TWO RESULTS ON MAXIMAL SUBSEMIGROUPS OF LIE GROUPS

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Abstract: It is proved here that the subsemigroup of positive matrices in $S\ell(n,\mathbb{R})$ is maximal connected. Also, let g be a simple non-compact Lie algebra and $\mathsf{k} \subset \mathsf{g}$ a maximal compactly embedded subalgebra. If G is a connected Lie group with Lie algebra g, and K the connected subgroup whose Lie subalgebra is k, then any coset $Kg, g \notin K$, generates G as a semigroup. Therefore K is maximal as a subsemigroup of G. This result was already obtained by Neeb [3], through different methods.

1. Introduction

The two results presented in this paper are about subsemigroups of Lie groups, more specifically of semi-simple Lie groups. Both of them were proved in [2] for $S\ell(2,\mathbb{R})$, the special linear group in dimension 2. In Theorem 1 below we show that the subsemigroup $S\ell(n)^+$ of positive matrices in $S\ell(n,\mathbb{R})$ is essentially maximal: It is maximal connected, and the only proper subsemigroup containing it is $S\ell(n)^{\pm} = S\ell(n)^+ \cup JS\ell(n)^+$, where J is some " negative" matrix in $S\ell(n,\mathbb{R})$. This result extends Proposition V.4.30 in [2].

For the second one, let g be a simple noncompact Lie algebra and k a maximal compactly embedded subalgebra. Let G be a connected Lie group whose Lie algebra is g and denote by K the connected subgroup of G whose Lie algebra is k. In Theorem 3 bellow we show that any coset $Kg, g \notin K$ generates G as a semigroup. Of course, the same result holds for cosets of the type $gK, g \notin K$. An immediate consequence of this is that K is maximal as a subsemigroup of G , that is, any subsemigroup of G containing K properly is G itself. From this maximality of K when G is simple one gets quickly the subsemigroups of G containing K when G is only semi-simple (see Corollary

1). This result was already obtained by Neeb [3], through different methods then ours. In [2], Theorem V.4.41 a proof is given for the case when $\mathbf{g}$ is $s\ell(2, \mathbb{R})$, the three dimensional simple Lie algebra.

The technique for proving the above mentioned theorems comes from the results in [4], concerning invariant control sets for subsemigroup actions on homogeneous spaces. We record the relevant facts in section 2 bellow. The main one is in Proposition 2b, where a criteria is given in order that a subsemigroup of a semi-simple Lie group is the group itself. Based on this, the essential of the proofs stays on checking that certain subsemigroups satisfy the given conditions . This method leads, in the $S\ell(2, \mathbb{R})$ case, to alternative proofs for those offered in [2].

2. Background

For latter use we record here some facts about transitivity and invariant control sets for semigroup actions. We restrict attention here to semigroups in Lie groups. Thus let G be a Lie group and S a subsemigroup of G. Make S act on $M = G/H, H$ a closed subgroup, as a semigroup of diffeomorphisms.

An invariant control set for S on M (abbreviated S-i.c.s.) is a subset $C \subset M$ which satisfies

i) $c\ell Sx = c\ell C$ for all $x \in C$,

ii) C is maximal with property i).

Here $c\ell$ means closure whereas $Sx = \{gx : g \in S\}$ stands for the orbit of x under S.

It is not hard to show (see e.g. [1], Lemma 3.1) the existence of invariant control sets if M is compact, which we assume from now on. Also, it is easy to see that ii) is a consequence of i) if C is known to be closed.

We say that S acts transitively on M provided $Sx = M$ for every $x \in M$.

Our main concern, in what follows, with the invariant control sets comes from the information which can be obtained regarding the transitivity of S. Such information is easier to get when S has non empty interior in G (which is the only case we consider). The reason is that when $\text{int}S \neq \emptyset$, both Sx

and $S^{-1}x$ have non empty interior in M, for every $x \in M$, so the approximate transitivity stated in property i) above can be turned into transitivity.

For S with int $S \neq \emptyset$, it is possible to prove the following facts:

1) Any S-i.c.s. is closed and has non empty interior,

2) S has just one invariant control set iff $C = \cap_{x \in M} c\ell(Sx) \neq \emptyset$. In this case C is the invariant control set.

As a consequence we have: Let $S_i, i = 1,2$, be subsemigroups with non void interior and suppose that $C(S_i), i = 1,2$, is the only S_i-i.c.s. on M. Then $S_1 \subset S_2 \Rightarrow C(S_1) \subset C(S_2)$

3) Let C be a S-i.c.s. on M, and set

$$C_0 = \{x \in C : \exists g \in int\, S \quad \text{with} \quad g^{-1}x \in C\}. \qquad (*)$$

Then C_0 is an open and dense subset of C. Moreover C_0 is S-invariant (i.e. $Sx \subset C_0$ if $x \in C_0$) and S is transitive inside C_0 (i.e. for every $x, y \in C_0$, there exists $g \in S$ with $gx = y$). Also, $C_0 = int\, C$ in case $1 \in c\ell\, int\, S$ (see [4] Proposition 2.1).

A consequence of this fact - which is not hard to show directly - is that S is transitive if M itself is an invariant control set for S. In fact, in this case, $C_0 = M$, as follows readily from $(*)$.

Regarding the transitivity of S on M, the following statement gives a sufficient condition in terms of invariant control sets

Proposition 1: Suppose G is connected and let $S \subset G$ be a subsemigroup with non empty interior. Let C be a S-i.c.s. on M. Let also C' be an invariant control set on M for $S^{-1} = \{g : g^{-1} \in S\}$. Suppose that int$C \cap$int$C' \neq \emptyset$. Then $C = C' = M$.

Proof: Let C_0 (resp. C'_0) be the subset of C (resp. C') that is invariant by S (resp. S^{-1}) as in $(*)$ above. By assumption, $C_0 \cap C'_0 \neq \emptyset$.

Take $x \in C_0 \cap C'_0$, and $y \in C'_0$. Then there exists $g \in S$ such that $g^{-1}y = x$, i.e., $gx = y$. This shows that $y \in C_0$ and hence that $C'_0 \subset C_0$. Using the same argument we get the reverse inclusion and thus that $C_0 = C'_0$.

We have then that C_0 is invariant by both S and S^{-1} and hence by the group generated by S, whi ch is G because G is connected and $\text{int}S \neq \emptyset$. Since G is transitive on M, we have the result. $\square$

The results to follow are specifc for subsemigroups of non-compact semi-simple Lie groups and their action on the Furstenberg boundaries of the group. They were proved in [4].

Proposition 2: a) Let G be a connected, semi-simple and non-compact Lie group, and supose that P is a parabolic subgroup. Let $M = G/P$ be the corresponding Furstenberg boundary. Then any subsemigroup $S \subset G$ with non empty interior has one and only one invariant control set on M; denoted by $C(S, G/P)$ or just $C(S)$.

b) Suppose P_{min} is a minimal parabolic subgroup and let $B = G/P_{min}$ be the maximal boundary . Suppose moreover that G has finite centre. Then G itself is the only of its subsemigroups with non void interior which is transitive on B. $\square$

Finally, we have the following statement about transitivity of semigroups on fibre bundles.

Proposition 3: Let $\pi : M_1 \to M_2$ be a fibre bundle with G acting on M_1 and M_2 transitively and equivariantly (i.e. $\pi \circ g = g \circ \pi$). Let $S \subset G$ be a subsemigroup and suppose that

a) S is transitive on M_2, and

b) for some $x_0 \in M_2$, S is transitive on the fibre $(M_1)_{x_0} = \pi^{-1}\{x_0\}$. This means that for all $p, q \in (M_1)_{x_0}$ there exists $g \in S$ with $gp = q$.

Then S is transitive on M_1.

Proof: Take $y \in M_2$ and $p, q \in (M_2)_y$. Choose $g_1, g_2, g_3 \in S$ with

$$g_1 y = x_0, \quad g_3 x_0 = y \quad \text{and} \quad g_2 g_1 p = g_3^{-1} q.$$

The existence of g_1 and g_3 follows from the transitivity of S on M_2. The choice of g_2 is possible because S is transitive on $(M_1)_{x_0}$ and $g_1 p, g_3^{-1} q \in (M_1)_{x_0}$. We have

$$g_3g_2g_1p = q,$$

showing that S is transitive on any fibre $(M_1)_y$. Take now $p_1, p_2 \in M_1$ and put $x_i = \pi(p_i), i = 1, 2$. There exists $h_1 \in S$ with $h_1x_1 = x_2$, because of the transitivity of S on M_2, and there exists $h_2 \in S$ with $h_2h_1p_1 = p_2$, because S is transitive on the fibre $(M_1)_{x_2}$. This shows the claim. □

3. Maximality of $S\ell(n)^+$

In order to state precisely what we mean by $S\ell(n)^+$ being maximal, we let

$$O^+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0\}$$

be the positive orthant in $\mathbb{R}^n$, and put $O^- = -O^+$ as for the negative one. By definition $S\ell(n)^+$ is the subsemigroup of those elements of $S\ell(n, \mathbb{R})$ which preserves O^+

$$S\ell(n)^+ = \{g \in S\ell(n, \mathbb{R}) : gO^+ \subset O^+\}.$$

We denote by $S\ell(n)^\pm$ the subsemigroup of $S\ell(n, \mathbb{R})$ which preserves $O^+ \cup O^-$. It decomposes into two connected componnents as

$$S\ell(n)^\pm = S\ell(n)^+ \cup JS\ell(n)^+$$

where J is any negative matrix in $S\ell(n, \mathbb{R})$, that is to say $J(O^+) \subset O^-$ and $J(O^-) \subset O^+$. If n is even, one can take $J = -1$ and in case n is odd, one can take for instance J with diagonal bloks

$$J = \begin{pmatrix} A & \\ & B \end{pmatrix}$$

with sizes 2 and $n - 2$ respectively, and with

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

and $B = -1$. To see this note that for $g \in S\ell(n)^\pm$ one has either $g(O^\pm) \subset O^\pm$ or $g(O^\pm) \subset O^\mp$, because O^+ and O^- are the connected components of $O^+ \cup O^-$.

In the first case $g \in S\ell(n)^+$ and in the second one $Jg \in S\ell(n)^+$.

Alternatively $S\ell(n)^{\pm}$ can be defined as the subsemigroup of $S\ell(n, \mathbb{R})$, which in its action on the projective space $\mathbb{R}P^{n-1}$, preserves the subset $\mathbb{P}O^+$, which corresponds to the positive orthant through the canonical map $\mathbb{R}^n - \{0\} \to \mathbb{R}P^{n-1}$.

Theorem 1: Let S be a subsemigroup of $S\ell(n, \mathbb{R})$ containing $S\ell(n)^+$ properly. Then $S = S\ell(n)^{\pm}$ or $S\ell(n, \mathbb{R})$. Therefore $S\ell(n)^+$ is a maximal connected subsemigroup.

This result was proved for $n = 2$ in [2] (see Proposition V. 4.30). The proof offered there is specific for $S\ell(2, \mathbb{R})$. Before going on into the general case we present here an alternative proof for $n = 2$, which is a guide for the general one.

The maximal Furstenberg boundary of $S\ell(2, \mathbb{R})$ is the projective space $\mathbb{R}P^1$, with canonical action. Taking into account Proposition 2b, in order to get the theorem (for $n = 2$) it is enough to show that S is transitive on $\mathbb{R}P^1$ provided S contains $S\ell(2)^+$ properly and S does not leave invariant $\mathbb{P}O^+$. To see this, note that the invariant control sets on $\mathbb{R}P^1$ for $S\ell(2)^+$ and $\Gamma = \{g^{-1} : g \in S\ell(2)^+\}$ are respectively $\mathbb{P}O^+$ and the subset of $\mathbb{R}P^1$ corresponding to $\{(x_1, x_2) \in \mathbb{R}^2 - \{0\} : x_1 x_2 \leq 0\}$. On the other hand, let $C(S)$ be the S-i.c.s. on $\mathbb{R}P^1$. Then $\mathbb{P}O^+ \subset C(S)$ because $S\ell(2)^+ \subset S$, and $C(S)$ is not contained in $\mathbb{P}O^+$ if S is not contained in $S\ell(2)^{\pm}$. Since $\Gamma \subset S^{-1}$, we see that $int C(S) \cap int\, C(S^{-1}) \neq \emptyset$. Therefore, by Proposition 1, S is transitive on $\mathbb{R}P^1$, and the result follows.

The proof for arbitrary n is similar. The maximal Furstenberg boundary of $S\ell(n, \mathbb{R})$ is the flag manifold $B = \mathbb{F}^n(1, \ldots, n-1)$, so Theorem 2 follows by showing that S is transitive on B provided S contains $S\ell(n)^+$ properly and is not contained in $S\ell(n)^{\pm}$.

In order to show this transitivity of S we describe first the invariant control sets on $\mathbb{R}P^{n-1}$ and on $\mathbb{F}^n(1, \ldots, n-1)$ for $S\ell(n)^+$ and for its inverse semigroup

$$\Gamma = \{g^{-1} : g \in S\ell(n)^+\}.$$

We have that $C(S\ell(n)^+, \mathbb{R}P^{n-1}) = \mathbb{P}O^+$, because $\mathbb{P}O^+$ is invariant by $S\ell(n)^+$, and $S\ell(n)^+$ contains the subgroup of diagonal matrices which is transitive on the interior of $\mathbb{P}O^+$. The other invariant control sets we need are given by the following lemmas.

Lemma 1: Put $B = \mathbb{F}^n(1, \ldots, n-1)$ and let $\pi : B \to \mathbb{R}P^{n-1}$ be the canonical projection. Then

$$C(S\ell(n)^+, B) = \pi^{-1}(\mathbb{P}O^+)$$

Lemma 2: $C(\Gamma, \mathbb{R}P^{n-1})$ is the complement of the interior of $\mathbb{P}O^+$.

From these two lemmas the proof of Theorem 1 follows almost the same way as in the 2-dimensional case: Let S be a subsemigroup containing $S\ell(n)^+$ properly and not contained in $S\ell(n)^\pm$. Then $\mathbb{P}O^+ \subset C(S, \mathbb{R}P^{n-1})$ because $S\ell(n)^+ \subset S$, and $C(S, \mathbb{R}P^{n-1})$ is not contained in $\mathbb{P}O^+$ because otherwise we would have $S \subset S\ell(n)^\pm$. Now, let $x \in C(S, \mathbb{R}P^{n-1}) - \mathbb{P}O^+$. By Lemma 2 we have that $x \in intC(\Gamma, \mathbb{R}P^{n-1})$. Since the interior of $S\ell(n)^+$ meets every neighborhood of the identity in $S\ell(n, \mathbb{R})$, and

$$int(S\ell(n)^+ x) \subset int(Sx) \subset intC(S, \mathbb{R}P^{n-1}),$$

we conclude that

$$intC(S, \mathbb{R}P^{n-1}) \cap intC(S^{-1}, \mathbb{R}P^{n-1}) \supset intC(S, \mathbb{R}P^{n-1}) \cap intC(\Gamma, \mathbb{R}P^{n-1}) \neq \emptyset,$$

Hence by Proposition 1, we see that S is transitive on $\mathbb{R}P^{n-1}$.

We apply now Proposition 3 with the fibration $\mathbb{F}^n(1, \ldots, n-1) \to \mathbb{R}P^{n-1}$. By Lemma 1, $S\ell(n)^+$ is transitive on at least one fibre of this fibre bundle (actually on every fibre over the interior points of $\mathbb{P}O^+$). Of course, the same statement holds for S. Since S is transitive on $\mathbb{R}P^n$ we conclude that S is transitive on the maximal flag manifold $\mathbb{F}^n(1, \ldots, n-1)$. Therefore $S = Sl(n, \mathbb{R})$. □

Lemma 1 was proved in [4] (see example 4.5).

proof of Lemma 2: For $v \in \mathbb{R}^n - \{0\}$ we denote by $[v]$ its class in $\mathbb{R}P^{n-1}$. Let $\{e_1, \ldots, e_n\}$ be the standard basis in $\mathbb{R}^n$. We start by noting that $[e_i] \in C(\Gamma, \mathbb{R}P^{n-1}), i = 1, \ldots, n$. In fact, taking e.g. $i = 1$, let $H = \text{diag}\,(\lambda_1, \ldots, \lambda_n)$

with $\lambda_1 + \cdots + \lambda_n = 0$ and $\lambda_1 > \cdots > \lambda_n$. Then $\exp tH \in Sl(n)^+ \cap \Gamma$, all $t \in I\!R$. Also, in projective space we have

$$\lim_{t\to\infty} (\exp tH)x = [e_1]$$

for x in an open and dense subset of $I\!RP^{n-1}$. Since int $\Gamma \neq \phi$, this shows that $[e_1] \in cl\Gamma x$, every $x \in I\!RP^{n-1}$, and therefore that $[e_1] \in C(\Gamma)$. Applying a similar argument to the other basic elements we see that $[e_i] \in C(\Gamma), i = 1, \ldots, n$.

Now, let A be the group of diagonal matrices in $Sl(n, I\!R)$. It has 2^{n-1} open orbits in $I\!RP^{n-1}$. These are the sets corresponding to the interior of the orthants and their union is dense in $I\!RP^{n-1}$. Since $A \subset \Gamma, C(\Gamma)$ is a union of orthants. So in order to prove the Lemma it is enough to show that $\Gamma[e_1]$ meets the interior of every orthant other than the positive one.

In order to check this we put, for $x \in I\!R^{n-1}$,

$$v_x = \begin{pmatrix} 1 \\ x \end{pmatrix} \in I\!R^n.$$

The set $\{[v_x] \in I\!RP^{n-1} : x \in I\!R^{n-1}\}$ is open dense in $I\!RP^{n-1}$, so it meets every orthant in $I\!RP^{n-1}$. We must show that for any orthant O in $I\!R^{n-1}$, aside from the positive one, there are $g \in \Gamma$ and $x \in O$ such that $ge_1 = v_x$. This is done by induction on the number $i = 0, \ldots, n-2$ of positive entries of the elements of O.

For $i = 0$, take $x \in I\!R^{n-1}$ with all entries strictly negative. Then

$$g = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \in \Gamma$$

because

$$g^{-1} = \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} \in Sl(n)^+,$$

and we have $ge_1 = v_x$.

To see that the induction goes on, take $x \in I\!R^{n-1}$ with non zero entries such that at least two of then, say x_r and x_s, are strictly negative. Assume that there exists $g \in \Gamma$ with $ge_1 = v_x$.

Now, for $k, l = 1, \dots, n$ let E_{kl} stand for the $n \times n$ matrix whose (k,l)–entry is 1 and the other ones are zero. We have

$$(1 + aE_{kl})^{-1} = 1 - aE_{kl}, k \neq l.$$

Therefore $1 + aE_{kl} \in \Gamma$ if $a \leq 0, k \neq l$. Taking x as above, we have

$$(1 + aE_{r+1,s+1})v_x = v_y$$

where the entries of y are the same as those of x apart from the $r-th$ one which is $x_r + ax_s$. Since $x_r, x_s < 0$, there exists $a < 0$ such that $x_r + ax_s > 0$, showing that the induction procedure works. This proves the Lemma. □

Remark: Contrary to the two dimensional case, the invariant control set for Γ on $\mathbb{F}^n(1, \dots, n-1), n \geq 3$, does not complement the invariant control set for $Sl(n)^+$. To see this, let as before, $\pi : \mathbb{F}^n(1, \dots, n-1) \to \mathbb{R}P^{n-1}$ be the canonical projection. For $x \in C(\Gamma, \mathbb{R}P^{n-1}), \pi^{-1}\{x\}$ is the set of flags $V_2 \subset \cdots \subset V_{n-1}$ with $\dim V_i = i$ and $x \subset V_2$. If V_2 does not meet the positive orthant, the same happens with $gV_2, g \in \Gamma$, because the complement of the positive orthant is invariant by Γ (as shown by Lemma 2). Therefore $\pi^{-1}(C(\Gamma, \mathbb{R}P^{n-1}))$ contains a proper Γ-invariant subset, so $C(\Gamma, \mathbb{F}^n(1, \dots, n-1))$ does not complement the invariant control set for $Sl(n)^+$ which is $\pi^{-1}(\mathbb{P}O^+)$. Despite this, the invariant control sets for $Sl(n)^+$ and Γ are complementary to each other on $\mathbb{R}P^{n-1}$, and the proof of Theorem 2 is achieved with the aid of the fibration technique stated in Proposition 3.

4. Maximality K

Let G be a connected, noncompact, semi-simple Lie group and $\mathfrak{g}$ its Lie algebra. Let also $\mathfrak{k}$ be a maximal compactly embedded subalgebra of $\mathfrak{g}$ and denote by K the connected subgroup of G with Lie algebra $\mathfrak{k}$. The following result and its corollary extends to arbitrary semi-simple Lie groups Theorem V.4.41 of [2].

Theorem 2: Suppose $\mathfrak{g}$ is simple and let S be a subsemigroup of G containg K.

Then either $S = K$ or $S = G$.

Corollary 1: Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ be the decomposition of semi-simple $\mathfrak{g}$ into simple ideals. Let $\mathfrak{k}_i \subset \mathfrak{g}_i$ be maximal compact in $\mathfrak{g}_i$ such that $\mathfrak{k} = \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_s$, and denote by K_i the connected subgroup with Lie algebra $\mathfrak{k}_i$. Let S be a semigroup containing K.

We have,

a) If G is simply connected then S is of the form

$$S = A_1 \dots A_s$$

with $A_i = K_i$ or G_i, where G_i is the subgroup corresponding to $\mathfrak{g}_i$.

b) In general S is a normal subgroup with Lie algebra $\mathfrak{a}_i \oplus \cdots \oplus \mathfrak{a}_s$ with $\mathfrak{a}_i = \mathfrak{k}_i$ or $\mathfrak{g}_i$.

Part a) of this Corollary is immediate from the Theorem, whereas part b) follows from a).

We prove next Theorem 2. Actually the technique for proving it shows the following slightly more general result.

Theorem 3: Suppose G is simple and let $g \notin K$. Then the coset Kg generates G as a semigroup. Similarly for the coset gK.

Of course, any semigroup which contains K properly must constain a coset $Kg, g \notin K$, so Theorem 2 follows from Theorem 3.

In order to prove Theorem 3, we start by reducing it to the case where G has finite centre.

Denote by $Z(G)$ the centre of G, and let S be the semigroup generated by $Kg, g \notin K$. We have that $S/Z(G)$ is a semigroup in $G/Z(G)$ which contains the coset $(K/Z(G))g', g' = gZ(G) \notin K/Z(G)$. Assuming the result for G with finite centre, we have that $S/Z(G) = G/Z(G)$. Now, $gZ(G) = Z(G)g \subset S$ because $Z(G) \subset K$. Also, $g^{-1}Z(G) \cap S \neq \phi$ because $S/Z(G) = G/Z(G)$. Let $m_0 \in Z(G)$ be such that $g^{-1}m_0 \in S$. Pick $m \in Z(G)$. Then

$$m = (g^{-1}m_0)(gm_0^{-1}m) \in S,$$

so $Z(G) \subset S$. This together with $S/Z(G) = G/Z(G)$ shows that $S = G$ (e.g. by applying Proposition 3 to the fibration $G \to G/Z(G)$ which is equivariant with respect to the left action of G).

This been so, we assume from now on that G is a simple Lie group with *finite centre*, and let S be the semigroup generated by $Kg, g \notin K$.

Let $B = G/P$ be the maximal Furstenberg boundary of G and take $b_1, b_2 \in B$. Then there exists $u \in K$ such that $u(gb_1) = b_2$, because K is transitive on B. It follows that S is transitive on B.

By virtrue of Proposition 2b, Theorem 3 will be proved as soon as it is checked that S has non empty interior in G.

We show next that $(Kg)^n$ has non empty interior in G, for some integer n. This will follow from the implicit function theorem after noting that $(Kg)^n$ is the image of $Kg \times \cdots \times Kg$ under the product map

$$\begin{array}{rcl} p_n : G^n & \longrightarrow & G. \\ (g_1, \ldots, g_n) & \longrightarrow & g_1 \cdots g_n \end{array}$$

The subset $Kg \times \cdots \times Kg$ is a submanifold of G^n, and the restriction q_n of p_n to it defines a differentiable map $q_n : Kg \times \cdots \times Kg \to G$. In order to show that its image has non empty interior, it is enough to check that its differential has full rank at some point. We have

Lemma 3: The image of the differential of q_n at $(s_1, \ldots, s_n) \in Kg \times \cdots \times Kg$ is the subspace

$$R_s(\mathsf{k} + Ad(s_1)(\mathsf{k}) + \cdots + Ad(s_1 \ldots s_n)(\mathsf{k})),$$

where $s = s_1 \ldots s_n$. Here R_s stands for the right action as well as its differential and Ad denotes the adjoint representation of G in g.

Proof: The image of the $i - th$ partial derivative of q_n at $(s_1, \ldots, s_n)$ is

$$L_{s_1\dots s_{i-1}} \circ R_{s_i\dots s_n}(\mathsf{k})$$

where L stands for the left action and its differential. Writing $L_h = R_h \circ Ad(h)$ and taking into account the commutativity of the right and left actions, the above subspace is rewritten as

$$R_s Ad(s_1 \dots s_{i-1})(\mathsf{k}).$$

Adding up on i we get the Lemma. □

The assumption in Theorem 3 that G is simple is required for the proof of the next statement.

Lemma 4: Suppose $g \notin K$. Then there exists are integer m and and m-tuple $(t_1, \dots, t_m) \in Kg \times \cdots \times Kg$ such that

$$\mathsf{g} = \mathsf{k} + Ad(t_1)\mathsf{k} + \cdots + Ad(t_1 \dots t_m)\mathsf{k}.$$

Proof: For $(s_1, \dots, s_n) \in Kg \times \cdots \times Kg$ put

$$V(s_1, \dots, s_n) = \mathsf{k} + Ad(s_1)\mathsf{k} + \cdots + Ad(s_1 \dots s_n)\mathsf{k}.$$

Choose $(t_1, \dots, t_m) \in Kg \times \cdots \times Kg$ such that $V(t_1, \dots, t_m)$ has maximal dimension between the subspaces of the type $V(s_1, \dots, s_n)$, arbitrary n. We have

$$V(s, t_1, \dots, t_m) = \mathsf{k} + Ad(s)V(t_1, \dots, t_m)$$

for $s \in Kg$. Since $V(t_1, \dots, t_m)$ is of maximal dimension, $\mathsf{k} \subset Ad(s)V(t_1, \dots, t_m)$ or equivalently

$$Ad(s^{-1})\mathsf{k} \subset V(t_1, \dots, t_m).$$

Repeating this argument successively, with $V(s_1, \dots, s_r, t_1, \dots, t_m)$ instead of $V(s, t_1, \dots, t_m)$, we get that

$$Ad(s_r^{-1} \dots s_1^{-1})\mathsf{k} \subset V(t_1, \dots, t_m),$$

for arbitrary $s_1, \dots, s_r \in Kg$. This shows that the subspace, say W, spanned by k and $\{Ad(s^{-1})\mathsf{k} : s \in (Kg)^n\}$ is contained in $V(t_1, \dots, t_m)$.
Clearly, $Ad(g^{-1})W \subset W$, or equivalently, by the finite dimensionality

$$Ad(g)W = W.$$

Similarly, for $u \in K$, $Ad(g^{-1}u^{-1})W = W$. It follows that

$$Ad(u)W = W, u \in K.$$

This shows that $\mathsf{g} = W$. In fact, let $\mathsf{g} = \mathsf{k} \oplus \mathsf{s}$ be the Cartan decomposition of g having k as subalgebra. We have

$$W = \mathsf{k} \oplus W \cap \mathsf{s}$$

because $\mathsf{k} \subset W$. Since $g \notin K$ and K is the normalizer of k in $G, W \cap \mathsf{s} \neq \{0\}$. Therefore $W \cap \mathsf{s} = \mathsf{s}$ because $W \cap \mathsf{s}$ is $Ad(K)$-invariant and the adjoint representation of K on s is irreducible, due to the assumed simplicity of g. We conclude that $\mathsf{g} \subset V(t_1, \dots, t_m)$, which shows the Lemma. □

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