GENERALIZED ZELDOVICH'S REGULARIZATION OF THE VACUUM ENERGY

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ABSTRACT – We generalize and improve the Zeldovich's regularization of the vacuum energy density and pressure. To do this we reinterpret his divergent sums. Also we introduce a scaling mass parameter m^* in order to cure a dimensionality inconsistence. It turns out that vacuum energy and pressure are scale dependent but not the state equation $\varepsilon = -P$.

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1. Introduction

One of the most interesting puzzles concerning the physical vacuum is how to conciliate its quantum (microscopic) structure with its structure at large. Ya Zeldovich took this puzzle seriously [1].

In appendix II of reference [1] he first used special relativity to show that, macroscopically, the physical vacuum can be regarded as a perfect fluid with a state equation given by

$$\varepsilon = -P \tag{1}$$

where ε is the energy density and P is the pressure. There is no argument from quantum physics in such appendix.

Equation (1) is well-known, mainly by cosmologists. It comes from the fact that the vacuum energy-momentum tensor is given by

$$T_{\mu\nu} = \Lambda g_{\mu\nu} = -P g_{\mu\nu} \tag{2}$$

where Λ is the so-called cosmological constant. Eq(s) (1) and (2) were, for the first time, ascribed to physical vacuum by Gliner [2]. Zeldovich deduced Eq(2), in the same paper, using the Lorentz Invariance of $T_{\mu\nu}$. A more explicit proof was given by Grøn [3]. The difference between the theoretical (in QFT) and experimental estimates of the cosmological constant Λ is, nowadays, a puzzling chalenge to the theoretical physicists [4].

In the same remarkable paper [1], appendix VIII, Zeldovich gave a proof of eq(1) using a kind of Pauli-Villars regularization [6-8] for energy density and pressure. In this work we point out some deficiences of Zeldovich's regularization proof and we propose a more general one that we also believe to be more physical and mathematically correct.

In section 2 we resume, for the sake of completeness, Zeldovich's regularization. In section 3 we introduce our proposal of regularization. In section 4 we introduce a scaling mass m^* in order to cure the dimensionality inconsistence of Zeldovich's regularization. In section 5 we make some conclusions.

2. Zeldovich's regularization

The zero point energy (ZPE) (or vacuum energy) of a scalar field with mass m is given by

$$\begin{aligned}\varepsilon &= \frac{1}{2}\left(\frac{1}{2\pi\hbar}\right)^3 \int_0^\infty c\sqrt{p^2+\mu^2}\; 4\pi p^2 dp \\ &= k\int_0^\infty \sqrt{p^2+\mu^2}\; p^2 dp \equiv kI(\mu)\end{aligned} \tag{3}$$

where $\mu = m\ c$.

For a spin one half field, the vacuum energy is negative

$$\varepsilon = -4kI(\mu)\ . \tag{4}$$

The vacuum pressure is given by

$$P = k\frac{1}{3}\int \frac{p^2}{\sqrt{p^2+\mu^2}}\, p^2 dp \equiv kF(\mu) \tag{5}$$

for spin zero field, and

$$P = -4k\ F(\mu) \tag{6}$$

for spin one half fields, where $I(\mu)$ and $F(\mu)$ are obvious from (3) and (5).

Zeldovich considers the physical vacuum as a perfect fluid which, in turn, is a mixture of zero-point oscillations of bosons and fermions fields. So, Zeldovich proposes that the vacuum's energy and pressure be defined by

$$\varepsilon = \sum_i C_i I(\mu_i) \tag{7a}$$

$$P = \sum_i C_i F(\mu_i) \tag{7b}$$

where the coefficients C_i are positive for bosons and negative for fermions. Now, "generalizing further", he writes, in the case of a continuous dependence on the mass of the fields,

$$\varepsilon = \int_0^\infty f(\mu) I(\mu) d\mu \tag{8a}$$

$$P = \int_0^\infty f(\mu) F(\mu) d\mu \tag{8b}$$

where the function $f(\mu)$ is the continuum limit of C_i in Eq(s) (7).

Now, he introduces a cut-off for the momentum p_0 and use a kind of Pauli-Villars regularization [6] to conclude that, if the function $f(\mu)$ satisfies the constraint equations

$$\int f(\mu) d\mu = 0 \tag{9a}$$

$$\int f(\mu) \mu^2 d\mu = 0 \tag{9b}$$

$$\int f(\mu) \mu^4 d\mu = 0 \tag{9c}$$

then the renormalized energy and pressure are given by

$$\varepsilon = \frac{1}{8} \int f(\mu) \mu^4 \ell n \ \mu d\mu \tag{10a}$$

$$P = -\frac{1}{8} \int f(\mu) \mu^4 \ell n \ \mu d\mu \tag{10b}$$

and therefore the equation (1) is satisfied.

The reader can make acquaintance with the details of this last part in the Zeldovich's paper. We don't repeat all the calculations here because ours follows his closely.

3. Generalized Zeldovich's regularization

Zeldovich's regularization has to do with the diverging integrals (8). Observe that integrals (8) implie that we have an infinite continuum of different fields (different masses) in the vacuum. If ones think the vacuum, in turn, as a kind of fluid, we have to conclude that this vacuum is a mixture of an infinite number (non countable) of fields, at their group states. Our approach does not eliminate this possibility, but, on the other hand, it allows any number of fields: finite, infinite countable and infinite non-countable.

Another shortcoming of integrals (8) is that f is a function only of the mass of the field and then it can not be a well defined function. For example, if we have two massless fields (e.g. neutrinos and photons), then we have an ambiguity to assign the value $f(\mu)$, with $\mu = 0$ since, for neutrinos (fermions), its vacuum energy is negative and for photons, it is positive.

Our proposal is very simple. We just have to modify Eq(s) (7) to:

$$\varepsilon = \sum_{\varphi \in \Phi} C(\varphi) I(\mu_\varphi) \tag{11a}$$

$$P = \sum_{\varphi \in \Phi} C(\varphi) F(\mu_\varphi) \tag{11b}$$

where the sums are taken on the all vacuum fields, μ_φ is the mass of the field φ and $C(\varphi)$ is a real function defined on the set of fields $\Phi = \{\varphi\}$. Observe that, formally, Eq(s) (11) are identical to Zeldovich's (7), but now the sum is on the set of fields Φ which can have any number of fields.

The integrals in (3) and (5) can be calculated for a cut-off p_0 and using the expansion

$$\sqrt{1 + \frac{\mu_\varphi^2}{p^2}} = 1 + \frac{\mu_\varphi^2}{2p^2} - \frac{1}{8}\frac{\mu_\varphi^4}{p^4} + \cdots \tag{12}$$

we find

$$I(\mu_\varphi, p_0) = \frac{\mu_\varphi^4 \ell n\ \mu_\varphi}{8} + \frac{\mu_\varphi^4}{32} + \frac{p_0^4}{4} + \frac{\mu_\varphi^2 p_0^2}{4} - \frac{\mu_\varphi^4 \ell n\ 2p_0}{8} + o\left(\frac{\mu_\varphi^6}{p_0^2}\right) \tag{13a}$$

$$F(\mu_\varphi, p_0) = \frac{\mu_\varphi^4 \ell n\ \mu_\varphi}{8} + \frac{7\mu_\varphi^4}{96} + \frac{p_0^4}{12} - \frac{\mu_\varphi^2 p_0^2}{12} + \frac{\mu_\varphi^4 \ell n\ 2p_0}{8} + o\left(\frac{\mu_\varphi^6}{p_0^2}\right). \tag{13b}$$

Substituting these expressions in (11) we obtain

$$\begin{aligned} \varepsilon(P_0) &= \sum_\varphi \frac{1}{8} C(\varphi)\mu_\varphi^4 \ell n\ \mu_\varphi + \frac{1}{32}\sum_\varphi C(\varphi)\mu_\varphi^4 \\ &+ \frac{p_0^4}{4}\sum_\varphi C(\varphi) + \frac{p_0^2}{4}\sum_\varphi C(\varphi)\mu_\varphi^2 - \frac{1}{8}\ell n(2p_0)\sum_\varphi C(\varphi)\mu_\varphi^4 \end{aligned} \tag{14a}$$

$$\begin{aligned} P(P_0) &= -\frac{1}{8}\sum_\varphi C(\varphi)\mu_\varphi^4 \ell n\ \mu_\varphi - \frac{7}{96}\sum_\varphi C(\varphi)\mu_\varphi^4 \\ &+ \frac{p_0^4}{12}\sum_\varphi C(\varphi) - \frac{p_0^2}{12}\sum_\varphi C(\varphi)\mu_\varphi^2 + \frac{1}{8}\ell n(2p_0)\sum_\varphi C(\varphi)\mu_\varphi^4 \,. \end{aligned} \tag{14b}$$

To get finite (renormalized) ε and P, with $p_0 \to \infty$, we must impose the constraint equations:

$$\sum_\varphi C(\varphi) = 0 \tag{15a}$$

$$\sum_\varphi C(\varphi)\mu_\varphi^2 = 0 \tag{15b}$$

$$\sum_\varphi C(\varphi)\mu_\varphi^4 = 0 \tag{15c}$$

and then the renormalized energy and pressure turns out to be:

$$\varepsilon = \frac{1}{8} \sum_{\varphi} C(\varphi) \mu_{\varphi}^4 \ell n \; \mu_{\varphi} \tag{16a}$$

$$P = -\frac{1}{8} \sum_{\varphi} C(\varphi) \mu_{\varphi}^4 \ell n \; \mu_{\varphi} \tag{16b}$$

and again Eq.(1) is obtained.

Now we can see in what sense our approach is a generalization of the Zeldovich's. If the set of fields Φ has a non-countable number of fields in a "non-degenerated state of mass", then the two definitions, namely (8) and (11), coincide. In a more mathematical language, if the application

$$\begin{array}{ccc} \Phi & \longrightarrow & \mathbb{R} \\ \varphi & \longmapsto & \mu_{\varphi} \in \mathbb{R} \end{array} \tag{17}$$

is injective and onto, we can write $f(\varphi) \equiv \tilde{f}(\mu_{\varphi})$ for any function f defined on Φ. Application (17) is nothing but a change of variables. Taking in account (17) we obtain

$$\sum_{\varphi} f(\varphi) = \int \tilde{f}(\mu_{\varphi}) d\mu_{\varphi} \tag{18}$$

and, therefore, the Zeldovich regularization is just a particular case of ours.

If we have a mass degenerescence then application (17) fails to be injective and Zeldovich's early regularization can not be applied. Of course, Zeldovich assumed correspondence (17) in some way without, apparentely, taking into account this kind of mass degenerescence described above.

Observe, in addition, that the Lorentz Invariance implies only Eq.(1) but QFT, in turn, provides a further mathematical expression for the renormalized energy and pressure of the vacuum.

4. Scaling the Generalized Zeldovich's Regularization

It is easy to show that to get Eq(s) (16) for vacuum energy and pressure the constraint equation (15c) is superfluous. The proof is as follows. We can rewrite Eq(s) (14a) and (14b) as

$$\begin{aligned} \varepsilon(p_0) &= \frac{1}{8} \sum_{\varphi} C(\varphi) \mu_{\varphi}^4 \ell n \left(\frac{\mu_{\varphi}}{2 p_0} \right) + \frac{1}{32} \sum_{\varphi} C(\varphi) \mu_{\varphi}^4 \\ &+ \frac{p_0^2}{4} \sum_{\varphi} C(\varphi) \mu_{\varphi}^2 + \frac{p_0^4}{4} \sum_{\varphi} C(\varphi) \,. \end{aligned} \tag{19}$$

$$
\begin{aligned}
P(p_0) &= \frac{1}{8}\sum_{\varphi} C(\varphi)\mu_\varphi^4 \ell n\left(\frac{2p_0}{\mu_\varphi}\right) - \frac{7}{96}\sum_{\varphi} C(\varphi)\mu_\varphi^4 \\
&+ \frac{p_0^2}{12}\sum_{\varphi} C(\varphi)\mu_\varphi^2 + \frac{p_0^4}{4}\sum_{\varphi} C(\varphi)\,.
\end{aligned} \tag{20}
$$

Now the constraints eq(s) (15a) and (15b) implies:

$$
\frac{\varepsilon(p_0)}{P(p_0)} = \frac{\frac{1}{8}\sum_{\varphi} C(\varphi)\mu_\varphi^4\, \ell n\left(\frac{\mu_\varphi}{2p_0}\right) + \frac{1}{32}\sum_{\varphi} C(\varphi)\mu_\varphi^4}{\frac{1}{8}\sum_{\varphi} C(\varphi)\mu_\varphi^4\, \ell n\left(\frac{2p_0}{\mu_\varphi}\right) - \frac{7}{96}\sum_{\varphi} C(\varphi)\mu_\varphi^4}\,. \tag{21}
$$

Taking the limit $p_0 \to \infty$

$$
\frac{\varepsilon}{P} = \lim_{p_0\to\infty} \frac{\varepsilon(p_0)}{P(p_0)} = -1\,. \tag{22}
$$

Therefore, the state equation $\varepsilon = -P$ is independent on the logarithmic divergence and the constraint equation (15c) is not necessary to prove it.

Nevertheless Eq.(15c) will be necessary to get a mathematically well defined expression for the vacuum energy and pressure. Observe that in Zeldovich's renormalized energy and pressure (Eq(s) 10), as well as in our Eq(s) (16), the logarithm has an argument which is not adimensional. In fact, μ_φ has dimension of momentum.

This is because constraint Eq(15c) spoils the adimensionality as it is easy to see in Eq(s) (14).* To remedy this situation we use a procedure often used in Zeta-Function regularization [5]. We introduce an arbitrary scaling mass parameter m^* and then $\mu^* = m^* c$. The mass m^* is a constant independent on the fields φ.

Inserting $\frac{\mu^*}{\mu^*} = 1$ in the argument of the logarithm in Eq(s) (19) and (20) we get

$$
\begin{aligned}
\varepsilon(p_0,\mu^*) &= \frac{1}{8}\sum_{\varphi} C(\varphi)\mu_\varphi^4 \ell n\left(\frac{\mu_\varphi}{\mu^*}\right) + \frac{1}{8}\sum_{\varphi} C(\varphi)\mu_\varphi^4 \ell n\left(\frac{\mu^*}{2p_0}\right) \\
&+ \frac{1}{32}\sum_{\varphi} C(\varphi)\mu_\varphi^4 + \frac{p_0^2}{4}\sum_{\varphi} C(\varphi)\mu_\varphi^2 + \frac{p_0^4}{4}\sum_{\varphi} C(\varphi)
\end{aligned} \tag{23}
$$

and

$$
\begin{aligned}
P(p_0,\mu^*) &= \frac{1}{8}\sum_{\varphi} C(\varphi)\mu_\varphi^4 \ell n\left(\frac{\mu_*}{\mu_\varphi}\right) + \frac{1}{8}\sum_{\varphi} C(\varphi)\mu_\varphi^4 \ell n\left(\frac{2p_0}{\mu^*}\right) \\
&- \frac{7}{96}\sum_{\varphi} C(\varphi)\mu_\varphi^4 - \frac{p_0^2}{12}\sum_{\varphi} C(\varphi)\mu_\varphi^2 + \frac{p_0^4}{12}\sum_{\varphi} C(\varphi)\,.
\end{aligned} \tag{24}
$$

*Observe that, above, we have proved that $\varepsilon = -P$ keeping the adimensionality of the argument of logarithm.

It follows immediately from Eq(s) (23) and (24) that all constraint equations (15) are necessary to get a vacuum energy and pressure independent on the cut-off p_0. We obtain

$$\varepsilon(\mu^*) = \frac{1}{8}\sum_{\varphi} C(\varphi)\mu_{\varphi}^4 \ell n\left(\frac{\mu_{\varphi}}{\mu^*}\right) \tag{25}$$

$$P(\mu^*) = -\frac{1}{8}\sum_{\varphi} C(\varphi)\mu_{\varphi}^4 \ell n\left(\frac{\mu_{\varphi}}{\mu^*}\right) \tag{26}$$

$$\varepsilon(\mu^*) = -P(\mu^*)\ . \tag{27}$$

Although now the expressions for vacuum energy and pressure are scale dependent, they are cut-off independent an don't suffer of inconsistence of dimensionality. Observe also that Eq.(27) is independent on the scale mass parameter. In the wording of reference [5], "the physics does not change, only our way of interpreting the constants".

5. Conclusions

In this work we generalize the Zeldovich's regularization of vacuum energy and pressure by Eq(s) (11). This avoids ambiguities to sum contributions from fields of same mass. In addition we show that the Zedolvich condition Eq(15c) is superfluous to prove state equation of the vacuum $\varepsilon = -P$, or in other words, this equation is independent on the logarithmic divergence.

However the Zeldovich expressions for vacuum energy and pressure, Eqs(10), suffer of dimensionality inconsistence. To cure this we introduce a scaling mass parameter m^*. Now the vacuum energy and pressure are scale dependent but not the state equation $\varepsilon = -P$.

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References

[1] Ya. B. Zeldovich, Sov. Phys. Uspekhi **11** (3), 381 (1968)

[2] E. B. Gliner, Sov. Phys. JETP **22** (2), 378 (1966)

[3] ϕ. Grϕn, Am. J. Phys. **54** (1), 46 (1986)

[4] S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989)

[5] P. Ramond, Field Theory: A Modern Primer, 2$^{\underline{nd}}$ Edition p. 81-84, Addison-Wesley Pub. Co. (1989).

[6] W. Pauli, and F. Villars, Rev. Mod. Phys. **21**, 434 (1949)

[7] R. Jost and J. Rayski, Helv. Phys. Acta **22**, 457 (1949)

[8] W. Umezawa, J. Yukawa and E. Yamada, Prog. Theor. Phys. **3**, 317 (1948)

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