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The cyclicity and isochronicity problems for monodromic tangential singularities in Filippov systems

Os problemas de ciclicidade e isocronicidade para singularidades monodrômicas tangenciais em sistemas de Filippov

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"The important thing is to never stop questioning." — Albert Einstein

Resumo

Em campos vetoriais analíticos planares, uma singularidade monodrômica pode ser distinguida entre um foco ou um centro por meio dos coeficientes de Lyapunov, que são dados em termos dos coeficientes da série de potências do mapa de primeiro retorno definido em torno da singularidade. Nesta tese, estamos interessados em um problema análogo para singularidades tangenciais monodrômicas de campos vetoriais analíticos por partes $Z = (Z^+, Z^-)$, explorando os coeficientes de Lyapunov para singularidades tangenciais monodrômicas em sistemas de Filippov e também explorando problemas relacionados aos coeficientes de Lyapunov, como os problemas de isocronicidade e ciclicidade.

Primeiro, demonstramos que o mapa de primeiro retorno, definido em uma vizinhança de uma singularidade tangencial monodrômica, é analítico, o que permite a definição dos coeficientes de Lyapunov. Em seguida, como consequência de uma propriedade geral para pares de involuções, obtemos que o índice do primeiro coeficiente de Lyapunov não nulo é sempre par. Além disso, é obtida uma fórmula recursiva geral juntamente com um algoritmo do *Mathematica* para calcular os coeficientes de Lyapunov. Como aplicação dos coeficientes de Lyapunov, temos os problemas de isocronicidade e ciclicidade, que são problemas clássicos na teoria qualitativa de campos vetoriais planares. O problema de ciclicidade consiste em estimar o número de ciclos limite que surgem de singularidades monodrômicas. Tradicionalmente, esta estimativa baseia-se nos coeficientes de Lyapunov. No entanto, em sistemas não suaves, além dos ciclos limite que bifurcam variando os coeficientes de Lyapunov, o número de ciclos limite pode aumentar em um por meio do fenômeno de bifurcação conhecido como bifurcação pseudo-Hopf. Neste estudo, vamos além da bifurcação pseudo-Hopf, demonstrando que a destruição de $(2k, 2k)$-singularidades tangenciais monodrômicas gera pelo menos k ciclos limite em torno de segmentos de deslize. O problema de isocronicidade consiste em caracterizar se um centro é isócrono ou não, isto é, se todas as trajetórias em uma vizinhança do centro têm o mesmo período. Este problema é geralmente investigado por meio da chamada função período. Neste trabalho, exploramos o problema de isocronicidade para centros tangenciais de campos vetoriais de Filippov planares. Ao calcular a função período para campos vetoriais de Filippov planares em torno de centros tangenciais, mostramos que tais centros nunca são isócronos.

Palavras-chave: Sistemas de Filippov, singularidades tangenciais, coeficientes de Lyapunov, ciclos limite, isocronicidade.

Abstract

In planar analytic vector fields, a monodromic singularity can be distinguished between a focus or a center by means of the Lyapunov coefficients, which are given in terms of the power series coefficients of the first-return map defined around the singularity. In this thesis, we are interested in an analogous problem for monodromic tangential singularities of piecewise analytic vector fields $Z = (Z^+, Z^-)$, exploring the Lyapunov coefficients for monodromic tangential singularities in Filippov systems and also exploring problems concerning the Lyapunov coefficients as the isochronicity and cyclicity problems.

First, we prove that the first-return map, defined in a neighborhood of a monodromic tangential singularity, is analytic, which allows the definition of the Lyapunov coefficients. Then, as a consequence of a general property for a pair of involutions, we obtain that the index of the first non-vanishing Lyapunov coefficient is always even. In addition, a general recursive formula together with a *Mathematica* algorithm for computing the Lyapunov coefficients is obtained. As an application of the Lyapunov coefficients, we have the isochronicity and cyclicity problems, which are classical problems in the qualitative theory of planar vector fields. The cyclicity problem consists in estimating the number of limit cycles emanating from monodromic singularities. Traditionally, this estimation relies on Lyapunov coefficients. However, in nonsmooth systems, besides the limit cycles bifurcating by varying the Lyapunov coefficients, the number of limit cycles can be increased by one via the bifurcation phenomenon, known as pseudo-Hopf bifurcation. In this study, we push beyond the pseudo-Hopf bifurcation, demonstrating that the destruction of $(2k, 2k)$-monodromic tangential singularities yields at least k limit cycles surrounding sliding segments. The isochronicity problem consists of characterizing whether a center is isochronous or not, that is if all the trajectories in a neighborhood of the center have the same period. This problem is usually investigated by means of the so-called period function. In this work, we explore the isochronicity problem for tangential centers of planar Filippov vector fields. By computing the period function for planar Filippov vector fields around tangential centers, we show that such centers are never isochronous.

Keywords: Filippov systems, tangential singularities, Lyapunov coefficients, limit cycles, isochronicity.

Contents

1 Introduction

Filippov vector fields are a class of dynamical systems that model a wide range of real-world phenomena, particularly those with non-smooth behavior or discontinuities. These vector fields are named after the Russian mathematician Andrey Filippov, who made significant contributions to the study of piecewise smooth dynamical systems.

Filippov vector fields are defined differently within distinct regions, separated by a discontinuity manifold. In this discontinuity manifold is where the dynamics of the system can abruptly change, which introduces a complexity in the analysis of these systems.

The study of Filippov vector fields is crucial in various fields, including control theory, robotics, ecological modeling, and celestial mechanics, where non-smooth behaviors and abrupt transitions are prevalent. Understanding these vector fields often involves techniques such as the Filippov solution concept, Poincaré maps, and the analysis of sliding and switching dynamics. Filippov planar vector fields have various applications in different fields. They are used to model and control systems with abrupt transitions or non-smooth behavior.

In this thesis, we will explore results concerning the center, cyclicity, and isochronicity problems in Filippov systems in the plane, that is, Filippov vector fields defined in a two-dimensional space, where the discontinuity manifold, in this case, will be a one-dimensional manifold.

The *center-focus problem* and the *cyclicity problem* stand as classical problems in the qualitative theory of smooth planar vector fields, tracing back to the studies of Poincaré and Lyapunov (see, for instance, (45)).

The *center-focus problem*, also known as the *center problem* consists in characterizing whether a monodromic singularity of a planar vector field is a center or a focus. A singularity is called a center, if in a small neighborhood around the singularity, all orbits are closed. Conversely, it is called a stable (or unstable) focus if nearby orbits spiral towards (or outwards from) the singularity. The center problem can be studied by analyzing the first-return map defined in a section within the monodromic singularity. Indeed, the monodromic singularity is a center if, and only if, the first-return map equals the identity. In the case of an analytic vector field, both the first-return map and the displacement function - defined as the difference between the first-return map and the identity- are analytic. The coefficients of the power series expansion of the displacement function around the singularity yield the so-called *Lyapunov Coefficients*, denoted as V_n's.

Hence, the monodromic singularity is a center if, and only if, V_n vanishes for all $n \in \mathbb{N}$. This immediately provides sufficient conditions for a monodromic singularity to be a focus. For polynomial vector fields, Poincaré and Lyapunov reduced the problem of solving the infinite system of equations $V_n = 0, n \in \mathbb{N}$, to an equivalent problem of finding a specific first integral for the vector field.

On the other hand, the cyclicity problem seeks to estimate the number of limit cycles that can bifurcate from a monodromic singularity, a matter that can also be studied by means of the Lyapunov coefficients. Further insights into the center and cyclicity problems can be found in the referred book (45).

Differential equations with discontinuities represent a very important class of dynamical systems due to their applications across various areas of applied science. It is worth mentioning the classical book of Andronov (3) and, for more contemporary perspectives, the books (20, 23, 30). Filippov, in his acclaimed book (25), provided a rigorous mathematical formalization for non-smooth differential equations, now termed Filippov systems. Researches into the center problem for non-smooth planar vector fields were extended as well. For instance, Pleškan and Sibirskiĭ (referenced as (44)) considered the center-focus problem for monodromic singularities of focus-focus type of piecewise analytic vector fields. Filippov, in Chapter 4 of his book (25), computed several Lyapunov coefficients for a monodromic singularity of fold-fold type in piecewise smooth vector fields. In (15), Coll et al. obtained the first seven Lyapunov constants for monodromic singularities of focus-focus type within discontinuous Liénard differential equations. Subsequently, in (19), using an algebraic approach introduced by Cima et al. in (14), they derived general expressions for the Lyapunov constants concerning monodromic singularities of the focus-focus type for some families of discontinuous Liénard differential equations. The same authors in (18) addressed both the center problem and cyclicity problems for monodromic singularities of focus-focus, fold-fold, and focus-fold types, explicitly computing the first three Lyapunov coefficients for these three types of singularities. In (27), Gasull and Torregrosa also addressed the center and cyclicity problems for monodromic singularities of focus-focus type of several classes of piecewise smooth systems. The generic unfolding of a monodromic singularity of fold-fold type was explored in (29, 33) (see also (26) on this matter). Recent studies, such as those cited in (6, 28), explored the problem of bifurcation of limit cycles from monodromic singularities in discontinuous systems by means of Lyapunov coefficients.

In this introduction, we will explore the characteristics of Filippov systems in the plane with a monodromic tangential singularity at the origin. Additionally, we will introduce a canonical form for this type of singularity, which will be of major importance for the results that will be presented later concerning the center, isochronicity, and cyclicity problems.

1.1 Filippov vector fields

The basic notions of smooth Dynamical Systems can be directly translated to piecewise smooth systems, but they need to be reformulated. The first step to studying non-smooth systems is to establish the notions of orbits and singularities.

Our main goal is to study planar vector fields, therefore we will discuss the ideas of orbits and singularities in this context. The definitions with more details can be found in (25).

Let Z^+ and Z^- be vector fields defined in an open and connected subset $U \in \mathbb{R}^2$. Without loss of generality, assume that $0 \in U$. Firstly, we introduce some assumptions and fix some notations. Assume that the discontinuity manifold is always an unidimensional differentiable manifold Σ that is given as $\Sigma = h^{-1}(0) \cap U$, for $U \subset \mathbb{R}^2$, where h is a function $h \in C^r$ (with $r > 0$) that has 0 as a regular value. Then, the curve Σ divides the set U into two open subsets

$$\Sigma^+ = \{(x,y) \in U : h(x,y) > 0\} \text{ and } \Sigma^- = \{(x,y) \in U : h(x,y) < 0\}.$$

A *Filippov system* is the piecewise vector field defined as follow

$$Z(x,y) = \begin{cases} Z^+(x,y), & \text{if } (x,y) \in \Sigma^+, \\ Z^-(x,y), & \text{if } (x,y) \in \Sigma^-, \end{cases} \tag{1.1}$$

where we will denote $Z^\pm = (X^\pm, Y^\pm)$ to indicate the components of the vector field. We also assume that the vector fields are C^k, for $k > 1$ in $\overline{\Sigma^+}$ and $\overline{\Sigma^-}$.

The notion of the local trajectories is the solutions of the following differential inclusion

$$p \in \mathcal{F}_Z(p) = \frac{Z^+(p) + Z^-(p)}{2} + sign(h(p))\frac{Z^+(p) - Z^-(p)}{2}, \tag{1.2}$$

where

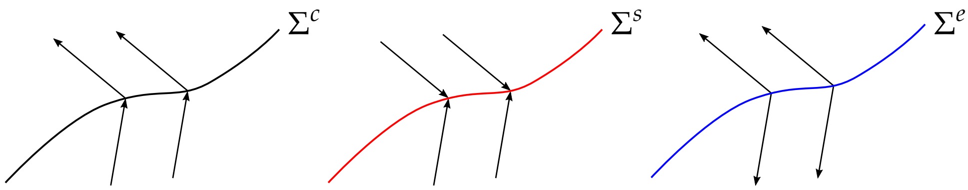
$$sign(s) = \begin{cases} -1, \ if\, s < 0, \\ [-1,1], \ if\, s = 0, \\ 1, \ if\, s > 0. \end{cases}$$

This approach is referred to as Filippov's convention. The vector field (1.1) is designated as a Filippov system when it adheres to the Filippov's convention. For further insights into differential inclusions, see (25).

The literature offers comprehensive descriptions of solutions to the differential inclusion, along with a straightforward geometric interpretation. When dealing with points located in the regions Z^+ and Z^-, we will apply the standard dynamics represented by Z^+ and Z^-, respectively. To understand the behavior of the local trajectories passing through a point $p \in \Sigma$, we establish the following open regions on Σ:

- *Crossing region:* $\Sigma^c = \{p \in \Sigma : Z^+h(p) \cdot Z^-h(p) > 0\}$, *see Figure 1a,*
- *Sliding region:* $\Sigma^s = \{p \in \Sigma : Z^+h(p) < 0, Z^-h(p) > 0\}$, *see Figure 1b,*
- *Escaping region:* $\Sigma^e = \{p \in \Sigma : Z^+h(p) > 0, Z^-h(p) < 0\}$, *see Figure 1c,*

where $Z^{\pm}h(p) = Z^{\pm}(p)\nabla h(p)$ is the *Lie derivative* of f with respect to the vector field $Z^{\pm}$ in p. These three regions are open relatively to the induced topology of Σ and they can have several connected components.



(a) Crossing Region (b) Sliding Region (c) Escaping Region

Figure 1 – Filippov Systems

We define the orbit passing through a point p, located on Σ^c, Σ^s and Σ^e. In Σ^c, where both vector fields Z^+ and Z^- point simultaneously to the same region, it is sufficient to concatenate the trajectories of Z^+ and Z^- passing through p.

Now, considering a point $p \in \Sigma^s \cup \Sigma^e$, the vector fields point to opposite directions, making it impossible to concatenate the trajectories. In this case, the trajectory on either side of the discontinuity Σ that reach p can be smoothly connected to trajectories following the sliding vector field Z_s, which is the convex linear combination of Z^+ and Z^- tangent to Σ, that is given by

$$Z_s(p) = \frac{1}{Yh(p) - Xh(p)} F_Z(p) = \frac{1}{Yh(p) - Xh(p)} (Yh(p)X(p) - Xh(p)Y(p)). \quad (1.3)$$

In the context of Filippov theory, the concept of singular points also comprehends the tangential points denoted as Σ^t. These tangential points are formed by the contact points between Z^+ and Z^- with Σ, that is,

$$\Sigma^t = \{p \in \Sigma : \ Z^+h(p) \cdot Z^-h(p) = 0\},$$

where $Fh(p) = \langle \nabla h(p), F(p) \rangle$ denotes the Lie derivative of h at p in the direction of the vector field F.

By taking local coordinates, we may assume, without loss of generality, that $p = (0,0)$ and also that $h(x,y) = y$. Therefore, denoting $Z^{\pm} = (X^{\pm}(x,y), Y^{\pm}(x,y))$, the Filippov vector field (1.1) writes as

$$Z(x,y) = \begin{cases} (X^{+}(x,y), Y^{+}(x,y)), & y > 0, \\ (X^{-}(x,y), Y^{-}(x,y)), & y < 0. \end{cases} \tag{1.4}$$

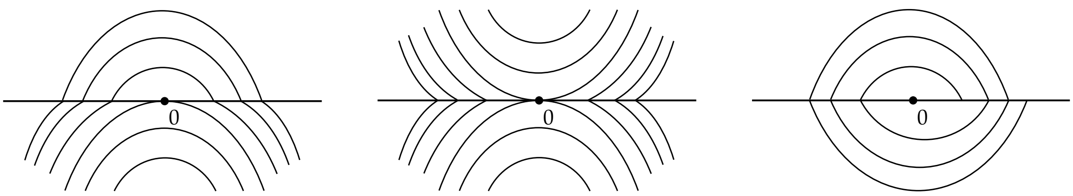
1.2 The monodromic tangential singularities

Some tangential singularities give rise to local monodromic behavior. In what follows, we shall introduce the concept of a $(2k^{+}, 2k^{-})$-monodromic tangential singularity for Filippov vector fields (39), that happens to be more degenerated than a monodromic fold-fold singularity already considered in the research literature.

Firstly, we recall that p is a *contact of multiplicity* k (or *order* $k-1$) between a smooth vector field F and Σ if 0 is a root of multiplicity k of $f(t) := h \circ \varphi_F(t,p)$, where $t \mapsto \varphi_F(t,p)$ is the trajectory of F starting at p. Equivalently,

$$Fh(p) = F^2h(p) = \ldots = F^{k-1}h(p) = 0, \text{ and } F^kh(p) \neq 0, \tag{1.5}$$

where the higher Lie derivative $F^nh(p)$ is recursively defined as $F^nh(p) = F(F^{n-1}h)(p)$, for $n > 1$. In addition, when considering Filippov vector fields (1.4), an even multiplicity contact, say $2k$, is called *invisible* for Z^{+} (resp. Z^{-}) when $(Z^{+})^{2k}h(p) < 0$ (resp. $(Z^{-})^{2k}h(p) > 0$). Otherwise, it is called *visible*.



(a) Invisible and visible (b) Both visible (c) Both invisible

Figure 2 – Singularities with even contact

In our particular case, our focus lies on monodromic singularities, and that is why we are working with invisible contacts. For a monodromic singularity to exist, it is essential not only that the singularities have an invisible contact, but also that the trajectories behave in such a way that in a neighborhood of the origin the discontinuity manifold has to be a crossing line. Keeping that into consideration, we proceed to define our singularity as follows.

Definition 1. *A tangential singularity $p \in \Sigma^t$ of a Filippov vector field Z (1.4) is called a $(2k^+, 2k^-)$-monodromic tangential singularity provided that p is simultaneously an invisible $2k^+$-multiplicity contact of Z^+ with Σ and an invisible $2k^-$-multiplicity contact of Z^- with Σ, and Z has a first-return map defined on Σ around p.*

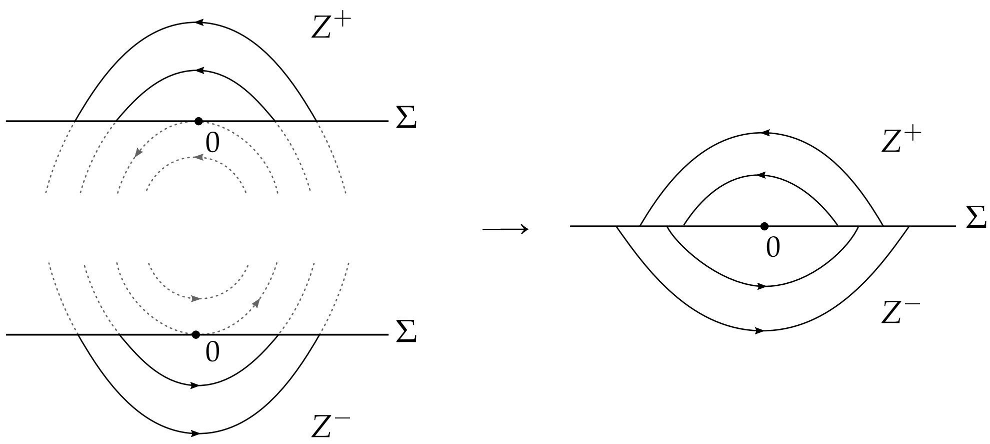


Figure 3 – Invisible $2k$-multiplicity contact between Σ and the vector fields Z^+ and Z^-. The concatenation of the orbits of Z^+ and Z^- through Σ allows the definition of a first-return map. The origin is a $(2k, 2k)$-monodromic tangential singularity of Z.

Computing the higher Lie derivatives (1.5) for $Z^\pm$, we get

$$(Z^\pm)^n h(0,0) = X^\pm(0,0)^{n-1} \frac{\partial^{n-1} Y^\pm}{\partial x^{n-1}}(0,0),$$

provided that $(Z^\pm)^i h(0,0) = 0$ for $i = 1, \ldots, n-1$. Thus, by Definition 1, one can see that the origin is a $(2k^+, 2k^-)$-monodromic tangential singularity for (1.4) provided that the following three conditions are satisfied:

C1. $X^\pm(0,0) \neq 0$, $Y^\pm(0,0) = 0$, $\dfrac{\partial^i Y^\pm}{\partial x^i}(0,0) = 0$ for $i = 1, \ldots, 2k^\pm - 2$, and $\dfrac{\partial^{2k^\pm-1} Y^\pm}{\partial x^{2k^\pm-1}}(0,0) \neq 0$;

C2. $X^+(0,0)\dfrac{\partial^{2k^+-1} Y^+}{\partial x^{2k^+-1}}(0,0) < 0$ and $X^-(0,0)\dfrac{\partial^{2k^--1} Y^-}{\partial x^{2k^--1}}(0,0) > 0$;

C3. $X^+(0,0)X^-(0,0) < 0$.

Condition **C1** imposes that the origin is a contact of multiplicity $2k^+$ (resp. $2k^-$) between Z^+ (resp. Z^-) and Σ. Condition **C2** imposes that both contacts are invisible. Finally, condition **C3** imposes that the orientation of the orbits of Z^+ and Z^- around the origin agrees in such a way that the trajectories of both vector fields can be concatenated at Σ in order to a first-return map be well defined around the origin. Denote

$$\delta = \text{sign}(X^+(0,0)) = -\text{sign}(X^-(0,0)). \tag{1.6}$$

Notice that Z is turning around the origin in the clockwise direction if $\delta > 0$, and in the anticlockwise direction if $\delta < 0$.

1.3 Canonical form

In this section, we provide a simpler expression for Filippov vector fields around a $(2k^+, 2k^-)$-monodromic tangential singularity. This canonical expression has been introduced in (1) and it will be important for proving our main results and will be used throughout this thesis. In what follows, for completeness, we briefly explain how to obtain it.

Assuming that the Filippov vector field (1.4) has a $(2k^+, 2k^-)$-monodromic tangential singularity at the origin (see conditions **C1**, **C2**, and **C3** from Definition 1), we have that $X^\pm(0,0) \neq 0$. Therefore, there exists a small neighborhood U of the origin such that $X^\pm(x,y) \neq 0$ for all $(x,y) \in U$. Taking into account that $|X^\pm(x,y)| = \pm\delta X^\pm(x,y)$, for all $(x,y) \in U$, a time rescaling can be performed in order to transform the Filippov vector field (1.4) restricted to U into

$$(\dot{x}, \dot{y}) = \tilde{Z}(x,y) = \begin{cases} (\delta, \eta^+(x,y)), & y > 0, \\ (-\delta, \eta^-(x,y)), & y < 0, \end{cases}$$

where

$$\eta^+(x,y) = \delta \frac{Y^+(x,y)}{X^+(x,y)} \text{ and } \eta^-(x,y) = -\delta \frac{Y^-(x,y)}{X^-(x,y)}.$$

In addition, we can show that

$$(\tilde{Z}^\pm)^i h(0,0) = 0 \text{ if, and only if, } (Z^\pm)^i h(0,0) = 0, \text{ for all } i = 1, 2, \ldots, 2k^\pm, \tag{1.7}$$

and

$$\tilde{Z}^\pm h(x,0) = \eta^\pm(x,0) \text{ and } (\tilde{Z}^\pm)^i h(x,0) = \frac{\partial^{i-1}}{\partial x^{i-1}} \eta^\pm(0,0), \text{ for all } i = 1, \ldots, 2k^\pm. \tag{1.8}$$

Since $(Z^\pm)^i h(0,0) = 0$, for $i = 1, 2, \ldots, 2k^\pm - 1$, and $(Z^\pm)^{2k^\pm} h(0,0) \neq 0$, by combining (1.7) and (1.8), we can expand $\eta^\pm(x,0)$ around $x = 0$ as follows:

$$\eta^\pm(x,0) = \sum_{i=0}^{2k^\pm - 1} \frac{1}{i!} \frac{\partial^i \eta^\pm}{\partial x^i}(0,0) x^i + x^{2k^\pm} f^\pm(x) = a^\pm x^{2k^\pm - 1} + x^{2k^\pm} f^\pm(x),$$

where, the functions $f^{\pm}$ are defined as

$$f^{\pm}(x) = \frac{\pm\delta Y^{\pm}(x,0) - a^{\pm}x^{2k^{\pm}-1}X^{\pm}(x,0)}{x^{2k^{\pm}}X^{\pm}(x,0)}, \tag{1.9}$$

and the values $a^{\pm}$ as

$$\begin{aligned} a^{\pm} =& \frac{1}{(2k^{\pm}-1)!}\frac{\partial^{2k^{\pm}-1}\eta^{\pm}}{\partial x^{2k^{\pm}-1}}(0,0) = \frac{\pm\delta}{(2k^{\pm}-1)!}\frac{\partial^{2k^{\pm}-1}}{\partial x^{2k^{\pm}-1}}\left(\frac{Y^{\pm}(x,0)}{X^{\pm}(x,0)}\right)\Bigg|_{x=0} \\ =& \frac{1}{(2k^{\pm}-1)!|X^{\pm}(0,0)|}\frac{\partial^{2k^{\pm}-1}Y^{\pm}}{\partial x^{2k^{\pm}-1}}(0,0). \end{aligned} \tag{1.10}$$

Considering

$$g^{\pm}(x,y) = \frac{\pm X^{\pm}(x,0)Y^{\pm}(x,y) \mp X^{\pm}(x,y)Y^{\pm}(x,0)}{y\delta X^{\pm}(x,y)X^{\pm}(x,0)}, \tag{1.11}$$

the function $\eta^{\pm}(x,y)$ writes as

$$\eta^{\pm}(x,y) = a^{\pm}x^{2k^{\pm}-1} + x^{2k^{\pm}}f^{\pm}(x) + yg^{\pm}(x,y).$$

Consequently, the Filippov vector field (1.4) on U is equivalent to

$$(\dot{x},\dot{y}) = \begin{cases} (\delta, a^{+}x^{2k^{+}-1} + x^{2k^{+}}f^{+}(x) + yg^{+}(x,y)), & y > 0, \\ (-\delta, a^{-}x^{2k^{-}-1} + x^{2k^{-}}f^{-}(x) + yg^{-}(x,y)), & y < 0. \end{cases} \tag{1.12}$$

1.4 Structure of the thesis and main results

In Chapter 2, we give our first main result Theorem 1, which gives us results about the regularity of the half-return maps that define the first-return map. Employing the regularity of the half-return maps, we define the Lyapunov coefficients for tangential singularities. In Section 2.3, we present Theorem 2 which states that the first non-vanishing Lyapunov coefficient for tangential singularity is always even, its proof follows directly from Proposition 1. In Section 2.4, we give Theorem 3 and Proposition 2 which give us recursive formulae for the Lyapunov coefficients, and Proposition 2 which gives us the first four Lyapunov coefficients explicitly. In addition, in Section 2.5, we provide an implemented list of *Mathematica* algorithms for computing the Lyapunov coefficients.

In Chapter 3, Section 3.2, we give Theorems 5 and 6 that explore the cyclicity problem for monodromic tangential singularities. Theorem 5 gives us results about Hopf-like bifurcation, and Theorem 6 gives us results about Bautin-like bifurcations. In Section 3.3, we give Propositions 3 and 4 which are a formalization of the pseudo-Hopf bifurcation and a version of the pseudo-Hopf bifurcation for more degenerate singularities, respectively. In Section 3.4, we present Theorem 7 which pushes beyond the pseudo-Hopf bifurcation bringing results of the appearance of limit cycles with the destruction of more degenerate tangential singularities. The proof of Theorem 7 is done through Subsections 3.4.1, 3.4.2 and 3.4.3. In addition, in Section 3.5, we give several examples exploring the cyclicity problem using Theorems 5, 6, and 7.

In Chapter 4, Section 4.1, we give Lemma 1 which provides us the formula for the period function for the canonical form for a system with a tangential singularity. In Section 4.2, we present Theorem 8 which gives us the formula for the period function for a general system with a tangential singularity. The proof of Theorem 8 is done in Subsection 4.2.1. In Section 4.3, we give the formulae for computing the *period constants*. In Section 4.4, we give Corollary 1 and Theorem 9 which give us the first *period constants* and state that tangential centers are never isochronous and do not admit critical periods, respectively.

2 Lyapunov coefficients for tangential singularities

The main results of this chapter are based on the paper (39). One of the main goals of this chapter is to obtain the formulae for the Lyapunov coefficients for monodromic tangential singularities of piecewise analytic vector fields $Z = (Z^+, Z^-)$. In order to do that, first we prove that the first-return map, defined in a neighborhood of a monodromic tangential singularity, is analytic, which allows the definition of the Lyapunov coefficients. Then, as a consequence of a general property for a pair of involutions, we obtain that the index of the first non-vanishing Lyapunov coefficient is always even. Then, we provide a recursive formula for the Lyapunov coefficients. In addition, in Section 2.5 of this chapter, a *Mathematica* algorithm for computing the Lyapunov coefficients is provided.

In (18), the authors studied the Lyapunov coefficients for parabolic-parabolic points, which in light of Definition 1, correspond to the $(2, 2)$-monodromic tangential singularities. In this chapter, our main goal consists of extending the previous results for $(2k^+, 2k^-)$-monodromic tangential singularity. It is worth mentioning that in (18) the Lyapunov coefficients were obtained by means of generalized polar coordinates (see (7)). Here, motivated by Teixeira's works (49, 50), we propose a different way of obtaining it by considering auxiliary sections, which are transversal to both the flow and the discontinuity manifold. This method allows us to provide a general recursive formula for the Lyapunov coefficients.

2.1 Half-return maps and their regularity

Considering (1.4) with a $(2k^+, 2k^-)$-monodromic tangential singularity, we have that the flows of Z^+ and Z^- restricted, respectively, to $\Sigma^+ = \{(x, y) : y \geqslant 0\}$ and $\Sigma^- = \{(x, y) : y \leqslant 0\}$ define half-return maps φ^+ and φ^- on Σ around 0, which are known to be **involutions** (38) satisfying $\varphi^+(0) = \varphi^-(0) = 0$, that is, $\varphi^+ \circ \varphi^+(x) = x$ and $\varphi^- \circ \varphi^-(x) = x$ whenever they are defined (see Figure 4).

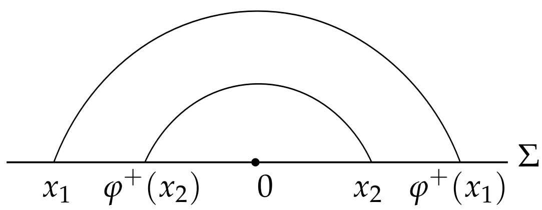


Figure 4 – Illustration of the half-return map φ^+.

The main tool we shall employ to obtain the analyticity of the half-return maps $\varphi^{\pm}$ is the *generalized polar coordinate transformation* (16, 34). This coordinate transformation was introduced by Lyapunov in (34) and was originally conceived for studying degenerate singularities of vector fields. Afterward, this tool has shown to be very useful in the study of degenerate singularities of smooth planar vector fields (see, for instance, (8, 16, 24)).

In (18), Coll et al. employed the generalized polar coordinate transformation to study several types of monodromic singularities of discontinuous piecewise analytic systems, one of them, the fold-fold type, coincides with our $(2,2)$-monodromic tangential singularity, according to Definition 1. For this case, they showed that, in the transformed space $\mathbb{S}^1 \times \mathbb{R}^+$, the monodromic singularity blows-up into $\{0\} \times \mathbb{S}^1$, which does not have singularities in the closure of the semi-plane of interest (see Figure 5). As we shall see, this implies the analyticity of half-return maps around $(2,2)$-monodromic tangential singularities. The same result for $(2,2)$-monodromic tangential singularities can be obtained by using an analytic version of Vishik's normal form (see (10)), however, this does not work for more degenerate tangential singularities.

Our first main result states that such half-return maps are analytic provided that the vector fields Z^+ and Z^- are analytic. In (18), the authors proved the analyticity of the half-return maps blowing-up the origin (see Figure 5) by means of *Generalized Polar Coordinates* (16, 34) assuming $k^{\pm} = 1$. Here, we shall follow the ideas from (18) to obtain the analyticity of half-return maps around $(2k^+, 2k^-)$-monodromic tangential singularities, adapting their proof in order to obtain the analyticity for the general case. In the proof, it will be clear that if we impose Z^+ (resp. Z^-) to be C^r, $1 \leqslant r \leqslant \infty$, instead of analytic, then the half-return map φ^+ (resp. φ^-) would be C^r around $x = 0$.

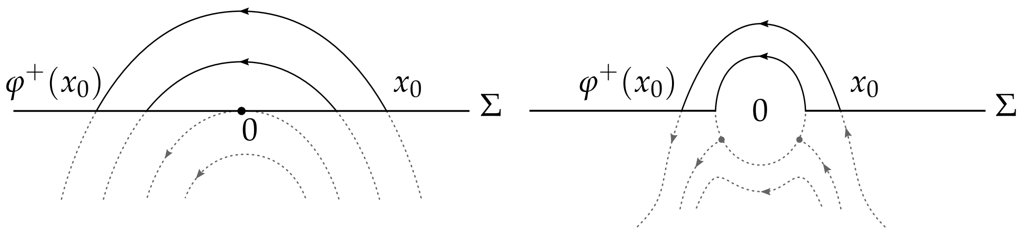


Figure 5 – Blow-up of Z^+ at the monodromic tangential singularity. In the transformed space $\mathbb{S}^1 \times \mathbb{R}^+$, the monodromic singularity blows-up into $\{0\} \times \mathbb{S}^1$, which does not have singularities in the closure of the semi-plane of interest.

In order to present and prove our first main result, firstly we recall the definition of generalized polar coordinates. For positive real numbers p and q, the (R, θ, p, q)-generalized polar coordinates are given by

$$(x, y) = \big(R^p \mathrm{Cs}(\theta), R^q \mathrm{Sn}(\theta)\big), \tag{2.1}$$

where $R > 0$, $\theta \in \mathbb{S}^1$, and the function $\big(\mathrm{Sn}(\theta), \mathrm{Cs}(\theta)\big)$ is the solution of the following Cauchy problem:

$$\begin{cases} \dot{\mathrm{Cs}} = -\mathrm{Sn}^{2p-1}, \\ \dot{\mathrm{Sn}} = \mathrm{Cs}^{2q-1}, \end{cases} \quad \mathrm{Cs}(0) = \sqrt[2q]{\frac{1}{p}}, \quad \mathrm{Sn}(0) = 0.$$

The functions $\mathrm{Cs}(\theta)$ and $\mathrm{Sn}(\theta)$ are also called (p,q)-*trigonometric functions* (34). In (16), it is proven that the functions $\mathrm{Cs}(\theta)$ and $\mathrm{Sn}(\theta)$ are analytic, T-periodic with

$$T = 2p^{\frac{-1}{2q}} q^{\frac{-1}{2p}} \int_0^1 (1-s)^{\frac{1-2p}{2p}} s^{\frac{1-2q}{2q}} ds > 0,$$

and satisfy the following properties:

- $p(\mathrm{Cs}(\theta))^{2q} + q(\mathrm{Sn}(\theta))^{2p} = 1$;
- Cs is an even function and Sn is an odd function;
- $\mathrm{Cs}(\frac{T}{2} - \theta) = -\mathrm{Cs}(\theta)$ and $\mathrm{Sn}(\frac{T}{2} - \theta) = \mathrm{Sn}(\theta)$.

Taking the above properties into account, one can see that

$$\begin{aligned} &\mathrm{Cs}(-\frac{T}{4}) = \mathrm{Cs}(\frac{T}{4}) = 0, \\ &\mathrm{Cs}(\theta) > 0, \text{ for } \theta \in (-\frac{T}{4}, \frac{T}{4}), \text{ and } \mathrm{Cs}(\theta) < 0, \text{ for } \theta \in [-\frac{T}{2}, -\frac{T}{4}) \cup (\frac{T}{4}, \frac{T}{2}], \\ &\mathrm{Sn}(\frac{T}{2}) = \mathrm{Sn}(0) = \mathrm{Sn}(\frac{T}{2}) = 0, \\ &\mathrm{Sn}(\theta) > 0, \text{ for } \theta \in (0, \frac{T}{2}), \text{ and } \mathrm{Sn}(\theta) < 0, \text{ for } \theta \in (-\frac{T}{2}, 0). \end{aligned} \tag{2.2}$$

Theorem 1. *Consider the Filippov vector field* (1.4) *and suppose that the vector field* Z^+ *(resp.* Z^-*) is analytic and has an invisible* $2k^+$*-multiplicity (resp.* $2k^-$*-multiplicity) contact at the origin with* $\Sigma = \{(x,0) : x \in \mathbb{R}\}$*, for a positive integer* k^+ *(resp.* k^-*). Then, the half-return map* φ^+ *(resp.* φ^-*) is analytic around* $x = 0$.

Proof. We shall prove the analyticity of the φ^+. The analyticity of φ^- will follows analogously.

Using the (R, θ, p, q)-generalized polar change of coordinates (2.1) for $p = 1$ and $q = 2k^+$ and rescaling the time by taking $\tau = \dfrac{t}{R}$, the vector field (1.12) restricted to $y \geqslant 0$ is transformed into (as shown in Figure 5)

$$(\theta', R') = (F^+(R,\theta), G^+(R,\theta)), \quad \theta \in [0, \frac{T}{2}] \text{ and } R > 0, \tag{2.3}$$

where

$$
\begin{aligned}
F^+(R,\theta) =& a^+\mathrm{Cs}(\theta)^{2k^+} - 2\delta k^+\mathrm{Sn}(\theta)\\
&+ R\mathrm{Cs}(\theta)\left(\mathrm{Sn}(\theta)g^+\left(R\mathrm{Cs}(\theta), R^{2k^+}\mathrm{Sn}(\theta)\right) + \mathrm{Cs}(\theta)^{2k^+}f^+(R\mathrm{Cs}(\theta))\right),\\
G^+(R,\theta) =& R\mathrm{Cs}(\theta)^{2k^+-1}\left(\delta\mathrm{Cs}(\theta)^{2k^+} + a\mathrm{Sn}(\theta)\right)\\
&+ R^2\mathrm{Sn}(\theta)\left(\mathrm{Sn}(\theta)g^+\left(R\mathrm{Cs}(\theta), R^{2k^+}\mathrm{Sn}(\theta)\right) + \mathrm{Cs}(\theta)^{2k^+}f^+(R\mathrm{Cs}(\theta))\right).
\end{aligned}
$$

Notice that, for $R = 0$, $F^+(0,\theta) = a^+\mathrm{Cs}^{2k^+}(\theta) - 2\delta k^+\mathrm{Sn}(\theta)$. Thus, since $\delta a^+ < 0$, and taking (2.2) into account, we conclude that any root of $F^+(0,\theta) = 0$ must satisfy $-\frac{T}{2} < \theta < 0$. Consequently, for $R > 0$ sufficiently small, $\theta' > 0$, for all $\theta \in [0, \frac{T}{2}]$. This means that θ can be taken as the independent variable in (2.3). Indeed, denoting

$$
H^+(R,\theta) = \frac{G^+(R,\theta)}{F^+(R,\theta)},
$$

the differential equation (2.3) writes

$$
\frac{dR}{d\theta} = H^+(R,\theta), \quad \theta \in [0, \frac{T}{2}] \text{ and } R > 0. \tag{2.4}
$$

Since $H^+(R,\theta)$ is analytic in a neighborhood of $\{0\} \times [0, \frac{T}{2}]$, the differential (2.4) can be analytically extended to $R = 0$. Accordingly, let $r^+(\theta, x_0)$ denote the solution of such an extension satisfying $r^+(0, x_0) = x_0$. From the above comments, we get that $r^+(\theta, x_0)$ is analytic in a neighborhood of $[0, \frac{T}{2}] \times \{0\}$. Finally, notice that $\varphi^+(x_0) = r^+(\frac{T}{2}, x_0)Cs(\frac{T}{2})$. Therefore, we conclude that $\varphi^+(x_0)$ is analytic in a neighborhood of $x_0 = 0$, which concludes the proof of Theorem 1 for the analytic case. The C^r case is analogous. □

2.2 Displacement function and the Lyapunov coefficients

Here, in order to obtain the so-called *Lyapunov coefficients*, instead of working with the first-return map, we consider the *displacement function* (see Figure 6)

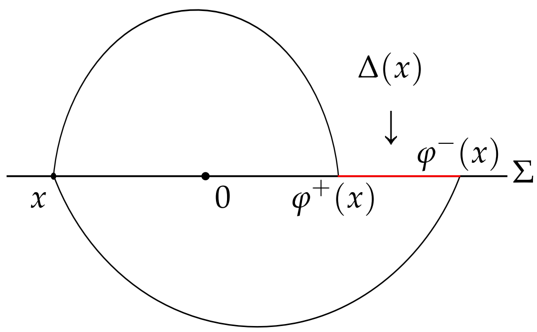


Figure 6 – Illustration of the displacement function Δ.

$$\Delta(x) = \delta(\varphi^{+}(x) - \varphi^{-}(x)). \tag{2.5}$$

Assuming the analyticity of the vector fields Z^{+} and Z^{-}, from Theorem 1 and taking into account that φ^{+} and φ^{-} are involutions, we have

$$\varphi^{\pm}(x) = -x + \alpha_2^{\pm}x^2 + \alpha_3^{\pm}x^3 + \cdots = -x + \sum_{n=2}^{\infty} \alpha_n^{\pm}x^n. \tag{2.6}$$

Thus,

$$\Delta(x) = \sum_{n=2}^{\infty} V_n x^n, \tag{2.7}$$

where $V_n = \delta(\alpha_n^{+} - \alpha_n^{-})$, for $n \geqslant 2$. Notice that (1.4) has a center at the origin if, and only if, the displacement function is identically zero, equivalently, $V_n = 0$, for all integer $n \geqslant 2$. Hence, if there exists $n \in \mathbb{N}$ such as $V_n \neq 0$, then (1.4) has a focus at the origin. In addition, if n_0 is the first index such that $V_{n_0} \neq 0$, then the origin is asymptotically stable (resp. unstable) provided that $V_{n_0} < 0$ (resp. $V_{n_0} > 0$).

Accordingly, it is natural to introduce the following definition:

Definition 2. *Fixing a system of the form* (1.4) *with a* $(2k^{+}, 2k^{-})$*-monodromic tangential singularity, we have:*

- *The coefficient* $V_n, n \in \mathbb{N}$*, in* (2.7)*, is called the n-th Lyapunov coefficient.*
- *The origin is called a* $(2k^{+}, 2k^{-})$*-tangential center provided that the displacement function* (2.5) *is identically zero, otherwise the origin is called a* $(2k^{+}, 2k^{-})$*-tangential focus.*

2.3 First non-vanishing Lyapunov coefficient

For non-degenerate monodromic singularities of planar smooth vector fields, the index of the first non-vanishing Lyapunov coefficient is always odd (see (45)). Here, as a consequence of an involutive property of the half-return maps, our second main result establishes that the index of the first non-vanishing Lyapunov coefficient of a $(2k^{+}, 2k^{-})$-monodromic tangential singularity is always even:

Theorem 2. *Consider the Filippov vector field Z given by* (1.4) *and suppose that the vector fields* Z^{+} *and* Z^{-} *are analytic. Assume that Z has a* $(2k^{+}, 2k^{-})$*-monodromic tangential singularity at the origin, for positive integers* k^{+} *and* k^{-}*. If* $V_n = 0$*, for all* $n = 2, \ldots, 2\ell$*, then* $V_{2\ell+1} = 0$*.*

Theorem 2 is a direct consequence of the following property of a pair of involutions:

Proposition 1. *Let $\varphi, \psi : I \to \mathbb{R}$ be $C^{2\ell+1}$ involutions around 0. If $\varphi(0) = \psi(0)$ and $\varphi^{(i)}(0) = \psi^{(i)}(0)$, for $i = 1, 2 \ldots, 2\ell$, then $\varphi^{(2\ell+1)}(0) = \psi^{(2\ell+1)}(0)$.*

For the proof of Proposition 1 and the next results we will use the concept of *partial Bell polynomials* $B_{p,q}$ and *ordinary Bell polynomials* $\hat{B}_{p,q}$.

Definition 3. *The (exponential) partial Bell polynomials are the polynomial*

$$B_{p,q}(x_1, \ldots, x_{p-q+1}) = \sum \frac{p!}{b_1!\, b_2! \cdots b_{p-q+1}!} \prod_{j=1}^{p-q+1} \left(\frac{x_j}{j!}\right)^{b_j} \tag{2.8}$$

and the ordinary Bell polynomials (or complete) are

$$\hat{B}_{p,q}(x_1, \ldots, x_{p-q+1}) = \sum \frac{p!}{b_1!\, b_2! \cdots b_{p-q+1}!} \prod_{j=1}^{p-q+1} x_j^{b_j}, \tag{2.9}$$

where the summations are taken over all the $(p-q+1)$-tuple of non-negative integers $(b_1, b_2, \cdots, b_{p-q+1})$ satisfying $b_1 + 2b_2 + \cdots + (p-q+1)b_{p-q+1} = p$, and $b_1 + b_2 + \cdots + b_{p-q+1} = q$.

Notice that

$$\hat{B}_{p,q}(x_1, \ldots, x_{p-q+1}) = \frac{q!}{p!} B_{p,q}(1!x_1, \ldots, (p-q+1)!x_{p-q+1}).$$

It is worth mentioning that partial Bell polynomials are implemented in algebraic manipulators as *Mathematica* and *Maple*.

We have the following relations for *Bell polynomials* (21):

(R1) $B_{p,q}(0, 0, \ldots, 0, x_j, 0, \ldots, 0) = 0$, except $B_{pq,q} = \dfrac{(pq!)}{q!(j!)^k} x_j^k$,

(R2) $B_{n,n-a}(x_1, x_2, \ldots, x_{a+1}) = \displaystyle\sum_{j=a+1}^{2a} \binom{n}{j} B_{j,j-a}(0, x_2, x_3, \ldots, x_{a+1})$.

Now we follow with the proof of Proposition 1.

Proof. In order to prove Proposition 1, first, we recall the useful *Faà di Bruno's Formula* for higher derivatives of a composite function (see (31))

$$\frac{d^l}{d\alpha^l} g(h(\alpha)) = \sum_{m=1}^{l} g^{(m)}(h(\alpha)) B_{l,m}\big(h'(\alpha), h''(\alpha), \ldots, h^{(l-m+1)}(\alpha)\big), \tag{2.10}$$

where $B_{l,m}$ denotes the partial Bell polynomials as defined in (3).

Since $\varphi \circ \varphi(x) = x$ and $\psi \circ \psi(x) = x$, then $\varphi'(0) = -1$ and $\psi'(0) = -1$.

Now, applying Faà di Bruno's Formula (2.10) for computing the n-th derivative of the composition $\varphi \circ \varphi(x) = x$, we get

$$\sum_{i=1}^{n} \varphi^{(i)}(0) \mathrm{B}_{n,i}(\varphi'(0), \varphi''(0), \ldots, \varphi^{(n-i+1)}(0)) = 0, \;\; n \geqslant 2. \tag{2.11}$$

Denote

$$S(\varphi'(0), \ldots, \varphi^{(n-1)}(0)) := -\sum_{i=2}^{n-1} \mathrm{B}_{n,i}(\varphi'(0), \ldots, \varphi^{(n-i+1)}(0)).$$

Thus, from (2.11), we have

$$\varphi'(0)\mathrm{B}_{n,1}(\varphi'(0), \ldots, \varphi^{(n)}(0)) + \varphi^{(n)}(0)\mathrm{B}_{n,n}(-1) = -\sum_{i=2}^{n-1} \mathrm{B}_{n,i}(\varphi'(0), \ldots, \varphi^{(n-i+1)}(0)),$$

which implies that

$$((-1)^n - 1)\varphi^{(n)}(0) = S(\varphi'(0), \ldots, \varphi^{(n-1)}(0)). \tag{2.12}$$

Analogously, we obtain that

$$((-1)^n - 1)\psi^{(n)}(0) = S(\psi'(0), \ldots, \psi^{(n-1)}(0)). \tag{2.13}$$

Now, assume that $\varphi^{(i)}(0) = \psi^{(i)}(0) = \alpha_i$, for $i = 1, 2, \ldots, 2\ell$. From (2.12) and (2.13), taking $n = 2\ell + 1$, we get that

$$-2\varphi^{(2\ell+1)}(0) = S(\varphi'(0), \ldots, \varphi^{(2\ell)}(0)) = S(\alpha_1, \ldots, \alpha_{2\ell}) = -2\psi^{(2\ell+1)}(0),$$

which concludes the proof. □

Proof of Theorem 2. The proof of Theorem 2 follows directly from Proposition 1 by taking $\varphi = \varphi^+$ and $\psi = \varphi^-$. □

2.4 Formulae for the Lyapunov coefficients

Our third main result provides a recursive formula for computing the coefficients α_n^+ and α_n^- of the series (2.6) of the half-return maps φ^+ and φ^- and, consequently, the Lyapunov coefficients V_n's (2.7).

Theorem 3. *Consider the Filippov vector field Z given by (1.4) and suppose that the vector fields Z^+ and Z^- are analytic. Assume that Z has a $(2k^+, 2k^-)$-monodromic tangential singularity at the origin, for positive integers k^+ and k^-. Then, the functions $f^\pm$ (1.9) and $g^\pm$ (1.11) are analytic in a neighborhood of $x = y = 0$. Also, consider the sequence of functions y_i^+ and y_i^- defined, in a neighborhood of $x = 0$, recursively by*

$$
\begin{aligned}
y_1^\pm(x) =& a^\pm x^{2k^\pm-1} + x^{2k^\pm} f^\pm(x),\\
y_i^\pm(x) =& (\pm\delta)^{i-1}\left(a^\pm \frac{(2k^\pm-1)!}{(2k^\pm-i)!}x^{2k^\pm-i} + \sum_{l=0}^{i-1}\binom{i-1}{l}\frac{(2k^\pm)!}{(2k^\pm-l)!}x^{2k^\pm-l}f^{\pm(i-1-l)}(x)\right)\\
&+\sum_{l=1}^{i-1}\sum_{j=1}^{l} j\binom{i-1}{l}(\pm\delta)^{i-l-1}B_{l,j}(y_1^\pm(x),\dots,y_{l-j+1}^\pm(x))\frac{\partial^{j+i-l-2}g^\pm}{\partial x^{i-l-1}\partial y^{j-1}}(x,0),\ \text{if } 2\leqslant i\leqslant 2k^\pm,\\
y_i^\pm(x) =& (\pm\delta)^{i-1}\left(\binom{i-1}{2k^\pm}(2k^\pm)!f^{\pm i-1-2k^\pm}(x) + \sum_{l=0}^{2k^\pm-1}\binom{i-1}{l}\frac{(2k^\pm)!}{(2k^\pm-l)!}x^{2k^\pm-l}f^{\pm(i-l-1)}(x)\right)\\
&+\sum_{l=1}^{i-1}\sum_{j=1}^{l} j\binom{i-1}{l}(\pm\delta)^{i-l-1}B_{l,j}(y_1^\pm(x),\dots,y_{l-j+1}^\pm(x))\frac{\partial^{j+i-l-2}g^\pm}{\partial x^{i-l-1}\partial y^{j-1}}(x,0),\ \text{if } i > 2k^\pm.
\end{aligned}
\tag{2.14}
$$

Then, the coefficients $\alpha_n^\pm$ of the series (2.6) of the half-return maps $\varphi^\pm$ are given recursively by

$$
\begin{cases}
\alpha_1^\pm = -1,\\
\alpha_n^\pm = \dfrac{p_{n,k^\pm}^\pm(\alpha_1^\pm, \alpha_2^\pm, \cdots \alpha_{n-1}^\pm) - \mu_{n+2k^\pm-1}^\pm}{2k^\pm \mu_{2k^\pm}^\pm},
\end{cases}
\tag{2.15}
$$

where

$$
p_{n,k^\pm}^\pm(\alpha_1,\dots,\alpha_{n-1}) = \mu_{2k^\pm}^\pm \hat{B}_{n+2k^\pm-1,2k}(\alpha_1,\dots,\alpha_{n-1},0) + \sum_{i=2k^\pm+1}^{n+2k^\pm-1}\mu_i^\pm \hat{B}_{n+2k^\pm-1,i}(\alpha_1,\dots,\alpha_{n+2k^\pm-i}),
$$

and

$$
\mu_i^\pm = \frac{1}{i!}\sum_{j=1}^{i}(\mp\delta)^j\binom{i}{j}(y_j^\pm)^{(i-j)}(0).
\tag{2.16}
$$

Proof. From here, in order to prove Theorem 3, we shall use some additional identities, namely the well-known *General Leibniz Rule* for higher derivatives of product of functions

$$
\frac{d^l}{d\alpha^l}(g(\alpha)h(\alpha)) = \sum_{k=0}^{l}\binom{l}{k}g^{(l-k)}(\alpha)h^{(k)}(\alpha);
\tag{2.17}
$$

and the following *Multinomial Formula*.

The multinomial theorem is the generalization of the well known binomial theorem for more than two variables and it describes how to expand a power of a sum in terms of powers of the terms in that sum.

The multinomial theorem provides a formula, which is described as follow:

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{n_1,n_2,\ldots,n_k \geqslant 0} \frac{n!}{n_1!n_2!\ldots n_k} \prod_{t=1}^{k} x_t^{n_t}, \tag{2.18}$$

where $n_1 + n_2 + \cdots + n_k = n$. The *Multinomial Theorem* can also be written in terms of ordinary Bell polynomials as

$$\left(\sum_{j=1}^{\infty} \alpha_j x^j\right)^n = \sum_{i=n}^{\infty} \hat{B}_{i,n}(\alpha_1, \ldots, \alpha_{i-n+1}) x^i, \tag{2.19}$$

where $\hat{B}_{l,m}$ denotes the ordinary Bell polynomials (2.9).

Denote by $\phi^{\pm}(t, x_0) = (x^{\pm}(t, x_0), y^{\pm}(t, x_0))$ the solutions of

$$(\dot{x}, \dot{y}) = (\pm\delta, a^{\pm} x^{2k^{\pm}-1} + x^{2k^{\pm}} f^{\pm}(x) + y g^{\pm}(x, y)),$$

with initial condition $\phi^{\pm}(0, x_0) = (x^{\pm}(0, x_0), y^{\pm}(0, x_0)) = (x_0, 0) \in \Sigma$. Notice that $x^{\pm}(t, x_0) = x_0 \pm \delta t$, so that $x^{\pm}(t, x_0) = 0$ if, and only if, $t = \mp\delta x_0$. Accordingly, we define (see Figure 7)

$$\mu^{\pm}(x_0) = y^{\pm}(\mp\delta x_0, x_0). \tag{2.20}$$

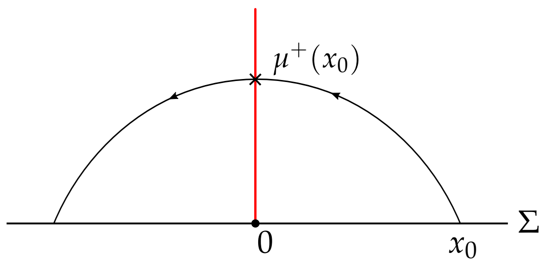


Figure 7 – Map μ

As commented before, we compute the series of $\mu^{\pm}(x_0)$ around the origin:

$$\mu^{\pm}(x_0) = \sum_{n=1}^{\infty} \mu_n^{\pm} x_0^n,$$

where

$$\mu_n^{\pm} = \frac{\mu^{\pm(n)}(0)}{n!} = \sum_{i+j=n} (\mp\delta)^i \binom{n}{i} \frac{\partial^n y^{\pm}(0,0)}{\partial t^i \partial x_0^j} = \frac{1}{n!} \sum_{i=1}^{n} (\mp\delta)^i \binom{n}{i} \frac{\partial^{n-i}}{\partial x_0^{n-i}} \left(\frac{\partial^i y^{\pm}}{\partial t^i}(0,0)\right).$$

In the last equality above, we are using that $y^{\pm}(0, x_0) = 0$ and, consequently, $\dfrac{\partial^n y^{\pm}}{\partial x_0^n}(0,0) = 0$. Now, denoting $y_i^{\pm}(x) = \dfrac{\partial^i y^{\pm}}{\partial t^i}(0, x)$, we get

$$\mu_i^{\pm} = \frac{1}{i!} \sum_{j=1}^{i} (\mp\delta)^j \binom{i}{j} (y_j^{\pm})^{(i-j)}(0).$$

Notice that this last expression coincides with the one presented in (2.16).

Suppose that $Z^{\pm}$ are analytic vector fields and assume that the piecewise analytic Filippov vector field (1.4) has a $(2k^+, 2k^-)$-monodromic tangential singularity at the origin, for positive integers k^+ and k^-. From the comments of Section 1.3, we know that there exists a small neighborhood $U \subset \mathbb{R}^2$ of the origin such that (1.4) is equivalent to the canonical form (1.12) through a time rescaling, where δ, $a^{\pm}$, $f^{\pm}(x)$, and $g^{\pm}(x,y)$ are given by (1.6), (1.10), and (1.9), respectively. In addition, since $X^{\pm}(x,y) \neq 0$, for all $(x,y) \in U$, we get that the functions $f^{\pm}$ and $g^{\pm}$ are analytic in a neighborhood of $x = y = 0$.

Now, working out the identity $\mu^{\pm}(x_0) = \mu^{\pm}(\varphi^{\pm}(x_0))$, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \mu_n^{\pm} x_0^n &= \sum_{n=1}^{\infty} \mu_n^{\pm} (\sum_{j=1}^{\infty} \alpha_j^{\pm} x_0^j)^n \\ &= \sum_{n=1}^{\infty} \mu_n^{\pm} \sum_{i=n}^{\infty} \hat{B}_{i,n}(\alpha_1^{\pm}, \dots, \alpha_{i-n+1}^{\pm}) x_0^i \\ &= \sum_{n=1}^{\infty} \sum_{i=n}^{\infty} \mu_n^{\pm} \hat{B}_{i,n}(\alpha_1^{\pm}, \dots, \alpha_{i-n+1}^{\pm}) x_0^i.\end{aligned} \tag{2.21}$$

In the second equality above, we are using the multinomial formula (2.19).

First, we claim that (2.14) provides a recursive formula for $y_i^{\pm}(x)$.

Claim 3.1. *The functions $y_i^{\pm}(x)$, for $i = 1, 2, \dots$, are defined recursively by* (2.14).

Proof of Claim 3.1. *First of all, notice that the derivative of the second component of the solution is equal to the second component of the vector field in the cannonical form* (1.12), *so*

$$\begin{aligned}\frac{\partial y^{\pm}}{\partial t}(t,x) =& \eta^{\pm}(x \pm \delta t, y^{\pm}(t,x)) \\ =& a^{\pm}(x \pm \delta t)^{2k^{\pm}-1} + (x \pm \delta t)^{2k^{\pm}} f^{\pm}(x \pm \delta t) + y^{\pm}(t,x) g^{\pm}(x \pm \delta t, y^{\pm}(t,x)).\end{aligned}$$

Then,

$$y_1^{\pm}(x) = \frac{\partial y^{\pm}}{\partial t}(0,x) = a^{\pm} x^{2k^{\pm}-1} + x^{2k^{\pm}} f^{\pm}(x),$$

which coincides with the initial condition for $i = 1$ of the recursive formula (2.14).

Now, denoting

$$g_i^{\pm}(x) = \frac{1}{(i-1)!} \frac{\partial^{i-1}}{\partial y^{i-1}} g^{\pm}(x,0), \tag{2.22}$$

we get that

$$y g^{\pm}(x,y) = \sum_{m=1}^{\infty} y^m g_m^{\pm}(x).$$

Thus, for $i \geqslant 2$,

$$\begin{aligned}\frac{\partial^i y^{\pm}}{\partial t^i}(t,x) =& \frac{\partial^{i-1}}{\partial t^{i-1}}\left(a^{\pm}(x\pm\delta t)^{2k^{\pm}+1} + (x\pm\delta t)^{2k^{\pm}} f^{\pm}(x\pm\delta t) + \sum_{j=1}^{\infty} y^j(t,x) g_j^{\pm}(x\pm\delta t)\right)\\ =& a^{\pm}\frac{\partial^{i-1}}{\partial t^{i-1}}(x\pm\delta t)^{2k^{\pm}+1} + \frac{\partial^{i-1}}{\partial t^{i-1}}\left((x\pm\delta t)^{2k^{\pm}} f^{\pm}(x\pm\delta t)\right)\\ &+ \frac{\partial^{i-1}}{\partial t^{i-1}}\left(\sum_{j=1}^{\infty} y^j(t,x) g_j^{\pm}(x\pm\delta t)\right).\end{aligned}$$

Clearly,

$$\frac{\partial^{i-1}}{\partial t^{i-1}}(x\pm\delta t)^{2k^{\pm}+1}\Bigg|_{t=0} = \begin{cases}(\pm\delta)^{i-1}\dfrac{(2k^{\pm}-1)!}{(2k^{\pm}-i)!}x^{2k-i}, \text{ if } i \leqslant 2k^{\pm},\\ 0, \text{ if } i > 2k^{\pm}.\end{cases} \tag{2.23}$$

Now, using the Leibniz general rule (2.17), we get that

$$\frac{\partial^{i-1}}{\partial t^{i-1}}\left((x\pm\delta t)^{2k^{\pm}} f^{\pm}(x\pm\delta t)\right)\Bigg|_{t=0} = \begin{cases}(\pm\delta)^{i-1}\displaystyle\sum_{l=0}^{i-1}\binom{i-1}{l}\frac{(2k^{\pm})!}{(2k^{\pm}-l)!}x^{2k^{\pm}-l} f^{\pm\,(i-1-l)}(x), & \text{if } i \leqslant 2k^{\pm},\\ (\pm\delta)^{i-1}\displaystyle\sum_{l=0}^{2k^{\pm}-1}\binom{i-1}{l}\frac{(2k^{\pm})!}{(2k^{\pm}-l)!}x^{2k^{\pm}-l} f^{\pm\,(i-l-1)}(x), & \text{if } i > 2k^{\pm}.\end{cases} \tag{2.24}$$

and

$$\begin{aligned}&\frac{\partial^{i-1}}{\partial t^{i-1}}\left(\sum_{j=1}^{\infty}(y^{\pm}(t,x))^j g_j^{\pm}(x\pm\delta t)\right)\Bigg|_{t=0} =\\ &\sum_{j=1}^{\infty}\sum_{l=0}^{i-1}\binom{i-1}{l}(\pm\delta)^{i-l-1}\frac{\partial^l}{\partial t^l}\left(y^{\pm}(t,x)^j\right)\Bigg|_{t=0} g_j^{\pm\,(i-l-1)}(x).\end{aligned} \tag{2.25}$$

In addition, denoting $P_j(y) = y^j$, we get from the Faà di Bruno's Formula (2.10) that

$$\begin{aligned}\frac{\partial^l}{\partial t^l}(y^{\pm}(t,x)^j)\Bigg|_{t=0} =& \frac{\partial^l}{\partial t^l}P_j(y^{\pm}(t,x))\Bigg|_{t=0} = \sum_{m=1}^{l} P_j^{(m)}(0) B_{l,m}(y_1^{\pm}(x),\dots,y_{l-m+1}^{\pm}(x))\\ =& \begin{cases}0, \text{ if } l < j,\\ j! B_{l,j}(y_1^{\pm}(x),\dots,y_{l-j+1}^{\pm}(x)), \text{ if } l \geqslant j.\end{cases}\end{aligned} \tag{2.26}$$

Therefore, substituting (2.26) into (2.25), and taking (2.22) into account, we obtain

$$\begin{aligned}&\frac{\partial^{i-1}}{\partial t^{i-1}}\left(\sum_{j=1}^{\infty}(y^{\pm}(t,x))^j g_j^{\pm}(x\pm\delta t)\right)\Bigg|_{t=0}\\&=\sum_{j=1}^{\infty}\sum_{l=j}^{i-1}\binom{i-1}{l}(\pm\delta)^{i-l-1}j!B_{l,j}(y_1^{\pm}(x),\dots,y_{l-j+1}^{\pm}(x))\frac{1}{(j-1)!}\frac{\partial^{j+i-l-2}g^{\pm}}{\partial x^{i-l-1}\partial y^{j-1}}(x,0)\\&=\sum_{l=1}^{i-1}\sum_{j=1}^{l}j\binom{i-1}{l}(\pm\delta)^{i-l-1}B_{l,j}(y_1^{\pm}(x),\dots,y_{l-j+1}^{\pm}(x))\frac{\partial^{j+i-l-2}g^{\pm}}{\partial x^{i-l-1}\partial y^{j-1}}(x,0).\end{aligned} \tag{2.27}$$

Finally, putting (2.23), (2.24), and (2.27) together we get the recursive formula (2.14) for $y_i^{\pm}(x)$, $i\geqslant 2$, which concludes the proof of Claim 3.1.

We also claim that the coefficients $\mu_i^{\pm}$ vanish for $i\leqslant 2k^{\pm}-1$.

Claim 3.2. *The value μ_i vanishes for $i=1,\dots,2k^{\pm}-1$.*

Proof of Claim 3.2. *First of all, we proceed by induction in order to prove that*

$$y_i^{\pm}(x)=x^{2k^{\pm}-i}R_i^{\pm}(x)\ \textit{for}\ i\leqslant 2k^{\pm}, \tag{2.28}$$

where $R_i^{\pm}(x)$ is a smooth function. For $i=1$, (2.28) holds. Indeed,

$$y_1^{\pm}(x)=a^{\pm}x^{2k^{\pm}-1}+x^{2k^{\pm}}f^{\pm}(x)=x^{2k^{\pm}-1}(a^{\pm}+xf^{\pm}(x)).$$

Now, let $i\leqslant 2k^{\pm}$. Recall that, from (2.14),

$$\begin{aligned}y_i^{\pm}(x)=&a^{\pm}(\pm\delta)^{i-1}\frac{(2k^{\pm}-1)!}{(2k^{\pm}-i)!}x^{2k-i}+\sum_{l=0}^{i-1}\binom{i-1}{l}\frac{(2k^{\pm})!}{(2k^{\pm}-l)!}x^{2k^{\pm}-l}f^{\pm(i-1-l)}(x)\\&+\sum_{l=1}^{i-1}\sum_{j=1}^{l}j\binom{i-1}{l}(\pm\delta)^{i-l-1}B_{l,j}(y_1^{\pm}(x),\dots,y_{l-j+1}^{\pm}(x))\frac{\partial^{j+i-l-2}g^{\pm}}{\partial x^{i-l-1}\partial y^{j-1}}(x,0).\end{aligned}$$

Suppose that (2.28) holds for all $s\leqslant i-1$, that is, $y_s^{\pm}(x)=x^{2k^{\pm}-s}R_s(x)$. Then, taking into account that $B_{l,j}$ is a homogeneous polynomial of degree j with $l-j+1$ variables, we have that

$$\sum_{l=1}^{i-1}\sum_{j=1}^{l}j\binom{s-1}{l}(\pm\delta)^{s-l-1}B_{l,j}(y_1^{\pm}(x),\dots,y_{l-j+1}^{\pm}(x))\frac{\partial^{j+i-l-2}g^{\pm}}{\partial x^{i-l-1}\partial y^{j-1}}(x,0)=x^{2k^{\pm}-i+1}T(x),$$

where T is a smooth function. Then,

$$\begin{aligned}y_i^{\pm}(x)=&a^{\pm}(\pm\delta)^{i-1}\frac{(2k^{\pm}-1)!}{(2k^{\pm}-i)!}x^{2k^{\pm}-i}\\&+\sum_{l=0}^{i-1}\binom{i-1}{l}\frac{(2k^{\pm})!}{(2k^{\pm}-l)!}x^{2k^{\pm}-l}f^{\pm(i-1-l)}(x)+x^{2k^{\pm}-i+1}T(x)\\=&x^{2k^{\pm}-i}R_i^{\pm}(x),\end{aligned}$$

which implies that (2.28) holds for all $i \leqslant 2k^{\pm}$.

From (2.16) and (2.28), we conclude that

$$\mu_i^{\pm} = \frac{1}{i!}\sum_{j=1}^{i}(\pm\delta)^j\binom{i}{j}\frac{\partial^{i-j}y_j^{\pm}}{\partial x^{i-1}}(0) = \frac{1}{i!}\sum_{j=1}^{i}(\pm\delta)^j\binom{i}{j}\frac{\partial^{i-j}}{\partial x^{i-1}}\left(x^{2k^{\pm}-j}R_j(x)\right)\Big|_{x=0} = 0,$$

for $i \leqslant 2k^{\pm}-1$, *which proves Claim 3.2.*

By comparing the coefficients of x_0^i in both sides of equality (2.21) and taking Claim 3.2 into account, we conclude that

$$\mu_i^{\pm} = \sum_{j=2k^{\pm}}^{i}\mu_j^{\pm}\hat{B}_{i,j}(\alpha_1^{\pm},\dots,\alpha_{i-j+1}^{\pm}), \quad i \geqslant 2k^{\pm}.$$

Doing some computations, we have that

$$\begin{aligned}\mu_i^{\pm} =& \mu_{2k^{\pm}}^{\pm}\hat{B}_{i,2k^{\pm}}(\alpha_1^{\pm},\dots,\alpha_{i-2k^{\pm}+1}^{\pm}) + \sum_{j=2k^{\pm}+1}^{i}\mu_j^{\pm}\hat{B}_{i,j}(\alpha_1^{\pm},\dots,\alpha_{i-j+1}^{\pm})\\ =& \mu_{2k^{\pm}}^{\pm}\hat{B}_{i,2k^{\pm}}(\alpha_1^{\pm},\dots,\alpha_{i-2k^{\pm}}^{\pm},0) + \mu_{2k^{\pm}}^{\pm}\alpha_1^{\pm\,2k^{\pm}-1}\alpha_{i-2k^{\pm}+1}^{\pm}2k^{\pm}\\ &+ \sum_{j=2k^{\pm}+1}^{i}\mu_j^{\pm}\hat{B}_{i,n}(\alpha_1^{\pm},\dots,\alpha_{i-j+1}^{\pm}).\end{aligned} \tag{2.29}$$

Therefore, isolating $\alpha_{i-2k^{\pm}+1}^{\pm}$ in (2.29), we get

$$\alpha_{i-2k^{\pm}+1}^{\pm} = \frac{\mu_i^{\pm} - \mu_{2k^{\pm}}\hat{B}_{i,2k^{\pm}}(\alpha_1^{\pm},\dots,\alpha_{i-2k^{\pm}}^{\pm},0) - \sum_{j=2k^{\pm}+1}^{i}\mu_j\hat{B}_{i,j}(\alpha_1^{\pm},\dots,\alpha_{i-j+1}^{\pm})}{\alpha_1^{\pm\,2k^{\pm}-1}2k^{\pm}\mu_{2k^{\pm}}}. \tag{2.30}$$

Finally, taking into account that $\alpha_1^{\pm} = -1$ and doing a change of the index in (2.30), we obtain the recurrence (2.15) for $\alpha_i^{\pm}$. This concludes the proof of Theorem 3. □

In Section 2.5, we present *Mathematica*'s codes that offer straightforward algorithms for computing all the Lyapunov coefficients when k is known. Code 1 represents the recursive formulae for $y_i^{\pm}$ (2.14), Code 2 represents the formulae for the values $\mu_i^{\pm}$ (2.16) and Code 3 represents the recursive formulae for the coefficients $\alpha^{\pm}$ (2.15).

In the following proposition, applying Theorem 3, we compute $\alpha_n^{\pm}$, for $n = 1,2,3,4$, for a general $(2k^+,2k^-)$-monodromic tangential singularity. Recall that the i-th Lyapunov coefficient is given by $V_n = \delta(\alpha_n^+ - \alpha_n^-)$.

Proposition 2. *Assume that the Filippov vector field* (1.4) *has a* $(2k^+, 2k^-)$*-monodromic tangential singularity at the origin, for positive integers* k^+ *and* k^-*, and denote*

$$f^{\pm}(x) = \sum_{i=0}^{\infty} f_i^{\pm} x^i \quad \text{and} \quad g^{\pm}(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} g_{i,j}^{\pm} x^i y^j.$$

Then, the first four coefficients $\alpha_n^{\pm}$*'s of the series* (2.6) *of the half-return maps* $\varphi^{\pm}$ *are given by*

$$\begin{aligned}
\alpha_1^{\pm} &= -1, \quad \alpha_2^{\pm} = \frac{-2f_0 \pm 2\delta a g_{0,0}}{2ak+a}, \quad \alpha_3^{\pm} = -(\alpha_2^{\pm})^2,\\
\alpha_4^{\pm} &= \frac{4(k(2k+3)+7)(-f_0 \pm \delta a g_{0,0})^3}{3a^3(2k+1)^3}\\
&\mp \frac{12\delta a(f_0 \mp \delta a g_{0,0})\left(a\left(g_{1,0} \mp \delta {g_{0,0}}^2\right) + 2f_0 g_{0,0} \mp 2\delta f_1\right)}{3a^3(8k+4)}\\
&\pm \frac{4\delta a^2\left(a\left(2g_{2,0} + {g_{0,0}}^3\right) + 6g_{0,0}f_1 + 3f_0 g_{1,0} \mp 3\delta\left(a g_{1,0} g_{0,0} + f_0 {g_{0,0}}^2 + 2f_2\right)\right)}{3a^3(8k+12)} + \xi_k,
\end{aligned}$$

where $\xi_1 = -\dfrac{4a g_{0,1}}{15}$ *and* $\xi_k = 0$ *for* $k > 1$*. For the sake of simplicity, in the above expressions we are dropping the sign* $\pm$ *from* $a^{\pm}, k^{\pm}, f_i^{\pm}$*, and* $g_{i,j}^{\pm}$*.*

Proof. We start this proof by computing the coefficients $\alpha_n^{\pm}$, $n = 1, 2, 3, 4$, for $k^{\pm} = 1$:

$$\begin{aligned}
\alpha_1^{\pm} &= -1, \quad \alpha_2^{\pm} = \frac{-2f_0 \pm \delta 2 g_{0,0}}{3a}, \quad \alpha_3^{\pm} = -{\alpha_2^{\pm}}^2,\\
\alpha_4^{\pm} &= \frac{1}{135a^3}\Big(80f_0^3 + (\mp\delta)150 a f_0^2 g_{0,0} + 132a^2 f_0 g_{0,0}^2 + (\mp\delta)44a^3 g_{0,0}^3\\
&\quad - 90 a f_0 f_1 \pm \delta 36 a^2 g_{0,0} f_1 + 54a^2 f_2 + 36a^4 g_{0,1} \pm \delta 18 a^2 f_0 g_{1,0}\\
&\quad - 18a^3 g_{0,0} g_{1,0} + (\mp\delta)18a^3 g_{2,0}\Big).
\end{aligned} \tag{2.31}$$

From now on, we shall consider $k \geqslant 2$. In order to compute the remaining coefficients, the following partial Bell polynomials (Definition 3) are needed:

$$\begin{aligned}
B_{n,n}(x_1) &= (x_1)^n,\\
B_{n,n-1}(x_1,x_2) &= \binom{n}{2}(x_1)^{n-2}x_2,\\
B_{n,n-2}(x_1,x_2,x_3) &= \binom{n}{3}(x_1)^{n-3}x_3 + 3\binom{n}{4}(x_1)^{n-4}(x_2)^2,\\
B_{n,n-3}(x_1,x_2,x_3,x_4) &= \binom{n}{4}(x_1)^{n-4}x_4 + 10\binom{n}{5}(x_1)^{n-5}x_2x_3 + 15\binom{n}{6}(x_1)^{n-6}x_2^3.
\end{aligned} \tag{2.32}$$

Now, developing the recursive formula given by (2.15) for α_i and using (2.32), we get that

$$\begin{aligned} &\alpha_1^{\pm} = -1, \quad \alpha_2^{\pm} = \frac{-\mu_{2k+1}^{\pm}}{k^{\pm}\mu_{2k}^{\pm}}, \quad \alpha_3^{\pm} = -{\alpha_2^{\pm}}^2, \\ &\alpha_4^{\pm} = \frac{(7+k(3+2k)){\mu_{2k+1}^{\pm}}^3 - 6k(1+k)\mu_2^{\pm}k\mu_{2k+1}^{\pm}\mu_{2k+2}^{\pm} + 6k^2{\mu_{2k}^{\pm}}^2\mu_{2k+3}^{\pm}}{6k^3{\mu_{2k}^{\pm}}^3}. \end{aligned} \tag{2.33}$$

From (2.33), we have to compute $\mu_{2k}^{\pm}$, $\mu_{2k+1}^{\pm}$, $\mu_{2k+2}^{\pm}$, and $\mu_{2k+3}^{\pm}$. Recall that we are denoting

$$f^{\pm}(x) = \sum_{i=0}^{\infty} f_i^{\pm} x^i \quad \text{and} \quad g^{\pm}(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} g_{i,j}^{\pm} x^i y^j.$$

Throughout the proof, we shall also drop the sign $\pm$ from $k^{\pm}$, $a^{\pm}$, $f_i^{\pm}$, and $g_{i,j}^{\pm}$.

Computation of $\mu_{2k}^{\pm}$. From (2.16), we have that

$$\mu_{2k}^{\pm} = \frac{1}{(2k)!}\sum_{j=1}^{2k}(\mp\delta)^j\binom{2k}{j}(y_j^{\pm})^{(2k-j)}(0). \tag{2.34}$$

Taking (2.14) into account, it follows that

$$(y_j^{\pm})^{(2k-j)}(0) = (\pm\delta)^{j-1}a(2k-1)!. \tag{2.35}$$

Then, substituting (2.35) in (2.34), we obtain

$$\mu_{2k}^{\pm} = \frac{1}{(2k)!}\sum_{j=1}^{2k}(\mp\delta)^j\binom{2k}{j}(\mp\delta)^{j-1}a(2k-1)! = \frac{(\mp\delta)a}{2k}. \tag{2.36}$$

Computation of $\mu_{2k+1}^{\pm}$. From (2.16), we have that

$$\mu_{2k+1}^{\pm} = \frac{1}{(2k+1)!}\sum_{j=1}^{2k+1}(\mp\delta)^j\binom{2k+1}{j}(y_j^{\pm})^{(2k+1-j)}(0). \tag{2.37}$$

Taking (2.14) into account, it follows that

$$(y_j^{\pm})^{(2k+1-j)}(0) = \begin{cases} (2k)!f_0, & \text{if } j=1, \\ (\pm\delta)^{j-1}(2k)!f_0 + (\pm\delta)^j a(2k-1)!g_{0,0}, & \text{if } 2 \leqslant j \leqslant 2k+1. \end{cases} \tag{2.38}$$

Although formula (2.14) distinguishes the cases $j < 2k+1$ and $j = 2k+1$, when developing this formula we see that these cases can be put together as (2.38). Now, substituting (2.38) into (2.37), we obtain

$$\begin{aligned} \mu_{2k+1}^{\pm} =& \frac{1}{(2k+1)!}\Bigg((\mp\delta)\binom{2k+1}{1}(2k)!f_0 + \sum_{j=2}^{2k+1}(\mp\delta)^j\binom{2k+1}{j}\big((\pm\delta)^{j-1}(2k)!f_0 \\ &+ (\pm\delta)^j a(2k-1)!g_{0,0}\big)\Bigg) = \frac{(\mp\delta)f_0 + ag_{0,0}}{2k+1}. \end{aligned} \tag{2.39}$$

Computation of $\mu^{\pm}_{2k+2}$. From (2.16), we have that

$$\mu^{\pm}_{2k+2} = \frac{1}{(2k+2)!}\sum_{j=1}^{2k+2}(\mp\delta)^j\binom{2k+2}{j}(y^{\pm}_j)^{(2k+2-j)}(0). \tag{2.40}$$

Taking (2.14) into account, it follows that

$$\begin{aligned}
(y^{\pm}_1)^{(2k+1)}(0) =&(2k+1)!f_1,\\
(y^{\pm}_2)^{(2k)}(0) =&(\pm\delta)\Big(\frac{(2k)!}{(2k-1)!}2f_1+2f_1\Big)+2\Big(f_0g_{0,0}+ag_{1,0}\Big),\\
(y^{\pm}_j)^{(2k+2-j)}(0) =&\frac{2k+2-j}{2!}\Bigg((\pm\delta)^{j-1}\Big(\frac{(2k)!}{(2k+1-j)!}2f_1\\
&+\binom{j-1}{j-2}\frac{(2k)!}{(2k+2-i)!}2f_1\Big)+2\Big(\big((\pm\delta)^{j-2}\frac{(2k)!}{(2k+2-j)!}f_0\\
&+a(\pm\delta)^{j-3}\frac{(2k-1)!}{(2k+2-j)!}g_{0,0}\big)g_{0,0}\\
&+a(\pm\delta)^{j-2}\frac{(2k-1)!}{(2k+1-j)!}g_{1,0}\Big)\\
&+\Big(2a\binom{j-1}{j-2}(\pm\delta)^{j-4}\frac{(2k-1)!}{(2k+2-j)!}g_{1,0}\Big)\Bigg),\ \text{if } 3\leqslant j\leqslant 2k,\\
(y^{\pm}_{2k+1})'(0) =&(2k)!f_1+\frac{(2k)!}{(2k-1)!}(2k)!f_1+\big((\pm\delta)(2k)!f_0\\
&+a(2k-1)!g_{0,0}\big)g_{0,0}\\
&+(\pm\delta)a(2k-1)!g_{1,0}+\binom{2k}{2k-1}(\pm\delta)a^{\pm}(2k-1)!g_{1,0},\\
y^{\pm}_{2k+2}(0) =&\binom{2k+1}{2k}(2k)!(\pm\delta)f_1+\big((2k)!f_1+(\pm\delta)a(2k-1)!g_{0,0}\big)g_{0,0}\\
&+\binom{2k+1}{2k}a(2k-1)!g_{1,0}.
\end{aligned} \tag{2.41}$$

Substituting (2.41) in (2.40) and proceeding with algebraic manipulations, we obtain

$$\mu^{\pm}_{2k+2} =\frac{2f_0g_{0,0}+(\mp\delta)ag^2_{0,0}+2(\mp\delta)f_1+ag_{1,0}}{4k+4}. \tag{2.42}$$

Computation of $\mu^{\pm}_{2k+3}$. From (2.16), we have that

$$\mu^{\pm}_{2k+3} = \frac{1}{(2k+3)!}\sum_{j=1}^{2k+3}(\mp\delta)^j\binom{2k+3}{j}(y^{\pm}_j)^{(2k+3-j)}(0). \tag{2.43}$$

Taking (2.14) into account, it follows that

$$
\begin{aligned}
(y_1^{\pm})^{(2k+2)}(0) =&(2k+2)!f_2,\\
(y_2^{\pm})^{(2k+1)}(0) =&(\pm\delta)(2k+1)!\big(2kf_2+12f_2\big)+(y_1^{\pm})^{(2k+1)}(0)g_{0,0}\\
&+2(y_1^{\pm})^{(2k)}(0)g_{1,0}+2(y_1^{\pm})^{(2k-1)}(0)g_{2,0},\\
(y_3^{\pm})^{(2k)}(0) =&\frac{(2k)!}{3!}\big(\frac{(2k)!}{(2k-2!)}6f_2+3\frac{(2k)!}{(2k-1)!}12f_2+12f_2\big)+(y_2^{\pm})^{(2k)}(0)g_{0,0}\\
&+2(y_2^{\pm})^{(2k-1)}(0)g_{1,0}+2(y_2^{\pm})^{(2k-2)}(0)g_{2,0}\\
&+3(\pm\delta)\big((y_1^{\pm})^{(2k)}(0)g_{1,0}+2(y_1^{\pm})^{(2k-1)}(0)g_{2,0}\big),\\
(y_j^{\pm})^{(2k+3-j)}(0) =&\frac{(2k+3-j)!}{3!}\Bigg((\pm\delta)^{j-1}\Big(\frac{(2k)!}{(2k+1-j)!}6f_2+\binom{j-1}{j-2}\frac{(2k)!}{(2k+2-i)!}12f_2\\
&+\binom{j-1}{j-3}\frac{(2k-1)!}{(2k+3-j)!}12f_2\Big)\Bigg)+(y_{j-1}^{\pm})^{(2k+3-j)}(0)g_{0,0}\\
&+2(y_{j-1}^{\pm})^{(2k+2-j)}(0)g_{1,0}+2(y_{j-1}^{\pm})^{(2k+1-j)}(0)g_{2,0}\\
&+\binom{j-1}{j-2}(\pm\delta)\big((y_{j-2}^{\pm})^{(2k+3-j)}(0)g_{1,0}+2(y_{j-2}^{\pm})^{(2k+2-j)}(0)g_{2,0}\big)\\
&+2\binom{j-1}{j-3}(y_{j-3}^{\pm})^{(2k+3-j)}(0)g_{2,0},\ \text{if } 4\leqslant j\leqslant 2k,\\
(y_{2k+1}^{\pm})''(0) =&(2k)!2f_2+\binom{2k}{2k-1}(2k)!4f_2+\binom{2k}{2k-2}\frac{(2k)!}{2!}4f_2\\
&+(y_{2k}^{\pm})''(0)g_{0,0}+2(y_{2k}^{\pm})'(0)g_{1,0}+2y_{2k}^{\pm}(0)g_{2,0}\\
&+2\binom{2k}{2k-1}(\pm\delta)\Big(\big(\frac{(2k)!}{2!}f_0+a(\pm\delta)\frac{(2k-1)!}{2!}g_{0,0}\big)g_{1,0}\\
&+2a(2k-1)!g_{2,0}\Big)+4\binom{2k}{2k-2}(\pm\delta)a\frac{(2k-1)!}{2!}g_{2,0},\\
(y_{2k+2}^{\pm})'(0) =&\binom{2k+1}{2k}(2k)!(\pm\delta)2f_2+(\pm\delta)\binom{2k+1}{2k-1}(2k)!2f_2\\
&+\Big((2k)!f_1+\binom{2k}{2k-1}(2k)!f_1+\big((\pm\delta)(2k)!f_0+a(2k-1)!g_{0,0}\big)g_{0,0}\\
&+(\pm\delta)a(2k-1)!g_{1,0}+\binom{2k}{2k-1}(2k-1)!g_{1,0}\Big)g_{0,0}\\
&+\big((2k)!f_0+(\pm\delta)a(2k-1)!g_{0,0}\big)g_{1,0}\\
&+\binom{2k+1}{2k}(\pm\delta)\Big(\big((\pm\delta)(2k)!f_0+a(2k-1)!g_{0,0}\big)g_{1,0}\\
&+2(\pm\delta)a(2k-1)!g_{2,0}\Big)+2\binom{2k+1}{2k-1}a(2k-1)!g_{2,0},
\end{aligned}
$$

$$
\begin{aligned}
(y^{\pm}_{2k+3})'(0) =& \binom{2k+2}{2k}(2k)!2f_2 + \left(\binom{2k+1}{2k}(2k)!(\pm\delta)f_1\right. \\
&+ \big((2k)!f_0 + a(\pm\delta)(2k-1)!g_{0,0}\big)g_{0,0} + \left.\binom{2k+1}{2k}a(2k-1)!g_{1,0}\right)g_{0,0} \\
&+ \binom{2k+2}{2k+1}(\pm\delta)\big((2k)!f_0 + (\pm\delta)a(2k-1)!g_{0,0}\big)g_{1,0} \\
&+ 2\binom{2k+2}{2k}(\pm\delta)a(2k-1)!g_{2,0}.
\end{aligned}
$$

Notice that, at this point, the computations start to become more cumbersome. Also, in the formulae above, there are some values of $y_i^{(j)}(0)$ which are not explicitly computed. However, except for $(y^{\pm}_{j-1})^{(2k+1-j)}(0) = a(\pm\delta)^{j-2}\dfrac{(2k-1)!}{(2k+1-j)!}$, the others can be computed by using the formulae (2.35), (2.38), and (2.41). Then, substituting all these values into (2.43) and proceeding with algebraic manipulations, we obtain

$$
\begin{aligned}
\mu^{\pm}_{2k+3} = \quad & \frac{1}{6(3+3k)}\Big(6g_{0,0}f_1 + 6(\mp\delta)f_2 + 3f_0\big((\mp\delta)g_{0,0}^2 + g_{1,0}\big) \\
& + a\big(g_{0,0}^3 + 3(\mp\delta)g_{0,0}g_{1,0} + 2g_{2,0}\big)\Big).
\end{aligned} \tag{2.44}
$$

Finally, substituting (2.36), (2.39), (2.42), and (2.44) into (2.33) and proceeding with algebraic manipulations, we conclude that

$$
\begin{aligned}
\alpha_1^{\pm} &= -1, \quad \alpha_2^{\pm} = \frac{-2f_0 \pm 2\delta a g_{0,0}}{2ak+a}, \quad \alpha_3^{\pm} = -(\alpha_2^{\pm})^2, \\
\alpha_4^{\pm} &= \frac{4(k(2k+3)+7)(-f_0 \pm \delta a g_{0,0})^3}{3a^3(2k+1)^3} \\
&\mp \frac{12\delta a(f_0 \mp \delta a g_{0,0})\left(a\left(g_{1,0} \mp \delta g_{0,0}{}^2\right) + 2f_0 g_{0,0} \mp 2\delta f_1\right)}{3a^3(8k+4)} \\
&\pm \frac{4\delta a^2\left(a\left(2g_{2,0} + g_{0,0}{}^3\right) + 6g_{0,0}f_1 + 3f_0 g_{1,0} \mp 3\delta\left(a g_{1,0}g_{0,0} + f_0 g_{0,0}{}^2 + 2f_2\right)\right)}{3a^3(8k+12)},
\end{aligned} \tag{2.45}
$$

for $k \geqslant 2$. This proof follows by comparing expressions (2.31) and (2.45) with the ones provided in the statement of Proposition 2. □

2.5 List of source codes

In this section, based on Theorem 3, we present an implemented *Mathematica* algorithm for computing the coefficients α_n^+ and α_n^- of the series (2.6) of the half-return maps φ^+ and φ^- and, consequently, the Lyapunov coefficients V_n's.

In what follows, we are denoting $k^+ =$ kp, $k^- =$ kn, $a^+ =$ ap , $a^- =$ an, $\mu^+ = \mu$p, and $\mu^- = \mu$n. In addition, yp0[i] and yp1[i] denote y_i^+, respectively, for $i \leqslant 2k^+$

and for $i > 2k^+$. Analogously, yn0[i] and yn1[i] denote y_i^-, respectively, for $i \leqslant 2k^-$ and for $i > 2k^-$.

In order to run the codes for computing the first N Lyapunov coefficients, we have to specify the values for kp, kn, δ, and imax=N.

Source code 1 – Mathematica's algorithm for computing $y_i^{\pm}$(2.14).

```
yp0[1] = ap x^(2 kp - 1) + x^(2 kp) fp[x]
yp0[i_] := δ^(i -1) (ap (2 kp - 1)!/(2 kp - i)! x^(2 kp - i) + Sum[Binomial[i - 1, l] (2 kp)!/(2 kp - l)! x^(2 kp - l) D[fp[x], {x, i - 1 - l}], {l, 0, i - 1}]) + Sum[Sum[j Binomial[i - 1, l] δ^(i - l - 1) BellY[l, j,Yp[l - j + 1, x]] (D[D[gp[x, y], {y, j - 1}], {x, i - 1 - l}] /. y -> 0), {j, 1, l}], {l, 1, i - 1}]
yp1[i_] := δ^(i - 1) (Binomial[i - 1, 2 kp] (2 kp)! D[fp[x], {x, i - 1 - 2 kp}] + Sum[Binomial[i - 1, l] (2 kp)!/(2 kp - l)! x^(2 kp - l) D[fp[x], {x, i - 1 - l}], {l, 0, 2 kp - 1}]) + Sum[Sum[j Binomial[i - 1, l] δ^(i - l - 1) BellY[l, j, Yp[l - j + 1,x]] (D[D[gp[x, y], {y, j - 1}], {x, i -1 - l}] /. y -> 0), {j,1, l}], {l, 1, i - 1}]
Yp[1] = {yp0[1]};
For[i = 2, i <= 2 kp, i++, Yp[i] = Join[Yp[i - 1],{yp0[i]}];]
For[i = 2 kp + 1, i <= 2 kp + imax, i++, Yp[i] = Join[Yp[i - 1], {yp1[i]}];]
For[i = 1, i <= 2 kp + imax, i++, yp[i] = Yp[2 kp + imax][[i]]]
yn0[1] = an x^(2 kn - 1) + x^(2 kn) fn[x]
yn0[i_] := (-δ)^(i - 1) (an (2 kn - 1)!/(2 kn - i)! x^(2 kn - i) + Sum[Binomial[i - 1, l] (2 kn)!/(2 kn - l)! x^(2 kn - l) D[fn[x], {x, i - 1 - l}], {l, 0, i - 1}]) + Sum[Sum[j Binomial[i - 1, l] (-δ)^(i - 1 - l) BellY[l, j, Yn[l - j + 1,x]] (D[D[gn[x, y], {y, j - 1}], {x, i - 1 - l}] /. y -> 0)
```

```
, {j,1, l}], {l, 1, i - 1}]
yn1[i_] := (-δ)^(i -1) (Binomial[i - 1, 2 kn] (2 kn)! D[fn[x], {x, i - 1 - 2 kn}] + Sum[Binomial[i - 1, l] (2 kn)!/(2 kn - l)! x^(2 kn - l) D[fn[x], {x, i - 1 - l}], {l, 0, 2 kn - 1}]) + Sum[Sum[j Binomial[i - 1, l] (-δ)^(i - 1 - l) BellY[l, j,Yn[l - j + 1, x]] (D[D[gn[x, y], {y, j - 1}], {x, i - 1 - l}] /. y -> 0), {j, 1, l}], {l, 1, i - 1}]
Yn[1] = {yn0[1, x]};
For[i = 2, i <= 2 kn, i++, Yn[i] = Join[Yn[i - 1], {yn0[i]}];]
For[i = 2 kn + 1, i <= 2 kn + imax, i++, Yn[i] = Join[Yn[i - 1], {yn1[i]}];]
For[i = 1, i <= 2 kn+ imax, i++, yn[i] = Yn[2 kn + imax][[i]]]
```

Source code 2 – Mathematica's algorithm for computing $\mu_n^{\pm}$(2.16).

```
μp[n_] := 1/n! Sum[(-δ)^j Binomial[n, j] D[yp[j, x], {x, n - j}] /. x -> 0, {j, 1, n}]
μn[n_] := 1/n! Sum[δ^j Binomial[n, j] D[yn[j, x], {x, n - j}] /.x -> 0, {j, 1, n}]
```

Source code 3 – Mathematica's algorithm for computing $\alpha_n^{\pm}$ and V_n(2.15).

```
αp[1] = -1;
Ap[1] = {αp[1]};
For[n = 2, n <= imax, n++, αp[n] = Factor[(μp[2 kp] (2 kp)!/(n + 2 kp - 1)! BellY[n + 2 kp - 1, 2 kp, Join[Ap[n - 1], {0}]] + Sum[μp[i] i!/(n + 2 kp - 1)! BellY[n + 2 kp - 1, i, Ap[n + 2 kp - i]], {i, 2 kp + 1, n + 2 kp - 1}] - μp[ n + 2 kp - 1])/(2 kp μp[2 kp])]; Ap[n] = Join[Ap[n - 1], {n! αp[n]}];]
αn[1] = -1;
An[1] = {αn[1]};
For[n = 2, n <= imax, n++, αn[n] = Factor[(μn[2 kn] (2 kn)!/(n + 2 kn - 1)! BellY[n + 2
```

```
kn - 1, 2 kn, Join[An[n - 1], {0}]] + Sum
    [μn[i] i!/(n + 2 kn - 1)! BellY[n + 2 kn -
    1, i, An[n + 2 kn - i]], {i, 2 kn + 1, n
    + 2 kn - 1}] - μn[n + 2 kn - 1])/(2 kn μn
    [2 kn])]; An[n] = Join[An[n - 1], {n! αn[n
    ]}];]
For[n = 1, n <= imax, j++, V[n] = δ(αp[n] - αn
    [n]);]
```

3 The cyclicity problem for monodromic tangential singularities

The cyclicity problem in the context of vector fields is a critical topic in mathematics and physics, particularly within the realm of differential equations and dynamical systems. This problem arises when one seeks to comprehend the presence of periodic orbits, or closed trajectories within vector fields. This chapter is based on the papers (39, 41), we will explore the cyclicity problem using the Lyapunov coefficients from Chapter 2 and we will show results with a standard view of bifurcation of limit cycles and also present an alternative look considering a pseudo-Hopf bifurcation.

It is well known that Lyapunov coefficients can be used to study the appearance of small amplitude limit cycles in smooth and non-smooth vector fields around weak foci (see, for instance, (45) for smooth vector fields and (18, 27, 28) for non-smooth vector fields). In this section, we apply classical ideas to study the appearance of limit cycles around monodromic tangential singularities. We start by providing Hopf and Bautin-like bifurcation theorems for monodromic tangential singularities.

3.1 Malgrange preparation theorem

A fundamental tool for studying the cyclicity problem is the Malgrange preparation theorem. The Malgrange preparation theorem is a counterpart version to the Weierstrass preparation theorem in the realm of smooth functions, initially proposed by René Thom and later proved by B. Malgrange (37). In (37), we can find an algebraic version of the theorem with a perspective of modules over rings of smooth, real-valued germs. To elaborate, we consider a manifold X, with $p \in X$. Denote $C_p^\infty(X)$ as the ring of real-valued germs of smooth functions at p on X. Let $M_p(X)$ denote the unique maximal ideal of $C_p^\infty(X)$, characterized by germs which vanish at p. Let A be a C_p^∞-module, and let $f : X \to Y$ be a smooth function between manifolds and consider $q = f(p)$. Through composition on the right, f induces a ring homomorphism $f^* : C_q^\infty(Y) \to C_p^\infty(X)$. Thus, A can be seen as a $C_q^\infty(Y)$-module. Shortly, the Malgrange preparation theorem concludes that if A is a finitely-generated $C_p^\infty(X)$-module, then A is a finitely-generated $C_q^\infty(Y)$-module if and only if $A/M_q(Y)A$ is a finite-dimensional real vector space.

Here, instead of working with the algebraic version, we will present a version for C^∞ functions (37).

Theorem 4 ((37)). *Let $f(t,x)$ be a C^∞ function of $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$ near $(0,0)$ satisfying*

$$f(0,0) = 0, \frac{\partial f}{\partial t}(0,0) = 0, \frac{\partial^2 f}{\partial t^2}(0,0) = 0, \ldots, \frac{\partial^{k-1} f}{\partial t^{k-1}}(0,0) = 0, \frac{\partial^k f}{\partial t^k}(0,0) \neq 0.$$

Then, there exists a factorization

$$f(t,x) = c(t,x)(t^k + a_{k-1}(x)t^k - 1 + \cdots + a_1(x)t + a_0(x))$$

where a_j and c are C^∞ functions near 0 and $(0,0)$ respectively, $c(0,0) \neq 0$ and $a_j(0) = 0$.

3.2 Hopf and Bautin-like bifurcations

This section is based on the paper (39). It is worth noticing that in the literature, the Hopf bifurcation, also known as the Poincaré-Andronov-Hopf bifurcation (2) refers to the appearance of a limit cycle from an equilibrium as a parameter crosses a critical value. The next theorem we will present is a Hopf-like bifurcation result that shows us the local birth of a limit cycle from a 1-parameter family in the context of Filippov systems with a $(2k^+, 2k^-)$-monodromic tangential singularity.

Theorem 5. *Let k^+ and k^- be positive integers and let Z_λ be an 1-parameter family of Filippov vector fields (1.4) having a $(2k^+, 2k^-)$-monodromic tangential singularity at the origin for all λ in an interval I. Let $V_2(\lambda)$ and $V_4(\lambda)$ be, respectively, the second and the forth Lyapunov coefficients. Assume that, for some $\lambda_0 \in I$, $V_2(\lambda_0) = 0$, $d := V_2'(\lambda_0) \neq 0$, and $\ell := V_4(\lambda_0) \neq 0$. Then, there exists a neighborhood $J \subset I$ of λ_0 such that, for all $\lambda \in J$ satisfying $d\ell(\lambda - \lambda_0) < 0$, the Filippov vector field Z_λ admits a hyperbolic limit cycle in a $\sqrt{|\lambda - \lambda_0|}$-neighborhood of the origin. In addition, such a limit cycle is asymptotically stable (resp. unstable) provided that $\ell < 0$ (resp. $\ell > 0$).*

Proof. Let Z_λ be a 1-parameter family of Filippov vector fields (1.4) having a $(2k^+, 2k^-)$-monodromic tangential singularity at the origin for all λ in an interval I. Let $V_2(\lambda)$, $V_3(\lambda)$, and $V_4(\lambda)$ be, respectively, the second, the third, and the forth Lyapunov coefficients. Accordingly, the displacement function of Z_λ around the origin writes

$$\Delta(x;\lambda) = V_2(\lambda)x^2 + V_3(\lambda)x^3 + V_4(\lambda)x^4 + O(x^5) = x^2\Gamma(x;\lambda) \tag{3.1}$$

where

$$\Gamma(x;\lambda) = V_2(\lambda) + V_3(\lambda)x + V_4(\lambda)x^2 + O(x^3). \tag{3.2}$$

By hypothesis, there exists $\lambda_0 \in I$ such that $V_2(\lambda_0) = 0$, $V_2'(\lambda_0) = d \neq 0$, and $V_4(\lambda_0) = \ell \neq 0$. Thus,

$$\Gamma(0;\lambda_0) = V_2(\lambda_0) = 0, \quad \frac{\partial^2 \Gamma}{\partial x^2}(0;\lambda_0) = 2V_4(\lambda_0) \neq 0,$$

and, from Theorem 2,

$$\frac{\partial \Gamma}{\partial x}(0;\lambda_0) = V_3(\lambda_0) = 0.$$

Therefore, as a consequence of the Malgrange Preparation Theorem there exists a small neighborhood $W \subset \mathbb{R}^2$ of $(0,\lambda_0)$ and smooth functions $c(x;\lambda)$, $a_0(\lambda)$, and $a_1(\lambda)$ such that $c(x,\lambda) \neq 0$ and

$$\Gamma(x;\lambda) = c(x,\lambda)(x^2 + a_1(\lambda)x + a_0(\lambda)), \tag{3.3}$$

for every $(x,\lambda) \in W$.

From (3.2) and (3.3) we obtain that

$$a_0(\lambda) = \frac{V_2(\lambda)}{c(0,\lambda)} \text{ and } a_1(\lambda) = \frac{V_3(\lambda) - \partial_x c(0,\lambda)a_0(\lambda)}{c(0,\lambda)}.$$

Consequently, $a_0(\lambda_0) = a_1(\lambda_0) = 0$. In addition, one can see that

$$c(0,\lambda_0) = V_4(\lambda_0) = \ell \text{ and } a_0'(\lambda_0) = \frac{V_2'(\lambda_0)}{c(0,\lambda_0)} = \frac{d}{\ell}.$$

Now, taking the hypothesis $d\ell(\lambda - \lambda_0) < 0$ into account, we can easily compute the unique positive root of (3.3) in W as

$$\begin{aligned} x^* =& \frac{-a_1(\lambda) + \sqrt{a_1(\lambda)^2 - 4a_0(\lambda)}}{2} = \sqrt{\frac{-V_2'(\lambda_0)(\lambda-\lambda_0)}{V_4(\lambda_0)}} + \mathcal{O}(\lambda - \lambda_0) \\ =& \sqrt{\frac{-d(\lambda-\lambda_0)}{\ell}} + \mathcal{O}(\lambda - \lambda_0). \end{aligned}$$

Thus, there exists a unique limit cycle bifurcating from the origin, which intersects the discontinuity manifold for $x > 0$ at $(x^*(\lambda), 0)$, which lies $\sqrt{|\lambda - \lambda_0|}$-close to the origin. Moreover, the stability of such a limit cycle coincides with the stability of the monodromic singularity at the origin for $\lambda = \lambda_0$, i.e., it is asymptotically stable (resp. unstable) provided that $\ell < 0$ (resp. $\ell > 0$). This information could also be obtained by computing the derivative of (3.1) at $x = x^*$. □

In the literature, the Bautin bifurcation is described as a generalized Hopf bifurcation (4). The next result is a generalization of Theorem 5, showcasing the birth of multiple limit cycles from a n-parameter family. Hence, this result can be seen as a Bautin-like bifurcation.

Theorem 6. *Let k^+ and k^- be positive integers and let Z_Λ be an n-parameter family of Filippov vector fields (1.4) having a $(2k^+, 2k^-)$-monodromic tangential singularity at the origin for every Λ in an open set $U \subset \mathbb{R}^n$. Let $V_{2i}(\Lambda)$ be the $2i-$th Lyapunov coefficient, for $i = 1, 2 \ldots, n+1$, and denote $\mathcal{V}_n = (V_2, V_4, \ldots, V_{2n}) : U \to \mathbb{R}^n$. Assume that, for some $\Lambda_0 \in U$, $\mathcal{V}_n(\Lambda_0) = 0$, $\det(D\mathcal{V}_n(\Lambda_0)) \neq 0$, and $V_{2n+2}(\Lambda_0) \neq 0$. Then, there exists an open set $W \subset U$ such that Z_Λ has n hyperbolic limit cycles for every $\Lambda \in W$. In addition, all the limit cycles converge to the origin as Λ goes to Λ_0.*

Proof. Let Z_Λ be an n-parameter family of Filippov vector fields (1.4) having a $(2k^+, 2k^-)$-monodromic tangential singularity at the origin for every Λ in an open set $U \subset \mathbb{R}^n$. Let $V_i(\Lambda)$ be the $i-$th Lyapunov coefficient, for $i = 1, 2 \ldots, 2n+2$. Accordingly, the displacement function of Z_λ around the origin writes

$$\Delta(x;\Lambda) = \sum_{i=2}^{2n+2} V_i(\Lambda)x^i + O(x^{2n+3}) = x^2\Gamma(x;\Lambda),$$

where

$$\Gamma(x;\Lambda) = \sum_{i=2}^{2n+2} V_i(\Lambda)x^{i-2} + O(x^{2n+1}). \tag{3.4}$$

Notice that

$$\frac{\partial^i \Gamma}{\partial x^i}(0;\Lambda) = i!V_{i+2}(\Lambda), \text{ for } i = 0, \ldots, 2n.$$

By hypothesis, there exists $\Lambda_0 \in U$ such that $\mathcal{V}_n(\Lambda_0) = 0$, $\det(D\mathcal{V}_n(\Lambda_0)) \neq 0$, and $V_{2n+2}(\Lambda_0) \neq 0$, where $\mathcal{V}_n = (V_2, V_4, \ldots, V_{2n}) : U \to \mathbb{R}^n$. Thus,

$$\frac{\partial^{2i}\Gamma}{\partial x^{2i}}(0;\Lambda_0) = 0, \text{ for } i = 1, \ldots, n-1, \text{ and } \frac{\partial^{2n}\Gamma}{\partial x^{2n}}(0;\Lambda_0) = (2n)!V_{2n+2}(\Lambda_0) \neq 0.$$

In addition, from Theorem 2,

$$\frac{\partial^{2i+1}\Gamma}{\partial x^{2i+1}}(0;\Lambda_0) = 0, \text{ for } i = 1, \ldots, n-1.$$

Therefore, as a consequence of the Malgrange Preparation Theorem (see (37)), there exists a small neighborhood $W \subset \mathbb{R} \times \mathbb{R}^n$ of $(0, \Lambda_0)$ and smooth functions $c(x;\Lambda)$, and $a_i(\Lambda)$, for $i = 0, \ldots, 2n-1$, such that $c(x,\Lambda) \neq 0$ and

$$\Gamma(x;\Lambda) = c(x,\Lambda)(x^{2n} + a_{2n-1}(\Lambda)x^{2n-1} + \cdots + a_1(\Lambda)x + a_0(\Lambda)), \tag{3.5}$$

for all $(x,\Lambda) \in W$.

From (3.4) and (3.5), we have that

$$i!V_{i+2}(\Lambda) = \frac{\partial^i\Gamma}{\partial x^i}(0;\Lambda) = \sum_{j=0}^{i}\binom{i}{j}\partial^{i-j}c(0,\Lambda)j!a_j(\Lambda).$$

Hence, denoting $\mathcal{A}(\Lambda) = \Big(a_0(\Lambda), \ldots, a_{2n-1}(\Lambda)\Big)$ and $\mathcal{V}(\Lambda) = \Big(2!V_2(\Lambda), \ldots, (2n-1)!V_2(\Lambda)\Big)$, we see that

$$M(\Lambda)\mathcal{A}(\Lambda) = \mathcal{V}(\Lambda),$$

where $M(\Lambda)$ is a lower triangular matrix with every entry in the diagonal given by $c(0,\Lambda)$. Therefore, $M(\Lambda)$ is invertible for all Λ in a small neighborhood of Λ_0 and

$$\mathcal{A}(\Lambda) = M(\Lambda)^{-1}\mathcal{V}(\Lambda).$$

Consequently, $\mathcal{A}(\Lambda_0) = (0,\dots,0)$ and $D\mathcal{A}(\Lambda_0) = M^{-1}(\Lambda_0)D\mathcal{V}(\Lambda_0)$. Thus, since by hypothesis $D\mathcal{V}_n(\Lambda_0)$ is invertible, we conclude that $D\mathcal{V}(\Lambda_0)$ and, consequently, $D\mathcal{A}(\Lambda_0)$ are full rank matrices, that is, $\text{rank}(D\mathcal{V}(\Lambda_0)) = \text{rank}(D\mathcal{A}(\Lambda_0)) = n$.

Now, for $(x,\Lambda) \in W$, denote

$$P(x,\Lambda) = \frac{\Gamma(x;\Lambda)}{c(x,\Lambda)} = x^{2n} + a_{2n-1}(\Lambda)x^{2n-1} + \cdots + a_1(\Lambda)x + a_0(\Lambda).$$

Let x_i, $i = 1,2,\dots,n$, be distinct n positive values and $\varepsilon > 0$. In what follows, we will conclude the proof of Theorem 6 by showing that for $\varepsilon > 0$ sufficiently small there exists $\Lambda^*(\varepsilon)$ sufficiently close to Λ_0 such that

$$P(\varepsilon x_i, \Lambda^*(\varepsilon)) = 0, \quad \text{for} \quad i = 1,2,\dots,n.$$

This will imply that $Z_{\Lambda^*(\varepsilon)}$ has n limit cycles bifurcating from the origin for $\varepsilon > 0$ sufficiently small.

First, consider the system of equations

$$P(\varepsilon x_i, \Lambda) = 0, \quad \text{for} \quad i = 1,2,\dots,n, \tag{3.6}$$

which is equivalent to

$$N(\varepsilon)\mathcal{A}(\Lambda) = b(\varepsilon), \tag{3.7}$$

where

$$N(\varepsilon) = \begin{pmatrix} 1 & \varepsilon x_1 & \dots & (\varepsilon x_1)^{2n-1} \\ 1 & \varepsilon x_2 & \dots & (\varepsilon x_2)^{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \varepsilon x_n & \dots & (\varepsilon x_n)^{2n-1} \end{pmatrix} \quad \text{and} \quad b(\varepsilon) = -\begin{pmatrix} (\varepsilon x_1)^{2n} \\ (\varepsilon x_2)^{2n} \\ \vdots \\ (\varepsilon x_n)^{2n} \end{pmatrix}.$$

Notice that the matrix $N(\varepsilon)$ is composed by two blocks, $N(\varepsilon) = \begin{pmatrix} T(\varepsilon) & S(\varepsilon) \end{pmatrix}$, where $T(\varepsilon)$ and $S(\varepsilon)$ are square matrices given by

$$T(\varepsilon) = \begin{pmatrix} 1 & \varepsilon x_1 & \dots & (\varepsilon x_1)^{n-1} \\ 1 & \varepsilon x_2 & \dots & (\varepsilon x_2)^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \varepsilon x_n & \dots & (\varepsilon x_n)^{n-1} \end{pmatrix} \quad \text{and} \quad S(\varepsilon) = \begin{pmatrix} (\varepsilon x_1)^{n} & \dots & (\varepsilon x_1)^{2n-1} \\ (\varepsilon x_1)^{n} & \dots & (\varepsilon x_2)^{2n-1} \\ \vdots & \ddots & \vdots \\ (\varepsilon x_1)^{n} & \dots & (\varepsilon x_n)^{2n-1} \end{pmatrix}.$$

Notice that the matrix $T(\varepsilon)$ is a matrix of type

$$V = V(\Lambda) = \begin{pmatrix} x_1 & x_1^2 & \dots & x_1^n \\ x_2 & x_2^2 & \dots & x_2^n \\ \vdots & & & \vdots \\ x_n & x_n^2 & \dots & x_n^n \end{pmatrix}, \tag{3.8}$$

with entries $V_{i,j} = x_i^j$, for $\Lambda = (x_1, x_2, \ldots, x_n)$. This type of matrix is of special importance, and it is called the Vandermonde's matrix (32). A Vandermonde matrix is a matrix with the terms of a geometric progression in each row.

The determinant of the Vandermonde's matrix (3.8) (see (32)) is

$$\det(V(\Lambda)) = \prod_{1 \leqslant j \leqslant n} x_j \prod_{1 \leqslant i < j \leqslant n} (x_j - x_i). \tag{3.9}$$

Since $T(\varepsilon)$ is a *Vandermonde Matrix* (3.8), from (3.9) we know that

$$\det(T(\varepsilon)) = \varepsilon^n \prod_{1 \leqslant i < j \leqslant n} (x_j - x_i)$$

and, therefore, invertible for $\varepsilon \neq 0$. Thus, consider the smooth matrix-valued functions $\widetilde{N}(\varepsilon) = T(\varepsilon)^{-1} N(\varepsilon)$ and $\tilde{b}(\varepsilon) = T(\varepsilon)^{-1} b(\varepsilon)$ which, because of the factor ε^n in both $S(\varepsilon)$ and $b(\varepsilon)$, can be smoothly extended for $\varepsilon = 0$ as $\widetilde{N}(0) = \begin{pmatrix} I_n & 0_n \end{pmatrix}$ and $\tilde{b}(0) = 0$. Now, define the function $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ as

$$F(\Lambda, \varepsilon) = \widetilde{N}(\varepsilon) \mathcal{A}(\Lambda) - \tilde{b}(\Lambda).$$

Clearly, the systems of equations (3.6) and (3.7) are equivalent to $F(\Lambda, \varepsilon) = 0$. Notice that $F(\Lambda_0, 0) = 0$ and, since $\widetilde{N}(0)$ and $D\mathcal{A}(\Lambda_0)$ are full rank matrices, we conclude that the square matrix $\dfrac{\partial F}{\partial \Lambda}(\Lambda_0, 0) = \widetilde{N}(0) D\mathcal{A}(\Lambda_0)$ has full rank and, therefore, is non-singular. Then, from the implicit function theorem, we obtain for $\varepsilon > 0$ sufficiently small a smooth function $\Lambda^*(\varepsilon)$ such that $\Lambda(0) = \Lambda_0$ and $F(\Lambda^*(\varepsilon), \varepsilon) = 0$ for $\varepsilon > 0$ sufficiently small. This concludes the proof of Theorem 6. □

3.3 The pseudo-Hopf bifurcation

This section is based on the preprint (41). In the nonsmooth context, besides the limit cycles bifurcating by varying the Lyapunov coefficients, monodromic singularities lying on the switching curve can always be split apart generating, under suitable conditions, a sliding region and an extra limit cycle surrounding it (see Figure 8). This last bifurcation phenomenon is called pseudo-Hopf bifurcation and it was first reported by Filippov in his book (25) (see item b of page 241, see also the paper (47), which provides 20 geometric mechanisms by which limit cycles are created locally in planar piecewise smooth vector fields). This bifurcation phenomenon has been used to investigate the cyclicity of monodromic singularities in Filippov vector fields, which allows to increase in the obtained lower bounds for the cyclicity at least by one (see, for instance, (22, 28, 39)).

As said before, the pseudo-Hopf bifurcation is a useful method to increase by one the number of limit cycles when investigating the cyclicity of monodromic singularities in Σ of the Filippov vector field (1.4) (see, for instance, (22, 28)). In what follows we present the formal statement of the pseudo-Hopf bifurcation in the case of the so-called invisible two-fold singularity which, in our terminology, corresponds to a $(2,2)$-monodromic tangential singularity. Such a bifurcation phenomenon is briefly commented on in (39, Remark 1), and here we first present a formalization of this phenomenon for a $(2,2)$-monodromic tangential singularity.

Proposition 3. *Assume that the Filippov vector field Z, given by (1.4), has a $(2,2)$-monodromic tangential singularity at the origin with non-vanishing second Lyapunov coefficient V_2. Consider the 1-parameter family of Filippov vector fields*

$$Z_b(x,y) = \begin{cases} Z^+(x-b,y), & y > 0, \\ Z^-(x,y), & y < 0. \end{cases} \tag{3.10}$$

Then, given a neighborhood $U \subset \mathbb{R}^2$ of $(0,0)$, there exists a neighborhood $I \subset \mathbb{R}$ of 0 such that the following statements hold for all $b \in I$:

- *If $\operatorname{sign}(b) = -\operatorname{sign}(\delta V_2)$, then Z_b has a hyperbolic limit cycle in U surrounding a sliding segment (see Figure 8). In addition, the hyperbolic limit cycle is stable (resp. unstable) provided that $V_2 < 0$ (resp. $V_2 > 0$).*
- *If $\operatorname{sign}(b) = \operatorname{sign}(\delta V_2)$, then Z_b does not have limit cycles in U.*

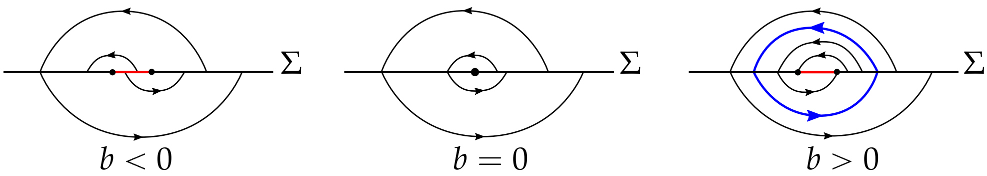


Figure 8 – In this figure, the origin is a repelling two-fold singularity of Z_0 which undergoes a pseudo-Hopf bifurcation as b varies. For $b \neq 0$ the two-fold singularity is split into two regular-fold singularities and between them a sliding segment is created, which is repelling for $b < 0$ and attracting for $b > 0$. In the last case, an attracting hyperbolic limit cycle surrounding the attracting sliding segment is created. The red segment on Σ represents the sliding region.

The following proposition is an original and more general version of Proposition 3 that establishes the bifurcation of a hyperbolic limit cycle from a $(2k^+, 2k^-)$-monodromic tangential singularities provided that some Lyapunov coefficient does not vanish. We prove a degenerate version of the pseudo-Hopf bifurcation for $(2k^+, 2k^-)$-monodromic tangential singularities, which is characterized by the birth of a hyperbolic limit cycle when a sliding segment is created provided that there exists a first non-vanishing Lyapunov coefficient $V_{2\ell}$ (recall from Theorem 2 that the first non-vanishing Lyapunov coefficient is always even). However, we have noticed that in the literature a proof for this result is only given for $k^+ = k^- = \ell = 1$, which is precisely the statement of Proposition 3.

Proposition 4. *Assume that the Filippov vector field Z, given by (1.4), has a $(2k^+, 2k^-)$-monodromic tangential singularity at the origin. Let $V_{2\ell}$ be its first non-vanishing Lyapunov coefficient. Consider the 1-parameter family of Filippov vector fields*

$$Z_b(x,y) = \begin{cases} Z^+(x-b,y), & y > 0, \\ Z^-(x,y), & y < 0. \end{cases}$$

Then, given a neighborhood $U \subset \mathbb{R}^2$ of $(0,0)$, there exists a neighborhood $I \subset \mathbb{R}$ of 0 such that the following statements hold for every $b \in I$:

- *If $\operatorname{sign}(b) = -\operatorname{sign}(\delta V_{2\ell})$, then Z_b has a hyperbolic limit cycle in U surrounding a sliding segment. In addition, the hyperbolic limit cycle is stable (resp. unstable) provided that $V_{2\ell} < 0$ (resp. $V_{2\ell} > 0$).*
- *If $\operatorname{sign}(b) = \operatorname{sign}(\delta V_{2\ell})$, then Z_b does not have limit cycles in U.*

Proof. First, let $\Delta_0(x)$ be the displacement function of Z defined in a neighborhood of $x = 0$. From (2.7), we know that

$$\Delta_0(x) = V_{2\ell} x^{2\ell} + \mathcal{O}(x^{2\ell+1}).$$

Now, recall that $\Delta(x;b) := \delta(\varphi_b^+(x) - \varphi^-(x))$ (see (2.5)), where φ_b^+ and φ^- are the half-return maps of $Z^+(x-b,y)$ and $Z^-(x,y)$ associated with Σ. It is easy to see that if φ^+ is the half-return map of $Z^+(x,y)$, then $\varphi_b^+(x) = \varphi^+(x-b) + b$. Therefore, taking into account that $(\varphi^+)'(0) = -1$, we get

$$\begin{aligned} \Delta(x;b) &= \delta(\varphi_b^+(x) - \varphi^-(x)) \\ &= \Delta_0(x) + 2\delta b + bO(x) + O(b^2) \\ &= V_{2\ell} x^{\ell} + 2\delta b + O(x^{2\ell+1}) + bO(x) + O(b^2). \end{aligned}$$

Notice that solutions of

$$\Delta_b(x) = 0, \quad x > |b|, \tag{3.11}$$

correspond to crossing periodic solutions of Z_b. In addition, simple solutions of (3.11) correspond to hyperbolic limit cycles of Z_b.

Denote

$$\widetilde{\Delta}(y;\beta) := \frac{\Delta(\beta y;\mu\beta^{2\ell})}{\beta^{2\ell}}, \tag{3.12}$$

where $\mu = -\text{sign}(\delta V_{2\ell})$. Notice that

$$\begin{aligned}\widetilde{\Delta}(y;\beta) =& \frac{V_{2\ell}\beta^{2\ell}y^{2\ell} + 2\delta\mu\beta^{2\ell} + O(\beta^{2\ell+1}y^{2\ell+1}) + \mu\beta^{2\ell}O(\beta y)}{\beta^{2\ell}}\\ =& V_{2\ell}y^{2\ell} + 2\delta\mu + O(\beta y^{2\ell+1}) + O(\beta y)\\ =& V_{2\ell}y^{2\ell} + 2\delta\mu + O(\beta).\end{aligned} \tag{3.13}$$

Then, for

$$y_0 = \sqrt[2\ell]{\left|\frac{2\delta}{V_{2\ell}}\right|}, \tag{3.14}$$

we have

$$\widetilde{\Delta}(y_0, 0) = 0 \;\text{ and }\; \frac{\partial\widetilde{\Delta}}{\partial y}(y_0, 0) = \pm 2\ell V_{2\ell}\left(\sqrt[2\ell]{\left|\frac{2\delta}{V_{2\ell}}\right|}\right)^{2\ell-1} \neq 0.$$

Then, from the Implicit Function Theorem, there exists $y(\beta)$ such that $y(0) = y_0$ and $\widetilde{\Delta}(y(\beta);\beta) = 0$, for all β in a neighborhood of 0.

Therefore, from (3.12), $\Delta(\beta y(\beta);\mu\beta^{2\ell}) = \beta^{2\ell}\widetilde{\Delta}(y(\beta);\beta) = 0$. Then, by taking $\beta = \sqrt[2\ell]{\mu b}$, we have that $x(b) = \sqrt[2\ell]{\mu b}\, y(\sqrt[2\ell]{\mu b})$ is a solution of $\Delta(x;b) = 0$, for $\mu b > 0$, that is, $\text{sign}(b) = \text{sign}(\mu) = -\text{sign}(\delta V_{2\ell})$. In this case, $x(b) = \sqrt[2\ell]{\mu b}\, y_0 + \mathcal{O}(b)$ with $y_0 \neq 0$ give in (3.14) and, therefore, $x(b) > |b|$, for $|b| \neq 0$ sufficiently small. Hence $x(b)$ corresponds to a crossing periodic solution of Z_b.

In addition, from (3.13),

$$\begin{aligned}\frac{1}{b^{\frac{2\ell-1}{2\ell}}}\frac{\partial\Delta}{\partial x}(x(b);b) =& \frac{2\ell V_{2\ell}\left(\sqrt[2\ell]{\mu b}\,y(\sqrt[2\ell]{\mu b})\right)^{2\ell-1} + O(x^{2\ell}) + bO(1) + O(b^2)}{b^{\frac{2\ell-1}{2\ell}}}\\ =& 2\ell V_{2\ell}O(b^{\frac{2\ell-1}{2\ell}}) + O(b) \neq 0.\end{aligned} \tag{3.15}$$

Therefore, the periodic solution associated to $x(b)$ is actually a limit cycle which is contained in U for $|b|$ small enough.

Finally, from (3.15), $\text{sign}\left(\frac{\partial\Delta}{\partial x}(x(b);b)\right) = \text{sign}(V_{2\ell})$, then the hyperbolic limit cycle is stable (resp. unstable) provided that $V_{2\ell} < 0$ (resp. $V_{2\ell} > 0$), which finishes the proof. □

3.4 Insights beyond the pseudo-Hopf bifurcation

Our third major result is based on the preprint (41). This result enhances Proposition 3 by demonstrating that at least k limit cycles bifurcate from a $(2k, 2k)$-monodromic tangential singularity when it is destroyed by considering a perturbation that adds some "missing" lower degree terms to Z. This implies a cyclicity of at least k for such singularities. Consequently, any number of limit cycles obtained without destroying the monodromic singularity (for instance, by varying the Lyapunov coefficients) can be increased at least by k.

In what follows, the concept of *norm of a polynomial* refers to the conventional notion, that is, the norm of the parameter-vector that defines it.

Theorem 7. *Let k be a positive integer. Assume that the Filippov vector field Z, given by (1.4), has a $(2k, 2k)$-monodromic tangential singularity at the origin with non-vanishing second Lyapunov coefficient V_2. Then, given $\lambda > 0$ and a neighborhood $U \subset \mathbb{R}^2$ of $(0, 0)$, there exist polynomials P^+ and P^-, with degree $2k-2$ and norm less than λ, and a neighborhood $I \subset \mathbb{R}$ of 0 such that, for every $b \in I$ satisfying $\operatorname{sign}(b) = -\operatorname{sign}(\delta V_2)$, the Filippov vector field*

$$\widetilde{Z}_b(x,y) = \begin{cases} \widetilde{Z}^+(x+b,y), & y>0, \\ \widetilde{Z}^-(x,y), & y<0, \end{cases} \quad \text{with } \widetilde{Z}^\pm(x,y) = \begin{pmatrix} X^\pm(x,y) \\ Y^\pm(x,y) + X^\pm(x,y)P^\pm(x) \end{pmatrix}, \tag{3.16}$$

has k hyperbolic limit cycles inside U such that each one of these limit cycles surrounds a single sliding segment (see Figure 9). In addition, the hyperbolic limit cycles are stable (resp. unstable) provided that $V_2 < 0$ (resp. $V_2 > 0$).

Remark 1. *In Theorem 7, notice that the perturbation terms $\widetilde{P}^+(x,y) := X^+(x,y)P^+(x,y)$ and $\widetilde{P}^-(x,y) := X^-(x,y)P^-(x,y)$ adds some "missing" lower degree terms to $Y^\pm$. Indeed, suppose that $X^\pm$ and $Y^\pm$ are polynomial vector fields. Since the origin is a $(2k, 2k)$-monodromic tangential singularity, one can see that $\deg(Y^\pm) \geqslant (2k-1)\deg(X^\pm)$. Therefore, $\deg(\widetilde{P}^\pm) = (2k-2)\deg(X^\pm) < \deg(Y^\pm)$. In other words, the perturbation $\widetilde{P}^\pm$ adds to the polynomial $Y^\pm$ the monomials of degree strictly lower than the degree of $Y^\pm$.*

The role of the perturbation terms, $\widetilde{P}^+(x,y)$ and $\widetilde{P}^-(x,y)$, consists in destroying the $(2k, 2k)$-monodromic tangential singularity of Z at the origin and unfold from it $2k-1$ two-fold singularities for $\widetilde{Z}_0(x,y)$. Among those singularities, k of them are $(2, 2)$-monodromic tangential singularities, from which limit cycles are created via pseudo-Hopf bifurcation for $\widetilde{Z}_b(x,y)$, with $b \in I$ satisfying $\operatorname{sign}(b) = -\operatorname{sign}(\delta V_2)$ (see Figure 9).

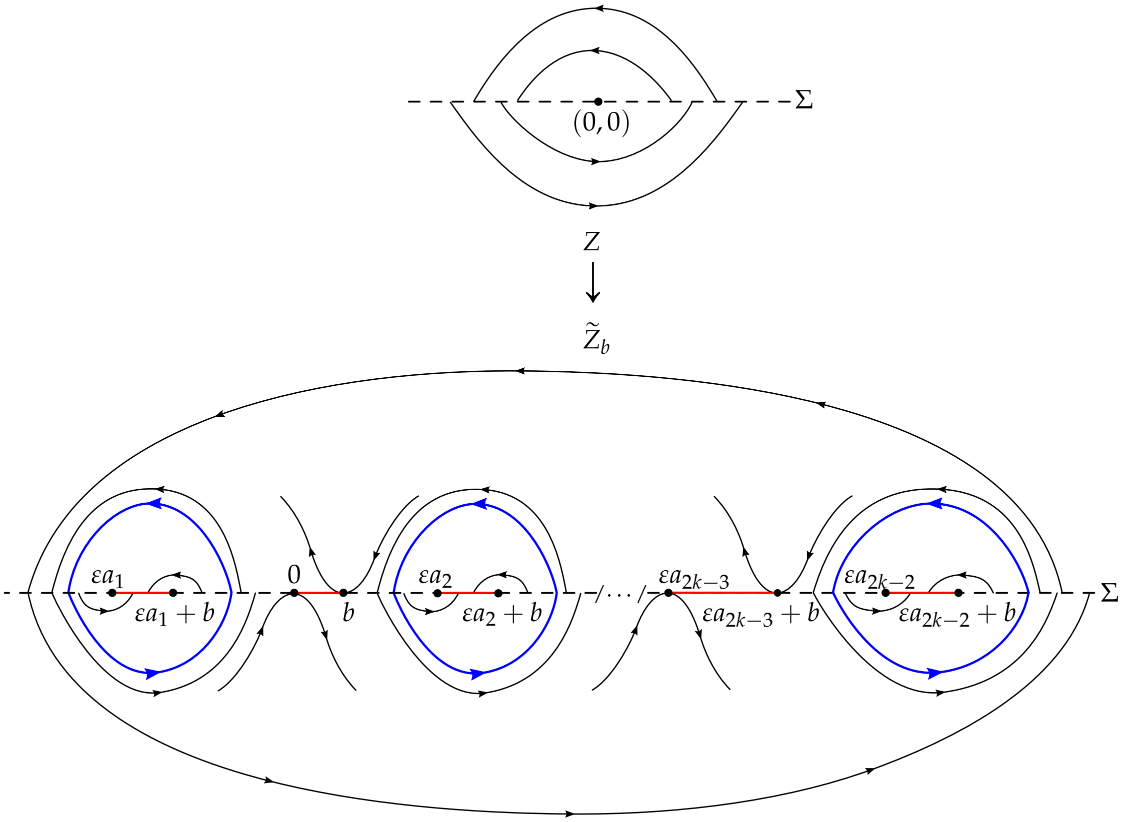


Figure 9 – Illustration of k limit cycles unfolding from a $(2k^+, 2k^-)$-monodromic tangential singularity and surrounding k sliding segments predicted by Theorem 7. Continuous and dashed segments on Σ represent sliding and crossing regions, respectively.

The proof of Theorem 7 will be done through a series of steps and it is going to be concluded in Section 3.4.3. The idea is to construct polynomials P_Λ^+ and P_Λ^- that unfold from the origin $2k-1$ two-fold singularities of which k of them are $(2,2)$-monodromic tangential singularities with non-vanishing second Lyapunov coefficients. Then, the proof will follow by applying Proposition 3 which ensures the creation of a hyperbolic limit cycle surrounding a sliding segment from each one of the $(2,2)$-monodromic tangential singularities when they are destroyed.

We start by considering the following $(2k-2)$-parameter family of perturbations of Z:

$$Z_\Lambda(x,y) = \begin{cases} Z_\Lambda^+(x,y) = \begin{pmatrix} X^+(x,y) \\ Y_\Lambda^+(x,y) \end{pmatrix}, & y > 0, \\ Z_\Lambda^-(x,y) = \begin{pmatrix} X^-(x,y) \\ Y_\Lambda^-(x,y) \end{pmatrix}, & y < 0, \end{cases} \tag{3.17}$$

where

$$Y_\Lambda^\pm(x,y) = Y^\pm(x,y) + X^\pm(x,y)P_\Lambda^\pm(x), \tag{3.18}$$

P_Λ^+ and P_Λ^- are continuous $(2k-2)$-parameter families of polynomials of degree $2k-2$ satisfying $P_0^+ = P_0^- = 0$ with $\Lambda \in \mathcal{L}$ and

$$\mathcal{L} = \{(a_1,\ldots,a_n) \in \mathbb{R}^n; a_i \neq 0\,\forall i \text{ and } a_i \neq a_j\,\forall\, i \neq j\}. \tag{3.19}$$

In Subsection 3.4.1, under the hypotheses of Theorem 7, we construct the perturbation polynomials P_Λ^+ and P_Λ^- so that the origin and the points $(\varepsilon a_i, 0)$, $i \in \{1,2,\ldots,2k-2\}$, are contact points between Σ and the vector fields $Z_{\varepsilon\Lambda}^+$ and $Z_{\varepsilon\Lambda}^-$, for $\varepsilon > 0$ sufficiently small. In Subsection 3.4.2, we show that all these contact points have multiplicity two (see Proposition 5) and that, in addition, k of these contact points are actually $(2,2)$-monodromic tangential singularities of $Z_{\varepsilon\Lambda}$ with non-vanishing second Lyapunov coefficient (see Proposition 6). Finally, in Subsection 3.4.3, we conclude that, by considering an additional perturbation, each one of these $(2,2)$-monodromic tangential singularities undergoes a pseudo-Hopf bifurcation, which creates k hyperbolic limit cycle surrounding sliding segments.

3.4.1 Construction of the perturbation terms

Here, given $\Lambda = (a_1,\ldots,a_{2k-2}) \in \mathcal{L}$, we shall construct polynomials P_Λ^+ and P_Λ^- such that

$$Y_{\varepsilon\Lambda}^+(\varepsilon a_i, 0) = 0 \text{ and } Y_{\varepsilon\Lambda}^-(\varepsilon a_i, 0) = 0 \text{ for } i \in \{1,\ldots,2k-2\}. \tag{3.20}$$

The coefficient ε will be chosen later on small enough.

Assuming that the Filippov vector field (1.4) has a $(2k,2k)$-monodromic tangential singularity at the origin (see conditions **C1**, **C2**, and **C3** from Definition 1), we have that $X^\pm(0,0) \neq 0$. Therefore, there exists a small neighborhood U of the origin such that $X^\pm(x,y) \neq 0$, for all $(x,y) \in U$. By following (39, Section 2), condition **C1** implies that

$$\frac{Y^\pm(x,y)}{|X^\pm(x,y)|} = \pm\delta\left(a^\pm x^{2k-1} + x^{2k} f^\pm(x) + y g^\pm(x,y)\right), \quad (x,y) \in U, \tag{3.21}$$

where the values $a^\pm$ and the functions $f^\pm(x)$ and $g^\pm(x,y)$ are given by (1.10) and (1.9), respectively. Therefore, by taking into account the expression for $Y_\Lambda^\pm$ given by (3.18), the identities in (3.20) are equivalent to

$$P_{\varepsilon\Lambda}^\pm(\varepsilon a_i) = \mp\delta\varepsilon^{2k-1}(a^\pm a_i^{2k-1} + \varepsilon a_i^{2k} f^\pm(\varepsilon a_i)), \;\; i \in \{1,\ldots,2k-2\}, \tag{3.22}$$

which is an interpolation problem that can be investigated with the help of a Vandermonde matrix (32). Indeed, by denoting

$$\xi_i^\pm(\varepsilon) := \mp\delta\varepsilon^{2k-1}(a^\pm a_i^{2k-1} + \varepsilon a_i^{2k} f^\pm(\varepsilon a_i)) \tag{3.23}$$

and considering $P_\Lambda^\pm(x) = \sum_{j=1}^{2k-2} c_j^\pm(\Lambda)x^j$, the identity (3.22) becomes

$$H(\Lambda,\varepsilon)\begin{pmatrix} c_1^\pm(\varepsilon\Lambda) & \dots & c_{2k-2}^\pm(\varepsilon\Lambda)\end{pmatrix}^T = \begin{pmatrix} \xi_1^\pm(\varepsilon) & \dots & \xi_{2k-2}^\pm(\varepsilon)\end{pmatrix}^T,$$

where $H(\Lambda,\varepsilon)$ is the following matrix:

$$H(\Lambda,\varepsilon) = \begin{pmatrix} \varepsilon a_1 & \dots & \varepsilon^{2k-2}a_1^{2k-2} \\ \vdots & & \vdots \\ \varepsilon a_{2k-2} & \dots & \varepsilon^{2k-2}a_{2k-2}^{2k-2}, \end{pmatrix}. \tag{3.24}$$

Notice that the matrix (3.24) is a matrix of type

$$V = V(\Lambda) = \begin{pmatrix} x_1 & x_1^2 & \dots & x_1^n \\ x_2 & x_2^2 & \dots & x_2^n \\ \vdots & & & \vdots \\ x_n & x_n^2 & \dots & x_n^n \end{pmatrix}, \tag{3.25}$$

with entries $V_{i,j} = x_i^j$, for $\Lambda = (x_1, x_2, \dots, x_n)$, which is a Vandermonde's matrix (32).

The determinant of the Vandermonde's matrix (3.25) (see (32)) is

$$\det(V(\Lambda)) = \prod_{1\leqslant j\leqslant n} x_j \prod_{1\leqslant i<j\leqslant n} (x_j - x_i). \tag{3.26}$$

From (3.26) we know that the determinant of the matrix (3.25) is non-zero if and only if all x_i are distinct. So, if x_i are all distinct, we have that the Vandermonde's matrix is invertible and its inverse is given by $V(\Lambda)^{-1} = (b_{ij})_n$ (see (32)), where

$$b_{ij} = \left(\frac{\displaystyle\sum_{\substack{1\leqslant m_1<\dots<m_{n-i}\leqslant n \\ m_1,\dots,m_{n-i}\neq j}} (-1)^{i-1} x_{m_1}\dots x_{m_{n-i}}}{\displaystyle x_j \prod_{\substack{1\leqslant m\leqslant n \\ m\neq j}} (x_m - x_j)} \right). \tag{3.27}$$

The sum in the numerator looks complicated, but it is just the coefficient of x^{j-1} in the polynomial

$$\frac{(x_1 - x)\dots(x_n - x)}{x_i - x}.$$

Hence, the matrix (3.24) is invertible provided that $a_i \neq a_j$, for $i \neq j$, being ensured by the fact that $\Lambda \in \mathcal{L}$. Then,

$$\begin{pmatrix} c_1^\pm(\varepsilon\Lambda) & \dots & c_{2k-2}^\pm(\varepsilon\Lambda)\end{pmatrix}^T = [H(\Lambda,\varepsilon)]^{-1}\begin{pmatrix} \xi_1^\pm(\varepsilon) & \dots & \xi_{2k-2}^\pm(\varepsilon)\end{pmatrix}^T. \tag{3.28}$$

Taking into account the inverse of the Vandermond matrix (3.27) and (3.24), we have $[H(\Lambda,\varepsilon)]^{-1} = [b_{ij}]_{2k-2}$ where

$$b_{ij} = \left(\frac{\sum\limits_{\substack{1\leqslant m_1<\cdots<m_{n-i}\leqslant n \\ m_1,\dots,m_{n-i}\neq j}} (-1)^{j-1}\varepsilon^{2k-2-i} a_{m_1}\dots a_{m_{n-i}}}{\varepsilon a_j \prod\limits_{\substack{1\leqslant m\leqslant n \\ m\neq j}} (\varepsilon a_m - \varepsilon a_j)} \right)$$
$$= \varepsilon^{-i} \left(\frac{\sum\limits_{\substack{1\leqslant m_1<\cdots<m_{n-i}\leqslant n \\ m_1,\dots,m_{n-i}\neq j}} (-1)^{j-1} a_{m_1}\dots a_{m_{n-i}}}{a_j \prod\limits_{\substack{1\leqslant m\leqslant n \\ m\neq j}} (a_m - a_j)} \right).$$

This means that the ith row of $[H(\Lambda,\varepsilon)]^{-1}$ has order ε^{-i}. Thus, from (3.23) and (3.28), $c_i^{\pm}(\varepsilon\Lambda) = \varepsilon^{2k-1-i} C_j^{\pm}(\Lambda,\varepsilon)$, where C_j is a smooth function, and so

$$P^{\pm}_{\varepsilon\Lambda}(x) = \sum_{j=1}^{2k-2} \varepsilon^{2k-1-j} C_j^{\pm}(\Lambda,\varepsilon)x^j. \tag{3.29}$$

Notice that the norm of the polynomial $P^{\pm}_{\varepsilon\Lambda}(x)$ goes to zero as $\varepsilon \to 0$.

3.4.2 Monodromic tangential singularities appearing in the unfolding

In what follows, we are going to show that, for $\Lambda \in \mathcal{L}$, the Filippov vector field $Z_{\varepsilon\Lambda}$, given by (3.17), has k $(2,2)$-monodromic tangential singularities with non-vanishing second Lyapunov coefficient.

First, in the next result, we will see that, for $\varepsilon > 0$ sufficiently small, the origin and the points $(0,\varepsilon a_i)$, $i \in \{1,\dots,2k-2\}$, are contact points of multiplicity 2 between Σ and the vector fields $Z^+_{\varepsilon\Lambda}$ and $Z^-_{\varepsilon\Lambda}$ of which k of them are invisible. For the sake of simplicity, we will denote $a_0 = 0$.

Proposition 5. *Let $\varepsilon > 0$, $a_0 = 0$, and $\Lambda = (a_1,\dots,a_{2k-2}) \in \mathcal{L}$, with $\mathcal{L}$ given by (3.19). Consider the vector fields $Z^+_{\varepsilon\Lambda}$ and $Z^-_{\varepsilon\Lambda}$ provided by (3.17). Then, for $\varepsilon > 0$ sufficiently small, the points $(0,\varepsilon a_i)$, $i \in \{0,\dots,2k-2\}$, are contact points of multiplicity 2 between Σ and the vector fields $Z^+_{\varepsilon\Lambda}$ and $Z^-_{\varepsilon\Lambda}$. In addition, if $a_1 < 0 < a_2 < \cdots < a_{2k-2}$, these contact points are invisible for $i \in \{1\} \cup \{2,4,\dots,2k-2\}$ (see Figure 10).*

Proof. First of all, notice that the construction of the polynomials $P^{\pm}_{\Lambda}$ implies that the points $(\varepsilon a_i, 0)$, for $i \in \{0,\dots,2k-2\}$, are contact points between Σ and the vector fields $Z^+_{\varepsilon\Lambda}$ and $Z^-_{\varepsilon\Lambda}$. Indeed, the identities (3.20) implies that $Y^{\pm}_{\varepsilon\Lambda}(\varepsilon a_i, 0) = 0$ and, in addition, $X^{\pm}_{\varepsilon\Lambda}(\varepsilon a_i, 0) = X^{\pm}(0,0) + \mathcal{O}(\varepsilon)$ which, by condition **C1**, is non-vanishing for $\varepsilon > 0$ sufficiently small.

Now, in order to see that these contact points have multiplicity 2, consider the function $h^{\pm}(x,\varepsilon) = Y^{\pm}_{\varepsilon\Lambda}(x,0)$. Notice that, condition **C1** implies that

$$h^{\pm}(0,0) = 0, \ \frac{\partial^i h^{\pm}}{\partial x^i}(0,0) = 0, \text{ for } i \in \{1,\dots,2k-2\}, \text{ and } \frac{\partial^{2k-1} h^{\pm}}{\partial x^{2k-1}}(0,0) \neq 0.$$

Thus, by Malgrange Preparation Theorem 4, there exists a neighborhood V of $(x,\varepsilon) = (0,0)$ such that $h^{\pm}|_V(x,\varepsilon) = A^{\pm}(x,\varepsilon)B^{\pm}_{\varepsilon}(x)$, where $A(x,\varepsilon) > 0$ for $(x,\varepsilon) \in V$ and

$$B^{\pm}_{\varepsilon}(x) = \pm\delta a^{\pm} x^{2k-1} + \ell^{\pm}_{2k-2}(\varepsilon)x^{2k-2} + \cdots + \ell^{\pm}_1(\varepsilon)x + \ell^{\pm}_0(\varepsilon).$$

Notice that $Y^{\pm}_{\varepsilon\Lambda}(\varepsilon a_i, 0) = 0$ implies that $B^{\pm}_{\varepsilon}(\varepsilon a_i) = 0$, for all $i \in \{0,\dots,2k-2\}$. Since $B^{\pm}_{\varepsilon}$ is a polynomial in x of degree $2k-1$, we conclude that those roots are simple provided that $a_i \neq a_j$, for all $i \neq j$ in $\{0,\dots,2k-2\}$, which is ensured by the fact that $\Lambda = (a_1,\dots,a_{2k-2}) \in \mathcal{L}$ (see (3.19)). Hence,

$$\frac{\partial Y^{\pm}_{\varepsilon\Lambda}}{\partial x}(\varepsilon a_i, 0) = A^{\pm}(\varepsilon a_i, \varepsilon)(B^{\pm}_{\varepsilon})'(\varepsilon a_i) \neq 0, \tag{3.30}$$

implying that the contact points $(\varepsilon a_i, 0)$, for $i \in \{0,\dots,2k-2\}$, have multiplicity 2.

Finally, since $B^{\pm}_{\varepsilon}$ has odd degree, its derivative at its smallest root has the same sign as the leading coefficient $\pm\delta a^{\pm}$. Thus, by assuming that $a_1 < 0 < a_2 < a_3 < \cdots < a_{2k-2}$ and taking into account that $\text{sign}(\pm\delta a^{\pm}) = \mp 1$ (see (1.10) and **C2**), it follows that $\text{sign}((B^{\pm}_{\varepsilon})'(\varepsilon a_1)) = \mp 1$ and, since the derivative at the roots has alternate signs, $\text{sign}((B^{+}_{\varepsilon})'(\varepsilon a_{2j})) = \mp 1$ for $j = 1,\dots k-1$. Taking (3.30) into account, we conclude that the contact points $(\varepsilon a_i, 0)$, for $i \in \{1\} \cup \{2,4,\dots,2k-1\}$, are invisible for both vector fields $Z^{+}_{\varepsilon\Lambda}$ and $Z^{-}_{\varepsilon\Lambda}$.The remaining contact points are visible (see Figure 10). □

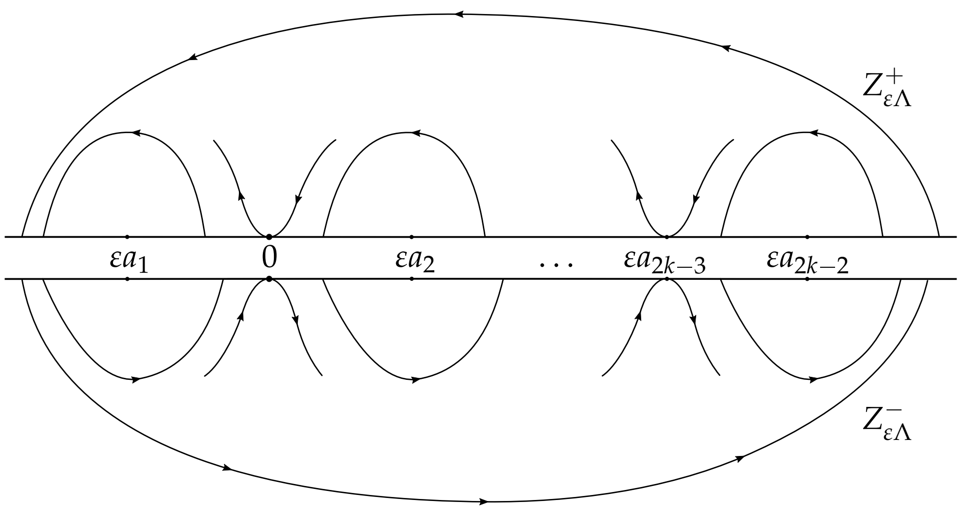


Figure 10 – Illustration of the contact points of multiplicity 2 between Σ and the vector fields $Z^{+}_{\varepsilon\Lambda}$ and $Z^{-}_{\varepsilon\Lambda}$ for $\Lambda = (a_1,\dots,a_{2k-2}) \in \mathcal{L}$ with the configuration $a_1 < 0 < a_2 < a_3 < \cdots < a_{2k-2}$. They alternate between invisible and visible contact points.

Proposition 5 implies that, for $\Lambda \in \mathcal{L}$ and $\varepsilon > 0$ sufficiently small, the points $(0, \varepsilon a_i)$, for $i \in \{1\} \cup \{2, 4, \ldots, 2k-1\}$, are $(2,2)$-monodromic tangential singularities of the Filippov vector field $Z_{\varepsilon\Lambda}$ given by (3.17). In what follows, we are going to compute the second Lyapunov coefficient for each one of these singularities.

Proposition 6. *Let $\varepsilon > 0$ and $\Lambda = (a_1, \ldots, a_{2k-2}) \in \mathcal{L}$, with $\mathcal{L}$ given by (3.19), satisfying $a_1 < 0 < a_2 < a_3 < \cdots < a_{2k-2}$. Consider the Filippov vector field $Z_{\varepsilon\Lambda}$ provided in (3.17). Then, for each $i \in \{1\} \cup \{2, 4, \ldots, 2k-2\}$ and $\varepsilon > 0$ sufficiently small, the second Lyapunov coefficient $V_{2,i}(\varepsilon)$ associated with the $(2,2)$-monodromic tangential singularity $(\varepsilon a_i, 0)$ satisfies*

$$V_{2,i}(\varepsilon) = \frac{2k+1}{3} V_2 + \mathcal{O}(\varepsilon). \tag{3.31}$$

Proof. We start by fixing an index $i \in \{1\} \cup \{2, 4, \ldots, 2k-2\}$ and translating the point $(\varepsilon a_i, 0)$ to the origin by means of the change of coordinates $u = x - \varepsilon a_i$. Thus, the vector field $Z_{\varepsilon\Lambda}$, given by (3.17), becomes

$$\widetilde{Z}_\varepsilon(u,y) = \begin{cases} \begin{pmatrix} \widetilde{X}_\varepsilon^+(u,y), \\ \widetilde{Y}_\varepsilon^+(u,y) \end{pmatrix}, & y > 0, \\ \begin{pmatrix} \widetilde{X}_\varepsilon^-(u,y), \\ \widetilde{Y}_\varepsilon^-(u,y) \end{pmatrix}, & y < 0, \end{cases} \tag{3.32}$$

where

$$\widetilde{X}_\varepsilon^\pm(u,y) = X^\pm(u + \varepsilon a_i, y) \text{ and } \widetilde{Y}_\varepsilon^\pm(u,y) = Y^\pm(u + \varepsilon a_i, y) + X^\pm(u + \varepsilon a_i, y) P_{\varepsilon\Lambda}^\pm(u + \varepsilon a_i).$$

From Proposition 2, we have that the second Lyapunov coefficient for a Filippov vector field with a $(2k^+, 2k^-)$–monodromic tangential singularity is given by

$$V_2 = \delta(\alpha_2^+ - \alpha_2^-), \text{ where } \alpha_2^\pm = \frac{-2f_0^\pm \pm 2\delta a^\pm g_{0,0}^\pm}{a^\pm(2k^\pm + 1)}. \tag{3.33}$$

Hence, the 2nd Lyapunov coefficient for the singularity at the origin of (3.32) is given by (3.33) as

$$V_{2,i}(\varepsilon) = \delta(\alpha_{2,i}^+(\varepsilon) - \alpha_{2,i}^-(\varepsilon)), \ \alpha_{2,i}^\pm(\varepsilon) = \frac{-2\tilde{f}_{0,\varepsilon}^\pm \pm 2\delta \tilde{a}_\varepsilon^\pm \tilde{g}_{0,0,\varepsilon}^\pm}{3\tilde{a}_\varepsilon^\pm}, \tag{3.34}$$

where $\delta = \text{sign}(X_\varepsilon^+(0,0)) = \text{sign}(X^+(0,0))$, for $\varepsilon > 0$ sufficiently small,

$$\tilde{a}_\varepsilon^\pm = \frac{1}{|\widetilde{X}_\varepsilon^\pm(0,0)|} \frac{\partial \widetilde{Y}_\varepsilon^\pm}{\partial u}(0,0), \ \tilde{f}_{0,\varepsilon}^\pm = \tilde{f}_\varepsilon^\pm(0), \text{ and } \tilde{g}_{0,0,\varepsilon}^\pm = \tilde{g}_\varepsilon^\pm(0,0),$$

with

$$\tilde{f}_\varepsilon^\pm(u) = \frac{\pm\delta \widetilde{Y}_\varepsilon^\pm(u,0) - \tilde{a}_\varepsilon^\pm u \widetilde{X}_\varepsilon^\pm(u,0)}{u^2 \widetilde{X}_\varepsilon^\pm(u,0)} \text{ and}$$

$$\tilde{g}_\varepsilon^\pm(u,y) = \frac{\pm \widetilde{X}_\varepsilon^\pm(u,0)\widetilde{Y}_\varepsilon^\pm(u,y) \mp \widetilde{X}_\varepsilon^\pm(u,y)\widetilde{Y}_\varepsilon^\pm(u,0)}{y\delta \widetilde{X}_\varepsilon^\pm(u,y)\widetilde{X}_\varepsilon^\pm(u,0)}.$$

Using that (see (3.21))

$$Y^{\pm}(x,y) = \pm\delta X^{\pm}(x,y)\left(a^{\pm}x^{2k-1} + x^{2k}f^{\pm}(x) + yg^{\pm}(x,y)\right), \quad (x,y) \in U,$$

we compute

$$\begin{aligned}
\tilde{a}^{\pm}_{\varepsilon} &= \pm\delta(P^{\pm}_{\varepsilon\Lambda})'(\varepsilon a_i) + \varepsilon^{2k-2}(2k-1)a^{\pm}a_i^{2k-2} + 2\varepsilon^{2k-1}ka_i^{2k-1}f^{\pm}(\varepsilon a_i) \\
&\quad + \varepsilon^{2k}a_i^{2k}(f^{\pm})'(a_i\varepsilon), \\
\tilde{f}^{\pm}_{0,\varepsilon} &= \frac{\pm\delta(P^{\pm}_{\varepsilon\Lambda})''(\varepsilon a_i)}{2} + \varepsilon^{2k-3}a^{\pm}(k-1)(2k-1)a_i^{2k-3} + \varepsilon^{2k-2}k(2k-1)a_i^{2k-2}f^{\pm}(a_i\varepsilon) \\
&\quad + 2\varepsilon^{2k-1}ka_i^{2k-1}(f^{\pm})'(a_i\varepsilon) + \varepsilon^{2k}\frac{a_i^{2k}(f^{\pm})''(a_i\varepsilon)}{2}, \\
\tilde{g}^{\pm}_{0,0,\varepsilon} &= g^{\pm}(\varepsilon a_i, 0).
\end{aligned} \tag{3.35}$$

From (3.29), we have that

$$(P^{\pm}_{\varepsilon\Lambda})'(\varepsilon a_i) = \sum_{j=1}^{2k-2} ja_i^{j-1}\varepsilon^{2k-2}C_j^{\pm}(\Lambda,\varepsilon) \text{ and } (P^{\pm}_{\varepsilon\Lambda})''(\varepsilon a_i) = \sum_{j=2}^{2k-2} j(j-1)a_i^{j-2}\varepsilon^{2k-3}C_j^{\pm}(\Lambda,\varepsilon).$$

Thus, by denoting

$$\begin{aligned}
s_1^{\pm} &= \sum_{j=1}^{2k-2} ja_i^{j-1}C_j^{\pm}(\Lambda,0), \\
s_2^{\pm} &= \sum_{j=1}^{2k-2} ja_i^{j-1}\frac{\partial C_j^{\pm}}{\partial\varepsilon}(\Lambda,0), \\
s_3^{\pm} &= \sum_{j=2}^{2k-2} j(j-1)a_i^{j-2}C_j^{\pm}(\Lambda,0), \\
s_4^{\pm} &= \sum_{j=2}^{2k-2} j(j-1)a_i^{j-2}\frac{\partial C_j^{\pm}}{\partial\varepsilon}(\Lambda,0),
\end{aligned} \tag{3.36}$$

we get that

$$P'_{\Lambda}(\varepsilon a_i) = \varepsilon^{2k-2}s_1^{\pm} + \varepsilon^{2k-1}s_2^{\pm} + O(\varepsilon^{2k}) \text{ and } P''_{\Lambda}(\varepsilon a_i) = \varepsilon^{2k-3}s_3^{\pm} + \varepsilon^{2k-2}s_4^{\pm} + O(\varepsilon^{2k-1}). \tag{3.37}$$

Substituting (3.37) into (3.35) we obtain

$$\begin{aligned}
\tilde{a}_{\varepsilon}^{\pm} =&\varepsilon^{2k-2}\Big((2k-1)a^{\pm}a_i^{2k-2} \pm \delta s_1^{\pm}\Big) \\
&+ \varepsilon^{2k-1}\Big(2ka_i^{2k-1}f^{\pm}(\varepsilon a_i) \pm \delta s_2^{\pm}\Big) + \mathcal{O}(\varepsilon^{2k}), \\
\tilde{f}_{0,\varepsilon}^{\pm} =&\varepsilon^{2k-3}\left(a^{\pm}(k-1)(2k-1)a_i^{2k-3} \pm \frac{\delta s_3^{\pm}}{2}\right) \\
&+ \varepsilon^{2k-2}\left(k(2k-1)a_i^{2k-2}f^{\pm}(a_i\varepsilon) \pm \frac{\delta s_4^{\pm}}{2}\right) + \mathcal{O}(\varepsilon^{2k-1}), \\
\tilde{g}_{0,0,\varepsilon}^{\pm} =&g^{\pm}(\varepsilon a_i, 0).
\end{aligned} \tag{3.38}$$

From here, in order to compute $V_{2,i}(\varepsilon)$, we proceed with the following steps. Firstly, we substitute the expressions in (3.38) into the formula (3.34) to get

$$\begin{aligned}
\alpha_{2,i}^{\pm}(\varepsilon) =&\frac{\mp\delta a_i^3 s_3^{\pm} - (2k-2)(2k-1)a^{\pm}a_i^{2k}}{\varepsilon\left(3(2k-1)a^{\pm}a_i^{2k+1} \pm 3\delta a_i^3 s_1^{\pm}\right) + \varepsilon^2\left(6kf_0^{\pm}a_i^{2k+2} \pm 3\delta a_i^3 s_2^{\pm}\right) + \mathcal{O}(\varepsilon^3)} \\
&+ \frac{\varepsilon\Big((2g_{0,0}^{\pm}s_1^{\pm} \mp \delta s_4^{\pm})a_i^3 - (4k-2)(f_0^{\pm}k \mp \delta a^{\pm}g_{0,0}^{\pm})a_i^{2k+1}\Big) + \mathcal{O}(\varepsilon^2)}{\varepsilon\left(3(2k-1)a^{\pm}a_i^{2k+1} \pm 3\delta a_i^3 s_1^{\pm}\right) + \varepsilon^2\left(6kf_0^{\pm}a_i^{2k+2} \pm 3\delta a_i^3 s_2^{\pm}\right) + \mathcal{O}(\varepsilon^3)} \\
=&\varepsilon^{-1}\frac{\mp\delta a_i^3 s_3^{\pm} - (2k-2)(2k-1)a^{\pm}a_i^{2k}}{3a^{\pm}(2k-1)a_i^{2k+1} \pm 3\delta a_i^3 s_1^{\pm}} + A_i^{\pm} + \mathcal{O}(\varepsilon).
\end{aligned}$$

The expression of $A_i^{\pm}$ is a little cumbersome, so we shall omit it here. Secondly, taking into account the definition of the coefficients of the polynomials $P_{\Lambda}^{\pm}$ in (3.28) and expression (3.23), we get the following identities

$$s_1^- = -\frac{a^-}{a^+}s_1^+, \; s_2^- = -\frac{f_0^-}{f_0^+}s_2^+, \; s_3^- = -\frac{a^-}{a^+}s_3^+, \text{and } s_4^- = -\frac{f_0^-}{f_0^+}s_4^+,$$

which imply that the coefficients of ε^{-1} in the expansions above for $\alpha_{2,i}^{+}(\varepsilon)$ and $\alpha_{2,i}^{-}(\varepsilon)$ coincide. Thus, from (3.34), we have that

$$V_{2,i}(\varepsilon) = \delta(A_i^+ - A_i^-) + \mathcal{O}(\varepsilon),$$

implying that $V_{2,i}(\varepsilon)$ is continuous at $\varepsilon = 0$. Moreover, the identities above allow to get rid of the terms s_1^-, s_2^-, s_3^-, and s_4^- in A_i^-.

Before concluding our result, we require additional identities, as provided by the following claim.

Claim 7.1. *Consider the values defined in* (3.36). *Then, the following identities hold:*

$$\begin{aligned}
s_2^{\pm} &= \frac{f_0^{\pm}}{a^{\pm}}\left[(a_i - \alpha)s_1^{\pm} \mp \delta a^{\pm}a_i^{2k-1} \mp \delta(2k-1)a^{\pm}\alpha a_i^{2k-2}\right], \\
s_4^{\pm} &= \frac{f_0^{\pm}}{a^{\pm}}\left[(a_i - \alpha)s_3^{\pm} + 2s_1^{\pm} \mp \delta(2k-2)(2k-1)a^{\pm}\alpha a_i^{2k-3}\right],
\end{aligned} \tag{3.39}$$

where

$$\alpha = -\sum_{j=1}^{2k-2} a_j. \tag{3.40}$$

Proof of Claim 7.1. *Firstly, from* (3.29), *we have*

$$\frac{P_\Lambda^\pm(\varepsilon a_i)}{\varepsilon^{2k-1}} = \sum_{j=1}^{2k-2} C_j^\pm(\Lambda,\varepsilon)a_i^j.$$

Thus, from (3.22), *the following identity holds for all* $\varepsilon > 0$ *sufficiently small:*

$$\sum_{j=1}^{2k-2} C_j^\pm(\Lambda,\varepsilon)a_i^j = \mp\delta a_i^{2k-1}(a^\pm + a_i\varepsilon f^\pm(\varepsilon a_i)). \tag{3.41}$$

Now, define the polynomial

$$T_\Lambda^\pm(x) = \sum_{j=1}^{2k-2} C_j^\pm(\Lambda,0)x^j \pm \delta a^\pm x^{2k-1}. \tag{3.42}$$

Notice that $deg(T_\Lambda^\pm) = 2k-1$ *and, by taking* $\varepsilon = 0$ *in* (3.41), *we get*

$$T_\Lambda^\pm(0) = 0 \;\; and \;\; T_\Lambda^\pm(a_i) = 0, \;\; for \;\; i \in \{1,\dots,2k-2\}.$$

Thus, the polynomial $T_\Lambda^\pm$ *can be factorized as follows:*

$$T_\Lambda^\pm(x) = \pm\delta a^\pm x(x-a_1)(x-a_2)\dots(x-a_{2k-2}). \tag{3.43}$$

Also, define the polynomial

$$U_\Lambda^\pm(x) = \sum_{j=1}^{2k-2} \frac{\partial C_j^\pm}{\partial\varepsilon}(\Lambda,0)x^j \pm \delta f_0^\pm x^{2k}. \tag{3.44}$$

Notice that $deg(U_\Lambda^\pm) = 2k$ *and, by taking the derivative of* (3.41) *at* $\varepsilon = 0$, *we get*

$$U_\Lambda^\pm(0) = 0, U_\Lambda^\pm(a_i) = 0, \; for \; i \in \{1,2,\dots,2k-2\}.$$

Thus, the polynomial $U_\Lambda^\pm$ *can be factorized as follows:*

$$U_\Lambda^\pm(x) = \pm\delta f_0^\pm x(x-\alpha^\pm)(x-a_1)(x-a_2)\dots(x-a_{2k-2}), \tag{3.45}$$

where $\alpha^\pm$ *is, a priori, an unknown root of* $U_\Lambda^\pm$. *Nevertheless, since the coefficient of the monomial* x^{2k-1} *of* $U_\Lambda^\pm$ *is zero, we know that the sum of its roots must vanish, which provides that* $\alpha^+ = \alpha^- = \alpha$, *where* α *is given in* (3.40).

From the equations (3.43) *and* (3.45), *we have*

$$U_\Lambda^\pm(x) = \frac{f_0^\pm}{a^\pm}(x-\alpha)T_\Lambda^\pm(x)$$

and, therefore, from (3.42) and (3.44), we get the following recursive identities:

$$\begin{aligned}
\frac{\partial C_1^{\pm}}{\partial \varepsilon}(\Lambda,0) &= -\frac{f_0^{\pm}}{a^{\pm}}\alpha C_1^{\pm}(\Lambda,0),\\
\frac{\partial C_j^{\pm}}{\partial \varepsilon}(\Lambda,0) &= \frac{f_0^{\pm}}{a^{\pm}}(C_{j-1}(\Lambda,0)-\alpha C_j(\Lambda,0)),\ \ 2\leqslant j\leqslant 2k-2,\\
C_{2k-2}^{\pm}(\Lambda,0) &= \pm\delta a^{\pm}\alpha.
\end{aligned} \tag{3.46}$$

For the sake of simplicity, in what follows, we denote $C_i^{\pm} = C_i^{\pm}(\Lambda,0)$*. By using* (3.36), (3.46), *and* (3.41), *we obtain* s_2 *as follows:*

$$\begin{aligned}
s_2^{\pm} &= \frac{\partial C_1^{\pm}}{\partial \varepsilon}(\Lambda,0) + \sum_{j=2}^{2k-2} a_i^{j-1} j \frac{\partial C_j^{\pm}}{\partial \varepsilon}(\Lambda,0)\\
&= \frac{f_0^{\pm}}{a^{\pm}}\left[-\alpha C_1^{\pm} - \alpha\sum_{j=2}^{2k-2} a_i^{j-1} j C_j^{\pm} + \sum_{j=2}^{2k-2} a_i^{j-1} j C_{j-1}^{\pm}\right]\\
&= \frac{f_0^{\pm}}{a^{\pm}}\left[-\alpha s_1^{\pm} + \sum_{j=2}^{2k-2} a_i^{j-1} j C_{j-1}^{\pm}\right]\\
&= \frac{f_0^{\pm}}{a^{\pm}}\left[-\alpha s_1^{\pm} + \sum_{j=1}^{2k-3} a_i^{j} (j+1) C_{j}^{\pm}\right]\\
&= \frac{f_0^{\pm}}{a^{\pm}}\left[-\alpha s_1^{\pm} + \sum_{j=1}^{2k-2} a_i^{j} (j+1) C_{j}^{\pm} - a_i^{2k-2}(2k-1)C_{2k-2}^{\pm}\right]\\
&= \frac{f_0^{\pm}}{a^{\pm}}\left[-\alpha s_1^{\pm} + \sum_{j=1}^{2k-2} a_i^{j} j C_{j}^{\pm} + \sum_{j=1}^{2k-2} a_i^{j} C_{j}^{\pm} - a_i^{2k-2}(2k-1)C_{2k-2}^{\pm}\right]\\
&= \frac{f_0^{\pm}}{a^{\pm}}\left[-\alpha s_1^{\pm} + a_i s_1^{\pm} \mp \delta a^{\pm} a_i^{2k-1} \mp \delta(2k-1)a^{\pm}\alpha a_i^{2k-2}\right].
\end{aligned}$$

Hence, we conclude that the first identity of (3.39) *holds.*

Analogously, by using (3.36) and (3.46), we compute $s_4^\pm$ as follows:

$$
\begin{aligned}
s_4^\pm &= \frac{f_0^\pm}{a^\pm}\left[\sum_{j=2}^{2k-2} a_i^{j-2} j(j-1)C_{j-1}^\pm - \alpha \sum_{j=2}^{2k-2} a_i^{j-2} j(j-1)C_j^\pm\right] \\
&= \frac{f_0^\pm}{a^\pm}\left[\sum_{j=2}^{2k-2} a_i^{j-2} j(j-1)C_{j-1}^\pm - \alpha s_3^\pm\right] \\
&= \frac{f_0^\pm}{a^\pm}\left[\sum_{j=1}^{2k-3} a_i^{j-1} (j+1)jC_{j}^\pm - \alpha s_3^\pm\right] \qquad (3.47) \\
&= \frac{f_0^\pm}{a^\pm}\left[\sum_{j=1}^{2k-2} a_i^{j-1} (j+1)jC_{j}^\pm - a_i^{2k-3}(2k-1)(2k-2)C_{2k-2}^\pm - \alpha s_3^\pm\right] \\
&= \frac{f_0^\pm}{a^\pm}\left[\sum_{j=1}^{2k-2} a_i^{j-1} (j+1)jC_{j}^\pm \mp \delta(2k-1)(2k-2)a^\pm \alpha a_i^{2k-3} - \alpha s_3^\pm\right].
\end{aligned}
$$

Finally, notice that

$$\sum_{j=1}^{2k-2} a_i^{j-1}(j+1)jC_j^\pm = \sum_{j=1}^{2k-2} a_i^{j-1}(j-1)jC_j^\pm + 2\sum_{j=1}^{2k-2} a_i^{j-1}jC_j^\pm = a_i s_3^\pm + 2s_1^\pm. \qquad (3.48)$$

Hence, the second identity of (3.39) follows by putting (3.47) and (3.48) together.

Therefore, by using the identities

$$
\begin{aligned}
s_2^+ &= \frac{f_0^+}{a^+}\left[(a_i-\alpha)s_1^+ - \delta a^+ a_i^{2k-1} - \delta(2k-1)a^+\alpha a_i^{2k-2}\right] \quad \text{and} \\
s_4^+ &= \frac{f_0^+}{a^+}\left[(a_i-\alpha)s_3^+ + 2s_1^+ - \delta(2k-2)(2k-1)a^+\alpha a_i^{2k-3}\right],
\end{aligned} \qquad (3.49)
$$

provided by Claim 7.1, and taking into account that $\delta^{2n} = 1$ and $\delta^{2n+1} = \delta$ for any integer n, we obtain

$$V_{2,i}(\varepsilon) = \frac{2}{3}\left(\frac{a^+ g_{0,0}^+ - \delta f_0^+}{a^+} + \frac{a^- g_{0,0}^- + \delta f_0^-}{a^-}\right) + \mathcal{O}(\varepsilon) = \frac{2k+1}{3}V_2 + \mathcal{O}(\varepsilon),$$

which concludes this proof. □

3.4.3 Concluding the appearance of the limit cycles

This section is devoted to conclude the proof of Theorem 7. Consider the Filippov vector field $Z_{\varepsilon\Lambda}$, given by (3.17), and let $U \subset \mathbb{R}^2$ be a neighborhood of $(0,0)$ and $\lambda > 0$.

From Propositions 5 and 6, we can fix $\Lambda = (a_1, \ldots, a_{2k-2}) \in \mathcal{L}$, with $a_1 < 0 < a_2 < \ldots a_{2k-2}$, such that, for $\varepsilon > 0$ sufficiently small, the Filippov vector field

$Z_{\varepsilon\Lambda}$ has k $(2,2)$-monodromic tangential singularities, namely, the points $(\varepsilon a_i, 0)$ for $i \in \{1\} \cup \{2,4,\ldots,2k-2\}$. Moreover, the second Lyapunov coefficients of each one of these singularities writes like (3.31) and, therefore, for $\varepsilon > 0$ sufficiently small, they are all non-vanishing and have the same sign as V_2. Thus, we fix $\varepsilon^* > 0$ such that the norms of the polynomials $P^+_{\varepsilon^*\Lambda}$ and $P^-_{\varepsilon^*\Lambda}$ are smaller than λ, $(\varepsilon^* a_i, 0) \in U$ and $\text{sign}(V_{2,i}(\varepsilon^*)) = \text{sign}(V_2)$ for $i \in \{1\} \cup \{2,4,\ldots,2k-2\}$.

Now, denote $P^+(x) = P^+_{\varepsilon^*\Lambda}$ and $P^-(x) = P^-_{\varepsilon^*\Lambda}$ and consider the one-parameter family of Filippov vector fields $\widetilde{Z}_b(x,y)$, given by (3.16). For each $i \in \{1\} \cup \{2,4,\ldots,2k-2\}$, consider the translated Filippov vector field $Z^i_b(u,x) = \widetilde{Z}_b(u + \varepsilon^* a_i, y)$. We notice that Z^i_b writes like (3.10) and satisfies all the hypotheses of Proposition 3. Thus, given a neighborhood $U_i \subset \mathbb{R}^2$ of the origin satisfying $U_i + (\varepsilon^* a_i, 0) \subset U$, there exists an interval $I_i \subset \mathbb{R}$ containing 0 such that, for every $b \in I_i$ satisfying $\text{sign}(b) = -\text{sign}(\delta V_2)$, the Filippov vector field Z^i_b has a hyperbolic limit cycle inside U surrounding a sliding segment. In addition, the hyperbolic limit cycle is stable (resp. unstable) provided that $V_2 < 0$ (resp. $V_2 > 0$). Since the neighborhoods U_i, $i \in \{1\} \cup \{2,4,\ldots,2k-2\}$, can be arbitrarily chosen, we can impose that $(U_i + \varepsilon^* a_i) \cap (U_j + \varepsilon^* a_j) = \varnothing$, for $i \neq j$ in $\{1\} \cup \{2,4,\ldots,2k-2\}$.

Hence, by taking $I = I_1 \cap (I_2 \cap I_4 \cap \cdots \cap I_{2k-2})$, which is an interval containing the origin, we conclude that, for each $i \in \{1\} \cup \{2,4,\ldots,2k-2\}$ and for every $b \in I$ satisfying $\text{sign}(b) = -\text{sign}(\delta V_2)$, the Filippov vector $\widetilde{Z}_b(x,y)$ has a limit cycle contained in $U_i + (\varepsilon^* a_i, 0) \subset U$ surrounding an sliding segment.

3.5 Examples

In this section, we will exhibit several examples that meet the conditions of the theorems discussed in this chapter.

Example 1. *Let k^+ and k^- be positive integers, $\lambda \in \mathbb{R}$, and consider the following 1-parameter family of Filippov vector fields:*

$$Z_\lambda(x,y) = \begin{cases} \Big(1, -x^{2k^+-1}(\lambda\, x + 1)\Big), & y > 0, \\ \Big(-1, x^{2k^- -1}(x-1)\Big), & y < 0. \end{cases} \tag{3.50}$$

Notice that the origin is a $(2k^+, 2k^-)$-monodromic tangential singularity for every $\lambda \in \mathbb{R}$. From Proposition 2, we compute

$$V_2(\lambda) = -\frac{2}{1+2k^+}\lambda + \frac{2}{1+2k^-} \text{ and}$$
$$V_4(\lambda) = -\frac{4(7+k^+(3+2k^+))\lambda^3}{3(1+2k^+)^3}\lambda^3 - \frac{4(7+k^-(3+2k^-))}{3(1+2k^-)^3}.$$

Thus, for $\lambda_0 = \dfrac{1+2k^+}{1+2K^-}$*, we have*

$$V_2(\lambda_0) = 0, \; d = V_2'(\lambda_0) = -\frac{2}{1+2k^+} < 0, \; \text{and} \; \ell = V_4(\lambda_0) = -\frac{8(7+k^+(3+2k^-))}{3(1+2k^-)^3} < 0.$$

Since $\text{sign}(\ell) = -1$*, Theorem 5 implies that the Filippov vector field (3.50) admits an asymptotically stable hyperbolic limit cycle for every* λ *sufficiently close to* λ_0*. Such a limit cycle converges to the origin as* λ *goes to* λ_0*.*

The obtained hyperbolic limit cycle coexists with the monodromic tangential singularity. In other words, the limit cycle was obtained without destroying the singularity. Therefore, by performing the small perturbations of Proposition 3

$$Z_{\lambda,b}(x,y) = \begin{cases} Z_\lambda^+(x-b,y), & y > 0, \\ Z_\lambda^-(x,y), & y < 0, \end{cases}$$

one can see that $Z_{\lambda,b}$ *and* $Z_{\Lambda,b}$ *undergo a pseudo-Hopf bifurcation at* $b = 0$*, which creates a sliding segment and an additional hyperbolic limit cycle (see Figure 8), increasing for* 6 *the number of limit cycles.*

Example 2. *Let* $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_5) \in \mathbb{R}^5$ *and consider the following 5-parameter family of Filippov vector fields:*

$$Z_\Lambda(x,y) = \begin{cases} \Big(1, -x + \lambda_1 x^2 + \lambda_2 xy + \lambda_3 y^2\Big), & y > 0, \\ \Big(-1, -x + x^2 + \lambda_4 xy + \lambda_5 y^2\Big), & y < 0. \end{cases} \tag{3.51}$$

Notice that the origin is a $(2,2)$*-monodromic tangential singularity for every* $\Lambda \in \mathbb{R}^5$*. From Theorem 3, we compute* $\mathcal{V}_5 = (V_2(\Lambda), V_4(\Lambda), \ldots, V_{10}(\Lambda))$ *and* $V_{12}(\Lambda)$*. Thus, for*

$$\Lambda_0 = \Big(1, \frac{5(-1+\sqrt{109})}{2}, -\frac{5(-7+\sqrt{109})}{4}, \frac{5(1+\sqrt{109})}{2}, \frac{5(7+\sqrt{109})}{4}\Big),$$

we have

$$\mathcal{V}_5(\Lambda_0) = 0, \; \det(D\mathcal{V}_5(\Lambda_0)) = \frac{1520768}{74263959}, \; \text{and} \; V_{12}(\Lambda_0) = \frac{20030\sqrt{109}}{9009}.$$

Therefore, from Theorem 6, there exists an open set $W \subset \mathbb{R}^5$ *such that the Filippov vector field (3.51),* Z_Λ*, has* 5 *hyperbolic limit cycles for every* $\Lambda \in W$*. In addition, all the limit cycles converge to the origin as* Λ *goes to* Λ_0*.*

The obtained hyperbolic limit cycle coexists with the monodromic tangential singularity. In other words, the limit cycle was obtained without destroying the singularity. Therefore, by performing the small perturbations of Proposition 3

$$Z_{\Lambda,b}(x,y) = \begin{cases} Z_\Lambda^+(x-b,y), & y > 0, \\ Z_\Lambda^-(x,y), & y < 0, \end{cases}$$

one can see that $Z_{\lambda,b}$ and $Z_{\Lambda,b}$ undergo a pseudo-Hopf bifurcation at $b = 0$, which creates a sliding segment and an additional hyperbolic limit cycle (see Figure 8), increasing for 2 the number of limit cycles.

Example 3. *Let k be a positive integer, $\lambda \in \mathbb{R}$, and consider the following 1-parameter family of Filippov vector fields:*

$$Z_\lambda(x,y) = \begin{cases} \Big(1, x^{2k-1}(\lambda\, x - 1) + y\Big), & y > 0, \\ \Big(-1, x^{2k-1}(x-1)\Big), & y < 0, \end{cases} \tag{3.52}$$

Notice that the origin is a $(2k, 2k)$-monodromic tangential singularity for every $\lambda \in \mathbb{R}$.

From Proposition 2, we compute

$$\begin{aligned} V_2(\lambda) =& \frac{2\lambda}{1+2k} \quad \text{and} \\ V_4(\lambda) =& \frac{1}{3(3+2k)(1+2k)^3}\Big(-4(2+k)(1+2k)^2 + 12(19+14k)\lambda + \\ & 6(3+2k)(13+2k)\lambda^2 + 4(3+2k)(7+k(3+2k))\lambda^3\Big). \end{aligned}$$

Thus, for $\lambda_0 = 0$, we have

$$V_2(\lambda_0) = 0, \; d = V_2'(\lambda_0) = \frac{2}{1+2k} > 0, \; \text{and } \; \ell = V_4(\lambda_0) = \frac{-4(2+k)}{9+12k(2+k)} < 0.$$

Theorem 5 implies that the Filippov vector field (3.52) admits an asymptotically stable hyperbolic limit cycle for every $\lambda > \lambda_0$ sufficiently close to λ_0. Such a limit cycle converges to the origin as λ goes to λ_0.

Knowing that

$$V_2 = \frac{2\lambda}{1+2k}.$$

Now, let $\lambda > 0$ be fixed such that (3.52) has an asymptotically stable hyperbolic limit cycle. Applying Theorem 7, we conclude that the Filippov system (3.52) can be perturbed within the space of polynomial Filippov systems of degree $2k-1$ in such a way that k extra unstable limit cycles emerge from the origin. More specifically, given $\lambda > 0$ and a neighborhood $U \subset \mathbb{R}^2$ of $(0,0)$, there exist polynomials P^+ and P^-, with degree $2k-2$ and norm less than λ, and a neighborhood $I \subset \mathbb{R}$ of 0 such that, for every $b \in I$ satisfying $\operatorname{sign}(b) = -\operatorname{sign}(\eta)$, i.e. $b < 0$, the Filippov system

$$\widetilde{Z}(x,y) = \begin{cases} \Big(1, (x+b)^{2k-1}(\lambda\,(x+b) - 1) + y + P^+(x+b)\Big), & y > 0, \\ \Big(-1, x^{2k-1}(x-1) - P^-(x)\Big), & y < 0, \end{cases}$$

has an asymptotically stable limit cycle and k unstable hyperbolic limit cycles. These k limit cycles are inside U.

4 Period constants for tangential singularities, isochronicity, and criticality

In planar vector fields, the *isochronicity problem* concerns about distinguishing whether a center is isochronous or not. Recall that a center of a planar vector field is called *isochronous* if the *period function* $T : S \to \mathbb{R}$ is constant. The period function is defined in a Poincaré section S transverse to the period annulus and corresponds to the period of the trajectory starting at a point in S. In other words, a center is isochronous provided that every trajectory in a neighborhood of it has the same period. The isochronicity problem goes back to C. Huygens with his studies on the pendulum clock that oscillates isochronously (5). For smooth planar vector fields, Poincaré and Lyapunov showed that the isochronicity of a center is directly connected with its linearizability (45). Their discovery has driven the subsequent studies on the isochronicity problem, which has attracted considerable attention ever since (see, for instance, (11, 43, 46, 48)).

More recently, the isochronicity problem has also been considered for planar non-smooth vector fields of type

$$Z(x,y) = \begin{cases} Z^+(x,y), & h(x,y) > 0, \\ Z^-(x,y), & h(x,y) < 0, \end{cases} \tag{4.1}$$

where $h : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function having 0 as a regular value, $Z^\pm$ are smooth vector fields, and $\Sigma = h^{-1}(0)$ is the discontinuity manifold, that is, we are considering Filippov systems, as defined in Section 1.1. Regarding the existence of isochronous centers in Filippov vector fields, conditions on a family of piecewise quadratic systems were provided in (17) to ensure that the origin is an isochronous center. Equally important, one can find results about the non-existence of isochronous centers, for instance, in (36), where it was shown that the origin of the non-smooth oscillator $\ddot{x} + g(x)\,\mathrm{sign}\,\dot{x} + x = 0$ is never an isochronous center for any analytic function g satisfying $g(0) = g'(0) = 0$. Finally, in (9), conditions were obtained for piecewise linear vector fields to have an isochronous center at infinity. The papers above, but the last one, investigated the isochronicity problem around a focus-focus center.

Another problem related to the isochronicity problem is the *criticality problem*, which has been gaining attention over the last decades. The *criticality problem* was introduced by Chicone and Jacobs (13) and it explores the presence of oscillations or *critical points* of the *period function*, also known as a *critical period*. A *critical period* is defined as a critical point of the period function, that is, a point $p > 0$ satisfying

$T'(p) = 0$. In addition, it is called *simple* provided that $T''(p) \neq 0$. The number of critical periods and the number of simple critical periods give, respectively, an upper and a lower bound for the number of oscillations of the period function. Analogously to the center and cyclicity problems, the isochronicity and criticality problems share a strong similarity, in this case, the period constants play the same role as the Lyapunov coefficients. For some examples of works about the criticality problem for some families see (35, 42).

In Chapter 2 we obtained a general recursive formula for the Lyapunov coefficients, which control whether a monodromic singularity is a center or a focus. Here, using the tools given in Chapter 2, we study the period function and the isochronicity problem for planar Filippov vector fields around tangential centers.

This chapter is based on the paper (40) and we will present a formula for the *period function* and also a way to compute the *period constants* for the system (4.1). Finally, we will show that in Filippov vector fields, isochronous tangential centers cannot exist, that is, considering tangential centers in Filippov vector fields those centers are always not isochronous.

To construct the period function of planar Filippov vector fields around tangential centers, firstly we suppose that the Filippov vector field (4.1) has a $(2k^+, 2k^-)$-tangential center at $p \in \Sigma$. Recall from Definition 2 that a $(2k^+, 2k^-)$-monodromic tangential singularity of a planar Filippov vector field is called $(2k^+, 2k^-)$-*tangential center* provided that it has a neighborhood of where the first return map is the identity. Also, we recall that, without loss of generality, by taking local coordinates, we can consider $p = (0, 0)$ and $h(x, y) = y$, therefore we will consider the system (4.1) as

$$Z(x,y) = \begin{cases} (X^+(x,y), Y^+(x,y)), & y > 0, \\ (X^-(x,y), Y^-(x,y)), & y < 0. \end{cases} \tag{4.2}$$

4.1 Period function for the canonical form

The *period function* is a mathematical function that associates a period with each point in the phase space of the system (see Figure 11). The period function describes the periodic behavior of trajectories in the vector field, specifically in cases where the system has closed orbits or periodic solutions.

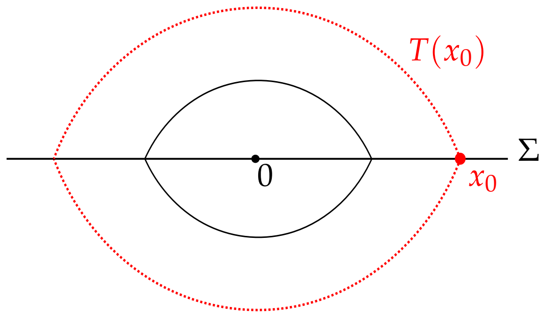


Figure 11 – $(2k^+, 2k^-)$-tangential center and the period function.

In Section 2.4, the formulae obtained for the Lyapunov coefficients for a $(2k^+, 2k^-)$-monodromic tangential singularity were obtained based on the canonical form developed in Section 1.3. Therefore, to compute the *period function* for (4.2) with a $(2k^+, k^-)$-monodromic tangential singularity, we will first compute the period function for the canonical form.

As commented in Section 1.3, assuming that the Filippov vector field (4.2) has a $(2k^+, 2k^-)$-tangential center at the origin (see conditions **C1**, **C2**, and **C3** from Definition 1), there exists a small neighborhood U of the origin such that $X^{\pm}(x,y) \neq 0$, for all $(x,y) \in U$. Taking into account that $|X^{\pm}(x,y)| = \pm\delta X^{\pm}(x,y)$, for all $(x,y) \in U$, the canonical form is obtained via a time-reparametrization that transforms the Filippov vector field (4.2) restricted to U into

$$\widetilde{Z}(x,y) = \begin{cases} \widetilde{Z}^+(x,y) = (\delta, \eta^+(x,y)), & y > 0, \\ \widetilde{Z}^-(x,y) = (-\delta, \eta^-(x,y)), & y < 0, \end{cases} \tag{4.3}$$

where

$$\eta^+(x,y) = \delta\frac{Y^+(x,y)}{X^+(x,y)}, \quad \eta^-(x,y) = -\delta\frac{Y^-(x,y)}{X^-(x,y)},$$

and

$$\delta = \operatorname{sign}(X^+(0,0)) = -\operatorname{sign}(X^-(0,0)). \tag{4.4}$$

Notice that

$$\widetilde{Z}^{\pm}(x,y) = \frac{Z^{\pm}(x,y)}{|X^{\pm}(x,y)|} \tag{4.5}$$

and that $\delta > 0$ (resp. $\delta < 0$) implies that the flow of Z turns around the origin in the clockwise (resp. anti-clockwise) direction.

It is worth mentioning that the time-reparametrization above was the first step in (38) for obtaining a canonical form for Filippov vector fields around a $(2k^+, 2k^-)$-monodromic tangential singularity.

The following lemma provides us with the period function for the canonical form (4.3).

Lemma 1. *Assume that the Filippov vector field* (4.2) *has a* $(2k^+, 2k^-)$*-tangential center at the origin and let* φ *be the associated half-return map. Then, the period function of its time-reparametrization* (4.3) *is given by* $\widetilde{T}(x) = 2(x - \varphi(x))$.

Proof. In order to compute the period function of (4.3), we shall consider the half-period functions $\widetilde{T}^+$ and $\widetilde{T}^-$ defined, respectively, as the flight-times taken for the trajectories of $\widetilde{Z}^+$ and $\widetilde{Z}^-$, starting at $(x, 0) \in \Sigma \cap U$, for $x \geqslant 0$, to reach Σ again. Accordingly, for $x \geqslant 0$ small, the period function $\widetilde{T}$ of (4.3) is defined as $\widetilde{T}(x) = \delta(\widetilde{T}^-(x) - \widetilde{T}^+(x))$. Notice that $\widetilde{T}(x) \geqslant 0$. Indeed, for $\delta > 0$ (resp. $\delta < 0$) one has that the flow of Z turns around the origin in the clockwise (resp. anti-clockwise) direction and, therefore, $\widetilde{T}^+(x) \leqslant 0 \leqslant \widetilde{T}^-(x)$ (resp. $\widetilde{T}^-(x) \leqslant 0 \leqslant \widetilde{T}^+(x)$) for all $x \geqslant 0$ such that $(x, 0) \in \Sigma \cap U$.

Now, the trajectories of $\widetilde{Z}^\pm$ with initial condition $(x, 0) \in \Sigma \cap U$ are given by

$$\gamma^\pm(t, x) = (x \pm \delta t, y^\pm(t, x)), \tag{4.6}$$

where $t \mapsto y^\pm(t, x)$ is the solution of the initial value problem

$$\frac{dy}{dt} = \eta^\pm(x, y), \quad y(0) = 0. \tag{4.7}$$

Consider the transversal sections $\Sigma_+^\perp = \{(x, y) \in U : x = 0, y > 0\}$ and $\Sigma_-^\perp = \{(x, y) \in U : x = 0, y < 0\}$ (see Fig. 12). It is clear that the flight-times taken for the trajectories of Z^+, starting at the points $(x, 0)$ and $(\varphi(x), 0)$, to reach the section $\Sigma_+^\perp$ are given by $t_1^+ = -\delta x$ and $t_2^+ = -\delta\varphi(x)$, respectively. Analogously, the flight-times taken for the trajectories of Z^-, starting at the points $(x, 0)$ and at $(\varphi(x), 0)$, to reach the section $\Sigma_-^\perp$ are given by $t_1^- = \delta x$ and $t_2^- = \delta\varphi(x)$, respectively. Therefore, the half-period functions are given by $\widetilde{T}^\pm(x) = t_1^\pm - t_2^\pm = \pm\delta(\varphi(x) - x)$, which yields $\widetilde{T}(x) = \delta(\widetilde{T}^-(x) - \widetilde{T}^+(x)) = 2(x - \varphi(x))$. □

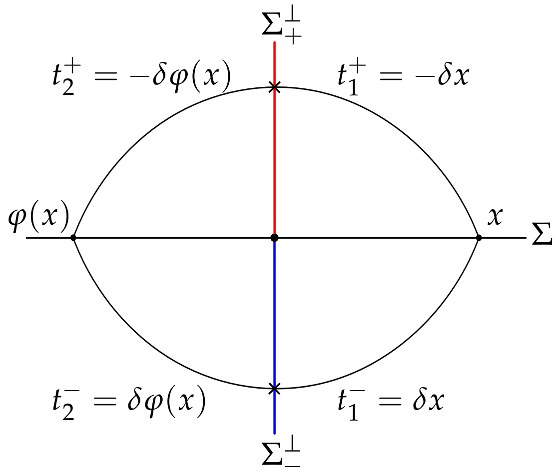


Figure 12 – Transversal sections $\Sigma_-^\perp$ and $\Sigma_+^\perp$ and the flight-times taken for the trajectories of $\widetilde{Z}^+$ and $\widetilde{Z}^-$, starting at the points $(x, 0)$ and $(\varphi(x), 0)$, to reach $\Sigma_-^\perp$ and $\Sigma_+^\perp$, respectively.

4.2 Period function for tangential centers

The next result provides an expression for the period function of the Filippov vector field (4.1) around the tangential center at the origin in terms of the half-return map φ and the functions $y^\pm$.

Theorem 8. *Assume that the Filippov vector field (4.1) has a $(2k^+, 2k^-)$-tangential center at the origin and let φ be the associated half-return map. Then, the period function is given by*

$$T(x) = \delta(T^-(x) - T^+(x)), \tag{4.8}$$

where

$$T^\pm(x) = (\varphi(x) - x)\int_0^1 \frac{1}{X^\pm(x + (\varphi(x) - x)t, y^\pm(\pm\delta(\varphi(x) - x)t, x))}\, dt. \tag{4.9}$$

In the proof of Theorem 8, we shall see that $T^-(x)T^+(x) \leqslant 0$ and that the constant δ corrects the sign of $T^-(x) - T^+(x)$ in such way that $T(x) \geqslant 0$.

To establish Theorem 8, we begin by presenting some essential preliminary results.

The proof of Theorem 8 is based on a time-reparametrization of Filippov vector fields around a $(2k^+, 2k^-)$-monodromic tangential singularity, for which the period function can be easily computed. Thus, the following result will be of major importance for recovering the period function of the original Filippov vector field:

Proposition 7 ((12, Proposition 1.14))**.** *Let $U \in \mathbb{R}^n$ be an open set, $F : U \to \mathbb{R}^n$ a smooth vector field, and $g : U \to \mathbb{R}$ a positive smooth function. Consider the following differential* *equations*

$$\dot{x} = F(x) \tag{4.10}$$

and

$$\dot{x} = g(x)F(x). \tag{4.11}$$

If $J \subset \mathbb{R}$ is an open interval containing the origin and $\gamma : J \to \mathbb{R}^n$ is a solution of the differential equation (4.10) with $\gamma(0) = x_0 \in U$, then the function $B : J \to \mathbb{R}$ given by

$$B(t) = \int_0^t \frac{1}{g(\gamma(s))} ds$$

is invertible on its range $K \subset \mathbb{R}$. If $\rho : K \to J$ denotes the inverse of B, then the identity

$$\rho'(t) = g(\gamma(\rho(t)))$$

holds for all $t \in K$ and the function $\sigma : K \to \mathbb{R}^n$ given by $\sigma(t) = \gamma(\rho(t))$ is the solution of the differential equation (4.11) with initial condition $\sigma(0) = x_0$.

4.2.1 Establishing the period function

This section is dedicated to the proof of Theorem 8.

Now, Lemma 1 can be applied together with Proposition 7 in order to compute the period function of the Filippov vector field (4.2) by assuming that it has a $(2k^+, 2k^-)$-tangential center.

Analogously to the proof of Lemma 1, the period function T of (4.2) is given by

$$T(x) = \delta(T^-(x) - T^+(x)) \geqslant 0,$$

where $T^\pm$ are the half-period functions of $Z^\pm$.

From (4.5), $Z^\pm|_U = g^\pm(x,y)\widetilde{Z}^\pm$, where $g^\pm(x,y) = |X^\pm(x,y)| = \pm\delta X^\pm(x,y)$. Hence, if $\sigma^\pm(t,x)$ denotes the trajectory of $Z^\pm$ with initial condition $(x,0) \in \Sigma \cap U$, then, from Proposition 7,

$$\sigma^\pm(t,x) = \gamma^\pm(\rho_x^\pm(t), x), \tag{4.12}$$

where $\gamma^\pm(t,x)$, given by (4.6), is the trajectory of $\widetilde{Z}^\pm$ with initial condition $(x,0) \in \Sigma \cap U$, $\rho_x^\pm = (B_x^\pm)^{-1}$, and

$$B_x^\pm(\tau) = \int_0^\tau \frac{1}{g^\pm(\gamma^\pm(s,x))} ds = \int_0^\tau \frac{\pm\delta}{X^\pm(x \pm \delta s, y^\pm(s,x))} ds.$$

Thus, from (4.12),

$$\gamma^\pm(\rho_x^\pm(T^\pm(x)), x) = \sigma^\pm(T^\pm(x), x) \in \Sigma \cap U.$$

Consequently, from the characterization of the half-period functions $\widetilde{T}^{\pm}$ of the cannonical system (4.3), we have that

$$\rho_x^{\pm}(T^{\pm}(x)) = \widetilde{T}^{\pm}(x) = \pm\delta(\varphi(x) - x),$$

which implies that

$$T^{\pm}(x) = B_x^{\pm}(\pm\delta(\varphi(x) - x)).$$

Hence,

$$T^{\pm}(x) = \int_0^{\pm\delta(\varphi(x)-x)} \frac{\pm\delta}{X^{\pm}(x \pm \delta s, y^{\pm}(s,x))} ds.$$

We conclude this proof by performing the change of variables $s = \pm\delta(\varphi(x) - x)t$ in the integrals above, which yields

$$T^{\pm}(x) = (\varphi(x) - x) \int_0^1 \frac{1}{X^{\pm}(x + (\varphi(x) - x)t, y^{\pm}(\pm\delta(\varphi(x) - x)t, x))} dt.$$

4.3 Period constants

This section provides the formulae for the computation of the period constants.

From Theorem 1 we know that the half-return map φ is analytic in a neighborhood of $x = 0$ provided that Z^+ and Z^- are analytic in a neighborhood of the origin. Therefore, one can easily see that the period function $T(x)$ given by Theorem 8 is also analytic in a neighborhood of $x = 0$ provided that Z^+ and Z^- are analytic in a neighborhood of the origin. In this case, the *period constants* $\widehat{T}_i$, $i \in \mathbb{N}$, are defined as the coefficients of the power series of $T(x)$ around $x = 0$, that is,

$$T(x) = \sum_{i=0}^{\infty} \widehat{T}_i x^i. \tag{4.13}$$

Notice that, from (4.13) and (4.8),

$$\widehat{T}_i = \frac{1}{i!} T^{(i)}(0) = \frac{1}{i!} \delta\big((T^-)^{(i)}(0) - (T^+)^{(i)}(0)\big).$$

In addition, from (4.9),

$$(T^{\pm})^{(i)}(0) = \int_0^1 \frac{\partial^i}{\partial x^i} \left(\frac{\varphi(x) - x}{X^{\pm}(x + (\varphi(x) - x)t, y^{\pm}(\pm\delta(\varphi(x) - x)t, x))} \right) \Bigg|_{x=0} ds.$$

Thus, in order to compute $\widehat{T}_i$, it only remains to know how to compute the higher derivatives of the functions $x \mapsto y^{\pm}(\pm\delta(\varphi(x) - x)t, x))$ and $\varphi(x)$ at $x = 0$. This has been done in Section 2.4.

Let

$$y^{\pm}(t,x) = \sum_{i=1}^{\infty} \frac{y_i^{\pm}(x)}{i!} t^i \tag{4.14}$$

and

$$\varphi(x) = -x + \sum_{n=2}^{\infty} \alpha_n x^n. \tag{4.15}$$

Recall that $\varphi'(0) = -1$.

The coefficient functions y_i of the series (4.14) are given by (2.14) of Theorem 3.

4.4 Tangential centers: Non-isochronicity and non-existence of critical periods

The isochronicity problem in dynamical systems examines whether the orbits of a system take the same time to complete, regardless of their starting point. For Filippov systems, as described in (4.1), it is equivalent to say that the orbits starting at the discontinuity manifold Σ take the same amount of time to return to Σ.

The criticality problem is a problem, related to the isochronicity problem, deals with the bifurcation of critical periods in perturbed centers.

As a consequence of Theorem 8 we have the following corollary:

Corollary 1. *Assume that Filippov vector field* (4.1) *has a* $(2k^+, 2k^-)$*-tangential center at the origin and let* $T(x)$ *be the period function given by* (4.8)*. Then,*

$$\widehat{T}_0 := T(0) = 0 \text{ and } \widehat{T}_1 := T'(0) = 2\delta\Big(\frac{X^-(0,0) - X^+(0,0)}{X^+(0,0)X^-(0,0)}\Big) > 0.$$

Clearly, an isochronous center must satisfy $\widehat{T}_0 \neq 0$ and $\widehat{T}_i = 0$, for all $i \in \mathbb{N}\backslash\{0\}$. Accordingly, it follows from Corollary 1, as well as from (4.8) and (4.9), along with the observation that $\varphi'(0) = -1$ and taking into account condition **C3** (see Definition 1) and identity (4.4) that a $(2k^+, 2k^-)$-tangential center of a planar Filippov vector field is not isochronous. Also as a consequence of Corollary 1, a perturbed tangential center does not admit critical periods in a neighborhood of $x = 0$, because in order to a $(2k^+, 2k^-)$-tangential center to have critical periods we need that $\widehat{T}_1 := T'(0) = 0$ and, also from Corollary 1 we have that $\widehat{T}_1 := T'(0) \neq 0$ for a $(2k^+, 2k^-)$-tangential center and, therefore, the period function does not oscillate in this neighborhood. Consequently, we arrive at the following theorem.

Theorem 9. *A* $(2k^+, 2k^-)$*-tangential center of a planar Filippov vector field is not isochronous and does not admit critical periods in a neighborhood of* $x = 0$*.*

Final considerations

In this thesis, we essentially consider the cyclicity and isochronicity problems for monodromic tangential singularities. To address these problems, after giving some properties and results about the half return maps we provided recursive formulae for computing the Lyapunov coefficients for monodromic tangential singularities, which we consider a significant result for exploring the cyclicity problem and other related problems. Using these recursive formulae, we also present, through some cumbersome computation, a proposition with the first four coefficients of the half-return maps, thereby explicitly establishing the first four Lyapunov coefficients for any given system, simplifying their computation.

Our studies have led us to employ various techniques from the qualitative theory of ordinary differential equations, such as generalized blow-up, canonical form, and the Malgrange preparation theorem.

We also implemented the algorithm for computing the Lyapunov coefficients in the algebra system *MATHEMATICA* being an useful opportunity to explore results related to the cyclicity problem and we believe that there is still a lot to be done in this area.

Regarding the *cyclicity problem* for monodromic tangential singularities, we believe there is still much to explore, such as searching for lower bounds for the number of limit cycles in certain families of planar systems with monodromic tangential singularities, similar to the study of the cyclicity problem in the regular discontinuous case. This work can be done using the formulae for the Lyapunov coefficients provided in this thesis. We intend to investigate these lower bounds for some polynomial families with monodromic tangential singularities, utilizing our formulae.

In Chapter 3, besides classical cyclicity results for monodromic tangential singularities, we also provide a different perspective on the pseudo-Hopf bifurcation, demonstrating that for a more degenerate singularity, a $(2k, 2k)$-monodromic tangential singularity, the pseudo-Hopf bifurcation leads to the appearance of additional k limit cycles. We plan to investigate if these findings hold true when considering a $(2k, 2k)$-monodromic tangential singularity with l limit cycles, proving then the appearance of $k \cdot l$ limit cycles.

In Chapter 4, we address period constants for tangential singularities, exploring the isochronicity and criticality problems. We concluded that isochronous tangential centers do not exist for planar systems with monodromic tangential singularities, and hence neither do critical periods. Despite this, we believe there is room for further investigation, for instance, when considering a mix of an invisible contact at the origin for Z^+ and a center or focus for Z^-.

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