STRICTLY ABSOLUTELY SUMMING MULTILINEAR MAPPINGS

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Abstract. The space of the strictly absolutely $(s;r_1,\ldots,r_n)$-summing multilinear mappings between Banach spaces is introduced along with a natural (quasi-) norm on it. For Hilbert spaces and $s=r_1=\ldots=r_n\in[2,+\infty)$ it is shown that such space is isomorphic to the space of the Hilbert-Schmidt multilinear mappings. If $s,r_k\in[1,+\infty]$, $k=1,\ldots,n$ this space is characterized as the topological dual of a space of quasi-nuclear mappings. Other properties are considered and a relationship with a topological tensor product is stablished.

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ABSTRACT - The space of the strictly absolutely $(s;r_1,\ldots,r_n)$-summing multilinear mappings between Banach spaces is introduced along with a natural (quasi-) norm on it. For Hilbert spaces and $s = r_1 = \ldots = r_n \in [2,+\infty)$ it is shown that such space is isomorphic to the space of the Hilbert-Schmidt multilinear mappings. If $s, r_k \in [1,+\infty]$, $k = 1,\ldots,n$ this space is characterized as the topological dual of a space of quasi-nuclear mappings. Other properties are considered and a relationship with a topological tensor product is stablished.

1. INTRODUCTION

In [9] A. Pietsch introduced the space of absolutely $(s;r_1,\ldots,r_n)$-summing n-linear functionals on Banach spaces and asked if it would coincide with the space of the Hilbert-Schmidt n-linear functionals on Hilbert spaces for some values of s and r_k, $k = 1,\ldots,n$. Motivated by this question we introduce the space of the strictly absolutely $(s;r_1,\ldots,r_n)$-summing n-linear mappings between Banach spaces endowed with a natural norm for $s \geq 1$(s-norm for $s \in (0,1)$)and show that it is isomorphic to the space of Hilbert-Schimidt n-linear mappings between Hilbert spaces when $r_1 = \ldots = r_n = s \in [2,+\infty)$ (see section 5). It is obvious that this result does not answer the problem posed by Pietsch, but shows that, under a particular point of view, the absolutely summing *linear* mappings have as their natural n-linear generalizations the strictly absolutely summing mappings. These mappings are considered in section 2 along with several examples and properties.

In section 3 we consider Banach spaces $E_1, \ldots, E_n, F$ and endow $E_1 \otimes \ldots \otimes E_n \otimes F$ with a (quasi-) norm in such a way that its topological dual is isometric to the space of the strictly absolutely $(s; r_1, \ldots, r_n)$-summing n-linear mappings from $E_1 \times \ldots \times E_n$ into F', when $s \in [1, +\infty]$.

Section 4 is dedicated to the study of the $(s; r_1, \ldots, r_n)$-quasi-nuclear mappings from $E_1 \times \ldots \times E_n$ into F. If $E_1', \ldots, E_n'$ have the bounded approximation property and $s, r_k \in [1, +\infty]$, $k = 1, \ldots, n$, we show that the vector space of these mappings endowed with a natural linear topology has its topological dual isometric to the space of all strictly absolutely $(s'; r_1', \ldots, r_n')$-summing mappings from $E_1' \times \ldots \times E_n'$ into F'. This result is analogous to the connection between absolutely summing n-linear mappings and multilinear mappings of nuclear type stablished in [8].

In section 5 we study the space of the Hilbert-Schmidt n-linear mappings between Hilbert spaces, its properties and, as already mentioned, its relationship with spaces of strictly absolutely summing mappings. The multilinear Hilbert-Schmidt mappings were introduced by Dwyer in his doctoral dissertation [2].

For results on linear operators between Banach spaces there are some very good texts. We mention Pietsch [10] as one of them.

Now we fix some the notations we use in this paper . For Banach spaces $E_1, \ldots, E_n$ and F over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) we denote by $\mathcal{L}(E_1, \ldots, E_n; F)$ the Banach space of all continuous n-linear mappings from $E_1 \times \ldots \times E_n$ into F, under the norm

$$\|T\| = \sup\{\|T(x_1, \ldots, x_n)\|; x_k \in B_{E_k}, k = 1, \ldots, n\}$$

Here B_{E_k} denotes the closed unit ball of E_k centered at 0. If φ_k is in the topological dual E_k' of E_k, $k = 1, \ldots, n$ and $b \in F$ we denote by $\varphi_1 \times \ldots \times \varphi_n b$ the element of $\mathcal{L}(E_1, \ldots, E_n; F)$ defined as being $\varphi_1(x_1) \ldots \varphi_n(x_n) b$ at the point $(x_1, \ldots, x_n)$. These mappings generate the vector space $\mathcal{L}_f(E_1, \ldots, E_n; F)$ of the *n-linear mappings of finite type.*

If $s \in (0, +\infty)$ we denote by $\ell_s(\mathbb{N}^n; F)$ (or $\ell_s(\mathbb{N}^n)$ for $F = \mathbb{K}$) the vector space of all families $(y_j)_{j \in \mathbb{N}^n}$ of elements of F such that

$$\|(y_j)_{j \in \mathbb{N}^n}\|_s = [\sum_{j \in \mathbb{N}^n} \|y_j\|^s]^{\frac{1}{s}} < +\infty.$$

For $s \geq 1$ $\|\cdot\|_s$ is a norm and for $s \in (0,1)$ a s-norm. In any case we have a complete metrizable topological vector space. We denote by $\ell_\infty(\mathbb{N}^n; F)(\ell_\infty(\mathbb{N}^n)$ for $F = \mathbb{K})$ the Banach space of all bounded families $(y_j)_{j \in \mathbb{N}^n}$ of elements of F under the norm

$$\|(y_j)_{j \in \mathbb{N}^n}\|_\infty = \sup_{j \in \mathbb{N}^n} \|y_j\|.$$

The Banach subspace of $\ell_\infty(\mathbb{N}^n; F)$ of the families $(y_j)_{j \in \mathbb{N}^n}$ such that

$$\lim_{\substack{j_k \to \infty \\ k=1,\ldots,n}} \|y_j\| = 0$$

is denoted by $c_0(\mathbb{N}^n; F)$ (or $c_0(\mathbb{N}^n)$ for $F = \mathbb{K}$). Here as usual we write $j = (j_1, \ldots, j_n) \in \mathbb{N}^n$. For $n = 1$ it is usual to omit $\mathbb{N}^n$ in all the preceding notations. In some cases we consider finite families $(y_j)_{j \in \mathbb{N}_m^n}$ of elements of a Banach space. Here $\mathbb{N}_m = \{1, \ldots, m\}$ and we apply the symbol $\|\cdot\|_s$ to these families as we have done in the non-finite case. The vector space of all sequences $(y_j)_{j \in \mathbb{N}}$ of elements F such that

$$\|(y_j)_{j \in \mathbb{N}}\|_{w,s} = \sup_{\varphi \in B_{F'}} \|(\varphi(y_j))_{j \in \mathbb{N}}\|_s < +\infty$$

is denoted by $\ell_s^w(F)$. It is a complete metrizable topological linear space under $\|\cdot\|_{w,s}$ for $s \in (0, +\infty]$.

In Hilbert spaces $< x, y >$ denotes the inner product of the vectors x and y.

As usual, if $s \in [1, +\infty]$, s' is the element of $[1, +\infty]$ such that $s^{-1} + (s')^{-1} = 1$.

2. STRICTLY ABSOLUTELY SUMMING MULTILINEAR MAPPINGS

In this section we consider $s, r, r_k \in (0, +\infty]$ such that $s \geq r, s \geq r_k, k = 1, \ldots, n$.

2.1. DEFINITION - A mapping $T \in \mathcal{L}(E_1, \ldots, E_n; F)$ is *strictly absolutely* $(s; r_1, \ldots, r_n)$*-summing* if there is $C \geq 0$ such that

$$\|(T(x_{1,j_1}, \ldots, x_{n,j_n})_{j \in \mathbb{N}^n}\|_s \leq C \prod_{k=1}^{n} \|(x_{k,j})_{j=1}^{m}\|_{w,r_k} \tag{1}$$

for $m \in \mathbb{N}, x_{k,j} \in E_k$, $k = 1, \ldots, n$ and $j = 1, \ldots, m$.

The vector space of all such mappings is denoted by $\mathcal{L}_{sas}^{(s;r_1,\ldots,r_n)}(E_1, \ldots, E_n; F)$ and the smallest C satisfying (1) is indicated by $\|T\|_{sas,(s;r_1,\ldots,r_n)}$. This defines a s-norm for $s \in (0,1)$ and a norm for $s \geq 1$. In any case we have a complete metrizable topological vector space.

We recall that the vector space $\mathcal{L}_{as}^{(s;r_1,\ldots,r_n)}(E_1, \ldots, E_n; F)$ of all absolutely $(s; r_1, \ldots, r_n)$ -summing mappings from $E_1 \times \ldots \times E_n$ into F was introduced by A. Pietsch in [9] and consists of the $T \in \mathcal{L}(E_1, \ldots, E_n; F)$ such that there is $D \geq 0$ satisfying:

$$\|(T(x_{1,j}, \ldots, x_{n,j}))_{j=1}^{m}\|_s \leq D \prod_{k=1}^{n} \|(x_{k,j})_{j=1}^{m}\|_{w,r_k} \tag{2}$$

for $m \in \mathbb{N}, x_{k,j} \in E_k, k = 1, \ldots, n$ and $j = 1, \ldots, m$. The smallest D with the preceding property is denoted by $\|T\|_{as,(s;r_1,\ldots,r_n)}$. This gives a s-norm for $s \in (0,1)$ and a norm for $s \in [1,+\infty]$ making the space metrizable and complete. We note that in this case it is enough to consider $s, r_1, \ldots, r_n \in (0,+\infty]$ such that

$$\frac{1}{s} \leq \frac{1}{r_1} + \ldots + \frac{1}{r_n} \quad .$$

When $r_1 = \ldots = r_n = r$ we may replace $(s; r_1, \ldots, r_n)$ by $(s; r)$ in all the preceding notations. If $s = r$ we replace $(r; r)$ by r and, when $r = s = 1$, we omit (1,1) in the previous notations.

It is clear that every strictly absolutely $(s; r_1, \ldots, r_n)$ -summing mapping is absolutely $(s; r_1, \ldots, r_n)$-summing and

$$\|T\| \leq \|T\|_{as,(s,r_1,\ldots,r_n)} \leq \|T\|_{sas,(s;r_1,\ldots,r_n)}$$

for each $T \in \mathcal{L}_{sas}^{(s;r_1,\ldots,r_n)}(E_1, \ldots, E_n; F)$.

A result of Defant and Voigt (see [1] for a proof) states that $\mathcal{L}_{as}(E_1, \ldots, E_n; I\!K)$ and $\mathcal{L}(E_1, \ldots, E_n; I\!K)$ are identically isometric.

2.2. EXAMPLES

(1) There is $T \in \mathcal{L}(c_0, c_0; I\!K) = \mathcal{L}_{as}(c_0, c_0; I\!K)$ such that

$$\sum_{j,k=1}^{\infty} |T(e_j, e_k)| = +\infty$$

where $(e_j)_{j\in I\!N} \in \ell_1^w(c_0)$ is the canonical Schauder basis of c_0 (see [7]). Hence T is not strictly absolutely summing.

(2) For an infinite dimensional Banach space E we fix $\varphi \in E', \varphi \neq 0$, and define $T\varphi$ from $E \times E$ into E by

$$T\varphi(x, y) = \varphi(x)y \qquad\qquad (\forall x, y \in E).$$

(a) $T\varphi \in \mathcal{L}_{as}^{(s;r_1,r_2)}(E,E;E)$ for $s \geq r_k, k=1,2$.

$$\|(T(x_j,y_j))_{j=1}^m\|_s \leq \|(y_j)_{j=1}^m\|_\infty \ \|(x_j)_{j=1}^m\|_{w,r_1} \ \|\varphi\|$$

$$\leq \|\varphi\| \ \|(x_j)_{j=1}^m\|_{w,r_1} \ \|(y_j)_{j=1}^m\|_{w,r_2}$$

hence $\|T\varphi\|_{as,(s;r_1,r_2)} \leq \|\varphi\|$.

(b) $T\varphi \notin \mathcal{L}_{sas}^{(r_2;r_1,r_2)}(E,E;E)$ for $1 \leq r_1 \leq r_2$.

We choose $(y_j)_{j=1}^\infty \in \ell_{r_2}^w(E)\backslash \ell_{r_2}(E)$ and $(x_j)_{j=1}^\infty \in \ell_{r_1}^w(E)$ with $\varphi(x_1) \neq 0$. Hence

$$\left[\sum_{j,k=1}^\infty \|T\varphi(x_j,y_k)\|^{r_2}\right]^{\frac{1}{r_2}} = \left[\sum_{j=1}^\infty |\varphi(x_j)|^{r_2}\right]^{\frac{1}{r_2}} \left[\sum_{k=1}^\infty \|y_k\|^{r_2}\right]^{\frac{1}{r_2}} = +\infty$$

(3) Every n-linear mapping of finite type is strictly absolutely $(s;r_1,\ldots,r_n)$-summing and

$$\|\varphi_1 \times \ldots \times \varphi_n b\|_{sas,(s;r_1,\ldots,r_n)} = \|\varphi_1\| \cdots \|\varphi_n\| \ \|b\|$$

as a consequence of Hölder's inequality.

2.3. PROPOSITION - For a continuous n-linear mapping T from $E_1 \times \ldots \times E_n$ into F the following conditions are equivalent:

(1) T is strictly absolutely $(s;r_1,\ldots,r_n)$-summing.

(2) For every $(x_{k,j})_{j=1}^\infty \in \ell_{r_k}^w(E_k), k=1,\ldots,n$

$$\|(T(x_{1,j_1},\ldots,x_{n,j_n}))_{j\in\mathbb{N}^n}\|_s < +\infty$$

(3) The mapping

$$T_w : \ell_{r_1}^w(E_1) \times \ldots \times \ell_{r_n}^w(E_n) \longrightarrow \ell_s(\mathbb{N}^n;F)$$

given by

$$T_w((x_{1,j})_{j=1}^{\infty}, \ldots, (x_{n,j})_{j=1}^{\infty}) = (T(x_{1,j_1}, \ldots, x_{n,j_n}))_{j \in N^n}$$

is well defined, n-linear and continuous

In this case

$$\|T\|_{sas,(s;r_1,\ldots,r_n)} = \|T_w\|.$$

PROOF - It is clear that (3) implies (2) and (3) implies (1) with

$$\|T\|_{sas,(s;r_1,\ldots,r_n)} \leq \|T_w\| .$$

Since we can casily prove that (1) implies (2) and (1) implies (3) with

$$\|T_w\| \leq \|T\|_{sas,(s;r_1,\ldots,r_n)} ,$$

it is enough to show that (2) implies (3). But this is a consequence of the Closed-Graph Theorem since we show easily that T_w is separately continuous, hence continuous, when we assume (2). ■

The following result has some interesting consequences.

2.4. PROPOSITION - If $s \geq r_k, r_1 \geq r_k$ for $k = 1, \ldots, n$ and $T \in \mathcal{L}(E_1, \ldots, E_n; F)$ are such that

$$T_1 \in \mathcal{L}_{as}^{(s;r_1)}(E_1; \mathcal{L}_{sas}^{(r_1;r_2,\ldots,r_n)}(E_2, \ldots, E_n; F))$$

with

$$T_1(x_1)(x_2, \ldots, x_n) = T(x_1, x_2, \ldots, x_n)$$

for each $x_k \in E_k, k = 1, \ldots, n$, then T is strictly absolutely $(s; r_1, \ldots, r_n)$-summing and

$$\|T\|_{sas,(s;r_1,\ldots,r_n)} \leq \|T_1\|_{as,(s;r_1)}$$

PROOF - For $m \in \mathbb{N}, x_{k,j} \in E_k, k = 1, \ldots, n$ and $j = 1, \ldots, m$

$$\Big[\sum_{\substack{j_k=1 \\ k=1,\ldots,n}}^{m} \|T(x_{1,j_1}, \ldots, x_{n,j_n})\|^s \Big]^{\frac{1}{s}}$$

$$\leq \Big\{ \sum_{j_1=1}^{m} \Big[\sum_{\substack{j_k=1 \\ k=2,\ldots,n}}^{m} \|T_1(x_{1,j_1})(x_{2,j_2}, \ldots, x_{n,j_n})\|^{r_1} \Big]^{\frac{s}{r_1}} \Big\}^{\frac{1}{s}}$$

$$\leq \Big\{ \sum_{j_1=1}^{m} \Big[\|T_1(x_{1,j_1})\|_{sas,(r_1;r_2,\ldots,r_n)} \prod_{k=2}^{n} \|(x_{k,j})_{j=1}^{m}\|_{w,r_k} \Big]^s \Big\}^{\frac{1}{s}}$$

$$\leq \|T_1\|_{as,(s;r_1)} \prod_{k=1}^{n} \|(x_{k,j})_{j=1}^{m}\|_{w,r_k} \quad .$$

The proof for $s = +\infty$ is analogous. ■

2.5. CONSEQUENCES

(1) $\mathcal{L}_{sas}(\ell_1, \ell_2; \mathbb{K}) = \mathcal{L}(\ell_1, \ell_2; \mathbb{K})$.

(2) $\mathcal{L}^2_{sas}(c_0, \ell_p; \mathbb{K}) = \mathcal{L}(c_0, \ell_p; \mathbb{K})$ for $p \in [2, +\infty)$.

(3) $\mathcal{L}^2_{sas}(c_0, c_0; \mathbb{K}) = \mathcal{L}(c_0, c_0; \mathbb{K})$.

(4) $\mathcal{L}^r_{sas}(c_0, \ell_p; \mathbb{K}) = \mathcal{L}(c_0, \ell_p; \mathbb{K})$ for $1 < r' < p < 2$.

In fact: (1) follows from 2.4 and the Grothendieck's Theorem stating that $\mathcal{L}_{as}(\ell_1; \ell_2) = \mathcal{L}(\ell_1; \ell_2)$ (see [3]). On the other hand (2) and (3) are consequences of 2.4 and the result of Lindenstrauss and Pelczynski of the equality between $\mathcal{L}^2_{sas}(c_0; \ell_p)$ and $\mathcal{L}(c_0; \ell_p)$ for $p \in [1, 2]$ (see [6]). Finally (4) follows from 2.4 and the following result proved by Schwartz and Kwapien $\mathcal{L}(c_0; \ell_p) = \mathcal{L}^r_{as}(c_0; \ell_p)$ for $2 < p < r < +\infty$ (see [11] and [5]). ■

The following two propositions are proved easily and give ways of constructing new examples of strictly absolutely summing mappings.

2.6. PROPOSITION - If $T \in \mathcal{L}_{sas}^{(s;r_1,\ldots,r_n)}(E_1,\ldots,E_n;F), S \in \mathcal{L}(F;G)$ and $R_k \in \mathcal{L}(D_k;E_k), k=1,\ldots,n$, then $S \circ T \circ (R_1,\ldots,R_n)$ is strictly absolutely $(s;r_1,\ldots,r_n)$-summing and

$$\|S \circ T \circ (R_1,\ldots,R_n)\|_{sas,(s;r_1,\ldots,r_n)} \leq \|S\| \ \|T\|_{sas,(s;r_1,\ldots,r_n)} \prod_{k=1}^{n} \|R_k\| \ .$$

2.7. PROPOSITION - If $T \in \mathcal{L}(E_1,\ldots,E_n;F), S_k \in \mathcal{L}_{sas}^{(s_k;r_k)}(D_k;E_k), k = 1,\ldots,n$, then $T\circ(S_1,\ldots,S_n)$ is strictly absolutely $(s;r_1,\ldots,r_n)$-summing for $s \geq \max\{s_1,\ldots,s_n\}$ and

$$\|To(S_1,\ldots,S_n)\|_{sas,(s;r_1,\ldots,r_n)} \leq \|T\| \prod_{k=1}^{n} \|S_k\|_{as,(s;r_k)} \ .$$

2.8. COROLLARY - If E_k has the Orlicz property for $k=1,\ldots,n$, then $\mathcal{L}(E_1,\ldots,E_n;F) = \mathcal{L}_{sas}^{(2;1)}(E_1,\ldots,E_n;F)$ and

$$\|T\|_{sas,(2;1)} \leq \|T\| \prod_{k=1}^{n} O(E_k)$$

for every T n-linear continuous. Here $O(E_k) = \| \operatorname{id}_{E_k}\|_{as,(2;1)}$ is the Orlicz constant for $k=1,\ldots,n$.

PROOF - It follows from 2.7. and the fact that $\operatorname{id}_{E_k} \in \mathcal{L}_{as}^{(2;1)}(E_k;E_k)$ if E_k has the Orlicz property, $k=1,\ldots,n$. ■

As a consequence of 2.7 and the results of Grothendieck, Lindenstrauss Pelczynski and Schwartz-Kwapien mentioned in the proof of 2.5 we have

2.9. CONSEQUENCES - (1) If $T \in \mathcal{L}(\ell_2,\ldots,\ell_2;F)$ and $S_k \in \mathcal{L}(\ell_1;\ell_2), k = 1,\ldots,n$ then $T\circ(S_1,\ldots,S_n) \in \mathcal{L}_{sas}^{(s;1)}(\ell_1,\ldots,\ell_1;F)$ for $s \geq 1$.

(2) If $p \in [1,2], T \in \mathcal{L}(\ell_p, \ldots, \ell_p; F)$ and $S_k \in \mathcal{L}(c_0; \ell_p)$ for $k = 1, 2, \ldots, n$, then $T \circ (S_1, \ldots, S_n) \in \mathcal{L}_{sas}^{(s;2)}(c_0, \ldots, c_0; F)$ for every $s \geq 2$.

(3) If $2 < p < r < +\infty, T \in \mathcal{L}(\ell_p, \ldots, \ell_p; F)$ and $S_k \in \mathcal{L}(c_0; \ell_p), k = 1, \ldots, n$, then $T \circ (S_1, \ldots, S_n) \in \mathcal{L}_{sas}^{(s;r)}(c_0, \ldots, c_0; F)$ for $s \geq r$.

3. STRICTLY ABSOLUTELY SUMMING MAPPINGS VERSUS TENSOR PRODUCTS

For $s \in [1, +\infty], 0 < r_k \leq s, k = 1, \ldots, n$ and $u \in E_1 \otimes \ldots \otimes E_n \otimes F$ we consider

$$\rho_{(s;r_1,\ldots,r_n)}(u) = \inf \|(\lambda_j)_{j \in \mathbb{N}_m^n}\|_{s'} \ \|(b_j)_{j \in \mathbb{N}_m^n}\|_\infty \prod_{k=1}^{n} \|(x_{k,j})_{j=1}^m\|_{w,r_k}$$

where the infimum is taken over all representations of u of the form

$$u = \sum_{j \in \mathbb{N}_m^n} \lambda_j x_{1,j_1} \otimes \ldots \otimes x_{n,j_n} \otimes b_j$$

with $\lambda_j \in \mathbb{K}, x_{k,j} \in E_k, b_j \in F, k = 1, \ldots, n$, $j = 1, \ldots, m$ and $m \in \mathbb{N}$.

We denote by t_n the element of [0,1] given by

$$\frac{1}{t_n} = \frac{1}{s'} + \frac{1}{r_1} + \ldots + \frac{1}{r_n}.$$

3.1. PROPOSITION - If ε denotes the injective tensor norm, $\rho_{(s;r_1,\ldots,r_n)}$ is a t_n-norm and $\varepsilon \leq \rho_{(s;r_1,\ldots,r_n)}$.

PROOF - If

$$u = \sum_{j \in \mathbb{N}_m^n} \lambda_j x_{1,j_1} \otimes \ldots \otimes x_{n,j_n} \otimes b_j$$

we have

$$\begin{aligned}
\varepsilon(u) &= \sup_{\substack{\varphi_k \in B_{E'_k} \\ k=1,\ldots,n}} \Big|\sum_{j\in N_m^n} \lambda_j \varphi_1(x_{1,j_1}) \ldots \varphi_n(x_{n,j_n}) b_j\Big| \\
&\leq \|(\lambda_j)_{j\in N_m^n}\|_{s'} \|(\varphi_1(x_{1,j_1}) \ldots \varphi_n(x_{n,j_n}))_{j\in N_m^n}\|_s \|(b_j)_{j\in N_m^n}\|_\infty \\
&\leq \|(\lambda_j)_{j\in N_m^n}\|_{s'} \prod_{k=1}^{n} \|(x_{k,j})_{j\in N_m}\|_{w,r_k} \|(b_j)_{j\in N_m^n}\|_\infty
\end{aligned}$$

and $\varepsilon(u) \leq \rho_{(s;r_1,\ldots,r_n)}(u)$.

For u, v in $E_1 \otimes \ldots \otimes E_n \otimes F$ and $\delta > 0$ it is possible to find representations of u and v of the form

$$u = \sum_{j\in N_m^n} \lambda_j x_{1,j_1} \otimes \ldots \otimes x_{n,j_n} \otimes b_j$$

$$v = \sum_{j\in N_p^n} \eta_j y_{1,j_1} \otimes \ldots \otimes y_{n,j_n} \otimes c_j$$

such that

$$\|(\lambda_j)_{j\in N_m^n}\|_{s'} \leq \Big[(1+\delta)\rho_{(s;r_1,\ldots,r_n)}(u)\Big]^{\frac{t_n}{s'}}$$

$$\|(\eta_i)_{i\in N_p^n}\|_{s'} \leq \Big[(1+\delta)\rho_{(s;r_1,\ldots,r_n)}(v)\Big]^{\frac{t_n}{s'}}$$

$$\|(x_{k,j})_{j=1}^m\|_{w,r_k} \leq \Big[(1+\delta)\rho_{(s;r_1,\ldots,r_n)}(u)\Big]^{\frac{t_n}{r_k}}$$

$$\|(y_{k,i})_{i=1}^p\|_{w,r_k} \leq \Big[(1+\delta)\rho_{(s;r_1,\ldots,r_n)}(v)\Big]^{\frac{t_n}{r_k}}$$

$$\|(b_j)_{j\in N_m^n}\|_\infty = 1 = \|(c_i)_{i\in N_p^n}\|_\infty \quad .$$

we have

$$\begin{aligned}
\varepsilon(u) &= \sup_{\substack{\varphi_k \in B_{E'_k} \\ k=1,\ldots,n}} \Big| \sum_{j \in N_m^n} \lambda_j \varphi_1(x_{1,j_1}) \ldots \varphi_n(x_{n,j_n}) b_j \Big| \\
&\leq \|(\lambda_j)_{j \in N_m^n}\|_{s'} \|(\varphi_1(x_{1,j_1}) \ldots \varphi_n(x_{n,j_n}))_{j \in N_m^n}\|_s \|(b_j)_{j \in N_m^n}\|_\infty \\
&\leq \|(\lambda_j)_{j \in N_m^n}\|_{s'} \prod_{k=1}^{n} \|(x_{k,j})_{j \in N_m}\|_{w,r_k} \|(b_j)_{j \in N_m^n}\|_\infty
\end{aligned}$$

and $\varepsilon(u) \leq \rho_{(s;r_1,\ldots,r_n)}(u)$.

For u, v in $E_1 \otimes \ldots \otimes E_n \otimes F$ and $\delta > 0$ it is possible to find representations of u and v of the form

$$u = \sum_{j \in N_m^n} \lambda_j x_{1,j_1} \otimes \ldots \otimes x_{n,j_n} \otimes b_j$$

$$v = \sum_{j \in N_p^n} \eta_j y_{1,j_1} \otimes \ldots \otimes y_{n,j_n} \otimes c_j$$

such that

$$\|(\lambda_j)_{j \in N_m^n}\|_{s'} \leq \Big[(1+\delta)\rho_{(s;r_1,\ldots,r_n)}(u)\Big]^{\frac{t_n}{s'}}$$

$$\|(\eta_i)_{i \in N_p^n}\|_{s'} \leq \Big[(1+\delta)\rho_{(s;r_1,\ldots,r_n)}(v)\Big]^{\frac{t_n}{s'}}$$

$$\|(x_{k,j})_{j=1}^m\|_{w,r_k} \leq \Big[(1+\delta)\rho_{(s;r_1,\ldots,r_n)}(u)\Big]^{\frac{t_n}{r_k}}$$

$$\|(y_{k,i})_{i=1}^p\|_{w,r_k} \leq \Big[(1+\delta)\rho_{(s;r_1,\ldots,r_n)}(v)\Big]^{\frac{t_n}{r_k}}$$

$$\|(b_j)_{j \in N_m^n}\|_\infty = 1 = \|(c_i)_{i \in N_p^n}\|_\infty \quad .$$

Hence we have

$$\left[\rho_{(s;r_1,\dots,r_n)}(u+v)\right]^{t_n}$$

$$\leq \left[\sum_{j\in N_m^n} |\lambda_j|^{s'} + \sum_{i\in N_p^n} |\eta_i|^{s'}\right]^{\frac{t_n}{s'}} \cdot \prod_{k=1}^{n}\left[\sup_{\varphi\in B_{E_k'}}\left(\sum_{j=1}^{m}|\varphi(x_{k,j})|^{r_k} + \sum_{i=1}^{p}|\varphi(y_{k,i})|^{r_k}\right)\right]^{\frac{t_n}{r_k}}$$

$$\leq (1+\delta)^{t_n}\left[(\rho_{(s;r_1,\dots,r_n)}(u))^{t_n} + (\rho_{(s;r_1,\dots,r_n)}(v))^{t_n}\right] .$$

For $s=+\infty$ we have an analogous inequality.

Hence the triangular inequality is proved for the t_n power of $\rho_{(s;r_1,\dots,r_n)}$. The other conditions are easily verified. ■

3.2. PROPOSITION - The topological dual of $(E_1\otimes\dots\otimes E_n\otimes F\ ,\ \rho_{(s;r_1,\dots,r_n)})$ is isometric to $\mathcal{L}_{sas}^{(s;r_1,\dots,r_n)}(E_1,\dots,E_n;F')$ through the mapping B defined by

$$B(\psi)(x_1,\dots,x_n)(b) = \psi(x_1\otimes\dots\otimes x_n\otimes b)$$

for every $\rho_{(s;r_1,\dots,r_n)}$-continuous linear functional ψ on $E_1\otimes\dots\otimes E_n\otimes F, x_k \in E_k, k=1,\dots,n$ and $b\in F$.

PROOF - (1) First we consider $B(\psi)$ defined as above. It is clear that $B(\psi)\in\mathcal{L}(E_1,\dots,E_n;F')$. For $\varepsilon>0, m\in \mathbb{N}$ and $x_{k,j}\in E_k, k=1,\dots,n$ and $j=1,\dots,m$ we can find $b_j = b_{(j_1,\dots,j_m)}\in F, \|b_j\|=1$ such that

$$\sum_{j\in N_m^n} \|B(\psi)(x_{1,j_1},\dots,x_{n,j_n})\|^s$$

$$\leq \varepsilon + \sum_{j\in N_m^n} |B(\psi)(x_{1,j_1},\dots,x_{n,j_n}(b_j)|^s = \otimes$$

For a convenient choice of $\lambda_j\in \mathbb{K}, |\lambda_j|=1$ we can write

Hence we have

$$\left[\rho_{(s;r_1,\dots,r_n)}(u+v)\right]^{t_n}$$

$$\le \left[\sum_{j\in N_m^n} |\lambda_j|^{s'} + \sum_{i\in N_p^n} |\eta_i|^{s'}\right]^{\frac{t_n}{s'}} \cdot \prod_{k=1}^{n}\left[\sup_{\varphi\in B_{E'_k}} \left(\sum_{j=1}^{m} |\varphi(x_{k,j})|^{r_k} + \sum_{i=1}^{p} |\varphi(y_{k,i})|^{r_k}\right)\right]^{\frac{t_n}{r_k}}$$

$$\le (1+\delta)^{t_n}\left[(\rho_{(s;r_1,\dots,r_n)}(u))^{t_n} + (\rho_{(s;r_1,\dots,r_n)}(v))^{t_n}\right] .$$

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$$\sum_{j\in N_m^n} \|B(\psi)(x_{1,j_1},\dots,x_{n,j_n})\|^s$$

$$\le \varepsilon + \sum_{j\in N_m^n} |B(\psi)(x_{1,j_1},\dots,x_{n,j_n}(b_j)|^s = \otimes$$

For a convenient choice of $\lambda_j\in\mathbb{K}, |\lambda_j|=1$ we can write

$$\otimes = \varepsilon + \sum_{j\in N_m^n} |\lambda_j \psi(|\psi(x_{1,j_1} \otimes \ldots \otimes x_{n,j_n} \otimes b_j|^{s-1} x_{1,j_1} \otimes \ldots \otimes x_n, j_n \otimes b_j)|$$

$$\leq \varepsilon + \|\psi\| \Big[\sum_{j\in N_m^n} |\psi(x_{1,j_1} \otimes \ldots \otimes x_{n,j_n} \otimes b_j)|^{(s-1)s'} \Big]^{\frac{1}{s'}} \cdot$$

$$\cdot \prod_{k=1}^{n} \|(x_{k,j})_{j=1}^m\|_{w,r_k} \quad \|(b_j)_{j\in N_m^n}\|_\infty$$

Since $(s-1)s' = s$ and $\varepsilon > 0$ is arbitrary the preceding inequalities give

$$\|(B(\psi)(x_{1,j_1}, \ldots, x_{n,j_n}))_{j\in N_m^n}\|_s \leq \|\psi\| \prod_{k=1}^{n} \|(x_{k,j})_{j=1}^m\|_{w,r_k} \ .$$

For $s = +\infty$ we have analogous inequality.
Hence $B(\psi)$ is strictly absolutely $(s; r_1, \ldots r_n)$-summing and

$$\|B(\psi)\|_{sas,(s;r_1,\ldots,r_n)} \leq \|\psi\| \ .$$

(2) If T is strictly absolutely $(s; r_1, \ldots, r_n)$-summing from $E_1 \times \ldots \times E_n$ into F' we define a linear functional on $E_1 \otimes \ldots \otimes E_n \otimes F$ by

$$\psi_T(u) = \sum_{j\in N_m^n} \lambda_j T(x_{1,j_1}, \ldots, x_{n,j_n}) b_j$$

for

$$u = \sum_{j\in N_m^n} \lambda_j x_{1,j_1} \otimes \ldots \otimes x_{n,j_n} \otimes b_j \ .$$

We have

$$|\psi_T(u)| \leq \|(\lambda_j)_{j\in N_m^n}\|_{s'} \ \|(T(x_{1,j_1}, \ldots, x_{n,j_n})_{j\in N_m^n}\|_s \ \|(b_j)_{j\in N_m^n}\|_\infty$$

$$\leq \|T\|_{sas,(s;r_1,\ldots,r_n)} \|(\lambda_j)_{j\in N_m^n}\|_{s'} \prod_{k=1}^{n} \|(x_{k,j})_{j=1}^m\|_{w,r_k} \|(b_j)_{j\in N_m^n}\|_\infty$$

Hence ψ_T is $\rho_{(s;r_1,\ldots,r_n)}$-continuous and

$$\|\psi_T\| \leq \|T\|_{sas,(s;r_1,\ldots,r_n)}. \quad \blacksquare$$

3.3. REMARK - The t_n-norm $\rho_{(s;r_1,\ldots,r_n)}$ is a norm if

$$\frac{1}{s} = \frac{1}{r_1} + \ldots + \frac{1}{r_n}$$

In this case we have $\rho_{(s;r_1\ldots,r_n)} \leq \pi$, where π denotes the projetive tensor norm on $E_1 \otimes \ldots \otimes E_n \otimes F$.

4. QUASI-NUCLEAR MAPPINGS

In this section, unless it is stated explicitly otherwise, we consider $s \in (0,+\infty]$ and $r_k \in [1,+\infty]$ such that $s \leq r_k, k = 1,\ldots,n$. If we take

$$\frac{1}{t_n} = \frac{1}{s} + \frac{1}{r'_1} + \ldots + \frac{1}{r'_n}$$

we have $t_n \in (0,1]$.

4.1. DEFINITION - A mapping $T \in \mathcal{L}(E_1,\ldots,E_n;F)$ is $(s;r_1,\ldots,r_n)$-*quasi-nuclear* if it has a representation of the form

$$T = \sum_{j\in\mathbb{N}^n} \lambda_j \varphi_{1,j_1} \times \ldots \times \varphi_{n,j_n} b_j$$

where $(\lambda_j)_{j\in\mathbb{N}^n} \in \ell_s(\mathbb{N}^n)$ if $s < +\infty$ and in $c_0(\mathbb{N}^n)$ if $s = +\infty$, $(\varphi_{k,j})_{j\in\mathbb{N}} \in \ell^w_{r'_k}(E'_k)$ for $k = 1,\ldots,n$ and $(b_j)_{j\in\mathbb{N}^n} \in \ell_\infty(\mathbb{N}^n;F)$.

The vector space of all such mappings is denoted by $\mathcal{L}^{(s;r_1,\ldots r_n)}_{qN}(E_1,\ldots,E_n;F)$ and we consider on it the following t_n-norm

$$\|T\|_{qN,(s;r_1,\ldots,r_n)} = \inf \|(\lambda_j)_{j\in\mathbb{N}^n}\|_s \prod_{k=1}^{n} \|(\varphi_{k,j})_{j=1}^{\infty}\|_{w,r'_k} \|(b_j)_{j\in\mathbb{N}^n}\|_\infty$$

Hence ψ_T is $\rho_{(s;r_1,\ldots,r_n)}$-continuous and

$$\|\psi_T\| \leq \|T\|_{sas,(s;r_1,\ldots,r_n)}. \quad \blacksquare$$

3.3. REMARK - The t_n-norm $\rho_{(s;r_1,\ldots,r_n)}$ is a norm if

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$$T = \sum_{j\in \mathbb{N}^n} \lambda_j \varphi_{1,j_1} \times \ldots \times \varphi_{n,j_n} b_j$$

where $(\lambda_j)_{j\in\mathbb{N}^n} \in \ell_s(\mathbb{N}^n)$ if $s < +\infty$ and in $c_0(\mathbb{N}^n)$ if $s = +\infty, (\varphi_{k,j})_{j\in\mathbb{N}} \in \ell^w_{r_k'}(E_k')$ for $k = 1,\ldots,n$ and $(b_j)_{j\in\mathbb{N}^n} \in \ell_\infty(\mathbb{N}^n;F)$.

The vector space of all such mappings is denoted by $\mathcal{L}_{qN}^{(s;r_1,\ldots r_n)}(E_1,\ldots,E_n;F)$ and we consider on it the following t_n-norm

$$\|T\|_{qN,(s;r_1,\ldots,r_n)} = \inf \|(\lambda_j)_{j\in\mathbb{N}^n}\|_s \prod_{k=1}^{n} \|(\varphi_{k,j})_{j=1}^{\infty}\|_{w,r_k'} \|(b_j)_{j\in\mathbb{N}^n}\|_\infty$$

where the infimum is taken over all the possible representations as described in 4.1. As usual we replace $(s; r_1, \ldots, r_n)$ by $(s; r)$ if $r_1 = \ldots = r_n = r$ and $(s; r)$ by r if $s = r$ in the preceding notations. When $s = r = 1$ we omit 1 in the notations. In all cases we have complete metrizable topological vector spaces.

In order to justify the use of the term "quasi-nuclear" we recall that for $s \in (0, +\infty]$ and $r_k \in [1, +\infty], k = 1, \ldots, n$ such that

$$1 \leq \frac{1}{t_n} = \frac{1}{s} + \frac{1}{r_1'} + \ldots + \frac{1}{r_n'}$$

we considered in [8] the following concept.

4.2. DEFINITION - A mapping $T \in \mathcal{L}(E_1, \ldots, E_n; F)$ is of *nuclear type* $(s; r_1, \ldots, r_n)$ if it has a representation of the form

$$T = \sum_{j=1}^{\infty} \lambda_j \varphi_{1,j} \times \ldots \times \varphi_{n,j} b_j$$

where $(\lambda_j)_{j=1}^{\infty} \in \ell_s$ for $s < +\infty$ and is in c_0 if $s = +\infty, (\varphi_{k,j})_{j=1}^{\infty} \in \ell_{r_k'}^w(E_k'), k = 1, \ldots, n$ and $(b_j)_{j=1}^{\infty} \in \ell_\infty(F)$.

The vector space of all these mappings is denoted by $\mathcal{L}_N^{(s;r_1,\ldots r_n)}(E_1, \ldots, E_n; F)$ and it is a complete metrizable topological vector space under the t_n-norm

$$\|T\|_{N,(s;r_1,\ldots,r_n)} = \inf \|(\lambda_j)_{j=1}^{\infty}\|_s \prod_{k=1}^{n} \|(\varphi_{k,j})_{j=1}^{\infty}\|_{w,r_k'} \|(b_j)_{j=1}^{\infty}\|_\infty$$

where the infimum is taken for all possible representations of T as described in 4.2. The simplification of the notations is made as in the quasi-nuclear case.

4.3. REMARKS

(1) $\mathcal{L}_N^{(s;r_1,\dots,r_n)}(E_1,\dots,E_n;F) \subset \mathcal{L}_{qN}^{(s;r_1,\dots,r_n)}(E_1,\dots,E_n;F)$

with

$$\|T\| \leq \|T\|_{qN,(s;r_1,\dots,r_n)} \leq \|T\|_{N,(s;r_1,\dots,r_n)}$$

for every T of nuclear type $(s;r_1,\dots r_n)$.

(2) $\mathcal{L}_f(E_1,\dots,E_n;F)$ is dense in $\mathcal{L}_{qN}^{(s;r_1,\dots,r_n)}(E_1,\dots,E_n;F)$ and

$$\|\varphi_1 \times \dots \times \varphi_n b\|_{qN,(s;r_1,\dots,r_n)} = \|\varphi_1\| \cdots \|\varphi_n\| \ \|b\|$$

for $\varphi_k \in E_k'$, $k = 1,\dots,n$ and $b \in F$.

(3) $\mathcal{L}_N(E_1,\dots,E_n;F) = \mathcal{L}_{qN}(E_1,\dots,E_n;F)$ isometrically

(4) If $T \in \mathcal{L}_{qN}^{(s;r_1,\dots,r_n)}(E_1,\dots,E_n;F), S_k \in \mathcal{L}(D_k;E_k), k = 1,\dots,n$ and $R \in \mathcal{L}(F;G)$, then $R \circ T \circ (S_1,\dots,S_n)$ is $(s;r_1,\dots,r_n)$-quasi-nuclear and

$$\|R \circ T \circ (S_1,\dots,S_n)\|_{qN,(s;r_1,\dots,r_n)} \leq \|R\| \prod_{k=1}^{n} \|S_k\| \ \|T\|_{qN,(s;r_1,\dots,r_n)}$$

(5) If $(\lambda_j)_{j\in\mathbb{N}^n}$ is in $\ell_s(\mathbb{N}^n)$ for $s < +\infty$ or in $c_0(\mathbb{N}^n)$ for $s = +\infty$, the n-linear mapping $D_{(\lambda)_{j\in\mathbb{N}^n}}$ defined on $\ell_{r_1'} \times \dots \times \ell_{r_n'}$ with values in $\ell_1(\mathbb{N}^n)$ by

$$D_{(\lambda_j)_{j\in\mathbb{N}^n}}((\xi_{1,j})_{j=1}^{\infty},\dots,(\xi_{n,j})_{j=1}^{\infty}) = (\lambda_j \xi_{1,j_1} \dots \xi_{n,j_n})_{j\in\mathbb{N}^n}$$

is $(s;r_1,\dots,r_n)$-quasi-nuclear and

$$\|D_{(\lambda_j)_{j\in\mathbb{N}^n}}\|_{qN,(s;r_1,\dots,r_n)} \leq \|(\lambda_j)_{j\in\mathbb{N}^n}\|_s \ .$$

The following result gives another characterization of quasi-nuclear mappings.

4.4. PROPOSITION - For $T \in \mathcal{L}(E_1, \ldots, E_n; F)$ the following conditions are equivalent

(1) T is $(s; r_1, \ldots, r_n)$-quasi-nuclear.

(2) There are $A_k \in \mathcal{L}(E_k; \ell_{r'_k}), k = 1, \ldots, n, Y \in \mathcal{L}(\ell_1(\mathbb{N}^m); F)$ and $(\lambda_j)_{j \in \mathbb{N}^n} \in \ell_s(\mathbb{N}^n)$ such that

$$T = Y \circ D_{(\lambda_j)_{j \in \mathbb{N}^n}} \circ (A_1, \ldots, A_n)$$

In this case

$$\|T\|_{qN,(s;r_1,\ldots,r_n)} \leq \inf \|Y\| \prod_{k=1}^{n} \|A_k\| \ \ \|(\lambda_j)_{j \in \mathbb{N}^n}\|_s$$

with the infimum taken over all possible factorizations as described in (2).

PROOF - It is clear that (2) implies (1) by 4.3.(4) and 4.3.(5).

In order to show that (1) implies (2) we consider a representation of T as in 4.1 and define

$$A_k(x) = (\varphi_{k,j}(x))_{j=1}^{\infty} \qquad (\forall x \in E_k, k = 1, \ldots, n)$$

and

$$Y((\xi_j)_{j \in \mathbb{N}^n}) = \sum_{j \in \mathbb{N}^n} \xi_j b_j \qquad (\forall (\xi_j)_{j \in \mathbb{N}^n} \in \ell_1(\mathbb{N}^n))$$

and the result follows by Holder's inequality. ■

4.5. REMARK - It is clear that every $T \in \mathcal{L}_f(E_1, \ldots, E_n; F)$ has a finite representation

$$T = \sum_{j \in \mathbb{N}_m^n} \lambda_j \varphi_{1,j_1} \times \ldots \times \varphi_n, j_n b_j \ .$$

It is also clear that we have a t_n-norm on $\mathcal{L}_f(E_1,\ldots,E_n;F)$ defined by

$$\|T\|_{qNf,(s;r_1,\ldots,r_n)} = \inf \|(\lambda_j)_{j\in N_m^n}\|_s \prod_{k=1}^{n} \|(\varphi_{k,j})_{j=1}^m\|_{w,r_k'} \|(b_j)_{j\in N_m^n}\|_\infty$$

where the infimum is taken over all finite representation of T as above. We know that

$$\|T\|_{qN,(s;r_1,\ldots,r_n)} \leq \|T\|_{qNf,(s;r_1,\ldots,r_n)}$$

for every $T \in \mathcal{L}_f(E_1,\ldots,E_n;F)$. We would like to know cases where there is equality.

4.6. PROPOSITION - If $E_1,\ldots,E_n$ are finite dimensional, then

$$\|T\|_{qNf,(s;r_1,\ldots,r_n)} = \|T\|_{qN,(s;r_1,\ldots,r_n)}$$

for every $T \in \mathcal{L}_f(E_1,\ldots,E_n;F)$.

PROOF - In this case $\mathcal{L}(E_1,\ldots,E_n;F) = \mathcal{L}_f(E_1,\ldots,E_n;F)$ is complete for both t_n-norms. Hence by the open mapping theorem these t_n-norms are equivalent and there is $C \geq 0$ such that

$$\|T\|_{qNf,(s;r_1,\ldots,r_n)} \leq C\|T\|_{qN,(s;r_1,\ldots,r_n)}$$

for every T in $\mathcal{L}_f(E_1,\ldots,E_n;F)$. For each $\varepsilon > 0$ we choose a representation

$$T = \sum_{j\in N^n} \sigma_j \varphi_{1,j_1} \times \ldots \times \varphi_{n,j_n} y_j$$

such that

$$\|(\sigma_j)_{j\in N^n}\|_s \|(y_j)_{j\in N^n}\|_\infty \prod_{k=1}^{n} \|(\varphi_{k,j})_{j\in N}\|_{w,r_k'} \leq (1+\varepsilon)\|T\|_{qN,(s;r_1,\ldots,r_n)}.$$

We have

$$\left[\|T\|_{qNf,(s;r_1,\dots,r_n)}\right]^{t_n} \leq \left[\|\sum_{j\in N_m^n} \sigma_j \varphi_{1,j_1} \times \dots \times \varphi_{n,j_n} y_j\|_{qNf,(s;r_1,\dots,r_n)}\right]^{t_n}$$

$$+ \left[\|\sum_{\substack{j_k>m \\ k=1,\dots,n}} \sigma_j \varphi_{1,j_1} \times \dots \times \varphi_{n,j_n} y_j\|_{qNf,(s;r_1,\dots,r_n)}\right]^{t_n}$$

$$\leq (1+\varepsilon)^{t_n}\left[\|T\|_{qN,(s;r_1,\dots,r_n)}\right]^{t_n}$$

$$+C^{t_n}\left[\|\sum_{\substack{j_k>m \\ k=1,\dots,n}} \sigma_j \varphi_{1,j_1} \times \dots \times \varphi_{n,j_n} y_j\|_{qN,(s;r_1,\dots,r_n)}\right]^{t_n}$$

$$\leq \left[(1+\varepsilon)^{t_n}+\varepsilon^{t_n}\right]\left[\|T\|_{qN,(s;r_1,\dots,r_n)}\right]^{t_n}$$

for m large enough. ∎

4.7. PROPOSITION - If $T \in \mathcal{L}_{qN}^{(s;r_1,\dots,r_n)}(E_1,\dots,E_n;F)$ and $S_k \in \mathcal{L}_f(D_k;E_k)$ for $k=1,\dots,n$, then

$$\|T\circ(S_1,\dots,S_n)\|_{qNf,(s;r_1,\dots,r_n)} \leq \|T\|_{qN,(s;r_1,\dots,r_n)} \prod_{k=1}^{n} \|S_k\|\,.$$

PROOF - If J_k denotes the natural injection from $S_k(D_k)$ into E_k we can write $S_k = J_k \circ \tilde{S}_k$ with $\|\tilde{S}_k\| = \|S_k\|$. Hence $T\circ(J_1,\dots,J_n)$ is in $\mathcal{L}_f(S_1(D_1),\dots,S_n(D_n);F)$. Now we apply 4.6 and 4.3.(4) to have the result. ∎

4.8. PROPOSITION - If $E_1',\dots,E_n'$ have the bounded approximation property, then

$$\|T\|_{qNf,(s;r_1,\dots,r_n)} = \|T\|_{qN,(s;r_1,\dots,r_n)}$$

for every $T\in\mathcal{L}_f(E_1,\dots,E_n;F)$.

PROOF - We prove the result for $n = 2$. For $n > 2$ and $n = 1$ the proofs are analogous. Since $T_1 \in \mathcal{L}_f(E_1; \mathcal{L}(E_2; F))$, where $T_1(x_1)(x_2) = T(x_1, x_2)$ for $x_k \in E_k, k = 1,2$ and T is of finite type, for every $\varepsilon > 0$ there is $S_1 \in \mathcal{L}_f(E_1; E_1)$ such that $T_1 \circ S_1 = T_1$ and $\|S_1\| \leq (1+\varepsilon)\lambda_1$ (because E_1' has the λ_1-approximation property for some $\lambda_1 > 0$). Hence $T(S_1(x_1), x_2) = T(x_1, x_2)$ for $x_k \in E_k, k = 1,2$. By the same type of reasoning $T_2 \in \mathcal{L}_f(E_2; \mathcal{L}(E_1; F))$, when $T_2(x_2)(x_1) = T(x_1, x_2)$ for $x_k \in E_k, k = 1,2$ and there is $S_2 \in \mathcal{L}_f(E_2; E_2)$ such that $T_2 \circ S_2 = T_2$ with $\|S_2\| \leq (1+\varepsilon)\lambda_2$. We have $T(x_1, S_2(x_2)) = T(x_1, x_2)$ for $x_k \in E_k, k = 1,2$. Thus $T = T \circ (S_1, S_2)$ and by 4.7. we have

$$\|T\|_{qNf,(s;r_1,r_2)} \leq \|T\|_{qN,(s;r_1,r_2)} \|S_1\| \, \|S_2\|$$
$$\leq (1+\varepsilon)^2 \lambda_1 \lambda_2 \|T\|_{qN,(s;r_1,r_2)} .$$

Hence

$$\|T\|_{qNf,(s;r_1,r_2)} \leq \lambda_1 \lambda_2 \|T\|_{qN,(s;r_1,r_2)}.$$

With the same argument used in the proof of 4.6 we have

$$\|T\|_{qNf,(s;r_1,r_2)} \leq \|T\|_{qN,(s;r_1,r_2)}. \quad \blacksquare$$

4.9. COROLLARY - If $E_1', \ldots, E_n'$ have the bounded aproximation property, then $\mathcal{L}_{qN}^{(s;r_1,\ldots r_n)}(E_1, \ldots, E_n; F)$ is isometric to the completion of $(E_1' \otimes \ldots \otimes E_n' \otimes F, \rho_{(s',r_1',\ldots,r_n')})$ for $s, r_k \in [1, +\infty], k = 1, \ldots, n$

4.10. PROPOSITION - If $E_1', \ldots, E_n'$ have the bounded approximation property, then the topological dual of $\mathcal{L}_{qN}^{(s;r_1,\ldots,r_n)}(E_1, \ldots, E_n; F)$ is isometric to $\mathcal{L}_{qN}^{(s';r_1',\ldots,r_n')}(E_1', \ldots, E_n'; F')$ for $s, r_k \in [1, +\infty] k = 1, \ldots, n$, through the mapping

$$\mathcal{B}(\psi)(\varphi_1, \ldots, \varphi_n)(b) = \psi(\varphi_1 \times \ldots \times \varphi_n b)$$

for ψ in the topological dual of $\mathcal{L}_{qN}^{(s;r_1,\ldots,r_n)}(E_1,\ldots,E_n;F)$, $\varphi_k \in E'_k, k = 1,\ldots,n$ and $b \in F$.

PROOF - It is a consequence of 4.9 and 3.2. ■

We recall that in [8] we proved that the topological dual of $\mathcal{L}_{N}^{(s;r_1,\ldots,r_n)}(E_1,\ldots,E_n;F)$ is isometric to $\mathcal{L}_{as}^{(s';r'_1,\ldots,r'_n)}(E'_1,\ldots,E'_n;F')$ through the mapping $\mathcal{B}$ as defined in 4.10, when $E'_1,\ldots,E'_n$ have the bounded approximation property and $s, r_k \in [1,+\infty], k = 1,\ldots,n$. This fact, 4.10 and 2.2.(2) show that in general the spaces $\mathcal{L}_{N}^{(s;r_1,\ldots,r_n)}(E_1,\ldots,E_n;F)$ and $\mathcal{L}_{qN}^{(s;r_1,\ldots,r_n)}(E_1,\ldots,E_n;F)$ are different.

5. HIBERT-SCHMIDT MULTILINEAR MAPPINGS

In this section $E_1,\ldots,E_n$ and F are Hilbert spaces. In this case, as we are going to show that, there is a close relationship between the Hilbert-Schmidt and the strictly absolutely summing mappings.

5.1. PROPOSITION - If $T \in \mathcal{L}(E_1,\ldots,E_n;F)$ and $(u_{k,j})_{j\in J_k}$ is an orthonormal basis for $E_k, k = 1,\ldots,n$, the value

$$\sum_{\substack{j_k \in J_k \\ k=1,\ldots,n}} \|T(u_{1,j_1},\ldots,u_{n,j_n})\|^2$$

(finite or not) is independent of the orthonormal basis chosen for E_k, $k = 1,\ldots,n$.

PROOF - For $n = 1$ Parseval's equality gives

$$\sum_{j\in J_1} \|T(u_{1,j})\|^2 = \sum_{j\in J} \|T^*(v_j)\|^2$$

where $(v_j)_{j\in J}$ is an orthonormal basis for F. The case $n > 1$ is proved by fixing $n-1$ variables and applying the linear result to the remaining variable. ∎

5.2. DEFINITION - A mapping $T \in \mathcal{L}(E_1,\ldots,E_n;F)$ is said to be *Hilbert-Schmidt* is there is an orthonormal basis $(u_{k,j})_{j\in J_k}$ for $E_k, k=1,\ldots,n$ such that

$$\|T\|_{HS} = \Big[\sum_{\substack{j_k\in J_k\\k=1,\ldots,n}} \|T(u_{1,j_1},\ldots,u_{n,j_n})\|^2\Big]^{\frac{1}{2}} < +\infty$$

We denote by $\mathcal{L}_{HS}(E_1,\ldots,E_n;F)$ the vector space of all such mappings. It is easy to show that it is a Hilbert space under the norm $\|\cdot\|_{HS}$ defined by the inner product

$$\langle T,S\rangle = \sum_{\substack{j_k\in J_k\\k=1,\ldots,n}} \langle T(u_{1,j_1},\ldots,u_{n,j_n}), S(u_{1,j_1},\ldots,u_{n,j_n})\rangle$$

5.3. PROPOSITION - The Hilbert spaces $\mathcal{L}_{HS}(E_1,\ldots,E_n;F)$ is isometric to $\mathcal{L}_{HS}(E_1;\mathcal{L}_{HS}(E_2,\ldots,E_n;F))$.

PROOF -

For $T \in \mathcal{L}(E_1,\ldots,E_n;F)$ we consider $T_1 \in \mathcal{L}(E_1;\mathcal{L}(E_2,\ldots,E_n;F))$. If T is Hilbert-Schmidt, $(u_{k,j})_{j\in J_k}$ is an orthonormal basis for $E_k, k=1,\ldots,n$ and $(v_j)_{j\in J}$ is an orthonormal basis for F, we can write for each $x\in E_1$

$$\sum_{\substack{j_k\in J_k\\k\geq 2}} \|T_1(x)(u_{2,j_2},\ldots,u_{n,j_n})\|^2$$

$$= \sum_{\substack{j_k\in J_k\\k\geq 2\\j\in J}} \Big|\sum_{j_1\in J_1} \langle x,u_{1,j_1}\rangle\langle T(u_{1,j_1},\ldots,u_{n,j_n}),v_j\rangle\Big|^2$$

$$\leq \|x\|^2 \, [\|T\|_{HS}]^2 \quad .$$

Hence $T_1(x)$ is Hilbert-Schmidt and $\|T_1(x)\|_{HS} \leq \|T\|_{HS}\|x\|$. Now it is clear that

$$\sum_{j_1 \in J_1} [\|T_1(u_{1,j_1})\|_{HS}]^2 = [\|T\|_{HS}]^2$$

and $T_1 \in \mathcal{L}_{HS}(E_1; \mathcal{L}_{HS}(E_2, \ldots, E_n; F))$ with $\|T_1\|_{HS} = \|T\|_{HS}$. If $S \in \mathcal{L}(E_1, \ldots, E_n; F)$ is such that $S_1 \in \mathcal{L}_{HS}(E_1; \mathcal{L}_{HS}(E_2, \ldots, E_n; F))$ it is easy to see that

$$\sum_{j_1 \in J_1} [\|S_1(u_{1,j_1})\|_{HS}]^2 = \sum_{\substack{j_k \in J_k \\ k=1,\ldots,n}} \|S(u_{1,j_1}, \ldots, u_{n,j_n}\|_{HS}]^2$$

Hence S is Hilbert-Schmidt and $\|S\|_{HS} = \|S_1\|_{HS}$. ∎

5.4. COROLLARY -

(a) $\mathcal{L}_{HS}(E_1, \ldots, E_n; F)$ and $\mathcal{L}_{HS}(E_1, \ldots, E_k; \mathcal{L}_{HS}(E_{k+1}, \ldots, E_n; F))$ are isometric.

(b) $\mathcal{L}_{HS}(E_1, \ldots, E_n, F'; \mathbb{K})$ and $\mathcal{L}_{HS}(E_1, \ldots, E_n; F)$ are isometric.

5.5. PROPOSITION - $\mathcal{L}_{HS}(E_1, \ldots, E_n; F)$ and $\mathcal{L}^2_{sas}(E_1, \ldots, E_n; F)$ are identically isometric.

PROOF - (a) If $T \in \mathcal{L}^2_{sas}(E_1, \ldots, E_n; F)$ and $(u_{k,j})_{j \in J_k}$ is an orthonormal basis for E_k, $k = 1, \ldots, n$, then $\|(u_{k,j})_{j \in J_k}\|_{w,2} = 1$ for each k and we have T Hilbert-Schmidt with $\|T\|_{HS} \leq \|T\|_{sas,2}$.

(b) We assume $T \in \mathcal{L}_{HS}(E_1, \ldots, E_n; F)$. For $n = 1$ we consider $x_j \in E_1$, $j = 1, \ldots, m$ and an orthonormal basis $(v_k)_{k \in I}$ for F. Then

$$\left[\sum_{j=1}^{m} ||T(x_j)||^2\right]^{\frac{1}{2}} = \left[\sum_{k \in I} \sum_{j=1}^{m} |\langle x_j, T^*(v_k)\rangle|^2\right]^{\frac{1}{2}}$$

$$\leq \left[\sum_{k \in I} ||T^*(v_k)||^2\right]^{\frac{1}{2}} \sup_{\varphi \in B_{E_1'}} \left[\sum_{j=1}^{m} |\langle x_j, \varphi\rangle|^2\right]^{\frac{1}{2}}$$

$$= ||T||_{HS} ||(x_j)_{j=1}^{m}||_{w,2} .$$

Hence $T \in \mathcal{L}^2_{sas}(E_1; F)$ and $||T||_{sas,2} \leq ||T||_{HS}$.

For $n > 1$ we assume the result true for $k \leq n - 1$. Since $T_1 \in \mathcal{L}_{HS}(E_1; \mathcal{L}_{HS}(E_2, \ldots, E_n; F)$ by 5.3., we have T_1 in $\mathcal{L}^2_{sas}(E_1; \mathcal{L}_{HS}(E_2, \ldots, E_n; F)) \subset \mathcal{L}^2_{sas}(E_1; \mathcal{L}^2_{sas}(E_2, \ldots, E_n; F))$ with $||T_1||_{sas,2} \leq ||T_1||_{HS} = ||T||_{HS}$. By 2.4. we have T in $\mathcal{L}^2_{sas}(E_1, \ldots, E_n; F)$ and $||T||_{sas,2} \leq ||T_1||_{sas,2} \leq ||T||_{HS}$. ■

For $p \in [1, +\infty)$ we can show some interesting connections between Hilbert-Schmidt and strictly absolutely p-summing multilinear mappings.

5.6. PROPOSITION - There is $d > 0$ such that for $p \in [1, +\infty)$ every T in $\mathcal{L}_{HS}(E_1, \ldots, E_n; F)$ is strictly absolutely p-summing and

$$d^n ||T||_{sas,p} \leq ||T||_{HS} .$$

PROOF - We use induction on n.

For $n = 1$ there are $(x_i)_{i \in \mathbb{N}}$ orthonormal in $E_1, (y_i)_{i \in \mathbb{N}}$ orthonormal in F and $(\lambda_i)_{i \in \mathbb{N}} \in \ell_2$ such that $T(x_i) = \lambda_i y_i$ for each $i \in \mathbb{N}$ and

$$||T||_{HS} = ||(\lambda_i)_{i \in \mathbb{N}}||_2 .$$

If $(u_j)_{j=1}^{m}$ is a finite sequence in E_1 and $(r_i)_{i \in \mathbb{N}}$ is the sequence of Rademacher functions, we consider

$$||T||_{HS}v(t) = \sum_{i=1}^{\infty} r_i(t)\lambda_i x_i \in E_1$$

for every $t \in [0,1]$. Now, by the Khintchine's inequality (see [4]; see also [10], page 41), we have

$$||T||_{HS}||(u_j)_{j=1}^m||_{w,1} \geq ||T||_{HS} \sup_{t\in[0,1]} \sum_{j=1}^{m} |\langle u_j, v(t)\rangle|$$

$$\geq \sum_{j=1}^{m} \int_0^1 |\sum_{i=1}^{\infty} r_i(t)\overline{\lambda}_i \langle u_j, x_i\rangle| dt$$

$$\geq d\sum_{j=1}^{m}\left[\sum_{i=1}^{\infty}|\lambda_i\langle u_j, x_i\rangle|^2\right]^{\frac{1}{2}} = d\sum_{j=1}^{m}||T(u_j)||.$$

Hence $T \in \mathcal{L}_{sas}(E_1;F)$ and $d||T||_{sas} \leq ||T||_{HS}$. Also $T \in \mathcal{L}^p_{sas}(E_1;F)$ with $d||T||_{sas,p} \leq d||T||_{sas} \leq ||T||_{HS}$ for $p \geq 1$.

Now we assume the result true for $n \leq k$, $k \geq 1$ and prove it for $k+1$.

If $T \in \mathcal{L}_{HS}(E_1,\dots,E_{k+1};F)$, then $T_1 \in \mathcal{L}_{HS}(E_1;\mathcal{L}_{HS}(E_2,\dots,E_{k+1};F))$ by 5.3. By our induction hypothesis

$$E_1 \xrightarrow{T_1} \mathcal{L}_{HS}(E_1,\dots,E_{k+1};F) \xrightarrow{J} \mathcal{L}^p_{sas}(E_2,\dots,E_{k+1};F)$$

with $J\circ T_1 \in \mathcal{L}^p_{sas}(E_1;\mathcal{L}^p_{sas}(E_2,\dots,E_{k+1};F))$ and

$$d^{k+1}||J\circ T_1||_{sas,p} \leq d[d^k||J||]||T||_{sas,p}$$

$$\leq d||T_1||_{sas,p} \leq ||T_1||_{HS}.$$

Hence, by 2.4, $T \in \mathcal{L}^p_{sas}(E_1,\dots,E_{k+1};F)$

$$d^{k+1}||T||_{sas,p} \leq d^{k+1}||J\circ T_1||_{sas,p} \leq ||T_1||_{HS} = ||T||_{HS}\ . \ \blacksquare$$

Before we prove next result we consider $m \in \mathbb{N}$ and $D_m = \{-1,1\}^m$ with a measure μ defined by $\mu(e) = 2^{-m}$ for every $e = (e_1, \ldots, e_m)$ in D_m. We denote by π_k the k-th projection from D_m onto $\{-1,1\}$. It follows that

$$\int_{D_m} \pi_j(e)\pi_k(e)d\mu(e) = \delta_{jk} = \begin{cases} 1 & \text{if} \quad j = k \\ 0 & \text{if} \quad j \neq k \end{cases}$$

5.7. PROPOSITION - For $p \in [2, +\infty)$ there is $b_p > 0$ such that $\mathcal{L}^p_{sas}(E_1, \ldots, E_n; F) = \mathcal{L}_{HS}(E_1, \ldots, E_n; F)$ and

$$d^n ||T||_{sas,p} \leq ||T||_{HS} \leq (b_p)^n ||T||_{sas,p}$$

for every T strictly absolutely p-summing.

PROOF - Part of this result follows from 5.6. Now we consider $T \in \mathcal{L}^p_{sas}(E_1, \ldots, E_n; F)$ and an orthonormal basis $(u_{k,j})_{j \in I_k}$, for $E_k, k = 1, \ldots, n$. For each finite subset J_k of I_k with m elements we consider $(u_{k,j})_{j \in J_k}$ ordered linearly and write $u_{k,1}, \ldots, u_{k,m}$, $k = 1, \ldots, n$. We take

$$w_k(e) = \sum_{j=1}^{m} e_j u_{k,j}$$

for $e \in D_m$ and $k = 1, \dots, n$, and write

$$\left[\sum_{\substack{j_k=1 \\ k=1,\dots,n}}^{m} \|T(u_{1,j_1,\dots,u_{n},j_n})\|^2 \right]^2$$

$$= \left[\int_{D_m^n} \|T(w_1(e^{(1)}), \dots, w_n(e^{(n)}))\|^2 d\mu(e^{(1)} \dots d\mu(e^{(n)}) \right]^{\frac{1}{2}}$$

$$\leq \left[\int_{D_m^n} \|T(w_1(e^{(1)}), \dots, w_n(e^{(n)}))\|^p d\mu(e^{(1)} \dots d\mu(e^{(n)}) \right]^{\frac{1}{p}}$$

$$\leq \|T\|_{sas,p} \prod_{k=1}^n \sup_{\varphi \in B_{E'_k}} \left[\int_{D_m} |\varphi(w_k(e))|^p d\mu(e) \right]^{\frac{1}{p}}$$

$$\leq \|T\|_{sas,p} \prod_{k=1}^n b_p \sup_{\varphi \in B_{E'_k}} \left[\sum_{j=1}^m |\varphi(u_{k,j})|^2 \right]^{\frac{1}{2}} = \|T\|_{sas,p} (b_p)^n$$

where the last inequality was obtained through the Khintchine's inequality. Hence $T \in \mathcal{L}_{HS}(E_1, \dots, E_n; F)$ and

$$\|T\|_{HS} \leq (b_p)^n \|T\|_{sas,p}. \quad \blacksquare$$

5.8. REMARKS

(1), Pietsch in [9] asked if there would be some $(s; r_1, \dots, r_n)$ such that

$$\mathcal{L}_{as}^{(s;r_1,\dots,r_n)}(E_1, \dots, E_n; I\!K) = \mathcal{L}_{HS}(E_1, \dots, E_n; I\!K)$$

when $n \geq 2$. It is easy to see that $T\varphi$ in 2.2(2) is not Hilbert-Schmidt but is $\mathcal{L}_{as}^{(s;r_1,r_2)}(E, E; E)$ whenever $s \geq r_1$. If we consider *strictly* absolutely summing mappings 5.7 gives an affirmative answer to the Pietsch question.

(2) For $n = 1$, since $\mathcal{L}_{as}^p(E; F) \subset \mathcal{L}_{as}^2(E; F)$ if $p \in [1, 2]$ we have 5.7 true even for $p \in [1, +\infty)$. For $n \geq 2$ and $p \in [1, 2)$ we do not know if 5.7 is still true.

5.9. DEFINITION - The following inner product is considered on $E_1 \otimes \ldots \otimes E_n$

$$\langle u, v\rangle_H = \sum_{j=1}^{p} \sum_{k=1}^{q} \langle x_{1,j}, y_{1,j}\rangle \ldots \langle x_{n,j}, y_{n,k}\rangle$$

where

$$u = \sum_{j=1}^{p} x_{1,j} \otimes \ldots \otimes x_{n,j} \quad \text{and} \quad v = \sum_{j=1}^{q} y_{1,j} \otimes \ldots \otimes y_{n,j} .$$

The space $E_1 \otimes \ldots \otimes E_n$ with this inner product is denoted by $E_1 \otimes_H \ldots \otimes_H E_n$ and its completion by $E_1 \widehat{\otimes}_H \ldots \widehat{\otimes}_H E_n$. The corresponding norm is denoted by $\|\cdot\|_H$.

5.10. REMARK - If $(e_{k,j})_{j\in J_k}$ is an orthonormal basis for E_k, $k = 1, \ldots, n$, then $(e_{1,j_1} \otimes \ldots \otimes e_{n,j_n})_{\substack{j_k\in J_k \\ k=1,\ldots,n}}$ is an orthonormal basis for $E_1 \widehat{\otimes}_H \ldots \widehat{\otimes}_H E_n$.

As consequence of this remark we can prove

5.11. PROPOSITION - If $T \in \mathcal{L}(E_1, \ldots, E_n; F)$ and $T_\otimes$ denotes the corresponding linear mapping from $E_1 \otimes \ldots \otimes E_n$ into F, then the following conditions are equivalent:

(1) T is Hilbert-Schmidt.

(2) $T_{\widehat{\otimes}} \in \mathcal{L}_{HS}(E_1 \widehat{\otimes}_H \ldots \widehat{\otimes}_H E_n; F)$.

Here $T_{\widehat{\otimes}}$ denotes the extension of $T_\otimes$ to $E_1 \widehat{\otimes}_H \ldots \widehat{\otimes}_H E_n$. In this case $\|T\|_{HS} = \|T_{\widehat{\otimes}}\|_{HS}$.

5.12. PROPOSITION - The Hilbert spaces $\mathcal{L}_{HS}(E'_1, \ldots, E'_n; F')$ and $[\mathcal{L}_{HS}(E_1, \ldots, E_n; F)]'$ are isometric through the mapping $\mathcal{B}$ given by

$$\mathcal{B}(\psi)(x'_1, \ldots, x'_n) = \sum_{j\in J} \psi(x'_1 \times \ldots \times f_j) f'_j$$

for $x'_k \in E'_k$, $k = 1, \ldots, n$, where $(f_j)_{j\in J}$ is an orthonormal basis of F and $(f'_j)_{j\in J}$ is the corresponding dual basis for F'.

PROOF - If we prove the result for $F = I\!K$ we use it and 5.4(b) to have the isometries

$$[\mathcal{L}_{HS}(E_1, \dots, E_n; F)]' \cong [\mathcal{L}_{HS}(E_1, \dots, E_n, F'; I\!K)]'$$

$$\cong \mathcal{L}_{HS}(E_1', \dots, E_n', F; I\!K) \cong \mathcal{L}_{HS}(E_1', \dots, E_n'; F').$$

Hence we have to prove only the case $F = I\!K$.

It is clear that

$$(e'_{1,j_1} \times \dots \times e'_{n,j_n})_{\substack{j_k \in J_k \\ k=1,\dots,n}}$$

is an orthonormal basis for $\mathcal{L}_{HS}(E_1, \dots, E_n; I\!K)$ if $(e'_{k,j})_{j \in J_k}$ denotes the dual basis of an orthonormal basis $(e_{k,j})_{j \in J_k}$ for E_k, $k = 1, \dots, n$. We have

$$T = \sum_{\substack{j_k \in J_k \\ k=1,\dots,n}} T(e_{1,j_1}, \dots, e_{n,j_n})(e'_{1,j_1} \times \dots \times e'_{n,j_n})$$

for every $T \in \mathcal{L}_{HS}(E_1, \dots, E_n; I\!K)$. Hence for $\psi \in [\mathcal{L}_{HS}(E_1, \dots, E_n; I\!K)]'$

$$\sum_{\substack{j_k \in J_k \\ k=1,\dots,n}} |\mathcal{B}(\psi)(e'_{1,j_1}, \dots, e'_{n,j_n})|^2 = ||\psi||^2.$$

This given $\mathcal{B}(\psi)$ Hilbert-Schmidt and $||\mathcal{B}(\psi)||_{HS} = ||\psi||$.

On the other hand if $S \in \mathcal{L}_{HS}(E_1', \dots, E_n'; I\!K)$ we define $\psi_S \in [\mathcal{L}_{HS}(E_1, \dots, E_n, F; I\!K)]'$ by

$$\psi_S(T) = \sum_{\substack{j_k \in J_k \\ k=1,\dots,n}} T(e_{1,j_1}, \dots, e_{n,j_n})\, S(e'_{1,j_1} \times \dots \times e'_{n,j_n})$$

and have $\mathcal{B}(\psi_S) = S$ with

$$|\psi_S(T)| \leq ||T||_{HS} ||S||_{HS}. \blacksquare$$

5.13. **COROLLARY** - The Hilbert spaces $(E_1 \hat{\otimes}_H \dots \hat{\otimes}_H E_n)'$ and $(E_1' \hat{\otimes}_H \dots \hat{\otimes}_H E_n')$ are isometric.

PROOF - By 5.12 and 5.11 we have the isometries

$$(E_1\hat{\otimes}_H \ldots \hat{\otimes}_H E_n)'' \cong [\mathcal{L}_{HS}(E_1, \ldots, E_n; \mathbb{K})]'$$

$$\cong \mathcal{L}_{HS}(E_1', \ldots, E_n'; \mathbb{K}) \cong (E_1'\hat{\otimes}_H \ldots \hat{\otimes}_H E_n')' . \blacksquare$$

5.14. PROPOSITION - The following Hilbert spaces are isometric: $\mathcal{L}_{HS}(E_1, \ldots, E_n; F), \mathcal{L}_{HS}(E_1, \ldots, E_n \mathbb{K})\hat{\otimes}_H F$ and $(E_1'\hat{\otimes}_H \ldots \hat{\otimes}_H E_n')\hat{\otimes}_H F$.

PROOF - By 5.4(b), 5.13 and 5.11 we have the isometries

$$\mathcal{L}_{HS}(E_1, \ldots, E_n; F) \cong \mathcal{L}_{HS}(E_1, \ldots, E_n, F'; \mathbb{K})$$

$$\cong (E_1\hat{\otimes}_H \ldots \hat{\otimes}_H E_n \hat{\otimes}_H F)' \cong (E_1\hat{\otimes}_H \ldots \hat{\otimes}_H E_n)'\hat{\otimes}_H F$$

and this last Hilbert space is isometric to $\mathcal{L}_{HS}(E_1, \ldots, E_n; \mathbb{K})\hat{\otimes}_H F$ and $E_1'\hat{\otimes}_H \ldots \hat{\otimes}_H E_n'\hat{\otimes}_H F$. $\blacksquare$

5.15. REMARK - The results of this section and section 4 give a linear homeomorphism between $[\mathcal{L}_{HS}(E_1, \ldots, E_n; F)]'$ and $[\mathcal{L}^p_{qN}(E_1, \ldots, E_n; F)]'$ for $p \in (1, 2]$. But we must note that $\mathcal{L}^p_{qN}(E_1, \ldots, E_n; F)'$ is not normed for $n \geq 2$.

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