NON FINITE AXIOMATIZABILITY OF FINITELY GENERATED QUASIVARIETIES OF GRAPHS

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Janeiro RP 03/93

Relatório de Pesquisa

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ABSTRACT – Appart from four trivial cases, a universal Horn class of graphs generated by finitely many finite graphs can not be finitely axiomatizable..

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Janeiro – 1993

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Campinas, October 1992

Appart from four trivial cases, a universal Horn class of graphs generated by finitely many finite graphs can not be finitely axiomatizable. This is an almost immediate consequence of a theorem of Nešetřil and Pultr [NP], plus the observation of Tarski [T] that any universally axiomatizable class of structures is characterized by a family of forbidden finite substructures; however, it does not seem to have been noticed before. It contrast strongly with the case of algebras, since by a result of Pigozzi [P] any finitely generated quasivariety of algebras with relative distributive congruences is finitely axiomatizable. Also, we may exhibit infinitely many finitely axiomatizable, finitely generated, quasivarieties of directed graphs, so that the result is characteristic of undirected graphs.

§1. Forbidden substructures

Let σ be a finite relational vocabulary, St_σ the class of structures of type σ, and $\mathcal{F} \subseteq St_\sigma$. Following [NP], define:

$$\mathcal{F}\neg = \{A \in St_\sigma \mid \text{no substructure of } A \text{ is isomorphic to an element of } \mathcal{F}\}$$

Many interesting classes at structures are of the above form for a class $\mathcal{F}$ of finite structures, for example the class of graphs having all its countable subgraphs planar (by Kuratowski's theorem). In fact, this is typical of universally axiomatizable classes.

LEMMA. *C is (finitely) axiomatizable by universal sentences if and only if $C = \mathcal{F}\neg$ for some (finite) class $\mathcal{F}$ of finite structures*

Proof. "$\Rightarrow$" Given a universal sentence φ:

$$\forall x_1 \ldots \forall x_n \theta(x_1, \ldots, x_n)$$

let $\mathcal{F}_\varphi = \{B \in St_\sigma \mid B \subseteq \{1, 2, \ldots, n\},\ B \not\models \varphi\}$. Then $A \models \varphi$ implies $B \models \varphi$ for any of its finite substructures and so $A \in \mathcal{F}_\varphi\neg$. Conversely, if $A \in \mathcal{F}_\varphi\neg$ then the substructure

Supported by FAPESP, Fundação de Amparo à Pesquisa do Estado de São Paulo.

induced by any sequence of elements $a_1, \ldots, a_n$ of $\mathcal{A}$ must satisfy φ and so $\theta[a_1, \ldots, a_n]$; hence, $A \models \varphi$. If $\mathcal{F} = \cup_\varphi \mathcal{F}_\varphi$ where φ runs through some universal axiomatization of C, then $C = \mathcal{F}\neg$. As $\mathcal{F}_\varphi$ is obviously finite, $\mathcal{F}$ will be finite whenever the axiomatization is finite.

" $\Leftarrow$ " If $C = \mathcal{F}\neg$ where $\mathcal{F}$ consists of finite structures, then C is axiomatized by the sentences:

$$\forall x_1 \ldots \forall x_{|B|} \neg D_B(x_1, \ldots x_{|B|}), \quad B \in \mathcal{F},$$

where D_B is the conjunction of the diagram of B. □

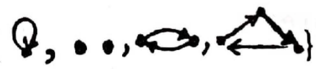
The lemma holds for infinite σ taking for $\mathcal{F}$ a class of finite structures defined in finite subtypes of σ, and defining $\mathcal{F}\neg$ as the class of structures not having substructures with reducts isomorphic to elements of $\mathcal{F}$.

If the structures in $\mathcal{F}$ are not finite, $\mathcal{F}\neg$ is not necessarily axiomatizable, but the following are equivalent (compare with [A]):

i) $C = \mathcal{F}\neg$ where $\mathcal{F}$ consists of structures of power less than κ, $\omega \leq \kappa \leq \infty$.

ii) C is axiomatizable by universal sentences of $L_{\kappa\kappa}$.

Examples. 1) Let $\sigma = \{R\}$, R a binary relation symbol, and let

$\mathcal{F}_0 = \{$  $\}$

then $\mathcal{F}_0\neg$ is the class linear orders. Notice that the effect of prohibiting the substructure {• •} forces trichotomy, and prohibiting {  } forces transivity. Consider now:

$$\mathcal{F}_1 = \mathcal{F}_0 \cup \{\cdots\cdots \rightarrow\bullet\rightarrow\bullet\rightarrow\bullet\rightarrow\bullet\},$$

then $\mathcal{F}_1\neg$ is the class of well ordered sets, obviously axiomatizable in $L_{\omega_1\omega_1}$.

2) Lemma 1 holds for classes of algebras if we allow partial algebras as substructures. For example, the semigroup axiom of associativity may be expressed:

$$\forall x_1 \ldots \forall x_6((f(x_1, x_2) = x_4 \wedge f(x_2, x_3) = x_5 \wedge f(x_1, x_5) = x_6) \rightarrow f(x_4, x_3) = x_6),$$

and so the class of semigroups coincides with $\mathcal{F}\neg$ when $\mathcal{F}$ consist of all the partial binary algebras (S, f) with $S \subseteq \{1, 2, \ldots, 6\}$ for which there are triples

$$(a_1, a_2, a_4), \ (a_2, a_3, a_5), \ (a_1, a_5, a_6), \ (a_4, a_3, a_6)$$

such that the first three appear in the table of the operation, and the fourth does not.

§2. Universal Horn Classes

If $\mathcal{A}$ is a non-empty class of structures, of type σ, then the *quasivariety* generated by $\mathcal{A}$, $ISP(\mathcal{A})$, is the closure of $\mathcal{A}$ under cartesian products, substructures, and isomorphic images. In general, $ISP(\mathcal{A})$ does not need to be first order axiomatizable, unless it is closed under ultraproducts, or equivalently under directed limits (cf. [BS]). In case it is axiomatizable, it will be axiomatized by universal Horn sentences, those of the form:

$$\forall x_1 \dots \forall x_n (\bigwedge \Phi \rightarrow \Theta)$$

where Φ consists of atomic formulae and Θ is atomic or negated atomic. An axiomatizable quasivariety is called also a *universal Horn class.*

If $\mathcal{A}$ consists of finitely many finite structures, $ISP(\mathcal{A})$ is said to be *finitely generated.* In such case $ISP(\mathcal{A})$ is first order axiomatizable. This may be readily seen by showing that the class is closed under ultraproducts, method which works also when $\mathcal{A}$ is any axiomatizable class, or just a projective class of structures. In the finitely generated case we may produce explicitely a recursive universal Horn axiomatization: $A \in ISP(\mathcal{A})$ if and only if A is a substructure of a product of elements of $\mathcal{A}$. Equivalently, for any (n-any) predicate $P \in \sigma \cup \{=\}$ and n-tuple $a_1, \dots, a_n \in A$ such that $A \not\models P(a_1, \dots, a_n)$, there is an homomorphism $f : A \rightarrow B \in \mathcal{A}$ such that $B \not\models P(f(a_1), \dots, f(a_n))$. If A and its elements are finite, it may be seen by a compactness argument that it is enough to have an homomorphism f with the above property for each finite substructure of A. But this may be expressed by the first order sentences: $\Theta_{p,k}$, $P \in \sigma \cup \{=\}$, $k = 1, 2, 3, \dots$:

$$\forall x_1 \dots \forall x_n \forall x_{n+1} \dots \forall x_{n+k} [\neg P(x_1, \dots, x_n) \rightarrow \bigvee_{B \in \mathcal{A}} \varphi_B(x_1, \dots, x_{n+k})]$$

where $\varphi_B(x_1, \dots x_{n+k})$ is the formula:

$$\bigvee_{\left(\substack{f:\{x_1,\dots,x_{n+k}\} \to B \\ B \not\models P(f(x_1),\dots,f(x_n))}\right)} \{\bigwedge_{\left(\substack{Q \in \sigma \\ B \not\models Q(f(x_{i1}),\dots,f(x_{ij}))}\right)} \neg Q(x_{i1}, \dots, x_{ij})\}$$

saying that there is an homomorphism from the substructure generated by $\{x_1, \dots x_{n+k}\}$ into B such that $B \not\models P(f(x_1), \dots, f(x_n))$.

The sentences $\Theta_{P,k}$ are note universal Horn, but they have the form $\forall \bar{x} (\bigvee_i \bigwedge_j \sigma_{ij})$ where the σ_{ij} are all negated atomic, except for one atomic. By distributing $\bigvee$ over $\bigwedge$ and then $\forall$ over $\bigwedge$ we get a conjunction of universal Horn sentences.

After Lemma 1, a universal Horn class is characterized by a class $\mathcal{F}$ of finite forbidden substructures. Given a universal Horn sentence

$$\varphi : \forall x_1 \dots \forall x_n (\bigwedge \Phi \rightarrow \Theta(x_1, \dots, x_n))$$

let B_φ be the structure:

$$B_\varphi = (\{1, 2, \dots, n\},\ Q^\varphi)_{Q \in \sigma}$$

where $Q^\varphi = \{(i_1, \dots, i_k) : Q(x_{i1}, \dots, x_{ik}) \in \Phi\}$. Then the set of finite forbidden structures associated to φ is

$$\mathcal{F}_\varphi = \{B \mid \exists f : B_\varphi \to B \text{ onto homomorphism such that } B \not\models \theta(f(1), \dots, f(n))\}.$$

§3. Finitely Axiomatizable Quasivarieties of Graphs

A *graph* is a set with an irreflexive, symmetric binary relation, that is, a model of the universal Horn sentences:

$G1 : \forall x(x = x \to \neg Rxx)$
$G2 : \forall x \forall y(Rxy \to Ryx)$

If it is asked to satisfy $G1$ only, it will be a *directed graph*. Consider the following quasivarieties of graphs:

$D_1 = ISP(\{\bullet\})$ = One vertex graphs
$D_2 = ISP(\{\bullet \quad \bullet\})$ = Discret graphs
$D_3 = ISP(\{\bullet\!-\!\!-\!\bullet\})$ = Disjoint sums of $\{\bullet\!-\!\!-\!\bullet\}$ and $\{\bullet\}$
$D_4 = ISP(\{\bullet\!-\!\!-\!\bullet\!-\!\!-\!\bullet\})$ = Disjoint sums of complete bipartite graphs and $\{\bullet\}$.

Then D_i is axiomatized by $\{G_1, G_2, \varphi_i\}$, $i = 1, 2, 3, 4$, respectively, where:

$\varphi_1 : \forall x \forall y(x = x \to x = y)$
$\varphi_2 : \forall x \forall y(x = x \to \neg xRy)$
$\varphi_3 : \forall x \forall y \forall z(Rxy \wedge Rxz \to y = z)$
$\varphi_4 : \forall x \forall y \forall z \forall w(Rxy \wedge Ryz \wedge Rzw \to Rxw)$.

These are the only finitely axiomatizable, finitely generated quasivarieties of graphs because of the following result and the Lemma.

PROPOSITION. (Nešetřil and Pultr [NP], Th. 3.2). *D_1, D_2, D_3 and D_4 are the only finitely generated quasivarieties of graphs, of the form $\mathcal{F}\neg$ for a finite family $\mathcal{F}$ of finite graphs.*

We have, for example that the quasivariety C_n of n-colourable graphs, $n \geq 2$, which is finitely generated (cf. [NP], [W]), is not finitely axiomatizable. This was shown first by

W. Taylor [T] utilizing a theorem of Erdős.

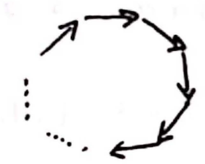
In fact, the argument in [NP], based in the same theorem of Erdős, shows the more general result:

THEOREM. *A finitely axiomatizable quasivariety of graphs distinct from D_1, D_2, D_3, D_4 has graphs of arbitrarily large chromatic number. In particular, it can not be finitely generated.*

Hence, no subquasivariety of C_n is finitely axiomatizable for $n \geq 2$.

As we have mentioned, the non finitely axiomatizable result does not hold for quasivarieties of directed graphs. Consider the directed cycle of length n:

C_n



the class $\mathcal{H}_n = ISP(C_n)$ consists of those graphs which are disjoint sums of copies of C_n and chains of lenght less than n because any power C_n^I consists exactly of $|I|$ copies of C_n. On the other hand, $\mathcal{H}_n$ is axiomatized by the sentences

$$\forall x \forall y \forall z (Rxy \wedge Rxz \rightarrow y = z)$$
$$\forall x \forall y \forall z (Rxy \wedge Rzy \rightarrow x = z)$$
$$\forall x_1 \forall x_2 \ldots \forall x_n (Rx_1x_2 \wedge Rx_2x_3 \wedge \ldots \wedge Rx_{n-1}x_n \rightarrow Rx_nx_1)$$
$$\forall x_1 \ldots \forall x_k (Rx_1x_2 \wedge \ldots \wedge Rx_{k-1}x_k \rightarrow \neg R(x_kx_1)), \quad k < n.$$

Similarly, if $A_n = (\{1, 2, \ldots, n\}, s)$ with $S(x) = x + 1 (mod\ n)$, then the quasivarieties of algebras $ISP(A_n)$ are all distinct and axiomatized each by the finite set of universal Horn sentences:

$$\forall x \forall y (f(x) = f(y) \rightarrow x = y).$$
$$\forall x (f^n(x) = x)$$
$$\forall x \neg (f^k(x) = x), \quad 1 \leq k < n.$$

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