APPROXIMATION PROCESSES FOR VECTOR-VALUED CONTINUOUS FUNCTIONS

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Abstract - A quantitative Bohman-Korovkin type theorem is established for continuous normed-space-valued functions defined on compact convex subsets, and sequences of operators which are dominated by positive operators. A qualitative result on Korovkin systems is established for sequences of linear operators which are monotonically regular, i.e., sequences $\{T_n\}$ such that for some sequence of positive linear operators $\{S_n\}$ we have $T_n(g \otimes v) = S_n(g) \otimes v$, for every continuous real-valued function g and every vector v.

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§1. Introduction

Let X be a compact Hausdorff space and let $(E, ||\cdot||_E)$ be a real normed space. The space $C(X;E)$ of all continuous functions from X into E is equipped with the supremum norm

$$||f|| = \sup\{||f(x)||_E \ ; \ x \in X\}.$$

Our objective is to establish convergence results $T_n f \to f$ for sequences of linear operators $T_n : C(X;E) \to C(X;E)$ and also derive estimates on the rate of convergence of $T_n f \to f$, similar to those valid in the case $E = \mathbb{R}$ and T_n is positive. Our results generalize those obtained by M.Weba [10, 11, 12] for stochastic processes, i.e., the case $E = L_p(\Omega, \Sigma, \tau)$, $1 \le p < \infty$, where (Ω, Σ, τ) is a probability space. Our monotonically regular operators defined below (see Definition 6) share features of the stochastically simple operators defined by Weba and our regular operators (see [7] and [8]). For them the usual Bohman-Korovkin type results are true and estimates can be obtained. All the operators $T : C(X;E) \to C(X;E)$ of the type

$$(Tf, x) = \sum_{k \in J} \varphi_k(x) f(t_k), \quad x \in X,$$

where J is a finite set, $t_k \in X$, and $\varphi_k \ge 0$, are monotonically regular in our sense. This includes the Bernstein operators and the Hermite-Fejér operators, besides many others.

We use the notation $g \otimes v$, where g belongs to $C(X;\mathbb{R})$ and $v \in E$, to denote the function $t \mapsto g(t)v$, $t \in X$. But, according to standard convention, $C(X;\mathbb{R}) \otimes E$ denotes the vector subspace of $C(X;E)$ generated by all such mappings $g \otimes v$. If $f \in C(X;E)$, then $||f||_E$ denotes the function $t \mapsto ||f(t)||_E$. Clearly, $||f||_E$ belongs to $C(X;\mathbb{R})$. If U is a bounded linear operator on the normed space E, i.e., an element of $\mathcal{L}(E)$, we denote its value on $v \in E$ by $\langle U, v\rangle$, and its operator norm by $||U||$:

$$||U|| = \sup\{||\langle U, v\rangle||_E; \ ||v||_E \le 1\}.$$

If for every $x \in X$, the operator $U(x)$ belongs to $\mathcal{L}(E)$ and $f \in C(X;E)$, then, for every $t \in X$, we may apply $U(x)$ to $f(t) \in E$ and

$$||\langle U(x), f(t)\rangle||_E \le ||U(x)|| \cdot ||f(t)||_E.$$

Definition 1. A linear operator S on $C(X;\mathbb{R})$ is called **positive** (or **monotone**) if $f \ge 0$ implies $Sf \ge 0$.

Definition 2. Let S be a linear operator on $C(X;\mathbb{R})$. We say that a linear operator T on $C(X;E)$ is **dominated by** S if, for every $f \in C(X;E)$ and $x \in X$, we have

(*) $$||(Tf,x)||_E \leq (S(||f||_E),x).$$

Definition 3. Let (X,d) be a compact metric space, and let S be a linear operator on $C(X;\mathbb{R})$. Define two functions α_S and β_S on X by

$$\alpha_S(x) = (S(\rho_x^2),x)$$
$$\beta_S(x) = (S(\rho_x),x)$$

for all $x \in X$, where $\rho_x(t) = d(x,t)$, for all $t \in X$.

Notice that when $S(e_0) = e_0$, where $e_0(t) = 1$ for all $t \in X$, the positivity of S will imply $[S(f)]^2 \leq S(f^2)$, for all $t \in C(X;\mathbb{R})$. Applying this to ρ_x one obtains

$$[\beta_S(x)]^2 \leq \alpha_S(x)$$

for all $x \in X$.

Remark. If S is a linear operator on $C(X;\mathbb{R})$ and $S(e_0) = e_0$, where $e_0(t) = 1$ for all $t \in X$, then $S(f) = f$ for all constant functions f in $C(X;\mathbb{R})$. We say in this case that S *preserves the constants*. Similarly, if T is a linear operator on $C(X;E)$ and $Tf = f$, for all constant functions $f \in C(X;E)$, then we say that T preserves the constants.

§2. Quantitative Estimates For Continuous Functions

Lema 1. *Let (X,d) be a compact metric space, and let $f \in C(X;E)$. For each $\varepsilon > 0$, there is some constant $K > 0$ such that*

$$||f(t) - f(x)||_E < \varepsilon + K[d(x,t)]^2$$

for every pair, t and x, of elements of X.

Proof: We omit the easy proof. □

Theorem 1. *Let (X,d) be a compact metric space. Let $\{S_n\}_{n\geq 1}$ be a sequence of positive linear operators on $C(X;\mathbb{R})$, with $S_n(e_0) = e_0$, for each $n \geq 1$, and $\alpha_{S_n}(x) \to 0$ uniformly on $x \in X$. For each $n \geq 1$, let T_n be a linear operator on $C(X;E)$ which is dominated by S_n and assume that each T_n preserves the constants. Then $T_n f \to f$, for each $f \in C(X;E)$.*

Proof. Let $\varepsilon > 0$ be given. By Lemma 1, there is $K > 0$ such that

$$||f(t) - f(x)||_E \leq \frac{\varepsilon}{2} + K[d(x,t)]^2$$

for all $t, x \in X$. Consider the function $g = f - f(x)$. Then

$$||g(t)||_E \leq \frac{\varepsilon}{2} \cdot e_0(t) + K\rho_x^2(t),$$

for all $t \in X$. Since each S_n in positive, we get

$$(S_n(||g||_E), x) \leq \frac{\varepsilon}{2} + K \cdot \alpha_{S_n}(x),$$

because $S_n(e_0) = e_0$. By Definition 2 we get

$$||(T_n g, x)||_E \leq \frac{\varepsilon}{2} + K \cdot \alpha_{S_n}(x).$$

Choose now n_0 so that $n \geq n_0$ implies $\alpha_{S_n}(x) < \varepsilon/2K$, for all $x \in X$, and notice that $(T_n g, x) = (T_n f, x) - f(x)$. Then

$$||(T_n f, x) - f(x)||_E < \varepsilon,$$

holds for all $x \in X$ and $n \geq n_0$. □

Let H be a real Hilbert space with scalar product $\langle u, v \rangle$, for $u, v \in H$, and let X be a compact subset of H. Define the following functions:

(i) $\varphi_x(t) = \langle t, x \rangle$
(ii) $\varphi(t) = \langle t, t \rangle$

for all $t \in H$, where x is some fixed element in H. Notice that, in this case,

$$\begin{aligned} \rho_x^2(t) &= [d(x,t)]^2 = \langle x - t, x - t \rangle = \\ &= \langle x, x \rangle e_0(t) - 2\varphi_x(t) + \varphi(t) \end{aligned}$$

for every $t \in H$. Let $\{S_n\}_{n \geq 1}$ be a sequence of positive linear operators on $C(X; \mathbb{R})$. The sequence $\{\alpha_{S_n}\}_{n \geq 1}$ satisfies

$$\alpha_{S_n}(x) = \varphi(x) - 2(S_n(\varphi_x), x) + (S_n(\varphi), x),$$

if we assume that $S_n(e_0) = e_0$.

Corollary 1. *Let X be a compact subset of some real Hilbert space H. Let $\{S_n\}_{n \geq 1}$ be a sequence of positive linear operators on $C(X; \mathbb{R})$ such that*

(1) $S_n(e_0) = e_0$;
(2) $(S_n(\varphi_x), x) \to \varphi(x)$, *uniformly on* $x \in X$;
(3) $S_n(\varphi) \to \varphi$.

Let $\{T_n\}_{n\geq 1}$, *be a sequence of linear operators on* $C(X; E)$. *Assume that each* T_n *preserves the constants and is dominated by* S_n. *Then* $T_n f \to f$, *for each* $f \in C(X; E)$.

Proof. We have seen that

$$\alpha_{S_n}(x) = \varphi(x) - 2(S(\varphi_x), x) + (S_n(\varphi), x).$$

By (2) and (3), it follows that $\alpha_{S_n}(x) \to 0$, uniformly on $x \in X$. It remains to apply Theorem 1. □

Let us now give the analogues of the quantitative estimates obtained by Shisha and Mond [9] for positive linear operators on $C(X; I\!R)$.

First recall that, when (X, d) is a compact metric space, every $f \in C(X; E)$ is uniformly continuous and its *modulus of continuity* is defined by

$$\omega(f;\delta) = \sup\{||f(s) - f(t)||_E; \quad d(s,t) \leq \delta\}$$

for each $\delta > 0$. Notice that $\omega(f;\delta)$ is monotonically increasing, i.e., $\delta_1 \leq \delta_2$ implies $\omega(f;\delta_1) \leq \omega(f;\delta_2)$. Moreover, $\omega(f;\delta) \to 0$ as $\delta \to 0$.

Proposition 1. *Let* X *be a compact and convex subset of some normed space* G. *Then*

$$\omega(f;\lambda\delta) \leq (1+\lambda)\omega(f;\delta)$$

for every $f \in C(X; E)$ *and* $\lambda \geq 0$, $\delta \geq 0$.

Proof. We omit the standard proof. □

Lemma 2. *Let* X *be compact and convex subset of some normed space* G. *Let* $f \in C(X; E)$ *and* $\delta > 0$ *be given. Then*

$$||f(t) - f(x)||_E \leq (1 + \frac{1}{\delta^2} \cdot ||t - x||_G^2)\omega(f;\delta)$$

for every pair, t *and* x, *of elements of* X.

Proof. If $\delta \le ||t-x||_G$, then

$$\begin{aligned} ||f(t)-f(x)||_E &\le \omega(f;||t-x||_G) \\ &\le (1+\frac{1}{\delta}\cdot||t-x||_G)\omega(f;\delta) \\ &\le (1+\frac{1}{\delta^2}\cdot||t-x||_G^2)\omega(f;\delta). \end{aligned}$$

If $||t-x||_G \le \delta$, then

$$||f(t)-f(x)||_E \le \omega(f;\delta) \le (1+\frac{1}{\delta^2}\cdot||t-x||_G^2)\omega(f;\delta).$$

□

Theorem 2. *Let X be a compact and convex subset of some normed space G. Let S be a positive linear operator on $C(X;\mathbb{R})$ such that $S(e_0)=e_0$ and let T be a linear operator on $C(X;E)$, which preserves the constants, and is dominated by S. Then*

$$||(Tf,x)-f(x)||_E \le [1+\frac{1}{\delta^2}\cdot\alpha_S(x)]\cdot\omega(f;\delta),$$

for every $f\in C(X;E),\delta>0$ and $x\in X$.

Proof. Let $f\in C(X;E),\delta>0$ and $x\in X$ be given. By Lemma 2, we have

$$||f(t)-f(x)||_E \le [1+\frac{1}{\delta^2}\cdot||t-x||_G^2]\omega(f;\delta)$$

for all $t\in X$. Consider the function $g=f-v$, where $v=f(x)$. Let $\rho_x(t)=||t-x||_G, t\in X$. Then

$$||g(t)||_E \le [e_0(t)+\frac{1}{\delta^2}\rho_x^2(t)]\omega(f;\delta)$$

for all $t\in X$. By monotonicity of S and the fact that $S(e_0)=e_0$, we get

(1) $$(S(||g||_E),x) \le [1+\frac{1}{\delta^2}\alpha_S(x)]\cdot\omega(f;\delta).$$

Now T is dominated by S and therefore, by relation (*) of Definition 2,

(2) $$||(Tg,x)||_E \le (S(||g||_E),x)$$

for all $x \in X$. On the other hand, $Tg = Tf - Tv = Tf - v$, and so $(Tg, x) = (Tf, x) - f(x)$. Hence by (1) and (2) above, we get

$$||(Tf, x) - f(x)||_E \leq [1 + \frac{1}{\delta^2} \cdot \alpha_S(x)] \cdot \omega(f; \delta).$$

□

Theorem 3. *Let X be as in Theorem 2, and let $\{S_n\}_{n \geq 1}$ be a sequence of positive linear operators on $C(X; \mathbb{R})$, such that $S_n(e_0) = e_0$, for each $n \geq 1$. Let $\{T_n\}_{n \geq 1}$ be a sequence of linear operators on $C(X; E)$ such that each T_n preserves the constants and is dominated by S_n. Suppose that*

$$\alpha_{S_n}(x) \leq \varphi(x) n^{-\beta}, \quad x \in X,$$

holds, for each $n \geq 1$, and for some $\varphi \in C(X; \mathbb{R})$ and some $\beta > 0$. Then

$$||(T_n f, x) - f(x)||_E \leq [1 + \varphi(x)] \omega(f; n^{-\beta/2})$$

holds, for each $n \geq 1$, for every $f \in C(X; E)$ and $x \in X$.

Proof. Make $\delta = n^{-\beta/2}$ in Theorem 2, applied to each pair S_n and T_n. □

Corollary 2. *Let $\{T_n\}_{n \geq 1}$ be as in Theorem 3. Then*

$$||T_n f - f|| \leq [1 + ||\varphi||] \cdot \omega(f; n^{-\beta/2})$$

holds for every $n \geq 1$ and $f \in C(X; E)$.

Proof. Take supremum on both sides of the estimate obtained in Theorem 3. □

§3. Examples

Let us give examples of operators satisfying the hypothesis of Theorems 1, 2 and 3. In our first example, let $X = [0, 1] \subset \mathbb{R}$. For each $n \geq 1$, the n^{th} Bernstein operator B_n on $C([0, 1]; \mathbb{R})$ is defined by

$$(B_n g, x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} g\left(\frac{k}{n}\right)$$

for each $g \in C([0, 1]; \mathbb{R})$ and $x \in [0, 1]$. It is clear that B_n is a positive linear operator, and since

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} = 1,$$

for every $x \in [0,1]$, it follows that $B_n(e_0) = e_0$, where $e_0(t) = 1$, for all $t \in [0,1]$. Consider now the corresponding n^{th} *Bernstein operator* T_n on $C(X;E)$, defined analogously as

$$(T_n f, x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$$

for each $f \in C([0,1];E)$ and $x \in [0,1]$. Now T_n preserves the constants and

$$\begin{aligned} \|(T_n f, x)\|_E &\leq \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \|f\left(\frac{k}{n}\right)\|_E \\ &= (B_n(\|f\|_E), x) \end{aligned}$$

holds for every $x \in [0,1]$. Hence T_n is dominated by B_n.

The classical estimates for the Bernstein operators on $C([0,1];\mathbb{R})$ give

$$\alpha_{S_n}(x) = (B_n(\rho_x^2), x) = x(1-x) \cdot \frac{1}{n}$$

since $\rho_x^2(t) = (t-x)^2$, for all $t \in [0,1]$. Hence the following estimates hold for the Bernstein operators T_n on $C([0,1];E)$:

$$\|(T_n f, x) - f(x)\|_E \leq [1 + x(1-x)]\omega(f; n^{-1/2})$$

for each $x \in [0,1]$ and so

$$\|T_n f - f\| \leq \frac{5}{4}\omega(f; n^{-1/2})$$

for all $n \geq 1$, since $x(1-x) \leq 1/4$ holds for all $x \in [0,1]$.

To generalize this example, let X be a compact Hausdorff space. Let J be a finite set, and for each $k \in J$, let $t_k \in X$ and $\Phi_k \in C(X; \mathcal{L}(E))$ be given. Define an operator $T : C(X;E) \to C(X;E)$ by setting

$$(1) \qquad (Tf, x) = \sum_{k \in J} \langle \Phi_k(x), f(t_k) \rangle$$

for every $f \in C(X;E)$ and $x \in X$. Then T is dominated by the linear operator $S : C(X;\mathbb{R}) \to C(X;\mathbb{R})$, where

$$(2) \qquad (Sg, x) = \sum_{k \in J} \varphi_k(x) g(t_k)$$

for every $g \in C(X;\mathbb{R})$ and $x \in X$, and $\varphi_k \in C(X;\mathbb{R})$ is the function

$$\varphi_k(x) = \|\Phi_k(x)\|, x \in X.$$

Since $\varphi_k \geq 0$, the linear operator S is positive and

$$\begin{aligned}
\|(Tf,x)\|_E &\leq \sum_{k\in J} \|\langle \Phi_k(x), f(t_k)\rangle\|_E \\
&\leq \sum_{k\in J} \|\Phi_k(x)\| \cdot \|f(t_k)\|_E \\
&= (S(\|f\|_E), x),
\end{aligned}$$

for all $x \in X$, the operator T is indeed dominated by S. If we assume that

$$\sum_{k\in J} \|\Phi_k(x)\| = 1, \quad x \in X, \tag{3}$$

holds true, then $Se_0 = e_0$, where $e_0(t) = 1$, for all $t \in X$. Now, if we also assume that

$$\sum_{k\in J} \Phi_k(x) = id_E, \quad x \in X, \tag{4}$$

holds true, then T preserves the constants. For example, let $\psi_k \in C(X; \mathbb{R})$ be a non-negative continuous function, and let

$$\langle \Phi_k(x), v\rangle = \psi_k(x)v$$

for every $v \in E$. Then $\Phi_k \in C(X; \mathcal{L}(E))$, and $\varphi_k(x) = \|\Phi_k(x)\| = \psi_k(x)$ and therefore

$$\sum_{k\in J} \psi_k(x) = 1, \quad x \in X, \tag{5}$$

implies both (3) and (4). Such operators are called of *interpolation type*. Clearly, the Bernstein operators are examples of such operators. Notice that, for each $x \in X$, where (X,d) is a compact metric space,

$$\begin{aligned}
\alpha_S(x) &= \sum_{k\in J} \varphi_k(x) \cdot \rho_x^2(t_k) \\
&= \sum_{k\in J} \varphi_k(x)[d(x,t_k)]^2.
\end{aligned}$$

When X a compact subset of some normed space G, then

$$\alpha_S(x) = \sum_{k\in J} \varphi_k(x) \cdot \|t_k - x\|_G^2, \quad x \in X.$$

If we have a sequence of such operators, then

$$\alpha_{S_n}(x) = \sum_{k\in J(n)} \varphi_{k,n}(x) \cdot \|t_{k,n} - x\|_G^2, \quad x \in X,$$

where, for each $k \in J(n), t_{k,n} \in X$ and $\varphi_{k,n} \in C(X; \mathbb{R})$ with $\varphi_{k,n} \geq 0$. By Theorem 1, $\alpha_{S_n}(x) \to 0$, uniformly on $x \in X$, implies $T_n f \to f$ for each $f \in C(X; E)$. When X is also convex, then we may apply Theorem 3 to obtain estimates on the rate of convergence of $||(T_n f, x) - f(x)||_E \to 0$ in terms of the modulus of continuity of f. For example, suppose X is the standard m-simplex

$$X = \{(x_1, \ldots, x_m) \in \mathbb{R}^m; \sum_{i=1}^{m} x_i \leq 1, \ \ x_i \geq 0, \ \ i = 1, \ldots, m\}.$$

The n^{th} Bernstein operator B_n on the simplex X is an operator of interpolation type defined as follows. Let $J(n)$ be the finite set of all m-tuples of non-negative integers $k = (k_1, \ldots, k_m)$ such that $k_1 + \ldots + k_m \leq n$. Now if $k \in J(n)$, the point

$$t_{k,n} = \frac{k}{n} = \left(\frac{k_1}{n}, \ldots, \frac{k_m}{n}\right)$$

belongs to X. The functions $\varphi_{k,n}$ are defined as follows

$$\varphi_{k,n}(x) = \binom{n}{k} x^k (1 - |x|)^{n-|k|}$$

where $x^k = x_1^{k_1} \cdot x_2^{k_2} \cdot \ldots \cdot x_m^{k_m}$ and

$$\binom{n}{k} = \frac{n!}{(k_1)!(k_2)! \ldots (k_m)!(n - |k|)!}$$

$$|k| = k_1 + k_2 + \ldots + k_m$$

$$|x| = x_1 + x_2 + \ldots + x_m.$$

Then (see Ditzian [5], p. 297),

(i) $\sum_{k \in J(n)} \varphi_{k,n}(x) = 1,$

(ii) $\sum_{k \in J(n)} ||\frac{k}{n} - x||^2 \varphi_{k,n}(x) = \frac{1}{n} \sum_{i=1}^{m} x_i (1 - x_i),$

the norm in (ii) being the Euclidean norm on $\mathbb{R}^m$. Hence, if we define

$$(T_n f, x) = \sum_{k \in J(n)} \varphi_{k,n}(x) f\left(\frac{k}{n}\right)$$

then, by Theorem 3, we get

$$||(T_n f, x) - f(x)||_E \leq \left[1 + \frac{1}{\delta^2} \cdot \frac{1}{n} \sum_{i=1}^{m} x_i (1 - x_i)\right] \omega(f; \delta)$$

and

$$\|(T_n f, x) - f(x)\|_E \leq \left[1 + \sum_{i=1}^{m} x_i(1 - x_i)\right]\omega(f; n^{-1/2}),$$

$$\|T_n f - f\| \leq \left(1 + \frac{m}{4}\right)\omega(f; n^{-1/2}).$$

To give further examples, let us consider the case of integral operators. Assume that E is a Banach space, and let μ be a positive Radon measure on the compact metric space (X, d) and let $K_n : X \times X \to \mathbb{R}$ be a positive continuous function such that

$$\int_X K_n(x,t)\mu(dt) = 1, \text{ for all } x \in X.$$

Define a positive linear operator S_n on $C(X; \mathbb{R})$ by setting

$$(S_n g, x) = \int_X K_n(x,t)g(t)\mu(dt).$$

Then S_n is a positive linear operator such that $S_n(e_0) = e_0$. By means of the Bochner integral, define a linear operator T_n on $C(X; E)$ by setting

$$(T_n f, x) = \int_X K_n(x,t)f(t)\mu(dt),$$

for every $f \in C(X; E)$. By the properties of the Bochner integral, we have

$$\begin{aligned} \|(T_n f, x)\|_E &\leq \int_X K_n(x,t)\|f(t)\|_E \cdot \mu(dt) \\ &= (S_n(\|f\|_E), x) \end{aligned}$$

for every $x \in X$. Hence (*) of Definition 2 is satisfied and T_n is dominated by S_n. In this case

$$\alpha_{S_n}(x) = \int_X K_n(x,t)[d(x,t)]^2\mu(dt), \quad x \in X.$$

§4. Quantitative estimates for differentiable functions

If Ω is an open subset of some normed space G, then $C^1(\Omega; E)$ denotes the set of all functions $f : \Omega \to E$ which are continuously differentiable on Ω. This means that for each $x \in \Omega$, the derivative of f at the point x, written $Df(x)$, exists and the mapping $x \in \Omega \mapsto Df(x)$ is continuous. Recall that $Df(x)$ is a continuous linear mapping of G into E, i.e. $Df(x) \in \mathcal{L}(G,E)$, and $\|Df(x)\|$ denotes its norm as an operator:

$$\|Df(x)\| = \sup_{\|v\|_G \le 1} \|\langle Df(x), v\rangle\|_E.$$

We shall need the Mean Value Theorem for such mappings. It states that

$$\|f(t) - f(x)\|_E \le \|t - x\|_G \cdot \sup\{\|Df(v)\|; v \in J\}$$

where J is the segment joining the two points x and t, i.e., $J = \{v \in G;\ v = \lambda t + (1-\lambda)x, 0 \le \lambda \le 1\}$. (See Theorem (8.5.4) of Dieudonné [4].)

Theorem 4. *Let X, S and T be as in Theorem 2. Let Ω be an open subset of G containing X and let $f \in C^1(\Omega; E)$. Then*

$$\|(Tf,x) - f(x)\|_E \le \|Df(x)\| \cdot \beta_S(x) + [\beta_S(x) + \frac{1}{\delta}\alpha_S(x)]\omega(Df,\delta),$$

for every $x \in X$ and $\delta > 0$.

Proof. Since $\|(Tf,x) - f(x)\|_E \le (S(\|f - f(x)\|_E), x)$, it suffices to estimate $(S(\|f - f(x)\|_E), x)$. Now, for each $t \in X$,

$$(1) \quad \|f(t) - f(x)\|_E \le \|\langle Df(x), t - x\rangle\|_E + \|f(t) - f(x) - \langle Df(x), t - x\rangle\|_E.$$

Consider the mapping $g(t) = \langle Df(x), t - x\rangle$, $t \in X$. We have $\|g(t)\|_E \le \|Df(x)\| \cdot \|t - x\|_G$, for all $t \in X$. The positivity of S implies

$$(2) \quad (S(\|g\|_E), x) \le \|Df(x)\| \cdot \beta_S(x).$$

Consider the mapping $h(t) = f(t) - f(x) - \langle Df(x), t - x\rangle$, $t \in X$. We have by the Mean Value Theorem

$$\|h(t)\|_E \le \|t - x\|_G \cdot \sup_{v \in J} \|Df(v) - Df(x)\|,$$

where $J \subset X$ is the segment joining the two vectors t and x in X. For every $v \in J$ we obtain

$$\begin{aligned} \|Df(v) - Df(x)\| &\le \omega(Df; \|v - x\|_G) \\ &\le \omega(Df; \|t - x\|_G) \le \left[1 + \frac{1}{\delta}\|t - x\|_G\right]\omega(Df;\delta), \end{aligned}$$

for any $\delta > 0$. Hence

$$||f(t) - f(x) - g(t)||_E \leq \left[||t - x||_G + \frac{1}{\delta}||t - x||_G^2\right]\omega(Df; \delta),$$

and therefore

$$(3) \quad (S(||f - f(x) - g||_E), x) \leq [\beta_S(x) + \frac{1}{\delta}\alpha_S(x)]\omega(Df; \delta).$$

From (1), (2) and (3) we get

$$(S(||f - f(x)||_E), x) \leq ||Df(x)|| \cdot \beta_S(x) + [\beta_S(x) + \frac{1}{\delta}\alpha_S(x)]\omega(Df; \delta).$$

□

Definition 4. Let S be a linear operator on $C(X; I\!R)$, and let T be a linear operator on $C(X; E)$. Let $\varphi \in E^*$ be a continuous linear functional. We say that T and S are **φ-commutative** if

$$\langle \varphi, (Tg, x)\rangle = (S(\varphi \circ g), x)$$

for every $x \in X$ and $g \in C(X; E)$. When T and S are φ-commutative for every $\varphi \in E^*$, then we say that T and S are **E^*-commutative**. When S is positive, this implies that T is dominated by S.

Theorem 5. *Let X, S and T be as in Theorem 2. Assume that*
(i) T and S are E^*-commutative,
(ii) $S\psi = \psi$, for every $\psi \in G^*$.
Let Ω be an open subset of G containing X and let $f \in C^1(\Omega; E)$. Then

$$||(Tf, x) - f(x)||_E \leq [\beta_S(x) + \frac{1}{\delta}\alpha_S(x)]\omega(Df; \delta)$$

for every $x \in X$ and every $\delta > 0$.

Proof. For every t and x of X we may write

$$f(t) - f(x) = \langle Df(x), t - x\rangle + [f(t) - t(x) - \langle Df(x), t - x\rangle].$$

Let $\varphi \in E^*, ||\varphi|| \leq 1$ be given. Consider the mapping $\psi = \varphi \circ [Df(x)]$. Then $\psi \in G^*$, since $Df(x) \in \mathcal{L}(G, E)$. Let $g = \psi - \psi(x) = \psi - \psi(x)e_0$, where $e_0(t) = 1$ for all $t \in X$. By (ii), and $S(e_0) = e_0$, we get

$$(Sg, x) = (S\psi, x) - \psi(x) = \psi(x) - \psi(x) = 0.$$

Let $h = \varphi \circ f \in C(X; \mathbb{R})$. Then

$$\begin{aligned}(Sh, x) - h(x) &= (S(h - h(x) - g), x) \\ &= (S(h - h(x) - \psi + \psi(x)), x).\end{aligned}$$

Now

$$\begin{aligned}|(Sh, x) - h(x)| &= |S(h - h(x) - \psi + \psi(x)), x)| \\ &\leq (S(|h - h(x) - \psi + \psi(x)|), x),\end{aligned}$$

since S is a positive linear operator. For every $t \in X$ we obtain

$$\begin{aligned}&|h(t) - h(x) - \psi(t) + \psi(x)| = |\varphi(f(t) - f(x) - \langle Df(x), t - x\rangle)| \\ &\leq ||f(t) - f(x) - \langle Df(x), t - x\rangle||_E \leq ||t - x||_G . \sup_{v \in j} ||Df(v) - Df(x)||,\end{aligned}$$

where $J \subset X$ is the segment joining the two vectors t and x in X. For every $v \in J$ we have

$$\begin{aligned}||Df(v) - Df(x)|| &\leq \omega(Df; ||v - x||_G) \leq \omega(Df; ||t - x||_G) \\ &\leq \left(1 + \frac{1}{\delta}||t - x||_G\right)\omega(Df; \delta)\end{aligned}$$

for any $\delta > 0$. Hence, for all $t \in X$ we can conclude

$$|h(t) - h(x) - \psi(t) + \psi(x)| \leq \left[||t - x||_G + \frac{1}{\delta}||t - x||_G^2\right]\omega(Df; \delta).$$

Therefore,

$$(S(|h - h(x) - \psi + \psi(x)|), x) \leq [\beta_S(x) + \frac{1}{\delta}.\alpha_S(x)]\omega(Df; \delta)$$

and so

$$|(Sh, x) - h(x)| \leq [\beta_S(x) + \frac{1}{\delta}\alpha_S(x)]\omega(Df; \delta).$$

Since $h = \varphi \circ f$, we have by (i),

$$\begin{aligned}(Sh, x) - h(x) &= (S(\varphi \circ f), x) - \varphi(f(x)) \\ &= \langle\varphi, (Tf, x) - f(x)\rangle\end{aligned}$$

and therefore

$$|\langle\varphi, (Tf, x) - f(x)\rangle| \leq [\beta_S(x) + \frac{1}{\delta}\alpha_S(x)]\omega(Df; \delta)$$

for all $\varphi \in E^*, ||\varphi|| \le 1$. By the Hahn-Banach Theorem, it follows that

$$||(Tf,x) - f(x)||_E \le [\beta_S(x) + \frac{1}{\delta}\alpha_S(x)]\omega(Df;;\delta).$$

Remark. Both the operators of interpolation type and the integral operators satisfy the hypothesis (i) of Theorem 5. Indeed, if $x \in X$ and

$$(Tf,x) = \sum_{k\in J} \varphi_k(x) f(t_k)$$

and $\varphi \in E^*$, then

$$\begin{aligned}\langle \varphi, (Tf,x)\rangle &= \sum_{k\in J} \varphi_k(x)\langle \varphi, f(t_k)\rangle \\ &= \sum_{k\in J} \varphi_k(x)(\varphi \circ f)(t_k) \\ &= (S(\varphi \circ f), x),\end{aligned}$$

for every $f \in C(X;E)$. On the other hand, if

$$(Tf,x) = \int_X K(x,t) f(t)\mu(dt)$$

then, by the properties of the Bochner integral

$$\begin{aligned}\langle \varphi, (Tf,x)\rangle &= \langle \varphi, \int_X K(x,t) f(t)\mu(dt)\rangle \\ &= \int_X K(x,t)\langle \varphi, f(t)\rangle \mu(dt) \\ &= (S(\varphi \circ f), x).\end{aligned}$$

Hence T and S are E^*-commutative.

Let us now consider hypothesis (ii) of Theorem 5. For the operators of interpolation type it means that

$$\sum_{k\in J} \varphi_k(x)\psi(t_k) = \psi(x)$$

for every $x \in X$ and $\psi \in G^*$. This is the case with the Bernstein operators on $C([0,1];\mathbb{R})$, since in this case each $\psi \in G^*$ is of the form

$$\psi(t) = at, t \in \mathbb{R}$$

for some $a \in I\!R$, and if we set $e_1(t) = t$ for every $t \in I\!R$, it is known that $B_n e_1 = e_1$ (see, e.g. Lorentz [6]). The same is true for the Bernstein operators on $C(X; I\!R)$, where X is the standard simplex on $I\!R^m$. Indeed, it is known that

$$B_n \pi_j = \pi_j \ \ (j = 1, \ldots, m),$$

where π_j is the j-th projection of $I\!R^m$ onto $I\!R$. (See Ditzian [5].) Now each $\psi \in (I\!R^m)^*$ is of the form

$$\psi(t) = \sum_{j=1}^{m} a_j t_j = \sum_{j=1}^{m} a_j \pi_j(t),$$

for all $t \in I\!R^m$. Hence $B_n \psi = \psi$ for all linear mappings $\psi : I\!R^m \longrightarrow I\!R$.

For the integral operators, hypothesis (ii) of Theorem 5 means that

$$\int_X K(x,t)\psi(t)\mu(dt) = \psi(x)$$

for all $x \in X$ and $\psi \in G^*$. For example, if $X \subset I\!R$, we must have

$$\int_X K(x,t)t\,dt = x$$

for all $x \in X$, or

$$\int_X K(x,t)(t-x)dt = 0. \quad \square$$

Theorem 6. *Let X be a compact and convex subset of some normed space G, and let $\{S_n\}_{n\geq 1}$ and $\{T_n\}_{n\geq 1}$ be as in Theorem 3. Assume that, for each $n \geq 1$,*
(i) T_n and S_n are E^*-commutative,
(ii) $S_n\psi = \psi$, for every $\psi \in G^*$,
(iii) $\alpha_{S_n}(x) \leq A(x)n^{-2\alpha}, \beta_{S_n}(x) \leq B(x)n^{-\alpha}$, holds for every $x \in X$ and $n \geq 1$, and for some functions $A, B \in C(X; I\!R)$ and for some $\alpha > 0$.
Then

$$||(T_n f, x) - f(x)||_E \leq [A(x) + B(x)]n^{-\alpha}\omega(Df; n^{-\alpha})$$

holds for every $x \in X$ and $n \geq 1$, and for every $f \in C^1(\Omega; E)$, where Ω is some open subset of G containing X.

Proof. Make $\delta = n^{-\alpha}$ in Theorem 5, applied to each pair S_n and $T_n, n \geq 1$. $\square$

Corollary 3. *For the Bernstein operators on $C([0,1]; E)$ we have*

$$||(B_n f, x) - f(x)||_E \leq [(x(1-x))^{1/2} + x(1-x)]n^{-1/2}\omega(Df; n^{-1/2})$$

for every $f \in C^1(\Omega; E)$, and $x \in [0,1]$, where Ω is some open subset of $\mathbb{R}$ containing $[0,1]$. Consequently,

$$\|B_n f - f\| \leq \frac{3}{4} n^{-1/2} \omega(Df, n^{-1/2}).$$

Proof. We know that $(\beta_{B_n}(x))^2 \leq \alpha_{B_n}(x)$, so $\beta_{B_n}(x) \leq (\alpha_{B_n}(x))^{1/2} = (x(1-x))^{1/2}.n^{-1/2}$. □

Corollary 4. *For the Bernstein operators on $C(X; E)$, where X is the standard m-simplex on $\mathbb{R}^m$, we have*

$$\|(B_n f, x) - f(x)\|_E \leq \left[\left(\sum_{i=1}^{m} x_i(1-x_i)\right)^{1/2} + \sum_{i=1}^{m} x_i(1-x_i)\right] n^{-1/2} \omega(Df; n^{-1/2})$$

for every $f \in C^1(\Omega; E)$ and $x \in X$, where Ω is some open subset of $\mathbb{R}^m$ containing X. Consequently,

$$\|B_n f - f\| \leq \left[\left(\frac{m}{4}\right)^{1/2} + \frac{m}{4}\right] n^{-1/2} \omega(Df, n^{-1/2}).$$

Proof. As in Corollary 3 we have

$$\beta_{B_n}(x) \leq (\alpha_{B_n}(x))^{1/2} \leq \left(\frac{1}{n} \sum_{i=1}^{m} x_i(1-x_i)\right)^{1/2} \quad \square$$

Remark. Notice that in the proof of Theorem 4 we get the estimate (2), which involves $\beta_S(x) = (S(\rho_x), x)$, where $\rho_x(t) = \|t - x\|_G$ for all $t \in X$. When $X \subset \mathbb{R}$, sometimes a better estimate can be obtained, since in this case $\rho_x(t) = |t - x|$ and if we denote by e_1 the identify function on $\mathbb{R}$, then $\rho_x(t) = |e_1(t-x)|$ and

$$|(Se_1, x) - x| \leq (S\rho_x, x) = \beta_S(x).$$

For example, if S is the n^{th} Hermite-Fejér operator H_n on $C([-1,1]; \mathbb{R})$, defined as

$$H_n(f) = \sum_{k=1}^{n} (1 - x t_{k,n}) \left[\frac{T_n(x)}{n(x - t_{k,n})}\right]^2 f(t_{k,n}),$$

where $t_{k,n} = \cos\left(\frac{2k-1}{2n}\pi\right), k = 1, 2, \ldots, n$, are the zeros of T_n, the Chebyshev polynomial of degree n

$$T_n(x) = \cos(n \; arc \; cos \; x)$$

for $x \in [-1, 1]$. We have, by De Vore [3, pg. 43],

$$(1) \qquad |(H_n e_1, x) - x| \leq 2n^{-1}|T_n(x)|$$

whereas

$$(2) \qquad \beta_{H_n}(x) = (H_n(\rho_x), x) \leq (\alpha_{H_n}(x))^{1/2} \leq n^{-1/2}.|T_n(x)|$$

and so (1) is a better estimate for $n \geq 4$.
Hence the better estimate for the sequence of Hermite-Fejér operators H_n on $C([-1,1]; E)$, is as follows:

$$||(H_n f, x) - f(x)||_E \leq 2n^{-1}|T_n(x)|.||Df(x)|| +$$
$$+ [|T_n(x)| + |T_n(x)|^2]n^{-1/2}.\omega(Df; n^{-1/2}). \quad \square$$

§5. Korovkin Systems

Definition 5. Let $\mathcal{A}$ be a class of linear operators on $C(X; E)$. A subset K of $C(X; E)$ is called a **Korovkin system for** $\mathcal{A}$ if, for every uniformly equicontinuous sequence $\{T_n\}_{n \geq 1}$ of linear operators $T_n \in \mathcal{A}$ the following holds:

$(*)$ $T_n g \to g$ for all $g \in K$, implies $T_n f \to f$ for all $f \in C(X; E)$.

When $E = \mathbb{R}$ and $\mathcal{A}$ is the class of all positive linear operators on $C(X; \mathbb{R})$ we obtain the usual definition of Korovkin systems in $C(X; \mathbb{R})$.

Definition 6. Let S be a linear operator on $C(X; \mathbb{R})$. A linear operator T on $C(X; E)$ is said to be S-**regular** if

$(**)$ $T(g \otimes v) = S(g) \otimes v$, for all $g \in C(X; \mathbb{R})$ and $v \in E$.

We say that

(a) T is **regular**, if it is S-regular for some linear operator S on $C(X; \mathbb{R})$;

(b) T is **monotonically regular**, if it is S-regular for some positive linear operator S on $C(X;\mathbb{R})$.

Remark. If T is **monotonically regular**, then it is S-regular for some positive linear operator S, and T and S are φ-commutative for each $\varphi \in E^*$. Hence T is dominated by S.

For example, all operators of interpolation type and all Bochner integral operators defined in §3 are monotonically regular.

Theorem 7. *Let $K \subset C(X;E)$ be a non-empty subset such that for some continuous linear functional $\varphi \in E^*$, the set $K(\varphi) = \{\varphi \circ g; g \in K\}$ is a Korovkin system for positive linear operators on $C(X;\mathbb{R})$. Then K is a Korovkin system for linear operators on $C(X;E)$ which are monotonically regular .*

Proof. Let $\{T_n\}_{n\geq 1}$ be a uniformly equicontinuous sequence of monotonically regular linear operators on $C(X;E)$. By Definition 6, for each $n \geq 1$, there is a positive linear operator S_n on $C(X;\mathbb{R})$ such that $T_n(h \otimes u) = S_n(h) \otimes u$, for all $h \in C(X;\mathbb{R})$ and $u \in E$, and moreover $\langle \varphi, (T_n f, x)\rangle = (S_n(\varphi \circ f), x)$, for every $x \in X$ and $f \in C(X;E)$. It is easy to see that $\{S_n\}_{n\geq 1}$ is then a uniformly equicontinuous sequence. Assume that $T_n g \to g$ for every $g \in K$. Let $h = \varphi \circ g$, for $g \in K$. Then $(S_n(h), x) = (S_n(\varphi \circ g), x) = \langle \varphi, (T_n g, x)\rangle$ for every $x \in X$. Now $(T_n g, x) \to g(x)$, uniformly on $x \in X$. Hence $(S_n(h), x) \to h(x)$, uniformly on $x \in X$. Since $K(\varphi)$ is a Korovkin system for positive linear operators on $C(X;\mathbb{R})$, we conclude that $S_n h \to h$, for all $h \in C(X;\mathbb{R})$. Let $f \in C(X;E)$ and $\varepsilon > 0$ be given. By uniform equicontinuity of $\{T_n\}_{n\geq 1}$, there exists $\delta > 0$, which we may assume to satisfy $\delta < \varepsilon/3$, such that $||f_1 - f_2|| < \delta$ implies $||T_n f_1 - T_n f_2|| < \varepsilon/3$, for $f_1, f_2 \in C(X;E)$. Since $C(X;\mathbb{R}) \otimes E$ is uniformly dense in $C(X;E)$, there exists $h \in C(X;\mathbb{R}) \otimes E$ such that $||f - h|| < \delta$. Suppose h is of the form

$$h = \sum_{i=1}^{m} g_i \otimes v_i$$

where $g_i \in C(X;\mathbb{R})$ and $v_i \in E, i = 1,2,\ldots,m$. Since $S_n(g_i) \to g_i$ for each $i = 1,2,\ldots,m$, it follows that

$$T_n(h) = \sum_{i=1}^{m} S_n(g_i) \otimes v_i \to \sum_{i=1}^{m} g_i \otimes v_i = h.$$

Hence, for some n_0 we have $||T_n(h) - h|| < \varepsilon/3$, for all $n \geq n_0$. Notice that $||f - h|| < \delta$ implies $||T_n(f) - T_n(h)|| < \varepsilon/3$ for all n. Therefore

$$||T_n(f) - f|| \leq ||T_n(f) - T_n(h)|| + ||T_n(h) - h|| + ||h - f|| < \varepsilon$$

for all $n \geq n_0$. This ends the proof that $T_n f \to f$, for all $f \in C(X;E)$ and so K is a Korovkin system for the class of all monotonically regular linear operators on $C(X;E)$. $\square$

As a corollary we obtain the following generalization of a result of M. Weba. (See Theorem 3.2, [10])

Theorem 8. *Let $K \subset C(X;E)$ be a subset containing a constant function $v_0 \in E, v_o \neq 0$. Let $\varphi \in E^*$ be such that $\varphi(v_0) \doteq 1$. Assume that, for each $t_0 \in X$ there exists g in the linear span of K such that $\varphi \circ g \geq 0$ and $\varphi(g(t)) = 0$ if, and only if, $t = t_0$. Then K is a Korovkin system for linear operators on $C(X;E)$ which are monotonically regular.*

Proof. Let $K(\varphi) = \{\varphi \circ g; g \in K\}$. Let H be the linear span of $K(\varphi)$ in $C(X;\mathbb{R})$. Then H is point separating and contains the constant function 1. The hypothesis made implies that each $t_0 \in X$ belongs to the Choquet boundary $\partial_H X$. Hence $X = \partial_H X$ and so $K(\varphi)$ is a Korovkin system in $C(X;\mathbb{R})$ for positive linear operators. (See [1] or [2].) It remains to apply Theorem 7. $\square$

Theorem 9. *Let $K \subset C(X;\mathbb{R})$ be a Korovkin system for positive linear operators on $C(X;\mathbb{R})$ and let $v_0 \in E, v_0 \neq 0$. Then $\{g \otimes v_0; g \in K\}$ is a Korovkin system for the class of all monotonically regular linear operators on $C(X;E)$.*

Proof. Choose $\varphi \in E^*$ such that $\varphi(v_0) = 1$. Then $\{\varphi \circ (g \otimes v_0); g \in K\} = K$, and therefore we may apply Theorem 7 to conclude that $\{g \otimes v_0; g \in K\}$ is a Korovkin system for the class of all monotonically regular linear operators on $C(X;E)$. $\square$

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