KLEIN-GORDON WAVE EQUATION IN THE DE SITTER UNIVERSE

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ABSTRACT – We discuss the Klein-Gordon wave equation in the de Sitter Universe. This equation is obtained for the second order Casimir invariant operator, using Fantappié-Arcidicono methods.

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Abstract

We discuss the Klein-Gordon wave equation in the de Sitter Universe. This equation is obtained for the second order Casimir invariant operator, using Fantappié-Arcidicono methods.

Introduction

There are some papers published many years ago discussing wave equations (we call wave equation; D'Alembert, Klein-Gordon and Dirac wave equations) in the de Sitter Universe. Some of then study the unified description of elementary particles[1,2] and some others study the spherically symmetric in general reativity[3,4]. One of then presents a five-simensional representation of the electromagnetic and electron field equations in a curved spacetime which is then compared with the formalism proposed by Dirac in the case of the de Sitter spacetime[5]. In another paper Takeno[6] discussed a generalization of special Lorentz transformations in the Sitter spacetime. Raje[7] discussed the linear meson wave equation in the Sitter spacetime and obtained the second order wave equation using a method proposed by Kemmer[8] where he eliminates matrices which appear in the linear wave equation. The equation obtained by Raje is essentially the same proposed formely by Goto[9]. We also mention Snyder's paper[10] where he discussed a quantized space time.

More recently there are the papers by Gürsey[11,12] where in the first of then he present an introduction to the de Sitter group which a discussion of the structure of the group (de Sitter group), commutation relations, invariants and the generators of the de Sitter group which are rotation operators in a five-dimensional euclidean space. In the second paper he presented the Casimir operators for the de Sitter group and concludes the paper showing that a particle in a de Sitter Universe does not have a definite mass and spin, but definite eigenvalues of the two invariant Casimir operators of the group. He did not discuss the equations obtained from the invariant Casimir operators.

Tagirov[13] solved the D'Alembert wave equations in the de Sitter spacetime obtaining the solution of a second order differential equation+ in terms of the spherical harmonics and Gegenbauer polynomials. The fourth order differential wave equation is not mentioned. This theme was also the subject of studies by some other authors. Specifically we mention firstly, the paper by Börner and Dürr[14] where they discussed the de Sitter spacetime and derived an eigenvalue equations for the Casimir operator; Roman and Aghassi[15] constructed quantum mechanical equations of motion associated with particles of a given spin; and Fulling's book[16] where the author presents and discusses the aspects of quantum field theory in curved spacetime, where the de Sitter spacetime is a particular spacetime and secondly, Birrell & Davies book[17] where they discusses the same subject where many parametrizations of the de Sitter spacetime are presented, for example; the Steady-State Universe of Bondi & Gold[18] and Hoyle[19] which covers the half of the de Sitter manifold, in contradiction to Tagirov[13] which studies the Einstein Universe using a conformal time, where the coordinates cover the whole of the Sitter manifold.

Now, an original way to study the cosmological problem is the theory of Hyperspherical Models Universe (tied to the integers numbers) proposed by Fantappié[20] and perfected by Arcidiacono[21,22]. In this theory it is necessary to distinguish the *absolute* spacetime (with constant curvature) effective seat of the physic events from the infinite *relative* spacetime (tangents) where each observer localize and see the phenomena. Then we use a flat representation of the de Sitter Universe on one of their tangents spaces. Among the infinite representations we use the Beltrami[23] geodetic representation where the geodetics of the Hyperspherical spacetime corresponds to the straightlines of the flat tangents spacetime of the observer's localization. It follows the group of motion in itself of the de Sitter Universe is represented by the so called Fantappié-de Sitter group, isomorphic to the five-dimensional pseudo-rotation group, i.e. by the projectives of the tangents space which change in itself the Cayley-Klein absolute[22].

Recently, Arcidiacono and Capelas de Oliveira have discussed the Laplace equation[24] and D'Alembert wave equation[25,26] in the de Sitter Universe, using the technique proposed by Fantappié-Arcidiacono, in terms of the ultraspherical polynomials. More recently, we have discussed the homogeneous D'Alembert generalized wave equation[27] when we have a small distance situation (a local problem) using the same technique. In another recent paper[28] we have proposed a new construction of the Casimir invariant operators for the Fantappié-de Sitter group using the same technique mentioned above, in that paper we obtained the commutation relations and Casimir invariant operators for the Fantappié-de Sitter group but the result equations (differentials wave equations) are not solved, this is done in the present paper which is organized as follow: in section one we present a brief review of the technique proposed by Fantappié-Arcidiacono; in section two we briefly discuss the Fantappié-de Sitter group, presenting the operators

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0+For second order differential wave equation we understand the equation obtained by the Casimir invariant operator of second order. The fourth order differential wave equation is an equation obtained by the second Casimir invariant operator of fourth order.

and the Casimir invariant operators; in section three we discuss and solve the second order differential wave equation in terms of the spherical harmonics and hypergeometric function (ultraspherical polynomial in some cases); in section four we discuss and solve the fourth order differential wave equation; we note that in all sections above mentioned the classical results obtained in Relativistic Quantum Theory in Minkowski spacetime are recovered when the radius of the de Sitter Universe goes to infinite. Finally we present ours comments and conclusions.

I. A Brief Review of the Method

In this section we present a brief review of the method proposed by Fantappié-Arcidiacono. Consider a fivedimensional space E_5 with the homogeneous coordinates and the fourdimensional Beltrami coordinates in the de-Sitter space.

Therefore the five dimensional homogeneous coordinates denoted by $\xi_A (A = 0,1,2,3,4)$ and the four dimensional coordinates denoted by $x_\mu (\mu = 0,1,2,3)$, are related by

$$x_\mu = R\,\frac{\xi_\mu}{\xi_4} \tag{1.1}$$

satisfying the relation of normalization $\xi_A \xi_A = R^2$ where R is the radius of the de Sitter Universe.

Introcuding the following notation we have for the Cayley-Klein absolute

$$A^2 = 1 + \alpha^2 - \gamma^2 = 1 + \alpha_\mu \alpha_\mu \tag{1.2}$$

where

$$\alpha_\mu = \frac{1}{R} x_\mu \qquad \text{and} \qquad \gamma = \frac{1}{R} ct \tag{1.3}$$

and now, we can remove the ξ_4 coordinate and we obtain

$$\xi_4 = \frac{R}{A} \qquad \text{and} \qquad \xi_\mu = \frac{x_\mu}{A} \tag{1.4}$$

where A is given by equation (1.2).

To obtain the relation for the partial derivatives we consider a function $\varphi(\xi_A)$ being an homogeneous function of degree N in all five variable ξ_A, and using Euler's theorem for homogeneous function we have

$$\xi_A \partial_A \varphi(\xi_A) = N \varphi(\xi_A) \tag{1.5}$$

where we have put $\partial_A \equiv \partial/\partial\xi_A$.

Using the definition of homogeneous function we can write

$$R^N \varphi(\xi_A) = (\xi_4)^N \varphi(R, x_\mu) \tag{1.6}$$

where the function which appear in the right side of the above equation is a function obtained from $\varphi(\xi_A)$ with the substitutions $\xi_4 \rightarrow R$ and $\xi_\mu \rightarrow x_\mu$.

Deriving the above equation, firstly in relation to ξ_4 and secondly in relation to ξ_μ and introducting a function $\psi(x_\mu)$ by

$$\psi(x_\mu) = A^{-N} \varphi(R, x_\mu) \tag{1.7}$$

we obtain, respectively

$$R\frac{\partial}{\partial \xi_4}\varphi(\xi_A) = \left(\frac{N}{A} - Ax_\mu \frac{\partial}{\partial x_\mu}\right)\psi(x_\mu) \tag{1.8a}$$

and

$$\frac{\partial}{\partial \xi_\mu}\varphi(\xi_A) = \left(A\frac{\partial}{\partial x_\mu} + \frac{N}{AR^2}x_\mu\right)\psi(x_\mu) \tag{1.8b}$$

which is the link between the two formulations. Then, we have solved the problem to pass of the five dimensional formulation, ξ_A, to spacetime formulation, x_μ, i.e. in orthogonal Cartesian coordinates.

II. Casimir Invariant Operators

In this section we present the Fantappié-de Sitter group; the respective operators and the Casimir invariant operators.

The Fantappié-de Sitter group, isomorphic to the five-dimensional pseudo rotation group, is the group of motions admited by a cosmological space with line element given by

$$-ds^2 = A^2 dx_\mu dx_\mu = A^2[(dx_1)^2 + (dx_2)^2 + (dx_3)^2 + (dx_0)^2] \tag{2.1}$$

where $x_0 = i\,ct$ and $R^2A^2 = R^2 + \rho^2 - x_0^2$ and $\rho^2 = (x_1)^2 + (x_2)^2 + (x_3)^2$. This space can be embedded in a flat five-dimensional space time, being the x_μ, the Beltrami projection from the "sphere" with equation

$$\sum_{A=0}^{4} \xi_A \xi_A = (\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2 + (\xi_4)^2 - (\xi_0)^2 = R^2$$

The coordinates are related by the equation (1.4) and the differential operators by the equation (1.8a) and equation (1.8b).

The representations for the Fantappié-de Sitter group[28] is given by the five-dimensional angular momentum operators

$$J_{AB} = -i\hbar\left(\xi_A \frac{\partial}{\partial \xi_B} - \xi_B \frac{\partial}{\partial \xi_A}\right) \equiv L_{AB} \tag{2.2}$$

where $A, B = 0, 1, 2, 3, 4$ which in terms of the Beltrami coordinates are given by

$$L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu \tag{2.3a}$$

and

$$\pi_\lambda \equiv \frac{1}{R} L_{0\lambda} = A^2 p_\lambda + \frac{1}{R^2} x_\mu L_{\lambda\mu} \tag{2.3b}$$

where $\nu, \mu, \lambda = 0, 1, 2, 3$.

We note that in the above equations (where π_μ are the analogous of the momentum operators in the Minkowski spacetime) that the linear momentum, p_μ, and the angular momentum, $L_{\mu\nu}$, mix in an unique tensor. This mixing is due to the fact that transformation of *displacement* are the analogous of the *translation* and therefore the energy-momentum operators are not conserved in relation to the Fantappié-de Sitter group.

Introducing the spherical coordinates and defining the following operators:

T_0 = temporal translations, with	$T_0 \equiv (-ic/R)L_{40}$
T_μ = spatial translations, with	$T_\mu \equiv (1/R)L_{\mu 4}$
V_μ = center of mass inertia momentum, with	$V_\mu \equiv (-i/c)L_{0\mu}$
L_μ = spatial rotation, with	$L_\mu \equiv L_{\nu\lambda}$

where $\mu, \nu, \lambda = 1, 2, 3$, we obtain ten differential operators corresponds to the generators in the explicit forms given by

$$T_0 = -\hbar c\left[\left(1 + \frac{t^2}{R^2}\right)\frac{\partial}{\partial t} + \frac{tr}{R^2}\frac{\partial}{\partial r}\right] \tag{2.4a}$$

$$T_1 = -\frac{i\hbar}{R^2}\left[(r^2 + R^2)\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{R^2}{r}\left(\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right) + \right.$$
$$\left. + rt\sin\theta\cos\phi\frac{\partial}{\partial t}\right] \tag{2.4b}$$

$$T_2 = -\frac{i\hbar}{R^2}\left[(r^2 + R^2)\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{R^2}{r}\left(\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right) + \right.$$
$$\left. + rt\sin\theta\sin\phi\frac{\partial}{\partial t}\right] \tag{2.4c}$$

$$T_3 = -\frac{i\hbar}{R^2}\left[(r^2 + R^2)\cos\theta\frac{\partial}{\partial r} - \frac{R^2}{r}\sin\theta\frac{\partial}{\partial\theta} + rt\cos\theta\frac{\partial}{\partial\theta}\right] \tag{2.4d}$$

$$V_1 = \frac{\hbar}{c}\left[t\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\phi\cos\theta\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right) - r\sin\theta\cos\phi\frac{\partial}{\partial t}\right] \tag{2.4e}$$

$$V_2 = \frac{\hbar}{c}\left[t\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right) - r\sin\theta\sin\phi\frac{\partial}{\partial t}\right] \tag{2.4f}$$

$$V_3 = \frac{\hbar}{c}\left[t(\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}) - r\cos\theta\frac{\partial}{\partial t}\right] \tag{2.4g}$$

$$L_1 = i\hbar\left(-\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right) \tag{2.4h}$$

$$L_2 = i\hbar\left(+\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right) \tag{2.4i}$$

$$L_3 = i\hbar\frac{\partial}{\partial\phi} \tag{2.4j}$$

where in the above expressions $\hbar$ and c have the usual meaning.

Considering a ciclic permutation of the index μ, ν and λ we can obtain the commutation relations for the above differential operators of the Fantappié-de Sitter group[28].

Using the above expressions for the operators we have for the second order Casimir differential operator the following expression

$$I_2 = -\hbar^2 A^2\left\{\left(1 + \frac{r^2}{R^2}\right)\frac{\partial^2}{\partial r^2} + \frac{2rt}{R^2}\frac{\partial^2}{\partial r\partial t} - \left(1 - \frac{t^2c^2}{R^2}\right)\frac{\partial^2}{\partial(ct)^2} + \right.$$
$$\left. + \frac{2}{r}\left(1 + \frac{r^2}{R^2}\right)\frac{\partial}{\partial r} + \frac{2t}{R^2}\frac{\partial}{\partial t} + \frac{1}{r^2}\mathcal{L}^2\right\} \tag{2.5}$$

where the $\mathcal{L}^2$ operator is given by

$$\mathcal{L}^2 \equiv \frac{\partial^2}{\partial\theta^2} + \cot\theta\,\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\phi^2} \tag{2.6}$$

it is the usual angular momentum operator.

We note that, when $R \to \infty$ the equation (2.5) reduces to the D'Alembert wave operator, i.e.

$$\lim_{R\to\infty} I_2 \equiv \Box = -\hbar^2\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) \tag{2.7}$$

where Δ is the Laplacian operator writen in spherical coordinates.

For the fourth order Casimir differential operator we have

$$I_4 = \frac{4\hbar^4}{c^2R^4}\mathcal{L}^2\mathcal{H}^2 \tag{2.8}$$

where

$$\mathcal{H}^2 \equiv t^2r^2\frac{\partial^2}{\partial r^2} + (R^2-t^2)^2\frac{\partial^2}{\partial t^2} - 2rt(R^2-t^2)\frac{\partial^2}{\partial r\partial t} + r(2t^2-R^2)\frac{\partial}{\partial r} - 2t(R^2-t^2)\frac{\partial}{\partial t} \tag{2.9}$$

and $\mathcal{L}^2$ given above. When $R \to \infty$ we have the usual relation, i.e.

$$I_4 = \left(\frac{4\hbar^4}{c^2}\right)\mathcal{L}^2\frac{\partial^2}{\partial t^2} \tag{2.10}$$

The second order Casimir differential operator is associated with a mass and the fourth order Casimir differential operator is associated with a spin. In consequence, a particle in a Fantappié-de Sitter Universe has not a well defined mass and a spin but has constant eigenvalues of the I_2 and I_4 Casimir invariant operators.

III Second Order Invariant Operator

In this section we discuss the second order differential equation obtained for the Casimir invariant operator of second order, which is the Klein-Gordon wave equation i.e., the following differential equation

$$I_2\psi(r,\theta,\phi,t) = M^2\psi(r,\theta,\phi,t) \tag{3.1}$$

where M^2 is a constant which is associated with a mass of the particle and the I_2 operator is given by equation (2.5)$^+$.

Then, introducing the I_2 operator in equation (3.1) we obtain the following partial differential equation

$$A_0^2\Big\{(1+\xi^2)\frac{\partial^2}{\partial\xi^2} + 2\xi\eta\frac{\partial^2}{\partial\xi\partial\eta} - (1-\eta^2)\frac{\partial^2}{\partial\eta^2} + \frac{2}{\xi}(1+\xi^2)\frac{\partial}{\partial\xi} + 2\eta\frac{\partial}{\partial\eta} + \frac{1}{\xi^2}\mathcal{L}^2\Big\}\psi(\xi,\eta,\theta,\phi) =$$

$$= -M_0^2\psi(\xi,\eta,\theta,\phi) \tag{3.2}$$

where we have put

$$\xi = \frac{r}{R}, \qquad \eta = \frac{ct}{R} \qquad \text{and} \qquad \frac{RM}{\hbar} = M_0 \tag{3.3}$$

and $A_0^2 = 1 + \xi^2 - \eta^2$.

$^{0+}$we note that, in the limit $R \to \infty$ we have the classical D'Alembert wave equation[24,25,26].

To solve this partial differential equation we use the method of separation of variable. Introducing a function as a product

$$\psi(\xi,\eta,\theta,\phi) = \sum_{\lambda} T_\lambda(\xi,\eta) S_\lambda(\theta,\phi) \tag{3.4}$$

we have an equation in the angular variables (θ,ϕ), and another equation in (ξ,η) variables, as follows

$$\Big\{(1+\xi^2)\frac{\partial^2}{\partial\xi^2} + 2\xi\eta\frac{\partial^2}{\partial\xi\partial\eta} - (1-\eta^2)\frac{\partial^2}{\partial\eta^2} + \frac{2}{\xi}(1+\xi^2)\frac{\partial}{\partial\xi} + $$
$$+ 2\eta\frac{\partial}{\partial\eta} + \frac{M_0^2}{A_0^2} - \frac{\lambda^2}{\xi^2}\Big\} T(\xi,\eta) = 0 \tag{3.5}$$

and

$$\left\{\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \lambda^2\right\} S(\theta,\phi) = 0 \tag{3.6}$$

where λ^2 is a constant.

The second of the above differential equations, for $\lambda^2 = -l(l+1)$ with $l = 0,1,2\ldots$ has the usual solution given by

$$S(\theta,\phi) = (-1)^m \left[\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} P_\ell^m(\cos\theta) e^{im\phi} \tag{3.7}$$

with $\ell \geq m$ (m is an integer) where $P_\ell^m(\cos\theta)$ are the associated Legendre functions*. This solution is exactly the spherical harmonics which is the same obtained with the classical treatments of the Klein-Gordon differential equation writen in spherical coordinates.

Finally, we must solve the follow differential equation

$$\Big\{(1+\xi^2)\frac{\partial^2}{\partial\xi^2} + 2\xi\eta\frac{\partial^2}{\partial\xi\partial\eta} - (1-\eta^2)\frac{\partial^2}{\partial\eta^2} + \frac{2}{\xi}(1+\xi^2)\frac{\partial}{\partial\xi} + $$
$$+ 2\eta\frac{\partial}{\partial\eta} + \frac{\ell(\ell+1)}{\xi^2} + \frac{M_0^2}{A_0^2}\Big\} T(\xi,\eta) = 0. \tag{3.8}$$

For this purpose we introduce firstly the changement of independent variable defined by

$$\xi = \rho\ ch\tau \quad \text{and} \quad \eta = \rho\ sh\tau \tag{3.9}$$

and we obtain the following partial differential equation

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[0]*Our notation for special function is the same of ref. 29.

$$\left\{(1+\rho^2)\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}(3+2\rho^2)\frac{\partial}{\partial\rho}-\frac{1}{\rho^2}\frac{\partial^2}{\partial\tau^2}-2\frac{th\tau}{\rho^2}\frac{\partial}{\partial\tau}+\right.$$

$$\left.+\frac{M_0^2}{1+\rho^2}-\frac{\ell(\ell+1)}{\rho^2 ch^2\tau}\right\}T(\rho,\tau)=0 \tag{3.10}$$

Now, we introduce

$$T(\rho,\tau)=F(\rho)G(\tau)$$

in the above equation and we have two ordinary differential equation given by

$$G''+2th\tau\;G'+\frac{\ell(\ell+1)}{ch^2\tau}G-\lambda_0^2G=0 \tag{3.11a}$$

$$\rho^2(1+\rho^2)F''+\rho(3+2\rho^2)F'+\frac{\rho^2M_0^2}{1+\rho^2}F-\lambda_0^2F=0 \tag{3.11b}$$

where λ_0^2 is a constant and the comma denote differentiation.

Taking $\lambda_0^2 = k(k+2)$ where $k = 0,1,2\ldots$ we obtain the solution of the equation (3.11a) as follow (in r and t variables)

$$G(r,t)=\left(1-\frac{c^2t^2}{r^2}\right)^{\frac{k+2}{2}}C_{l-k+1}^{k+3/2}\left(\frac{ct}{r}\right) \tag{3.12}$$

where $C_\mu^\nu(x)$ are the Gegenbauer polynomials.

The solution of equation (3.11b) is given by

$$F(\rho)=e^{-i\pi/2\,(k+\nu+3)}\left(\frac{A_0^2}{A_0^2-1}\right)^{3/4}\frac{\Gamma(k-\nu)}{\Gamma(k+\nu+3)}2^{\nu+3/2}\Gamma(\nu+5/2).$$

$$\cdot\left\{P_{k+1/2}^{\nu+3/2}\left(\frac{1}{\sqrt{1-A_0^2}}\right)+\frac{2i}{\pi}Q_{k+1/2}^{\nu+3/2}\left(\frac{1}{\sqrt{1-A_0^2}}\right)\right\} \tag{3.13}$$

where $M_0^2=\nu(\nu+3)$ and $A_0^2=1+\xi^2-\eta^2=1+\rho^2$ and $P_k^\mu(x)$ and $Q_k^\mu(x)$ are the Legendre associated functions of the first kind and the second kind, respectively.

Finally, we can write the solution of the second order Casimir operator (equation 3.1) as follow:

$$\psi(r,\theta,\phi,t)=\sum_{\nu=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}S(\theta,\phi)T(\xi,\eta)$$

where $S(\theta,\phi)$ is given by equation (3.7); $T(\xi,\eta)$ is given by a product of equations (3.12) and (3.13) and ξ and η are given by equations (3.3).

IV Fourth Order Invariant Operator

In this section we discuss and solve the fourth order differential equation obtained for the Casimir invariant operator of fourth order, i.e. the following differential equation

$$I_4\,\Omega(r,\theta,\phi,t) = N^2\Omega(r,\theta,\phi,t) \tag{4.1}$$

where N^2 is a constant which is associated with a spin of the particle and I_4 operator is given by equation (2.8).

Introducting the I_4 operator in equation (4.1) we have the following partial differential equation

$$\left(\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\cdot$$
$$\cdot\left[t^2r^2\frac{\partial^2}{\partial r^2}+(R^2-t^2)^2\frac{\partial^2}{\partial t^2}-2rt(R^2-t^2)\frac{\partial^2}{\partial r\partial t}+r(2t^2-R^2)\frac{\partial}{\partial r}-2t(R^2-t^2)\frac{\partial}{\partial t}\right]\Omega(r,\theta,\phi,t)=$$
$$= N_0^2\Omega(\varphi,\theta,\phi,t) \tag{4.2}$$

where $N_0^2=\dfrac{c^2R^4}{4\hbar^4}N^2$.

We note that, when we take the limit $R\to\infty$ in equation (4.2) we have exactly the classical results.

To solve the above equation (fourth order differential equation) we note that, this equation can be write in the following way

$$\mathcal{L}^2\Omega_1(\theta,\phi)\mathcal{H}^2\Omega_2(r,t)=N_0^2\Omega_1(\theta,\phi)\Omega_2(r,t)$$

and therefore we have two partial differential equations of second order,

$$\mathcal{L}^2\Omega_1(\theta,\phi)=N_1^2\ \Omega_1(\theta,\phi) \tag{4.3a}$$

$$\mathcal{H}^2\Omega_2(r,t)=N_2^2\ \Omega_2(r,t) \tag{4.3b}$$

where $N_1^2\ N_2^2=N_0^2$.

The equation (4.3a) is the same equation solved above (equation 3.6) when we have $N_1^2=\ell(\ell+1)$ with $l=0,1,\ldots$ and the solution is given by equation (3.7).

Then, we must solve, finally, the following partial differential equation (parabolic equation)

$$\left\{t^2r^2\frac{\partial^2}{\partial r^2}+(R^2-t^2)^2\frac{\partial^2}{\partial t^2}-2rt(R^2-t^2)\frac{\partial^2}{\partial r\partial t}+\right. \tag{4.4}$$
$$\left.+r(2t^2-R^2)\frac{\partial}{\partial r}-2t(R^2-t^2)\frac{\partial}{\partial t}\right\}\Omega_2(r,t)=N_2^2\ \Omega_2(r,t)$$

Therefore we introduce the variables τ and ρ defined as follow

$$t = R\tau \qquad \text{and} \qquad r = R\rho$$

and introducing the following changement of independent variable

$$\xi = \frac{1}{\rho}(\tau^2 - 1)^{1/2} \qquad \text{and} \qquad \eta = \rho$$

we get

$$\left\{\eta^2(1 + \xi^2\eta^2)\frac{\partial^2}{\partial\eta^2} + \eta(1 + 2\xi^2\eta^2)\frac{\partial}{\partial\eta}\right\}\Omega_2(\xi,\eta) = N_0^2\ \Omega_2(\xi,\eta) \tag{4.5}$$

where $N_0 = N_2/R$.

We note that, in this equation the ξ variable does not appear explicit in the solution $\Omega_2(\xi,\eta)$. To see this fact we introduce a new variable defined by

$$\xi\eta = u$$

and we obtain the following ordinary differential equation[+]

$$\left\{u^2(1 + u^2)\frac{d^2}{du^2} + u(1 + 2u^2)\frac{d}{du} - N_0^2\right\}\Omega_2(u) = 0. \tag{4.6}$$

The solution, for N_0 non integer, of the above differential equation is given by (in t variable)

$$\Omega_2(r,t) = \left(1 - \frac{t^2}{R^2}\right)^{N_0/2}\left\{A\,{}_2F_1\left(\frac{N_0}{2}, \frac{N_0+1}{2}; \frac{1}{2}; \frac{t^2}{R^2}\right) + B\frac{t}{R}\,{}_2F_1\left(\frac{N_0+1}{2}, \frac{N_0+2}{2}; \frac{3}{2}; \frac{t^2}{R^2}\right)\right\} \tag{4.7}$$

where ${}_2F_1(a, b; c; x)$ is the usual hypergeometric function and A and B are constants. For N_0 an integer the solution is given by

$$\Omega_2(r,t) = t^\lambda(1 - t)^\mu P_n(t)$$

where λ and μ are parameters and $P_n(t)$ is a polynomial.

Finally, the solution of fourth order differential equation is given as follow

$$\Omega(r,\theta,\phi,t) = \Omega_1(\theta,\phi)\Omega_2(t,r) \tag{4.8}$$

where $\Omega_1(\theta,\phi)$ is given by equation (3.7) and $\Omega_2(t,r)$ is given by equation (4.7).

[0+] This equation is the same obtained from the equation (4.4) taking $r = 0$.

V. Conclusions

In this paper we discussed and solved, by an alternative way, (Fantappié-Arcidiacono method) firstly the second order differential wave equation obtained for the second order Casimir invariant operator and secondly the fourth order differential wave equation obtained for the fourth order Casimir invariant operator.

For the second order differential equation our result are the same obtained by several authors using a different techniques but the fourth order differential equation our results seens to be new and must be interpreted.

The next point is solve the generalized Dirac wave equation for the scalar field but this topic are present in another paper[30].

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