SOME APPLICATIONS OF THE GENERALIZED KHINTCHINE'S INEQUALITY

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ABSTRACT – The generalized Khintchine's inequality is applied to the theory of homogeneous polynomials between Banach spaces in order to prove some non-trivial inclusions (e.g.: the Banach space of the n-homogeneous polynomials of r-dominated type from a Banach space into a Hilbert spaces is continuously included in the Banach space of the absolutely 2-summing polynomials for each $r \in (0, +\infty)$; the Banach spaces of the 2-homogeneous complex polynomials of r-dominated type on Banach spaces are isomorphic for $r \in [2, +\infty)$.)

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The generalized Rademacher functions, introduced by Aron and Globevnik in [1] were used in several applications by Aron, Lacruz, Ryan and Tonge in [2]. In this same article the authors mention that standard type Khintchine inequalities can be obtained with the generalized Rademacher functions replacing the Rademacher functions and making an adaptation of the usual proof the classical inequality. However they do not make any applications of it. In this paper we apply this inequality in order to obtain results of the theory of polynomial mappings between Banach spaces.

The generalized Rademacher functions are described in the following way. For a fixed natural number $n \geq 2$ we take the n-th roots of unity $1 = \alpha_1, \alpha_2, \ldots, \alpha_n$ considered in the order of their increasing arguments. The closed interval $[0, 1]$ is divided in n intervals of equal length $I_1, \ldots, I_n$ described in the order they appear from the left to the right side of the original interval. We consider s_1 from $[0, 1]$ into $\mathbb{C}$ given by $s_1(t) = \alpha_j$ if t is in the interior of the interval $I_j, j = 1, \ldots, n$ and $s_1(t) = 1$ if t is one of the endpoints of the subintervals $I_1, \ldots, I_n$. If $k \in \mathbb{N}, k \geq 1$ and we suppose that $s_1, \ldots, s_k$ are defined, we construct s_k in the following way: each interval I used in the definition of s_k is divided in n intervals $I_1, \ldots, I_n$ of equal length writen in the order they appear on I from the left to the right side of it. Then we consider s_{k+1} equal to α_j on the interior of I_j and equal to 1 at the endpoints of $I_j, j = 1, \ldots, n$.

The following lemma appears in [2].

1. LEMMA - If $(s_k)_{k=1}^{\infty}$ are the generalized Rademacher functions associated to $n \in I\!N, n \geq 2$, then

(1) $|s_k(t)| = 1$ for $k \in I\!N$ and $t \in [0,1]$.

(2) $\int_0^1 s_{j_1}(t) \ldots s_{j_n}(t) dt = \begin{cases} 0 & \text{if } j_1 = \ldots = j_n \\ 0 & \text{otherwise} \end{cases}$

(3) If $j_1 < \ldots < j_k$ are natural numbers and $\sigma_j(t)$ equal either to $s_j(t)$ or $\overline{s_j(t)}$, then

$$\int_0^1 \sigma_{j_1}(t)^{m_1} \ldots \sigma_{j_k}(t)^{m_k} dt = \begin{cases} 1 & \text{if } m_j \underset{j=1,\ldots,k}{\equiv} 0 (\text{mod } n) \\ 0 & \text{otherwise.} \end{cases}$$

This result plays an important role in the proof of the

2. GENERALIZED KHINTCHINE INEQUALITIES - If $n \in I\!N, n \geq 2$ is fixed and $(s_k)_{k=1}^{\infty}$ are the generalized Rademacher functions associated to n, for every $p \in (0, +\infty)$ there are $\alpha(n,p) > 0$ and $\beta(n,p) > 0$ such that for each $m \in I\!N$ and $a_j \in \mathbb{C}, j = 1, \ldots, m$

$$\alpha(n,p) \Big[\sum_{j=1}^{m} |a_j|^2 \Big]^{\frac{1}{2}} \leq \Big[\int_0^1 | \sum_{j=1}^{m} a_j s_j(t) |^p dt \Big]^{\frac{1}{p}} \leq$$

$$\leq \beta(n,p) \Big[\sum_{j=1}^{m} |a_j|^2 \Big]^{\frac{1}{2}}$$

In order to apply this result to the theory of polynomial mappings between Banach spaces we fix notations and recall some concepts.

$\wp(^nE;F)$ denotes the Banach space of all continuous n-homogeneous polynomials from the complex Banach space E into the complex Banach space F under the norm

$$\|P\| = \sup_{\|x\| \leq 1} \|P(x)\| \qquad (\forall\, P \in \wp(^nE;F)).$$

If $p \in (0, \infty)$ we denote by $\ell_p^w(E)$ the set of all sequences $(x_j)_{j=1}^{\infty}$ of elements of E such that

$$\|(x_j)_{j=1}^{\infty}\|_{w,p} = \sup_{\varphi \in B_{E'}} \Big[\sum_{j=1}^{\infty} |\varphi(x_j)|^p \Big]^{\frac{1}{p}} < +\infty.$$

Here $B_{E'}$ is the closed unit ball of E' centered at the origin. If $r \in (0,+\infty)$ we consider $\ell_r(F)$ as the set of all sequences $(y_j)_{j=1}^{\infty}$ of elements of F such that

$$\|(y_j)_{j=1}^{\infty}\|_r = \left[\sum_{j=1}^{\infty} \|y_j\|^r\right]^{\frac{1}{r}} < +\infty.$$

If $P \in \wp(^nE;F)$, $s,r \in (0,+\infty)$ and $ns \geq r$, P is said to be *absolutely* $(s;r)$*-summing* if $(P(x_j))_{j=1}^{\infty} \in \ell_s(F)$ for each $(x_j)_{j=1}^{\infty} \in \ell_r^w(E)$. It can be proved that P is absolutely $(s;r)$-summing if and only if there is $C \geq 0$ such that for each $m \in I\!N$ and $x_j \in E$, $j = 1,\ldots,m$

$$\|(P(x_j))_{j=1}^{m}\|_s \leq C\left[\|(x_j)_{j=1}^{m}\|_{w,r}\right]^n \qquad (*)$$

We denote by

$$\|P\|_{as,(s;r)} = \inf_{(*)} C = \min_{(*)} C$$

and by $\wp_{as}^{(s;r)}(^nE;F)$ the vector space all absolutelly $(s;r)$-summing polynomials from E into F. This space is complete s-normed by $\|\ \|_{as,(s;r)}$ if $s \in [0,1]$ and a Banach space under $\|\ \|_{as,(s;r)}$ for $s \geq 1$. See [3] for the linear case and [4] for $n \geq 2$.

If $ns = r$ it can be proved that $P \in \wp(^nE;F)$ is absolutely $(s;r)$-summing if and only if there are $D \geq 0$ and a regular probability measure μ on the Borel subsets of $B_{E'}$ (with the weak star topology) (we denote this: $\mu \in W(B_{E'})$) such that

$$\|P(x)\| \leq D\left[\int_{B_{E'}} |\varphi(x)|^r d\mu(\varphi)\right]^{\frac{n}{r}} \qquad (**)$$

for every $x \in E$. In this case

$$\|P\|_{as,(s;r)} = \min_{(**)} D = \inf_{(**)} D$$

is denoted by $\|P\|_{d,r}$. This motivates the use of the name r-*dominated* for these polynomials. We denote by $\wp_d^r(^nE;F)$ the vector space of all r-dominated polynomials from E into F.

3. THEOREM - If F is a Hilbert space and $r \in (0,+\infty)$, then $\wp_d^r(^nE;F) \subset \wp_{as}^{(2;2)}(^nE;F) \overset{\text{not.}}{=} \wp_{as}^2(^nE;F)$ and

$$\|P\|_{as,2} \overset{\text{not.}}{=} \|P\|_{as,(2,2)} \leq (\beta(2n;r))^n \|P\|_{d,r}.$$

PROOF - Since for $0 < r_1 \leq r_2 < +\infty$ we have $\wp_d^{r_1}({}^nE;F) \subset \wp_d^{r_2}({}^nE;F)$ and $\|P\|_{d,r_2} \leq \|P\|_{d,r_1}$ for each P r_1-dominated, we can suppose $r \geq 2n$ without loss of generality.

We consider $(s_j)_{j=1}^{\infty}$, the generalized Rademacher functions associated to $2n$. If $P \in \wp_d^r({}^nE;F)$ we take $\mu \in W(B_{E'})$ corresponding to $\|P\|_{d,r}$ by (**) and consider the continuous symmetric n-linear mapping T from E^n into F such that $P(x) = T(x,\ldots,x) \overset{\text{not.}}{=} Tx^n$ for each $x \in E$. Thus for $m \in I\!N$ and $x_j \in E, j = 1,\ldots,m$ we have:

$$\sum_{j=1}^{m} \|P(x_j)\|^2 = \sum_{j=1}^{m} (Tx_j^n/Tx_j^n)$$

$$\overset{\text{Lemma 1}}{\leq} \sum_{\substack{j_k=1 \\ k=1,\ldots,n}}^{m} \sum_{i_k=1}^{m} (T(x_{j_1},\ldots,x_{j_n})/T(x_{i_1},\ldots,x_{i_n})).$$

$$\cdot \int_0^1 s_{j_1}(t)\ldots s_{j_n}(t)\overline{s_{i_1}(t)}\ldots\overline{s_{i_n}(t)}dt$$

$$= \int_0^1 (T(\sum_{j=1}^{m} s_j(t)x_j)^n / T(\sum_{j=1}^{m} s_j(t)x_j)^n)dt$$

$$= \int_0^1 \|P(\sum_{j=1}^{m} s_j(t)x_j)\|^2 dt$$

$$\leq [\|P\|_{d,r}]^2 \int_0^1 \left[\int_{B_{E'}} |\sum_{j=1}^{m} s_j(t)\varphi(x_j)|^r d\mu(\varphi)\right]^{\frac{2n}{r}} dt$$

$$\leq [\|P\|_{d,r}]^2 \left[\int_0^1 \int_{B_{E'}} |\sum_{j=1}^{m} s_j(t)\varphi(x_j)|^r d\mu(\varphi)dt\right]^{\frac{2n}{r}}$$

$$\leq [\|P\|_{d,r}]^2 \left[\int_{B_{E'}} (\beta(2n,r))^r (\|(\varphi(x_j))_{j=1}^m\|_2)^r d\mu(\varphi)\right]^{\frac{2n}{r}}$$

$$\leq [\|P\|_{d,r}]^2 \left[\beta(2n,r)\right]^{2n} \left(\|(x_j))_{j=1}^m\|_{w,2}\right)^{2n}. \quad \blacksquare$$

4. THEOREM - If $p \in [2,+\infty)$ and $r \in (0,+\infty)$, then $\wp_d^r({}^nE;L_p([a,b])) \subset \wp_{as}^{(p;2)}({}^nE;L_p([a;b]))$ and

$$\|P\|_{as,(p;2)} \leq (\beta(2n;r))^n \|P\|_{d,r}$$

for each P r-dominated.

PROOF - As we have done in the proof of 3 we can take $r \geq pn$ without loss generality. We consider the generalized Rademacher functions $(s_j)_{j=1}^{\infty}$ associated to 2n and, for $P \in \wp_d^r({}^nE; L_p([a,b]))$ we take $\mu \in W(B_{E'})$ corresponding to $\|P\|_{d,r}$ by (**). If T is the continuous symmetric n-linear mapping from E^n into $L_p([a,b])$ such that $P(x) = Tx^n$ for each $x \in E$, we can write for $m \in \mathbb{N}$ and $x_j \in E, \ j = 1, \dots, m$

$$\sum_{j=1}^{m} \|P(x_j)\|^p = \int_a^b \sum_{j=1}^{m} |P(x_j)(\theta)|^p d\theta$$

$$\leq \int_a^b \left[\sum_{j=1}^{m} |P(x_j)(\theta)|^2\right]^{\frac{p}{2}} d\theta$$

$$\overset{\text{Lemma 1}}{\leq} \Big[\sum_{\substack{j_k=1 \\ k=1,\dots,m}}^{m} \sum_{i_k=1}^{m} T(x_{j_1}, \dots, x_{j_n})(\theta)\overline{T(x_{i_1}, \dots, x_{i_n})(\theta)}.$$

$$\cdot \int_0^1 s_{j_1}(t) \dots s_{j_n}(t)\overline{s_{i_1}(t)} \dots \overline{s_{i_n}(t)} \, dt\Big]^{\frac{p}{2}} d\theta$$

$$= \int_a^b \left[\int_0^1 |P(\sum_{j=1}^{m} s_j(t)x_j)(\theta)|^2 dt\right]^{\frac{p}{2}} d\theta$$

$$\leq \int_a^b \int_0^1 \left(|P(\sum_{j=1}^{m} s_j(t)x_j)(\theta)|^2\right)^{\frac{p}{2}} d\theta$$

$$= \int_0^1 \|P(\sum_{j=1}^{m} s_j(t)x_j)\|^p dt$$

$$\leq (\|P\|_{d,r})^p \int_0^1 \left[\int_{B_{E'}} |\sum_{j=1}^{m} s_j(t)\varphi(x_j)|^r d\mu(\varphi)\right]^{\frac{pn}{r}} dt$$

$$\leq (\|P\|_{d,r}]^p \left[\int_0^1 \int_{B_{E'}} |\sum_{j=1}^{m} s_j(t)\varphi(x_j)|^r d\mu(\varphi) dt\right]^{\frac{pn}{r}}$$

$$\leq (\|P\|_{d,r})^p \left[\int_{B_{E'}} (\beta(2n,r))^r (\|(\varphi(x_j))_{j=1}^m\|_2)^r d\mu(\varphi)\right]^{\frac{pn}{r}}$$

$$\leq (\|P\|_{d,r}]^p (\beta(2n,r))^{pn} (\|(x_j)_{j=1}^m\|_{w,2})^{pn}. \ \blacksquare$$

5. THEOREM - If $n \geq 2, r \in (0, +\infty)$, then $\wp_d^r({}^nE;\mathbb{C}) \subset \wp_{as}^{(1;2)}({}^nE;\mathbb{C})$ and

$$\|P\|_{as,(1;2)} \leq (\beta(n;r))^n \|P\|_{d,r}$$

for every P r-dominated.

PROOF - Without loss of generality we may suppose $r \geq n$. For $P \in \wp_d^r({}^nE;\mathbb{C})$ we consider $\mu \in W(B_{E'})$ cooresponding to $\|P\|_{d,r}$ by (**) and the continuous symmetric n-linear mapping T from E^n into $\mathbb{C}$ such that $Tx^n = P(x)$ for every $x \in E$. For $m \in \mathbb{N}, x_j \in E, j = 1, \ldots, m$, a convenient choice of $\lambda_j \in \mathbb{C}, |\lambda_j| = 1, j = 1, \ldots, m$, and $(s_k)_{k=1}^{\infty}$ associated n, we write:

$$\begin{aligned}
&\sum_{j=1}^{m} |P(x_j)| = |\sum_{j=1}^{m} P(\lambda_j x_j)| \\
= \ & \left| \int_0^1 \sum_{\substack{j_k=1 \\ k=1,\ldots,n}}^{m} \lambda_{j_1} \ldots \lambda_{j_n} T(x_{j_1}, \ldots, x_{j_n}) s_{j_1}(t), \ldots, s_{j_n}(t) dt \right| \\
= \ & \left| \int_0^1 P(\sum_{j=1}^{m} \lambda_j s_j(t) x_j) dt \right| \\
\leq \ & \|P\|_{d,r} \int_0^1 \left[\int_{B_{E'}} |\sum_{j=1}^{m} \lambda_j s_j(t) \varphi(x_j)|^r d\mu(\varphi) \right]^{\frac{n}{r}} dt \\
\leq \ & \|P\|_{d,r} \left[\int_0^1 \int_{B_{E'}} |\sum_{j=1}^{m} \lambda_j s_j(t) \varphi(x_j)|^r d\mu(\varphi) dt \right]^{\frac{n}{r}} \\
\leq \ & \|P\|_{d,r} \left[\int_{B_{E'}} (\beta(n,r))^r (\|(\lambda_j \varphi(x_j))_{j=1}^{m}\|_2)^r d\mu(\varphi) \right]^{\frac{n}{r}} \\
\leq \ & \|P\|_{d,r} (\beta(n,r))^n (\|(x_j)_{j=1}^{m}\|_{w,2})^n. \ \blacksquare
\end{aligned}$$

6. COROLLARY - If $r \in [2, +\infty)$, then $\wp_d^r({}^2E;\mathbb{C}) = \wp_d^2({}^2E;\mathbb{C})$ and

$$\|P\|_{d,2} \leq \|P\|_{d,r} \leq (\beta(n,r))^2 \|P\|_{d,2}$$

for every P 2-dominated from E into $\mathbb{C}$.

Since $\wp_{as}^{(1;2)}({}^2E;\mathbb{C}) = \wp_d^2({}^2E;\mathbb{C}), \wp_d^r({}^2E;\mathbb{C})$ increases with r and Theorem 5 is true, we have this corollary all right.

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