# SIMULTANEOUS APPROXIMATION AND INTERPOLATION IN $p$-ADIC ANALYSIS

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# Simultaneous approximation and interpolation in $p$-adic Analysis

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**Abstract:** Let $\mathbb{Q}_p$ denote the field of $p$-adic numbers. Let $S$ be a zero-dimensional compact Hausdorff space and let $C(S; \mathbb{Q}_p)$ be the Banach space of all continuous functions from $S$ into $\mathbb{Q}_p$ equipped with the supremum norm. In this paper we prove a Weierstrass-Stone type theorem for subsets of $C(S; \mathbb{Q}_p)$ and apply it to the problem of simultaneous approximation and interpolation.

Let $S$ be a zero-dimensional compact Hausdorff space and let $\mathbb{Q}_p$ be the field of $p$-adic numbers. Recall that $\mathbb{Q}_p$ is the completion of the rational field $\mathbb{Q}$ with the $p$-adic absolute value: $|0|_p = 0$, and if $x \in \mathbb{Q}$, $x \neq 0$, then $|x|_p = p^{-k}$, where $k \in \mathbb{Z}$ is such that $x = p^{-k}ab$, and $a$ and $b$ cannot be divided by $p$ ($p$ is a fixed prime number). We denote $C(S; \mathbb{Q}_p)$ the vector space of all continuous functions $f$ from $S$ into $\mathbb{Q}_p$, equipped with the topology of uniform convergence given by the supremum norm

$$||f|| = \sup\{|f(x)|_p;\ x \in S\}$$

for every $f \in C(S; \mathbb{Q}_p)$. In this paper we prove a Weierstrass-Stone type theorem for subsets of $C(S; \mathbb{Q}_p)$ which generalizes the results of Dieudonné [2]. In a forthcoming paper [5] we will extend our present results to the case of any non-Archimedean absolute valued division ring ($\mathbb{K}$, $|\cdot|$).

Let us start recalling the definition of a **multiplier** of $A$, where $A \subset C(S; \mathbb{Q}_p)$. A function $\varphi \in C(S; \mathbb{Q}_p)$ is called a multiplier of $A$ if $|\varphi(x)|_p \leq 1$, for all $x \in S$, and $\varphi f + (1 - \varphi)g$ belongs to $A$ for every pair, $f$ and $g$, of elements of $A$. The set $M$ of all multipliers of $A$ contains the constant functions 0 and 1, and moreover,

(1) $\varphi \in M$ implies $1 - \varphi$ belongs to $M$;
(2) $\varphi \in M$ and $\psi \in M$ implies $\varphi\psi \in M$.

A set $M$ satisfying properties (1) and (2), and $|\varphi(x)|_p \leq 1$, for al $x \in S$ and $\varphi \in M$, is said to have **property** $V$.

We say that a subset $M \subset C(S; \mathbb{Q}_p)$ **separates the points of** $S$ if given any two distinct points $s$ and $t$ of $S$, there is a function $\varphi \in M$ such that $\varphi(s) \neq \varphi(t)$.On the other hand, we say that $M$ **strongly separates the points of** $S$ if for every ordered pair $(s, t) \in S \times S$, with $s \neq t$, there exists $\varphi \in M$ such that $\varphi(s) = 0$, $\varphi(t) = 1$, and $|\varphi(x)|_p \leq 1$, for all $x \in S$.

The following result, known as Kaplansky's Lemma, will play a fundamental rôle in what follows. (See Proposition 1 below.)

**Lemma 1.** *Let $K$ be a compact subset of $\mathbb{Q}_p$ and let $a \neq 0$ be given in $\mathbb{Q}_p$. There exists a polynomial $q$ with coefficients in $\mathbb{Q}_p$ such that $q(0) = 0$, $q(a) = 1$, and $|q(x)|_p \leq 1$ for all $x \in K$.*

**Proof.** See Lemma 1, Kaplansky [3]. □

**Proposition 1.** *If $A$ is a unitary subalgebra of $C(S;\mathbb{Q}_p)$ which is separating over $S$, then $A$ is strongly separating over $S$.*

**Proof.** Let $(s,t) \in S \times S$ be given with $s \neq t$. Since a subalgebra is a vector subspace, $A$ is a vector subspace containing the constants and therefore there is $a \in A$ such that $a(s) = 1$ and $a(t) = 0$. By continuity, the set $K = a(S)$ is a compact subset of $\mathbb{Q}_p$. By Kaplansky's Lemma, there is a polynomial $q$ such that $q(1) = 1$, $q(0) = 0$ and $|q(x)|_p \leq 1$, for all $x \in K$. The function $\varphi = p \circ a$ belongs to $A$ and satisfies $\varphi(s) = 1$, $\varphi(t) = 0$ and $|\varphi(y)|_p \leq 1$ for all $y \in S$. □

**Lemma 2.** *Let $M \subset C(S;\mathbb{Q}_p)$ be a non-empty subset with property $V$, which contains the constant function 1 and is strongly separating over $S$. Let $N$ be a clopen subset of $S$. For each $\delta > 0$, there is $\varphi \in M$ such that $\|\varphi - \xi_N\| < \delta$, where $\xi_N$ is the characteristic function of $N$, i.e., $\xi_N(t) = 1$ for all $t \in N$, and $\xi_N(t) = 0$ for all $t \notin N$.*

**Proof.** If $N = S$, the constant function $\varphi(t) = 1$, for all $t \in S$, satisfies our requirements. Assume $K = S \backslash N$ is non-empty. Fix $x \in S$, $x \notin N$. For each $t \in N$, there is $\varphi_t \in M$ such that $\varphi_t(t) = 0$, $\varphi_t(x) = 1$ and $|\varphi_t(s)|_p \leq 1$, for all $s \in S$. By continuity, there exists a neighborhood $W(t)$ of $t$ such that $|\varphi_t(s)|_p < \delta$, for all $s \in W(t)$. By compactness of $N$, there are $t_1, \ldots, t_n \in N$ such that $N \subset W(t_1) \cup \ldots \cup W(t_n)$. Let

$$\varphi_x = 1 - \varphi_{t_1} \cdot \varphi_{t_2} \cdot \ldots \cdot \varphi_{t_n}.$$

Then $\varphi_x \in M$, $\varphi_x(x) = 0$ and $|1 - \varphi_x(t)|_p < \delta$ for all $t \in N$. By continuity, there exists a neighborhood $W(x)$ of $x$ such that $|\varphi_x(t)| < \delta$ for all $t \in W(x)$. By compactness of $K$, there are $x_1, \ldots, x_m \in K$ such that $K \subset W(x_1) \cup \ldots \cup W(x_m)$. Let $\varphi = \varphi_{x_1} \cdot \varphi_{x_2} \cdot \ldots \cdot \varphi_{x_m}$. Clearly $\varphi \in M$. We claim that

$$(1) \qquad |1 - \varphi_{x_1}(t) \cdot \ldots \cdot \varphi_{x_k}(t)|_p < \delta$$

for all $t \in N$, $k = 1, 2, 3, \ldots, m$. For $k = 1$, inequality (1) is clear. Assume that (1) has been proved for $k$. Then, for each $t \in N$,

$$\begin{aligned}
&|1-\varphi_{x_1}(t)\dots\varphi_{x_{k+1}}(t)|_p = \\
= \ & |1-\varphi_{x_{k+1}}(t)+\varphi_{x_{k+1}}(t)-\varphi_{x_1}(t)\dots\varphi_{x_k}(t)\varphi_{x_{k+1}}(t)|_p \\
= \ & |1-\varphi_{x_{k+1}}(t)+\varphi_{x_{k+1}}(t)(1-\varphi_{x_1}(t)\dots\varphi_{x_k}(t))|_p \\
\leq \ & \max\{|1-\varphi_{x_{k+1}}(t)|_p, |\varphi_{x_{k+1}}(t)|_p \cdot |1-\varphi_{x_1}(t)\dots\varphi_{x_k}(t)|_p\} \\
\leq \ & \max\{|1-\varphi_{x_{k+1}}(t)|_p, |1-\varphi_{x_{k+1}}(t)\dots\varphi_{x_k}(t)|_p\} < \delta.
\end{aligned}$$

This ends the proof of our claim (1). Making $k = m$, we get $|1-\varphi(t)|_p < \delta$ for all $t \in N$. On the other hand, if $t \notin N$, then $t \in K$ and $t \in W(x_i)$ for some $i = 1, \dots, m$. Hence $|\varphi_{x_i}(t)| < \delta$, while $|\varphi_{x_j}(t)| \leq 1$ for all $j \neq i$. Hence $|\varphi(t)| < \delta$. This completes the proof that $||\varphi - \xi_N|| < \delta$. □

**Theorem 1.** *Let $W$ be a non-empty subset of $C(S; \mathbb{Q}_p)$ such that the set of all multipliers of $W$ separates strongly the points of $S$. Let $f \in C(S; \mathbb{Q}_p)$ and $\varepsilon > 0$ be given. The following are equivalent:*

(1) *there is some $g \in W$ such that $||f - g|| < \varepsilon$,*

(2) *for each $x \in S$, there is some $g_x \in W$ such that $|f(x) - g_x(x)|_p < \varepsilon$.*

**Proof.** Clearly, (1) $\Rightarrow$ (2). Conversely, assume that (2) is true. For each $x \in S$, let

$$N(x) = \{t \in S; \ |f(t) - g_x(t)|_p < \varepsilon\}.$$

Then $N(x)$ is a clopen neighborhood of $x$ in $S$. By compactness of $S$ there are $x_1, x_2, \dots, x_m$ in $S$ such that $S = N(x_1) \cup N(x_2) \cup \dots \cup N(x_m)$. Let

$$k = \max\{||f - g_{x_1}||, \ ||f - g_{x_2}||, \dots, ||f - g_{x_m}||\}.$$

Let $N_2, N_3, \dots, N_m$ be clopen subsets defined as

$$\begin{aligned}
N_2 &= N(x_2)\backslash N(x_1), \\
N_3 &= N(x_3)\backslash (N(x_1) \cup N(x_2)), \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
N_m &= N(x_m)\backslash \left(\cup_{j=1}^{m-1} N(x_j)\right).
\end{aligned}$$

Choose $\delta > 0$ so small that $\delta k < \varepsilon$. By Lemma 2, there are $\varphi_2, \varphi_3, \dots, \varphi_m \in M$ such that $||\varphi_i - \xi_i|| < \delta$, where $\xi_i$ is the characteristic function of $N_i$ ($i =$

2, 3, ..., $m$). Define $N_1 = N(x_1)$ and

$$\psi_2 = \varphi_2,$$
$$\psi_3 = (1-\varphi_2)\varphi_3,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\psi_m = (1-\varphi_2)(1-\varphi_3)\dots(1-\varphi_{m-1})\varphi_m.$$

Clearly, $\psi_i \in M$, for all $i = 2, 3, \dots, m$. Now

$$\psi_2 + \psi_3 + \dots + \psi_m = 1 - (1-\varphi_2)(1-\varphi_3)\dots(1-\varphi_m).$$

Define $\psi_1 = (1-\varphi_2)(1-\varphi_3)\dots(1-\varphi_m)$. Then $\psi_1 \in M$ and $\psi_1 + \psi_2 + \dots + \psi_m = 1$. Notice that $|\psi_i(t)|_p < \delta$ for all $t \notin N_i (i = 1, 2, \dots, m)$. This is clear for $i = 2, 3, \dots, m$, since $|\varphi_i(t)|_p < \delta$ for all $t \notin N_i$. On the other hand, if $t \notin N_1$, then $t \in N_j$ for some $j = 2, \dots, m$. Hence $|1 - \varphi_j(t)|_p < \delta$ and therefore $|\psi_1(t)| = |1-\varphi_j(t)|_p \Pi_{j \neq i} |1 - \varphi_i(t)|_p < \delta$, because $|1 - \varphi_i(t)|_p \leq 1$ for all $i \neq j$.

Let $g = \psi_1 g_1 + \psi_2 q_2 + \dots + \psi_m g_m$, where we have written $g_i = g_{x_i} (i = 1, 2, \dots, m)$. Then

$$g = \varphi_2 g_2 + (1-\varphi_2)[\varphi_3 g_3 + (1-\varphi_3)[\varphi_4 g_4 + \dots + (1-\varphi_{m-1})[\varphi_m g_m + (1-\varphi_m) g_1]\dots]].$$

Hence $g \in W$. Let $x \in S$ be given. There is exactly one integer $1 \leq j \leq m$ such that $x \in N_j$. Then

$$|\psi_j(x)|_p \cdot |f(x) - g_j(x)|_p < \varepsilon$$

because $|\psi_j(x)|_p \leq 1$ and $N_j \subset N(x_j)$. For all $i \neq j$, we have $x \notin N_i$. Hence $|\psi_i(x)|_p < \delta$ and

$$|\psi_i(x)|_p \cdot |f(x) - g_i(x)|_p \leq \delta k < \varepsilon$$

for all indices $i \neq j$. Hence

$$|f(x) - g(x)|_p = |\sum_{i=1}^{m} \psi_i(x)(f(x) - g_i(x))|_p$$
$$\leq \max_{1 \leq i \leq m} \{|\psi_i(x)|_p \cdot |f(x) - g_i(x)|_p\} < \varepsilon.$$

□

Let us recall the definition of the distance of an element $f \in C(S; \mathbb{Q}_p)$ from $W$:

$$\operatorname{dist}(f; W) = \inf\{\|f - g\| \,;\, g \in W\}.$$

**Theorem 2.** *Let $W$ be a non-empty subset of $C(S; \mathbb{Q}_p)$ such that the set $M$ of all multipliers of $W$ strongly separates the points of $S$. For each $f \in C(S; \mathbb{Q}_p)$ there exists $x \in S$ such that*

$$\operatorname{dist}(f; W) = \operatorname{dist}(f(x); W(x)).$$

**Proof.** If $\operatorname{dist}(f; W) = 0$, then $\operatorname{dist}(f(x); W(x)) = 0$ for every $x \in S$. Suppose now that $\operatorname{dist}(f; W) = d > 0$. By contradiction, assume that $\operatorname{dist}(f(x); W(x)) < d$ for every $x \in S$. Hence, for each $x \in S$, there is some $g_x \in W$ such that $|f(x) - g_x(x)|_p < d$. Consequently, $f$ and $d > 0$ satisfy condition (2) of Theorem 1. By Theorem 1, there exists $g \in W$ such that $\|f - g\| < d$, a contradiction, since $d = \operatorname{dist}(f; W)$. □

**Theorem 3.** *Let $A$ be a unitary subalgebra of $C(S; \mathbb{Q}_p)$ which is separating over $S$. Then $A$ is uniformly dense in $C(S; \mathbb{Q}_p)$.*

**Proof.** Let $W = A$. Notice that every element $\varphi \in A$, such that $|\varphi(x)|_p \leq 1$ for all $x \in S$, is a multiplier of $W$. By Proposition 1, the set $M$ of all multipliers of $W$ is strongly separating over $S$. Let now $f \in C(S; \mathbb{Q}_p)$ be given. By Theorem 2, there exists $x \in S$ such that

$$\operatorname{dist}(f; A) = \operatorname{dist}(f(x); A(x)).$$

Since $A$ contains the constants, $A(x) = \mathbb{Q}_p$. Hence $\operatorname{dist}(f(x); A(x)) = 0$, and therefore $\operatorname{dist}(f; A) = 0$. This shows that $A$ is uniformly dense in $C(S; \mathbb{Q}_p)$. □

**Corollary 1.** (Weierstrass Theorem) *Let $S$ be a non-empty compact subset of $\mathbb{Q}_p$. For every $f \in C(S; \mathbb{Q}_p)$ and every $\varepsilon > 0$, there exists a polynomial $q$ with coefficients in $\mathbb{Q}_p$ such that $|f(x) - q(x)|_p < \varepsilon$, for all $x \in S$.*

**Remark.** Theorem 3 and its Corollary 1 were proved by J. Dieudonné in 1944. (See Dieudonné [2].) In 1958, K. Mahler gave a constructive proof of Dieudonné's Weierstrass theorem (Corollary 1 above) for the case $S$ is the ring of $p$-adic integers $\{\lambda \in \mathbb{Q}_p \,;\, |\lambda|_p \leq 1\}$. (See Mahler [3].) However, Mahler's proof is based on some properties of the cyclotomic extension of $\mathbb{Q}$. In 1974, R. Bojanic presented another proof of Mahler's result, which is entirely analytic. (See Bojanic [1].)

A non-empty subset $A \subset C(S; \mathbb{Q}_p)$ is called an **interpolating family for** $C(S; \mathbb{Q}_p)$ if, for every $f \in C(S; \mathbb{Q}_p)$ and every finite subset $F \subset S$, there exists $g \in A$ such that $f(x) = g(x)$ for all $x \in F$.

**Theorem 4.** *Let $A$ be a uniformly dense linear subspace of $C(S; \mathbb{Q}_p)$. Then, for every $f \in C(S; \mathbb{Q}_p)$, every $\varepsilon > 0$ and every finite subset $F \subset S$, there exists $g \in A$ such that $\|f - g\| < \varepsilon$ and $f(x) = g(x)$ for all $x \in F$.*

**Proof.** Let $F = \{x_1, \ldots, x_n\}$. Let $\mathbb{K} = \mathbb{Q}_p$. Define a linear mapping $T : C(S; \mathbb{K}) \to \mathbb{K}^n$ by

$$Tg = (g(x_1), \ldots, g(x_n))$$

for each $g \in C(S; \mathbb{K})$. By density of $A$ and continuity of $T$, we have

$$T(C(S; \mathbb{K})) = T(\overline{A}) \subset \overline{T(A)}.$$

Now $T(A)$ is a linear subspace of $\mathbb{K}^n$ and therefore $T(A)$ is closed. Hence

$$T(C(S; \mathbb{K})) = T(A)$$

and $A$ is an interpolating family for $C(S; \mathbb{K})$. Therefore $a_1, \ldots, a_n$ can found in $A$ such that

$$a_i(x_j) = \delta_{ij} \quad , \quad 1 \leq i, j \leq n\,.$$

Choose $\delta > 0$ so that $\delta < \varepsilon$ and $\delta k < \varepsilon$, where $k = \max\{\|a_i\| \,;\, 1 \leq i \leq n\}$. By density of $A$ there is some $g_1 \in A$ such that $\|f - g_1\| < \delta$. Let

$$v_i = f(x_i) - g_1(x_i) \quad , \quad 1 \leq i \leq n\,.$$

Define $g_2 = \sum_{i=1}^{n} v_i a_i$. Then $g_2 \in A$ and $g_2(x_j) = v_j$ for all $1 \leq j \leq n$. Finally, let $g = g_1 + g_2$. Then $g \in A$ and $g(x_j) = f(x_j) \;,\; 1 \leq j \leq n$. Moreover,

$$\|f - g\| \leq \max(\|f - g_1\|, \|g_2\|) < \varepsilon,$$

since $\|f - g_1\| < \varepsilon$ and $\|g_2\| \leq \delta \max\{\|a_i\|; 1 \leq i \leq n\}$. □

**Corollary 2.** *Let $A$ be a unitary subalgebra of $C(S; \mathbb{Q}_p)$ which is separating over $S$. Then, for every $f \in C(S; \mathbb{Q}_p)$, every $\varepsilon > 0$ and every finite subset $F \subset S$, there exists $g \in A$ such that $\|f - g\| < \varepsilon$ and $f(x) = g(x)$ for all $x \in F$.*

**Proof.** By Theorem 3, $A$ is a uniformly dense linear subspace of $C(S; \mathbb{Q}_p)$. It remains to apply Theorem 4. □

**Theorem 5.** *Let $A \subset C(S; \mathbb{Q}_p)$ be an interpolating family for $C(S; \mathbb{Q}_p)$ such that the set of multipliers of $A$ strongly separates the points of $S$. Then, for every $f \in C(S; \mathbb{Q}_p)$, every $\varepsilon > 0$ and every finite subset $F \subset S$, there exists $g \in A$ such that $\|f - g\| < \varepsilon$ and $f(x) = g(x)$ for all $x \in F$.*

**Proof.** Let $W = \{g \in A \; ; \; f(x) = g(x)$ for all $x \in F\}$. Since $A$ is an interpolating family , $W \neq \emptyset$. Notice that every multiplier of $A$ is also a multiplier of $W$. Let $x \in S$ be given. Consider the finite set $F \cup \{x\}$. Since $A$ is an interpolating family for $C(S; \mathbb{Q}_p)$, there exists $g_x \in A$ such that $f(t) = g_x(t)$ for all $t \in F \cup \{x\}$. Therefore $g_x \in W$. Notice that $|f(x) - g_x(x)|_p = 0 < \varepsilon$. By Theorem 1 there exists $g \in W$ such that $\|f - g\| < \varepsilon$. Notice that $g \in W$ implies $g \in A$ and $f(x) = g(x)$ for all $x \in F$. □

**Corollary 3.** *Let $A$ be the set of all functions $g \in C(S; \mathbb{Q}_p)$ of the form*

$$g(x) = \sum_{i=1}^{n} \varphi_i(x) a_i, \quad x \in S,$$

*where $\varphi_i$ is the characteristic function of some clopen subset $K_i \subset S$; $a_i \in \mathbb{Q}_p$; $i = 1, 2, \ldots, n$, and $n \in \mathbb{N}$. Given any $f \in C(S; \mathbb{Q}_p)$, any $\varepsilon > 0$ and any finite subset $F \subset S$, there exists $g \in A$ such that $\|f - g\| < \varepsilon$ and $f(x) = g(x)$ for all $x \in F$.*

**Proof.** Clearly, $A$ is an interpolating family for $C(S; \mathbb{Q}_p)$, admitting all characteristic functions of clopen subsets of $S$ as multipliers. It remains to apply Theorem 5. Or else reason as follows: $A$ is a unitary subalgebra which is separating over $S$ and apply Corollary 2. □

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