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ON THE NATURE OF THE GRAVITATIONAL FIELD

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ON THE NATURE OF THE GRAVITATIONAL FIELD

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ABSTRACT

The Clifford bundle approach to the geometry of a Riemann-Cartan-Weyl space developed in a previous paper suggests by itself to interpret Einstein's gravitational theory as a field theory in the sense of Faraday, i.e., with the gravitational field living in Minkowski spacetime. Here we present such a theory. For the variables playing the role of the gravitational field the lagrangiam density is of the Yang-Mills type with gauge fixing and auto interaction terms. A brief comparison of our theory with some others field theories of the gravitational field in Minkowski spacetime is given. Some misconceptions and misanderstandings are clarified.

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1. INTRODUCTION

Einstein's gravitational theory (General Relativity) is one of the most beautiful physical theories[1] Yet, many physicists are not happy with one or another aspect of the theory. Weinberg[2], e.g., is of the oppinion that the geometrical interpretation of the theory in terms of a Lorentzian spacetime[3,4,5,6] is a coincidence and a physical interpretation for the gravitational field in the sense of Faraday, i.e., with the gravitational field living and interacting with the other physical fields in Minkowski spacetime is desirable. One of the main criticisms to the geometrical interpretation comes from Logunov and collaborators[7,8]. In particular, in the geometrical interpretation there are no genuine conservation laws for energy-momentum and angular momentum in general. This point is indeed a crucial one and deserves a critical consideration by people supporting the geometrical interpretation. In their book "General Relativity for Mathematicians", Sachs and Wu[3] said: "It is a shame to loose the special relativistic total energy conservation laws in General Relativity. Many of the attempts to resurrect it are quite interesting, many are simply garbage" (see, e.g., ref.[9] for a mathemathicaly clear presentation of the reason for the non existence of conservation laws in the sense of special relativity in Einstein's gravitational theory).

These considerations among many others equally important inspired several physicists to build up a consistent theory of the gravitational field in flat Minkowski spacetime see, e.g.,[10–21] besides[7,8].

In this paper we present our approach to the subject which is possible due to the Clifford bundle formulation to the geometry of a Riemann-Cartan-Weyl space (RCWS) developed in[22] (hereafter called I).

To start we recall that, as it is well known, in General Relativity every gravitational field which is a solution of Einstein equations is modelled by a Lorentz spacetime[3,4,5,6], i.e., a quadruple $\langle M, g, \tau_g, \nabla\rangle$ where $\mathcal{M} = \langle M, g, \tau_g\rangle$ is a Lorentzian manifold, i.e., M is a Hausdorff, paracompact, C^∞, connected four dimensional manifold oriented by τ_g (the volume element 4 form) and time oriented. The tensor field $g \in T_0^2 M$ is a Lorenttz metric of signature (1,3) and ∇ is its Levi-Civita connection. The pair $\langle T_xM, g_x\rangle, x \in M$, is isomorphic to $\mathbb{R}^{1,3}$ the so called Minkowski vector space[4], which is not to be confounded with the Minkowski spacetime, i.e., the particular Lorentzian spacetime represented by the quadruple $\langle M \equiv \mathbb{R}^4, \eta, \tau_\eta, \widehat{D}\rangle$ where η is a Lorentzian metric, $\widehat{D}$ is its Levi-Civita connection such that $\widehat{D}\eta = 0$, $\boldsymbol{T}[D] = 0$ and $\boldsymbol{R}[D] = 0$, with $\boldsymbol{T}$ and $\boldsymbol{R}$ being respectively the torsion and curvature tensors. We denote by $\mathbb{M} = \langle \mathbb{R}^4, \eta, \tau_\eta\rangle$ the Minkowski manifold and by $T(\mathbb{R}^4)$ and $T^*(\mathbb{R}^4)$ respectively the tangent and the cotangent bundles of the Minkowski manifold.

The strategy for the present paper is then as follows. Using the Clifford bundle approach to the geometry of a RCWS developed in I we first formulate in section 2 Einstein's theory in the Clifford bundle $\mathcal{C}\ell(\mathcal{M})[\equiv \mathcal{C}\ell(M, g^{-1})]$ thereby identifying the set of linearly independent 1-form fields $\langle\theta^\mu\rangle, \mu = 0,1,2,3,$ $\theta^\mu \in \sec\Lambda^1(M) \subset \sec\mathcal{C}\ell(\mathcal{M})$ as the natural variables playing the role of the gravitational field in the formalism. We give the field equations satisfied by the $\langle\theta^\mu\rangle$ (that are equivalent to Einstein equations). The field equations for the $\langle\theta^\mu\rangle$ [eq.(6)] are non linear wave equations. We give also an equivalent form of the field equations [eq.(14)] and discuss its meaning.

In section 3 we introduce the Lagrangian density that gives eq.(6) through the action principle. This Lagrangian $\mathcal{L}_E^{(1)}$ turns to be of the Yang Mills type for the $\langle\theta^\mu\rangle$, with a gauge fixing term plus an autointeraction term. $\mathcal{L}_E^{(1)}$ turns out to be equivalent to the first order Lagrangian density first introduced by Einstein[23] and well discussed in, e.g., ref.[24]. In our formalism $\mathcal{L}_E^{(1)}$ is a functional of θ^μ and $d\theta^\mu$ and so it is intrinsic covariant. The action $\int \mathcal{L}_E^{(1)}$ is then by a well known result[25] invariant under the diffeomorphisms of the so called manifold mapping groups (general covariance). $\mathcal{L}_E^{(1)}$ possess also the restrict, homogeneous and orthocronous Lorentz group $\mathcal{L}_+^\uparrow$ as a *local* gauge group, but ours is not a gauge theory of the $\mathcal{L}_+^\uparrow$ group.

The functional form of $\mathcal{L}_E^{(1)}$ does not involves anymore the geometrical concepts of a connection ∇ and the associated Riemann and/or Ricci tensors, but includes the concept of the Lorentzian metric g through the Clifford inner product in eq.(21) or through the Hodge dual in eq.(24). This suggests the following. Take $M = I\!R^4$ in $\overline{\mathcal{M}}$ and consider the Minkowski manifold $I\!M\!I = \langle I\!R^4, \eta, \tau_\eta\rangle$ and the Minkowski spacetime $\langle I\!R^4, \eta, \tau_\eta, \widehat{D}\rangle$. Next recall the discussion of section 3.1.c of I where it is shown that in the Clifford bundle $\mathcal{C}\ell(I\!M\!I)$ (with $I\!M\!I = \mathcal{M}$) we can introduce through the theory of the symmetyric automorphims of a linear space infinitely many non degenerated symmetric bilinear forms fields on the metrical manifold $(I\!R^4, \eta)$. If we interpret $g \in \sec T_0^2 I\!R^4$ as a fixed nondegenerated positive$^{(*)}$ symmetric bilinear form field on $I\!M\!I$ then we have a chance to formulate Einstein's theory in flat Minkowski spacetime.

In section 4 we present then the gravitational theory as a field theory in the sense of Faraday. We show that $\mathcal{L}_E^{(1)}$ interpreted as a Lagrangian density in Minkowski spacetime has the Poincaré group $\mathcal{P} = \mathcal{L}_+^\uparrow \otimes T^4$ as invariance group in the usual sense of field theory and $\int \mathcal{L}_E^{(1)}$ is also generally covariant. Besides, $\mathcal{L}_E^{(1)}$ possess also $\mathcal{L}_+^\uparrow$ as a *local* gauge group and possess T^4 (the translation group in $I\!R^4$) as a local gauge group in the sense of gauge theory in flat spacetime[25−27]. This last statement can be proved directly using the results of[27], which are valid only after we identify the correct "Hodge dual operator" to be used in $\mathcal{L}_E^{(1)}$. In[27] the Lagrangian is constructed with the flat spacetime Hodge dual operator and thus the theory there presented is not equivalent to Einstein's theory.

Finally in section 5 we present our conclusion together with a brief comparison of our theory with other flat spacetime formulations of the gravitational theory.

2. EINSTEIN EQUATIONS IN THE CLIFFORD BUNDLE $\mathcal{C}\ell(\overline{\mathcal{M}})$

Let $\overline{\mathcal{M}} = \langle M, g, \tau_g\rangle$ be a Lorentzian manifold and $\langle M, g, \tau_g, \nabla\rangle$ a Lorentzian spacetime modelling the gavitational field g in Einstein's theory. Let $\mathcal{C}\ell(\overline{\mathcal{M}})[\equiv \mathcal{C}\ell(M, g^{-1})]$ be the Clifford bundle of $\overline{\mathcal{M}}$ [section 3 of I]. Let $\langle\theta^\mu\rangle \in sec(T^*M) \subset \sec \mathcal{C}\ell(M)$ be an arbitrary moving frame of T^*M, dual to the frame $\langle e_\mu\rangle$ of TM. Then, $g = g_{\mu\nu}\theta^\mu \otimes \theta^\nu \in \sec T_0^2(M)$ and $g^{-1} = g^{\mu\nu} e_\mu \otimes e_\nu \in \sec T_0^2(M)$ with $g^{\mu\alpha} g_{\alpha\nu} = \delta_\nu^\mu$. Taking into account the fact that our goal is to give a flat spacetime formulation of Einstein's theory and taking into account the results of section 3.1.c. of I we denote the Clifford product in $\mathcal{C}\ell(\overline{\mathcal{M}})$ by $\vee$ and the inner product by $\circ$. The θ^μ satisfy

$$\theta^\mu \vee \theta^\nu + \theta^\nu \vee \theta^\mu = 2g^{\mu\nu} \quad ; \quad \theta^\mu \circ \theta^\nu = g^{\mu\nu} \tag{1}$$

$^{(*)}$ for the reason for the use of the term positive see section 2.1.c of I.

Consider now Einstein equations written in local coordinates $\langle x^\mu \rangle$ for $U \subset M$ and referred to the basis $\{e_\mu \otimes \theta^\nu\}$ of $T_1^1(M)$,

$$R^\mu_\nu - \frac{1}{2} R \delta^\mu_\nu = T^\mu_\nu \qquad \text{or} \qquad R^\mu_\nu = T^\mu_\nu - \frac{1}{2}\delta^\mu_\nu T \tag{2}$$

where T^μ_ν are the components of the energy momentum tensor of matter and $T = T^\mu_\mu$ its trace. Multiplying both members of eq.(2) by θ^ν we get

$$R^\mu - \frac{1}{2} R \theta^\mu = T^\mu \qquad \text{or} \qquad R^\mu = T^\mu - \frac{T}{2}\theta^\mu \tag{3}$$

Recalling eq.(109) of I we see that $R^\mu = (\check{\partial} \wedge \check{\partial})\theta^\mu$ are the Ricci 1 form fields $\check{\partial} \wedge \check{\partial}$ being the Ricci operator and $\check{\partial}$ is the fundamental Dirac operator acting on section of $\mathcal{C}\ell(\overline{\mathcal{M}})$. $T^\mu = T^\mu_\nu \theta^\nu$ are said to be the energy momentum 1-form fields. Also recalling eq.(114) of I we see that $\Box\theta^\mu = R^\mu - \frac{1}{2} R\theta^\mu = G^\mu$, where $\Box$ is the Einstein operator and G^μ are said to be the Einstein 1-form fields, i.e., $G^\mu = G^\mu_\nu \theta^\nu$. Then eqs.(3) can be written

$$\Box\theta^\mu = T^\mu \quad \text{(a)} \qquad \text{or} \qquad (\check{\partial} \wedge \check{\partial})\theta^\mu = T^\mu = -\frac{T}{2}\theta^\mu \quad \text{(b)}. \tag{4}$$

Now, the Ricci operator $(\check{\partial} \wedge \check{\partial})$ can be written in terms of the D'Almbertian operator $\check{\partial} \circ \check{\partial}$ [eq. 105 of I] and other combinations of the Dirac operator as

$$\check{\partial} \wedge \check{\partial} = -\check{\partial} \circ \check{\partial} + \check{\partial} \circ \check{\partial} \circ + \check{\partial} \circ \check{\partial} \wedge \tag{5}$$

and eq.(4b) can be written

$$-(\check{\partial} \circ \check{\partial})\theta^\mu + \check{\partial} \wedge (\check{\partial} \circ \theta^\mu) + \check{\partial} \circ (\check{\partial} \wedge \theta^\mu) = T^\mu - \frac{1}{2} T\theta^\mu \Longleftrightarrow \star G^\mu = \star T^\mu . \tag{6}$$

Eq.(6) represents a wave equation for the $\langle \theta^\mu \rangle$ and it suggests us to take the $\langle \theta^\mu \rangle$ as the basic variables representing the gravitational field. Observe that eq.(6) is an intrinsic equation written for sections of the Clifford bundle $\mathcal{C}\ell(\overline{\mathcal{M}})$ and $\langle \theta^\mu \rangle$ does not need to be a coordinate basis. When $\theta^\mu = \check{\partial} \wedge x^\mu = dx^\mu$ and in the gauge $\check{\partial} \circ \theta^\mu = -\delta\theta^\mu = 0$ (δ being the Hodge codifferential) eq.(6) reduces to

$$(\check{\partial} \circ \check{\partial})\theta^\mu = -T^\mu + \frac{1}{2} T\theta^\mu \tag{7}$$

and taking into account eq.(105) of I we see that in this case $(\check{\partial} \circ \check{\partial})\theta^\mu = g^{\alpha\beta}\nabla_\alpha \nabla_\beta \theta^\mu$, and the name wave equation for eq.(6) is well justified. Observe also that $\partial \circ \theta^\mu \Longleftrightarrow L^\mu_{\alpha\beta} g^{\alpha\beta} = 0$ ($\nabla e_\mu \theta^\nu = -L^\nu_{\mu\alpha}\theta^\alpha$) which means that the coordinates are harmonic.

Taking into account that $\langle \theta^\mu \rangle$ is an arbitrary moving frame of T^*M we choose hereafter without loss of generality the orthonormal basis of $T^*M, \langle \vartheta^\mu \rangle$. Then $g = \eta_{\mu\nu}\vartheta^\mu \otimes \vartheta^\nu$, $g^{-1} = \eta^{\mu\nu} e_\mu \otimes e_\nu$, $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1) = \eta^{\mu\nu}$ and

$$\vartheta^\mu \vee \vartheta^\nu + \vartheta^\nu \vee \vartheta^\mu = 2\eta^{\mu\nu} \ ; \ \vartheta^\mu \circ \vartheta^\nu = \eta^{\mu\nu} . \tag{8}$$

If $\omega^\mu_\nu = L^\mu_{\alpha\nu}\vartheta^\alpha$ are the connection 1-form fields and if $\langle\vartheta_\alpha\rangle$ is the reciprocal basis $\langle\vartheta^\mu\rangle$ i.e., $\vartheta^\mu \circ \vartheta_\nu = \delta^\mu_\nu$ then Cartan's structure equations [eq.(42) of I] are

$$
\begin{aligned}
&d\vartheta^\rho + \omega^\rho_\sigma \wedge \vartheta^\sigma = 0 \qquad \text{or} \qquad d\vartheta_\sigma - \omega^\rho_\sigma \wedge \vartheta_\rho = 0 \qquad \text{(a)}\\
&d\omega^\rho_\sigma + \omega^\rho_\beta \wedge \omega^\beta_\sigma = \Omega^\rho_\sigma \qquad \text{(b)}
\end{aligned}
\tag{9}
$$

where $d \equiv \check{\partial}\wedge$ is the exterior part of the Dirac fundamental operator and Ω^ρ_σ are the curvature 2-forms [eq.(38) of I], i.e., $\Omega^\rho_\sigma = \frac{1}{2}R_\mu{}^\rho{}_{\alpha\beta}\vartheta^\alpha \wedge \vartheta^\beta$, $R_\mu{}^\rho{}_{\alpha\beta}$ being the components of the Riemann tensor of ∇.

We can now write Einstein equations in another suggestive form

Proposition. The Einstein 1-forms $G^\mu = \Box\vartheta^\mu = R^\mu - \frac{1}{2}R\vartheta^\mu$ can be written as

$$\star G^\sigma = d \star \mathcal{G}^\sigma + \star t^\sigma \tag{10}$$

where

$$\star^{-1}\mathcal{G}^\sigma = \frac{1}{2}\omega_{\alpha\beta} \wedge \star(\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\sigma) \tag{11}$$

$$\star^{-1}t^\sigma = -\frac{1}{2}\omega_{\alpha\beta} \wedge [\omega^\sigma_\rho \wedge \star(\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\sigma) + \omega^\beta_\alpha \wedge \star(\vartheta^\alpha \wedge \vartheta_\beta \wedge \vartheta^\sigma) \tag{12}$$

Proof: Taking into account eq.(113) of I namely

$$2 \star G^\sigma = \star\Box\vartheta^\sigma \quad = \quad \Omega^{\rho\sigma} \wedge i_\rho i_\sigma \star \vartheta^\mu$$

we get

$$2 \star G^\sigma \quad = \quad \Omega_{\alpha\beta} \wedge \star(\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\sigma) \tag{13}$$

where we used eqs.(20) of I.

Now, using Cartan's structure equations we can write of eq.(13) as $2 \star G^\sigma = \Omega_{\alpha\beta} \wedge \star(\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\sigma) = d\omega_{\alpha\beta} \wedge \star(\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\sigma) + \omega_{\alpha\rho} \wedge \omega^\rho_\beta \wedge \star(\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\sigma)$. But, $d\omega_{\alpha\beta} \wedge \star(\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\sigma) = d[\omega_{\alpha\beta} \wedge \star(\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\sigma) + \omega_{\alpha\beta} \wedge d\star(\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\sigma)] = d[\omega_{\alpha\beta} \wedge \star(\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\sigma) - \omega_{\alpha\beta} \wedge \omega^\alpha_\rho \wedge \star(\vartheta^\rho \wedge \vartheta^\beta \wedge \vartheta^\sigma) - \omega_{\alpha\beta} \wedge \omega^\beta_\rho \wedge \star(\vartheta^\alpha \wedge \vartheta^\rho \wedge \vartheta^\sigma) - \omega_{\alpha\beta} \wedge \omega^\sigma_\rho \wedge \star(\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\rho)$ where we used that

$$
\begin{aligned}
d\vartheta^{\alpha_1} \wedge \ldots \wedge \vartheta^{\alpha_r} &= -\omega^{\alpha_1}_\beta \wedge \vartheta^\rho \wedge \vartheta^{\alpha_2} \wedge \ldots \wedge \vartheta^{\alpha_r} - \ldots - \omega^{\alpha_r}_\beta \wedge \vartheta^{\alpha_1} \wedge \ldots \wedge \vartheta^{\alpha_{r-1}} \wedge \vartheta^\beta\\
d \star \vartheta^{\alpha_1} \wedge \ldots \wedge \vartheta^{\alpha_r} &= -\omega^{\alpha_1}_\beta \wedge \star\vartheta^{\beta_\rho} \wedge \vartheta^{\alpha_2} \wedge \ldots \wedge \vartheta^{\alpha_r} - \ldots - \omega^{\alpha_r}_\beta \wedge \star\vartheta^{\alpha_1} \wedge \ldots \wedge \vartheta^{\alpha_{r-1}} \wedge \vartheta^\beta .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
2 \star G^\sigma \quad &= \quad d[\omega_{\alpha\beta} \wedge \star(\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\sigma)]\\
&- \quad \omega_{\alpha\beta} \wedge [\omega^\sigma_\rho \wedge \star(\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\sigma) + \omega^\beta_\alpha \wedge \star(\vartheta^\alpha \wedge \vartheta_\beta \wedge \vartheta^\sigma) \quad \blacksquare
\end{aligned}
$$

With the result of the above proposition we can write Einstein equations as

$$d \star \mathcal{G}^\sigma + \star t^\sigma = - \star T^\sigma \tag{14}$$

a form that appears in[28]. From this equation it follows the following "conservation law",

$$d(\star T^\sigma + \star t^\sigma) = 0 \tag{15}$$

and t^σ can be thought as the energy momentum 1-forms of the gravitational field. However we must take care here since the gravitational field $\langle \vartheta^\mu \rangle$ is living in a general Lorentzian spacetime where there are no genuine conservation laws in general [3,9] and also $*t^\sigma$ is not uniquely defined and in general $t^\sigma = t^\sigma_\mu \vartheta^\mu$ is such that $t_{\mu\nu}$ is not symmetric [28] Observe that since $[(f_\mu) \in \sec TM$ is the dual frame of $\langle \vartheta^\mu \rangle]$ then $f_0 \in \sec TM$ is a reference frame field in $\mathcal{M}$ if and f_0 is geodesic[4] i.e., $D_{f_0}\vartheta^0 = 0$ and $\vartheta^0 \wedge d\vartheta^0 = 0$, then we can alaways put $t^\sigma = 0$ along a geodesic line that is an integral line of f_0 (equivalence principle) and thus gravitational energy has no localization in Einstein theory despite the beautiful "conservation equation" [eq.(15)] found above. We come back to this point in section 5.

3. THE LAGRANGIAN DENSITY

We want now a Lagrangian density that yields eq.(6) [or eq.(14)] for $\langle \vartheta^\mu \rangle \in \sec(T^*M) \subset \sec \mathcal{C}\ell(\overline{\mathcal{M}})$ taken as basic variables representing the gravitational field and where $\vartheta^\mu \vee \vartheta^\nu + \vartheta^\nu \vee \vartheta^\mu = 2\eta^{\mu\nu}$. Since eq.(6) [or eq.(14)] is equivalent to $\star G^\mu = \star T^\mu$, the Lagrangian density we are looking for (call it $\mathcal{L}_E^{(1)}$) must differs from the Einstein-Hilbert Lagrangian

$$\mathcal{L}_E = \frac{1}{2} R \tau_g \tag{16}$$

by almost an exact differential. Moreover $\mathcal{L}_E^{(1)}$ must depends on ϑ^μ and $\check{\partial} \wedge \vartheta^\mu = d\vartheta^\mu$ in such a way that the action results invariant for the diffeomorphisms of the so called manifold mapping group[13,14] (general covariance).

Taking into accout that,

$$\frac{1}{2} \star (G^\sigma \circ \vartheta_\sigma) = \frac{1}{2} \star [(\square \vartheta^\sigma) \circ \vartheta_\sigma] = -\frac{1}{2} R\, \tau_g$$

and by using eqs.(20) and eq.(112) of I we get

$$\mathcal{L}_E = \frac{1}{2} \Omega_{\mu\nu} \wedge \star(\vartheta^\mu \wedge \vartheta^\nu) \tag{17}$$

That eq.(17) is indeed correct can be seen trivially, since $2\Omega_{\alpha\beta} \wedge \star(\vartheta^\alpha \wedge \vartheta^\beta) = (\vartheta^\alpha \wedge \vartheta^\beta) \wedge \star\Omega_{\alpha\beta} = -\vartheta^\alpha \wedge \star(\vartheta^\beta \circ \Omega_{\alpha\beta}) = -\star[\vartheta^\alpha \circ (\vartheta^\beta \circ \Omega_{\alpha\beta})] = -\star[\vartheta^\alpha \circ (\frac{1}{2} R_{\alpha\beta\mu\nu} \vartheta^\beta \circ (\vartheta^\mu \wedge \vartheta^\nu)] = -\star[\vartheta^\alpha \circ (\frac{1}{2} R_{\beta\alpha\mu\nu} (\eta^{\beta\mu}\vartheta^\nu - \eta^{\beta\nu}\vartheta^\mu)] = -\star[\vartheta^\alpha \circ (-R_\alpha)] = \star R$ ∎

As can be easily checked using Cartan's second structure equation [eq.(9b)] we can obtain Einstein's free field equations $\star G_\sigma = 0$ by varying $\int \mathcal{L}_E$ with respect to ϑ^μ and $\omega_{\mu\nu}$. To obtain $\star G^\mu = \star T^\mu$ we need to use the total Lagrangian

$$\mathcal{L}_E^t = \mathcal{L}_E + \mathcal{L}_E^{\text{int}}, \quad \mathcal{L}_E^{\text{int}} = -\vartheta^\sigma \wedge \star T_\sigma \tag{18}$$

where the T_σ as in eq.(3) are the energy-momentum 1-forms of matter.

Our goal is to write $\mathcal{L}_E$ as a functional of ϑ^μ and $d\vartheta^\mu$. This can be done once we take into account the following formula for the connection 1-forms $\omega_{\mu\nu}$ which are trivially proved (using the formalism of I) by inverting Cartan's structure equation. We have

$$\begin{aligned}\omega^{\mu\nu} &= \frac{1}{2}[\vartheta^\nu \circ (\check{\partial} \wedge \vartheta^\mu) - \vartheta^\mu \circ (\check{\partial} \wedge \vartheta^\mu) + (\vartheta^\mu \circ (\vartheta^\nu \circ (\check{\partial} \wedge \vartheta_\sigma)))\vartheta^\sigma] \\ &= \frac{1}{2}[\vartheta^\nu \circ d\vartheta^\mu - \vartheta^\mu \circ d\vartheta^\nu + (\vartheta^\mu \circ (\vartheta^\nu \circ d\vartheta_\sigma))\vartheta^\sigma] \end{aligned} \tag{19}$$

This last equation shows explicitly that within $\mathcal{C}\ell(\overline{\mathcal{M}})$ the connection 1-forms are given in terms of internal products of ϑ^μ and $d\vartheta^\nu$, thereby implying that the concept of the non-flat connection ∇ can be expurgate from the formalism of the gravitation theory. Indeed, the Lagrangian density given by eq.(17) will be shown now to be written

$$\mathcal{L}_E = \mathcal{L}_E^{(1)} - \frac{1}{2}d[\vartheta^\mu \wedge \star(d\vartheta_\mu)] \tag{20}$$

where

$$\begin{aligned}\mathcal{L}_E^{(1)} &= -\frac{1}{2}\tau_g \vartheta^\alpha \circ \vartheta^\beta \circ (\omega_{\alpha\rho} \wedge \omega_\beta^\rho) && (21.a) \\ &= -\frac{1}{2}\tau_g \vartheta^\alpha \circ \vartheta^\beta \circ \{[\frac{1}{2}(\vartheta_\alpha \circ d\vartheta_\beta - \vartheta_\beta \circ d\vartheta_\alpha + (\vartheta_\alpha \circ (\vartheta_\beta \circ d\vartheta^\sigma))\vartheta_\sigma] \wedge \\ &\quad \wedge [\frac{1}{2}(\vartheta_\beta \circ d\vartheta^\rho - \vartheta^\rho \circ d\vartheta_\beta + (\vartheta_\beta \circ (\vartheta^\rho \circ d\vartheta^\sigma))\vartheta_\sigma]\} && (21.b)\end{aligned}$$

Indeed, since $2\Omega_{\alpha\beta} \wedge \star(\vartheta^\alpha \wedge \vartheta^\beta) = -\star[\vartheta^\alpha \circ (\vartheta^\beta \circ \Omega_{\alpha\beta})]$ we have,

$$\vartheta^\alpha \circ (\vartheta^\beta \circ \Omega_{\alpha\beta}) = \vartheta^\alpha \circ (\vartheta^\beta \circ d\omega_{\alpha\beta}) + \vartheta^\alpha \circ (\vartheta^\beta \circ (\omega_{\alpha\rho} \wedge \omega_\beta^\rho) \tag{22}$$

and

$$\begin{aligned}\vartheta^\alpha \circ (\vartheta^\beta \circ (\omega_{\alpha\rho} \wedge \omega_\beta^\rho) &= \vartheta^\alpha \circ [(\vartheta^\beta \circ \omega_{\alpha\rho})\omega_\beta^\rho - (\vartheta^\alpha \circ \omega_\beta^\rho)\omega_{\alpha\beta}] \\ &= (\vartheta^\alpha \circ \omega_{\alpha\rho})(\vartheta^\alpha \circ \omega_\beta^\rho) - (\vartheta^\rho \circ \omega_\beta^\rho)(\vartheta^\alpha \circ \omega_{\alpha\rho}) \\ &= \eta^{\beta\lambda}(L^\sigma_{\lambda\rho}L^\rho_{\sigma\beta} - L^\sigma_{\sigma\rho}L^\rho_{\lambda\beta})\end{aligned} \tag{23}$$

We can easily verify that the dual of the first term in eq.(20) is the exact differential $-\frac{1}{2}d[\vartheta^\mu \wedge \star(d\vartheta_\mu)]$ and eq.(20) is proved. We immediately recognize $\mathcal{L}_E^{(1)}$ as the first order lagrangian first introduced by Einstein[23]. Here in the form of eq.(21) we see that $\mathcal{L}_E^{(1)}$ is intrinsic covariant since it involves only the internal products of ϑ^μ and $d\vartheta^\mu$. In this way according to a well known result[25] the action $\int \mathcal{L}_E^{(1)}$ is manifestely invariant under the action of the manifold mapping group of M. $\int \mathcal{L}_E^{(1)}$ also has as it is clear the restrict orthocrounous Lorentz group $\mathcal{L}_+^\uparrow$ as a *local* gauge invariant group. This happens since (i) all sections of the Clifford bundle transforms in the same way[34] under the action of $\mathbf{Spin}_+(1,3)$, the structural group of the bundle, e.g., $\vartheta^\mu(x) \longmapsto R(x)\vartheta^\mu(x)R^+(x)$, $d\vartheta^\mu(x) \longmapsto R(x)d\theta^\mu(x)R^+(x)$, with $R(x)R^+(x) = R^+(x)R(x) =$

1 , $R(x) \in \mathbf{Spin}_+(1,3)$, $\forall\, x \in M$; (ii) $R(x) = e^{F(x)}$ with $F(x) \in \sec \Lambda^2(M) \subset \mathcal{C}\ell(\mathcal{M})$ and (iii) Lagrangian $\mathcal{L}$ are 4-forms and then commutes with 2-forms, i.e., $R(x)\mathcal{L}R^{+}(x) = \mathcal{L}$.

With some algebraic manipulation $\mathcal{L}_E^{(1)}$ can be written in the following suggestive form

$$\mathcal{L}_E^{(1)} = -\frac{1}{2}d\vartheta^\mu \wedge \star d\vartheta_\mu - \frac{1}{2}d \star \vartheta^\mu \wedge \star(d \star \vartheta_\mu) + \frac{1}{4}(d\vartheta^\mu \wedge \vartheta_\mu) \wedge \star(d\vartheta_\nu \wedge \vartheta^\nu) \tag{24}$$

which appears in [28]. The first term in eq.(24) is of the Yang-Mills type. The second term will be called the "gauge fixing" term, since it can be written as $-\frac{1}{2}\delta\vartheta^\mu \wedge \star\delta\vartheta_\mu$ and recalling that $\delta\vartheta^\mu = 0$ is equivalent to the harmonic gauge. The third term is the autointeration term.

Writting $\mathcal{L}_G = \mathcal{L}_E^{(1)} - \vartheta^\sigma \wedge \star T_\sigma$ we can obtain Einstein equations in the form of eq.(14) by varying $\int \mathcal{L}_G$ with respect to $\langle \vartheta^\sigma \rangle$ and $\langle d\vartheta^\mu \rangle$.

In resume, having arrived at $\mathcal{L}_E^{(1)}$ written as a functional of ϑ^σ and $d\vartheta^\nu$ we succeeded to expurgate from Einstein's theory the concept of nonflat connection ∇ and the associated geometrical objects (Riemann tensor, Ricci tensor, etc). However, the Lagrangian density $\mathcal{L}_E^{(1)}$ written as in eq.(24) makes use of the Hodge dual $\star$ associated with g. To formulate the theory in Minkowski spacetime we must expurgate g from the theory. This will be done in the next section.

4. THE GRAVITATIONAL FIELD IN MINKOWSKI SPACETIME

To achieve our main objective of obtaining a gravitational field theory in the sense of Faraday and yet equivalent to Einstein theory, we need to express $\mathcal{L}_E^{(1)}$ in $\mathcal{C}\ell(\mathbb{M})$ where $\mathbb{M} = \langle \mathbb{R}^4, \eta, \tau_\eta \rangle$ is the Minkowski manifold. This can be done once we remember that g can be represented in $\mathcal{C}\ell(\mathbb{M})$ by the field of linear transformations $h^{-1/2}$ (taking $M \equiv \mathbb{R}^4$) as discussed in section 3.1 of I. We have

$$\alpha \circ \beta = g^{-1}(\alpha,\, \beta) = h^{-1/2}(\alpha) \cdot h^{-1/2}(\beta) = \alpha \cdot h^{-1}(\beta) \tag{25}$$

where $h^{-1/2} : T^*\mathbb{R}^4 \to T^*\mathbb{R}^4$ is the square root of $h^{-1} : T^*\mathbb{R}^4 \to T^*\mathbb{R}^4$ ($T^*\mathbb{R}^4 \subset \mathcal{C}\ell(\mathbb{M})$) the field of linear tranformations which induces g^{-1}.

Eq. 25 shows explicitly that $g^{-1}(\theta^\mu, \theta^\nu)$ can be expressed in terms of the internal product associated to η (the metric of the Minkowski spacetime) in $\mathcal{C}\ell(\mathbb{M})$. To fix the ideas, observe that from Eq. 25 we can define the fields $\langle a^\mu \rangle$, $\mu = 0,1,2,3$, $a^\mu \in \sec(T^*\mathbb{R}^4) \subset \sec(\mathcal{C}\ell(\mathbb{M}))$ by

$$a^\mu = h^{-1}(\theta^\mu) = (h^{-1})^\mu_\nu \theta^\nu = \bar{h}^\mu_\nu \theta^\nu. \tag{26}$$

Then,

$$g^{-1}(\theta^\mu, \theta^\nu) = a^\mu \cdot a^\nu = \eta^{\mu\nu} = \eta(a^\mu, a^\nu). \tag{27}$$

It follows that the $\langle a^\mu \rangle$ are η-orthonormal and satisfy the defining property of the Clifford algebra $\mathbb{R}_{1,3}$: $a^\mu a^\nu + a^\nu a^\mu = 2\eta^{\mu\nu}$, $\forall x \in \mathbb{R}^4$. It is important to observe that in general the a^μ are not exact differentials and in the basis $\langle a^\mu \rangle$ of $T^*\mathbb{R}^4$ the expression of g is

$$g = g_{\alpha\beta} a^\alpha \otimes a^\beta, \quad g_{\alpha\beta} = \eta_{\mu\nu} h^\mu_\alpha h^\nu_\beta, \tag{28}$$

where $h^{\mu}_{\alpha} = (h^{-1})^{\mu}_{\alpha}$.

We observe also that due to the existence in the Minkowski spacetime of the metrical field η and the field of symmetric automorphisms g, we have two star operators $*$ and $\star$. The relation between these operators is:

$$\star = \Lambda_h * \Lambda_{h^{-1}}, \tag{29}$$

where $\Lambda_{h^{-1}}$ is defined in Eq.(3) of I once we put $\phi^{1/2} \equiv h^{-1}$.

With Eq. 29 we can write, if we want, the Lagrangian $\mathcal{L}^{(1)}_E$ in terms of the usual Hodge star operator of the Minkowski spacetime, but this results in an odd expression which adds nothing new to the theory.

To continue, we proceed as follows. We interpret the $\langle\vartheta^{\mu}\rangle$ as physical fields living in Minkowski spacetime, i.e., $\vartheta^{\mu} \in \sec \mathcal{C}\ell(I\!M)$, and thereby we interpret $\mathcal{L}^{(1)}_E$ [Eq. 21.b or Eq. 24] as the Lagrangian density for these fields in Minkowski spacetime, even if it is not written with the standard star operator $*$.

The resulting equations of motion for the ϑ^{μ} fields are, of course, Eqs. 6 or 14. In order to express Eq. 6 in terms of the fundamental Dirac operator $\partial = \theta^{\alpha} D_{e_{\alpha}}$ and the Hodge star operator $*$, we observe that it holds the following identity, which is proved in[20]:

$$(\check{\partial} \wedge \check{\partial})\omega = (\partial \wedge \partial)\check{\omega} + g^{\rho\sigma} J_{\alpha\sigma}\omega_{\rho}\theta^{\alpha}, \tag{30}$$

where $\omega \in \sec(T^*M) \subset \mathcal{C}\ell(I\!M)$ is an arbitrary 1-form field on M and $\check{\omega} = h^{-2}(\omega)$. Using this relation, Eq. 6 is written as:

$$(\partial \wedge \partial)\check{\theta}^{\rho} = -\check{J}^{\rho} + T^{\rho} - \frac{1}{2} T\theta^{\rho}, \tag{31}$$

where $\check{J}^{\rho} = g^{\rho\sigma} J_{\alpha\sigma}\theta^{\alpha}$ and $J_{\alpha\sigma}$ is defined in Eq.(95) of I.

By its turn, Eq. 14 can be written in Minkowski spacetime as:

$$d * \mathcal{G}^{\sigma}_M + *t^{\sigma}_M = *T^{\sigma}_M, \tag{32}$$

where

$$\begin{aligned} *\mathcal{G}^{\sigma}_M &= \Lambda_h * \Lambda_{h^{-1}} \mathcal{G}^{\sigma} \\ *t^{\sigma}_M &= \Lambda_h * \Lambda_{h^{-1}} t^{\sigma} \\ *T^{\sigma}_M &= \Lambda_h * \Lambda_{h^{-1}} T^{\sigma} \end{aligned} \tag{33}$$

and

$$\Lambda_h * \Lambda_{h^{-1}} \mathcal{G}^{\sigma} = \frac{1}{2}\omega_{\mu\nu} \wedge \Lambda_h * \Lambda_{h^{-1}}(\theta^{\mu} \wedge \theta^{\nu} \wedge \theta^{\sigma}) \tag{34}$$

$$\begin{aligned} \Lambda_h * \Lambda_{h^{-1}} t^{\sigma} &= -\frac{1}{2}\omega_{\mu\nu} \wedge \Big[\omega^{\sigma}_{\rho} \wedge \Lambda_h * \Lambda_{h^{-1}}(\theta^{\mu} \wedge \theta^{\nu} \wedge \theta^{\rho}) + \\ &+ \omega^{\nu}_{\rho} \wedge \Lambda_h * \Lambda_{h^{-1}}(\theta^{\mu} \wedge \theta^{\rho} \wedge \theta^{\sigma})\Big], \end{aligned} \tag{35}$$

where the $\omega_{\mu\nu}$ are given by Eq. 19, but they are not more connection 1-forms. Eq. 32 express the conservation law of energy and momentum for the gravitational plus matter fields in Minkowski spacetime. Since t^σ_M is expressed as the combinations of the internal product of θ_μ and $d\theta_\mu$, it is gauge invariant under $\mathbf{Spin}_+(1,3)$ in our theory. Thus this equation does not suffer the problems[28] associated with analogous equation $d(\star T^\sigma - \star t^\sigma) = 0$ interpreted as an equation valid in a general Lorentzian spacetime $\langle M, g, \tau_g, \nabla\rangle$ in Einstein's theory.

5. CONCLUSIONS

As we said the introduction, there are several attempts to formulate the gravitational theory as a field theory (*in the sense of Faraday*) in Minkowski spacetime. In particular, we said that Weinberg[2] is of the opinion that the geometrical interpretation in terms of a Lorentzian spacetime is a coincidence. We found above how the "coincidence" comes up. We said also in the introduction that one of the main criticisms to the geometrical theory comes from Logunov and collaborators[7, 8] and has to do with the fact that in particular, in the geometrical theory there are no genuine conservation laws for energy-momentum and angular momentum in general. In our theory we have a natural conservation law for energy and momentum of the gravitational plus matter fields that follows directly from eq.(32).

In his field theory of gravitation (RTG) Logunov[7,8] fixes the gauge by writting $D_\nu(\sqrt{-g}g^{\mu\nu}) = 0$ where $\sqrt{-g}g^{\mu\nu} = \sqrt{-\gamma}\gamma^\mu\nu + \sqrt{-\gamma}\phi^{\mu\nu}, \gamma^{\mu\nu}$ are the components of the Minkowski metric and ϕ^μ_ν are the components of the gravitational field. The gauge fixing equation is then interpreted as one of the field equations necessary to eliminate the spins 0' and 1 from the tensor field $\phi^{\mu\nu}$. According to Logunov this gauge makes it possible for RTG to predict without ambiguity gravitational phenomena like the "radar echo time-delay experiment." Also according to Logunov, RTG with the gauge fixing condition prohibits the existence of black-holes. We shall discuss these points in another paper. Here we obtained the conditions for Einstein's gravitational theory to be equivalent to a field theory in flat Minkowski spacetime without fixing any gauge a priori. We must emphasize here that the arena of physical phenomena in our theory is Minkowski spacetime $\langle \mathbb{R}^4, \eta, \tau_\eta, \widehat{D}\rangle$. The Lorentzian manifold $\langle \mathbb{R}^4, g, \tau_g, \nabla\rangle$ of our gravitational theory is an effective curved space of field origin. Then it must have a topology compatible with $\mathbb{R}^4$, being *non sequitur* Grischuk's[30] statement that in RTG it is possible to have a closed world (this puts a new restriction for the Weinberg's "coincidence"). A very simple interpretation of how measurements done by standard clocks and standard rods in a gravitational field give the effective nonflat Lorentzian manifold is given by Schwinger.[31]

We must say that our $\mathcal{L}^{(1)}_E$ suggests the interpretation of the gravitational field as a gauge field. Moving espressed the Lagrangian of all theory in flat Minkowski spacetime, then the arguments of[27] can be easily used to show that our Lagrangian has indeed T^4 as a gauge group. From our approach this seems natural, since the effective metric g is generated for strains in the cosmic lattice $\langle a^\mu\rangle$ as we have seen in section 2.1.c of I. Thus our presentation justifies Pommaret's[33] criticism to the usual presentation of General Relativity as a gauge theory of the Lorentz group (rotations) that is associated with energy-momentum tensor (that results in all field theories as coming from the translation group). We discuss further this point, as well as models of Einstein's theory in our formulation, in another publication. We would like to call the reader's attention to the fact that recently it has been showed that Maxwell and Dirac fields can be represented as sections of the Clifford bundle over Minkowski spacetime (see, e.g.,[34–36]). It is interesting that this is also the case for the gravitational field. Obviously, the representation of these different fields as objects of the same mathematical nature is the preliminary condition for any attempt to construct an unified field theory.

To end, we must recall here that there have been some applications of Clifford algebras in General Relativity, as, e.g., in[37–40]. However those presentations are not equivalent to the ours.

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References

[1] L. Landau and L. Lifishifts. Classical Field Theory, Pergamon Press, 1976.

[2] S. Weinberg. *Gravitation and Cosmology.* J. Wiley & Sons, New York , 1972.

[3] R. K. Sachs and H. Wu. *General Relativity for Mathematicians.* Springer-Verlag, New York, 1977.

[4] W. A. Rodrigues, Jr. and M. A. Faria-Rosa. *Found. of Phys.*, **19**, 705 (1989).

[5] B. O'Neill. *Elementary Differential Geometry.* Academic Press, New York, 1966.

[6] J. K. Beem and P. E. Ehrlich. *Global Lorentzian Geometry.* Marcel Dekker, Inc., New York, 1981.

[7] A. A. Logunov and M. A. Mestvirishvili. *Found. of Phys.*, **16**, 1 (1986).

[8] A. A. Logunov, Yu. M. Loskutov and M. A. Mestvirishvili. *Int. J. Mod. Phys.*, **43**, 2067 (1988).

[9] I. M. Benn. *Ann. Inst. Henri Poicaré*, XXXVIIA, 67 (1982).

[10] H. Poincaré. *Rend. C. Math. Palermo*, **21**, 129 (1906).

[11] N. Rosen. *Phys. Rev.*, **57**, 147 (1940).

[12] N. Rosen. *Ann. Phys.*, **22**, 1 (1963).

[13] A. Papapetrou. *Proc. R. Ir. Acad. Sect.*, **A52**, 11 (1948).

[14] R. H. Kraichnan. *Phys. Rev.*, **98**, 1118 (1955).

[15] S. N. Gupta. *Rev. Mod. Phys.*, **29**, 334 (1957).

[16] R. P. Feynman. *Acta Phys. Pol.*, **24**, 697 (1963).

[17] L. Halpern. *Bull. Acad. R. Belg. Cl. Sci.*, **49**, 226 (1963).

[18] V. I. Ogsevetsky and I. V. Polubarinon. *Ann. Phys.*, **35**, 167 (1965).

[19] S. Weinberg. *Phys. Lett.*, **B138**, 988 (1965).

[20] Ya. B. Zel'dovich and I. D. Novikov. *The Theory of Gravitation and the Evolution of Stars.* Nauka, Moscow, 1971, pp. 87.

[21] L. P. Grishchuk, A. N. Petrov and A. D. Popova. *Comm. Math. Phys.*, **94**, 379 (1984).

[22] Q. A. G. de Souza and W. A. Rodrigues, Jr. *The Dirac Operator and the Structure of Riemann-Cartan-Weyl Spaces.* Subm. for publ. (1992)

[23] A. Einstein. *Sitzungsber. Preuss. Akad. Wiss.*, Berlin, 1916, pp. 1111.

[24] M. Ferraris and J. Kijowski. *Gen. Rel. Grav.*, **14**, 165 (1982).

[25] M. Göckeler and T. Schücker. *Differential Geometry, Gauge Theories and Gravity.* Cambridge Univ. Press, Cambridge, 1988.

[26] Choquet-Bruhat, C. De Witt-Morewtte and M. Dillard-Bleick, *Analysis, Manifolds and Physics.* North Holland Publ. Co., Amsterdan, 1982.

[27] R. P. Wallner. *Acta Phys. Aust.*, **54**, 165 (1982).

[28] W. Thirring. *Classical Field Theory.* Springer-Verlag, New York, 1980.

[29] W. A. Rodrigues, Jr., M. Scanavini and L.P. de Alcantara. *Found. Phys. Lett.*, , (1990).

[30] L. P. Grishchuk and A. N. Petrov. *Sov. Astron. Lett.*, **12**, 179 (1986).

[31] J. Schwinger. *Particles, Sources and Fields.* A. Wesley Publ. Comp. Reading, Ma, 1970.

[32] H. Kleinert. *Gauge Fields in Condensed Matter*, vol. II. World Scientific Publ. Co., Singapore, 1989.

[33] J. F. Pommaret. *Lie Pseudo Groups and Mechanics*, Gordon and Breach, 1989.

[34] W. A. Rodrigues, Jr. and V. L. Figueiredo. *Int. J. Theor. Phys.*, **29**, 451 (1990).

[35] W. A. Rodrigues, Jr. and E. C. Oliveira. *Int. J. Theor. Phys.*, **29**, 435 (1990).

[36] A. Maia, Jr., E. Recami, M. A. Faria-Rosa and W. A. Rodrigues, Jr. *Journal Math. Phys.*, **31**, 502 (1990).

[37] D. Hestenes. *Spacetime Algebra.* Gordon and Breach, New York, 1966.

[38] D. Hestenes. *Journal Math. Phys.*, **8**, 1046 (1967).

[39] F. J. Chinea. *Gen. Rel. Grav.* **21** (1989).

[40] R. W. Tucker. in *Clifford Algebras and their Applications in Mathematical Physics* ed. J. S. R. Chisholm and A. K. Common D. Reidel Publ. Comp., Dordrecht, 1986.

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