TOPOLOGICAL DEFECTS: A DISTRIBUTION THEORY APPROACH

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ABSTRACT - The theory of distributions in Riemaniann spaces due to Lichnerowicz is used to obtain exact solutions to the Einstein equations for spacetimes that have null Riemann-Christoffel curvature tensor everywhere except on a hypersurface. The cases of spherically, cylindrically, plane and axially symmetric spacetimes in which the matter content of the singular surfaces can be described by a barotropic equation of state are treated in some detail. Solutions with null curvature tensor except on a) concentric spheres, b) concentric cylinders, c) parallel planes, and d) parallel discs, are exhibited and studied.

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The theory of distributions in Riemaniann spaces due to Lichnerowicz is used to obtain exact solutions to the Einstein equations for spacetimes that have null Riemann-Christoffel curvature tensor everywhere except on a hypersurface. The cases of spherically, cylindrically, plane and axially symmetric spacetimes in which the matter content of the singular surfaces can be described by a barotropic equation of state are treated in some detail. Solutions with null curvature tensor except on a) concentric spheres, b) concentric cylinders, c) parallel planes, and d) parallel discs, are exhibited and studied.

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1. INTRODUCTION

The prediction of the appearance of structures by theories of the early universe based in the spontaneous symmetry breaking of some unifying group has produced a great interest because of the cosmological as well as astrophysical implications[1]. In particular, cosmic strings are produced in the breaking of an $U(1)$ symmetry, and are good candidates to seed the formation of galaxies[2]. Cosmic walls are associated to the breaking of a discrete symmetry, and may decay later forming cosmic strings[3].

Geometrically, cosmic strings and domain walls are characterized by spacetimes with metrics that have null Riemann-Christoffel curvature tensor everywhere except on the lines that represent the strings and on the planes of the walls[1]. In other words, they are characterized by curvature tensors that are proportional to distributions with support on the defects. If we consider that in the universe there are other kinds of topological defects, we can characterize them in the same way as we do for cosmic strings and domain walls, i.e., by spacetimes having null Riemann-Christoffel curvature tensors everywhere except on the defects[4]. For a universe filled with usual matter and topological defects we shall have a curvature tensor that is not null even outside the defects.

The construction of a theory of distributions in curved spacetimes with support on one and two dimensional submanifolds is rather problematic. Thus, in general, the mathematical description of point particles (monopoles) and strings evolving in curved spacetimes as distributions is not on solid ground. This point has recently been stressed by Geroch and Traschen[5]. On the other hand, distributions in curved spaces with support on three dimensional submanifolds (hypersurfaces) are well defined, and the application of them to General Relativity is due to the pioneering work of Lichnerowicz[6]. This theory has been used in the study of propagation of different shock waves in curved spacetimes[6,7] and in the study of singularities of other well known models of the Einstein equations[8,9]. Recently, we have studied thin shells interacting with surrounding gravitational and matter fields by using this theory[10,11]. Our starting point is the "generalized" Bianchi identities.

The most used description of cosmic walls[12−15] is based in Israel's theory of thin shells of matter[16]. This theory takes as departure point the study of the extrinsic geometry of the surfaces that describe the shells of matter via Gauss-Codazzi equations. In principle, both approaches give the same results and are complementary[9].

In the present work we use Lichnerowicz theory of distributions to generate metrics that represent topological defects. This approach considers the metric that is discontinuous across the surface that represents the topological defect, but has discontinuous derivatives.

In Sec. 2, we present a summary of the Lichnerowicz theory of distribution in curved spaces[6]. We follow closely a presentation of Taub[9]. Our studies are restricted to solutions of the Einstein equations that present pure topological defects. In Sec. 3 we construct spacetimes that represent topological defects with spherical symmetry. Explicit examples of bubbles that satisfy a barotropic equation of state are given. The metric for a defect formed by several concentrical spherical surfaces is also given and studied. In Secs. 4, 5 and 6 we present similar considerations for cylindrically, plane and axially symmetric topological defects. In Sec. 7 we discuss the found results and study the possibility to have strings and monopoles as limiting cases of cylindrically and spherically symmetric thin shells.

All the notations and conventions to be used in this paper will closely follow the ones adapted in Refs. 10-11. So, some of them will be used without any further explanations.

2. DISTRIBUTION VALUED CURVATURE TENSOR

We shall study spacetimes whose curvature tensors contain Dirac delta functions with supports on submanifolds. The Riemann-Christoffel curvature tensor is linear in the second derivatives of the metric tensor and quadratic in the first derivatives. Hence a spacetime in which the first derivatives of the metric tensor have a finite jump across a submanifold will have Dirac delta functions with support on the submanifold appearing in the curvature tensor. The jump in the first derivative will be described by a Heaviside function which will enter in the curvature tensor quadratically. Fortunately, the product of such distributions is quite tractable[10]. In the present work we deal with three dimensional submanifolds in which the previous described situation has a complete realization. For lower dimensional submanifolds this situation may not be so simpler. Indeed, for points and lines we may end up with situations in which no consistent definition of distribution valued curvature tensor exists.

Following Lichnerowicz[6], we shall assume that there exists a hypersurface $\sum$ in which the metric tensor has a discontinuos derivative. Let $\sum$ be described by the equation

$$\varphi(x) = 0\,, \tag{2.1}$$

and the normal vector

$$\xi_\mu = \frac{\partial\varphi}{\partial x^\mu}. \tag{2.2}$$

We shall assume that the hypersurface $\sum$ divides a region Ω of the spacetime into two parts Ω^+ and Ω^- where $\varphi > 0$ and $\varphi < 0$, respectively.

The tensor g will be assumed to be continuous across $\sum$, i.e.,

$$[g_{\mu\nu}] = 0. \tag{2.3}$$

In the neighborhood of $\sum$ we may write

$$g^\pm_{\mu\nu} = \overset{0}{g}_{\mu\nu} + \varphi g'_{\mu\nu} + \frac{1}{2}\varphi^2 g''^{\pm}_{\mu\nu} + \dots, \tag{2.4}$$

where a prime denotes the partial derivative with respect to φ.

The discontinuities in the first derivative of the metric tensor is characterized by the tensor $b_{\mu\nu}$, defined via the relations[9]

$$[g_{\mu\nu,\,\lambda}] = \xi_\lambda b_{\mu\nu}. \tag{2.5}$$

Combining Eqs. (2.3) and (2.5) with the assumption that $g_{\mu\nu}$ is at least c^3 in regions $\Omega^\pm$, we find that the Riemann tensor takes the form

$$R^\rho_{\sigma\mu\nu} = \delta(\varphi)H^\rho_{\sigma\mu\nu} + (R^\rho_{\sigma\mu\nu})^D + \theta(1-\theta)J^\rho_{\sigma\mu\nu}, \tag{2.6}$$

with

$$2H^\rho_{\sigma\mu\nu} = b^\rho_\nu\xi_\sigma\xi_\mu - b^\rho_\mu\xi_\sigma\xi_\nu - b_{\sigma\nu}\xi^\rho\xi_\mu + b_{\sigma\mu}\xi^\rho\xi_\nu, \tag{2.7}$$

$$J^\rho_{\sigma\mu\nu} = [\Gamma^\tau_{\sigma\mu}]\,[\Gamma^\rho_{\tau\nu}] - [\Gamma^\tau_{\sigma\nu}][\Gamma^\rho_{\tau\mu}]. \tag{2.8}$$

θ denotes the Heaviside function, defined by[†]

$$\theta(\varphi) = \begin{cases} 1, & \varphi > 0\,, \\ \dfrac{1}{2}, & \varphi = 0\,, \\ 0, & \varphi < 0\,, \end{cases} \tag{2.9}$$

and $(R^\rho_{\sigma\mu\nu})^\pm$ are the usual Riemman-Cristoffel tensor defined in $\Omega^\pm$.

Hence,

$$R_{\sigma\mu} = R^\rho_{\sigma\rho\mu} = \delta(\psi)H_{\sigma\mu} + (R_{\sigma\mu})^D + \theta(1-\theta)J_{\sigma\mu}. \tag{2.10}$$

We shall generalize the Einstein equations by assuming that they involve distribution valued curvature and energy-momentum tensors. Thus we assume that the field equations are

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= \delta(\varphi)(H_{\mu\nu} - \frac{1}{2}g_{\mu\nu}H) + G^D_{\mu\nu} + \theta(1-\theta)(J_{\mu\nu} - \frac{1}{2}g_{\mu\nu}J) \\ &= Q_{\mu\nu} \\ &= \delta(\varphi)\Theta_{\mu\nu} + T^D_{\mu\nu} + \theta(1-\theta)T_{\mu\nu}, \end{aligned} \tag{2.11}$$

where all the symbols have their usual meaning and $T^\pm_{\mu\nu}$ is the energy-momentum tensor in $\Omega^\pm$; $\Theta_{\mu\nu}$ and $T_{\mu\nu}$ are the stress-energy tensors associated with the hypersurface Σ.

[†] Note that the definition of the Heaviside function adapted here is slightly different from the one used in Refs. 10-11. However, all the following results are valid for both of them if we just simply replace one by another.

The above equations are equivalent to

$$H_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}H = \Theta_{\mu\nu}\,, \tag{2.12a}$$

$$R^{\pm}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R^{\pm} = T^{\pm}_{\mu\nu}\,, \tag{2.12b}$$

$$J_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}J = \mathcal{T}_{\mu\nu}\,. \tag{2.12c}$$

Note that the last one of these equations in Eq. (2.11) appears as the terms multiplied by the distribution $\theta(1-\theta)$ that vanishes everywhere except on $\sum$. Thus, for finite $J_{\mu\nu}$ and $\mathcal{T}_{\mu\nu}$ the contribution of these terms in Eq. (2.11) is identically zero in the sense of distributions. In other words, finite $J_{\mu\nu\lambda\sigma}$ and $\mathcal{T}_{\mu\nu}$ do not contribute significantly to the curvature $R_{\mu\nu\lambda\sigma}$ and to the energy-momentum tensor $Q_{\mu\nu}$, respectively. We shall refer to these tensors as residual parts of the curvature and energy-momentum tensor.

As mentioned previously, our main purpose is to study topological defects, i.e., spacetimes that have null Riemann-Christoffel tensor everywhere except on the defects. If we assume that the submanifold $\sum$ is associated to a topological defect we have that

$$R^{\pm}_{\mu\nu\rho\sigma} = 0. \tag{2.13}$$

Hence, in this case, Eq. (2.12b) are satisfied identically.

To construct spacetimes that represent topological defects we may use Eq. (2.13) as starting point. Indeed, Eq. (2.13) are satisfied identically for spacetimes that are generated by coordinates transformations of the Minkowski metric

$$ds^2 = \eta_{\mu\nu}dZ^{\mu}dZ^{\nu}. \tag{2.14}$$

We shall assume that in the neighborhood of $\sum$ we have the coordinate transformation

$$Z^{\pm}_{\mu} = Z^{\pm}_{\mu}(x) = Z^{0}_{\mu} + \varphi(x)Z'^{\pm}_{\mu} + \frac{1}{2}\varphi(x)^2 Z''^{\pm}_{\mu} + \ldots, \tag{2.15}$$

such that the metric

$$g_{\alpha\beta}(x) = \eta_{\mu\nu}\frac{\partial Z^{\mu}}{\partial x^{\alpha}}\frac{\partial Z^{\nu}}{\partial x^{\beta}} \tag{2.16}$$

be continuous across $\sum$, which is provided by

$$\eta^{\mu\nu}(\partial_{\alpha}Z^{0}_{\mu})\,[Z'_{\nu}] = 0, \tag{2.17}$$

$$\eta^{\mu\nu}[Z'_{\mu}Z'_{\nu}] = 0. \tag{2.18}$$

The tensors $H_{\mu\nu\rho\sigma}$ and $J_{\mu\nu\rho\sigma}$ depend only on the direction ξ^{α} and on the tensor $b_{\mu\nu}$ that describes the jump on the derivative of the metric tensor. The coordinate transformation (2.15) must also be restricted by the condition that $b_{\mu\nu}$ be finite in order to have well defined tensor distributions.

The equations that govern the evolution of topological defects in a curved spacetime are Eqs. (2.12). In principle, using these equations we can find the metric that describes, let say, a spherical topological defect evolving in a Friedman-Robertson-Walker (FRW) universe. The evolution of a spherical domain wall in a fixed FRW metric is studied in Ref. 17.

3. WALLS WITH SPHERICAL SYMMETRY

In this section we shall use the formalism described in the preceding section to construct solutions to the Einstein equations that represent spherically symmetric topological defects. There is a variety of known solutions to the Einstein equations that represents spherical surfaces (bubbles) evolving in different spaces, in general, however they are not pure topological defects[13–17]. For instance, in the simplest bubble studied in Ref. 13 we have that the exterior space of the wall is represented by the Schwarzschild metric. In consequence, this bubble does not classify as a pure topological defect.

In order to end up with a spacetime with spherical symmetry we shall perform in Eq. (2.14) the transformation of coordinates

$$\begin{aligned} Z_0^{\pm} &= T_{\pm}(t,\, r,\, \varphi), \\ Z_1^{\pm} &= R_{\pm}(t,\, r,\, \varphi)\sin\theta\cos\phi, \\ Z_2^{\pm} &= R_{\pm}(t,\, r,\, \varphi)\sin\theta\sin\phi, \\ Z_3^{\pm} &= R_{\pm}(t,\, r,\, \varphi)\cos\theta, \end{aligned} \tag{3.1}$$

where φ is a function of t and r; T and R are functions of the indicate arguments such that when $\varphi = 0$ we have

$$\begin{aligned} T_{\pm}(t,\, r,\, 0) &= T(t,\, r), \\ R_{\pm}(t,\, r,\, 0) &= R(t,\, r). \end{aligned} \tag{3.2}$$

The conditions to have a continuous metric across $\varphi = 0$ [Cf. Eqs. (2.17) and (2.18)] reduce, in this case, to

$$\begin{aligned} [T']\partial_t T &= [R']\partial_t R, \\ [T']\partial_t T &= [R']\partial_r R, \end{aligned} \tag{3.3}$$

and

$$T_+'^2 + R_+'^2 = T_-'^2 + R_-'^2, \tag{3.4}$$

respectively. Note that Eq. (3.4) needs valid only on the surface $\varphi = 0$. The relations (3.3) tell us that $T = T(\sigma)$ and $R = R(\sigma)$, where σ is a function of t and r. Equation

(3.4) is fulfilled if we take

$$T_+ = [U(\sigma - \varphi) + V(\sigma + \varphi)]/2,$$

$$R_+ = [M(\sigma - \varphi) + N(\sigma + \varphi)]/2, \quad (3.5a)$$

and

$$T_- = [U(\sigma - \varphi) + V(\sigma - \varphi)]/2,$$

$$R_- = [M(\sigma + \varphi) + N(\sigma + \varphi)]/2, \quad (3.5b)$$

where U, V, M and N are arbitrary functions of the indicated arguments. for simplicity sake, in the following, we shall choose M and N such that $M = U$, and $N = -V$. Then, the metric that represents the spacetime with the singular surface $\varphi = 0$ is

$$ds^2_\pm = F^\pm dt^2 + 2G^\pm dtdr - H^\pm dr^2 - h^\pm(d^2\theta + \sin^2 Qdp^2), \quad (3.6)$$

where

$$F^\pm = \dot{U}_\pm \dot{V}_\mp(\sigma^2_{,t} - \varphi^2_{,t}),\ G^\pm = \dot{U}_\pm \dot{V}_\mp(\sigma_{,t}\sigma_{,r} - \varphi_{,t}\varphi_{,r}),$$
$$H^\pm = \dot{U}_\pm \dot{V}_\mp(\varphi^2_{,r} - \sigma^2_{,r}),\ h^\pm = [(U_\pm - V_\mp)/2]^2, \quad (3.7)$$

the dots indicate derivations with respect to the arguments and $U_\pm = U(\sigma \mp \varphi)$, etc. The spacetime with the index $+(-)$ represents the spacetime limited by the side $\varphi > 0 (\varphi < 0)$ of the surface $\varphi = 0$. In particular, we shall be interested in the surfaces described by $\varphi = \varphi(r)$. For these particular cases it is convenient to set $\sigma = t$ and $\varphi(r)_{,r} = f$ in Eq. (3.7). We get

$$F^\pm = H^\pm/f^2 = \dot{U}_\pm \dot{V}_\mp,\ G^\pm = 0, \quad (3.8)$$

while $h^\pm$ still takes the form given by Eq. (3.7).

The different tensors associated with the surface $\varphi = 0$ can be easily computed. The normal to the surface is

$$\xi_\lambda = (0,\ f,\ 0,\ 0), \quad (3.9)$$

where the tensor indices ($\lambda = 0,\ 1,\ 2,\ 3$) refer to the coordinates $\{t,\ r,\ \theta,\ \phi\}$. From Eqs. (2.5), (3.8) and (3.9), we find

$$b_{11} = -f^2 b_{00} = f^2 W_1,\ b_{33} = \sin^2\theta b_{22} = \sin^2\theta W_2, \quad (3.10)$$

where

$$W_1 = 2(\dot{U}\ddot{V} - \ddot{U}\dot{V}),$$
$$W_2 = (U - V)(\dot{U} + \dot{V}). \quad (3.11)$$

From Eqs. (3.9)-(3.11) and (2.8), we find that the non-vanishing components of the tensor $J_{\mu\nu\lambda\sigma}$ are given by

$$J_{0202} = J_{0303}/\sin^2\theta = W_1 W_2/(4F),$$
$$J_{1212} = J_{1313}/\sin^2\theta = -f^2[W_1 W_2/(4F) - W_2^2/(4h)],$$
$$J_{2323} = -(W_2 \sin\theta)^2/(4F), \tag{3.12}$$

Where the function F and h are the metric functions on the surface $\varphi = 0$.

Similarly, the non null components of the tensor $H_{\mu\nu\lambda\sigma}$ are

$$H_{0101} = f^2 W_1/2,$$
$$H_{1212} = H_{1313}/\sin^2\theta = -f^2 W_2/2. \tag{3.13}$$

From Eqs. (3.9)-(3.13) and (2.12) we get that the energy-momentum tensor associated to the surface $\varphi = 0$ is

$$Q_{\mu\nu} = \delta(\varphi)\Theta_{\mu\nu} + \theta(1-\theta)\mathcal{T}_{\mu\nu}, \tag{3.14}$$

where

$$\Theta_{\mu\nu} = \widehat{\rho} U_\mu U_\nu - \widehat{p}(\theta_\mu\theta_\nu + \phi_\mu\phi_\nu), \tag{3.15}$$
$$\mathcal{T}_{\mu\nu} = \varepsilon U_\mu U_\nu - \pi_\perp R_\mu R_\nu - \pi(\theta_\mu\theta_\nu + \phi_\mu\phi_\nu). \tag{3.16}$$

U_μ, R_μ, θ_μ, and ϕ_μ are the orthonormal vierbein

$$U_\mu = (\sqrt{F},\ 0,\ 0,\ 0), \quad R_\mu = (0,\ -f\sqrt{F},\ 0,\ 0),$$
$$\theta_\mu = (0,\ 0,\ -\sqrt{h},\ 0), \quad \phi_\mu = (0,\ 0,\ 0,\ -\sin\theta\sqrt{h}). \tag{3.17}$$

And $\widehat{\rho}$, $\widehat{p}$, ε, $\pi_\perp$ and π, are the scalars

$$\widehat{\rho} = 4(\dot{U} + \dot{V})/[\dot{U}\dot{V}(U - V)],$$
$$\widehat{p} = \frac{1}{\dot{U}\dot{V}}\left[\frac{2(\dot{U} + \dot{V})}{U - V} + \frac{\ddot{U}}{\dot{U}} - \frac{\ddot{V}}{\dot{V}}\right], \tag{3.18}$$

$$\varepsilon = -\pi = -\frac{4(\dot{U} + \dot{V})}{\dot{U}\dot{V}(U - V)^2}\left[\dot{U} + \dot{V} - \frac{\ddot{U}}{\dot{U}} + \frac{\ddot{V}}{\dot{V}}\right],$$
$$\pi_\perp = \frac{4(\dot{U} + \dot{V})}{\dot{U}\dot{V}(U - V)^2}\left[\dot{U} + \dot{V} + \frac{\ddot{U}}{\dot{U}} - \frac{\ddot{V}}{\dot{V}}\right]. \tag{3.19}$$

Note that Eq. (3.18) can also be obtained directly from Eq. (24) in Ref. 11, by noticing the difference in the definition of the normal vector ξ_μ.

From Eqs. (3.15)-(3.17) we have that the density and the tensions associated to the spherical surface $\varphi = 0$ are

$$\rho = \widehat{\rho}\delta(\psi) + v(1-v)\varepsilon,$$
$$p_\perp = \theta(1-\theta)\pi_\perp,$$
$$p = \widehat{p}\delta(\psi) + \theta(1-\theta)\pi. \tag{3.20}$$

The density ρ is formed by a Dirac delta type of distribution with support on the spherical surface and a part that vanishes everywhere except on $\varphi = 0$. Since this last part has a vanishing integral it does not contributes to the associated mass of the spherical wall. In principle, we can ignore this residual part. $p_\perp$ and p represent the radial tension and the surface tension. As in the case of the density, the parts that appear in the tensions containing the distribution $\theta(1-\theta)$ can be ignored. Hence, we have only tensions that are parallel to the surface.

To be more specific, let us consider the hypersurface defined by

$$\varphi = \prod_{i=1}^{n}(r - a_i) = 0, \tag{3.21}$$

where the non-negative constants will be chosen such that $a_1 < a_2 < \ldots < a_n$. Then, we have

$$\delta(\varphi) = \sum_{k=1}^{n} \delta(r - a_k)/|\prod_{i\neq k}^{n}(a_k - a_i)|. \tag{3.22}$$

It follows that the hypersurface defined by Eq. (3.21) represents n concentrical spheres of constant radii a_i.

The density and the tension now read

$$\rho = \sum_{k=1}^{n}\widehat{\rho}_k\delta(r - a_k), \qquad p = \sum_{k=1}^{n}\widehat{p}_k\delta(r - a_k), \tag{3.23a}$$

$$\widehat{\rho}_k = \widehat{\rho}/|\prod_{i\neq k}^{n}(a_k - a_i)|, \qquad \widehat{p}_k = \widehat{p}/|\prod_{i\neq k}^{n}(a_k - a_i)|. \tag{3.23b}$$

The functions $\widehat{\rho}$ and $\widehat{p}$ appearing in Eq. (3.23a) are the same functions as defined in Eq. (3.18). In Eq. (3.23a) we have omitted the part containing the distribution θ, practice that we shall adopt from now on. Thus, the density and the tension given by Eq. (3.23a) have support on n concentrical spherical surfaces of radii $a_1, \ldots, a_n$. Note that the actual values of these quantities on a given surface depend on the relative distances of the other spherical surfaces. The function f appearing in the metric now is

$$f = \sum_{k=1}^{n}\frac{\prod_{i=1}^{n}(r - a_i)}{(r - a_k)}. \tag{3.24}$$

We shall study the particular cases of bubbles with "barotropic" equation of state

$$p = \gamma \rho, \tag{3.25}$$

where $|\gamma| \leq 1$. From Eq. (3.18) we get

$$2(1-2\gamma)\frac{\dot{U}+\dot{V}}{U-V} + \frac{\ddot{U}}{\dot{U}} - \frac{\ddot{V}}{\dot{V}} = 0. \tag{3.26}$$

Assuming that $V = V(U)$, then the above equation reduces to

$$2(1-2\gamma)\,\frac{1+V'}{U-V} - \frac{V''}{V'} = 0, \tag{3.27}$$

where the primes now indicate derivations with respect to U.

A. Bubbles formed by Cosmic strings

When $\gamma = \frac{1}{2}$, from Eq. (3.27) we find

$$V = AU + B, \quad (A > 0), \tag{3.28}$$

where A and B are integration constants. Bubbles with $\gamma = 1/2$ have zero effective Newtonian mass, as cosmic strings. Thus, we can consider these bubbles as formed by cosmic strings.

The combination of Eqs. (3.2), (3.5) and (3.28) yields

$$R = U_0(T - \frac{1}{2}B) - \frac{1}{2}B, \tag{3.29}$$

Where $U_0 \equiv (1-A)/(1+A)$. It follows that all the bubbles belonging to this category move with a constant velocity U_0. Explicit examples of such bubbles are obtained by choosing the function U as

$$U(t) = \alpha t^m, \tag{3.30}$$

Where α and m are arbitrary constants. Then, from Eq. (3.8) we find that the non-vanishing metric coefficients now read

$$F = H/f^2 = A\alpha^2 m^2 (t^2 - \varphi^2)^{m-1},$$
$$h = \frac{1}{4}[\alpha(t-|\varphi|)^m - A\alpha(t+|\varphi|)^m - B]^2. \tag{3.31}$$

Inserting Eqs. (3.28) and (3.30), on the other hand, into Eq. (3.18), we obtain

$$\hat{\rho} = 2\hat{p} = \frac{4(1+A)}{A\alpha m t^{m-1}[\alpha(1-A)t^m - B]}. \tag{3.32}$$

To study this class of solutions, it is sufficient to consider only the following representative cases.

Case A-1): $n = m = \alpha = A = 1$, and $B = -8/\rho_0$: In this case we have

$$ds^2 = dt^2 - dr^2 - (|r - r_0| - 4/\rho_0)^2 \left(d^2\theta + \sin^2\theta\phi p^2\right), \tag{3.33}$$

and

$$\hat{\rho} = \rho_0, \quad \delta(\varphi) = \delta(r - r_0), \tag{3.34}$$

where $r_0 \equiv a_1$, and ρ_0 is a positive constant. Therefore, this metric represents a static bubble with center in the origim of the system of the coordinates, radius r_0 [in the Minkowski coordinates the radius is $R = 4/\rho_0$, see Eq. (3.29)], and constant density ρ_0.

Case A-2): $m = \alpha = A = 1$, $n = 1$, and $B = -8/\rho_0$. Then we find

$$ds^2 = dt^2 - 4(r - r_m)^2 dr^2 - [|(r - a_1)(r - a_2)| - 4/\rho_0]^2 [d^2\theta + \sin^2\theta d\phi^2], \tag{3.35}$$

and

$$\rho = 2p = \frac{\rho_0}{a_2 - a_1}[\delta(r - a_1) + \delta(r - a_2)], \tag{3.36}$$

Where $r_m \equiv (a_2 + a_1)/2$. Thus, in the present case the metric (3.34) represents two concentrical static surfaces with equation of state $p = \rho/2$.

Case A-3): $n = 1$, $\alpha = a^{-1}$, $A = a/b$, and $B = 0$: Then, we find

$$ds^2 = \frac{m^2}{ab}[t^2 - (r - r_0)^2]^{m-1}(dt^2 - dr^2) - \frac{1}{4}[(t - |r - r_0|)^m/a - (t + |r - r_0|)^m/b]^2 (d^2\theta + \sin^2\theta dp^2), \tag{3.37}$$

and

$$\hat{\rho} = 2\hat{p} = \frac{4ab(a + b)}{m(b - a)} t^{1-2m}, \quad \delta(\varphi) = \delta(r - r_0). \tag{3.38}$$

It follows that this solution represents a single bubble with its surface density given by Eq. (3.37), and moving with a constant velocity $U_0 = (b - a)/(a + b)$ in the Minkowski system of coordinates.

B. Cosmic domain wall

In this subsection, we shall consider the cases where $\gamma = 1$, i.e., bubbles with the equation of state of cosmic domain walls. For $\gamma = 1$, Eq. (3.27) has the solution

$$V = -\frac{A}{U}, \tag{3.39}$$

and the corresponding surface energy density and tension are

$$\hat{\rho} = \hat{p} = \frac{4U}{A\dot{U}}. \tag{3.40}$$

Inserting Eq. (3.39) into Eqs. (3.2) and (3.5), on the other hand, we find

$$R^2 - T^2 = A, \tag{3.41}$$

which means that all the bubbles with $\gamma = 1$ have radii which grow with constant accelerations. The simplest bubble with constant density $\hat{\rho} = \hat{p} = \rho_0$ is obtained with the choice

$$U(t) = \frac{4}{\rho_0} \ \mathrm{Exp}\ \{\rho_0 t/4\}, \quad A = (4/\rho_0)^2. \tag{3.42}$$

Then, the corresponding metric takes the form

$$ds^2 = \ \mathrm{Exp}\ [-\rho_0|r - r_0|/2]\ \{(dt^2 - dr^2) - (4/\rho_0)^2 \cosh^2(\rho_0 t/4)\ (d^2\theta + \sin^2\theta dp^2)\}. \tag{3.43}$$

In Refs. 17 and 18 solutions representing membrances with the same equations of state as cosmic domain walls were also studied.

4. WALLS WITH CYLINDRICAL SYMMETRY

Walls with cylindrical symmetry will be obtained in a similar way as that in the spherically symmetric case. We shall perform in Eq. (2.14) the transformation of coordinates,

$$\begin{aligned} Z_0^{\pm} &= T_{\pm}(t,\ r,\ \varphi), \\ Z_1^{\pm} &= R_{\pm}(t,\ r,\ \varphi)\cos\phi, \\ Z_2^{\pm} &= R_{\pm}(t,\ r,\ \varphi)\sin\phi, \\ Z_3^{\pm} &= z, \end{aligned} \tag{4.1}$$

where φ is a function of t and r. Note that in this case r indicates a cylindrical coordinate. T and R are functions given by the relations (3.5).

Thus, the metric that represents the spacetime with the singular surface $\varphi = 0$ is

$$ds^2_{\pm} = F^{\pm}dt^2 + 2G^{\pm}dtdr - H^{\pm}dr^2 - h^{\pm}d^2\phi - dz^2, \tag{4.2}$$

where the functions $F^{\pm}$, $G^{\pm}$, $H^{\pm}$, and $h^{\pm}$ are given by Eq. (3.7).

As before, we shall be particularly interested in the surfaces described by Eq. (3.21), which now represents n concentrical cylinders of radii a_i, centered on the $z-$axis. It is also convenient to set $\sigma = t$ and $f = \varphi_{,r}$ in Eqs. (4.2) and (3.7). Then, the metric coefficients reduce to the one given by Eq. (3.8).

The normal vector to the hypersurface $\varphi = 0$ now takes exactly the form given by Eq. (3.9), but with the coordinates being numbered as $\{x^{\mu}\} \equiv \{t,\ r,\ \phi,\ z\}$, $(\mu = 0,\ 1,\ 2,\ 3)$.

Following a similar procedure as we did in the last section,we find that the energy-momentum tensor is given by

$$Q_{\mu\nu} = \Theta_{\mu\nu}\delta(\varphi) = \{\widehat{\rho}U_\mu U_\nu - \widehat{p}_\phi \phi_\mu \phi_\nu - \widehat{p}_2 Z_p Z_\nu\}\delta(\varphi), \tag{4.3}$$

where U_μ, R_μ, ϕ_μ and Z_μ form an orthonormal vierbein, defined by

$$U_\mu = \sqrt{F}\delta^0_\mu,\ R_\mu = -f\sqrt{F}\delta^1_\mu,\ \phi_\mu = -\sqrt{h}\delta^2_\mu,\ \ Z_\mu = \delta^3_\mu, \tag{4.4}$$

and $\widehat{\rho}$, $\widehat{p}_\phi$, and $\widehat{p}_z$ are the surface energy density and tensions, given, respectively, by

$$\widehat{\rho} = 2(\dot{U} + \dot{V})/[\dot{U}\dot{V}(U - V)],$$

$$\widehat{p}_\phi = (\ddot{U}\dot{V} - \dot{U}\ddot{V})/(\dot{U}\dot{V})^2,$$

$$\widehat{p}_z = \widehat{\rho} + \widehat{p}_\phi. \tag{4.5}$$

In writing Eq. (4.3) we had omitted the residual part of the energy-momentum tensor, $\mathcal{T}_{\mu\nu}$, for the same reasons as explained in the last section. In this case we have that the tensions on the directions ∂_z and ∂_ϕ can be different.

For the functions φ defined by Eq. (3.21) we have that the density is given by Eq. (3.23) with the function $\widehat{\rho}$ defined in Eq. (4.5). The tensions now read

$$p_z = \sum_{k=1}^n \widehat{p}_{zk}\delta(r - a_k),\quad p_\phi = \sum_{k=1}^n \widehat{p}_{\phi k}\delta(r - a_k), \tag{4.6a}$$

$$\widehat{p}_{zk} = \widehat{p}_z/|\prod_{i\neq k}^n (a_k - a_i)|,\quad \widehat{p}_{\phi k} = \widehat{p}_\phi/|\prod_{i\neq k}^n (a_k - a_i)|. \tag{4.6b}$$

The functions $\widehat{p}_z$ and $\widehat{p}_\phi$ appearing in Eq. (4.6a) are the same functions as defined in Eq. (4.5). The density, as well as the tensions, have support on n concentrical cilindrical surfaces of radii $a_1, \ldots, a_n$. Again, as in the spherically symmetric case, we have that the actual values of these quantities on a given surface depend on the relative distances of the other cylindrical surfaces.

If we consider p_z and p_ϕ positive, in other words tensions, we have that Eq. (4.5) implies $\rho \leq p_z$. Thus, unless $p_\phi = 0$, we will have superluminal propagation of sound waves along the ∂_z direction. When $p_\phi = 0$ we have a cylindrical surface formed by cosmic strings aligned along the z-axis, with the equation of state $p_z = \rho$. If we allow negative values for p_z and p_ϕ, i.e., anisotropic pressures, we can have $p_\phi \neq 0$. But, thin shells of matter with pressure are in general unstable.

Particular cases of shells of cosmic strings are obtained by considering the functions, U and V related by

$$V = AU + B. \tag{4.7}$$

Then, we have

$$\hat{\rho} = \hat{p}_z = \frac{2(1+A)}{A\dot{U}[(1-A)U-B]}, \quad \hat{p}_\phi = 0. \tag{4.8}$$

It can be shown that in the present case Eq. (3.29) still holds. That is, all the cylindrical walls formed by cosmic strings aligned along the z-axis move with constant velocities, similar to spherically-symmetric bubbles formed by cosmic strings discussed in the proceeding section.

Corresponding to the choice of the function U of Eq. (3.36), we find that the metric takes the form

$$ds^2 = A\alpha^2 m^2 (t^2 - \varphi^2)^{m-1}(dt^2 - f^2 dr^2) - \frac{1}{4}[\alpha(t-|\varphi|)^m - A\alpha(t+|\varphi|)^m - B]^2 d\phi^2 - dz^2, \tag{4.9}$$

while Eq. (4.8) becomes

$$\hat{\rho} = \hat{p}_z = \frac{2(1+A)}{A\alpha m t^{m-1}[\alpha(1-A)t^m - B]}, \quad \hat{p}_\phi = 0, \tag{4.10}$$

where φ and f are given by Eqs. (3.24).

When $n = m = \alpha = A$, and $B = -4/\rho_o$, Eq (4.9) reads

$$ds^2 = dt^2 - dr^2 - (|r - r_0| - 2/\rho_0)^2 dp^2 - dz^2, \tag{4.11}$$

and Eq. (4.10) simply gives $\hat{\rho} = \hat{\rho}_z = \rho_0$, and $\hat{p}_\phi = 0$. Thus, it is concluded that the metric (4.11) represents a static cylindrically-symmetric shell of cosmic strings, with its radius r_0 and constant energy density ρ_0.

When $m = \alpha = A = 1$, $m = 2$, and $B = -4/\rho_0$, we find that Eqs. (4.9) and (4.10) give the following results

$$ds^2 = dt^2 - 4(r - r_m)^2 dr^2 - [|(r-a_1)(r-a_2)| - 2/\rho_0]^2 d\phi^2 - dz^2, \tag{4.12}$$

and

$$\rho = p_z = \frac{\rho_0}{a_2 - a_1}[\delta(r-a_1) + \delta(r-a_2)], \; p_\phi = 0. \tag{4.13}$$

Obviously, this solution represents two cylindrical walls of radii a_1 and a_2, centered on the z-axis.

When $\alpha = a^{-1}$, $A = a/b$, $n = 1$, and $B = 0$, we have

$$ds^2 = \frac{m^2}{ab}[t^2 - (r-r_0)^2]^{m-1}(dt^2 - dr^2) - \frac{1}{4}[(t - |r - r_0|)^m/a - (t + |r - r_0|)^m/b]^2 d\phi^2 - dz^2, \tag{4.14}$$

$$\hat{\rho} = \hat{p}_z = \frac{2ab(a+b)}{m(b-a)} t^{1-2m}, \quad \text{and} \quad \hat{p}_\phi = 0, \tag{4.15}$$

which corresponds to Case A-3) discussed in the last section, and represents a cylindrical thin wall moving with a constant velocity in the Minkowski system of coordinates.

In parallel to sect. II, we can also consider the choice of the function U and V related of each other by Eq. (3.38). Clearly, this corresponds to cylindrical shells with equation of state given by

$$\hat{\rho} = \hat{p}_\phi = \frac{2U}{A\dot{U}}, \; \hat{p}_z = 2\hat{\rho}. \tag{4.16}$$

In particular, corresponding to Eq. (3.41), we have

$$ds^2 = e^{-\rho_0|r-r_0|}\{dt^2 - dr^2 - (2/\rho_0)^2\cosh^2(\rho_0 t/2)d\phi^2\} - dz^2, \tag{4.17}$$

$$\hat{\rho} = \hat{p}_\phi = \frac{1}{2}\hat{p}_z = \rho_0, \tag{4.18}$$

which represents a single cylindrical shell with constant surface energy density and tensions, and moving with constant acceleration [Cf. Eq. (3.40)].

The relation of the solutions presented in this section with the metrics that represent single cosmic strings will be studied in the last section of this work.

The generation of cylindrical shells of matter via principal sigma models can be found in Ref. 19. Multiple cylindrical walls can be associated to multiple soliton solutions.

5. PLANE SYMMETRIC WALLS

In order to end up with a spacetime with plane symmetry we shall perform in Eq. (2.14) the transformation of coordinates

$$\begin{aligned} Z_0^{\pm} &= 2U_{\mp} + (x^2 + y^2 + 1/4)\exp(V_{\pm}/2), \\ Z_1^{\pm} &= 2U_{\mp} + (x^2 + y^2 - 1/4)\exp(V_{\pm}/2), \\ Z_2^{\pm} &= x\exp(V_{\pm}/2), \\ Z_3^{\pm} &= y\exp(V_{\pm}/2), \end{aligned} \tag{5.1}$$

where $U_{\pm}$, $V_{\pm}$ are defined as before, and σ and φ are functions of t and z. Thus, the metric that represents the spacetime with the singular surface $\varphi = 0$ is

$$ds^2 \pm = F^{\pm}dt^2 + 2G^{\pm}dtdz - H^{\pm}dt^2 - h^{\pm}(dx^2 + dy^2). \tag{5.2}$$

where

$$\begin{aligned} F^{\pm} &= \dot{V}_{\pm}\dot{U}_{\mp}\exp(V_{\pm}/2)(\sigma_{,t}^2 - \varphi_{,t}^2), \; G^{\pm} = \dot{V}_{\pm}\dot{U}_{\mp}\exp(V_{\pm}/2)(\sigma_{,t}\sigma_{,z} - \varphi_{,t}\varphi_{,z}), \\ H^{\pm} &= \dot{V}_{\pm}\dot{U}_{\mp}\exp(V_{\pm}/2)(\varphi_{,z}^2 - \sigma_{,z}^2), \; h^{\pm} = \exp(V_{\pm}), \end{aligned} \tag{5.3}$$

and, as usual, the dots indicate derivation with respect to the argument. Note that Eq. (5.2) is the metric studied by Taub[20] in the coordinates $\sigma = \sigma(t,\ z)$ and $\varphi = \varphi(t,\ z)$.

As before, we shall be interested in the spacetimes with singular surfaces described by $\varphi = \varphi(z)$. For these partical case it is also convenient to set $\sigma = t$ and $\varphi(z)_{,z} = f$ in Eq. (5.3). Then we have

$$F^{\pm} = H^{\pm}/f^2 = \dot{V}_{\pm}\dot{U}_{\mp}\exp(V_{\pm}/2),\ G^{\pm} = 0,\ h^{\pm} = \exp(V_{\pm}). \tag{5.4}$$

It is easy to show that corresponding to Eq. (5.4) the energy-momentum tensor is given by

$$Q_{\mu\nu} = \Theta_{\mu\nu}\delta(\varphi) = \{\hat{\rho}U_\mu U_\nu - \hat{p}(X_\mu X_\nu + Y_\mu Y_\nu)\}\delta(\varphi), \tag{5.5}$$

where U_μ, Z_μ, X_μ, and Y_μ are the orthonormal vierbein, given by

$$U_\mu = \sqrt{F}\delta^0_\mu,\ Z_\mu = f\sqrt{F}\delta^1_\mu,\ \ X_\mu\sqrt{h}\delta^2_\mu,\ \ Y_\mu = \sqrt{h}\delta^3_\mu, \tag{5.6}$$

with the coordinates being numbered as $\{x^\mu\} = \{t,\ z,\ x,\ y\}$, and $\hat{\rho}$ and $\hat{p}$ are the surface energy density and tensor given by

$$\begin{aligned} \hat{\rho} &= \frac{2}{\dot{U}}e^{-V/2}, \\ \hat{p} &= \frac{e^{-V/2}}{\dot{U}^2\dot{V}^2}[\dot{U}\ddot{V} - \dot{V}\ddot{U} + \frac{3}{2}\dot{U}\dot{V}^2]. \end{aligned} \tag{5.7}$$

As before, we had omitted the residual part in writing Eq. (5.5). Corresponding to Eq. (3.21), we may choose the function φ as

$$\varphi = \prod_{i=1}^{n}(z - a_i), \tag{5.8}$$

which obviously represents n parallel planes to the one with $z = 0$, and intersects the z-axis at $z = \{a_i\}$. Then, the effective energy density and tension are given by

$$\rho = \sum_{k=1}^{n}\hat{\rho}_k\delta(z - a_k)\ ,\quad p = \sum_{k=1}^{n}\hat{p}_k\delta(z - a_k), \tag{5.9}$$

where $\hat{\rho}_k$ and $\hat{p}_k$ are the same functions as defined in Eq. (3.23b) but with $\hat{\rho}$ and $\hat{p}$ now given by Eq. (5.7).

We shall study the particular cases of plane walls with "barotropic" equation of state (3.25). From Eq. (5.6) we get

$$\frac{\ddot{U}}{\dot{U}} = (\frac{3}{2} - 2\gamma)\dot{V} + \frac{\ddot{V}}{\dot{V}}. \tag{5.10}$$

Hence,

$$ln\ \dot{U} = ln\ \dot{V} + (\frac{3}{2} - 2\gamma)V + ln\ C_1, \tag{5.11}$$

where C_1 is an integration constant. The metric coefficients in this case read

$$F = C_1\dot{V}(t+|\varphi|)\dot{V}(t-|\varphi|)\exp\{[V(t-|\varphi|)+V(t+|\varphi|)]/2$$
$$+(1-2\gamma)V(t+|\varphi|)\},$$
$$h = \exp V(t-|\varphi|). \tag{5.12}$$

The usual cosmic domain wall is obtained by choosing $\gamma = 1$. When $\varphi = z$, the above solutions reduce to the ones first studied by Ipser in Ref. 13.

Inserting Eq. (5.11), on the other hand, into Eq. (5.7) we find

$$\hat{\rho} = \gamma^{-1}\hat{p} = \frac{2}{C_1\dot{V}}\exp\{2(\gamma-1)V\}. \tag{5.13}$$

From the above equation we can see that plane walls with constant density are given by

$$V(t) = \begin{cases} At + B, & \gamma = 1, \\ \\ \dfrac{1}{2(1-\gamma)}\ln t + V_0, & |\gamma| < 1, \end{cases} \tag{5.14}$$

where A, B and V_0 are integration constants.

Vilenkin's planar domain wall[21] corresponds to

$$A = \rho_0/2, \;\; B = 0, \;\; C_1 = (2/\rho_0)^2, \;\; \varphi = z. \tag{5.15}$$

The simplest case of multi walls is provided by two parallel walls located at $z = h_0$ and $z = -h_0$, i.e., $a_1 = h_0$ and $a_2 = -h_0$. For the case of domain walls we find

$$ds^2 = \exp(-\rho_0|z^2 - h_0^2|/2)[dt^2 - z^2dz^2 - \exp(\rho_0 t/2)(dx^2 + dy^2)]. \tag{5.16}$$

The density and the pressure in this case are

$$\rho = p = (\rho_0/2h_0)[\delta(r-h_0) + \delta(r+h_0)]. \tag{5.17}$$

This particular solution is studied in Ref. 22 wherein is derived using a slightly different method.

For two planes with equation of state $p = \gamma\rho$, $|\gamma| < 1$ and $\hat{\rho} = \rho_0$ we find that ρ is given by Eq. (5.17) and the metric functions are

$$F = (\beta/\rho_0)[t+|\varphi|)]^{-\frac{1}{4}\beta}[t-|\varphi|)]^{\frac{1}{4}\beta - 1},$$
$$h = [t-|\varphi|)]^{\frac{1}{2}\beta}, \tag{5.18}$$

where $\beta = 1/(1-\gamma)$, $|\gamma| < 1$, and $\varphi = z^2 - h_0^2$.

The usual domain wall solution is associated to the kink solution of the $\lambda\phi^4$ theory. In principle, multiwall solutions can be associated with multikink solutions. The exact relation between these solutions will be the matter of another paper. The evolution of the plane symmetric topological defects in the Minkowski spacetime can be found in Ref. 13. By using the same method studied in the paper quoted in Ref. 4, the multiwall solutions presented in this section can be generalized by the inclusion of multiple cosmic strings crossing the walls.

6. AXISYMMETRIC DISCS

In this section, let us turn to consider solutions which represent discs with axial symmetry. Following the same vein as we did in the proceeding sections, we first perform the coordinate transformations

$$\begin{aligned} Z_0^\pm &= \sqrt{h^\pm}\exp(P^\pm/2)\sinh t, \quad Z_1^\pm = \sqrt{h^\pm}\exp(P^\pm/2)\cosh t, \\ Z_2^\pm &= \sqrt{h^\pm}\exp(-P^\pm/2)\cos\phi, \quad Z_3^\pm = \sqrt{h^\pm}\exp(-P^\pm/2)\sin\phi, \end{aligned} \tag{6.1}$$

in Eq. (2.14), where $h^\pm$ and $P^\pm$ are functions of z, r and φ, with $\varphi = \varphi(r,\ z)$. Then, it is easy to show that the metric takes the form

$$ds_\pm^2 = -(F^\pm dr^2 + 2G^\pm dr dz + H^\pm dz^2) + h^\pm(e^{P^\pm}dt^2 - e^{-P^\pm}d\phi^2), \tag{6.2}$$

where

$$\begin{aligned} F^\pm &= \frac{1}{2}[h^{-1}\cosh P(h_{,r}^2 + 2h\tanh P h_{,r}P_{,r} + h^2P_{,r}^2)]^\pm, \\ G^\pm &= \frac{1}{2}[h^{-1}\cosh P[h_{,z}h_{,z} + h\tanh P(h_{,r}P_{,z} + h_{,z}P_{,r}) + h^2P_{,r}P_{,z}]\}^\pm, \\ H^\pm &= \frac{1}{2}[h^{-1}\cosh P(h_{,z}^2 + 2h\tanh P h_{,z}P_{,z} + h^2P_{,z}^2)]^\pm. \end{aligned} \tag{6.3}$$

The conditions given by Eq. (2.18) now read

$$[h'^2 + h^2P'^2 + 2h\tanh P h'P'] = 0, \tag{6.4}$$

Where a prime, as before, denotes the partial derivative with respect to φ.

One of the solutions of Eq. (6.4) is

$$\begin{aligned} h &= U(\sigma - |\varphi|) + V(\sigma + |\varphi|), \\ P &= M(\sigma - |\varphi|) + N(\sigma + |\varphi|), \end{aligned} \tag{6.5}$$

where $\sigma = \sigma(r,\ z)$, and U, V, M and N are arbitrary functions, but when Eq. (2.17) is concerned, they must satisfy the following equation on the surface $\varphi = 0$

$$(U'^2 - V'^2) + 2h\ \tanh P(U'M' - V'N') + h^2(M'^2 - N'^2) = 0. \tag{6.6}$$

Obviously, to obtain exact solutions, we need first to solve the above equation. However, because of its non-linearity, this is not an easy task. Therefore, we restrict ourselves only to consider some special cases.

A. When $U = 0 = N$

When $U = 0 = N$, Eq. (6.6) has the solution

$$V(\sigma) = \exp[M(\sigma)]. \tag{6.7}$$

For the cases where $\sigma = r$ and $\varphi = \varphi(z)$, assumption that from now on we shall adapt, the metric takes the form of Eq. (6.2) with the metric coefficients given by

$$\begin{aligned}
F &= \frac{1}{2}e^{M(r+|\varphi|)}\cosh M(r-|\varphi|)[\dot{M}^2(r-|\varphi|) + \dot{M}^2(r+|\varphi|) \\
&\qquad +2\tanh M(r-|\varphi|)\dot{M}(r-|\varphi|)\dot{M}(r+|\varphi|)], \\
G &= -\frac{1}{2}e^{M(r+|\varphi|)}\cosh M(r-|\varphi|)f[\dot{M}^2(r-\varphi) - \dot{M}^2(r+\varphi)], \\
H &= \frac{1}{2}e^{M(r+|\varphi|)}\cosh M(r-|\varphi|)f^2[\dot{M}^2(r-|\varphi|) + \dot{M}^2(r+|\varphi|) \\
&\qquad -2\tanh M(r-|\varphi|)\dot{M}(r-|\varphi|)\dot{M}(r+|\varphi|)], \\
h &= e^{M(r+|\varphi|)},\ \ P = M(r-|\varphi|),\ \ f = \varphi_{,z}\,.
\end{aligned} \tag{6.8}$$

Then, the corresponding energy-momentum tensor is given by

$$Q_{\mu\nu} = \Theta_{\mu\nu}\delta(\varphi) = (\hat{\rho}U_\mu U_\nu - \hat{p}_r R_\mu R_\mu)\delta(\varphi), \tag{6.9}$$

where the surface energy density and tension are given by

$$\hat{\rho} = \hat{p}_r = -\frac{2}{\dot{M}}, \tag{6.10}$$

and the unity vectors U_μ and R_μ are

$$U_\mu = e^M\delta^0_\mu,\ \ R_\mu = \dot{M}e^M\delta^1_\mu, \tag{6.11}$$

with the tensor indices 0, 1, 2, 3 indicating the coordinates t, r, z and ϕ.

If we choose the functions M and φ as

$$M(r) = -\frac{2}{\rho_0} r, \quad \varphi = \prod_{i=1}^{n} (z - a_i), \tag{6.12}$$

we find

$$ds^2 = e^{-4r/\rho_0}[dt^2 - (2/\rho_0)^2 dr^2] - e^{-4|\varphi|/\rho_0}[d\varphi^2 + (2/\rho_0)^2 f^2 dz^2], \tag{6.13}$$

with $\hat{\rho} = \rho_0$ and $\delta(\varphi)$ being given by Eq. (3.22) after r is replaced by z. Thus, the above solution describes n parallel discs with constant energy density and tensions.

Another choice of the function M is

$$M(r) = -\frac{2}{\rho_0 q_0} e^{q_0 r}, \quad (q_0 > 0). \tag{6.14}$$

Then, from Eq. (6.10) we find

$$\hat{\rho} = \hat{p}_r = \rho_0 e^{-q_0 r}. \tag{6.15}$$

It follows that in this case the solution can be approximately considered as representing n parallel discs with finite radii.

Instead of setting U and N equal to zero, one can choose $V = 0 = M$. In the latter case Eq. (6.6) has the solution $U(\sigma) = e^{N(\sigma)}$. Then the corresponding energy-momentum tensor takes exactly the form as given by Eq. (6.9) with

$$\hat{\rho} = \hat{p}_r = \frac{2}{\dot{N}}, \quad U_\mu = e^N \delta^0_\mu, \quad \text{and} \quad R_\mu = \dot{N} e^N \delta^1_\mu. \tag{6.16}$$

B. When $\mathrm{Exp}(P) = [Q + \sqrt{Q^2 + h^2}]/h$

In this case, it is easy to show that Eq. (6.6) reduces to

$$Q^+_{,r} Q^+_{,z} - f(\dot{U} - \dot{V})(\dot{U} + \dot{V}) = 0. \tag{6.17}$$

one of the solutions is given by

$$Q = U(r + |\varphi|) - V(r - |\varphi|). \tag{6.18}$$

Then, the corresponding metric coefficients are given by

$$\begin{aligned} F &= \frac{1}{2}(Q^2 + h^2)^{-\frac{1}{2}} \{[\dot{U}(r - |\varphi|) + \dot{V}(r + |\varphi|)]^2 + [\dot{U}(r + |\varphi|) - \dot{V}(r - |\varphi|)]^2\}, \\ G &= -\frac{f}{2}(Q^2 + h^2)^{-\frac{1}{2}} \{[\dot{U}^2(r - \varphi) - \dot{U}^2(r + \varphi)] + [\dot{V}^2(r - \varphi) - \dot{V}^2(r + \varphi)]\}, \\ H &= \frac{f^2}{2}(Q^2 + h^2)^{-\frac{1}{2}} \{[\dot{U}^2(r - |\varphi|) - \dot{V}(r + |\varphi|)]^2 + \\ &\quad + [\dot{U}(r + |\varphi|) + \dot{V}(r - |\varphi|)]^2\}. \end{aligned} \tag{6.19}$$

And the energy-momentum tensor reads

$$Q_{\mu\nu} = \Theta_{\mu\nu}\delta(\varphi) = (\hat{\rho}U_\mu U_\mu - \hat{p}_r R_\mu R_\nu - \hat{p}_\phi \phi_\mu \phi_\nu)\delta(\varphi), \tag{6.20}$$

where

$$\begin{aligned} U_\mu &= (Q + \sqrt{Q^2 + h^2})^{\frac{1}{2}}\delta^0_\mu, \; R_\mu = \left(\frac{\dot{U}^2 + \dot{V}^2}{\sqrt{Q^2 + h^2}}\right)^{\frac{1}{2}}\delta^1_\mu, \\ \phi_\mu &= h(Q + \sqrt{Q^2 + h^2})^{-\frac{1}{2}}\delta^3_\mu, \end{aligned} \tag{6.21}$$

and

$$\begin{aligned} \hat{\rho} &= \frac{2}{h(\dot{U}^2 + \dot{V}^2)^2\sqrt{Q^2 + h^2}}\{(Q + \sqrt{Q^2 + h^2})(\dot{U}^2 + \dot{V}^2)(U\dot{U} + V\dot{V}) \\ &\quad -2h(U^2 + V^2)(\dot{U}\ddot{V} - \dot{V}\ddot{U})\}, \\ \hat{p}_r &= \frac{2\sqrt{Q^2 + h^2}}{h(\dot{U}^2 + \dot{V}^2)}(\dot{U} - \dot{V}), \\ \hat{p}_\phi &= -\frac{2}{(\dot{U}^2 + \dot{V}^2)^2\sqrt{Q^2 + h^2}(Q + \sqrt{Q^2 + h^2})}\{h(\dot{U}^2 + \dot{V}^2)(U\dot{U} + V\dot{V}) \\ &\quad +2(Q + \sqrt{Q^2 + h^2})\{U^2 + V^2)(\dot{U}\ddot{V} - \dot{V}\ddot{U})\} \end{aligned} \tag{6.22}$$

If one chooses the functions U and V as

$$U(r) = Ar, \;\; V(r) = Br, \tag{6.23}$$

where A and B are constants. Then, the metric takes the form

$$ds^2 = -\frac{A_0^2}{\sqrt{r^2 + \varphi^2}}(dr^2 + f^2 dz^2) + h(e^P dt^2 - e^{-P} d\phi^2), \tag{6.24}$$

with

$$A_0^2 \equiv (\frac{A^2 + B^2}{2})^{\frac{1}{2}}, \;\; h = (A + B)r - (A - B)|\varphi|, \;\; P = (A - B)r + (A + B)|\varphi|, \tag{6.25}$$

and

$$\hat{\rho} = \frac{1}{A_0^2(A + B)}[(A - B) + 2A_0^2], \;\; \hat{p}_r = \frac{2(A - B)}{A_0^2(A + B)}, \; \hat{p}_\phi = -\frac{A + B}{A_0^2[(A - B) + 2A_0^2]}. \tag{6.26}$$

It follows that the metric (6.24) describes n parallel discs with non-vanishing tensions in the radial and tangential directions. When $n = 1$, i.e., $\varphi = z - a_1$, the corresponding solution is the Lemos-Letelier solution given in Ref. 23. In fact, in the later case if we make the coordinate transformations $r = \rho^{2k}\sin 2k\theta$, and $z = \rho^{2k}\cos 2k\theta$, we shall get exactly the form given by Eq. (6) in the above cited paper.

7. DISCUSSIONS

In the present work we have studied the generation of metrics that represent topological defects with spherical, cylindrical, plane and axial symmetries. Really, the spacetimes presented here are the simplest, albeit nontrivial, that can be obtained using the method presented in Sec. 2. In principle, the same method can be used for any symmetry, in particular, for spacetimes singular on surfaces with toroidal and spheroidal symmetry that have as limits spacetimes that are singular on circular loops, and finite lines and disks, respectively. Also, a few attempts to find a coordinate transformation like Eq. (3.1) for toroidal and spheroidal symmetry was unsuccessfully. We hope to come back to this subject in another time. Even more, for each symmetry there exists more than one canonical form for the line element. In the present work we have used the ones that privilege the characteristic coordinates associated to each symmetry. However, this is by no means to say that these coordinates are the most suitable ones for every case. As the matter of fact, to obtain the solution presented in Ref. 24, in addition to Eq. (4.1), one needs to make another simple coordinate transformation in the $(r,\ \phi)$ plane.

Along this work we have studied the generation of metrics that describe topological defects without inquiring too deeply about the possible field theory that generates the defects. In general, this is a very difficult and unsolved problem. Moreover, for the most studied particular case of cosmic wall (Vilenkin's wall[21]) the complete solution to the Einstein equations coupled with the self-interacting $\lambda\phi^4$ scalar field that gives as a limiting case the domain wall solution is an open problem. Progress in this direction has been recently reported[25]. It should be noted that Widrow[26] has studied the problem by using sine-Gordon model.

All the particular solutions presented in this work were found by assuming that the surface $\varphi = 0$ has no time dependence. If we assume explicit time dependence of φ we can describe oscillating spheres or cylinders. To be more precise let us consider in Eq. (3.21) that a_i are time dependent. This case is described by the general metric Eq. (3.6). We note that all the relations that are invariant are also valid in this case, e.g., Eqs. (3.18)-(3.21). The relations (3.12)-(3.13), and (3.17) need to be changed; these "time dependent" tensorial quantities can be found from the "static" ones with $f = 1$ by using a simple transformation of coordinates.

In general, the residual part of the energy-momentum tensor $\mathcal{I}_{\mu\nu}$ does not represent a physically acceptable energy-momentum tensor. In particular, for the usual domain wall given by Eq. (5.15) we find,

$$\mathcal{I}_{\mu\nu} = (\rho_0/2)^2(U_\mu U_\nu - Z_\mu Z_\nu + X_\mu X_\nu + Y_\mu Y_\nu). \tag{7.1}$$

Thus, we have a very thin wall with a pressure along the transversal direction.

Infinite cylinders and spheres have as limit cases infinite lines and points , respectively. We shall examine the possibility to represent cosmic strings and monopoles as limiting cases of our cylindrically and spherically symmetric metrics. In particular, in the limit $r_0 = 0$ the cylindrically symmetric metric (4.11) represents a cosmic string with

$$\rho = p_z = \rho_0 \delta(r). \tag{7.2}$$

It is interesting to note that the metric

$$ds^2 = dt^2 - dr^2 - (r - \alpha)^2 d\phi^2 - dz^2, \tag{7.3}$$

with $0 \leq t < \infty$, $0 \leq r < \infty$, $0 \leq \phi < 2\pi$, and $-\infty < z < \infty$, also represents a spacetime with a deficit angle. In this direction, let us consider a circle of radius R in a plane with a deficit angle δ and a circle of radius $R > \alpha$ on the plane $t = z = 0$ of the spacetime decribed by Eq. (7.3). If both circles are equal, $2\pi(R-\alpha) = (2\pi - \delta)R$, we have $\delta = 2\pi\alpha/R$. Thus, Eq. (7.3) represents a spacetime with a deficit angle that depends on the distance to the z-axis. The usual cosmic string is represented by a spacetime with a constant deficit angle. A cylindrical shell of strings that has as a limiting case the metric of a usual cosmic string can be found in Ref. 24.

A solution given by taking the limit $r_0 \to 0$ in Eq. (3.32) will represents a punctual topological defect that geometrically can be thought as a defect of solid angle that depends on the radius. Recently, Barriola and Vilenkin[27] studied the Einstein equation coupled to a $\lambda\phi^4$ theory with an $O(3)$ internal symmetry. They found a metric with a defect of solid angle that is not a topological defect; it rather represents a cloud of ordered strings with spherical symmetry[28].

The description of punctual topological defects using the metrics of Sec.3 have the usual problems of the description of a punctual object in continuous mechanics. For instance, in this case the concept of tension does not make sense, i.e., the same type problems found in the classical models of electrons that goes back to Poincare proposal of models with non zero size for the electron[29].

Along this work we used extensively the algebraic manipulation program MUTENSOR[30].

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