3171

EMBEDDING OF LEVEL CONTINUOUS FUZZY SETS AND APPLICATIONS

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ABSTRACT - Puri and Ralescu [10] given, recently, an embedding of the class $\mathbb{E}^n_L$ of fuzzy sets u with the level application $\alpha \longrightarrow L_\alpha u$ Lipschitzian on $C([0,1] \times S^{n-1})$. In this work we extend the above result to the class $\mathbb{E}^n_C$ of the continuous level applications. Moreover, we prove that $\mathbb{E}^n_C$ is a complete metric space while at the $\mathbb{E}^n_L$ is not, and that $\overline{\mathbb{E}^n_L} = \mathbb{E}^n_C$. Also, we deduce some properties in the fuzzy random variables.

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Embedding of Level Continuous Fuzzy Sets and Applications[(*)].

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Puri and Ralescu [10] given, recently, an embedding of the class $\mathbb{E}_L^n$ of fuzzy sets u with the level application $\alpha \longrightarrow L_\alpha u$ Lipschitzian on $C([0,1] \times S^{n-1})$. In this work we extend the above result to the class $\mathbb{E}_C^n$ of the continuous level applications. Moreover, we prove that $\mathbb{E}_C^n$ is a complete metric space while at the $\mathbb{E}_L^n$ is not, and that $\overline{\mathbb{E}_L^n} = \mathbb{E}_C^n$. Also, we deduce some properties in the fuzzy random variables.

Keywords: fuzzy sets, level applications, embedding of level continous fuzzy sets, fuzzy random variables

1. INTRODUCTION.

Recently, Puri and Ralescu [10] showed that there is an embedding $j : \mathbb{E}_L^n \rightarrow C([0,1] \times S^{1-1})$, where $\mathbb{E}_L^n$ is the subspace of $(\mathbb{E}^n, D)$ with Lipschitzian levels and $\mathbb{E}^n$ denote the class of normal convex fuzzy sets with compact support. This fact is very important, since $(\mathbb{E}^n, D)$ is not separable and this is an empeachment to develop clearly an integration theory for the fuzzy random variables. Unfortunatedly $(\mathbb{E}_L^n, D)$ isn't a complete subspace of $(\mathbb{E}^n, D)$ as to will be showed in the section 4. We observe that the application j can be defined by the same expression as in Puri and Ralescu ([9], [10]) for all $\mathbb{E}^n$ (obviosly with a different image space). So, it is raised the following question: is there some subspace of $(\mathbb{E}^n, D)$ that is separable and complete for the metric D and in such manner that it is still embeddies in $C([0,1] \times S^{n-1})$ through of j?. In this paper, we prove that the space that answers the above question is $\mathbb{E}_C^n$, which consist of the fuzzy sets with levels continuous; also we prove that $\mathbb{E}_L^n \subsetneq \mathbb{E}_C^n$ and that $\mathbb{E}_C^n$ is the maximal subspace of $\mathbb{E}^n$ with this propertie (see Section 3) and $\overline{\mathbb{E}_L^n} = \mathbb{E}_C^n$ (see Section 4). Also, we give some applications for the theory of the fuzzy random variables (see Section 5). Actually, it was only for simplicity that we derived our results in $\mathbb{R}^n$; they extend easily to the case of real separable Banach Spaces.

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To stress the importance of the embedding j, we recall that Kaleva [7] used the embedding j together with one characterization of the fuzzy compact subsets of $\mathbb{E}^n$, due a Diamond and Kloeden [5], in the subclass of $\mathbb{E}^n_L$, wich he called equi-Lipschitzian, to demonstrate the existence of the solutions of the Cauchy Problem for fuzzy differential equations with values in the equi-Lipschitzian subsets. Also, Puri and Ralescu used the embedding to study fuzzy random variables and the convergences of fuzzy martingales.

In a previous work [13], we proved the equivalence of the various notions of convergence in the class of fuzzy sets with continous levels, but not necessarily with convex levels. Obviously, these results are true for $\mathbb{E}^n_C$.

In a forthcoming paper we will describe applications to the problem of the convergence of fuzzy martingales.

2. Preliminaires

In the sequel $\mathbb{K}(\mathbb{R}^n)$ will denote the set of the nonempty compact subsets of $\mathbb{R}^n$. The Hausdorff metric H over this class is defined by

$$H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}$$

where d is usual distance, and $d(a,B) = \sup_{b\in B} d(a,b)$.

It is well known that $(\mathbb{K}(\mathbb{R}^n), H)$ is a separable complete metric space (see [4] and [9]).

If A_q is a sequence of subsets of $\mathbb{R}^n$, we define the lower and upper limits in the Kuratowski sense as

$$\lim_{q\to\infty} \inf A_q = \{x \in \mathbb{R}^n | x = \lim_{q\to\infty} x_q, x_q \in A_q\}$$

and,

$$\begin{aligned}\lim_{q\to\infty} \sup A_q &= \{x \in \mathbb{R}^n | x = \lim_{j\to\infty} x_{q_j}, x_{q_j} \in A_{q_j}\} \\ &= \bigcap_{q=1}^{\infty} \left(\overline{\bigcup_{m\geq q} A_m} \right),\end{aligned}$$

respectively.

We say that a sequence of sets A_q converges to a set $A, A \subseteq \mathbb{R}^n$, in the Kuratowski sense, if $\lim_{q\to\infty} \inf A_q = \lim_{q\to\infty} \sup A_q = A$; in this case, we write $A = \lim_{q\to\infty} A_q$ or $A_q \xrightarrow{K} A$, and we say that A_q K-converges to A.

The Kuratowski limits are closed sets. Moreover, the following rélation are true:

$$\begin{aligned}\lim_{q\to\infty} \inf A_q &\subseteq \lim_{q\to\infty} \sup A_q \\ \lim_{q\to\infty} \inf \overline{A_q} &= \lim_{q\to\infty} \inf A_q \quad \text{and} \\ \lim_{q\to\infty} \sup \overline{A_q} &= \lim_{q\to\infty} \sup A_q.\end{aligned}$$

The following result is well know (see [9])

Proposition 2.1. *A sequence $(A_q) \subseteq I\!K(I\!R^n)$ converges to a compact set A respect to the Hausdorff metric if and only if there is $K \in I\!K(I\!R^n)$ such that $A_q \subseteq K$ for all q and*

$$\liminf_{q\to\infty} A_q = \limsup_{q\to\infty} A_q = A.$$

We can understand a fuzzy set in $I\!R^n$ as a function $u : I\!R^n \to [0,1]$.

As an extension of $I\!K(I\!R^n)$, we define the space $I\!E^n$ of fuzzy sets $u : I\!R^n \to [0,1]$ with the following properties:

i) u is *normal*, i.e., $\{x \in I\!R^n | u(x) = 1\} \neq \phi$;

ii) u is *fuzzy-convex*, i.e., for all $x, y \in I\!R^n$ and $\lambda \in [0,1]$ we have,

$$u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\};$$

iii) u is *upper semicontinuous*;

iv) The clousure of the set $\{x \in I\!R^n | u(x) > 0\}$ is a nonempty compact subset in $I\!R^n$. This set is call the *support* of u and it's denoted by $L_0 u$.

The linear structure in $I\!E^n$ is defined by the operations

$(u+v)(x) = \sup_{j+z=x} \min[u(y), v(z)]$, and

$$(\lambda u)(x) = \begin{cases} u(\frac{x}{\lambda}) & \text{if } \lambda \neq 0 \\ \chi_{\{0\}}(x) & \text{if } \lambda = 0, \end{cases}$$

Where $u, v \in I\!E^n, \lambda \in I\!R$ and χ_A denote the characteristic function of A.

Recall that every fuzzy set is characterized by his family of α-level ($\alpha \in (0,1]$), where the α-level of u is defined by

$$L_\alpha u = \{x \in I\!R^n | u(x) \geq \alpha\}.$$

We observe that $L_0 u \supseteq L_\alpha u \supseteq L_\beta u$ for all $0 \leq \alpha \leq \beta$. So, if $u \in I\!E^n$, then $L_\alpha u \in I\!K(I\!R^n)$ for all $\alpha \in [0,1]$. Moreover, the linear structure in terms of the family $(L_\alpha u)$ is given by:

$$\begin{aligned} L_\alpha(u+v) &= L_\alpha u + L_\alpha v \quad \text{and} && (1) \\ L_\alpha(\lambda u) &= \lambda L_\alpha u && (2) \end{aligned}$$

for all $\alpha \in [0,1]$.

We extend the Hausdorff metric by defining the D-metric:

$$D(u,v) = \sup_{0<\alpha\leq 1} H(L_\alpha u, L_\alpha v).$$

Concerning the properties of this space, Puri and Ralescu [10] proved that $(\mathbb{E}^n, D)$ is a complete metric space; Kaleva [7] proved that $(\mathbb{E}^n, D)$ is not separable (Exemple 2.1).

The D-metric is homogeneous and invariant by translations under the operations (1) and (2), and consequently, applying the Theorem of Radström [12], Diamond and Kloeden [5], Kaleva [7] and, Puri and Ralescu [10] showed that $\mathbb{E}^n$ can be embedded as convex cones in certain Banach spaces.

We denote by $\mathbb{E}^n_C$ the subspace of $\mathbb{E}^n$ for which the elements u are such the mapping $\alpha \to L_\alpha u$ is H-continuous on $[0,1]$, i.e., given $\varepsilon > 0$, there is $\delta > 0$ such that

$$|\alpha - \beta| < \delta \Rightarrow H(L_\alpha u, L_\beta u) < \varepsilon.$$

Also, we denote by $\mathbb{E}^n_L$ the subspace of $\mathbb{E}^n_C$ for which the element u are such that the application $\alpha \to L_\alpha u$ is Lipschitz continuous, i.e., there is $\nu > 0$ such that, for all $\alpha, \beta \in [0,1]$

$$H(L_\alpha u, L_\beta u) \leq \nu|\alpha - \beta|.$$

The following exemple shows that $\mathbb{E}^n_L \subsetneq \mathbb{E}^n_C$.

Exemple 1. Let $u : \mathbb{R} \to [0,1]$ defined by $u(x) = x^2$ if $x \in [0,1]$ and $u(x) = 0$ if $x \in (\mathbb{R} \setminus [0,1])$. Then, $L_\alpha u = [\sqrt{\alpha}, 1]$ for all $\alpha \in [0,1]$, consequently,

$$H(L_\alpha u, L_\beta u) = |\sqrt{\alpha} - \sqrt{\beta}| = \frac{1}{\sqrt{\alpha} + \sqrt{\beta}}|\alpha - \beta|$$

for all $\alpha \neq \beta$. So, $u \in \mathbb{E}^n_C \setminus \mathbb{E}^n_L$.

By using the following properties of H-metric,

$$H(A + B, C + D) \leq H(A,C) + H(B,D),$$
$$H(\lambda A, \lambda B) = \lambda H(A,B)$$

for all $A, B \in \mathbb{K}(\mathbb{R}^n)$ and $\lambda > 0$, we deduce that $\mathbb{E}^n_C$ and $\mathbb{E}^n_L$ are closed under the operations (1) and (2).

Moreover, recall that the support function of a nonempty subset A of $\mathbb{R}^n$ is the function $s_A : S^{n-1} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$s_A(x) = \sup\{\langle x, a\rangle / a \in A\}$$

where $S^{n-1} = \{x \in \mathbb{R}^n / \|x\| = 1\}$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$. If we take $A \in \mathbb{K}(\mathbb{R}^n)$, then

$$s_A(x) = \max\{\langle x, a\rangle / a \in A\}.$$

Some properties of the function $s_A(\cdot)$ are

$$
\begin{aligned}
s_{A+B} &= s_A + s_B && (3)\\
s_{\lambda A} &= \lambda s_A && (4)
\end{aligned}
$$

s_A is Lipschitz continuous with constant: $||A|| = H(\{0\}, A)$. (5)

Moreover, the H-metric can be written as

$$H(A,B) = \max\{|s_A(x) - s_B(x)|; x \in S^{n-1}\}. \tag{6}$$

The above result can be seen in [1] or [3]. We consider $C(S^{n-1}) = \{f : S^{n-1} \to \mathbb{R}; f$ is continuous$\}$ with usual norm of uniforme covergence.

The following Theorem is due to Minkowski:

THEOREM 2.2. *The application $j : \mathbb{K}(\mathbb{R}^n) \to C(S^{n-1})$ defined by $j(A) = s_A$ is positively homogeneous, subadditive, and it is also an isometry.*

Puri and Ralescu [10] extended the definition of support functions to the fuzzy context setting

$$s_u(\alpha, x) = s_{L_\alpha u}(x)$$

for all $(\alpha, x) \in [0,1] \times S^{n-1}$

It is easily seen n that $s_{u+v} = s_u + s_v$ for all $u, v \in \mathbb{E}^n$ and $\lambda > 0$. We denote by $C([0,1] \times S^{n-1}) = \{f : [0,1] \times S^{n-1} \to R; f$ continuous$\}$ with the usual norm.

One of the principal results of [10] is:

THEOREM 2.3. *The application $j : \mathbb{E}^n_L \to C([0,1] \times S^{n-1})$ defined by $j(u) = s_u$ is positively homogeneous, subadditive, and it is also an isometry. Moreover, $j(u)$ is Lipschitz continuous.*

3. The Isometry j defined on $\mathbb{E}^n_C$.

Our purpose in this section is to show that $\mathbb{E}^n_C$ is a complete metric space and that j is an isometry when defined on $\mathbb{E}^n_C$ with values on $C([0,1] \times S^{n-1})$. Also, we will show that $\mathbb{E}^n_C$ is the maximal subspace of $\mathbb{E}^n$ with this property.

THEOREM 3.1. *$(\mathbb{E}^n_C, D)$ is a complete metric space.*

Proof. Let (u_p) a D-Cauchy sequence in $\mathbb{E}^n_C$. Then, by using the completeness of $\mathbb{E}^n$, we deduce that there exist $u \in \mathbb{E}^n$ such that $u_p \xrightarrow{D} u$.

In continuation, we prove that $u \in \mathbb{E}^n_C$. In fact, given $\varepsilon > 0$ there is $\overline{n} \in \mathbb{N}$ such that $D(u_p, u) < \varepsilon/3$ for all $p \geq \overline{n}$. For a fixed $p_0 > \overline{n}$, we have that there exist $\delta = \delta(\varepsilon, p_0) > 0$ such that

$$|\alpha - \beta| < \delta \Rightarrow H(L_\alpha u_{p_0}, L_{\beta u_{p_0}}) < \varepsilon/3,$$

since $u_{p_0} \in \mathbb{E}^n_C$. Consequently,

$$\begin{aligned} H(L_\alpha u, L_\beta u) &\le H(L_\alpha u, L_\alpha u_{p_0}) + H(L_\alpha u_{p_0}, L_\beta u_{p_0}) + H(L_\beta u_{p_0}, L_\beta u) \\ &\le D(u, u_{p_0}) + \varepsilon/3 + D(u_{p_0}, u) < \varepsilon, \end{aligned}$$

for all $|\alpha - \beta| < \delta$. So, $u \in \mathbb{E}^n_C$ and this completes the proof.

Now, we give an extension of the Theorem 2.3.

THEOREM 3.2. *The application $j : \mathbb{E}^n_C \to C([0,1] \times s^{n-1})$ defined by $j(u) = s_u$ is positively homogeneous, subadditive and it is also an isometry.*

Proof. The positively homogeneous and subadditivity of j are clear. We show that if $u \in \mathbb{E}^n_C$, then $j(u)$ is continuous. In fact, given $\varepsilon > 0$ we can to choose $\delta_1 > 0$ such that

$$|\alpha - \beta| < \delta_1 \Rightarrow H(L_\alpha u, L_\beta u) < \varepsilon/2.$$

Moreover, since $u \in \mathbb{E}^n_C$, we have $\sup_{\beta \ge 0} ||L_\beta u|| = q < +\infty$ and we can take $\delta_2 > 0$ such that $q\delta_2 < \varepsilon/2$.

Now, we consider $\alpha, \beta \in [0,1]$ and $x, y \in S^{n-1}$ such that $|\alpha - \beta| < \delta_1$ and $||x - y|| < \delta_2$, and it follows that

$$\begin{aligned} |s_u(\alpha, x) - s_u(\beta, y)| &= |s_{L_\alpha u}(x) - s_{L_\beta u}(y)| \\ &\le |s_{L_\alpha u}(x) - s_{L_\beta u}(x)| + |s_{L_\beta u}(x) - s_{L_\beta u}(y)| \end{aligned} \tag{7}$$

By using (5) and the chosen δ_2, we obtain

$$\begin{aligned} |s_{L_\beta u}(x) - s_{L_\beta u}(y)| &\le ||L_\beta u||\, ||x - y|| \\ &\le q\delta_2 \\ &< \varepsilon/2. \end{aligned} \tag{8}$$

Moreover, by using (6) and the chosen δ_1, we have

$$\begin{aligned} |s_{L_\alpha u}(x) - s_{L_\beta u}(x)| &\le \sup\{|s_{L_\alpha u}(z) - s_{L_\beta u}(z); \,|z \in S^{n-1}\} \\ &= H(s_{L_\alpha u}, s_{L_\beta u}) \\ &< \varepsilon/2. \end{aligned} \tag{9}$$

Consequently, (8) and (9) together (7) imply that $j(u)$ is continuous on $[0,1] \times S^{n-1}$.

Now, we proceed to prove that j is an isometry. We have

$$
\begin{aligned}
\|j(u)-j(v)\|_\infty &= \sup_{\alpha,x} |s_u(\alpha,x)-s_v(\alpha,x)| \\
&= \sup_{\alpha>0} \sup\{|s_u(\alpha,x)-s_v(\alpha,x)|; x\in S^{n-1}\} \\
&= \sup_{\alpha>0} \sup\{|s_{L_\alpha u}(x)-s_{L_\alpha v}(x)|; x\in S^{n-1}\} \\
&= \sup_{\alpha>0} H(L_\alpha u, L_\alpha v) \\
&= D(u,v).
\end{aligned}
$$

Therefore, the proof of the Theorem is completed.

Since $[0,1]\times S^{n-1}$ is compact, we can deduce imediately the following

COROLLARY 3.3. *$(\mathbb{E}^n_C, D)$ is a separable metric space.*

COROLLARY 3.4. *If $u_p, u \in \mathbb{E}^n_C$, then $u_p \xrightarrow{D} u$ iff $s_{u_p} \to s_u$ uniformly on $[0,1]\times S^{n-1}$.*

In what follows, we show that $\mathbb{E}^n_C$ is the maximal subspace of $\mathbb{E}^n$ that can be embedded in $C([0,1]\times S^{n-1})$ through the isometry j.

THEOREM 3.5. *Let $u\in \mathbb{E}^n \setminus \mathbb{E}^n_C$ be, then $j(u) \notin C([0,1]\times S^{n-1})$.*

Proof. Let $\alpha' \in [0,1]$ be such that $\alpha \to L_\alpha u$ is not continuous for α'. Then, there exists $\varepsilon > 0$ such that for each $p \in \mathbb{N}$, we can find $\alpha_p \in [0,1]$ such that $|\alpha_p - \alpha'| < \dfrac{1}{p}$ and $H(L_{\alpha_p}u, L_{\alpha'}u) \geq \varepsilon$. Thus, for all p, we have

$$\sup_{x\in S^{n-1}} |s_{L_{\alpha_p}}(x) - s_{L_{\alpha'}u}(x)| \geq \varepsilon,$$

and, consequently, $|s_u(\alpha_p,x) - s_u(\alpha',x)| \geq \varepsilon$ for all p; that is, $j(u)$ is not continuous for $(\alpha',x), x\in S^{n-1}$.

4. THE SPACE $\mathbb{E}^n_L$ IS NOT COMPLETE.

We begin with some definitions and preliminary results:

DEFINITION 4.1. We say that a sequence $(u_q) \subseteq \mathbb{E}^n$ *L-converges* to $u\in\mathbb{E}^n$, $(u_q \xrightarrow{L} u)$, if for all $\alpha\in(0,1]$ we have,

$$H(L_\alpha u_q, L_\alpha u) \to 0 \quad \text{as } q\to\infty.$$

Now we are going to use the following results that were proved in [13]

Proposition 4.2. $u \in I\!E^n_C$ if and only if

$$L_\alpha u = \overline{\{u > \alpha\}}, \quad \forall \alpha \in (0,1).$$

Proposition 4.3. Let $u_q, u \in I\!E^n$; if $u \in I\!E^n_C$, then the following are equivalent:

i) $u_q \xrightarrow{D} u$

ii) $u_q \xrightarrow{L} u$ and $L_0 u_q \xrightarrow{H} L_0 u$.

Applying the above propositions we have

Corollary 4.4. *Let $u_q, u \in I\!E^n$; if $u \in I\!E^n_C$, satisfying $u_q \nearrow u$ and $L_1 u_q \to L_1 u$, then $u_q \xrightarrow{D} u$.*

Proof. Let us consider $\alpha \in (0,1)$; since $u_q \nearrow u$, then $(L_\alpha u_q)_q$ is an increacing sequence and $\limsup_{q\to\infty} L_\alpha u_q \subseteq \bigcup_q L_\alpha u_q \subseteq L_\alpha u$.

Now, if $\alpha < u(x) \le 1$, then $u(x) = \alpha + \varepsilon$ with $\varepsilon > 0$. Consequently, there exists $N \in I\!N$ such that $u(x) - u_q(x) < \varepsilon$ for all $q \ge N$, i.e., $u_q(x) > u(x) - \varepsilon = \alpha$ for all $q \ge N$. In other words, $x \in L_\alpha u_q$ for all $q \ge N$ it that implies $x \in \liminf_{q\to\infty} L_\alpha u_q$. Whence, $\{u > \alpha\} \subseteq \liminf_{q\to\infty} L_\alpha u_q$. Since the lower limit is closed, the Proposition 4.2. implies that $\overline{\{u > \alpha\}} = L_\alpha u \subseteq \liminf_{q\to\infty} L_\alpha u_q$. Thus, we get : $\liminf_{q\to\infty} L_\alpha u_q = \limsup_{q\to\infty} L_\alpha u_q = L_\alpha u$ for all $\alpha \in (0,1)$.

Being $L_\alpha u$ compact and $L_\alpha u_q \subseteq L_\alpha u$ for all q, Proposition 2.3. implies $L_\alpha u_q \xrightarrow{H} L_\alpha u$ for all $\alpha \in (0,1)$. By hypothesis $L_1 u_q \xrightarrow{H} L_1 u$; consequently by definition, $u_q \xrightarrow{L} u$.

Now, we prove that $L_0 u_q \xrightarrow{H} L_0 u$. By using the same arguments above, it is enough to prove that $L_0 u_q \xrightarrow{K} L_0 u$. It is easy to see that

$$\limsup_{q\to\infty} L_0 u_q \subseteq \bigcup_q L_0 u_q \subseteq L_0 u$$

because $\{u_q > 0\} \subseteq L_0 u$ for all $q \in I\!N$. Thus, if we prove that $L_0 u \subseteq \liminf_{q\to\infty} L_0 u_q$, we can conclude the proof of Corollary. In fact, if $u(x) > 0$ then there is $N \in I\!N$ such that $u_q(x) > 0$ for all $q \in I\!N$ due to the fact that $u_q \nearrow u$. Thus, by definition, $x \in L_0 u_q$ for all $q \in I\!N$, and it follows that $x \in \liminf_{q\to\infty} L_0 u_q$. Consequently, $\{u > 0\} \subseteq \liminf_{q\to\infty} L_0 u_q$, and being this last set closed, we obtain $L_0 u = \overline{\{u > 0\}} \subseteq \liminf_{q\to\infty} L_0 u_q$.

In continuation, we will analyse of the case $n = 1$, and we will show that $I\!E^1_L$ is a noncomplete subspace of $(I\!E^1, D)$. To build an exemple, we will work with a special class of polygonal functions with support equal to interval $[0,1]$.

Lemma 4.5. *Let $x_0, t_0 \in [0,1)$ and define $p : I\!R \to [0,1]$ such that*

$$p(x) = \begin{cases} 1 & \text{if} \quad 0 \le x \le x_0 \\ 1 + (x - x_0)(t_0 - 1)/(1 - x_0) & \text{if} \quad x_0 < x \le 1 \\ 0 & \textit{elsewhere} \end{cases}$$

it Then $p \in \mathbb{E}_L^1$ with Lipschitz constant equal to $(1-x_0)/(1-t_0)$.

Proof: Easily we observe that the levels of p are:

$$L_\alpha p = \begin{cases} [0,1] & \text{if } \ 0 \le \alpha \le t_0 \\ [0,(1-x_0)(1-\alpha)/(1-t_0)+x_0] & \text{if } \ t_0 < \alpha \le 1, \end{cases}$$

consequently if $0 \le \alpha \le \beta \le t_0$ we have $H(L_\alpha p, L_\beta p) = 0$; also if $t_0 < \alpha \le \beta \le 1$ we deduce $H(L_\alpha p, L_\beta p) = |(1-x_0)(1-t_0)|\,|\alpha-\beta|$ and finally if $0 \le \alpha \le t_0 < \beta \le 1$ we obtain

$$\begin{aligned} H(L_\alpha p, L_\beta p) &= |1 - \{(1-x_0)(1-\beta)/(1-t_0)+x_0\}| \\ &\le |(1-x_0)/(1-t_0)|\,|\alpha-\beta|. \end{aligned}$$

This, in all cases we have

$$\begin{aligned} H(L_\alpha p, L_\beta p) &\le |(1-x_0)/(1-t_0)|\,|\alpha-\beta| \\ &= |(1-x_0)/(1-t_0)|\,|\alpha-\beta|, \end{aligned}$$

Since $x_0, t_0 \in [0,1)$. This complete the proof of Lemma.

Remark 4.6. If in the above Lemma $x_0 = 1$ or $t_0 = 1$, then $p = \chi_{[0,1]}$ and, it is a Lipschitzian with null constant.

We denote by $\mathcal{P}$ the set of all polygonals functions such that consider in above Lemma.

LEMMA 4.7. *If $p_1, p_2 \in \mathcal{P}$ with Lipschitz constant c_1 and c_2, respectively, then $p = \min\{p_1,p_2\} \in \mathbb{E}_L^1$ with Lipchitz constant equal to (c_1+c_2).*

Proof. The no trivials cases are two as to show the following figures a and b:

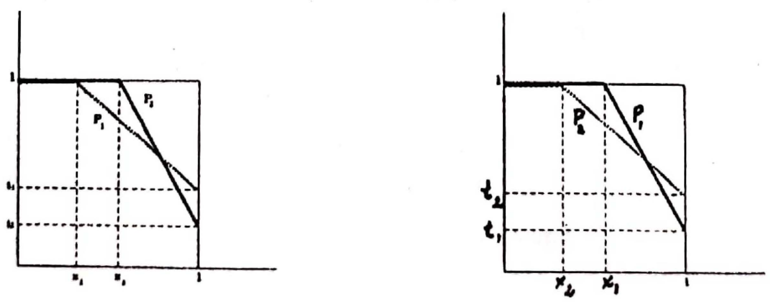

Figure a   Figure b

We prove only the case of the figure a, since the other case (figure b) is analogous. If $p = \min\{p_1,p_2\}$ we have

$$L_\alpha p = \begin{cases} L_\alpha p_2 & \text{if } \ 0 \le \alpha \le t \\ L_\alpha p_1 & \text{if } \ t < \alpha \le 1. \end{cases}$$

In continuation, we observe that $L_t p_1 = L_t p_2$, consequently we have,

$$H(L_\alpha p, L_\beta p) = \begin{cases} H(L_\alpha p_2, L_\beta p_2) \le c_2|\alpha - \beta| & \text{if } 0 \le \alpha \le \beta \le t \\ H(L_\alpha p_1, L_\beta p_1) \le c_1|\alpha - \beta| & \text{if } t \le \alpha \le \beta \le 1. \end{cases}$$

Finally, if $0 \le \alpha \le t < \beta \le 1$ then

$$\begin{aligned} H(L_\alpha p, L_\beta p) &= H(L_\alpha p_2, L_\beta p_1) \\ &\le H(L_\alpha p_2, L_t p_2) + H(L_t p_2, L_\beta p_1) \\ &= H(L_\alpha p_2, L_t p_2) + H(L_t p_1, L_\beta p_1) \\ &\le c_2|\alpha - \beta| + c_1|\alpha - \beta| \\ &\le (c_1 + c_2)|\alpha - \beta|. \end{aligned}$$

This conclude the proof of the Lemma.

LEMMA 4.8. *Let $f, g \in \mathbb{E}^1_L$ be Lipschitz functions with constant M and N, respectively. We assume that f and g are continuous and decreasing on their supports, and such that $L_0 f = L_0 g = [a, b]$.*

If $Gr(f) \cap Gr(g) \cap [a, b] \times [0, 1] = \{(x_0, t_0)\}$, then $q = \min(f, g)$ belongs to $\mathbb{E}^1_L$.

Proof. We remark that, in this case, we have

i) $L_{t_0} f = L_{t_0} g$;

ii) over the subintervals $[a, x_0]$ and $[x_0, b]$ we have $q = f$ or $q = g$ (see figure c) and the proof is essentialy the same that of the Lemma 4.7.

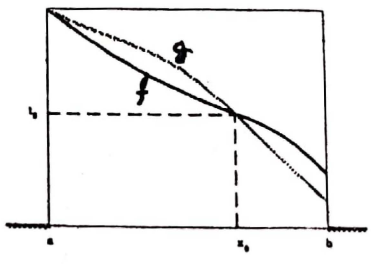

Figure c

THEOREM 4.9: *$(\mathbb{E}^1_L, D)$ is a noncomplete metric space.*

Proof: We consider $u(x) = 1 - x^2$ if $x \in [0, 1]$ and $u(x) = 0$ if $x \in \mathbb{R} \setminus [0, 1]$. Clearly, $u \in \mathbb{E}^1_C$ and $u \notin \mathbb{E}^1_L$.

In fact,

$$\begin{aligned} H(L_\alpha u, L_\beta u) &= H([0,\sqrt{1-\alpha}],[0,\sqrt{1-\beta}]) = |\sqrt{1-\alpha}-\sqrt{1-\beta}| \\ &= \frac{1}{\sqrt{1-\alpha}+\sqrt{1-\beta}}|\alpha-\beta| \text{ for } \alpha \neq \beta \end{aligned} \tag{10}$$

If we suppose that $u \in \mathbb{E}^1_L$, then there exists $C > 0$ such that

$$H(L_\alpha u, L_\beta u) \leq C|\alpha-\beta| \quad \text{for all } \alpha \neq \beta,$$

consequently, by using (10) we conclude that $C > (\sqrt{1-\alpha}+\sqrt{1-\beta})^{-1}$ for all $\alpha \neq \beta$ which is an absurd. This proves that $u \notin \mathbb{E}^1_L$.

Let us consider the sequence of polygonal functions where each p_q consist of the union of 2^{q-1} rectilineal segments which are obtained by sucessive divisions of the graph of u as shown in the following figures:

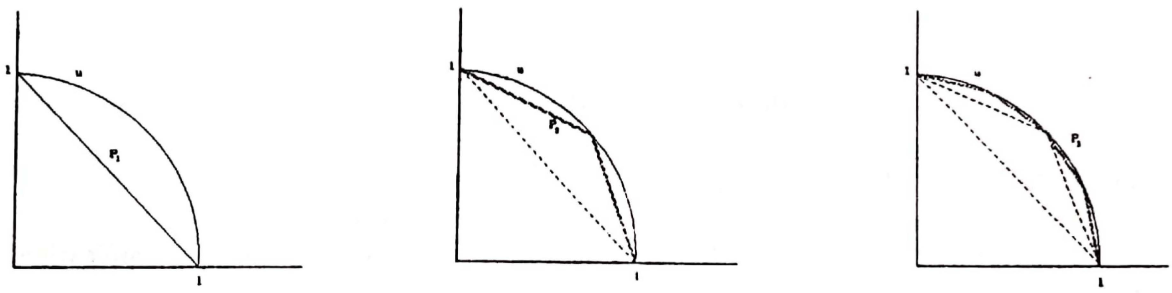

Figure d

Combining the Lemmas 4.5, 4.7 and 4.8 we have that $p_q \in \mathbb{E}^1_L$ for all $q \in \mathbb{N}$. On the other hand, since $u \in \mathbb{E}^n_C$, then $p_q \nearrow u$ and $L_1 p_q \xrightarrow{H} L_1 u$, we conclude that $p_q \xrightarrow{D} u$ thanks to the Corollary 4.4. It follows that the space $(\mathbb{E}^1_L, D)$ is not complete.

We and this section by characterizing the closure of $\mathbb{E}^n_L$.

Theorem 4.10. $\overline{\mathbb{E}^n_L} = \mathbb{E}^n_C$.

Proof. Let $u \in \mathbb{E}^n_C$, then the multifunction $F : [0,1] \to \mathbb{K}(\mathbb{R}^n)$ given by $F(\alpha) = L_\alpha u$ is H-continuous on $[0,1]$. We consider the q^{th} Bernstein polynomial $B_q(F;\alpha)$ associated with F:

$$B_q(F;\alpha) = \sum_{j=0}^{q} \binom{q}{j} F\left(\frac{j}{q}\right) \alpha^j (1-\alpha)^{q-j}, 0 \leq \alpha \leq 1$$

R.A. Vitale [15] has proved that

$$D(F, B_q(F,\cdot)) \to 0 \quad \text{as} \quad q \to +\infty.$$

We observe that $B_q(F;\alpha) \in \mathbb{K}(\mathbb{R}^n)$ for each $q \in \mathbb{N}$ and $\alpha \in [0,1]$. Now, we verify the hypothesis of the Representation Theorem given by Negoita and Ralescu [8] to show that the family $N_\alpha = B_q(F;\alpha)$, for each $q \in I\!N$, define an unique fuzzy set. If $\alpha \leq \beta$, then $F(\alpha) \supseteq F(\beta)$ and, consequently, $B_q(F;\alpha) \supseteq B_q(F;\beta)$ (see Vitale [15], p. 312). So, we only have to prove that, if $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_l \to \alpha \neq 0$ as $l \to \infty$, then

$$B_q(F;\alpha) = \bigcap_{l=1}^{\infty} B_q(F;\alpha_l).$$

We observe that $\alpha \to B_q(F;\alpha)$ is a H-continuous multifunction, consequently, for each fixed q, we have

$$B_q(F;\alpha_l) \xrightarrow{H} B_q(F;\alpha) \text{ as } l \to \infty,$$

as $B_q(F;\alpha) \in I\!K(I\!R^n)$, and we deduce from the Proposition 2.1 that

$$B_q(F;\alpha_l) \xrightarrow{K} B_q(F;\alpha) \text{ as } l \to \infty. \tag{11}$$

Being $\{B_q(F;\alpha_l) | l \in I\!N\}$, one decreasing sequence, we have

$$B_q(F;\alpha_l) \xrightarrow{K} \bigcap_{l=1}^{\infty} B_q(F;\alpha_l) \text{ as } l \to \infty.$$

j so it follows from (11) that it holds the required equality

This completes the hypothesis of the Negoita-Ralescu Theorem.

Finally, we prove that $\alpha \to B_q(F;\alpha)$ for each $q \in I\!N$ is a Lipschitzian application. By virtue of (6) is sufficient to show that

$$\max\{|s_{B_q(F;\alpha)}(x) - s_{B_q(F;\beta)}(x)|; x \in S^{n-1}\} \leq C|\alpha - \beta|$$

with $C > 0$ independent of α and β.

Note that the support function of Bernstein approximant of F is given by

$$s_{B_q(F;\alpha)}(x) = \sum_{j=0}^{q} \binom{q}{j} \alpha^j (1-\alpha)^{q-j} s_{F\left(\frac{j}{q}\right)}(x)$$

with $x \in S^{n-1}$, so that

$$\begin{aligned}
&|s_{B_q(F;\alpha)}(x) - s_{B_q(F;\beta)}(x)| \\
&\leq \sum_{j=0}^{q} \binom{q}{j} |s_{F\left(\frac{j}{q}\right)}(x)|\, |\alpha^j(1-\alpha)^{q-j} - \beta^j(1-\beta)^{q-j}| \\
&\leq \|S_{F(0)}\|_\infty \sum_{j=0}^{q} \binom{q}{j} |\alpha^j(1-\alpha)^{q-j} - \beta^j(1-\beta)^{q-j}| \\
&\leq C|\alpha - \beta|,
\end{aligned}$$

Since $F(0) \supseteq F\left(\dfrac{j}{q}\right) \supseteq F(1)$ for all $0 < j < q$ implies that

$$s_{F(1)}(x) \leq s_{F(\frac{i}{q})}(x) \leq s_{F(0)}(x)$$

for all $x \in S^{n-1}$. This completes the proof.

5. APPLICATIONS

In this section we give some applications of our previous results to the convergence of the fuzzy random variables in $\mathbb{E}^n_C$.

We will briefly go over some basic material on the measurability and integration of multifunctions that we will needed in the sequel. For more details we refer to Aumann [2], Castaing and Valadier [4], Hukuhara [6] and, Klein and Thompson [8].

Let $\mathbb{P}(\mathbb{R}^n)$ be the set of nonempty subsets of $\mathbb{R}^n$ and $(\Omega, \sum, \mu)$ a complete finite measure space. Let $F : \Omega \to \mathbb{P}(\mathbb{R}^n)$ be a multifunction from Ω into $\mathbb{R}^n$. Let $Gr(F) = \{(w,x) \in \Omega \times \mathbb{R}^n / x \in F(w)\}$ be the graph of F. We say that F is measurable if $Gr(F) \in \sum \times \mathbb{B}(\mathbb{R}^n)$, where $\mathbb{B}(\mathbb{R}^n)$ is the Borel σ-field of $\mathbb{R}^n$.

For any multifunction $F : \Omega \to \mathbb{P}(\mathbb{R}^n)$ we can define the set $S(F) = \{f \in L^1(\Omega, \mathbb{R}^n) | f(w) \in F(w), \mu - a.e.\}$, i.e., $S(F)$ contains all integrable selectors of F. The ***integral*** introduced by Aumann [2] as a generalization of the single-valued integral and of the Minkowski sum of sets is defined by

$$\int_\Omega F(w)d\mu(w) = \{\int_\Omega f(w)d\mu(w) | f \in S(F)\}.$$

and denoted simply by $\int F$.

It is natural to ask under what conditions $\int F$ (or equivalenty, $S(F)$) is nonempty. The multifunction F will be called *integrably bounded* if there exist $\varphi \in L^1(\Omega, \mathbb{R})$ such that $||x|| \leq \varphi(w)$ $\mu - a.e.$, almost all x and w such that $x \in F(w)$. The following results are in Aumann [2].

THEOREM 5.1. *If the measure μ on the σ-álgebra $\sum$ of Ω is atomless, then the integral $\int F$ is a convex set.*

THEOREM 5.2. *If F is integrably bounded and $F(w)$ is closed for almost all $w \in \Omega$, then $\int F \in \mathbb{K}(\mathbb{R}^n)$.*

Also, we mention the following generalization of Lebesgue's dominated convergence Theorem,

THEOREM 5.3. *If $F_p : \Omega \to \mathbb{P}(\mathbb{R}^n)$ are measurables and there is $f \in L^1(\Omega, \mathbb{R})$ such that $\sup_{p\geq 1} ||g_p(w)|| \leq f(w)$ for all $g_p \in S(F_p)$, then if $F_p(w) \xrightarrow{K} F(w)$ we have*

$$\int F_p \xrightarrow{H} \int F \quad \text{as} \quad p \to \infty$$

THEOREM 5.4. *Let $F : \Omega \to \mathbb{P}(\mathbb{R}^n)$ be a measurable and integrably bounded, if π is a linear form over $\mathbb{R}^n$, then*

$$\sup \pi(\int F d\mu) = \int \sup \pi(F(w)) d\mu(w).$$

A *fuzzy random variable* is a function $\Gamma : \Omega \to \mathbb{E}^n$ such that for every $\alpha \in [0,1]$ the multifunction $\Gamma_\alpha : \Omega \to \mathbb{P}(\mathbb{R}^n)$ defined by $\Gamma_\alpha(w) = L_\alpha \Gamma(w)$ is measurable [10]. Moreover, we say that Γ is *integrably bounded* if Γ_α is integrably bounded for all $\alpha \in [0,1]$. We observe that for Γ to be integrably bounded it is necessary and sufficient that Γ_0 be integrable bounded; this is a consequence of the following fact: $0 \leq \alpha \leq \beta$ implies $\Gamma_\beta(w) \subseteq \Gamma_\alpha(w) \subseteq F_0(w)$, for all $w \in \Omega$.

The following Theorem due a Puri and Ralescu [10] permit us to define the integral of a fuzzy random variable $\Gamma : \Omega \to \mathbb{E}^n$.

THEOREM 5.5. *If $\Gamma : \Omega \to \mathbb{E}^n$ is an integrably bounded fuzzy variable, there exists a unique fuzzy set $u \in \mathbb{E}^n$ such that $L_\alpha u = \int \Gamma_\alpha d\mu$, for every $\alpha \in [0,1]$.*

The element $u \in \mathbb{E}^n$ obtained in Theorem 5.5 define the integral of the fuzzy random variable Γ, i.e.,

$$\int \Gamma d\mu = u \Leftrightarrow L_\alpha u = \int \Gamma_\alpha d\mu, \quad \text{for every } \alpha \in [0,1].$$

THEOREM 5.6. *Let $\Gamma : \Omega \to \mathbb{E}^n_C$ be a fuzzy random variable integrably bounded, then $\int \Gamma \in \mathbb{E}^n_C$.*

Proof. We consider an sequence $(\alpha_p) \subseteq [0,1]$ such that $\alpha_p \to \alpha, \alpha \in [0,1]$. Since $F(w) \in \mathbb{E}^n_C$, it follows that $L_{\alpha_p}\Gamma(w) \to L_\alpha \Gamma(w)$ for all $w \in \Omega$ as $p \to \infty$. Thus, we deduce that for all $w \in \Omega, \Gamma_{\alpha_p}(w) \to \Gamma_\alpha(w)$ as $p \to \infty$. Moreover, being Γ integrably bounded, we conclude that each Γ_{α_p} is also integrably bounded, and, if $f \in L^1(\Omega, \mathbb{R})$ is such that for all $x \in \Gamma_0(w) : ||x|| \leq f(w)$, we also conclude that $\sup_{p \geq 1}\{||x|| x \in \Gamma_{\alpha_p}(w)\} \leq f(w)$. Consequently, using Theorem 5.3, we have $\int \Gamma_{\alpha_p} \to \int \Gamma_\alpha$ as $p \to \infty$. In other words, $L_{\alpha_p} \int \Gamma \to L_\alpha \int \Gamma$ as $p \to \infty$, and therefore $\int \Gamma \in \mathbb{E}^n_C$.

COROLLARY 5.7. *Let $\Gamma_p, \Gamma : \Omega \to \mathbb{E}^n_C$ be an integrably bounded fuzzy random variable. Then, $\int \Gamma_p \to \int \Gamma$ on $(\mathbb{E}^n_C, D) \Leftrightarrow s_{\int \Gamma_p} \to s_{\int \Gamma}$ on $C([0,1] \times S^{n-1}), ||.||_\infty)$.*

Also, we have

Theorem 5.8. *Let $\Gamma : \Omega \to \mathbb{E}^n_C$ be an integrably bounded fuzzy random variable. Then,*

$$s_{\int \Gamma}(\alpha, x) = \int s_{\Gamma(w)}(\alpha, x) d\mu(w).$$

Proof. It follows imediatelly from Theorem 5.4. In fact,

$$\begin{aligned} s_{\int \Gamma}(\alpha, x) &= s_{L_\alpha \int \Gamma}(x) = s_{\int \Gamma_\alpha}(x) = \int s_{\Gamma_\alpha(w)}(x) d\mu(w) \\ &= \int s_{L_\alpha \Gamma(w)}(x) d\mu(w) = \int s_{\Gamma(w)}(\alpha, x) d\mu(w). \end{aligned}$$

Remark 5.9. It is well know that if $A \in \mathbb{K}(\mathbb{R}^n)$ then

$$A = \bigcap_{y \in S^{n-1}} \{x \in \mathbb{R}^n \mid \langle x, y \rangle \leq s_A(y)\},$$

see [1] or [3].

If we apply this in the fuzzy context, we have that if $u \in \mathbb{E}^n_C$, then for each $\alpha \in [0,1]$

$$L_\alpha u = \bigcap_{y \in S^{n-1}} \{x \in \mathbb{R}^n \mid \langle x, y \rangle \leq s_u(\alpha, x)\}.$$

Thus, given some relations involving fuzzy sets in $\mathbb{E}^n_C$, we obtain the corresponing relations for the fuzzy support function s_u. On the other hand, from relations involving fuzzy support functions s_u we can obtain analogous relations for the α-Level of the fuzzy set $u \in \mathbb{E}^n_C$ and, consequently, for u. Thus, we can apply the duality theory between support functions and $\mathbb{K}(\mathbb{R}^n)$ in the fuzzy context.

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