{i(2\pi i x \cdot \xi - B_2^0(t,\xi)) - R(t,\xi)} \widehat{v}(t,\xi) d\xi \tag{3.26}$$

for $v \in \bigwedge^p \mathcal{D}'(\Omega; E)$. Notice U acts on $v(t,x)$ as a pseudo-differential operator for fixed x and as a Fourier integral operator for fixed t. If $\mathcal{L}(E)$ is endowed with the strong operator topology, then (U_t) is continuous respect to t. Finally, for reasons I do not understand, the original author states in [Tre77] that the cases $E \in \{H^{\pm\infty}\}$ are distinguished from the remaining ones, that is, $E = H^s$ with $s \in \mathbb{R}$, by the fact $t \mapsto U_t$ is smooth in former setting.

The remarks made so far lead to the following.

Proposition 3.1 ([Tre76, Prop.I.3.1]). *Let $E = \{H^{\pm\infty}\}$, the assignments $u \mapsto \langle U_t, u \rangle$ with formula (3.17) are automorphisms of E depending smoothly on $t \in \Omega$. Furthermore, they induce automorphisms $U \in \mathrm{Aut}(\bigwedge^p \mathcal{D}'(\Omega; E))$ by means of (3.26) for each level $p = 0, \ldots, \nu$. Finally, restrictions of U to the subspaces $\bigwedge^p C^\infty(\Omega; E)$ and $\bigwedge^p C_c^\infty(\Omega; E)$ are automorphisms as well.*

The whole reason for the preceding discussion is to allow a simplifying assumption about the coefficient b. Let $v \in \bigwedge^p \mathcal{D}'(\Omega; H^{\pm\infty})$, we compute the result of Uv when acted upon by $\mathbb{P}$ –

$$\mathbb{P}(Uv) = d_t(Uv) + b(t, D_x) \wedge (Uv) = \int e^{i(2\pi x \cdot \xi - B_2^0) - R}(-i \operatorname{Im} b_0 - r) \wedge \hat{v} \, d\xi \tag{3.27}$$

$$+ \int e^{i(2\pi x \cdot \xi - B_2^0) - R} d_t \widehat{v} \, d\xi + b(t, D_x) \wedge (Uv) \tag{3.28}$$

By definition,

$$b(t, D_x) \wedge (Uv) = \int e^{2\pi i x\cdot\xi} b(t,\xi) \wedge \widehat{Uv}\, d\xi \tag{3.29}$$

so considering $\langle U, v\rangle = \mathcal{F}^{-1}\{\widehat{v}(\xi)e^{-iB_2^0 - R}\}$, we have $\widehat{Uv} = \widehat{v}(\xi)e^{-iB_2^0-R}$ and thus, in conclusion,

$$\mathbb{P}(Uv) = \int e^{i(2\pi x\cdot\xi - B_2^0)-R}(d_t\widehat{v} + \operatorname{Re} b_0 \wedge \widehat{v})\, d\xi \tag{3.30}$$

Notice the expression in parenthesis inside the integral is none other than the operator $\widehat{\mathbb{P}}_{\operatorname{Re} b_0}$ applied to $\widehat{v}$, as we defined in (3.7). Since U is invertible, we may express $\mathbb{P}_b$ and $\mathbb{P}_{\operatorname{Re} b_0}$ as conjugates of each other:

$$\mathbb{P}_b(Uv) = U(\mathbb{P}_{\operatorname{Re} b_0} v) \implies \mathbb{P}_{\operatorname{Re} b_0} = U^{-1}\mathbb{P}_b U \tag{3.31}$$

This simplifies our problem considerably. A solution u to the equation $\mathbb{P}u = f$ is equivalent to a solution v to $\mathbb{P}_{\operatorname{Re} b_0} v = g$ with the bijective associations $v = U^{-1}u$ and $g = U^{-1}f$. To be sure,

$$\mathbb{P}u = f \iff \mathbb{P}(Uv) = f \iff \mathbb{P}_{\operatorname{Re} b_0} v = (U^{-1}\mathbb{P}U)v = g \tag{3.32}$$

Hence, to the end of determining solvability, it suffices to consider $B = B_1^0$, that is, both b and B can be assumed real-valued and positively homogeneous of degree 1 on $\xi \in \mathbb{R}^n$ (and we do so from now on).

Compatibility conditions

Having $b(\cdot,\xi)$ exact with primitive $B(\cdot,\xi)$, the differential operator defined in (3.7) can be rewritten

$$\widehat{\mathbb{P}} = e^{-B} d_t(e^B \cdot) \tag{3.33}$$

so $d_t \circ d_t = 0$ implies $(\widehat{\mathbb{P}}, \bigwedge^p \mathcal{D}'(\Omega; L^1_{\mathrm{loc}}))$ defines a chain complex on its own right. We look for minimal conditions on the x-Fourier transform of f which make a solution as in (3.13) feasible. Suppose given f there exists $u \in \mathcal{D}'(\Omega; \mathcal{F}^{-1}L^1_{\mathrm{loc}}(\mathbb{R}^n))$ such that $\mathbb{P}u = f$. Then since

$$\mathbb{P}u = f \iff \widehat{\mathbb{P}}\widehat{u} = \widehat{f} \text{ a.e. } \xi \tag{3.34}$$

it follows from expression (3.33) that

$$e^{B(\cdot,\xi)}\widehat{f}(\cdot,\xi) = d_t(\widehat{u}(\cdot,\xi)) \tag{3.35}$$

In other words, if f can be solved for, then $e^{-B(\cdot,\xi)}\widehat{f}(\cdot,\xi)$ is exact, by necessity. This motivates the following.

Definition 3.2. *A $(p+1)$-current $f \in \bigwedge^{p+1} C^\infty(\Omega; H^\infty)$ is said* ***compatible*** *if $e^{B(\cdot,\xi)}\hat{f}(\cdot,\xi)$ is t-exact for almost all $\xi \in \mathbb{R}^n$. The set of all $f \in \bigwedge^{p+1} C^\infty(\Omega; H^\infty)$ such that f is compatible will be denoted $\mathbb{E}_b^p$.*

Remark 3.3. *Conversely, if f is compatible, $\mathbb{P}u = f$ has a solution $u \in \bigwedge^p \mathcal{D}'(\Omega; \mathcal{F}^{-1}L^1_{loc})$. Indeed, for almost every ξ fixed we can integrate (3.35) (as we shall do soon) with respect to t. This leads to some $\widehat{u} \in \bigwedge^p \mathcal{D}'(\Omega; L^1_{loc})$ satisfying $\widehat{\mathbb{P}}\widehat{u} = \widehat{f}$ a.e. ξ. By (3.34) we have a formal solution.*

Now that solutions in the formal setting are thoroughly characterized, we turn to the problem of actual interest, which is solvability in the framework of distributions. As mentioned before, distributions can be localized in their appropriate domain, contrary to the functions in $\mathcal{D}'(\Omega; \mathcal{F}^{-1}L^1_{\mathrm{loc}})$ we considered so far. Making the value spaces Sobolev functions of x, we can localize distributions on $\mathbb{R}^n$ and thus make sense of semi-global solvability.

Definition 3.4. *Let Ω' be an open subset of Ω, then $\mathbb{P}^{(p)} = d_t + b(t, D_x) \wedge \cdot$ is said*

(i) semi-globally solvable with respect to Ω', if for any $U' \Subset \Omega'$,

$$\forall f \in \mathbb{E}_b^p,\ \exists u \in \bigwedge^p \mathcal{D}'(U'; H^{-\infty}) \text{ such that } \mathbb{P}u = f \text{ in } U' \tag{3.36}$$

(ii) smoothly semi-globally solvable with respect to Ω', if for any $U' \Subset \Omega'$,

$$\forall f \in \mathbb{E}_b^p,\ \exists u \in \bigwedge^p \mathcal{D}'(U'; H^{\infty}) \text{ such that } \mathbb{P}u = f \text{ in } U' \tag{3.37}$$

3.4 Condition (ψ) and proper decay on the coefficients

To better understand the phenomena at play, we investigate the conditions of solvability which arise when $\nu = 1$, $p = 0$ and Ω is connected. The inhomogeneous term f will then be a 1-form, which can be naturally identified with a scalar function. Doing so (and replacing d_t with ∂_t), we have a single pseudo-differential equation to solve for –

$$(\partial_t + b(t, D_x))u = f \tag{3.38}$$

Fourier transforming on x leads to a family of ODEs

$$(\partial_t + b(t, \xi))\hat{u} = \hat{f} \tag{3.39}$$

varying with $\xi \in \mathbb{R}^n$. Referring once again to (3.35), we have $\partial_t(e^B\hat{u}) = e^B\hat{f}$, so we can integrate with respect to t to solve for the frequency-domain function

$$\widehat{u}(t, \xi) = e^{-B(t,\xi)} \int_{t_0(\xi)}^t e^{B(s,\xi)} \hat{f}(s, \xi) ds \tag{3.40}$$

with each $t_0(\xi) \in \Omega$ a base point of our choice.

To be able to reconstruct a solution $u \in \mathcal{D}'(\Omega; H^{\pm\infty})$, the function of ξ defined by (3.40) must decay like a tempered distribution, which amounts to appropriate bounds on the growth of its derivatives. To this end, the behaviour of the real exponent $e^{B(s,\xi)-B(t,\xi)}$ can be a problem. Unless we can choose starting points $t_0(\xi)$ such that $B(t,\xi) - B(s,\xi)$ is bounded below for s in the interval of integration (and uniformly on ξ), we won't have the appropriate estimates to ensure we have a solution in the sense of distributions.

Say we wish to choose $t_0(\xi)$ so that

$$B(t,\xi) - B(s,\xi) \geqslant 0 \tag{3.41}$$

for each s between $t_0(\xi)$ and t. In this case, it is quite easy to come up with a condition which is both necessary and sufficient to allow such choice. They are given in a couple different ways below.

Proposition 3.5. *Suppose $\nu = 1$ and $U' = (a,b) \Subset \Omega$, then the following are equivalent*

(i) there exists $t_0 : \mathbb{R}^n \to \overline{U'}$ so that, for all $(t,\xi) \in U' \times \mathbb{R}^n$, $B(t,\xi) - B(s,\xi) \geqslant 0$ whenever s lies between $t_0(\xi)$ and t;

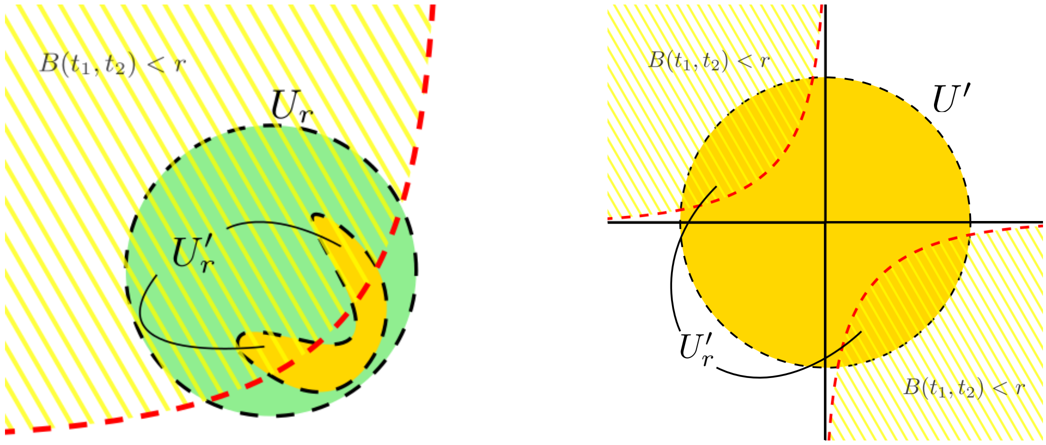
(ii) for every $(t,\xi) \in U' \times \mathbb{R}^n$, $b(t,\xi) > 0$ implies $b(s,\xi) \geqslant 0$ for every $s > t$;

(iii) for every $t \in U'$ and $r \in \mathbb{R}$, the sublevel sets $\{t \in U' : B(t,\xi) < r\}$ are connected.

Proof. Let ξ be fixed and $\alpha = B(\cdot,\xi)$. We can read the statement of (i) separately for the cases $t \leqslant t_0(\xi)$ and $t_0(\xi) \geqslant t$. In the former, (i) says α is non-increasing in $(a, t_0(\xi)]$ while in the latter it says α is non-decreasing in $[t_0(\xi), b)$. Consequently, $t_0(\xi)$ is necessarily a point of global minimum for α with derivative α' non-positive for $t \leqslant t_0(\xi)$ and non-negative elsewhere. Implication $(i) \implies (ii)$ then readily follows. With those same considerations, we can see the pre-images of $(-\infty, r)$ by α restricted to the monotonic intervals are both convex with common point $t_0(\xi)$ (if any) so $\{t \in U' : B(t,\xi) < r\}$ is again an interval and we conclude (iii) follows from (i).

The converse statements follow just as easily. Since this is a simple illustration for the more general case, we leave them without a proof. □

It turns out that the statement of (iii), with its condition of connectivity for sublevel sets of $B(\cdot,\xi)$, is better suited for a generalization to higher dimensions.



(a) Some $r \in \mathbb{R}$ split the precompact region U' into a disconnected set U'_r. However, there is $U \Subset \Omega$ containing U' such that each component of U'_r falls within the same component of $U_r = \{t \in U : B(t) < r\}$.

(b) Some $r \in \mathbb{R}$ split U' into disconnected regions. Since the sublevel $\{t : B(t) < r\}$ is itself disconnected, no choice of $U \Subset \Omega$ can possibly make the components of U'_r fall within a single component of U_r.

Figure 1 – Example and non-example for condition (ψ)

Condition (ψ)

For the remaining part of this section, we keep a shorthand for the sublevel sets determined by B. If $U \subset \Omega$ is open and $(\xi, r) \in \mathbb{R}^n \times \mathbb{R}$, we denote

$$U(\xi, r) \doteq \{t \in U : B(t, \xi) < r\} \tag{3.42}$$

Taking into account the formulation of Proposition 3.5 (iii), as well as the semi-global character of the solutions under consideration, we make the following generalization.

Definition 3.6. *Let $\Omega' \subset \Omega$ be an open connected set. Then Ω' satisfies condition (ψ) if it fits the following:*

For all $U' \Subset \Omega'$, there exists $U \Subset \Omega$ with $U' \subset U$ such that, for any $(\xi, r) \in \mathbb{R}^n \times \mathbb{R}$, the intersection $U'(\xi, r) \cap \Omega'$ is contained in a single component of $U(\xi, r)$.

Example 3.7. *Consider $\Omega = \mathbb{R}^2$ and B constant in ξ given by $B(t_1, t_2) = t_1 \cdot t_2$. Then $\Omega' = \{(x, y) : x, y < 0\}$ satisfies (ψ) in Definition 3.6, but $\Omega' = \mathbb{R}^2$ does not.*

Consider $U' \Subset \Omega$ given and $U'_r = \{t \in U' : B(t) < r\}$ for each $r \in \mathbb{R}$. Figure 1(a) illustrates how, for a given a non-convex pre-compact set U' (in golden), there exists $U \Subset \Omega$ (in green) as (ψ) requires. By contrast, no corresponding U fitting the requirement can be chosen for U' in Figure 1(b).

As our remarks in the proof of Proposition 3.5 suggest, the ideal choice of base points $t_0(\xi)$ to produce an appropriate bound on $B(t, \xi) - B(s, \xi)$ involves making $t_0(\xi)$ a point of minimum for $B(\cdot, \xi)$. We follow this same approach for the general case, but in doing so, we need to make sure some choice of $t_0(\xi)$ turns out measurable.

Lemma 3.8. *Let $K_1 \subset \mathbb{R}^\nu, K_2 \subset \mathbb{R}^n$ be compact subsets and $B : K_1 \times K_2 \to \mathbb{R}$ a continuous function.*

Then, there exists a measurable function $t^* : K_2 \to K_1$ such that

$$B(t^*(\xi), \xi) = \inf_{K_1} B(\cdot, \xi), \quad \forall \xi \in K_2 \tag{3.43}$$

Proof. Given B is continuous in a compact domain, $\alpha : \xi \mapsto \min_{K_1} B(\cdot, \xi)$ must be continuous as well. Now suppose the Lemma holds in the special case where the function B_0 non-negative and such that

$$\forall \xi \in K_2, \exists t \in K_1 \text{ with } B(t, \xi) = 0 \tag{3.44}$$

Then, given any B continuous, $B - \alpha$ is continuous, non-negative and also reaches a value of 0 for any $\xi \in K_2$ fixed. Therefore, our assumption implies the existence of t^* measurable such that $\forall \xi \in K_2, B(t^*(\xi), \xi) - \alpha(\xi) = 0$, which is precisely condition (3.43) with respect to B. This means it suffices to construct t^* for B in the special case to conclude.

As a further simplification, we will assume K_1 is a cartesian cube. Indeed, by Tietze's extension theorem, B can be continuously extended to a cube $Q \supset K_1$ where it remains non-negative. Adding to this extension a function which vanishes on K_1 and is strictly positive on $Q \backslash K_1$ (known to exists by Urysohn's lemma), we retain $\inf_Q B(\cdot, \xi) = \inf_{K_1} B(\cdot, \xi) = 0$, so there is no issue taking $Q = K_1$ from the start.

We now prove the existence of t^* by an inductive argument on ν. When $\nu = 1$, one has K_1 an interval and it suffices to let $t^*(\xi) = \inf_{K_1}\{t : B(t, \xi) = 0\}$, as that gives an upper semicontinuous function (Definition 1.20), which is clearly measurable.

Let $\rho_\nu : K_1 \to \mathbb{R}^{\nu-1}$ and $\pi_\nu : K_1 \to \mathbb{R}$ be the coordinate projections such that $\rho_\nu \times \pi_\nu$ is the identity on K_1. For $\nu > 1$, analogously to the case $\nu = 1$, we have semicontinuous scalar mappings $t_\nu : K_2 \to \pi_\nu(K_1)$, which are given by

$$t_\nu(\xi) = \inf_{t \in K_1} \{\pi_\nu(t) : B(t, \xi) = 0\} \tag{3.45}$$

Again, they are measurable and will be used in the construction of t^*.

To that end, we decompose K_2 into a disjoint countable union $\bigsqcup_j S_j \sqcup T$ which attains the following:

(i) The sets in (S_j) are compact

(ii) T has Lebesgue measure $\mu(T) = 0$

(iii) for each j, the restriction of t_ν to S_j is continuous

Those can be met by bringing together the compact sets obtained through an inductive iteration of Lusin's theorem (Theorem 1.21). Start by letting $\epsilon_1 = 1$, Lusin's theorem gives a compact $S_1 \subset K_2$ with $\mu(K_2 \backslash S_1) < \epsilon_1$ where $t_\nu|_{S_1}$ is continuous. Inductively, at the j-th step, we let $\epsilon_j = 2^{-j}$ and consider t_ν restricted to $K_2 \backslash \bigsqcup_{l=1}^{j-1} S_l$. Then, Lusin's theorem provides an S_j compact and disjoint from $\bigcup_{l=1}^{j-1} S_l$ where t_ν restricts to a continuous function and $\mu(K_2 \backslash \bigsqcup_{l=1}^{j} S_l) < \epsilon_j$. The remaining part of K_2, not covered by any S_j, will have measure 0: let $T = K_2 \backslash \bigcup_{l=1}^{\infty} S_j$, then the standard continuity property of measures for decreasing sequences (with the fact $\mu(K_2) < \infty$) implies

$$\mu(T) = \mu\left(\bigcap_j K_2 \backslash S_j\right) = \lim_j \mu\left(K_2 \backslash \bigcup_{l=1}^{j-1} S_l\right) = \lim \epsilon_j = 0 \tag{3.46}$$

so we accomplish what we intended.

Having that, notice for each $j \in \mathbb{N}$ the mapping

$$B_j : \rho_\nu(K_1) \times S_j \to \mathbb{R} \tag{3.47}$$

$$(\widetilde{t}, \xi) \mapsto B(\widetilde{t}, t_\nu(\xi), \xi) \tag{3.48}$$

is continuous and $\rho_\nu(K_1) \subset \mathbb{R}^{\nu-1}, S_j \subset \mathbb{R}^n$ are compact, so the inductive hypothesis gives measurable functions $\widetilde{t}_j^* : S_j \to \rho_\nu(K_1)$ such that

$$\forall \xi \in K_2, B(\widetilde{t}_j^*(\xi), t_\nu(\xi), \xi) = 0 \tag{3.49}$$

Then it suffices to take

$$t^*(\xi) = \begin{cases} (\widetilde{t}_j^*(\xi), t_\nu(\xi)), & \text{if } \xi \in S_j, \\ \text{arbitrarily in } \{t : B(t,\xi) = 0\} & \text{if } \xi \in T \end{cases} \tag{3.50}$$

This makes $B(t^*(\xi), \xi) = 0$ for all ξ and, given a measurable $M \subset K_2$

$$t^{*-1}(M) = \{\xi \in T : t^*(\xi) \in M\} \cup \bigcup_j \widetilde{t}_j^{*-1}(M) \cap t_\nu^{-1}(M) \tag{3.51}$$

is again measurable due to the basic properties concerning measurable functions and sets. □

Remark 3.9. *Making t^* upper semicontinuous in each entry would work if not for the fact such choice is not generally possible beyond the base case $\nu = 1$, as the following example illustrates:*

Let $B : [-1,1]^2 \times [-1,1] \to \mathbb{R}$ be a continuous non-negative function with

$$\{(t,\xi) : B(t,\xi) = 0\} = \{(1,0,\xi) : \xi \leqslant 0\} \cup \{(0,1,\xi) : \xi \geqslant 0\} \tag{3.52}$$

then t^* is uniquely determined by (3.43) except at $\xi = 0$, where the value of t^* is either $(1,0)$ *or* $(0,1)$. *In the first case, the second entry t^* fails to be upper semicontinuous, while in the other, the first entry does so.*

Going from the case $\nu = 1$ to higher dimensions, we are met with new difficulties. We can still solve for the Fourier coefficients with a line integral from $t_0(\xi)$ to t, but our choice of integration path $\gamma(t,\xi)$ will affect the estimates, as the length of the paths must be factored in. The growth of the coefficients would be suitable for Fourier inversion if we could find path choices so that $B(t,\xi) - B(s,\xi) \geqslant 0$ for $s \in \gamma(t,\xi)$ while the length of $\gamma(t,\xi)$ remains uniformly bounded on ξ. However, those requirements are incompatible in general. The solution is to loosen the constraints a bit – the length of $\gamma(t,\xi)$ is allowed to grow with $|\xi|$, but only asymptotically less than $|\xi|^{\nu-1}$.

Lemma 3.10. *Suppose (ψ) holds for $\Omega' \Subset \Omega$ open and connected, with U' and U pairs of precompact sets related as they appear in Definition 3.6. Also let $t^* : S^{n-1} \to \overline{U}$ be a measurable function such that*

$$B(t^*(\xi),\xi) = \inf_{\overline{U}} B(\cdot,\xi), \quad \forall \xi \in S^{n-1} \tag{3.53}$$

and $t_0(\xi) \doteq t^*(\xi/|\xi|)$ for $\xi \in \Omega \backslash \{0\}$.

Then there exists a constant $C > 0$ depending on U' such that for all $\xi \in \mathbb{R}^n, t \in U'$ we have an integration path $\gamma(t,\xi) \subset \Omega$ from $t_0(\xi)$ to t with the following properties

1. *$B(t,\xi) - B(s,\xi) \geqslant -1, \ \forall s \in \gamma(t,\xi)$*
2. *the length $|\gamma(t,\xi)|$ of $\gamma(t,\xi)$ is bounded by*

$$|\gamma(t,\xi)| \leqslant C(1 + |\xi|)^{\nu-1} \tag{3.54}$$

Remark 3.11. *Notice since B is positive homogeneous, the first condition can be reformulated as*

$$B(t,\xi/|\xi|) - B(s,\xi/|\xi|) \geqslant -\frac{1}{|\xi|}, \quad \forall s \in \gamma(t,\xi) \tag{3.55}$$

highlighting the fact that, although the endpoints $t_0(\xi)$ and t of $\gamma(t,\xi)$ do not change as ξ increases in module along the same direction, the bound for the values of B along $\gamma(t,\xi)$ gets increasingly strict.

Proof. Let $\mathcal{D}_0$ be the collection of closed unit cubes in $\mathbb{R}^\nu$ with vertices in $\mathbb{Z}^\nu$. Take $(\mathcal{D}_k)$ the sequence where each $\mathcal{D}_k$ is the image of $\mathcal{D}_0$ through the contraction $t \mapsto 2^{-k}t$.

For $(t,\xi) \in U' \times \mathbb{R}^n$ given, set $r_{t,\xi} = B(t,\xi) + 1/2$ and consider for each k the sub-collections $\mathcal{Q}_k(\xi, r_{t,\xi}) = \{D \in \mathcal{D}_k : D \cap U(\xi, r_{t,\xi}) \neq \varnothing\} \subset \mathcal{D}_k$ of those cubes that do intersect $U(\xi, r_{t,\xi})$ non-trivially. Also denote $U_k^+(\xi,r)$ the union of all $D \in \mathcal{Q}_k(\xi,r)$ and observe each $s \in U_k^+(\xi,r)$ belongs to a cube of diameter $2^{-k}\sqrt{\nu}$ which intersects $U(\xi,r)$ at some point s'.

Similarly, we take $U_k^+ = \bigcup\{D \in \mathcal{D}_k : D \cap U \neq \varnothing\}$. Since $U \Subset \Omega$, either Ω^c is empty or it sits within a positive distance of U. Either way, there exists $\delta > 0$ such that

$U + \mathcal{B}_\delta(0) \subset \Omega$. Then, taking $k_0 \in \mathbb{N}$ so that $2^{-k_0}\sqrt{\nu} < \delta$, we have $U_k^+ \subset \Omega$ for every $k \geqslant k_0$. We make use of the following global estimate:

$$M \doteq \max\{\sup_{\substack{t \in \overline{U_{k_0}^+} \\ |\xi|=1}} |d_t B(t,\xi)|, 1\} \tag{3.56}$$

Notice if $N \doteq \#\{D \in \mathcal{D}_0 : D \cap \overline{U} \neq \varnothing\}$ is the number of unit cubes intercepting $\overline{U}$ then for all k the amount of cubes in $\mathcal{Q}_k(\xi, r)$ is bounded above by $\#\mathcal{Q}_k(\xi, r) \leqslant 2^{\nu k} N$.

Now let (t, ξ) be fixed. Given $s \in U_k^+(\xi, r_{t,\xi})$ take $s' \in U(\xi, r_{t,\xi})$ such that s and s' partake a common cube in $\mathcal{Q}_k(\xi, r_{t,\xi})$. Then if $k \geqslant k_0$ we obtain

$$\begin{aligned} B(s,\xi) &\leqslant |B(s,\xi) - B(s',\xi)| + B(s',\xi) &(3.57)\\ &\leqslant M|\xi||s - s'| + r_{t,\xi} \leqslant 2^{-k} M\sqrt{\nu}|\xi| + r_{t,\xi} &(3.58)\end{aligned}$$

There is a unique integer $k_\xi \geqslant 2$ such that $2^{k_\xi - 2} < M\sqrt{\nu}(1 + |\xi|) \leqslant 2^{k_\xi - 1}$. Thus applying $k' \doteq k_0 + k_\xi$ to the previous we end up with

$$\begin{aligned} B(s,\xi) &\leqslant 2^{-(k_0+k_\xi)} M\sqrt{\nu}|\xi| + r_{t,\xi} &(3.59)\\ &\leqslant 2^{-(k_0+k_\xi)} 2^{k_\xi - 1} + r_{t,\xi} \leqslant 1/2 + r_{t,\xi}, \quad \forall s \in U_{k'}^+(\xi, r) &(3.60)\end{aligned}$$

On the other hand, the statement of (ψ) says points in $U'_{k'}(\xi, r)$ partake a common connected component of $U(\xi, r)$. Then since $U(\xi, r) \subset U_{k'}^+(\xi, r)$, we have a piece-wise linear path $\gamma(t, \xi) \subset U_{k'}^+(\xi, r)$ joining t and $t^*(\xi)$. [4] If we avoid traversing the same cube more than once, this may be achieved with length

$$\begin{aligned} |\gamma(t,\xi)| &\leqslant \#\mathcal{Q}_{k'}(2^{-k'}\sqrt{\nu}) = N\sqrt{\nu}\, 2^{k'(\nu-1)} &(3.61)\\ &\leqslant N\sqrt{\nu}(2^{k_0+2} M\sqrt{\nu}(1 + |\xi|))^{\nu-1} = C(1 + |\xi|)^{\nu-1} &(3.62)\end{aligned}$$

with C a constant independent of (t, ξ). Apply (3.59) in particular to points in the paths we constructed to get the uniform bounds

$$B(t,\xi) - B(s,\xi) \geqslant B(t,\xi) - B(s,\xi) \geqslant -1, \quad \forall s \in \gamma(t,\xi) \tag{3.63}$$

so both conditions are met and the proof is concluded. □

Theorem 3.12 (Sufficiency of (ψ)). *If (ψ) holds for an open connected $\Omega' \subset \Omega$, then $\mathbb{P}^{(0)}$ is smoothly semi-globally solvable with respect to Ω' at $p = 0$.*

Proof. Let $f \in \mathbb{E}_b^0$ and $U' \Subset \Omega'$, then (ψ) gives a suitable set U. Take $t^* : S^{n-1} \to \overline{U}$ obtained from Lemma 3.8 with $K_1 = S^{n-1} \subset \mathbb{R}^n$ and $K_2 = \overline{U}$, then we are set to apply Lemma 3.10, which provides paths $(\gamma(t,\xi))_{(t,\xi) \in U' \times \mathbb{R}^n}$ where the appropriate estimates hold.

[4] $t^*(\xi) \in U'(\xi, r)$ due to equation (3.53)

Given $(t, \xi) \in U' \times \mathbb{R}^n$, we let

$$v(t, \xi) \doteq \int_{\gamma(t,\xi)} e^{B(s,\xi)-B(t,\xi)} \widehat{f}(s, \xi) \tag{3.64}$$

Notice the values of v are not dependent on the particular choice of integration path, since the integrand is closed.

Properties *1.* and *2.* in Lemma 3.10 ensure v satisfies the estimates

$$|v(t, \xi)| \leqslant |\gamma(t, \xi)| \sup_{s \in \gamma(t,\xi)} |e^{B(s,\xi)-B(t,\xi)}||\widehat{f}(t, \xi)| \tag{3.65}$$

$$\leqslant e\, C(1 + |\xi|)^{\nu-1} \sup_{s \in \overline{U}} |\widehat{f}(s, \xi)| \tag{3.66}$$

Now given $f \in \bigwedge^1 C^\infty(\Omega; H^\infty)$, we have $\widehat{f} \in \bigwedge^1 C^\infty(\Omega; H^\infty)$ a function whose decay in the second variable is faster than arbitrary powers of $|\xi|^{-1}$ when $\xi \to \infty$. A similar estimate holds for the t-variable, and the standard inductive argument shows $\mathcal{F}^{-1}v$ is smooth in t. Therefore, we conclude $\mathcal{F}^{-1}v$ is in $\bigwedge^1 C^\infty(\Omega; H^\infty)$. $\square$

4 A model with real coefficients

4.1 Definition

Consider a system of vector fields in $\mathbb{T}^n \times \mathbb{T}$ as follows.

$$L_j = \partial_{t_j} + a_j(t)\partial_x, \quad j = 1, \dots, n, (t, x) \in \mathbb{T}^n \times \mathbb{T} \tag{4.1}$$

where $a_j \in C^\infty_{\mathbb{R}}(\mathbb{T}^n)$.

By direct computation, the Lie bracket between any two of those fields is

$$[L_j, L_k] = (\partial_{t_j} a_k - \partial_{t_k} a_j)\partial_x \tag{4.2}$$

so the requirement of commutativity is equivalent to the equalities

$$\partial_{t_j} a_k = \partial_{t_k} a_j \tag{4.3}$$

for $j, k = 1, \dots, \nu$. In such case, our introductory discussion determines we can naturally associate a chain complex of differential operators, given by

$$\mathbb{L}^p_a = d_t + a(t) \wedge \partial_x, \quad (p = 0, \dots, n-1) \tag{4.4}$$

with $a(t) \doteq \sum_j a_j dt_j \in C^\infty(\mathbb{T}^n, \bigwedge^1 \mathbb{T}^n)$ and d_t denoting the exterior derivative with respect to the t variable only. Notice (4.3) is precisely the condition of a being closed.

The first cohomology group of the de Rham complex for $\mathbb{T}^n$ is $H^1_{dR}(\mathbb{T}^n) \cong \mathbb{R}^n$, with the constant 1-forms dt_j giving representatives for a set of generators. Therefore, there exists $A \in C^\infty(\mathbb{T}^n)$ and $a_0 \in \mathrm{span}_{\mathbb{R}}(dt_j)_j$ such that

$$a = a_0 + dA \tag{4.5}$$

The constant 1-form a_0 will sometimes be identified with a constant in $\mathbb{R}^n$ by means of its real coordinates respect to the generators $(dt_j)_j$.

The action of the differential maps (4.4) will take place between ascending levels of **currents** over $\mathbb{T}^n \times \mathbb{T}$, as they are defined in Example 2.4. A slight difference, however, is that only p-currents spanned by dt_J with $|J| = p$ are considered. For this reason we denote the domains $\mathcal{D}'(\mathbb{T}^n \times \mathbb{T}; \bigwedge^{p,0})$. An element in it is therefore written uniquely as

$$u = \sum_{|J|=p} u_J dt_J, \quad u_J \in \mathcal{D}'(\mathbb{T}^n \times \mathbb{T}) \tag{4.6}$$

This modification should make sense in view of the fact (4.4) does not involve dx in any way. The subspace of p-forms with the same restriction is denoted, analogously, as $C^\infty(\mathbb{T}^n \times \mathbb{T}; \bigwedge^{p,0})$

Fourier series are adapted in a straight-forward manner: with $u \in \mathcal{D}'(\mathbb{T}^n \times \mathbb{T}; \bigwedge^{1,0})$ represented by coefficients u_J as before, the (total) Fourier series of u reads

$$u = \sum_{(j,k) \in \mathbb{Z}^n \times \mathbb{Z}} \widehat{u}(j,k) e^{i(j \cdot t + kx)} \tag{4.7}$$

where the coefficients are defined by

$$\widehat{u}(j,k) = \sum_{[J]=p} \widehat{u}_J(j,k) dt_J \tag{4.8}$$

with $\widehat{u}_J$ now the standard Fourier coefficients.[1] Similarly, we also have the partial Fourier series with respect to x

$$u = \sum_{k \in \mathbb{Z}} \widehat{u}^x(t,k) e^{ikx} \tag{4.9}$$

where $\widehat{u}^x(\cdot, k) \in \mathcal{D}'(\mathbb{T}^n)$. We will omit the x from $\widehat{u}^x$ if there is no risk of confusion.

The problem under consideration is to ascertain the solvability, and determine explicit solutions if it has, for the linear systems

$$\mathbb{L}_a^p u = f, \quad p = 0, \ldots, n-1 \tag{4.10}$$

where $f \in C^\infty(\mathbb{T}^n \times \mathbb{T}; \bigwedge^{p,0})$ is in a suitable subset of the codomain defined by compatibility conditions. The developments of [BP99] characterize the **global solvability** of $\mathbb{L}_a^p$ with an algebraic condition on the vector a, namely whether or not it is a Liouville form.

4.2 Liouville forms

Liouville numbers

The concept of a Liouville numbers goes back to the middle of the 19th century, when Joseph Liouville first showed all such numbers are transcendental, thus proving the existence of non-algebraic numbers for the first time. Liouville's constant, defined through its decimal representation

$$L_b = \sum_{l=1}^{\infty} 10^{-l!} = 10^{-1} + 10^{-2} + 10^{-6} + \ldots \tag{4.11}$$

is among the first known examples.

Liouville numbers are characterized by the existence of very tight approximations by rational numbers: $x \in \mathbb{R}$ is Liouville if for every $l \in \mathbb{N}$, there exists $(p,q) \in \mathbb{Z} \times \mathbb{Z}$ with $q \geqslant 2$ such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^l} \tag{4.12}$$

[1] to avoid confusion: the indices j and J are not related

In a more refined classification, we can assign to every number $x \in \mathbb{R}$ an irrationality exponent $\mu(x)$ as follows. Let $R_x \subset \mathbb{R}_{>0}$ be the set of positive reals μ such that

$$\left\{(p,q) \in \mathbb{Z} \times \mathbb{Z}_{>0} : 0 < |x - p/q| < q^{-\mu} \text{ and } \gcd(p,q) = 1\right\} \tag{4.13}$$

is finite and take $\mu(x) \doteq \inf_{\mu \in R_x} \mu$, with $\mu(x) = \infty$ if R_x is unbounded.

Comparing with (4.12), it is easy to see a Liouville number corresponds to the extreme case $\mu(x) = \infty$ and, by contrast, $\mu(x) = 1$ whenever $x \in \mathbb{Q}$.

It follows from an important theorem of Dirichlet in the subject of Diophantine approximations that for $x \notin \mathbb{Q}$, we have $2 \in R_x$ in the previous construction, and therefore $\mu(x) \geqslant 2$ for all irrationals. A rational number x has irrationality measure $\mu(x) = 1$, of course, so there is a 'gap' between 1 and 2 in $\mu(\mathbb{R})$. As Roth's theorem would later show, this is already the best lower bound one can get in the case of irrational algebraic numbers, since every such x has irrationality exponent $\mu(x) = 2$.

Global hypoellipticity

In regards to solvability of PDEs, Liouville numbers make their first appearance in [GW72] in the form of a necessary and sufficient algebraic condition for hypoellipticity. The name comes from the well-know regularity property of elliptic operators

$$\text{if } P \text{ is elliptic,} \quad Pu \in C^\infty \implies u \in C^\infty \tag{4.14}$$

which becomes the defining property of hypoellipticity. Indeed, the condition is much weaker than a trivial characteristic set.

To be more precise, what the authors found in [GW72] is that global hypoellipticity of a constant coefficients differential operator

$$P = \partial_t + c\,\partial_x, \quad (t,x) \in \mathbb{T} \times \mathbb{T} \tag{4.15}$$

is equivalent to c being a non-Liouville constant. This has to do with the necessary decay of Fourier coefficients to reconstruct a solution. Many conditions related to this idea have been proposed and proved since then.

Liouville vectors and forms

When dealing with systems of vector fields, as is our case, multiple coefficients appear, so we have to make sense of a Liouville vector.

Definition 4.1. *A given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ is said **Liouville** if $\alpha \notin \mathbb{Q}^n$ and there are sequences $(\boldsymbol{p}_j)_{l \in \mathbb{Z}_+} = (\boldsymbol{p}_{j,1}, \ldots, \boldsymbol{p}_{l,n})_{l \in \mathbb{N}} \subset \mathbb{Z}^n$ and $(q_j)_{j \in \mathbb{N}}$ such that*

$$\left\{\max_j \left|\alpha_j - \frac{\boldsymbol{p}_{l,j}}{q_l}\right| \cdot {q_l}^l : l \in \mathbb{N}\right\} \tag{4.16}$$

is bounded.

Remark 4.2. *Notice the denominator q_l used for the approximations is common to all numerators.*

Definition 4.3 ([BCM93, Def. 2.1])**.** *A closed 1-form $\alpha \in \bigwedge C^\infty_{\mathbb{R}}(\mathbb{T}^n)$ is said*

(i) integral if

$$\frac{1}{2\pi}\int_\sigma \alpha \in \mathbb{Z} \tag{4.17}$$

for all 1-cycles σ

(ii) rational if $q\alpha$ is integral for some $q \in \mathbb{Z}_{>0}$

(iii) Liouville if α is not rational and there exist sequences $(\alpha_j) \subset \bigwedge^1 C^\infty(\mathbb{T}^n), (q_l) \subset \mathbb{Z}_{\geqslant 2}$ such that

$$\left\{ {q_l}^l \left| \alpha - \frac{\alpha_j}{q_l} \right| \right\}_{l\in\mathbb{N}} \subset \bigwedge^1 C^\infty(\mathbb{T}^n) \tag{4.18}$$

is bounded.

Making a choice of 1-cycles $\sigma_1, \ldots, \sigma_n$ which represent generators of $H^1(\mathbb{T}^n)$ in the homology of chains, we can define a linear map $I : H^1_{dR}(\mathbb{T}^n) \to \mathbb{R}^n$

$$[\beta] \mapsto \frac{1}{2\pi}\left(\int_{\sigma_1} \beta, \ldots, \int_{\sigma_n} \beta\right) \tag{4.19}$$

We can then relate the previous definitions through the following.

Proposition 4.4 ([BCM93, Prop.2.2])**.** *Let $\alpha \in \bigwedge C^\infty_{\mathbb{R}}(\mathbb{T}^n)$ be a closed form, then*

(i) α is integral iff $I[\alpha] \in \mathbb{Z}^n$

(ii) α is rational iff $I[\alpha] \in \mathbb{Q}^n$

(iii) α is Liouville iff $I[\alpha]$ is Liouville (according to Definition 4.1)

In view of the fact we can decompose $a = a_0 + dA$ as in (4.5), a is homologous to a constant 1-form, which is Liouville iff a is. It's immediate to verify that the linear map I applied to $[a_0]$ is simply its identification in $\mathbb{R}^n$, therefore, rather than relying on Definition 4.3 to classify a, we can work solely with Definition 4.1 as applied to a_0.

There is one final remark regarding Diophantine approximations which will have an important role in the main proof. One may read it as saying that, if q is the smallest integer such that $q\alpha$ is integral, then rationals approximate α as if it were non-Liouville, provided that denominators in $q\mathbb{Z}$ are precluded in the approximations. The elementary proof is given below.

Lemma 4.5. *Let $\alpha \in \mathbb{Q}^n$ and $q = \min\{n \in \mathbb{N}_{>0} : n\alpha \in \mathbb{Z}^n\}$, then*

$$\|j\alpha - k\|_\infty \geqslant \frac{1}{q}, \quad \forall k \in \mathbb{Z}^n, j \in \mathbb{Z} - q\mathbb{Z} \tag{4.20}$$

Proof. Write $\alpha = \frac{1}{q}p$ with $p \in \mathbb{Z}^n$, then

$$j\alpha - k = \frac{jp - qk}{q} \implies \|j\alpha - k\|_\infty = \frac{1}{q}\|jp - qk\|_\infty \tag{4.21}$$

so the lower bound holds unless there are $k \in \mathbb{Z}^n$ and $j \in \mathbb{Z} - q\mathbb{Z}$ such that $jp = qk$.

If we did have the equality, it would entail $k = \frac{j}{q}p \in \mathbb{Z}^n$. Now since j is not divisible by q, it must be the case that q divides the gcd of the entries of p. This, however, would mean q is not minimal, a contradiction. □

4.3 Global solvability

To determine the appropriate notion of global solvability for $\mathbb{L}_a^p$, we must find a requirement on $f \in \operatorname{ran}(\mathbb{L}_a^p)$ so we can expect to solve $\mathbb{L}_a^p u = f$. The approach is similar to (3.35), where the differential equations for the x-Fourier coefficients lead to a requirement of exactness involving $\hat{f}(\cdot, \xi)$. Things are a bit trickier this time, however, because a is only assumed to be closed. To write the equations as exact differentials, we rely on a primitive defined in the universal covering of $\mathbb{T}^n$.

Lemma 4.6. *Let $f \in C^\infty(\mathbb{T}^n \times \mathbb{T}; \bigwedge^{p+1,0})$ be such that there exists $u \in \mathcal{D}'(\mathbb{T}^n \times \mathbb{T}; \bigwedge^{p+1,0})$ with $\mathbb{L}_a^p u = f$ and $\Pi : \mathbb{R}^n \to \mathbb{T}^n$ the universal covering map of $\mathbb{T}^n$. Then $\mathbb{L}_a^{p+1} f = 0$ and taking $\psi_j \in C^\infty(\mathbb{R}^n)$ such that $d\psi_j = \Pi^*(ja_0)$, we have $e^{i(\psi_j + jA)}\widehat{f}(\cdot, j)$ an exact form whenever ja_0 is integral.*

Proof. That $\mathbb{L}_a^{p+1} f = 0$ readily follows from $\mathbb{L}_a$ being a chain complex, i.e. $\mathbb{L}_a^{p+1}\mathbb{L}_a^p = 0$. For the next part, we use the Fourier series on x to represent $u = \sum_j \widehat{u}(t, j)e^{ijx}$ and $f = \sum_j \widehat{f}(t, j)e^{ijx}$, so that

$$\mathbb{L}_a^p u = f \iff (d_t + ija\wedge)\hat{u}(\cdot, j) = \hat{f}(\cdot, j), \quad \forall j \in \mathbb{Z} \tag{4.22}$$

$$\iff (d_t + ija_0\wedge)(e^{ijA}\hat{u}(\cdot, j)) = e^{ijA}\hat{f}(\cdot, j), \quad \forall j \in \mathbb{Z} \tag{4.23}$$

Now let $j \in \mathbb{Z}$ be such that ja_0 is integral. Then $\Pi^*(ja_0) \in \bigwedge^1(\mathbb{R}^n)$ gives a closed form in $\mathbb{R}^n$. Because $\mathbb{R}^n$ is simply connected, $\Pi^*(ja_0)$ is in fact an exact form, say $d\psi_j = \Pi^*(ja_0)$.

Since $\pi_1(\mathbb{T}^n)$ is abelian, Hurewicz's theorem implies each 1-cycle in $\mathbb{T}^n$ is homologous to a smooth loop. Thus, if $p, q \in \mathbb{R}^n$ are in the same fiber of Π (that is, $\Pi(p) = \Pi(q)$),

$$\psi_j(p) - \psi_j(q) = \int_p^q d\psi_j = \int_p^q \Pi^*(ja_0) \in 2\pi\mathbb{Z} \tag{4.24}$$

This implies $e^{i\psi_j} : \mathbb{R}^n \to \mathbb{R}$ factors through the quotient $\Pi : \mathbb{R}^n \to \mathbb{T}^n$, so it induces a smooth function $e^{i\psi_j} \in C^\infty(\mathbb{T}^n)$. Equation (4.23) can be rewritten as

$$d_t(e^{ijA}\hat{u}(\cdot, j)) = e^{ijA}(\hat{f}(\cdot, j) - ija_0\hat{u}(\cdot, j)) \tag{4.25}$$

thus

$$d_t\left(e^{i(\psi_j+jA)}\hat{u}(\cdot, j)\right) = e^{i\psi_j}(ija_0\hat{u}(\cdot, j) + d_t(e^{ijA}\hat{u}(\cdot, j))) = e^{i(\psi_j+jA)}\hat{f}(\cdot, j)$$

confirms exactness of the right-side. □

Definition 4.7. *Let $\mathbb{E}^p_a \subset C^\infty(\mathbb{T}^n \times \mathbb{T}; \bigwedge^{p+1,0})$ be given by*

$$\mathbb{E}^p_a = \{f \in \operatorname{Im}(\mathbb{L}^p_a) : ja_0 \textit{ integral implies } e^{i(\psi_j+jA)}\hat{f}(\cdot, j) \textit{ exact}\}$$

*We say $\mathbb{L}^p_a$ is **globally solvable** ((GS) for short) if given $f \in \mathbb{E}^p_a$ there exists $u \in \mathcal{D}'(\mathbb{T}^n \times \mathbb{T}; \bigwedge^{p+1,0})$ such that $\mathbb{L}^p_a u = f$.*

An automorphism of $\mathcal{D}'(\mathbb{T}^n \times \mathbb{T}; \bigwedge^{p+1,0})$ allows us establish a conjugation between operators $\mathbb{L}^p_a$ with homologous coefficients a. Indeed, let S_A act over $\mathcal{D}'(\mathbb{T}^n \times \mathbb{T}; \bigwedge^{p,0})$ by

$$u = \sum_{j\in\mathbb{Z}} \widehat{u}(t, j)e^{ijx} \mapsto \sum_{j\in\mathbb{Z}} \widehat{u}(t, j)e^{ij(x+A(t))} \tag{4.26}$$

then S_A is bounded with $(S_A)^{-1} = S_{-A}$, thus it defines an automorphism. Since

$$(d_t + ija_0\wedge)(e^{ijA}\widehat{u}(\cdot, j)) = e^{ijA}(d_t\widehat{u}(\cdot, j) + ij(a_0 + dA) \wedge \widehat{u}(\cdot, j)) \tag{4.27}$$

the conjugate relation

$$\mathbb{L}^p_{a_0}S_A = S_A\mathbb{L}^p_a \tag{4.28}$$

follows. Finally, given $f \in C^\infty(\mathbb{T}^n \times \mathbb{T}; \bigwedge^{p,0})$,

$$\widehat{S_A f}(\cdot, j) = e^{ijA}\widehat{f}(\cdot, j), \quad \forall j \in \mathbb{Z} \tag{4.29}$$

so we have

$$\mathbb{E}^p_a = \{f \in \operatorname{Im}(\mathbb{L}^p_a) : \forall\, ja_0 \text{ integral, } e^{i\psi_j}\widehat{S_A f}(\cdot, j) \text{ is exact}\} = S_A(\mathbb{E}^p_{a_0}) \tag{4.30}$$

Those considerations lead to the following.

Lemma 4.8. *Let $a, b \in C^\infty(\mathbb{T}^n)$ with a and b homologous, then $\mathbb{L}^p_a$ is (GS) iff $\mathbb{L}^p_b$ is.*

Proof. It suffices to verify the statement when b is the constant 1-form a_0 in the homology class of a. The result is almost immediate from our considerations – by (4.30), each $f \in \mathbb{E}^p_a$ corresponds bijectively to $g = S_A f \in \mathbb{E}^p_{a_0}$, therefore (4.29) leads to

$$\exists u : \mathbb{L}^p_a u = f \iff \exists g : \mathbb{L}^p_{a_0} v = g \tag{4.31}$$

where the solutions are related by $v = S_A u$. Global solvability of $\mathbb{L}^p_{a_0}$ is then equivalent to global solvability of $\mathbb{L}^p_a$. □

We are now in place to state the main result of the paper.

Theorem 4.9. *For each $p = 0, \dots, n-1$, $\mathbb{L}_a^p$ is globally solvable if and only if a is not a Liouville 1-form.*

Naturally, Lemma 4.8 reduces the problem to the cases $a = a_0$ a constant 1-form. We will only concern ourselves with the proof of sufficiency, that is, $\mathbb{L}_a^p$ is (GS) if a is non-Liouville, doing so in a constructive manner (that is, making an explicit choice of Fourier coefficients).

4.4 Exactness of $\xi \wedge \cdot$

The proof of solvability will involve the standard procedure of determining the Fourier coefficients of candidate solutions. In solving for the coefficients of the total Fourier series (4.40), the differential problem is replaced with a system of algebraic equations in the alternating algebra spanned by (dt_j), which is isomorphic to $\bigwedge^{\bullet} \mathbb{R}^n$. More precisely, $\mathbb{L}^p u = f$ leads to a system of equations of type

$$\xi_{j,k} \wedge \widehat{u}(j,k) = \widehat{f}(j,k) \tag{4.32}$$

indexed by $(j,k) \in \mathbb{Z}^n \times \mathbb{Z}$, where $(\xi_{j,k})_{(j,k)\in\mathbb{Z}^{n+1}}$ is a family of covectors in $\bigwedge^1 \mathbb{R}^n$.

Each equation in the system corresponds to a linear, non-homogeneous problem with regards to the linear map $\xi_{j,k} \wedge \cdot : \bigwedge^p \mathbb{R}^n \to \bigwedge^{p+1} \mathbb{R}^n$. Existence and general form of solutions are thus fully understood once kernel and image of the operators $\xi \wedge \cdot$, determined by some $\xi \in \bigwedge^1 \mathbb{R}^n$, are characterized – (4.32) is solvable iff $\widehat{f}(j,k) \in \mathrm{Im}(\xi_{j,k} \wedge \cdot)$ and, in such case, the general solution is a coset in $\bigwedge^p \mathbb{R}^n / \ker \xi_{j,k} \wedge \cdot$. Fortunately, complexes of such type are already familiar to us from Example 2.13, where we showed the Rham complex is elliptic. Indeed, the induced linear complexes (2.39), generated by $i\xi \bigwedge \cdot$ for some $\xi \in \bigwedge^1 \mathbb{F}^\nu$ were found to be exact, so we do know $\ker(\xi_{j,k} \wedge \cdot) = \mathrm{Im}(\xi_{j,k} \wedge \cdot)$ at each level in (4.32).

The present situation still requires a bit more. We are unable to make estimates on the decay of the Fourier coefficients to ensure they come from a distribution, unless coordinates of a particular solution are constructively specified. The next lemma complements our discussion by doing so.

Lemma 4.10 ([BP99, Lemma 2.1])**.** *Consider V a vector space of dimension n (over $\mathbb{R}$ or $\mathbb{C}$) and $\gamma = (e_1, \dots, e_n)$ a basis for $\bigwedge^1 V$. Let $\xi = \sum_j \xi_j e_j \in \bigwedge^1 V \backslash \{0\}$, then for any $g = \sum_{|J|=p+1} g_J e_J \in \bigwedge^{p+1} V$ chosen $(1 \leqslant p \leqslant n-1)$ the equation $(*)$ $\xi \wedge v = g$ has a solution $v \in \bigwedge^p V$ iff $\xi \wedge g = 0$ holds. Furthermore, when solutions do exist, take*

$r \in \{1, \dots n\}$ such that $\xi_r \neq 0$, then a particular solution of $(*)$ is

$$v_0 = \frac{1}{\xi_r} \sum_{\substack{|J|=p+1 \\ r \in J}} sgn(r, J - \{r\}) g_J e_{J-\{r\}} \tag{4.33}$$

where the summation is over ascending $(p+1)$-tuples which contain r, and $sgn(r, J-\{r\}) \in \{\pm 1\}$ denotes the sign of the permutation $(r, j_1, \dots, \hat{r}, \dots, j_{p+1})$. Finally, the full set of solutions for (*) is given by

$$v = v_0 + \xi \wedge w, \quad w \in \bigwedge\nolimits^p V$$

Proof. The only claim which cannot be derived directly from the preceding discussion is the fact v_0 as given by (4.33) is a solution, so we restrict ourselves to that.

Since $\xi_r \neq 0$, we can rewrite e_r as a linear combination of ξ and the remaining elements of the basis

$$e_r = \frac{1}{\xi_r}\xi - \sum_{l \neq r} \frac{\xi_l}{\xi_r} e_l \tag{4.34}$$

We split the terms of g according to whether or not $r \in J$. If it does, we can substitute $e_J = \mathrm{sgn}(r, J - \{r\}) e_r \wedge e_{J-\{r\}}$ to make e_r the leading covector in the expressions.

$$g = \sum_{\substack{|J|=p+1 \\ 1 \in J}} g_J e_{(1, J-\{1\})} + \sum_{\substack{|J|=p+1 \\ 1 \notin J}} g_J e_{J-\{1\}} \tag{4.35}$$

$$= \sum_{\substack{|J|=p+1 \\ 1 \in J}} g_J \,\mathrm{sgn}(r, J - \{r\}) e_r \wedge e_{J-\{r\}} + \sum_{\substack{|J|=p+1 \\ 1 \notin J}} g_J e_{J-\{1\}} \tag{4.36}$$

Then, substituting e_r for (4.34) so as to obtain g in terms of the base $\{\xi\} \cup (\gamma - \{r\}) \subset \bigwedge^1 V$, the expression for v_0 appears. We have

$$g = \sum_{\substack{|J|=p+1 \\ 1 \in J}} \frac{1}{\xi_1} \mathrm{sgn}(r, J - \{r\}) g_J \; \xi \wedge e_{J-\{1\}} \tag{4.37}$$

$$- \sum_{l=2}^{n} \sum_{\substack{|I|=p+1 \\ 1 \in I}} \frac{\xi_l}{\xi_1} \mathrm{sgn}(r, J - \{r\}) g_J e_l \wedge e_{I-\{1\}} + \sum_{\substack{|J|=p+1 \\ 1 \notin J}} g_J e_{J-\{1\}} \tag{4.38}$$

$$= \xi \wedge v_0 + g^* \tag{4.39}$$

where g^* is a vector spanned by $(e_2, \dots, e_n)$.

The boundary condition $\xi \wedge g = 0$ then leads to $\xi \wedge g^* = 0$. But since the restriction of $\xi \wedge \cdot$ to the span of $(e_2, \dots, e_n)$ is injective, $\xi \wedge g^* = 0$ implies $g^* = 0$, and therefore $g = \xi \wedge v_0$. □

It is worth pointing out that the denominator ξ_r in the expression obtained for v_0, which is a coordinate of ξ, is what later requires the Diophantine estimates on the coefficient a to obtain some bound on the growth of the Fourier coefficients.

4.5 Proof of sufficiency

All is set for the main proof.

Proposition 4.11. *If a is not a Liouville form, then $\mathbb{L}_a^p$ is globally solvable.*

Proof. Again, in view of Lemma 4.8, it suffices to show $\mathbb{L}_a^p$ is (GS) when $a = a_0 \in \bigwedge^1 \mathbb{T}^n$ is non-Liouville. Let $f \in \mathbb{E}_{a_0}$, we split the argument in regards to a_0 being rational or not.

First suppose $I[a_0] \notin \mathbb{Q}^n$. Let $(a_{0j}) \doteq I[a_0]$, a solution u for the equation $(d_t + a_0 \wedge \partial_x)u = f$ requires the total Fourier coefficients to be such that

$$i\left(\sum_{m=1}^{n} j_m dt_m + ka_0\right) \wedge \widehat{u}(j,k) = i\left(\sum_{m=1}^{n}(j_m + ka_{0m})dt_m\right) \wedge \widehat{u}(j,k) \tag{4.40}$$

$$= \widehat{f}(j,k), \quad (j,k) \in \mathbb{Z}^n \times \mathbb{Z} \tag{4.41}$$

Let $\xi_{j,k} \doteq \sum_{m=1}^{n}(j_m + ka_{0m})dt_m$. Since $I[a_0] \notin \mathbb{Q}^n$, ka_0 is never integral and therefore $\xi_{j,k}$ doesn't vanish - except, of course, if $(j,k) = 0$. The compatibility condition on f implies the partial Fourier coefficient $\widehat{f}^x(\cdot, 0)$ is exact, thus

$$\widehat{f}(0,0) = \int_{\mathbb{T}^n} \widehat{f}^x(\cdot,0) = 0 \tag{4.42}$$

in virtue of Stoke's theorem. Therefore, it suffices to set $\widehat{u}(0,0) = 0$ to solve (4.40) for $(j,k) = 0$. Otherwise, for any $(j,k) \neq 0$, we have $\xi_{j,k} \neq 0$ and, given $f \in \mathbb{E}_{a_0} \subset \ker \mathbb{L}_{a_0}^{p+1}$, the equation $\xi_{j,k} \wedge \widehat{f}(j,k) = 0$ holds and Lemma 4.10 applies, yielding solutions

$$\widehat{u}(j,k) = \frac{1}{i(j_M + ka_{0_M})} \sum_{\substack{|J|=p+1 \\ M \in J}} sgn(r, J - \{M\})(\widehat{f}(j,k))_J dt_{J-\{M\}} \tag{4.43}$$

to (4.40) with $M \doteq \arg\max\{|k_l + ja_{0_l}| : l \in \{1, \dots, n\}\}$.

Given a_0 is non-Liouville, Definition 4.1 implies there are constants $C > 0$ and $L \in \mathbb{Z}_+$ such that

$$\max_{1 \leqslant m \leqslant n} |qa_{0_m} - p_m| \geqslant C|q|^{-L}, \quad \forall (p,q) \in \mathbb{Z}^n \times \mathbb{Z}_{>0} \tag{4.44}$$

On the other hand, given f is a smooth function, the growth of its coefficients is sub-polynomial, so there exist $N \in \mathbb{N}$ and another constant $C > 0$ with

$$\left|\widehat{f}(j,k)\right| \leqslant C(1 + |(j,k)|)^{-N}, \quad \forall (j,k) \in \mathbb{Z}^{n+1} \tag{4.45}$$

Combining the estimates and changing the constants, we conclude

$$|\widehat{u}(j,k)| \leqslant C\left|\frac{\widehat{f}(j,k)}{j_M + ka_{0_M}}\right| \leqslant C(1 + |k|)^{-L}(1 + |(j,k)|)^{-N} \tag{4.46}$$

$$\leqslant C(1 + |(j,k)|)^{-(L+N)}, \quad \forall (j,k) \in \mathbb{Z}^{n+1} - \{0\} \tag{4.47}$$

and therefore the coefficients do come from a p-form $u \in \bigwedge^{p,0} C^\infty(\mathbb{T}^n \times \mathbb{T})$ which solves the differential problem. This concludes the proof for the case a_0 not rational.

Now suppose $a_0 \in \mathbb{Q}^n$ and take $q \in \mathbb{Z}^+$ minimum such that $qa_0 \in \mathbb{Z}^n$. We split the domain of the operator into two complementary subspaces

$$\mathcal{D}'(\mathbb{T}^n \times \mathbb{T}; \bigwedge\nolimits^{p,0}) = {\mathcal{D}'_{q\mathbb{Z}}}^{p,0} \oplus {\mathcal{D}'_{\mathbb{Z}-q\mathbb{Z}}}^{p,0}$$

where, given $A \subset \mathbb{Z}$, ${\mathcal{D}'_A}^{p,0}$ corresponds to the subset of $(p,0)$-currents whose partial Fourier coefficients with respect to x are supported on a subset A, so that $u \in {\mathcal{D}'_A}^{p,0}$ can be written

$$u(t,x) = \sum_{k \in A} \widehat{u}^x(t,k) e^{ikx} \tag{4.48}$$

The action of $\mathbb{L}^p_a$ corresponds to a linear action on the partial Fourier coefficients (which is given by (4.23)), so the operator acts invariantly over the complementary subspaces just defined. The linear subspace projections therefore induce differential complexes $(\mathbb{L}^p_{a,q\mathbb{Z}}, {\mathcal{D}'_{q\mathbb{Z}}}^{p,0})_p$, $(\mathbb{L}^p_{a,\mathbb{Z}-q\mathbb{Z}}, {\mathcal{D}'_{\mathbb{Z}-q\mathbb{Z}}}^{p,0})_p$ where the differential maps on each level add up to $\mathbb{L}^p_a$. Solving $\mathbb{L}^p_a u = f$ is therefore equivalent to solving the equations

$$\begin{cases} \mathbb{L}^p_{a,q\mathbb{Z}} u_1 = f_1 \\ \mathbb{L}^p_{a,\mathbb{Z}-q\mathbb{Z}} u_2 = f_2 \end{cases} \tag{4.49}$$

where (u_1, u_2) and (f_1, f_2) are the respective images of u and f through the natural isomorphisms ${\mathcal{D}'_{q\mathbb{Z}}}^{p,0} \oplus {\mathcal{D}'_{\mathbb{Z}-q\mathbb{Z}}}^{p,0} \to {\mathcal{D}'_{q\mathbb{Z}}}^{p,0} \times {\mathcal{D}'_{\mathbb{Z}-q\mathbb{Z}}}^{p,0}$

To solve the first equation, we show $\mathbb{L}^p_{a,q\mathbb{Z}}$ is a conjugate of d_t. Let $T \in Aut({\mathcal{D}'_{q\mathbb{Z}}}^{p,0})$ be given by

$$u = \sum_{N \in \mathbb{Z}} \widehat{u}^x(t,qN) e^{iqNx} \mapsto \sum_{N \in \mathbb{Z}} \widehat{u}^x(t,qN) e^{i(qNx - \psi_{qN}(t))} \tag{4.50}$$

with ψ_{qN} as in Lemma 4.6. Then, notice

$$\mathbb{L}^p_{a,q\mathbb{Z}} T u = \sum_{N \in \mathbb{Z}} (d_t + iqNa_0 \wedge \cdot - d_t\psi_{qN} \wedge \cdot) \widehat{u}^x(t,qN) e^{i(qNx - \psi_{qN}(t))} = T^{-1} d_t u \tag{4.51}$$

so indeed $d_t = T^{-1}\mathbb{L}^p_{a,q\mathbb{Z}}T$. Thus, the bijective correspondences $g \leftrightarrow f_1$ and $v \leftrightarrow u_1$ given by $g = T^{-1}f_1$ and $v = T^{-1}u_1$ are so that solutions in ${\mathcal{D}'_{q\mathbb{Z}}}^{p,0}$ for $\mathbb{L}^p_{a,q\mathbb{Z}} u_1 = f$ and $d_t v = g$ are equivalent.

Set $g \doteq T^{-1} f_1$, then since f_1 has partial Fourier coefficients supported in $q\mathbb{Z}$ with $\widehat{f_1}^x(\cdot, qN) = \widehat{f}^x(\cdot, qN)$ for $N \in \mathbb{Z}$, the compatibility conditions over $f \in \mathbb{E}_{a_0}$ imply the partial Fourier coefficients

$$\widehat{g}^x(\cdot,k) = \begin{cases} e^{i\psi_k} \widehat{f}^x(\cdot,k), & \text{if } k = qN \\ 0, & \text{otherwise} \end{cases} \tag{4.52}$$

are all exact. Then, the same argument we used to show $\widehat{f}(0,0) = 0$ in (4.42) works here to assure $\widehat{g}(0,k) = 0$ for all $k \in \mathbb{Z}$.

In solving $d_t v = g$, we obtain a system of equations

$$i \sum_{m=1}^{n} j_m dt_m \wedge \widehat{v}(j,k) = \widehat{g}(j,k), \quad \forall (j,k) \in \mathbb{Z}^n \times \mathbb{Z}$$

By (4.52), the coefficients

$$\widehat{g}(j,k) = \int_{\mathbb{T}^n} \widehat{g}^x(t,k) e^{it\cdot j} \tag{4.53}$$

are identically zero for $k \notin q\mathbb{Z}$, so the equations can be solved if we take $\widehat{v}(j,k) = 0$ for either $j = 0$ or $k \notin q\mathbb{Z}$, and the solutions provided by Lemma 4.10 in the remaining cases. The denominators (originally denoted ξ_r) for the particular solutions obtained from the Lemma are all non-zero integers, so instead of (4.46), the estimates for the coefficients become

$$|\widehat{u}(j,k)| \leqslant C|\widehat{f}(j,k)|, \quad (j,k) \in \mathbb{Z}^* \times \mathbb{Z} \tag{4.54}$$

which is already sufficient to verify the chosen coefficients have sub-polynomial growth. They are thus obtained from some $v \in \bigwedge^{p,0} C^\infty(\mathbb{T}^n \times \mathbb{T})$. Furthermore, since $\widehat{v}^x(t,k) = \sum_j \widehat{v}(j,k) e^{ij\cdot t} = 0$ for $k \notin q\mathbb{Z}$, v belongs to the appropriate subspace of smooth functions in ${\mathcal{D}'_{q\mathbb{Z}}}^{p,0}$. Hence the first equation in (4.49) is solvable.

The proof that $\mathbb{L}^p_{a_0,\mathbb{Z}-q\mathbb{Z}} u_2 = f_2$ is solvable goes analogously to the case of a_0 irrational non-Liouville. The equations for the Fourier coefficients are

$$i \left(\sum_{m=1}^{n} (j_m + k a_{0m}) dt_m \right) \wedge \widehat{u_2}(j,k) = \widehat{f_2}(j,k), \quad (j,k) \in \mathbb{Z}^n \times \mathbb{Z} \tag{4.55}$$

but since $f_2 \in {\mathcal{D}'_{\mathbb{Z}-q\mathbb{Z}}}^{p,0}$ the coefficients

$$\widehat{f_2}(j,k) = \int_{\mathbb{T}^n} \widehat{f_2}^x(t,k) e^{it\cdot j} \tag{4.56}$$

are all zero for $k \in q\mathbb{Z}$. Then, we can take $\widehat{u_2}(j,k) = 0$ for $k \in q\mathbb{Z}$ and a solution like (4.43) otherwise, since $\sum_m j_m + k a_{0m} dt_m \neq 0$ in the latter case. In view of Lemma 4.5, we have $C > 0$ such that

$$|q a_0 - k| \geqslant C, \quad \forall j \in \mathbb{Z} - q\mathbb{Z}, k \in \mathbb{Z}^n$$

so we have the estimate $\max_m |q a_{0m} - p_m| \geqslant C, \forall (q,p)$ in place of (4.46) to show

$$|\widehat{u_2}(j,k)| \leqslant C|\widehat{f_2}(j,k)|, \quad \forall (j,k) \in \mathbb{Z}^{n+1} \tag{4.57}$$

thus concluding the coefficients come from a smooth u_2 which solves the second equation in (4.49). □

Remark 4.12. *As shown in [BCM93], in more generality, with M a compact smooth manifold in place of $\mathbb{T}^n$, the differential operator $\mathbb{L}^0$ defined by (4.4) is globally hypoelliptic (cf. Section 4.2) iff a is neither rational nor Liouville. In particular, solutions in the case $I[a] \notin \mathbb{Q}$ non-Liouville are necessarily smooth. By contrast, although we were also able to determine smooth solutions in the case $I[a] \in \mathbb{Q}$, not all solutions are of that sort. The counter-example given in [BCM93, Theorem 2.4] is*

$$u(t,x) = \sum_{N=1}^{\infty} e^{-iN(qx-\psi_q(t))} \tag{4.58}$$

Indeed $\liminf_{|k|\to\infty} |\widehat{u}^x(t,k)| = 1$ *implies* $(\widehat{u}^x(t,k))_{k\in\mathbb{Z}}$ *are slowly increasing, but not rapidly decreasing sequences. Then,* $u \in \mathcal{D}'(\mathbb{T}^n \times \mathbb{T})\backslash C^\infty(\mathbb{T}^n \times \mathbb{T})$ *even though* $(d_t + a(t) \wedge \partial_x)u = 0 \in \bigwedge^1 C^\infty(\mathbb{T}^n \times \mathbb{T})$.

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