ON THE CHOICE OF THE SPACE DIMENSION IN ILL-POSED PROBLEMS

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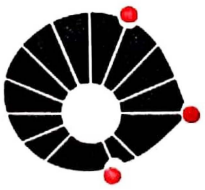

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ABSTRACT – We consider aspects of the problem of the control of dimensionality in projection methods for integral equations of the first kind. We show that by combining the error estimates of the regularized solution and the approximate results of the projection space, it is possible to choose a satisfatory dimension of this projection space. We apply the spectral method to traditional ill-posed problems to performe some numerical experiments.

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On the choice of the space dimension in ill-posed problems

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Abstract

We consider aspects of the problem of the control of dimensionality in projection methods for integral equations of the first kind. We show that by combining the error estimates of the regularized solution and the approximate results of the projection space, it is possible to choose a satisfatory dimension of this projection space. We apply the spectral method to traditional ill-posed problems to performe some numerical experiments.

1. Introduction.

Let H_1 and H_2 be Hilbert spaces and $K : H_1 \to H_2$ an integral operator. It is well known that linear Fredholm integral equations of the first order,

$$Kx = \int K(s,t)x(t)dt = g(s) \tag{1.1}$$

are ill-posed problems: even if a unique solution exists an arbitrarily small perturbation of the right-hand side can give rise to a large perturbation of the solution.

When the kernel $K(s,t)$ is compact the nature of the problem is completely specified by the Picard theorem (see Groetsch [5] for further details): there exist adjoint L_2-orthogonal functions $\{\phi_i(s)\}$, $\{\psi_i(t)\}$ and real scalars $\lambda_i \to 0$ (the singular system) such that

$$\int K(s,t)\phi_i(s)ds = \lambda_i\psi_i(t) \quad \text{and} \quad \int K(s,t)\psi_i(t)ds = \lambda_i\phi_i(s)\ .$$

Thus, if $g(s) = \sum \beta_i\psi_i(s)$ then $x(t) = \sum\left(\frac{\beta_i}{\lambda_i}\right)\phi_i(t)$. However $x \in L_2$ only if $\sum\left(\frac{\beta_i}{\lambda_i}\right)^2 < \infty$, this is the called Picard condition which specifies whether a nice solution exists to the problem.

Due to this ill-posed nature of the problem it is necessary to regularize (1.1). A well-known and effective procedure is the Tikhonov regularization method [5], [11]. This method, in an abstract sense, essentially replaces (1.1) with the problem of minimizing the functional

$$f_\alpha(x) = ||Kx - g||^2 + \alpha||Lx||^2 \tag{1.2}$$

where $\alpha > 0$ is the regularization parameter and L is a suitably defined differential operator (which "smoothes" the approximation).

Typically, the equation (1.1) is discretized at some stage of its solution, which reduces the original problem to solving a system of linear equations. Another well-known procedure for dealing with ill-posed problems is the truncated singular value decomposition (TSVD) of the discretized system matrix. In [6] Hansen showed that if there is a distinct gap in the singular value spectrum of finite dimensional approximation matrix

then the TSVD method is equivalent to the Tikhonov regularization. For matrices with ill-determined numerical rank Hansen also showed in [7] that the existence of a satisfactory approximate solution depends on the satisfation of a discrete Picard condition. Through theoretical arguments and numerical experiments, he showed that when the discret Picard condition is satisfied then TSVD and Tikhonov regularization always yield very similar solutions, even for problems with ill-determined numerical rank matrices.

The crux of the difficulty in solving ill-posed problems is that, in general, the data are only imprecisely known. In other words, only some garbled version g^δ is available. We will assume that, in this case,

$$\|g - g^\delta\| \leq \delta \tag{1.3}$$

where δ is an *a priori* known error level. In this inexact data case the regularization method consists of choosing space norms and $\alpha = \alpha(\delta)$ for the regularization parameter. This last choice may be made in either an *a priori* or an *a posteriori* way but in any case the matching of the regularization parameter with the noise is at the heart of the theory of regularization.

The more general *a posteriori* strategy for choosing the regularization parameter, α, as a function of the error level is the discrepancy principle. This principle was firstly clearly enunciated by Morozov [9] and it is based on the point that the quality of the results of any computation must be comparable to the quality of the input data. Several authors have analysed this compromise using different approachs.

On the other hand, the regularization approximation scheme turns into algorithms once a resolution strategy can be effectivelly implemented. Whichever the method chosen for solving the ill-posed problem, there is a critical parameter choice whose "optimal" value is crucial to the amenability and numerical implementation of the method. This is not a trivial problem since it involves a trade-off between accuracy and numerical stability, a situation that does not usually arise in the numerical solution of well-posed problems. In other words, it must be recalled that there are conflicting forces in attempts to solve ill-posed problems. The necessity of employing a finite basis set is always present, introducing a truncation error. In general, the larger is the set, the smaller is the truncation error. In contrast, the increase in the size of the set introduces the kind of conditioning problems that have been analysed by several authors [10], [12].

In this paper we analyse the relative effects of the mentioned errors when we use projection methods in orthogonal polynomial subspaces to solve the problem (1.1). We present some arguments that allow the choice of the subspace dimension based on the minimization of the overall error.

2. The discretization procedure.

The minimizer of $||Kv - g||$ in H_1 is called the least square solution of (1.1). It is easy to prove that if K is a bounded linear operator from H_1 into H_2, whose range $R(T)$ is not necessarily close, then a least square solution exists if and only if $g \in R(T) + R(T)^{\perp}$, which is a dense set in H_2. In this case the set of least square solutions is a closed convex set and (1.1) has a unique solution of smallest norm which is denoted by $x = K^{+}g$.

Let us consider a simple example of the regularization function (1.2):

$$f_\alpha(z) = ||Kz - g||^2 + \alpha||z||^2 . \tag{2.1}$$

A direct computation shows that x^α, the unique minimizer of (2.1) in H_1, is the solution of the equation

$$(K^*Kz + \alpha z, v) = (K^*g, v) \quad \forall v \in H_1 \tag{2.2}$$

where K^* is the adjoint of K. The rates of convergence of x^α, as $\alpha \to 0$, are known: (see Groetsch, chapter 3 for example):

$$\begin{aligned} &\text{i. if} \quad x = K^{+}g \in R(K^*) \quad \text{then} \quad ||x - x^\alpha|| = 0(\sqrt{\alpha}) \\ &\text{ii. if} \quad x = K^{+}g \in R([K^*K]^\nu) \quad \text{then} \quad ||x - x^\alpha|| = 0(\alpha^\nu). \end{aligned} \tag{2.3}$$

Now we will consider regularized Ritz approximations obtained by minimizing the Tikhonov functional over a finite dimensional supspace. We assume that $\{V_m\}$ is an expanding sequence of finite dimensional subspaces of H_1 whose union is dense in H_1, i.e.

$$V_1 \subseteq V_2 \subseteq \ldots \qquad \text{and} \qquad \overline{UV_m} = H_1 .$$

The finite dimensional problem corresponding to (1.1) is

$$K_m x = g.$$

Here $K_m = KP_m$ where P_m is the orthogonal projection of H_1 onto V_m. Let x_m^α be the minimizer of the Tikhonov functional (2.1) over V_m. If $\{v_1, v_2, \ldots, v_m\}$ is a basis for V_m we can use (2.2) with $v = v_j, j = 1, \ldots, m$ to find

$$x_m^\alpha = \sum_{i=1}^{m} y_i v_i \,. \tag{2.4}$$

In this way $y = (y_1, \ldots, y_m)$ is the unique solution of the linear system

$$(B + \alpha M)y = w \,, \tag{2.5}$$

where $w_j = (Kv_j, g)$, $M_{ij} = (v_i, v_j)$, $B_{ij} = (Kv_i, Kv_j), 1 \leq i, j \leq m$.

The sucess of this scheme depends on how well K_m approximates K or equivalently how quickly the number

$$\gamma_m = ||K(I - P_m)|| \tag{2.6}$$

becomes small. Since K is assumed compact, the properties of the subspaces $\{V_m\}$ guarantee that $\gamma_m \to 0$ as $m \to \infty$ (Groetsch pag. 74).

The convergence analysis for the approximation follows from the standard arguments used in this case. We will present some points for the sake of self sufficiency of this paper. The following results tell us about the rate of convergence in the error-free case and in the noisy data case.

Theorem 1: Let $x = K^+ g \in R(K^*)$ and x_m be the approximation defined by (2.4)-(2.5) with $\alpha(\gamma_m) = 0(\gamma_m^2)$.

Then

$$||x_m - x|| = 0(\gamma_m) \,. \tag{2.7}$$

Proof: Let $x = K^* w$. It is known that

$$||x^\alpha - x_m|| \leq \sqrt{1 + \frac{\gamma_m}{\alpha}}\gamma_m ||w|| \,.$$

Therefore, if we take $\alpha = 0(\gamma_m^2)$ we will have

$$||x^\alpha - x_m|| = 0(\gamma_m).$$

The desired result follows from the inequality

$$||x_m - x|| \leq ||x_m - x^\alpha|| + ||x^\alpha - x||$$

combined with (2.3). ∎

We now suppose that g is not exactly known but we have approximate data g^δ satisfying (1.3). We will denote by $\tilde{x}_m$ the minimizer of (1.2) with $g = g^\delta$.

Theorem 2: Let $x = K^+ g \in R(K^*)$ and $\tilde{x}_m$ be the approximation defined by (2.4)-(2.5) with $\alpha(\gamma_m) = 0(\gamma_m^2)$ and $\gamma_m = 0(\sqrt{\delta})$. Then

$$||\tilde{x}_m - x|| = 0(\gamma_m) \tag{2.8}$$

Proof: It is known (c.f. Groetsch) that

$$||x_m - \tilde{x}_m|| \leq \frac{\delta}{\sqrt{\alpha}}.$$

But

$$||x - \tilde{x}_m|| \leq ||x - x_m|| + ||x_m - \tilde{x}_m|| \tag{2.9}$$

and consequently (2.8) follows from (2.7) and the supposition $\gamma_m = 0(\sqrt{\delta})$. ∎

The inequality (2.9) is typical for approximations in ill-posed problems. Looking at the bounds of each term in this inequality we recognize two terms in the error estimate: one due to approximation and another due to the measurement error. The first term (responsible for the regularization error) tends to zero as $m \to \infty$ while the second term (responsible for the magnification of contamination error due to ill-posedness) tends to infinity as $m \to \infty$. The balance of these two terms gives an "optimal" approximation. In our case this balance is obtained if we take finite-dimensional spaces such that $\gamma_m = 0(\sqrt{\delta})$.

By (2.7) we can conclude that if the data are exact we can choose γ_m, the dimension of V_m, in accordance with the desired approximation. On the other hand we can see by Theorem 2 that we have more severe restrictions on the noise data case. In this case, given a value δ we must choose V_m such that $\gamma_m = k_1\sqrt{\delta}, k_1 > 0$, and solve (2.5) taking

$\alpha = k_2\delta$ for some $k_2 > 0$.

3. Numerical considerations

The theoretical results presented in the previous section yields some insight into the order of approximations of the combination of the Tikhonov regularization and Ritz approximation. Like any numerical method for an ill-posed problem, this one ultimately involves solutions of linear systems of equations in order to obtain numerical approximations. In our case the discretized version (2.5) must be solved. The increase in the size of the matrices introduces the kind of conditioning problems. In [12] Wing studied the condition number of the matrices corresponding to Galerkin and collocation methods for integral equations of the first order. It can be shown that condition number of B must grow with the dimension m, since the eigenvalues of B must decay to zero without any particular gap in the spectrum.

As a rough rule of thumb, we know that if we are working on a computer whose arithmetic carries, say, p significant digits, then the computed solution to a system with a condition number $\text{cond}(A) = 10^q$ may have only $p-q$ accurate significant digits. We must say "rule of thumb" and "may" because the error bound is only a bound and because different scalings of the matrix may alter what we mean by "significant digits". But in general, we should become increasingly concerned as q increases towards p, and alarmed whenever q exceeds p.

It is well-known that the condition number of a matrix A can be used in the sensitivity measure of the $Ax = b$ solution when there exists a perturbation Δb on the right-hand side:

$$\frac{||\Delta x||}{||x||} \leq \quad \text{cond}(A)\frac{||\Delta b||}{||b||}.$$

Therefore, even if the error in b is very small, the larger $\text{cond}(A)$ is, the less information about the relative error in x can be taken out.

Since we are using variable subspace dimensions, there is a practical question: how this invitable blow-up can be used on the dimension control? We try to answer this question with a reason suggested by the role of the Picard condition in the truncated singular value decomposition methods [6], [7].

Let $v_1, v_2, \ldots, v_m$ be an orthonormal basis for V_m and x_m the approximation calculated by (2.4)-(2.5) using $\alpha = \alpha(m)$ as in the previous section. By the orthogonality of the basis we have,

$$||x_m||^2 = \sum_{i=1}^{m} y_{i,m}^2 . \tag{3.1}$$

Let $s_j = \sum_{i=1}^{j} y_{i,j}^2$. Since the sequence of finite dimensional subspace is dense in H_1, by Theorem 1 the sequence s_j is convergent. Theoretically as a Cauchy sequence in the set of positive real numbers, $s_q - s_p \to 0$ if $p, q \to \infty$. Similar arguments can be used in the noisy data case if we assume that in Theorem 2, $\delta \to 0$. With this reason we can set up a practical rule for the choice of the dimension of V_m.

To test the convergence of the partial sums (i.e. the convergence of the sequence s_j) we can use the L-curve [7]. If we consider the Tikhonov regularization method (1.2), the L-curve is a plot of the side constraint $||Lx_{\text{reg}}||$ versus the residual norm $||Kx_{\text{reg}} - g||$. The L-curve is a useful tool in connection with ill-posed problems because it is a convenient way of displaying the inter-relationship between the norm of x^α and the corresponding residual. Also the L-shaped corner corresponds to a regularization parameter that balances the regularization error and the perturbation error from the noise data. In our case the parameter is the dimension of the subspace and we will consider set of points such that $(||Kx_m - g||, ||x_m||)$.

4. Some approximation spaces and their order of approximation

Methods based on the Fourier series of orthogonal polynomials have become increasingly popular in recent years. These are the spectral methods wich have showed to be competitive with the traditional methods of finite difference and finite elements in the numerical solution of differential equations. It seems to us that the use of orthogonal polynomial is also a good choice in dealing with integral equations [4]. Here we present the error estimates for some orthogonal systems.

As we observed in Section 2, given an operator K we need to know how quickly the number $\gamma_m = ||K(I - P_m)||$ goes to zero as $m \to \infty$. First we observe that

$$||K(I - P_m)|| = ||(I - P_m)K^*|| = \sup_{||v||=1} ||(I - P_m)K^* v|| .$$

Let $z(t) = K^*v$, $v \in H_2$ such that $||v|| = 1$. In this case we have

$$||(I - P_m)K^*v|| = ||(I - P_m)z|| = ||z - P_m z||$$

where $P_m : H_1 \to V_m$ is the orthogonal projection. Let $V_m = \text{span}[v_1, \ldots, v_m]$ where $v_i, i = 1, ...m$, are the orthogonal polynomials. We know that $P_m z = \sum_{i=1}^{m} b_i v_i(t)$ where $b_i = (z, v_i), i = 1, ..., m$, are the Fourier coefficients.

On the other hand,

$$z(t) = K^*v = \int_a^b K(t,s)v(s)ds\,.$$

If the differentiation is allowed we have

$$z'(t) = \int_a^b \frac{\partial K(t,s)}{\partial t} v(s)ds\,.$$

In fact, if the derivative $\frac{\partial^j K}{\partial t^j}$ exists and is an integrable function then $z(t) \in C^j(a,b)$. and

$$z^j(t) = \int_a^b \frac{\partial^j K(t,s)}{\partial t^j} v(s)ds\,.$$

The rate of convergence of the Legendre and Chebyshev expansions for a general function $z(t)$ have been studied by various authors [1], [2], [8]. Let Ω be an open bounded interval. The weight Sobolev spaces are

$$H_w^s(\Omega) = \{\phi \in L^2(\Omega)\ ;\ ||\phi||_{s,w} < \infty\}$$

where $||\phi||_{s,w}^2 = \int_\Omega w(s) \sum_{i=0}^{s} (D^i\phi)^2 dx$ and $w(x)$ is a weight function ($w(x) > 0$ and integrable function). In [2] Canuto and Quarteroni prove the following estimates.

i - Chebyshev polynomials. Let us take $\Omega = (-1,1)$ and $w(x) = (1-x^2)^{-1/2}$. Let V_m be the m-dimensional subspace of $H_w^\sigma(\Omega)$ spanned by the Chebyshev polynomials of degree least than or equal to m. If P_m is the orthogonal projection onto V_m then for any real μ and σ such that $0 \le \mu \le \sigma$ there exists a constant C such that

$$||z - P_m z||_{\mu,w} \le C\, m^{e(\mu,\sigma)} ||z||_{\sigma,w}, \qquad \forall z \in H_w^\sigma(\Omega) \tag{4.1}$$

where

$$e(\mu,\sigma) = \begin{cases} 2\mu - \sigma - 0.5, & \mu \ge 1 \\ 1.5\mu - \sigma\,, & 0 \le \mu \le 1 \end{cases} \tag{4.2}$$

ii - Legendre polynomials. In this case $\Omega = (-1,1)$ and $w(x) = 1$. If now we consider V_m as the m-dimensional subspace of $H^\sigma(\Omega)$ spanned by the Legendre polynomials of degree least than or equal to m, the same estimates (4.1)-(4.2) are achieved.

If the Laguerre functions are used as an orthogonal system, the following estimates can be used. The Laguerre functions are defined by

$$\phi_k^p(t) = \sqrt{2p}\, e^{-pt} L_k(2pt)\,, \quad p > 0\,,$$

where $L_k(t) = \sum_{i=0}^{k} \binom{k}{i} \frac{(-t)^i}{i!}$ are the classical Laguerre polynomials (orthogonal in $(0,\infty)$, weight function $w(x) = e^{-x}$). It can be showed [4] that if $K = L^2(0,\infty) \to L^2[(c,d)], c > 0$ then for $p^* = \sqrt{cd}$ there exists $C > 0$ such that

$$\|K - K_m\| \le C\, \theta^{m+1}\,,$$

$$\text{where} \qquad \theta = \frac{d - p^*}{d + p^*} < 1\,.$$

5. Numeral experiments.

Example 1: As a first example we consider the Laplace transform inversion problem. In the notation of this paper, let $H_1 = L^2(0,\infty)$, $H_2 = L^2(c,d)$, $0 < c < d$, and

$$g(s) = \int_0^\infty e^{-st} x(t) dt.$$

We take the data values $g(s) = \frac{1}{(s+1)^2}$, $s \in (1,4)$. In this case it can be showed that $x(t) = te^{-t}$. As the finite dimensional subspace V_m we consider the space spanned by the Laguerre functions, defined by

$$\phi_k^p(t) = \sqrt{2p}\, e^{-pt} L_k(2pt)\,, \quad k \le m,$$

where $L_k(\cdot)$ are the Laguerre polynomials, what means that

$$\int_0^\infty e^{-t} L_i(t) L_j(t) dt = \delta_{ij}$$

The parameter p can be appropriately chosen as a function of c and d. In our example $p = 2$. Also, it is showed in [4] that

$$(5.1) \qquad \gamma_m = \left\{ \frac{2p(d-c)}{(d+p)(c+p)} \right\}^{1/2} \left(\frac{d-p}{d+p} \right)^{m+1} = \sqrt{\frac{2}{3}} \cdot \left(\frac{1}{3} \right)^{m+1}$$

for these subspaces V_m. Some values of γ_m are presented in table 5.1, where the numbers between parenhesis indicate decimal exponents.

m	4	5	6	7	8	9	10
γ_m	3.37(-3)	1.16(-3)	3.75(-4)	1.25(-4)	4.16(-5)	1.39(-5)	4.63(-6)

Table 5.1

We must observe that, even in the exact data case, during the calculations we work with values that depend on the machine precision. Using double precision arithmetic, for example, the "exact data" means $\delta = 10^{-14}$. In this case, by Theorem 2, it is convenient to take V_m such that $\gamma_m = 0(10^{-7})$. Using (5.1) we obtain $m = 13$.

On the other hand, the condition number of the matrix B on (2.5) increases drastically with the increasing of the subspace dimension. Some of these values are presented Table 5.2.

m	5	6	7	8	9	10	11	12	13
$cond(B)$	1.1(6)	4.1(7)	1.5(9)	5.3(10)	1.9(12)	7.0(13)	2.6(15)	9.1(16)	3.4(18)

Table 5.2

Looking at these values we see that it is not be possible to use $m = 13$, because the condition numbers blew up before. In fact, working with double precision arithmetic, in accordance with the numerical consideration of Section 3, we can only rely on the calculations which come from $m = 8$ or $m = 9$. The results obtained for various values of m are presented in the table 5.3. The values of $||x_m||$ are, in fact, the square root of the partial sums defined in (3.1), i.e, they are approximations of $||x|| = 0.5$ in this example. The residuals are calculated by $||Kx_m - g||$. We see by this table that the partial sums grow out of control for $m \geq 10$.

m	$\|\|x_m\|\|$	$\|\|Kx_m - g\|\|$
2	4.4244(-1)	9.0090(-2)
3	4.9331(-1)	9.0083(-2)
4	4.9987(-1)	9.0083(-2)
5	5.0013(-1)	9.0083(-2)
6	5.0004(-1)	9.0083(-2)
7	5.0001(-1)	9.0083(-2)
8	5.0000(-1)	9.0083(-2)
9	5.0001(-1)	9.0083(-2)
10	5.0156(-1)	9.0083(-2)
11	7.9165(-1)	9.0083(-2)
12	8.8928	9.0083(-2)
13	1.2068(2)	9.0083(-2)
14	1.5529(3)	9.0083(-2)
15	1.9040(4)	9.0083(-2)
16	2.2360(5)	9.0083(-2)

Table 5.3

Example 2: In this example we consider again the Laplace transform inversion problem, but now we test the noisy data case. In this case we replace $g(s)$ by the noisy version

$$g^{\delta}(s) = \frac{1}{(s+1)^2} + \delta \sin\left(\frac{\pi s}{4}\right)$$

with $\delta = 10^{-4}$. Using Theorem 2 we can choose V_m such that $\gamma_m = 0(10^{-2})$. By the equation (5.1) we must take $m = 4$. Also by Theorem 2, the recommended regularization parameter must be $\alpha = 0(10^{-4})$. The condition numbers of the matrices of (2.5) with $\alpha = (10^{-4})$ are presented in Table 5.4. The effect of the regularization is now clear: the condition numbers are moderate if double precision arithmetic is used.

m	3	4	5	6	8	9
$cond(A)$	8.0(2)	5.5(3)	6.6(3)	6.7(3)	6.7(3)	6.7(3)

Table 5.4

Table 5.5 is the version of Table 5.3 for this example: it contains the values of $||x_m||$ and the residual measure $||Kx_m - g^{\delta}||$. As the exact solution is such that $||x|| = 0.5$ this example confirm the forecast: there is no improve in the results for $m > 4$.

m	$\|\|x_m\|\|$	$\|\|Kx_m - g^\delta\|\|$
1	2.3069(-1)	8.9686(-2)
2	4.4121(-1)	9.0089(-2)
3	4.8068(-1)	9.0083(-2)
4	4.8028(-1)	9.0083(-2)
5	4.8000(-1)	9.0083(-2)
6	4.8000(-1)	9.0083(-2)
7	4.8000(-1)	9.0083(-2)

Table 5.5

Example 3: Let us consider the integral equation

$$\int_0^1 k(s,t)x(t)dt = g(s)$$

with the kernel

$$k(s,t) = \begin{cases} s(1-t) & \text{if} \quad s \leq t \\ t(1-s) & \text{if} \quad s > t \end{cases}$$

This example have been used by several authors. In our particular case we choose $g(s) = \frac{\sin(\pi s)}{\pi^2}$. In this case the solution is $x(t) = \sin(\pi t)$. As we are looking for a solution defined in a bounded domain we use the Legendre polynomials of degree less than or equal to m to span the finite dimensional subspace V_m. For these polynomials the values of the condition number of the matrix B in (2.5) are not alarming (at least for $m \leq 13$) if we use double precision arithmetic. The bound for γ_m, in this case (4.1), depends on the regularity of the solution, σ in that expression, and on the constant C. However we do not have information about the values of C. Let us now consider the error estimates on L^2 (0,1), i.e. $\mu = 0$. The noisy data case will be simulated by

$$g^\delta(s) = \frac{\sin(\pi s)}{\pi^2} + \delta \sin\left(\frac{\pi s}{4}\right)$$

with $\delta = 10^{-4}$. By Theorem 2 we must take V_m such that $\gamma_m = 0(10^{-2})$ and $\alpha = 0(10^{-4})$. To choose m we assume $\sigma = 4$ and $C||z||_4 \leq 10$. In this case it is reasonable to take $m = 6$ to guarantee $\gamma_m = 0(10^{-2})$. The obtained values of $||x_m||$ and $||Kx_m - g^\delta||$, for some values of m, are presented in Table 5.6. The exact solution is such that $||x|| = \sqrt{0.5}$.

m	2	3	4	6	8	10
$\|\|x_m\|\|$	0.7753	0.7289	0.7289	0.7294	0.7294	0.7294
$\|\|Kx_m - g^\delta\|\|$	0.27(-4)	0.13(-4)	0.13(-4)	0.13(-4)	.46(-5)	.46(-5)

Table 5.6

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