A GEOMETRICAL INTERPRETATION OF THE EQUIVALENCE OF DIRAC AND MAXWELL EQUATIONS

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ABSTRACT – We prove, for non-null electromagnetic fields and for their respective free cases, that Maxwell and Dirac equations are equivalent. Our proof is based on the use of Rainich-Misner-Wheeler theorem and on general assumptions which are indeed satisfied for the case under consideration. This equivalence is discussed in terms of relationship between a non-linear Dirac-like equation (with is a spinorial representation of Maxwell equation) and Dirac equation. This relationship is interpreted by means of a Riemann-Cartan-Weyl geometry which is metric compatible and with the trace of the torsion 1-form playing the role of the Weyl 1-form. We also discuss the relationship between Maxwell and Dirac fields in the light of the above resuts. All calculations are performed in terms of the Clifford algebra of spacetime, the so-called spacetime algebra.

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Abstract: We prove, for non-null electromagnetic fields and for their respective free cases, that Maxwell and Dirac equations are equivalent. Our proof is based on the use of Rainich-Misner-Wheeler theorem and on general assumptions which are indeed satisfied for the case under consideration. This equivalence is discussed in terms of relationship between a non-linear Dirac-like equation (with is a spinorial representation of Maxwell equation) and Dirac equation. This relationship is interpreted by means of a Riemann-Cartan-Weyl geometry which is metric compatible and with the trace of the torsion 1-form playing the role of the Weyl 1-form. We also discuss the relationship between Maxwell and Dirac fields in the light of the above resuts. All calculations are performed in terms of the Clifford algebra of spacetime, the so-called spacetime algebra.

1. Introduction

There is a paradoxical situation in Physics: electrodynamics and relativistic quantum mechanics can be said to be well established theories, but the same is not true for the concepts of photon and electron. One can support this assertion by quoting the greats Einstein and Dirac: "You know, it would be sufficient to really understand the electron" (Einstein, quoted in [8]); "Every physicist thinks that he knows what a photon is. I spent my whole life to find out what a photon is, and I still don't know it" (Einstein, quoted in [3]); "I really spent my life mainly trying to find better equations for quantum-electrodynamics, and so far without success, but I continue to work on it" (Dirac, quoted in [3]). It is therefore a fundamental issue that one understands things like the (real) meaning of Dirac theory, the meaning of spinors, the range of applicability of Maxwell's electrodynamics, etc. Questions like these ones and others were studied, for example, by Barut [1–3], Campolattaro [4, 5], Hestenes [6-8], etc.

The problem we want to study here is related to the above one but it is a little more general: the relation between electrodynamics and quantum mechanics. More specifically, we shall prove, for non-null electromagnetic fields and for their respective *free* cases, that Maxwell and Dirac equations are equivalent. In order to prove our claim, we start showing by means of the theorem of Rainich-Misner-Wheeler [9, 10] how one can associate a spinor field to any non-null electromagnetic field, and by using this result we can find a spinor representation of Maxwell equations [11-13]. A pioneering work in this direction is the one of Campolattaro [4, 5], but our approach is different from his one since we do not use the traditional tensor and spinor calculus (which is often intricated, and sometimes does not help in the physical interpretation). We shall use instead the Clifford bundle formalism, which we briefly review in the next section. In our approach, pages of Campolattaro's calculations [4] are performed in a line, and the equivalence between our spinor representation of Maxwell equations and the one of Campolattaro is proved in [14]. We can show therefore that under general assumptions the spinor equation that represents Maxwell equations reduces to the Dirac equation [11-14].

Some of the results we shall present have already been discussed in [11-14], but these references left open some question to which we shall give an answer here. More specifically, the spinor representation of Maxwell equation is a non-linear equation, but Dirac equation is linear. Moreover, the non-linear term in that equation is of quantum-potential type. We have therefore to interpret the non-linear term and to clarify how one goes from a non-linear to a linear equation. Our interpretation of the non-linear term is very interesting; we shall exploit the fact that the spinor field we associate to that non-null electromagnetic field is the so-called Dirac-Hestenes (DH) spinor field (which has a well-defined geometrical meaning, to be discussed in the next section) and use the fact that DH spinor field can be viewed as generating an effective Riemann-Cartan-Weyl geometry in spacetime, as explained in [15]. The quantum potential type non-linear term will be interpreted as an effective torsion in spacetime (but in a way different from [15], which will be briefly reviewed for a self-contained presentation).

It is clear that equivalence of Maxwell and Dirac equations does not imply the equivalence of Maxwell and Dirac *fields*, but suggests indeed the existence of a deep relationship between them. We shall exploit the fact that a DH spinor field can represent a non-null electromagnetic field and that the amplitude of a DH spinor field can be used to define a conformal mapping in order to suggest that the difference between Maxwell and Dirac fields is due to different transformation properties of DH spinor fields under that conformal mapping. We shall also discuss in connection with this problem and in the light of our results the two gauge conditions used by Campolattaro [5] in order to eliminate the two additional degrees of freedom present in the spinor field if it is to represent a Maxwell field.

2. Dirac Equation in the Clifford Bundle.

Let $\mathcal{M} = (M,\ g,\ \tau)$ be a Lorentzian manifold, i.e., M is a Hausdorff, paracompact, C^∞, connected, four-dimensional manifold, oriented by the volume element 4-form τ and time-oriented, and the tensor field $g \in T^{0,2}(M)$ is a metric of signature (1, 3). Let TM and T^*M denote the tangent and cotangent bundles, respectively. Cross-sections $e \in \sec TM$ [$\gamma \in \sec T^*M$] are called 1-vector [1-form] fields. Let $\{e_a\} \in \sec TM$ ($a = 0,\ 1,\ 2,\ 3$) be an orthonormal basis of TM and $\{\gamma^a\} \in \sec T^*M$ be the dual basis: $g^{-1}(\gamma^a,\ \gamma^b) = \eta^{ab}$, $g(e_a,\ e_b) = \eta_{ab}$, with $\eta^{ab} = \mathrm{diag}\,(1,\ -1,\ -1,\ -1)$ and $g^{-1} \in T^{2,0}(M)$. If (x^μ) is a chart for $U \subset M$ and if $\{\frac{\partial}{\partial x^\mu}\}$ and $\{dx^\mu\}$ ($\mu = 0,\ 1,\ 2,\ 3$) are the natural coordinate basis of TU and T^*U, respectively, we have $e_a = h^\mu_a \partial_\mu$ and $\gamma^a = h^a_\mu dx^\mu$ with $\eta^{ab} = h^a_\mu h^b_\nu g^{\mu\nu}$, $g^{\mu\nu} = g^{-1}(dx^\mu,\ dx^\nu)$.

The Clifford bundle can be defined in different ways [16] which are equivalent for fields of characteristic $\neq 2$. We define the Clifford bundle of M as the bundle $C\ell(T^*M,\ g^{-1}) = C\ell(\mathcal{M}) = \cup_{x\in M} C\ell(T^*_x M,\ g_x^{-1}) = T^*(M)/J$, where $T^*(M)$ is the tensor bundle over the cotangent bundle of M and J is the ideal generated by the elements of $T^*(M)$ of the form $\alpha \otimes \beta + \beta \otimes \alpha - 2g^{-1}(\alpha,\ \beta)$, with $\alpha,\ \beta \in \sec T^*M \subset T^*(M)$. The Clifford bundle is a vector (algebra) bundle. It can be shown [17–19] that $C\ell(\mathcal{M}) = P_{SO_+(1,3)} X_{Ad}\, \mathbb{R}_{1,3}$, where $P_{SO_+(1,3)}$ is the principal bundle of orthonormal frames, Ad is the adjoint representation of $SO_+(1,3)$, $Ad : SO_+(1,3) \to Aut(\mathbb{R}_{1,3})$, and $\mathbb{R}_{1,3}$ (which is the typical fiber of the bundle) is the spacetime algebra [20]. Sometimes $C\ell M)$ is also called the Kähler-Atiyah bundle of differential forms [21].

Intuitively we have at each point of M a local spacetime algebra. As a vector space $\mathbb{R}_{1,3}$ is 16-dimensional and one basis is $\{1,\ \gamma^0, \ldots, \gamma^3;\ \gamma^0 \wedge \gamma^1, \ldots; \gamma^2 \wedge \gamma^3;\ \gamma^0 \wedge \gamma^1 \wedge \gamma^2, \ldots, \gamma^1 \wedge \gamma^2 \wedge \gamma^3;\ \gamma^5\}$ with $\tau = \gamma^5 = \gamma^0 \wedge \gamma^1 \wedge \gamma^2 \wedge \gamma^3$ being the volume element and $\wedge$ being the exterior product. We also have $\gamma^a \cdot \gamma^b = \eta^{ab}$, where $\cdot$ is the interior product. The Clifford (or geometrical) product of 1-forms is $\gamma^a\gamma^b = \gamma^a \cdot \gamma^b + \gamma^a \wedge \gamma^b$, with $\gamma^a \cdot \gamma^b = \frac{1}{2}(\gamma^a\gamma^b + \gamma^b\gamma^a)$ and $\gamma^a \wedge \gamma^b = \frac{1}{2}(\gamma^a\gamma^b - \gamma^b\gamma^a)$. For a general definition of these products see [22]. We note that, since the base manifold is always a metric one, the use of 1-forms or vectors is only a matter of convenience; and in this case one may replace γ^a by e_a in the above definitions and define the reciprocal basis $\{e^a\}$ by $e^a \cdot e_b = \delta^a_b$, use the boundary theorem [20, 22] to "integrate" vectors, etc., as explained by Hestenes. Therefore we shall use raised or lowered indices whenever convenient, like $\gamma_a\gamma_b = \gamma_a \cdot \gamma_b + \gamma_a \wedge \gamma_b$, and interpret these γ_a as vectors e_a, etc. (this, of course, does not cause confusion, despite its redundancy).

Now let ∇ be a connection on M; if ∇ is a Riemann-Cartan connection, i.e., non-null curvature and non-null torsion and null nonmetricity (i.e.: $\nabla g = 0$), then ∇ pass to the quotient bundle $C\ell(\mathcal{M}) = T^*(M)/J$. The Dirac operator acting on sections of $C\ell(\mathcal{M})$ is defined as $\partial = dx^\mu \nabla_\mu = \gamma^a \nabla_a$, where (for example) ∇_a is the covariant derivative in the direction of e_a. The connection coeficients Γ^c_{ab} in the $\{\gamma^a\}$ basis are defined by $\nabla_a\gamma^c = -\Gamma^c_{ab}\gamma^b$. Since $\nabla_a\eta^{bc} = \nabla_a(\gamma^b\cdot\gamma^c) = 0$ we have that $\Gamma^{bc}_a = -\Gamma^{cb}_a$, which enable us to define:

$$\Gamma_a = \frac{1}{2}\Gamma^{bc}_a(\gamma_b\wedge\gamma_c), \tag{1}$$

with $\gamma^a\cdot\gamma_b = \delta^a_b$. It follows that:

$$\nabla_a\gamma^b = \frac{1}{2}[\Gamma_a,\ \gamma^b], \tag{2}$$

where [,] is the commutator, with $\frac{1}{2}[\Gamma_a,\ \gamma^b] = -\gamma^b\cdot\Gamma_a = -\Gamma^{bc}_a\gamma_c = -\Gamma^b_{ac}\gamma^c$. Since the commutator of a 2-form [2-vector] and a multiform [multivector] preserves the grade of that multiform [multivector] [22], eq. (2) can be generalized to any multiform [multivector]: B as

$$\nabla_a(B) = \partial_a B + \frac{1}{2}[\Gamma_a,\ B] \tag{3}$$

where ∂_a is the Pfaff derivative.

It is easy to see that the above expression for the covariant derivative of a multiform is related to the fact that under a active Lorentz transformation described by $R\in Spin_+(1,3)\simeq SL(2,\mathbb{C})$ the multiforms transform like $\gamma^a\mapsto R\gamma^a\tilde{R}$, where $R\ \tilde{R}=\tilde{R}\ R = 1$ with $\sim$ (called reversion) is the principal anti-automorphism in $\mathbb{R}_{1,3}$: $(\alpha\beta)^\sim = \tilde{\beta}\tilde{\alpha}$ with $\tilde{\alpha}=\alpha$ for α scalar or 1-form. Note that $\tilde{R}=R^{-1}$ and that $R\in\mathbb{R}^+_{1,3}$ the even subalgebra of $\mathbb{R}_{1,3}$.

Now, let us consider the covariant derivative of a spinor. First of all, the existence of spinor structures on arbitrary manifolds is an intricated problem since the existence of either Milnor-Lichnerowicz spinor structure or algebraic spinor structure is restricted to a certain class of manifolds [17, 23]. Moreover, it is very difficult to see any geometrical meaning underlying these spinor fields. On the other hand, the Clifford bundle always exists (as we have seen) and its elements have a clear geometrical interpretation. It would be convenient, therefore, to define a spinor by means of the Clifford bundle. One such spinor has been introduced by Hestenes [24] and named operator spinors (see [25]). Although the existence of operator spinors in arbitrary manifolds is a question that deserves specific attention [26], we shall not be worried with this topic here since the manifolds we shall consider later admit a spinor structure. One great advantage of working with operator spinor is its geometrical interpretation described below.

An operator spinor ψ is an element of the even subalgebra of $\mathbb{R}_{1,3}$, i.e: $\mathbb{R}^+_{1,3}\ni\psi = a + B + \gamma^5 p$, where a and p are scalars and B a 2-form. Since $\tilde{\psi}= a - B + \gamma^5 p$ we have $\psi\ \tilde{\psi}= (a^2 - p^2 - B\cdot B) + \gamma^5(2ap + \gamma^5 B\wedge B) = \rho\cos\beta + \gamma^5\rho\sin\beta = \rho e^{\gamma^5\beta}$, and we can define a normalized spinor $\phi = \psi(\rho e^{\gamma^5\beta})^{-1/2}$ such that $\phi\ \tilde{\phi}= 1$ with $\phi\in\mathbb{R}^+_{1,3}$ with $\phi\in\mathbb{R}^+_{1,3}$. But since $\phi\in\mathbb{R}^+_{1,3}$ with $\phi\ \tilde{\phi}=\tilde{\phi}\ \phi = 1$, we can identify ϕ with R that describes a Lorentz transformation, and write ψ as:

$$\psi = \rho^{1/2} e^{\gamma^5 \beta/2} R. \tag{4}$$

This is the canonical decomposition of an operator spinor (valid whenever it is non-singular: $\psi\tilde{\psi} \neq 0$). It can be shown [16, 27] that ψ carries the same information as the standard covariant Dirac spinor $\psi_D \in \mathbb{C}^4$; the difference is that ψ does not involve complex numbers since it is a sum of *real* multiforms of $\mathbb{R}^+_{1,3}$. This great achievement of Hestenes enable him to develop a *real* formulation of Dirac theory within the spacetime algebra [6-8, 28, 29]. The Dirac algebra of matrices $\mathbb{C}(4)$ is isomorphic to $\mathbb{R}_{1,3}$ complexified: $\mathbb{C} \otimes \mathbb{R}_{1,3} \simeq \mathbb{R}_{4,1} \simeq \mathbb{C}(4)$, but there is no need for such complexification to formulate Dirac theory (see [16] for details). We call ψ a Dirac-Hestenes (DH) spinor.

DH spinor *fields* that are sum of even sections of $\mathcal{C}\ell(\mathcal{M})$ possess a very important property: its canonical decomposition (eq. (4)) which clearly shows its geometrical content: R is a Lorentz transformation, $e^{\gamma^5\beta/2}$ represents a duality rotation [30] and $\rho^{1/2}$ a dilation. The angle β is the so-called Takabayasi angle [31]. In view of this interpretation and since the composition of a Lorentz transformation R_1 and a Lorentz transformation R_2 is equivalent to a Lorentz transformation $R_3 = R_2R_1$, it is clear that a DH spinor ψ must transform under as active Lorentz transformation R as $\psi \mapsto R\psi$. Now, remember that the covariant derivative of a multiform A is, from eq(2):

$$\nabla_a A = \partial_a A + \frac{1}{2}[\Gamma_a, A], \tag{5}$$

where ∂_a is the Pfaff derivative [32]: $\partial_a A = \partial_a(A_b\gamma^b) = (\partial_a A_b)\gamma^b$, and this expression is related to the fact that A transforms as $A \mapsto RA\tilde{R}$. Since ψ transforms as $\psi \mapsto R\psi$ we must have:

$$\nabla_a\psi = \partial_a\psi + \frac{1}{2}\Gamma_a\psi. \tag{6}$$

This is not a proof, of course, but a justification which is enough for our purposes. A rigorous exposition of those topics (definition, transformation properties, covariant derivative, etc.) related to operator spinors is to be found in [26].

Now, the represenattive in $\mathcal{C}\ell(\mathcal{M})$ of the Free Dirac equation when M is *Minkowski* spacetime is [6-8, 27]:

$$\boxed{\partial\psi\gamma^1\gamma^2 + \frac{mc}{\hbar}\psi\gamma^0 = 0} \tag{7}$$

where $\partial = \gamma^\mu\partial_\mu$ and $\gamma^\mu = dx^\mu$. When M is a *Riemannian* spacetime, Hestenes [33] generalizes eq. (7) to:

$$\partial\psi\gamma^1\gamma^2 + \frac{mc}{\hbar}\psi\gamma^0 = 0, \tag{8}$$

where now $\{\gamma^a\}$ is an orthonormal basis and $\partial = \gamma^a\nabla_a = dx^\mu\nabla_\mu$ with $\nabla_a\psi$ given by eq. (6). When M is a *Riemann-Cartan* spacetime, we propose the following expression for the representative of free Dirac equation in $C\ell(\mathcal{M})$:

$$\boxed{[\partial\psi + \frac{1}{2}T\psi]\gamma^1\gamma^2 + \frac{mc}{\hbar}\psi\gamma^0 = 0} \tag{9}$$

where $T = dx^\mu T_\mu = \gamma^a T_a$ with, for example, T_μ being the trace $T^\nu_{\mu\nu}$ of the torsion $T^\sigma_{\mu\nu} = \Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\mu}$, and the other symbols with the same meaning as in eq. (8). Eq. (9) is just the translation in $C\ell(\mathcal{M})$ and in terms of DH spinor field of the equation given by Hehl et. al. [34], Ivanenko and Obukhov [35].

We shall need to consider the DH equation in Riemann-Cartan spacetime only in sec. 5; so, in the following two sections we restrict our attention to Minkowski spacetime.

3. Maxwell Equations and Rainich-Misner-Wheeler Theorem

Let us consider M to be Minkowski spacetime, so that the Dirac operator is $\partial = dx^\mu\nabla_\mu = \gamma^a\nabla_a$ with $\nabla_\mu = \partial_\mu$. The representative of Maxwell equations in $C\ell(\mathcal{M})$ is [20]:

$$\partial F = J. \tag{9}$$

It has been shown in [27] that all spinorial forms of Maxwell equations found in the literature can be deduced from the above one after a suitable choice of a global idempotent field. This form of Maxwell equations is due to Juvet & Schidlof [36] and Mercier [37], and reconsidered by Riesz [38]. In eq. (10) the electromagnetic field $F \in \sec \wedge^2 M \subset \sec C\ell(\mathcal{M})$ and the electric current $J \in \sec \wedge^1 M \subset \sec C\ell(\mathcal{M})$. When $J = 0$ the *free* Maxwell equations assume of course the simple form $\partial F = 0$.

The Rainich-Misner-Wheeler Theorem: Let an "extremal field" be any electromagnetic field for which the magnetic [electric] field is zero and the electric [magnetic] field is parallel to one of the spatial axis. Then at any point of Minkowski spacetime any non-null electromagnetic field can be reduced to an extremal field by a Lorentz transformation and a duality rotation.

An elementary proof of this theorem by using the spacetime algebra can be found in [14]. Now, under an active Lorentz transformation described by L the electromagnetic field F transforms into $F' = LF\tilde{L}$ and under a duality rotation by an angle α the field F' transforms into $F'' = e^{\gamma^5\alpha}F'$. Therefore, $F'' = e^{\gamma^5\alpha}LF\tilde{L}$ is an extremal field. Since we can always choose the extremal field to be a magnetic one along the z-direction we have:

$$e^{\gamma^5\alpha}LF\tilde{L} = h\gamma^1\wedge\gamma^2, \tag{10}$$

where $h > 0$ is the magnitude of the extremal field (note that the symbol h is not to be confused with Planck constant, since in this case we shall always use the reduced one $\hbar$). If we redefine $\alpha = -\beta$ and $L = \tilde{R}$, we have from eq. (11)

$$F = he^{\gamma^5\beta} R(\gamma^1 \wedge \gamma^2)\, \tilde{R}, \tag{11}$$

or:

$$\boxed{F = \phi(\gamma^1 \wedge \gamma^2)\, \tilde{\phi}} \tag{13}$$

where

$$\phi = h^{1/2} e^{\gamma^5/2} R \tag{14}$$

is a DH spinor field. This is a very important result since it permit us to interpret the DH spinor on the basis of the above discussion; that is: in the canonical decomposition given by eq. (4) of DH spinor, the Lorentz transformation R and the angle β are just the ones discussed above, and the dilation $\sqrt{h}$ is the square root of the extremal field magnitude. This is indeed a remarkable relation between a non-null electromagnetic field and a DH spinor.

4. Spinor Representation of Maxwell Equations and its Relationship with Dirac Equation

We have seen in the preceeding section that any non-null electromagnetic field can be written as $F = \phi\gamma^1\gamma^2\, \tilde{\phi}$, where ϕ is a DH spinor field and $\gamma^1\gamma^2 = \gamma^1 \wedge \gamma^2$ because $\{\gamma^a\}$ is an orthonormal basis. It is natural, therefore, to use this fact into Maxwell equations to find a spinor representation of them. For our purposes we need to consider only the *free* equations, but its extension to the general case is trivial. Then, introducing $F = \phi\gamma^1\gamma^2\, \tilde{\phi}$ into $\partial F = 0$ (the representative of Maxwell equations in the Clifford bundle), we get:

$$\gamma^\mu(\partial_\mu\phi)\gamma^1\gamma^2\, \tilde{\phi} + \gamma^\mu\phi\gamma^1\gamma^2\partial_\mu\, \tilde{\phi} = 0, \tag{15}$$

and multiplying eq. (15) on the right by $\phi(\phi\, \tilde{\phi})^{-1}$.

$$\gamma^\mu\partial_\mu\phi\gamma^1\gamma^2 + \gamma^\mu\phi\gamma^1\gamma^2(\partial_\mu\, \tilde{\phi})\phi(\phi\, \tilde{\phi})^{-1} = 0. \tag{16}$$

We must note that the spinor representation eq. (16) of the free Maxwell equations is equivalent to the one found by Campolattaro [4], as proved in [14]; however, while our approach is trivial Campolattaro's one is intricated. The simplicity of our approach is due to the fact that DH spinor ϕ has an inverse $\phi^{-1} = \tilde{\phi}\, (\phi\, \tilde{\phi})^{-1}$, while the standard covariant Dirac spinor used by Campolattaro does not have an inverse.

Before we continue, let us make a simplifying hypothesis, supposing β = constant. (We shall discuss later the meaning of this simplification) This assumption implies from eq. (14) that

$$\partial_\mu \phi = \frac{1}{2}[\partial_\mu \ln h + \Omega_\mu]\phi, \tag{17}$$

where we defined the 2-form

$$\Omega_\mu = 2(\partial_\mu R)\,\tilde{R}\,. \tag{18}$$

Eq. (16) then becomes:

$$\gamma^\mu \partial_\mu \phi \gamma^1 \gamma^2 + \frac{1}{2}\gamma^\mu (\partial_\mu)\ln h)\phi\gamma^1\gamma^2 - \frac{1}{2}\gamma^\mu \phi\gamma^1\gamma^2\,\tilde{\phi}\,\Omega_\mu \phi(\phi\,\tilde{\phi})^{-1} = 0 \tag{19}$$

where we used $\tilde{\Omega}_\mu = -\Omega_\mu$. If we define the spin 2-form S.

$$S = \frac{\hbar}{2}R\gamma^1\gamma^2\,\tilde{R} = \frac{\hbar}{2}\phi\gamma^1\gamma^2\,\tilde{\phi}\,(\phi\,\tilde{\phi})^{-1}, \tag{20}$$

eq. (19) assumes the form:

$$\boxed{\gamma^\mu \partial_\mu \phi \gamma^1 \gamma^2 - \frac{1}{\hbar}\gamma^\mu S \Omega_\mu \phi = -\frac{1}{2}(\partial \ln h)\phi\gamma^1\gamma^2} \tag{21}$$

This non-linear equation for ϕ is equivalent to the free Maxwell equations when the electromagnetic field F is non-null and the duality rotation is constant.

In eq. (21) the term $S\Omega_\mu$ is the Clifford product of 2-forms S and Ω_μ, which results in a sum of a scalar, a 2-form and a pseudo-scalar according to [22]:

$$S\Omega_\mu = S \cdot \Omega_\mu + \frac{1}{2}[S,\, \Omega_\mu] + S \wedge \Omega_\mu. \tag{22}$$

Now, Hestenes [6-8] has shown that the component p_μ of the momentum p in Dirac theory (for the free case) can be written as:

$$p_\mu = -S \cdot \Omega_\mu. \tag{23}$$

such that

$$p = \gamma^\mu p_\mu = e^{\gamma^5 \beta} mcv, \tag{24}$$

where $v = R\gamma^0\,\tilde{R}$. We note that eq. (23) differs by a sign from the one of Hestenes because our definition of S also differs by a sign from his one. Then, if we suppose that the product $S\Omega_\mu$ has only a scalar part $S \cdot \Omega_\mu$ (which is true in Dirac theory for its free case), it follows

$$-\frac{1}{\hbar}\gamma^\mu S \Omega_\mu \phi = \frac{1}{\hbar}p\phi = \frac{mc}{\hbar}\phi\gamma^0, \tag{25}$$

and eq. (21) finally becomes:

$$\boxed{\partial\phi\gamma^1\gamma^2 + \frac{mc}{\hbar}\phi\gamma^0 = -(\partial \ln\sqrt{h})\phi\gamma^1\gamma^2\,.} \tag{27}$$

Eq. (27) is a non-linear DH equation which is equivalent to the free Maxwell equations for the case where the electromagnetic field is non-null, the duality rotation is constant and the product $S\Omega_\mu$ has only scalar part. For the case where $S\Omega_\mu$ has also 2-form and pseudo-scalar parts, the generalized equation has been considered in [11]. Since Ω_μ describes [39] the infinitesimal rotation of $\{\lambda^\nu\}(\lambda^\nu = R\gamma^\nu\tilde{R})$ along the direction $\frac{\partial}{\partial x^\mu}$, the assumption that $S\Omega_\mu$ has only a scalar part means that this is a rotation in the plane $\lambda^1 \wedge \lambda^2$, which in view of def. (20) is just the spin plane. As a consequence, mass has a kinetic origin as suggested by Hestenes [6-8] – now from a different view point.

The nonlinearity of eq. (27) provides ϕ with those known properties of solutions of non-linear equations, which is a welcome fact. It is important to note that the nonlinearity of eq. (27) is just of quantum potential type, i.e., $\partial \ln\sqrt{h} = \frac{1}{2}\frac{\partial h}{h}$. In the next section we shall look for an interpretation of this term.

We must observe now that eq. (27) can be *linearized*. In fact, if we multiply it by $\sqrt{h}$ we have:

$$\gamma^\mu\sqrt{h}\partial_\mu\phi\gamma^1\gamma^2 + \gamma^\mu(\partial_\mu\sqrt{h})\phi\gamma^1\gamma^2 + \frac{mc}{\hbar}\sqrt{h}\phi\gamma^0 = 0, \tag{26}$$

and defining a DH spinor ψ:

$$\boxed{\psi = \sqrt{h}\phi} \tag{29}$$

we have that ψ satisfies:

$$\boxed{\partial\psi\gamma^1\gamma^2 + \frac{mc}{\hbar}\psi\gamma^0 = 0} \tag{30}$$

which is just DH equation, i.e., the representative of Dirac equation in $\mathcal{C}\ell(\mathcal{M})$. From eq. (14) and eq. (29), ψ is of the form:

$$\psi = \rho^{1/2}e^{\gamma^5\beta/2}R \tag{31}$$

with ρ given by:

$$\boxed{\rho = h^2} \tag{32}$$

i.e., ρ is the square of the magnitude of the extremal field, i.e., it is proportional to the intensity of the extremal field. We remember that ρ is assumed to be the probability density in Dirac theory, i.e., $\rho = \psi\tilde{\psi}$ for the cases $\beta = 0$ and $\beta = \pi$. Moreover, these values of β distinguish electrons ($\beta = 0$) from positrons ($\beta = \pi$) in Dirac theory [6-8]; in this way, our hypothesis that $\beta =$ constant is satisfied in these cases.

Finally, we observe that studies in the same direction were performed by Daviau [40,41]; however, in writing $F = \psi\gamma^1\gamma^2\tilde{\psi}$ that author attempted for associating an electromagnetic field with Dirac waves, which certainly is not the case in our approach. There are other differences also, but it is not our purpose to discuss them here.

5. Interpretation of the Non-Linearity as Torsion.

In this section we shall look for an interpretation of the nonlinear DH equation and its quantum potential type nonlinearity by means of the geometry of spacetime. We first discuss how one can do this for Dirac equation in terms of an effective Riemann-Cartan-Weyl geometry generated by a DH spinor field, as discussed in [15]. Then we show how to do the same with the nonlinear DH equation in terms of Riemann-Cartan geometry where the trace of the torsion plays the role of the effective Weyl gauge field.

In order to present that interpretation of DH equation, we first add a "twist" on spacetime; that is: we take orthonormal vectors and rotate them differently at each spacetime point along some direction; clearly the original and the rotated vectors are not parallel but we shall *take* these rotated vectors as being *parallel*. More specifically, let the rotated vectors be $\lambda_\mu = R\gamma_\mu\tilde{R}$; to say that the rotated vectors are parallel mathematically means to have a *new* connection $\overline{\nabla}$ on M such that $\overline{\nabla}_\mu\lambda_\mu = 0$, in place of the old connection ∇ such that $\nabla_\mu\gamma_\nu = 0$. Since the connection coeficients $\overline{\Gamma}^{\nu}_{\mu\sigma} = 0$ where $\nabla_\mu\gamma^\nu = \overline{\Gamma}^{\nu}_{\mu\sigma}\gamma^\sigma$, the curvature is null but torsion is non-null as a result of that "twist" because $[\lambda_\mu, \lambda_\nu] \neq 0$. (The Dirac commutator of 1-form fields is defined in [42, 43].) Let us calculate the torsion: from $\lambda^\mu = R\gamma^\mu\tilde{R}$ it follows $\nabla_\mu\lambda^\nu = \frac{1}{2}[\Omega_\mu, \lambda^\nu] = -\Omega^{\nu}_{\mu\sigma}\lambda^\sigma$, where $\Omega_\mu = 2(\partial_\mu R)\tilde{R}$ and $\Gamma^{\nu}_{\mu\sigma} = \Omega^{\nu}_{\mu\sigma}$. But for the connection ∇ the torsion is null: $T^{\nu}_{\mu\sigma} = \Gamma^{\nu}_{\mu\sigma} - \Gamma^{\nu}_{\sigma\mu} - C^{\nu}_{\mu\sigma} = 0$, which gives for the structure coeficients of $\{\lambda^\mu\} : C^{\nu}_{\mu\sigma} = \Gamma^{\nu}_{\mu\sigma} - \Gamma^{\nu}_{\sigma\mu} = \Omega^{\nu}_{\mu\sigma} - \Omega^{\nu}_{\sigma\mu}$. Consequently, for the new connection

$$\overline{T}^{\nu}_{\mu\sigma} = -(\Omega^{\nu}_{\mu\sigma} - \Omega^{\nu}_{\sigma\mu})\,; \quad \overline{R}^{\nu}_{\mu\sigma\rho} = 0. \tag{33}$$

Consider now DH equation (7) and take again $\beta = 0$ for simplicity, i.e: $\psi = \eta^{1/2}R$. In this case, DH equation can be written as :

$$\gamma^\mu[\partial_\mu\sqrt{\eta} + \frac{1}{2}\sqrt{\eta}\Omega_\mu] = \frac{mc}{\hbar}\sqrt{\eta}\lambda^0\lambda^1\lambda^2, \tag{34}$$

where $\lambda^\mu = R\gamma^\mu\tilde{R}$. After splitting eq. (34) into its 1-form and 3-form parts, we have:

$$\partial\sqrt{\eta} + \frac{1}{2}\sqrt{\eta}(\gamma^\mu \cdot \Omega_\mu) = 0, \tag{35}$$

$$\frac{1}{2}(\gamma^{\mu} \wedge \Omega_{\mu}) = \frac{mc}{\hbar} R\gamma^0\gamma^1\gamma^2 \tilde{R}\,. \tag{36}$$

Eq. (36) can be rewritten, after some manipulations described in details in [15], as:

$$[\partial R - \frac{1}{2}(\gamma^{\mu} \cdot \Omega_{\mu})R]\gamma^1\gamma^2 + \frac{mc}{\hbar} R\gamma^0 = 0. \tag{37}$$

that is, DH equation is equivalent to the two equations (35) and (37). But for $\Omega_{\mu} = \frac{1}{2}\Omega_{\mu}^{\alpha\beta}(\gamma_{\alpha} \wedge \gamma_{\beta})$ we have $(\gamma^{\mu} \cdot \Omega_{\mu}) = \Omega^{\mu}_{\mu\beta}\gamma^{\beta}$, or since $\Omega^{\mu}_{\beta\mu} = 0$, that

$$\gamma^{\mu} \cdot \Omega_{\mu} = (\Omega^{\mu}_{\mu\nu} - \Omega^{\mu}_{\nu\mu})\gamma^{\nu} = T_{\nu}\gamma^{\nu} = T. \tag{38}$$

where $T_{\nu} = T^{\mu}_{\nu\mu}$ is the trace of the torsion given by eq. (33). Eq. (35) and eq. (37) becomes:

$$\partial\sqrt{\eta} + \frac{1}{2}T\sqrt{\eta} = 0, \tag{39}$$

$$[\partial R - \frac{1}{2}TR]\gamma^1\gamma^2 + \frac{mc}{\hbar}R\gamma^0 = 0. \tag{40}$$

Solving eq. (39) fot T and introducing it into eq. (40) we have:

$$[\partial R + \frac{1}{2}(\partial \ln \eta)R]\gamma^1\gamma^2 + \frac{mc}{\hbar}R\gamma^0 = 0, \tag{41}$$

which is just DH equation for the unimodular spinor R in Riemann-Cartan spacetime (eq. 9), with the term $\partial \ln \eta = \dfrac{\partial \eta}{\eta}$ playing the role of a torsion 1-form. This result was interpreted in [15] as follows: first, note that the non-null torsion comes from the contribution of the structure coeficients; then think of a Lorentz vacuum characterized by $\{dx^{\mu}\}$ and which define a cosmic lattice [44]; then the presence of "matter" as described by DH spinor field induces dislocations in the cosmic lattice so that the dislocated lattice is characterized by $\{d\xi^{\mu}\}$ with the transformation $x^{\mu} \mapsto \xi^{\mu} = \xi^{\mu}(x)$ being singular.

Now, if we look to eq. (27) we see that the same arguments above outlined can be applied to that non-linear equation. However, we shall adopt a slightly different point of view here, which we believe is more interesting and promising than the above one. We can implement this idea by noting that the amplitude $\sqrt{\rho}$ of the DH spinor field ψ (see eq. (4)) can be seen as the generator of the *conformal* transformation of a basic set of 1-form fields on the original Minkowski space, i.e. $C_{\rho} : \gamma^{\mu} \mapsto \sqrt{\rho}\gamma^{\mu}\sqrt{\rho} = \rho\gamma^{\mu}$. This conformal transformation introduces a Weyl 1-form which is the trace of the torsion tensor, and it defines an effective Riemann-Cartan-Weyl geometry which is metric compatible, in distinction with Weyl's geometry [15, 45, 46].

Let us be more specific; for the conformal transformation:

$$\begin{aligned} C_{\rho} : T^{*}M &\rightarrow T^{*}M \\ \gamma^{\mu} &\mapsto \overline{\gamma}^{\mu} = \rho\gamma^{\mu} \end{aligned} \tag{42}$$

let us define a "new" metric $\bar{g}$ by

$$\bar{g}(\gamma^{\mu},\gamma^{\nu}) = g(\bar{\gamma}^{\mu},\bar{\gamma}^{\nu}) = g(\rho\gamma^{\mu},\rho\gamma^{\nu}) \tag{43}$$

and a "new" connection $\bar{\nabla}$ such that:

$$\bar{\nabla}_{\mu}\gamma^{\nu} = \frac{\nabla_{\mu}(\bar{\gamma}^{\nu})}{\rho} = \frac{\nabla_{\mu}(\rho\gamma^{\nu})}{\rho}. \tag{44}$$

Eq. (43) gives:

$$\bar{g} = \rho^2 g = e^{2\ln\rho} g, \tag{45}$$

which is the conformal transformation of the metric, i.e: $g \mapsto \bar{g} = e^{2\sigma} g$ with $\sigma = \ln\rho$, while eq. (44) gives for the connection coeficients:

$$\bar{\Gamma}^{\nu}_{\mu\sigma} = \Gamma^{\nu}_{\mu\sigma} - (\partial_{\mu}\ln\rho)\delta^{\nu}_{\sigma}, \tag{46}$$

where $\bar{\nabla}_{\mu}\gamma^{\nu} = -\bar{\Gamma}^{\nu}_{\mu\sigma}\gamma^{\sigma}$, etc. Note that:

$$\bar{\nabla}_{\mu}\bar{g}^{\nu\sigma} = \partial_{\mu}(\rho^2 g^{\nu\sigma}) + (\Gamma^{\nu}_{\mu\alpha} - \partial_{\mu}\ln\rho\delta^{\nu}_{\alpha})\rho^2 g^{\alpha\sigma} +$$

$$(\Gamma^{\sigma}_{\mu\alpha} - \partial_{\mu}\ln\rho\delta^{\sigma}_{\alpha})\rho^2 g^{\nu\alpha} = \rho^2\nabla_{\mu}g^{\nu\sigma} = 0, \tag{47}$$

which proves the metric compatibility of the connection $\bar{\nabla}$.

It is very important to note that for an orthonormal basis $\bar{\Gamma}^{a}_{\mu b} = \Gamma^{a}_{\mu b}$, and from eq. (1): $\bar{\Gamma}_{\mu} = \Gamma_{\mu}$. In fact, for $\gamma^{a} = h^{a}_{\mu}\gamma^{\mu}$ with $\eta^{ab} = h^{a}_{\mu}h^{b}_{\nu}g^{\mu\nu}$ we have $\gamma^{a} = h^{a}_{\mu}\rho^{-1}\rho\gamma^{\mu} = \rho^{-1}h^{a}_{\mu}\bar{\gamma}^{\mu}$, which gives for $\gamma^{a} = \bar{h}^{a}_{\mu}\bar{\gamma}^{\mu}$ that $\bar{h}^{a}_{\mu} = \rho^{-1}h^{a}_{\mu}$. But for $\Gamma^{a}_{\mu b}$ given by $\nabla_{\mu}\gamma^{a} = -\Gamma^{a}_{\mu b}\gamma^{b}$ we have $\Gamma^{a}_{\mu b} = h^{a}_{\nu}h^{\sigma}_{b}\Gamma^{\nu}_{\mu\sigma} - (\partial_{\mu}h^{a}_{\nu})h^{\nu}_{b}$; then, by using $\bar{h}^{a}_{\mu} = \rho^{-1}h^{a}_{\mu}$, $\bar{h}^{\mu}_{a} = \rho h^{\mu}_{a}$ and eq. (46), we have:

$$\begin{aligned}\bar{\Gamma}^{a}_{\mu b} &= \bar{h}^{a}_{\nu}\bar{h}^{\sigma}_{b}\bar{\Gamma}^{\nu}_{\mu\sigma} - (\partial_{\mu}\bar{h}^{a}_{\nu})\bar{h}^{\nu}_{b} = \\ &= h^{a}_{\nu}h^{\sigma}_{b}(\Gamma^{\nu}_{\mu\sigma} - (\partial_{\mu}\ln\rho)\delta^{\nu}_{\sigma}) - \partial_{\mu}(\rho^{-1}h^{a}_{\nu})\rho h^{\nu}_{b} = \\ &= h^{a}_{\nu}h^{\sigma}_{b}\Gamma^{\nu}_{\mu\sigma} - (\partial_{\mu}h^{a}_{\nu})\bar{h}^{\nu}_{b} = \Gamma^{a}_{\mu b}\end{aligned} \tag{48}$$

Moreover, note that while for γ^{μ} we have $\gamma^{\mu} \mapsto \rho\gamma^{\mu}$, for γ_{μ} we must have $\gamma_{\mu} \mapsto \rho^{-1}\gamma_{\mu}$; and, since for $\nabla_{\mu} = \nabla_{\gamma_{\mu}}$ we have $\nabla_{\rho^{-1}\gamma_{\mu}} = \rho^{-1}\nabla_{\gamma_{\mu}}$, then eq. (48) implied the invariance of the Dirac operator:

$$\bar{\partial} = \partial \tag{49}$$

It is clear that eq. (48) does not imply that curvature and torsion remain unchanged, since the structure coefficients are changed. In order to calculate the torsion and the curvature, it is easy to work with a natural coordinate basis; then, from eq. (46) we have:

$$\bar{T}^{\sigma}_{\mu\nu} = T^{\sigma}_{\mu\nu} + [(\partial_{\nu}\ln\rho)\delta^{\sigma}_{\mu} - (\partial_{\mu}\ln\rho)\delta^{\sigma}_{\nu}], \tag{50}$$

$$\overline{R}^{\tau}_{\mu\nu\sigma} = R^{\tau}_{\mu\nu\sigma} + [\partial_\sigma, \partial_\nu] \ln \rho \delta^{\tau}_{\mu}. \tag{51}$$

Eq. (50-51) imply that if our original spacetime is the Minkowski one, then as a result of that conformal transformation we have:

$$\overline{T}^{\sigma}_{\mu\nu} = (\partial_\nu \ln \rho)\delta^{\sigma}_{\mu} - (\partial_\mu \ln \rho)\delta^{\sigma}_{\nu}, \tag{52}$$

$$\overline{R}^{\tau}_{\mu\nu\sigma} = [\partial_\sigma, \partial_\nu] \ln \rho \delta^{\tau}_{\mu} = \overline{R}_{\nu\sigma}\delta^{\tau}_{\mu}. \tag{53}$$

In particular, the trace of the torsion is:

$$\overline{T}_\mu = \overline{T}_{\mu\nu}{}^{\nu} = -3(\partial_\mu \ln \rho), \tag{54}$$

and the torsion 1-form $\overline{T} = \overline{T}_\mu \gamma^\mu = -3\partial \ln \rho$ plays the rote of the Weyl 1-form in this geometry, which contrary to other Weyl's ones, is metric compatible and does not meet therefore the kind of problems we have when the connection is not metric compatible.

Now we are in position of interpreting the non-linear eq. (27) in the light of the geometry just introduced. The crucial step is to forget the (possible) identification given by eq. (32), but instead to define now

$$\boxed{\rho = \frac{1}{\hbar}.} \tag{55}$$

Moreover, we rewrite ϕ (remember that $\beta = 0$) as $\phi = \sqrt{\hbar}R = \frac{1}{\sqrt{\rho}}R = \frac{1}{\rho}\sqrt{\rho}R = \frac{1}{\rho}\Phi$, that is:

$$\boxed{\Phi = \rho\phi} \tag{56}$$

with

$$\Phi = \sqrt{\rho}R. \tag{57}$$

Eq. (56) implies that

$$\partial\phi = \frac{1}{\rho}[-(\partial \ln \rho)\Phi + \partial\Phi]. \tag{58}$$

If we use eq. (55) to rewrite the nonlinear term in eq. (27) as $(\partial \ln \sqrt{\hbar})\phi = -\frac{1}{2}(\partial \ln \rho)\frac{1}{\rho}\Phi$ and use eq. (58) in that eq. (27), it assumes the noticeable form:

$$\boxed{[\partial\Phi - \frac{(3\partial \ln \rho)}{2}\Phi]\gamma^1\gamma^2 + \frac{mc}{\hbar}\Phi\gamma^0 = 0\,.} \tag{59}$$

Eq. (59) is just the DH equation in Riemann-Cartan spacetime (eq. (9)) particularized for the geometry we have just introduced, where $T = \gamma^\mu T_\mu = -3\partial \ln \rho$ (eq. (54)).

Now, in order to DH equation to remain invariant under the conformal transformation (42), we must define a DH spinor field $\overline{\psi} = \rho^{3/2}\psi$ and take Φ in eq. (59) as this $\overline{\psi}$, i.e:

$$\boxed{\Phi = \rho^{3/2}\psi} \tag{60}$$

and use eq. (60) into eq. (59) we get:

$$\boxed{\partial\psi\gamma^1\gamma^2 + \frac{mc}{\hbar}\psi\gamma^0 = 0} \tag{61}$$

which is DH equation in Minkowski spacetime. Note that, from eq. (56) and eq. (60) that:

$$\boxed{\phi = \rho^{1/2}\psi} \tag{62}$$

which is just the transformation given by eq. (29) in view of our new definition of ρ given by eq. (55). The above results show that our "old" definition of ρ given by eq. (32) has only a first sight appealing interest, while the one of eq. (55) is related to a deep geometrical context.

It remains to discuss the relation between the electromagnetic (Maxwell) field and the Dirac field, which will be done in the next section.

6. The Relation Between Maxwell and Dirac Fields

Although we have shown the equivalence between Maxwell and Dirac equations for their free cases and for non-null electromagnetic fields, this does not imply the equivalence between Maxwell (electromagnetic) and Dirac fields,. In this section we shall look for a relationship between these fields, which, because of the Rainich-Misner-Wheeler theorem, can be discussed in terms of the relationship between those DH spinor fields ϕ and ψ, in the notation of the previous sections.

The natural way of introducing these fields is to use the results of the preceeding section for the geometry there discussed. Let us think in terms of the following scenary: suppose we have in Minkowski spacetime an unimodular DH spinor field R satisfying DH equation:

$$\partial R\gamma^1\gamma^2 + \frac{mc}{\hbar}R\gamma^0 = 0. \tag{63}$$

This spinor field, being unimodular, induces no conformal transformation on the basic 1-form fields. Now suppose we have a change in the amplitude of the DH spinor field, from $\rho_0 = 1$ to ρ. The natural, but speculative, way of interpreting this is to think of a change in the density of a "fluid". The amplitude of the DH spinor field is now $\sqrt{\rho}$, that is, $\Phi = \sqrt{\rho}R$, and ρ induces the conformal transformation described in the preceeding section, which give rise to the Riemann-Cartan-Weyl geometry there discussed. In this space, the DH equation for Φ is:

$$[\partial\Phi - \frac{(3\partial \ln \rho)}{2}]\gamma^1\gamma^2 + \frac{mc}{\hbar}\Phi\gamma^0 = 0. \tag{64}$$

One can define now two other DH spinor fields, namely:

$$\phi = \rho^{-1}\Phi\ , \tag{65}$$

and

$$\psi = \rho^{-3/2}\Phi\ . \tag{66}$$

The case of eq. (65), which is just eq. (56) when introduced into eq. (64) gives

$$\partial\phi\gamma^1\gamma^2 + \frac{mc}{\hbar}\phi\gamma^0 = \frac{1}{2}(\partial \ln \rho)\phi\gamma^1\gamma^2, \tag{67}$$

which is just the non-linear eq. (27) in view of eq. (55), and therefore equivalent to the free Maxwell equations under the hypothesis we assumed and for the non-null electromagnetic field $F = \phi\gamma^1\gamma^2\tilde{\phi}$. On the other hand, the case of eq. (66), which is just eq. (60), when introduced into eq. (64) gives the DH equation for ψ

$$\partial\psi\gamma^1\gamma^2 + \frac{mc}{\hbar}\psi\gamma^0 = 0. \tag{68}$$

Clearly the scenary outlined above is still speculative, but its *beauty* and *simplicity* suggest that it may be relevant from the physical point of view. We note that due to the definitions eq. (65-66) the spinor fields ϕ and ψ are related by

$$\phi = \rho^{1/2}\psi, \tag{69}$$

which is that eq. (62) and $\phi = \sqrt{\hbar}R$ after using eq.(55), which gives eq.(14). Note moreover for the electromagnetic field F:

$$F = \phi\gamma^1\gamma^2\tilde{\phi} = \rho\psi\gamma^1\gamma^2\tilde{\phi} = \rho M, \tag{70}$$

which suggest us to interpret M as a "density" of F, where $M = \psi\gamma^1\gamma^2\tilde{\phi}$ for ψ satisfying DH equation (68); and, in fact, M is interpret in Dirac theory as the density of magnetic moment.

Another thing that must be discussed is related to the fact that the electromagnetic field F has 6 real degrees freedom, while a DH spinor field has 8 real degrees of freedom; in another words, the spinor field ϕ that represents F via $F = \phi\gamma^1\gamma^2\tilde{\phi}$ has two additional degrees of freedom. Let us discuss a way of eliminating these two additional degrees of freedom.

First, consider a DH spinor field χ; which will be assumed to be either ϕ or ψ whenever convenient. Let us define the following 1-form fields:

$$j = \langle \partial_\mu \chi \gamma^2 \gamma^1 \tilde{\chi} \rangle_0 \gamma^\mu, \tag{71}$$

$$g = \langle \partial_\mu \chi \gamma^5 \gamma^2 \gamma^1 \tilde{\chi} \rangle_0 \gamma^\mu, \tag{72}$$

where $\langle \ \rangle_0$ indicates "the scalar part". Next, note that:

$$\langle \partial \chi \gamma^2 \gamma^1 \tilde{\chi} \rangle_1 = \gamma^\mu \langle \partial_\mu \chi \gamma^2 \gamma^1 \tilde{\chi} \rangle_0 + \gamma^\mu \cdot \langle \partial_\mu \chi \gamma^2 \gamma^1 \tilde{\chi} \rangle_2 \tag{73}$$

where $\langle \ \rangle_1$ and $\langle \ \rangle_2$ indicated "the 1-form part" and "the 2-form part", respectively. But for a 2-form B we have $B = -\tilde{B}$, while for a scalar $\alpha = \tilde{\alpha}$ and for a pseudo-scalar $(\alpha\gamma^5) = (\alpha\gamma^5)^\sim$; from this fact we can write

$$\langle \partial_\mu \chi \gamma^2 \gamma^1 \tilde{\chi} \rangle_2 = \frac{1}{2}[(\partial_\mu \chi \gamma^2 \gamma^1 \tilde{\chi}) - (\partial_\mu \chi \gamma^2 \gamma^1 \tilde{\chi})^\sim] =$$

$$= \frac{1}{2}[\partial_\mu \chi \gamma^2 \gamma^1 \tilde{\chi} + \chi \gamma^2 \gamma^1 \partial_\mu \tilde{\chi}] = \frac{1}{2}\partial_\mu(\chi \gamma^2 \gamma^1 \tilde{\chi}). \tag{74}$$

Then, using eq. (71) and eq. (74) into eq. (73) we get:

$$\boxed{j = \langle \partial \chi \gamma^2 \gamma^1 \tilde{\chi} \rangle_1 + \frac{1}{2}\partial \cdot (\chi \gamma^1 \gamma^2 \tilde{\chi})} \tag{75}$$

A similar calculation gives:

$$\langle \partial \chi \gamma^2 \gamma^1 \tilde{\chi} \rangle_3 = \frac{1}{2}\partial \wedge (\chi \gamma^2 \gamma^1 \tilde{\chi}) + \gamma^\mu \cdot \langle \partial_\mu \chi \gamma^2 \gamma^1 \tilde{\chi} \rangle_4, \tag{76}$$

and after using eq. (72):

$$\boxed{\gamma^5 g = \langle \partial \chi \gamma^2 \gamma^1 \tilde{\chi} \rangle_3 + \frac{1}{2}\partial \wedge (\chi \gamma^1 \gamma^2 \tilde{\chi})} \tag{77}$$

Now, χ is an arbitrary DH spinor field, which can be either that ψ which satisfies DH equation (30) or (68) or that ϕ which satisfies non-linear DH equation (27) or (67). Let us consider the two cases in separate:

(i) $\boxed{\chi = \psi}$ In this case, after using DH equation and noting that $\langle \psi \gamma^0 \tilde{\psi} \rangle_1 = \psi \gamma^0 \tilde{\psi}$, we have from eq. (75):

$$j = \frac{mc}{\hbar}\psi \gamma^0 \tilde{\psi} + \partial \cdot (\frac{1}{2}\psi \gamma^1 \gamma^2 \tilde{\psi}), \tag{78}$$

or

$$\psi\gamma^0\tilde{\psi} = \frac{\hbar}{mc}j + \partial\cdot(\frac{\hbar}{2mc}\psi\gamma^2\gamma^1\tilde{\psi}), \tag{79}$$

which is just Gordon decomposition for the Dirac current $\psi\gamma^0\tilde{\psi}$. The part $\partial\cdot M$ for $M = \frac{\hbar}{2mc}\psi\gamma^2\gamma^1\tilde{\psi}$ is the so-called Gordon current, due to the density of magnetic moment M, while the part $\frac{\hbar}{mc}j$ is due to the overall motion. On the other hand, since $\langle\psi\gamma^0\tilde{\psi}\rangle_3 = 0$, eq. (77) gives:

$$\gamma^5 g = \partial\wedge(\frac{1}{2}\psi\gamma^1\gamma^2\tilde{\psi}), \tag{80}$$

or

$$g = \partial\cdot(\gamma^5\frac{1}{2}\psi\gamma^1\gamma^2\tilde{\psi}), \tag{81}$$

which is a Gordon current for the density of electric moment $\gamma^5 M$. Finally, if we use the identify [22]:

$$a\cdot(b\cdot B) = (a\wedge b)\cdot B \tag{82}$$

for a, b 1-form and B a 2-form, we see that

$$\partial\cdot[\partial\cdot M] = (\partial\wedge\partial)\cdot M = 0, \tag{83}$$

$$\partial\cdot[\partial\cdot(\gamma^5 M)] = (\partial\wedge\partial)\cdot(\gamma^5 M) = 0, \tag{84}$$

since $\partial\wedge\partial = 0$, and these results when introduced into eq. (79) and eq. (81) give:

$$\partial\cdot j = 0, \tag{85}$$

$$\partial\cdot g = 0, \tag{86}$$

where in order to arrive at eq. (85) we have also used the conservation of Dirac current, that is: $\partial\cdot(\psi\gamma^0\tilde{\psi}) = 0$. Note that eq. (85) and eq. (86) are strict consequences of DH equation.

(ii) $\boxed{\chi = \phi}$ In this case, after using the non-linear DH equation (27), one obtains from eq. (75),

$$j = \frac{mc}{\hbar}\phi\gamma^0\tilde{\phi} + \frac{1}{2}(\partial\ln h)\cdot(\phi\gamma^1\gamma^2\tilde{\phi}) + \frac{1}{2}\partial\cdot(\phi\gamma^1\gamma^2\tilde{\phi}), \tag{87}$$

or

$$\phi\gamma^0\tilde{\phi} = \frac{\hbar}{mc}j - \frac{\hbar}{2mc}[(\partial\ln h)\cdot(\phi\gamma^1\gamma^2\tilde{\phi}) + \partial\cdot(\phi\gamma^1\gamma^2\tilde{\phi})], \tag{88}$$

while eq. (77) gives:

$$\gamma^5 g = \frac{1}{2}[(\partial\ln h)\wedge(\phi\gamma^1\gamma^2\tilde{\phi}) + \partial\wedge(\phi\gamma^1\gamma^2\tilde{\phi})], \tag{89}$$

One can also establish the following identities after using eq. (82) and $\partial \wedge \partial = 0$:

$$\partial \cdot [\partial \cdot (\phi\gamma^1\gamma^2 \tilde{\phi})] = 0, \tag{90}$$

$$\partial \cdot [(\partial \ln h) \cdot (\phi\gamma^1\gamma^2 \tilde{\phi})] = -(\partial \ln h) \cdot [\partial \cdot (\phi\gamma^1\gamma^2 \tilde{\phi})], \tag{91}$$

$$\partial \wedge [\partial \wedge (\phi\gamma^1\gamma^2 \tilde{\phi})] = 0, \tag{92}$$

$$\partial \wedge [(\partial \ln h) \wedge (\phi\gamma^1\gamma^2 \tilde{\phi})] = -(\partial \ln h) \wedge \partial \wedge (\phi\gamma^1\gamma^2 \tilde{\phi})], \tag{93}$$

If we use eq. (90-91), we have from eq. (88):

$$\partial \cdot (\phi\gamma^0 \tilde{\phi}) = \frac{\hbar}{mc}(\partial \cdot j) + \frac{\hbar}{2mc}(\partial \ln h) \cdot [\partial \cdot (\phi\gamma^1\gamma^2 \tilde{\phi})], \tag{94}$$

and from eq. (89), after using eq. (92-93) and the fact that $\partial \wedge (\gamma^5 g) = -\gamma^5(\partial \cdot g)$:

$$\gamma^5(\partial \cdot g) = \frac{1}{2}(\partial \ln h) \wedge [\partial \wedge (\phi\gamma^1\gamma^2 \tilde{\phi})]. \tag{95}$$

Note also that for non-linear DH equation, the Dirac current is not conserved; in fact, we have that $\partial \cdot (h\phi\gamma^0 \tilde{\phi}) = 0$, which gives:

$$\partial \cdot (\phi\gamma^0 \tilde{\phi}) = -(\partial \ln h) \cdot (\phi\gamma^0 \tilde{\phi}). \tag{96}$$

Now, eq. (88, 89, 94, 95, 96) are consequences of the non-linear equation for ϕ. This non-linear equation, on the other hand, follows from Maxwell equation for $F = \phi\gamma^1\gamma^2 \tilde{\phi}$ under certain assumptions which are, indeed, assumptions about ϕ. One can, of course, follow the steps of sec. 4 backwards and obtain (from the non-linear DH equations and under the same assumptions) the Maxwell equations $\partial F = 0$ for $F = \phi\gamma^1\gamma^2 \tilde{\phi}$, that is:

$$\partial \cdot (\phi\gamma^1\gamma^2 \tilde{\phi}) = 0, \tag{97}$$

$$\partial \wedge (\phi\gamma^1\gamma^2 \tilde{\phi}) = 0. \tag{98}$$

In this case, we have from eq. (88-89):

$$\phi\gamma^0 \tilde{\phi} = \frac{\hbar}{mc}j - \frac{\hbar}{2mc}(\partial \ln h) \cdot (\phi\gamma^1\gamma^2 \tilde{\phi}), \tag{99}$$

$$\gamma^5 g = \frac{1}{2}(\partial \ln h) \wedge (\phi\gamma^1\gamma^2 \tilde{\phi}), \tag{100}$$

and from eq. (94-95):

$$\partial \cdot (\phi\gamma^0 \tilde{\phi}) = \frac{\hbar}{mc}(\partial \cdot j), \tag{101}$$

$$\partial \cdot g = 0. \tag{102}$$

Eq. (99–102) are not anymore direct consequences of the non-linear DH equation but are now consequences of the assumptions we made about ϕ in order to the non-linear DH equation be a spinorial representation of Maxwell equation for $F = \phi\gamma^1\gamma^2\tilde{\phi}$. Eq. (99-100) are therefore two equations for ϕ which must be thought as "compatibility equations". One can use eq.(101–102) as a kind of "gauge conditions" to be used in order to eliminate the two additional degrees of freedom of the DH spinor field ϕ. Note moreover that if one assumes also the conservation of Dirac current $\phi\gamma^0\tilde{\phi}$, these two "gauge conditions" reduce to:

$$\partial \cdot j = 0\,, \quad \partial \cdot g = 0. \tag{103}$$

It is time now to observe that the "gauge conditions" (103) are just the ones discussed by Campolattaro in [5]. In fact, note first that

$$\begin{aligned}\langle \partial^\mu \phi\gamma^2\gamma^1\partial_\mu \tilde{\phi}\rangle_0 &= \frac{1}{2}[\langle \partial^\mu \phi\gamma^2\gamma^1\partial_\mu \tilde{\phi}\rangle_0 + \langle \partial^\mu \phi\gamma^2\gamma^1\partial_\mu \tilde{\phi}\rangle\tilde{_0}] \\ &= \frac{1}{2}[\langle \partial^\mu \phi\gamma^2\gamma^1\partial_\mu \tilde{\phi}\rangle_0 - \langle \partial_\mu \phi\gamma^2\gamma^1\partial^\mu \tilde{\phi}\rangle_0] = 0,\end{aligned} \tag{104}$$

and in the same way

$$\langle \partial^\mu \phi\gamma^5\gamma^2\gamma^1\partial_\mu \tilde{\phi}\rangle_0 = 0 \tag{105}$$

Then, from eq. (71) and using eq. (104), we have:

$$\begin{aligned}\partial \cdot j = \partial_\mu\langle \partial^\mu \phi\gamma^2\gamma^1\tilde{\phi}\rangle_0 &= \partial_\mu\langle \partial^\mu \phi\gamma^2\gamma^1\tilde{\phi}\rangle_0 - \langle \partial^\mu \phi\gamma^2\gamma^1\tilde{\phi}\rangle_0 \\ &= \langle \partial_\mu\partial^\mu \phi\gamma^2\gamma^1\tilde{\phi}\rangle_0 = \langle \Box\phi\gamma^2\gamma^1\tilde{\phi}\rangle_0,\end{aligned} \tag{106}$$

and from eq. (72) and using eq. (105), we have:

$$\begin{aligned}\partial \cdot g = \partial_\mu\langle \partial^\mu \phi\gamma^5\gamma^2\gamma^1\tilde{\phi}\rangle_0 &= \partial_\mu\langle \partial^\mu \phi\gamma^5\gamma^2\gamma^1\tilde{\phi}\rangle_0 - \langle \partial^\mu \phi\gamma^5\gamma^2\gamma^1\partial_\mu\tilde{\phi}\rangle_0 \\ &= \langle \partial_\mu\partial^\mu \phi\gamma^5\gamma^2\gamma^1\tilde{\phi}\rangle_0 = \langle \Box\phi\gamma^5\gamma^2\gamma^1\tilde{\phi}\rangle_0,\end{aligned} \tag{107}$$

and the two "gauge conditions" (103) can be written as:

$$\langle \Box\phi\gamma^2\gamma^1\tilde{\phi}\rangle_0 = 0, \tag{108}$$

$$\langle \Box\phi\gamma^5\gamma^2\gamma^1\tilde{\phi}\rangle_0. \tag{109}$$

In terms of the standard covariant Dirac spinor field $\phi_D(x) \in \mathbb{C}^4$, having as representative in the Clifford bundle $\phi_D = \phi\varepsilon$ for ε an appropriate idempotent field (see [16, 17, 25] for details), conditions (108) and (109) are written respectively as:

$$I_m[\bar{\phi}_D\Box\phi_D] = \frac{i}{2}[\bar{\phi}_D(\Box\phi_D) - (\Box\bar{\phi}_D)\phi_D] = 0, \tag{110}$$

$$I_m[\bar{\phi}_D\gamma^5\Box\phi_D] = \frac{i}{2}[\bar{\phi}_D\gamma^5(\Box\phi_D) - (\Box\bar{\phi}_D)\gamma^5\phi_D] = 0, \qquad (111)$$

which are eq. (28-29) of [5]. According to our approach, however, the gauge condition (110) is consistent only when the Dirac current is conserved, which is the case when $(\partial \ln h) \cdot (\phi\gamma^0 \tilde{\phi}) = 0$ according to eq. (96).

7. Concluding Remarks

The objective for this paper was to show the equivalence between Maxwell and Dirac equations for their respective free cases and when the electromagnetic field is non-null. The hypothesis we have assumed were very general and indeed satisfied for the cases considered. The geometrical interpretation we gave to the relationship between the non-linear DH equation and the DH equation, and to the quantum potential type non-linear term, suggests further developments which may be physically relevant. Finally, the relation we discussed between Maxwell and Dirac fields and its comparison with the one of Campolattaro may also lead to further interesting developments, especially when one confronts our analysis with the interpretation given by Campolattaro to his results. However, such possible developments are an issue to be considered elsewhere.

The generalization of this work for general cases (i.e., the non-free ones) requires the consideration of several issues outside the scope of this work. We do not need to consider them here, but let us quote an specific example: the role of the Takabayasi angle β. The way we introduced it exhibits in a clear manner its meaning in connection to electromagnetism, that is, it is identified with the "complexion" of the field [10]. However, its role in Dirac theory is still mysterious [30]. Indeed, it appears that when we consider external interactions, the angle β is variable. This is the case, for example, for the Darwin solutions of the hydrogen atom [30]. However, it has ~~been found recently~~ solutions - the Krüger solutions [30, 47] – of the hydrogen atom with $\beta = 0$. It may happen that a full understanding of the role of this angle in Dirac theory depends in generalizing this work for the cases with interactions.

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