ON A SYSTEM OF EVOLUTION EQUATIONS OF MAGNETOHYDRODYNAMIC TYPE

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ON A SYSTEM OF EVOLUTION EQUATIONS OF MAGNETOHYDRODYNAMIC TYPE

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ABSTRACT: *By using the spectral Galerkin method, we prove the existence and uniqueness of strong local solutions for a system of equations of magnetohydrodynamic type. Several estimates for the solution and their approximations are given. These estimates could be used for derivation of error bounds.*

KEY WORDS: Magnetohydrodynamic, Galerkin method, strong solution.

RESUMO: *SOBRE UM SISTEMA DE EQUAÇÕES DE EVOLUÇÃO DO TIPO DA MAGNETOHIDRODINÂMICA. Pelo uso do método de Galerkin espectral, provamos a existência local e unicidade de soluções fortes de um sistema de equações do tipo da magnetohidrodinâmica. Diversas estimativas para a solução e suas aproximações são obtidas. Tais estimativas poderiam ser usadas para obter estimativas de erro.*

PALAVRAS-CHAVE: Magnetohidrodinâmica, método de Galerkin, soluções fortes.

1. INTRODUCTION

In several situations the motion of incompressible electrical conducting fluids can be modelled by the so called equations of magnetohydrodynamics, which correspond to the Navier-Stokes' equations coupled with the Maxwell's equations. In the case where there is free motion of heavy ions, not directly due to the electric field (see Schlüter [9] and Pikelner [8]), these equations can be reduced to the following form:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\eta}{\rho_m}\Delta u + u.\nabla u - \dfrac{\mu}{\rho_m} h.\nabla h = f - \dfrac{1}{\rho_m}\nabla(p^* + \dfrac{\mu}{2}h^2), \\ \dfrac{\partial h}{\partial t} - \dfrac{1}{\mu\sigma}\Delta h + u.\nabla h - h.\nabla u = -\text{grad } w, \\ \text{div } u = 0, \\ \text{div } h = 0, \end{cases} \tag{1.1}$$

together with suitable boundary and initial conditions.

Here, u and h are respectively the unknown velocity and magnetic fields; p^* is the unknown hydrostatic pressure; w is an unknown function related to the motion of heavy ions (in such way that the density of electric current, j_0, generated by this motion satisfies the relation $rot\ j_0 = -\sigma\nabla w$); ρ_m is the density of mass of the fluid (assumed to be a positive constant); $\mu > 0$ is the constant magnetic permeability of the medium; $\sigma > 0$ is the constant electric conductivity; $\eta > 0$ is the constant viscosity of the fluid; f is an given external force field.

The stationary problem corresponding to (1.1) was considered by Chizhonkov [2]; while the question of (local) existence of solution of the evolution problem (1.1) was analysed by Lassner [7] by making the use of semigroup techniques as the ones in Fujita and Kato [5] (he also studied the asymptotic behavior of the solution as $t \to 0^+$).

In this paper we will consider the problem of local existence of strong solutions of (1.1) as in Lassner [7], with homogeneous boundary condition for u an h for simplicity of exposition, but under weaker assumptions on the initial data. Also, differently from [7], we will use the more constructive spectral Galerkin method of approximation. Thus, the results in this paper form the theoretical basis for future numerical analysis of the problem: here we will obtain estimates for the approximate solutions that will be fundamental in a forthcoming paper in which we will obtain optimal error estimates for such Galerkin approximations. These estimates will also play a role in the proof of global existence of solutions of (1.1).

Finally, the paper is organized as follows: in Section 2 we state the basic assumptions and results that will be used later in the paper; we also rewrite (1.1) in a more suitable weak form; we describe the approximation method and state the results of the paper (Theorems 2.1, 2.2 and 2.3). Each one of the following sections will be devoted to their proofs.

2. PRELIMINARIES AND RESULTS

Let $\Omega \subseteq \mathbb{R}^n$, $n = 2$ or 3, be a bounded domain with boundary $\partial\Omega$ of Class $C^{1,1}$. We denote by $H^s(\Omega)$ the Sobolev Spaces on Ω with norm $\|\cdot\|_s$, $(\cdot,\cdot)$ denotes the L^2-norm on Ω. $H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|_1$. Also, we denote by $L^p(\Omega)$ for $1 \leq p \leq \infty$ the usual Lebesgue Spaces and by $\|\cdot\|_{L^p}$ the L^p-norm on Ω. With the same symbols we denote the spaces of n-dimensional vector functions.

We put

$$\begin{aligned} C_{0,\sigma}^\infty &= \{v \in C_0^\infty(\Omega)/\text{div} v = 0\} \\ H &= \text{closure of } \ C_{0,\sigma}^\infty(\Omega) \ \text{ in } \ (L^2(\Omega))^n \\ V &= \text{closure of } \ C_{0,\sigma}^\infty(\Omega) \ \text{ in } \ (H^1(\Omega))^n . \end{aligned}$$

It is possible to show that

$$V = \{v \in H_0^1(\Omega) | \text{div } v = 0\}.$$

We recall the Helmholtz decomposition of vector field: $L^2(\Omega) = H \oplus G$, being $G = \{\phi | \phi = \nabla p \ , \ p \in H^1(\Omega)\}$.

Throughout the paper P will denote the orthogonal projection from $L^2(\Omega)$ onto H. Then, the operator $A : H \to H$ given by $A = -P\Delta$ with domain $D(A) = H^2(\Omega) \cap V$ is called the Stokes' operator. It is well know that A is a positive definite self-adjoint operator and it is characterized by the relation

$$(Aw, v) = (\nabla w, \nabla v) \quad \text{for all} \quad w \in D(A), \ v \in V.$$

The operator A^{-1} is linear continuous from H into $D(A)$, and, since the injection of $D(A)$ in H is compact, A^{-1} can be considered as a compact operator in H. As an operator in H, A^{-1} it is also self-adjoint. By a well know theorem of Hilbert, there exists a sequence of positive numbers $\mu_j > 0, \mu_{j+1} \leq \mu_j$ and an orthonormal basis of $H, (w_j)$ such that $A^{-1}w_j = \mu_j w_j$. We denote $\lambda_j = \mu_j^{-1}$. Since A^{-1} has range in $D(A)$ we obtain that $Aw_j = \lambda_j w_j$, $w_j \in D(A)$,$0 < \lambda_1 < \ldots \leq \lambda_j \leq \lambda_{j+1} \leq \ldots$ and $\lim_{j\to+\infty} \lambda_j = +\infty$. Also, $\{w_j\}_{j=1}^{\infty}$ is an orthonormal basis for H, and $\{w_j/\sqrt{\lambda_j}\}_{j=1}^{\infty}$ and $\{w_j/\lambda_j\}_{j=1}^{\infty}$ form orthonormal basis in V (with the inner product $(\nabla u, \nabla v), u, v \in V$) and $H^2(\Omega) \cap V$ (with the inner product $(Au, Av), u, v \in D(A)$), respectively.

By using the properties of P, we can reformulate the problem (1.1), with homogeneous boundary conditions, as follows: find u, h, in suitable spaces to be exactly defined later on, satisfying

$$\begin{cases} \alpha(u_t, \phi) + \alpha(u.\nabla u, \phi) - (h.\nabla h, \phi) + \nu(Au, \phi) = (\alpha f, \phi) \\ (h_t, \psi) + (u.\nabla h, \psi) - (h.\nabla u, \psi) + \gamma(Ah, \psi) = 0 \\ \text{for} \quad 0 < t < T \quad , \quad \forall \phi, \psi \in V \\ u(0) = u_0 \quad , \quad h(0) = h_0. \end{cases} \tag{2.1}$$

Here we have denoted

$$\alpha = \frac{\rho_m}{\mu} \quad , \quad \nu = \frac{\eta}{\mu} \quad , \quad \gamma = \frac{1}{\mu\sigma} .$$

Now, we define strong solutions of the problem (2.1).

Definition. Let $u_0, h_0 \in V$ and $f \in L^2(0,T; L^2(\Omega))$. By a strong solution of the problem (2.1), we mean a pair of vector-valued functions (u, h) such that $u, h \in L^\infty(0,T;V) \cap L^2(0,T;D(A))$ and that satisfies (2.1).

Remark. In what follows, we will prove that if (u, h) is a strong solution of (2.1) then $u_t, h_t \in L^2(0,T;H)$. This condition, together with $u, h \in L^2(0,T;D(A))$, implies by interpolation (see, Temam p. 260), that u, h are almost everywhere equal to a continuous function from $[0,T]$ into V, consequently the initial conditions $u(0) = u_0$ and $h(0) = h_0$ are meaningful.

To prove (local) existence of strong solutions we will use the spectral Galerkin method applied to (2.1). That is, we consider the finite dimensional subspaces

$V_k = \text{span}[w^1, \ldots, w^k]$, $k \in N$, the corresponding orthogonal projections $P_k : H \to V_k$ and the approximate solutions

$$u^k(x,t) = \sum_{i=1}^{k} c_{ik}(t) w^i(x) \,, \quad h^k(x,t) = \sum_{i=1}^{k} d_{ik}(t) w^i(x) \,,$$

developed in terms of eigenfunctions of Stokes' operator, satisfying the following equations:

$$\begin{cases} \alpha u_t^k + \nu A u^k + \alpha P_k(u^k.\nabla u^k) - P_k(h^k.\nabla h^k) - \alpha P_k f = 0, \\ h_t^k + \gamma A h^k + P_k(u^k.\nabla h^k) - P_k(h^k.\nabla u^k) = 0, \\ u^k(0) = P_k u_0 \quad , \quad h^k(0) = P_k h_0. \end{cases} \tag{2.2}$$

This is equivalent to the weak form

$$\begin{cases} \alpha(u_t^k, \phi) + \nu(\nabla u^k, \nabla \phi) + \alpha(u^k.\nabla u^k, \phi) - (h^k.\nabla h^k, \phi) = \alpha(f, \phi) \quad \forall \phi \in V_k \\ (h_t^k, \psi) + \gamma(\nabla h^k, \nabla \psi) + (u^k.\nabla h^k, \psi) - (h^k.\nabla u^k, \psi) = 0 \quad \forall \psi \in V_k \\ u^k(0) = P_k u_0 \quad , \quad h^k(0) = P_k h_0. \end{cases} \tag{2.3}$$

By using these approximations, we will prove the following results.

Theorem 2.1. Let the initial values $u_0, h_0 \in V$ and the external force $f \in L^2(0,T; L^2(\Omega))$. Then, on a (possibly small) time interval $[0, T_1]$, $0 < T_1 \le T$, problem (2.1) has a unique strong solution (u, h). This solution belongs to $C([0,T_1], V)$.

Moreover, there exist continuous functions F and G such that for any $t \in [0, T_1]$, there hold

$$||\nabla u(t,\cdot)||^2 + ||\nabla h(t,\cdot)||^2 + \int_0^t (||Au(\tau,\cdot)||^2 + ||Ah(\tau,\cdot)||^2) d\tau \le F(t),$$

and

$$\int_0^t (||u_t(\tau,\cdot)||^2 + ||h_t(\tau,\cdot)||^2) d\tau \le G(t).$$

Also, the same kind of estimates hold uniformly in $k \in N$ for the Galerkin approximations (u^k, h^k).

$$= 2\alpha(f, u^k) \le C_\varepsilon ||f||^2 + \varepsilon ||u^k||^2,$$

for any $\varepsilon > 0$ and a suitable positive constant C_ε.

By using Poincaré's inequality and by taking ϵ small enough, after an integration in time and an application of Gronwall's inequality, (3.1) implies the global existence of u^k and h^k and also that

$$u^k, h^k \quad \text{are uniformly bounded in} \quad L^\infty(0,T;H) \cap L^2(0,T;V). \tag{3.2}$$

By taking $\phi = Au^k$ and $\psi = Ah^k$ in (2.3) (i) and (ii), respectively, we have

$$\frac{\alpha}{2}\frac{d}{dt}||\nabla u^k||^2 + \nu ||Au^k||^2 = \alpha(f, Au^k) + (h^k.\nabla h^k, Au^k) - \alpha(u^k.\nabla u^k, Au^k), \tag{3.3}$$

$$\frac{1}{2}\frac{d}{dt}||\nabla h^k||^2 + \gamma ||Ah^k||^2 = (h^k.\nabla u^k, Ah^k) - (u^k.\nabla h^k, Ah^k). \tag{3.4}$$

As before, for any $\varepsilon > 0$ and suitable $C_\varepsilon > 0$ we have

$$|(f, Au^k)| \le C_\varepsilon ||f||^2 + \varepsilon ||Au^k||^2 .$$

On the other hand, we can use the estimate in Duff [4, p. 464] in the third term of the right-side of (3.3) to obtain

$$|(u^k \nabla u^k, Au^k)| \le C_\varepsilon ||\nabla u^k||^6 + \varepsilon ||Au^k||^2.$$

To estimate the second term in the right-side of (3.3), we use the Sobolev type inequality (see [10]) $||\varphi||_L^4 \le ||\varphi||^{1/4} ||\nabla\varphi||^{3/4}$ as follows

$$\begin{aligned}
|(h^k.\nabla h^k, Au^k)| &\le ||h^k||_{L^4} ||\nabla h^k||_{L^4} ||Au^k|| \\
&\le C||\nabla h^k|| (||\nabla h^k||^{1/4} ||Ah^k||^{3/4}) ||Au^k|| \\
&\le C||\nabla h^k||^{5/4} ||Ah^k||^{3/4} ||Au^k|| \\
&\le C_\varepsilon ||\nabla h^k||^{5/2} ||Ah^k||^{3/2} + \varepsilon ||Au^k||^2 \\
&\le C_\varepsilon C_\delta ||\nabla h^k||^{10} + \delta ||Ah^k||^2 + \varepsilon ||Au^k||^2,
\end{aligned}$$

where we have used Hölder's and Young's inequalities and $\delta > 0$ (with suitable $C_\delta > 0$).

Analogously, we can prove that the terms in the right hand side of (3.4) satisfy

$$|(u^k.\nabla h^k, Ah^k)| \le C_\delta ||\nabla u^k||^8 ||\nabla h^k||^2 + \delta ||Ah^k||^2,$$

Theorem 2.2. In addition to the assumptions of Theorem 2.1, assume that $u_0, h_0 \in V \cap H^2(\Omega)$ and that $f_t \in L^2(0,T; L^2(\Omega))$. Then, the functions u, h satisfy

$$||u_t(t,\cdot)||^2 + ||h_t(t,\cdot)||^2 + \int_0^t (||\nabla u_t(\tau,\cdot)||^2 + ||\nabla h_t(\tau,\cdot)||^2)d\tau \leq H(t);$$

$$||Au(t,\cdot)||^2 + ||Ah(t,\cdot)||^2 \leq L(t);$$

$$\int_0^t (||u_{tt}(\tau,\cdot)||^2_{V^*} + ||h_{tt}(\tau,\cdot)||^2_{V^*})d\tau \leq M(t),$$

for every $t \in [0, T_1]$, where $H(\cdot)$, $L(\cdot)$ and $M(\cdot)$ are continuous function in $t \in [0, T_1]$. Also, $u, h \in C^1([0,T_1]; H) \cap C([0,T_1], D(A))$.

Moreover, the same kind of estimates hold uniformly in k for the Galerkin approximations (u^k, h^k).

As a consequence of the above, by using the results of Amrouche and Girault [1], we conclude:

Theorem 2.3. Under the hypothesis of Theorem 2.1, there exist unique functions $p, w \in L^2(0,T; H^1(\Omega)/I\!R)$ such that by taking $p^* = p - \frac{\mu}{2}h^2$, (u, h, p^*, w) is solution of (1.1). Under the hypothesis of Theorem 2.2, $p, w \in L^\infty(0,T_1; H^1(\Omega)/I\!R) \cap C([0,T_1], L^2(\Omega)/I\!R)$.

3. THE PROOF OF THEOREM 2.1

The following estimates show the global existence (in t) of the approximate solutions u^k, h^k.

Setting $\phi = u^k$ and $\psi = h^k$ in (2.3) (i) and (ii), respectively, we have

$$\begin{aligned}
\frac{\alpha}{2}\frac{d}{dt}||u^k||^2 + \nu||\nabla u^k||^2 &= \alpha(f, u^k) + (h^k.\nabla h^k, u^k) \\
\frac{1}{2}\frac{d}{dt}||h^k||^2 + \gamma||\nabla h^k||^2 &= (h^k.\nabla u^k, h^k),
\end{aligned}$$

since $(u^k.\nabla u^k, u^k) = (u^k.\nabla h^k, h^k) = 0$.

Adding the above inequalities and observing that $(h^k.\nabla h^k, u^k) + (h^k.\nabla u^k, h^k) = 0$, we obtain

$$\frac{d}{dt}(\alpha||u^k||^2 + ||h^k||^2) + 2(\nu||\nabla u^k||^2 + \gamma||\nabla h^k||^2) \qquad (3.1)$$

$$|(h^k.\nabla u^k, Ah^k)| \leq C_\epsilon C_\delta ||\nabla h^k||^8 ||\nabla u^k||^2 + \epsilon||Au^k||^2 + \delta||Ah^k||^2.$$

If we add equalities (3.3) and (3.4) and use the above estimates with suitable small enough ε and δ, we are left with the following differential inequality,

$$\frac{d}{dt}(\alpha||\nabla u^k||^2 + ||\nabla h^k||^2) + \nu||Au^k||^2 + \gamma||Ah^k||^2 \tag{3.5}$$

$$\leq C||f||^2 + C(||\nabla u^k||^6 + ||\nabla h^k||^{10} + ||\nabla u^k||^8||\nabla h^k||^2 + ||\nabla h^k||^8||\nabla u^k||^2).$$

That is,

$$\frac{d}{dt}\xi^k(t) \leq \tau(t) + C\xi^k(t)^5$$

where $\xi^k(t) = \alpha||\nabla u^k(t)||^2 + ||\nabla h^k(t)||^2$ and $\tau(t) = C||f(t)||^2$. By using Lemma 3 in of Heywood [6, p. 656], we conclude that there exists $T_1 \in [0, T]$ such that

$$\alpha||\nabla u^k(t)||^2 + ||\nabla h^k(t)||^2 \leq F_0(t, \xi(0))$$

where $\xi(0) = \alpha||\nabla u_0||^2 + ||\nabla h_0||^2$, and F_0 is the solution of the initial value problem

$$F_0' = CF_0^5 + \tau(t) \quad , \quad F_0(0) = \xi(0).$$

Returning to (3.5), we are left with

$$\alpha||\nabla u^k(t)||^2 + ||\nabla h^k(t)||^2 + \int_0^t (\nu||Au^k||^2 + \gamma||Ah^k||^2)ds \tag{3.6}$$

$$\begin{aligned} &\leq \alpha||\nabla u_0||^2 + ||\nabla h_0||^2 + \int_0^t \tau(s)ds + CF_0^5(t, \xi(0)) \\ &\equiv F(t). \end{aligned}$$

Thus,

$$u^k, h^k \quad \text{are uniformly bounded in} \quad L^\infty(0, T_1; V) \cap L^2(0, T; D(A)) \; . \tag{3.7}$$

Now, by taking $\phi = u_t^k$ and $\psi = h_t^k$ in (2.3) (i) and (ii), respectively, we get

$$\begin{aligned} \alpha||u_t^k||^2 &= \alpha(f, u_t^k) + (h^k.\nabla h^k, u_t^k) - \alpha(u^k.\nabla u^k, u_t^k) - \nu(Au^k, u_t^k), \\ ||h_t^k||^2 &= (h^k.\nabla u^k, h_t^k) - (u^k.\nabla h^k, h_t^k) - \gamma(Ah^k, h_t^k) \; . \end{aligned}$$

From this, we have

$$\int_0^t ||u_t^k(s)||^2 ds \leq C\int_0^t (||f(s)||^2 + ||h^k.\nabla h^k||^2 + ||u^k.\nabla u^k||^2 + ||Au^k||^2)ds, \tag{3.8}$$

$$\int_0^t ||h_t^k(s)||^2 ds \leq C\int_0^t (||h^k.\nabla u^k||^2 + ||u^k.\nabla h^k||^2 + ||Ah^k||^2)ds. \tag{3.9}$$

Now, bearing in mind (3.7) and the Sobolev embedding $H^2 \hookrightarrow L^\infty$, we obtain the following estimates:

$$\begin{aligned} ||h^k.\nabla h^k||^2 &\leq ||h^k||^2_{L^\infty}||\nabla h^k||^2 \\ &\leq C||Ah^k||^2||\nabla h^k||^2 \\ &\leq C \sup_{0\leq t\leq T_1} F(t) \quad ||Ah^k||^2, \end{aligned}$$

Similarly,

$$\begin{aligned} ||u^k.\nabla u^k||^2 &\leq C \sup_{0\leq t\leq T_1} F(t)||Au^k||^2, \\ ||h^k.\nabla u^k||^2 &\leq C \sup_{0\leq t\leq T_1} F(t)||Ah^k||^2, \\ ||u^k.\nabla h^k||^2 &\leq C \sup_{0\leq t\leq T_1} F(t)||Au^k||^2. \end{aligned}$$

Since, $\int_0^t (\nu||Au^k(s)||^2 + \gamma||Ah^k(s)||^2)ds \leq F(t)$ for all $t \in [0, T_1]$ (see (3.6)), we deduce from the above estimates, together with (3.8) and (3.9), that

$$u_t^k; h_t^k \quad \text{are uniformly bounded in} \quad L^2(0, T_1; H). \tag{3.10}$$

Now, by standard methods (see for instance [3], [6], [10]), these estimates enable us to take the limit as $k \to +\infty$ in (2.3) and conclude that a solution for (2.1) exists in the stated class.

In the following we prove the uniqueness of such a solution. Consider that (u_1, h_1) and (u_2, h_2) are two solutions of the problem (2.1) with the same f and u_0, h_0 and define the diferences $w = u_1 - u_2$ and $v = h_1 - h_2$. They satisfy

$$\begin{aligned} \alpha(w_t, \phi) + \nu(Aw, \phi) &= -\alpha(w.\nabla u_1, \phi) - \alpha(u_2.\nabla w, \phi) + (v.\nabla h_1, \phi) + (h_2.\nabla v, \phi) \\ (v_t, \psi) + \gamma(Av, \psi) &= -(u_1.\nabla v, \psi) - (w.\nabla h_2, \psi) + (v.\nabla u_1, \psi) + (h_2.\nabla w, \psi) \end{aligned}$$

for any $\phi, \psi \in V$; also $w(0) = v(0) = 0$.

By setting $\phi = w$ and $\psi = v$ in the above in equalities, we obtain

$$\frac{\alpha}{2}\frac{d}{dt}||w||^2 + \nu||\nabla w||^2 = -\alpha(w.\nabla u_1, w) + (v.\nabla h_1, w) + (h_2.\nabla v, w), \tag{3.11}$$

$$\frac{1}{2}\frac{d}{dt}||v||^2 + \gamma||\nabla v||^2 = -(w.\nabla h_2, v) + (v.\nabla u_1, v) + (h_2.\nabla w, v). \tag{3.12}$$

Now, we observe that for any $\varepsilon > 0$ and suitable $C_\varepsilon > 0$

$$\begin{aligned}
|\alpha(w.\nabla u_1, w)| &\le ||w||||\nabla u_1||_{L^4}||w||_{L^4} \\
&\le C||w||||Au_1||||\nabla w|| \\
&\le C_\varepsilon||Au_1||^2||w||^2 + \varepsilon||\nabla w||^2,
\end{aligned}$$

thanks to the Sobolev embedding $H^2 \hookrightarrow L^4$ and Hölder and Yuong inequalities.

Similarly,

$$\begin{aligned}
|(v.\nabla h_1, w)| &\le C_\varepsilon||Ah_1||^2||v||^2 + \varepsilon||\nabla w||^2, \\
|(h_2.\nabla v, w)| &\le C_\delta||Ah_2||^2||w||^2 + \delta||\nabla v||^2, \\
|(w.\nabla h_2, v)| &\le C_\delta||Ah_2||^2||w||^2 + \delta||\nabla v||^2, \\
|(v.\nabla u_1, v)| &\le C_\delta||Au_1||^2||v||^2 + \delta||\nabla v||^2, \\
|(h_2.\nabla w, v)| &\le C_\varepsilon||Ah_2||^2||v||^2 + \varepsilon||\nabla w||^2,
\end{aligned}$$

for any positive ε and δ and suitable positive constants C_ε, C_δ.

Consequently, by taking $\varepsilon = \nu/6$ and $\delta = \gamma/6$, in the above inequalities, **(3.11)** and (3.12) imply the differential inequality

$$\begin{aligned}
&\frac{d}{dt}(\alpha||w||^2 + ||v||^2) + \nu||\nabla w||^2 + \gamma||\nabla v||^2 \\
\le\ & C(\alpha||w||^2 + ||v||^2)(||Ah_1||^2 + ||Ah_2|| + ||Au_1||^2 + ||Au_2||^2).
\end{aligned}$$

By integration in time, the use of Gronwall's inequality and (3.6), we obtain

$$\begin{aligned}
\alpha||w(t)|| &+ ||v(t)||^2 + \nu\int_0^t ||\nabla w(s)||^2 ds + \gamma\int_0^t ||\nabla v(s)||^2 ds \\
&\le (\alpha||w(0)||^2 + ||v(0)||^2)e^{CF(t)},
\end{aligned}$$

Since $w(0) = v(0) = 0$, we finally obtain $w(t) = v(t) = 0$ for all $t \in [0, T_1]$. Hence $u_1 = u_2$ and $h_1 = h_2$ and the uniqueness is proved. For the proof of the continuity of $u(t)$ and $h(t)$ in the $H^1(\Omega)$ -norm, we proceed as in the remark after the definition of strong solution. This completes the proof of the Theorem.

4. THE PROOF THE THEOREM 2.2

To prove Theorem 2.2 we will need further estimates for the approximations u^k, h^k. To this end, we differentiate (2.3) (i) and (ii) with respect to t and set $\phi = u^k_t$ and $\psi = h^k_t$. We are left with

$$\frac{\alpha}{2}\frac{d}{dt}||u^k_t||^2 + \nu||\nabla u^k_t||^2 = \alpha(f_t, u^k_t) - \alpha(u^k_t.\nabla u^k, u^k_t) \tag{4.1}$$

$$-\alpha(u^k \nabla u^k_t, u^k_t) + (h^k_t.\nabla h^k, u^k_t) + (h^k.\nabla h^k_t, u^k_t),$$

$$\frac{1}{2}\frac{d}{dt}||h^k_t||^2 + \gamma||\nabla h^k_t||^2 = -(u^k_t.\nabla h^k, h^k_t) - (u^k.\nabla h^k_t, h^k_t) \tag{4.2}$$

$$+(h^k_t.\nabla u^k, h^k_t) + (h^k.\nabla u^k_t, h^k_t)\ .$$

By using the Cauchy-Schwarz and Young's inequalities, we obtain

$$\alpha|(f_t, u^k_t)| \leq \frac{1}{2}||f_t||^2 + \frac{\alpha}{2}||u^k_t||^2\ .$$

Also, we observe that

$$(u^k.\nabla u^k_t, u^k_t) = (u^k.\nabla h^k_t, h^k_t) = 0\ .$$

The second term in (4.2) can be estimated in the following form

$$\begin{aligned} |\alpha(u^k_t.\nabla u^k, u^k_t)| &\leq \alpha||u^k_t||||\nabla u^k||_{L^4}||u^k_t||_{L^4} \\ &\leq C_\varepsilon||u^k_t||^2||Au^k||^2 + \varepsilon||\nabla u^k_t||^2\ , \end{aligned}$$

for any $\varepsilon > 0$ and suitable $C_\varepsilon > 0$. Here we have used the Hölder's and Young's inequalities, together with the Sobolev embedding $H^1 \hookrightarrow L^4$.

Similarly, we have

$$\begin{aligned} |(h^k_t.\nabla h^k, u^k_t)| &\leq C_\varepsilon||h^k_t||^2||Ah^k||^2 + \varepsilon||\nabla u^k_t||^2, \\ |(h^k_t.\nabla u^k, h^k_t)| &\leq C_\delta||h^k_t||^2||Au^k||^2 + \delta||\nabla h^k_t||^2, \\ |(u^k_t.\nabla h^k, h^k_t)| &\leq C_\delta||u^k_t||^2||Ah^k||^2 + \delta||\nabla h^k_t||^2, \end{aligned}$$

for any $\varepsilon, \delta > 0$ and suitable $C_\varepsilon, C_\delta > 0$.

To estimate the fifth term in (4.1), we use the Hölder's and Young's inequalities together with the Sobolev embedding $H^2 \hookrightarrow L^\infty$. We obtain for any $\delta > 0$ and suitable

$C_\delta > 0$,

$$\begin{aligned}|(h^k.\nabla h_t^k, u_t^k)| &\leq ||h^k||_{L^\infty}||\nabla h_t^k||||u_t^k|| \\ &\leq C_\delta||Ah^k||^2||u_t^k||^2 + \delta||\nabla h_t^k||^2 .\end{aligned}$$

Similary, for any $\varepsilon > 0$ and suitable $C_\varepsilon > 0$, we have

$$|(h^k.\nabla u_t^k, h_t^k)| \leq C_\varepsilon||Ah^k||^2||h_t^k||^2 + \varepsilon||\nabla u_t^k||^2 .$$

By taking small enough $\varepsilon > o$ and $\delta > 0$, by adding (4.1) and (4.2) and by using the above estimates, we are left with the following differential inequality

$$\frac{d}{dt}(\alpha||u_t^k||^2 + ||h_t^k||^2) + (\nu||\nabla u_t^k||^2 + \gamma||\nabla h_t^k||^2)$$

$$\leq ||f_t||^2 + C(\alpha||u_t^k||^2 + ||h_t^k||^2)(||Au^k||^2 + ||h_t^k||^2 + 1),$$

where $C > 0$ is a generic constant. Consequently, for $0 \leq t \leq T_1$, we obtain

$$\begin{aligned}\alpha||u_t^k||^2 &+ ||h_t^k||^2 + C\int_0^t(\nu||\nabla u_t^k||^2 + \gamma||\nabla h_t^k||^2)ds \\ &\leq \alpha||u_t^k(0)||^2 + ||h_t^k(0)||^2 + \int_0^t C||f_t||^2 ds \\ &+ C\int_0^t(\alpha||u_t^k||^2 + ||h_t^k||^2)(||Au^k||^2 + ||Ah^k||^2 + 1)ds .\end{aligned}$$

Now, analogously as in Heywood [6, p. 665], we can prove that

$$\alpha||u_t^k(0)||^2 + ||h_t^k(0)||^2 \leq L ,$$

where $L > 0$ is a constant independent of k.

Thus, by applying Gronwall's inequality to the above integral inequality, we obtain

$$\alpha||u_t^k||^2 + ||h_t^k||^2 + C\int_0^t(\nu||\nabla u_t^k||^2 + \gamma||\nabla h_t^k||^2)ds \tag{4.3}$$

$$\begin{aligned}&\leq C(\int_0^t(||f_t||^2 ds + L)e^{\int_0^t \theta(s)ds} \\ &\equiv G(t)\end{aligned}$$

for all $t \in [0, T_1]$, with $\theta(t) = C(||Au^k||^2 + ||Ah^k||^2 + 1)$. By the estimates in (3.6), we have

$$\int_0^t \theta(s)ds \leq Ct + CF(t)$$

for all $t \in [0,T_1]$, where $F(t)$ is a function independent of k. Consequently, we conclude that

$$u_t^k, h_t^k \quad \text{are uniformly bounded in} \quad L^\infty(0,T_1;H)\cap L^2(0,T_1;V). \tag{4.4}$$

Now, by taking $\phi = Au^k$ and $\psi = Ah^k$ in (2.3) (i) and (ii), respectively, we obtain

$$\nu||Au^k||^2 \le C(||f||^2 + ||u_t^k||^2 + ||u^k.\nabla u^k||^2 + ||h^k.\nabla h^k||^2), \tag{4.5}$$

$$\gamma||Ah^k||^2 \le C(||h_t^k||^2 + ||u^k.\nabla u^k||^2 + ||h^k.\nabla u^k||^2)\ . \tag{4.6}$$

We observe that

$$\begin{aligned} ||u^k.\nabla u^k||^2 &\le ||u^k||^2_{L^4}||\nabla u^k||^2_{L^4} \\ &\le C||\nabla u^k||^2(||\nabla u^k||^{1/2}||Au^k||^{3/2} \\ &\le C||\nabla u^k||^{5/2}||Au^k||^{3/2} \\ &\le C_\varepsilon||\nabla u^k||^{10} + \varepsilon||Au^k||^2 \end{aligned}$$

for any $\varepsilon > 0$ and suitable $C_\varepsilon > 0$.

Analogously as above, we obtain

$$\begin{aligned} ||h^k.\nabla h^k||^2 &\le C_\delta||\nabla h^k||^{10} + \delta||Ah^k||^2, \\ ||u^k.\nabla h^k||^2 &\le C_\delta||\nabla u^k||^8||\nabla h^k||^2 + \delta||Ah^k||^2, \\ ||h^k.\nabla u^k||^2 &\le C_\varepsilon||\nabla h^k||^8||\nabla u^k||^{10} + \varepsilon||Au^k||^2. \end{aligned}$$

Now, by adding the inequalities (4.5) and (4.6) and by using the above estimates, together with a suitable choice of small enough ε and δ, we are left with.

$$\begin{aligned} \nu||Au^k||^2 + \gamma||Ah^k||^2 &\le C(||f||^2 + ||u_t^k||^2 + ||h_t^k||^2 \\ &+ ||\nabla u^k||^{10} + ||\nabla h^k||^{10} + ||\nabla u^k||^8||\nabla h^k||^2 \\ &+ ||\nabla u^k||^8||\nabla u^k||^2) \\ &\le C(||f||^2 + F(t)^5 + G(t)) \\ &\equiv H(t) \end{aligned}$$

for all $t \in [0,T_1]$.

Here we have used the estimates obtained in the proof Theorem 1 and (4.3). Hence, we have

$$u^k, h^k \quad \text{are uniformly bounded in} \quad L^\infty(0,T_1;D(A))\ . \tag{4.7}$$

Now, by standard methods [3], [6], [10], these estimates enable us to assert that the solution (u,h) of the problem (2.1) satisfies the stated estimates.

To prove the continuity of $u_t(t)$ in the L^2-norm, we only need to show that u_{tt} is in $L^2(0,T_1;V^*)$. In fact, if $u_{tt}\in L^2(0,T_1;V^*)$ then the fact that u_t is in $L^2(0,T_1;V)$, implies that $u\in C^1([0,T];H)$ (Lemma 1.2, p. 260 in Temam [10]).

To prove that $u_{tt}\in L^2(0,T_1,V^*)$, we observe that it is enough to show the existence of $C>0$ (independent of k) such that

$$\int_0^{T_1}||u_{tt}^k(s)||_{V^*}^2 ds \le C\ .$$

To this end, we differentiate equation (2.2) with respect to t; we obtain

$$\begin{aligned} u_{tt}^k &= P_k(\alpha f_t + h_t^k.\nabla h^k + h^k.\nabla h_t^k - \alpha u_t^k.\nabla u^k - \alpha u^k.\nabla u_t^k) - \nu A u_t^k \\ &\equiv g^k\ . \end{aligned}$$

The above estimates for u^k and h^k imply that g^k is uniformly bounded in $L^2(0,T_1;V^*)$. In fact, we have

$$\begin{aligned} ||P_k(h_t^k.\nabla h^k)||_{V^*} &= \sup_{||v||_V\le 1} |(P_k h_t^k.\nabla h^k, v)| \\ &\le \sup_{||v||_V\le 1} |(h_t^k.\nabla h^k, P_k v)| \\ &\le C \sup_{||v||_V\le 1} ||h_t^k||_{L^4}||\nabla h^k||\,||v||_{L^4} \\ &\le C||\nabla h_t^k||\ . \end{aligned}$$

Here we have used the Sobolev embedding $H^1\hookrightarrow L^4$, the estimate (3.7) and the continuity of P_k in L^4 (von Wahl [11, p. XXIII]); C denotes a general constant depending only the previous estimates.

Consequently, due to estimates (4.4) we obtain

$$\int_0^{T_1}||P_k(h_t^k.\nabla h^k)||_{V^*}^2 ds \le C\int_0^{T_1}||\nabla h_t^k||^2 ds \le C,$$

where $C>0$ is independent of $k\in N$.

Also, we have

$$\begin{aligned} ||P_k u^k.\nabla u_t^k||_{V^*} &= \sup_{||v||_V\le 1} |(u^k\nabla u_t^k, P_k v)| \le ||u^k||_{L^\infty}||\nabla u_t^k|| \\ &\le C||\nabla u_t^k||. \end{aligned}$$

Consequently, $\int_0^{T_1} ||P_k u^k.\nabla u_t^k||^2_{V^*}.ds \leq C$ thanks to the above estimates.

Also, from

$$||Au_t^k||_{V^*} = \sup_{||v||_V \leq 1} |(Au_t^k, v)| = \sup_{||v||_V \leq 1} |(\nabla u_t^k, \nabla v)| \leq ||\nabla u_t^k||,$$

we conclude that $\int_0^{T_1} ||Au_t^k||^2_{V^*}.ds \leq C$.

The other terms in the g^k are analogously estimated.

To prove the continuity of $h_t(t)$ in the L^2-norm, we work exactly as before.

To finish the proof we have to show the continuity of $u(t)$ and $h(t)$ in the $H^2(\Omega)$-norm. We will only prove the continuity of $h(t)$; the proof for $u(t)$ is quite similar. Also, we will prove this continuity only at $t_0 = 0$; for other $t_0 > 0$ the argument is analogous.

We observe that $h \in L^\infty(0, T_1, D(A))$ (see (4.7)). Thus, given any sequence $\{t_k\}_{k=0}^{\infty} \subset I\!R^+$, with $t_k \to 0^+$ we can extract a subsequence such that $h(t_{n_k}) \to \bar{h}$ weakly in H^2 for some $\bar{h} \in H^2$. Since we konw (Theorem 2.1) that $h(t_{n_k}) \to h_0$ strongly, the above implies that $\bar{h} = h_0$. Moreover, since this holds for any sequence $\{t_k\}_{n=0}$, with $t_k \to 0^+$, we conclude that $h(t) \to h_0$ weakly in H^2 as $t \to 0^+$. Consequently, due to the lower semicontinuity with respect to the weak topology of the norm, we have $||Ah_0|| \leq \lim_{t\to 0^+} \inf ||Ah(t)||$. Now, if we are able to prove that

$$\lim_{t\to 0+} \sup ||Ah(t)|| \leq ||Ah_0||, \tag{4.8}$$

then we will have $\lim_{t\to 0+} ||Ah(t)|| = ||Ah_0||$, which together with the fact that $Ah(t) \to Ah_0$ weakly in L^2 will imply that $Ah(t) \to Ah_0$ strongly in L^2.

In order to prove (4.8) we proceed as follows: put $\psi = P\Delta h_t^k$ in (2.3) to obtain

$$\begin{aligned} ||\nabla h_t^k||^2 &+ \frac{1}{2}\gamma\frac{d}{dt}||Ah^k||^2 = (u^k.\nabla h^k - h^k.\nabla u^k,\ P\Delta h_t^k) \\ &= \frac{d}{dt}(u^k.\nabla h^k - h^k.\nabla u^k,\ P\Delta h^k) \\ &- (u_t^k.\nabla h^k + u^k.\nabla h_t^k - h_t^k.\nabla u^k - h^k.\nabla u_t^k, P\Delta h^k)\,. \end{aligned}$$

By integration with respect to time, and by using our previous estimates for u^k and h^k, we obtain

$$\begin{aligned} \gamma||Ah^k(t)||^2 &\leq \gamma||Ah_0||^2 + 2\{(u^k.\nabla h^k - h^k.\nabla u^k, P\Delta h^k) \\ &- (u_0^k.\nabla h_0^k - h_0^k.\nabla u_0^k, P\Delta h_0^k)\} + Mt, \end{aligned}$$

where M is a positive constant depending on the previous estimates in k. From this, we conclude

$$\begin{aligned}\gamma||Ah(t)||^2 &\leq \gamma||Ah_0||^2 + 2\{(u.\nabla h - h.\nabla u, P\Delta h) \\ &- (u_0.\nabla h_0 - h_0.\nabla u_0, P\Delta h_0)\} + Mt\end{aligned}$$

Now, since $u.\nabla h \to u_0.\nabla h_0$, $h.\nabla u \to h_0.\nabla u_0$ in L^2 and $P\Delta h \to P\Delta h_0$ weakly in L^2 as $t \to 0^+$. We obtain the desired result.

5. THE PROOF OF THE THEOREM 2.3

The observe that (2.1) (i) and (ii) are equivalent to $Au = P(\overline{f})$ and $Ah = P(\overline{g})$, where $\overline{f} = \alpha f - u_t - \alpha u.\nabla u + h.\nabla h$ and $\overline{g} = h.\nabla u - u.\nabla h - h_t$, respectively.

Now, we observe that under the hypothesis of the Theorem 2.1 (respectively of Theorem 2.2), we have $\overline{f}, \overline{g} \in L^2(0, T_1; L^2(\Omega))$ (respectively $\overline{f}, \overline{g} \in L^\infty(0, T_1; L^2(\Omega))$).

Therefore, Amrouch and Girault's results [1] imply that there are unique $p, w \in L^2(0, T_1; H^1(\Omega)/\mathbb{R})$ (respectively $p, w \in L^\infty(0, T_1; H^1(\Omega)/\mathbb{R} \cap C([0, T_1]; L^2(\Omega)/\mathbb{R}))$ such that

$$\begin{aligned}&-\nu\Delta u + \nabla p = \overline{f} \\ &\text{div } u = 0 \\ &u|_{\delta\Omega} = 0\end{aligned}$$

and

$$\begin{aligned}&-\gamma\Delta h + \nabla r = \overline{g} \\ &\text{div } h = 0 \\ &h|_{\delta\Omega} = 0.\end{aligned}$$

Now, it is enough to take $p^* = p - \frac{\mu}{2}h^2$ and Theorem 2.3 is proved.

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