ON THE MULTIPLICATIVE GENERATORS OF SEMI-FREE CIRCLE ACTIONS

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RELATÓRIO TÉCNICO Nº 48/91

Abstract. This paper deals with the bordism groups of manifolds with semi-free S^1-actions. These groups are denoted by $SF_n(S^1)$. We study the multiplicative structure by using a J-homomorphism map. We also study the construction K, which gives a set of multiplicative generators, and we present an algebraic interpretation of this geometric construction. Finally, as an application, we analyze the homomorphisms $r_p : SF_*(S^1) \to SF_*(\mathbb{Z}_p)$ from the bordism group of semi-free S^1-actions on the bordism group of $\mathbb{Z}_p$-actions induced by the restriction functors.

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Setembro – 1991

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1. Introduction

This paper deals with the bordism groups of manifolds with semi-free S^1-actions. These groups are denoted by $SF_n(S^1)$. We study the multiplicative structure by using a J-homomorphism map. We also study the construction K, which gives a set of multiplicative generators, and we present an algebraic interpretation of this geometric construction. Finally, as an application, we analyze the homomorphisms $r_p : SF_*(S^1) \to SF_*(\mathbb{Z}_p)$ from the bordism group of semi-free S^1-actions on the bordism group of $\mathbb{Z}_p$-actions induced by the restriction functors.

2. Semi-free S^1-actions.

Let $SF_n(S^1)$ be the bordism group of semi-free S^1-actions on the n-dimensional closed manifolds, and let $SF_*(S^1) = \oplus_{n\geq 0}\ SF_n(S^1)$ be the N_*-module obtained. Analogously, we have the bordism groups $N_n(S^1)$ of free S^1-actions on n-dimensional closed manifolds and $N_*(S^1) = \oplus_{n\geq 0} N_n(S^1)$.

There is the Smith homomorphism

$$\Delta : N_*(S^1) \to N_*(S^1)$$

which is a N_*-module homomorphism with degree -2. (see [3]).

Proposition 2.01. If $\Delta[M^{2n+1},\ T] = 0$ for $[M^{2n+1},\ T]$ in $N_{2n+1}(S^1)$, then there is an unique $[X^{2n}]$ in N_{2n} such that $[M^{2n+1},\ T] = [X^{2n}]\ [S^1,\ S^1]$.

Proposition 2.02. If $[M^{2n+1}, T]$ is a free S^1-action on the closed $(2n+1)$-manifold M^{2n+1} and if $W^{2n+1} \subset M^{2n+1}$ is a regular compact submanifold such that $W^{2n+1} \cup T(W^{2n+1}) = M^{2n+1}$ and $\partial W^{2n+1} = W^{2n+1} \cap T(W^{2n+1})$ then $\Delta[M^{2n+1}, T) = [\partial W^{2n+1}, T|\partial W^{2n+1}]$.

Let $SF_*(S^1) \otimes_{N_*} N_*(S^1) \to N_*(S^1)$ be the pairing given by: if $[V^m, \tau] \in SF_m(S^1)$ and $[M^n, T] \in N_n(S^1)$ then $[V^m \times M^n, \tau \times T]$ is a free S^1-action and we take $[V^m, \tau]\,[M^n, T] = [V^m \times M^n, \tau \times T]$ to define the pairing.

Lemma 2.03 If $\gamma \in SF_{2m}(S^1)$ and $\alpha \in N_{2n+1}(S^1)$ then $\Delta(\gamma\alpha) = \gamma\Delta(\alpha)$.
Proof. Let $[M^{2n+1}, T]$ be a semi-free S^1-action on a closed $(2n+1)-$manifold. So, by (2.02), if $W^{2n+1} \subset M^{2n+1}$ is a compact regular submanifold such that $W \cup TW = M$ and $W \cap TW = \partial W$ then $\Delta[M^{2n+1}, T] = [\partial W^{2n+1}, T|_{\partial W^{2n+1}}]$. Thus, since $V^{2m} \times W^{2n+1} \subset V^{2m} \times M^{2n+1}$, we have $(V \times W) \cup T'(V \times W) = (V \times W) \cup (\tau V \times TW) = (V \times W) \cup (V \times TW) = V \times (W \cup TW) = V \times M$, where $T' = (\tau, T)$ and $\partial(V \times W) = (\partial W \times W) \cup (V \times \partial W) = V \times \partial W = V \times (W \cap TW) = (V \times W) \cap (V \times TW) = (V \times W) \cap (\tau V \times TW)$. Therefore, we have $\Delta[V \times M, \tau \times T] = [\partial(V \times W), \tau \times T|_{\partial(V \times W)}] = [V \times \partial W, \tau \times T|_{V \times \partial W}] = [V, \tau]\,[\partial W, T|\partial W]$, i. e., $\Delta(\delta\alpha) = \gamma\Delta(\alpha)$. □

Now let $\mathcal{B}_n(S^1)$ be the bordism group of semi-free S^1- actions on n-manifolds with boundary wich are free on the boundary. We have the N_*-module homomorphisms $j : SF_n(S^1) \to \mathcal{B}_n(S^1)$ which assigns $[M^n, T]$ to $[M^n, T]$, since $\partial M^n = \emptyset$; and $\partial : \mathcal{B}_n(S^1) \to N_n(S^1)$ which associates $[V^n, T]$ to $[\partial V^n, T|_{\partial V^n}]$.

Theorem 2.04. The sequence

$$0 \to SF_n(S^1) \xrightarrow{j} \mathcal{B}_n(S^1) \to N_{n-1}(S^1) \to 0$$

is exact and it splits. (see [5], 8.3). □

Let $M_n(S^1) = \oplus_{k\geq 0}^{[n/2]} N_{n-2k}(BU_k)$ be the classifying space for the k-dimensional bundles. We denote $M_*(S^1) = \oplus_{n\geq 0} M_n(S^1)$.

Theorem 2.05. There is the isomorphism

$$F : \mathcal{B}_n(S^1) \to \oplus_{k\geq 0}^{[n/2]} N_{n-2k}(BU_k)$$

given by: $[M^n, T]$ is associated to $\sum_k [\nu^k \to F^{n-2k}]$ where F^{n-2k} is the fixed point set component and ν^k is the normal boundle corresponding. $\square$

Theorem 2.06. $M_*(S^1)$ is a graded polynomial algebra on N_* with generators the classes $[\lambda \to CP(n)]$, $n \geq 0$, i. e., the canonic line bundles over $CP(n)$ $\square$

Theorem 2.07. $N_*(S^1)$ is a N_* module with base given by the elements $\{[S^{2n+1}, S^1]\}_{n\geq 0}$. $\square$

Remark. Any element in $N_{2n+1}(S^1)$ can be written in an unique way as $\sum_{r=0}^{n}[X^{2r}][S^{2(n-r)+1}, S^1]$, and any element in $N_{2n}(S^1)$ can be written in an unique way as $\sum_{r=0}^{n-1}[X^{2r+1}]\,[S^{2(n-r)-1}, S^1]$.

Let i in $M_2(S^1)$ be a semi-free S^1-action on the unitary disk given by the scalar product, then $\partial(i) = [S^1, S^1]$ and we have the following lemma.

Lemma 2.08. The diagram

$$\begin{array}{ccc} M_{2k+2}(S^1) & \xrightarrow{\partial} & N_{2k+1}(S^1) \\ i\uparrow & & \uparrow \\ M_{2k}(S^1) & \xrightarrow{\partial} & N_{2k-1}(S^1) \end{array}$$

is commutative. $\square$

3. Products in $\oplus_{n\geq 0} N_{2n+1}(S^1)$.

We define a product in $\oplus_{n\geq 0} N_{2n+1}(S^1)$ the following way: suppose that $\alpha \in N_{2n+1}(S^1)$ and $\beta \in N_{2m+1}(S^1)$. Let α' be in $M_{2n+2}(S^1)$ and $\beta' \in M_{2m+2}(S^1)$ such that $\partial(\alpha') = \alpha$ and $\partial(\beta') = \beta$. Let be $K = \max(m, n)$ and $k = \min(m, n)$. We define

$$\alpha\beta = \Delta^{K+1}(\alpha'\beta') \in N_{2k+1}(S^1).$$

In case $n = m$, this define a ring structure with unity on $N_{2n+1}(S^1)$, where the unity element is $[S^{2n+1}, S^1]$.

Lemma 3.01. If $n < m$ then $\alpha\Delta\beta = \alpha\beta$ and if $n \geq m$ then $\Delta(\alpha\beta) = \alpha\Delta\beta$.

Proof. First consider $n < m$ and select $\beta'' \in M_{2m}(S^1)$ with $\partial(\beta') = \Delta\beta$. Thus, $\Delta(\partial(\beta''i) + \beta) = 0$. So, by (2.01), there is an unique $[V^{2m}]$ in N_{2m} such that $\partial(\beta''i) + \beta = [V^{2m}]\ [S^1,\ S^1]$. Then, $\partial(\beta''i) + [V^{2m}]\ [S^1,\ S^1] = \beta$. Since $\partial(i) = [S^1,\ S^1]$, we have $\partial(\beta''i + [V^{2m}]i) = \beta$. Thus, $\alpha\beta = \Delta^{m+1}\partial(\alpha'\beta''i + [V^{2m}]\alpha'i) = \Delta^{m+1}\partial(\alpha'\beta''i) + \Delta^{m+1}\partial([V^{2m}]\alpha'i) = \Delta^m\partial(\alpha'\beta'') + [V^{2m}]\Delta^m\partial(\alpha') = \alpha\Delta\beta + [V^{2m}]\Delta^m\alpha$. Now, since $n < m$ and $\alpha \in N_{2n+1}(S^1)$, we have that $\Delta^m\alpha = 0$.

Next, in case $n \geq m$, we have $\partial(\beta'') = \Delta\beta$. Then $\Delta(\partial(\beta''i + \beta)) = 0$. Therefore, there is $[V^{2m}] \in N_{2m}$ with $\partial(\beta''i) + [V^{2m}]\ [S^1,\ S^1] = \beta$. Since $\partial(i) = [S^1,\ S^1]$, we obtain $\partial(\beta''i + [V^{2m}]i) = \beta$. Thus, $\Delta(\alpha\beta) = \Delta^{n+2}\partial(\alpha'\beta''i + [V^{2m}]\alpha'i) = \Delta^{n+1}\partial(\alpha'\beta'' + [V^{2m}]\alpha') = \Delta^{n+1}\partial(\alpha'\beta'') + [V^{2m}]\Delta^{n+1}\partial(\alpha') = \alpha\Delta\beta + [V^{2m}]\Delta^{n+1}\alpha$. Since $n \geq m$ and $\alpha \in N_{2n+1}(S^1)$, we have that $\Delta^{n+1}\alpha = 0$. Therefore, $\Delta(\alpha\beta) = \alpha\Delta\beta$. □

Lemma 3.02. For any pair α, β we have $\Delta(\alpha\beta) = \Delta\alpha\Delta\beta$.
proof. Suppose that $n < m$. Then $\alpha\beta = \alpha\Delta\beta$ by (3.01). Thus, $m-1 \geq n$ and $\Delta(\alpha\Delta\beta) = \Delta\alpha\Delta\beta$. Therefore, $\Delta(\alpha\beta) = \Delta\alpha\Delta\beta$. Finally, if $n = m$, $\Delta(\alpha\beta) = \alpha\Delta\beta$ and $\Delta\alpha\Delta\beta = \alpha\Delta\beta$. Hence, $\Delta(\alpha\beta) = \Delta\alpha\Delta\beta$. □

Lemma 3.03. If α, β and $\delta \in N_{2n+1}(S^1)$ then $\alpha(\beta\delta) = (\alpha\beta)\delta$.
proof. Choose α', β' and $\delta' \in M_{2n+2}(S^1)$ such that $\partial(\alpha') = \alpha$, $\partial(\beta') = \beta$ and $\partial(\delta') = \delta$. Then $\alpha(\beta\delta) = \alpha\Delta^{n+1}\partial(\beta'\delta') = \alpha\Delta(\Delta^n\partial(\beta'\delta')) = \alpha\Delta^n\partial(\beta'\delta')$. (by (3.01)). Thus, successively, we obtain $\alpha(\beta\delta) = \alpha\partial(\beta'\delta')$. So, by definition, $\alpha\partial(\beta'\delta') = \Delta^{2n+1+1}\partial(\alpha'(\beta'\delta')) = \Delta^{2n+2}\Delta((\alpha'\beta')\delta') = \partial(\alpha'\beta')\delta = \Delta^{n+1}\partial(\alpha'\beta')\delta = (\alpha\beta)\delta$. □

Theorem 3.04. The product $\alpha\beta$ defines a ring structure on $\oplus_{n\geq 0} N_{2n+1}(S^1)$.
proof. We must show that the associative law is true in general for $\alpha \in N_{2n+1}(S^1)$, $\beta \in N_{2m+1}(S^1)$ and $\delta \in N_{2p+1}(S^1)$. We can suppose $m \leq p$, then, by (3.01), we have $\alpha(\beta\delta) = \alpha(\beta\Delta^{p-m}\delta)$, since $\beta\Delta^{p-m}\delta = \beta\Delta(\Delta^{p-m-1}\delta) = \beta\Delta^{p-m-1}\delta = \beta\Delta(\Delta^{p-m-2}\delta) = \beta\Delta^{p-m-2}\delta = \ldots = \beta\delta$.
Case I. If $n \leq m$, then $\alpha(\beta\delta) = \alpha\Delta^{m-n}(\beta\delta) = \alpha\Delta^{m-n}(\beta\Delta^{p-m}\delta) = \alpha(\Delta^{m-n}\beta\Delta^{m-n}\Delta^{p-m}\delta) = \alpha(\Delta^{m-n}\beta\Delta^{p-n}\delta) = (\alpha\Delta^{m-n}\beta)\Delta^{p-n}\delta = (\alpha\beta)\delta$.
Case II. If $n > m$, then $\alpha(\beta\delta) = \Delta\alpha(\beta\delta) = (\Delta^{n-m}\alpha)(\beta\delta) = (\Delta^{n-m}\alpha)(\beta\Delta^{p-m}\delta) = (\Delta^{n-m}\alpha\beta) = \Delta^{p-n}\delta = (\alpha\beta)\delta$. □

Remark. For $\alpha \in N_{2n+1}(S^1)$, since $\partial(i^{m+1}) = [S^{2m+1}, S^1]$ and $\Delta^{m+1}\partial(\alpha' i^{m+1}) = \partial(\alpha') = \alpha$, we have that $\alpha[S^{2m+1}, S^1] = \alpha$ for $n \leq m$. If $n > m$, we have $\alpha[S^{2m+1}, S^1] = \Delta^{n+1}\partial(\alpha' i^{m+1}) = \Delta^{n-m}\alpha$. In particular, $N_{2n+1}(S^1)$ is a subring of $\oplus_{n\geq 0} N_{2n+1}(S^1)$ with unity $[S^{2n+1}, S^1]$, and $\Delta : N_{2n+1}(S^1) \to N_{2n-1}(S^1)$ is a ring homomorphism.

4. The stable bordism homomorphism.

We are going to introduce the ring $\mathcal{R} = \xrightarrow{invlim} (N_{2n-1}(S^1) \xleftarrow{\Delta} N_{2n+1}(S^1))$. An element in $\mathcal{R}$ is a sequence $\{\alpha_n\}_0^\infty$ with $\alpha_n \in N_{2n+1}(S^1)$ and $\Delta(\alpha_n) = \alpha_{n-1}$, $\forall n \geq 1$.

Next, we define the homomorphism $\overline{J} : M_*(S^1) \to \mathcal{R}$, where $M_*(S^1) = \oplus_{k\geq 0}\oplus_{j=0}^{k} N_{2k-2j}(BU_j)$, in the following way: Let be $A \in M_{2k}(S^1)$, then $\overline{J}_n(A) = \Delta^k\partial(Ai^{n+1})$. Observe that $\overline{J}_n(A)$ belongs to $N_{2n+1}(S^1)$. The sequence $\{\overline{J}_n(A)\}_0^\infty$ belongs to $\mathcal{R}$, since $\Delta\overline{J}_n(A) = \Delta^{k+1}\partial(Ai^{n+1}) = \Delta^k\partial(Ai^n) = \overline{J}_{n-1}(A)$. Finally, we define $\overline{J}(A) = \{\overline{J}_n(A)\}_0^\infty$. Note that $\overline{J}_n(A) = \Delta^{k-n-1}\partial(A)$, if $k \geq n+1$.

The homomorphism $\overline{J}$ is stable with relation the multiplication by i, i. e., $\overline{J}(Ai) = \overline{J}(A)$.

Theorem 4.01. The stable homomorphism

$\overline{J} : \oplus_{k\geq 0} \oplus_{j=0}^{k} N_{2k-2j}(BU_j) \to \mathcal{R}$ is multiplicative.

proof. One needs only to consider pairs A, B in $M_{2k}(S^1)$ and components $0 \leq n < k$, since $\overline{J}$ is stable. In this case, $\overline{J}_n(A)\overline{J}_n(B) = \Delta^{k-n-1}\partial(A)\Delta^{k-n-1}\partial(B) = \Delta^{k-n-1}(\partial(A)\partial(B))$. By definition, $\partial(A)\partial(B) = \Delta^k\partial(AB) = \Delta^{2k}\partial(ABi^k) = \Delta^{2k}\partial(ABi^{n-1+1}) = \overline{J}_{k-1}(AB)$. Therefore, $\overline{J}_n(A)\overline{J}_n(B) = \Delta^{k-n-1}(\partial(A)\partial(B)) = \Delta^{k-n-1}(\overline{J}_{k-1}(AB)) = \overline{J}_n(AB)$. □

Taking the ring $M_*(S^1) = \oplus_{k\geq 0}\oplus_{j=0}^{k} N_{2k-2j}(BU_j)$, we are going to consider the quocient ring F obtained factoring the ideal constituted by the elements $A + A_i$, where A belongs to $M_*(S^1)$. There is a natural induced ring homomorphism $\overline{J} : F \to \mathcal{R}$.

5. The structure of $\mathcal{R}$.

Since $N_*(S^1)$ is a N_*-module with base given by $\{[S^{2n+1}, S^1]\}_{n\geq 0}$, we can describe the product in $N_{2n+1}(S^1)$ directly. If $\alpha = \sum_{r=0}^{n} [X^{2r}]\,[S^{2(n-r)+1}, S^1]$ and $\beta = \sum_{r=0}^{n} [Y^{2r}]\,[S^{2(n-r)+1}, S^1]$, consider $\alpha' = \sum_{r=0}^{n} [X^{2r}]\,i^{n-r+1}$, and $\beta' = \sum_{r=0}^{n} [Y^{2r}]\,i^{n-r+1}$, so

$$\alpha'\beta' = \sum_{r=0}^{n} \Big(\sum_{r+s=l} [X^{2r}]\,[Y^{2s}]\Big) i^{2(n+1)-l},$$

and

$$\alpha\beta = \Delta^{n+1}\partial(\alpha'\beta') = \sum_{r=0}^{n} \Big(\sum_{r+s=l} [X^{2r}]\,[Y^{2s}]\Big)[S^{2n+1-l}, S^1].$$

We can use this to identify $\mathcal{R}$ with the ring $N_*^{(2n)}(\theta)$ of formal power series on the graduated ring $N_*^{(2n)}$. We assign $\{\alpha_n\}_0^\infty$ to $\sum_0^\infty [X^{2r}]\theta^r$, where $\alpha_n = \sum_0^n [X^{2r}]\,[S^{2(n-r)+1}, S^1]$.

We have $\Delta\alpha_{n+1} = \alpha_n$, and the product $\alpha_n\beta_n$ is the rule to multiply formal power series. Thus, we have $\overline{J} : F \to N(\theta)$.

Lemma 5.01. The image of

$$SF_{2m}(S^1) \to M_{2m}(S^1) \to F \to N(\theta)$$

is the ideal generated by θ^m, and $\overline{J}([M^{2m}, T]) = [M^{2m}]\theta^m +$ terms with power bigger than m.

proof. Let $[M^{2m}, T]$ be given, consider the normal bundle to the fixed point set $[\xi \to F]$ in $M_{2m}(S^1)$. So, $\partial([\xi \to F]) = 0$ in $N_{2m-1}(S^1)$, and $\overline{J}_n([\xi \to F]) = \Delta^{m-n-1}\partial([\xi \to F]) = 0$ for $0 \le n \le m-1$. Thus, $\overline{J}_m([\xi \to F]) = \Delta^m \partial([\xi \to F] i^{m+1})$ in $N_{2m+1}(S^1)$. Consequently, $\overline{J}_m([\xi \to F]) = \partial([\xi \to F]i) = [S(\xi \oplus 1_C), S^1]$. Since $\Delta([S(\xi \oplus 1_C), S^1]) = 0$, then, by (2.01), there is an unique $[X^{2m}]$ such that $[S(\xi \oplus 1_C), S^1] = [X^{2m}]\,[S^1, S^1]$. Next, since $[CP(\xi \oplus 1_C)] = [M^{2m}]$, we have that $[S(\xi \oplus 1_C), S^1] = [M^{2m}]\,[S^1, S^1]$. Thus $\overline{J}([\xi \to F]) = [M^{2m}]\theta^m +$ terms with power bigger than m. □

Lemma 5.02. Let $[\xi \to F]$ be in $M_{2m}(S^1)$. If $\overline{J}([\xi \to F]) = \beta\theta^m +$ terms with power bigger than m, then there is a manifold with semi-free S^1-action

$[M^{2m},\ T]$ such that β is in the class of M^{2m} and $[\xi \to F]$ is the normal bundle to the fixed point set of $[M^{2m},\ T]$.
proof. We have $0 = \overline{J}_{m-1}([\xi \to F]) = \Delta^{m-m+1-1}\partial([\xi \to F]) = \partial([\xi \to F])$. On the other hand, $\partial([\xi \to F]) = [S(\xi),\ S^1]$. Therefore, $[S(\xi),\ S^1] = 0$. Now, suppose that $\partial[V^{2m}, \tau] = [S(\xi),\ S^1]$, where $[V^{2m}, \tau]$ is in $N_{2m}(S^1)$. Next, consider $M^{2m} = (D(\xi) \cup V^{2m})/S(\xi) \equiv \partial V^{2m}$ and $T = S \cup \tau$. The normal bundle to the fixed point set of $[M^{2m},\ T]$ is $\xi \to F$, hence $\beta = [M^{2m}]$. □

Lemma 5.03. Let 1_C be in $M_2(S^1)$. Then $\overline{J}(1_C) = 1$.
proof. Since $\overline{J}_n(1_C) = \Delta\partial(1_C i^n) = \Delta\partial(i^{n+2} = \Delta[S^{2n+3},\ S^1] = [S^{2n+1},\ S^1] \in N_{2n+1}(S^1)$, $\forall n \geq 0$, we have that $\overline{J}(1_C) = 1$. □

Lemma 5.04. Let $[V^{2s}]$ be in N_{2s} and $A \in M_{2k}$. Then $\overline{J}([V^{2s}]A) = [V^{2s}]\theta^s\overline{J}(A)$.
proof. We have $\overline{J}_n([V^{2s}]A) = \Delta^{s+k}\partial([V^{2s}])Ai^{n+1}) = \Delta^{s+k}[V^{2s}]\partial(Ai^{n+1}) = [V^{2s}]\Delta^{s+k}\partial(Ai^{n+1})$. Since $\Delta^{s+k}\partial(Ai^{n+1})$ is in $N_{2n+1-2s}(S^1)$, and $2(n-s)+1$ is odd, we have $\Delta^{s+k}\partial(Ai^{n+1}) = \sum_{r=0}^{n-s}[X^{2r}][S^{2(n-s-r)+1},\ S^1]$. Therefore,

$$\overline{J}_n([V^{2s}]A) = [V^{2s}]\sum_{r=0}^{n-s}[X^{2r}][S^{2(n-s-r)+1},\ S^1] = \sum_{r=0}^{n-s}[X^{2r}][V^{2s}][S^{2(n-s-r)+1},\ S^1].$$

Thus, $\overline{J}_n([V^{2s}]A) = \sum_{r=0}^{\infty}[X^{2r}][V^{2s}]\theta^{s+r} = [V^{2s}]\theta^s\sum_{r=0}^{\infty}[X^{2r}]\theta^r = [V^{2s}]\theta^s\overline{J}(A)$. □

Now, we are going to define the operator $\mathcal{K} : SF_*(S^1) \to SF_*(S^1)$. Let $[M^n,\ T]$ be a semi-free S^1-action. Consider $D^2 \times M^n$ and the actions T_1 and T_2 defined by:

$$T_1 : (t,\ (z,\ m)) \mapsto (tz,\ m), \quad \text{and}$$
$$T_2 : (t,\ (z,\ m)) \mapsto (tz,\ T(t,\ m)).$$

Restricting T_1 and T_2 to $S^1 \times M^n$, we get the induced actions $[S^1 \times M^n, T_1]$ and $[S^1 \times M^n, T_2]$. There exists a diffeomorphism $\varphi : [S^1 \times M^n,\ T_1] \to [S^1 \times M^n,\ T_2]$ given by $(s,\ x) \mapsto (s,\ T(s,\ x))$. Thus, the action $[S^1 \times M^n, T_1]$ is equivariantly diffeomorphic to $[S^1 \times M^n, T_2]$. Taking the disjoint union $[D^2 \times M^n, T_1] \cup [D^2 \times M^n, T_2]$, we can get the closed manifold M^{n+2} and the S^1-action τ_1 on M^{n+2}, using the identification of $[S^1 \times M^n, T_1]$ with $[S^1 \times M^n, T_2]$

through φ. Finally, we define

$$\mathcal{K}[M^n, T] = [M^{n+2}, \tau_1].$$

The fixed point set of τ_1 is $F_1 = \text{Fix}\,(T) \cup M^n$, where Fix (T) is the fixed point set of T and the normal bundle to Fix (T) is $\eta \oplus 1_C \to \text{Fix}\,(T)$, where η is the normal bundle to the Fix (T) on M^n. Moreover, the normal bundle to M^n is the trivial complex bundle 1_C.

Using an inductive process, one can assign to $[M^n, T]$ a sequence of semi-free S^1-actions $[V(n, k), \tau_k]$. To do this, consider $[V(n, 0), \tau_0] = [M^n, T]$ and $[V(n, 1), \tau_1] = \mathcal{K}[M^n, T]$. Now, we get $[V(n, 2), \tau_2]$ applying the above construction to $[V(n, 1), \tau_1]$. Thus, applying this construction, successively, k times, we get $[V(n, k), \tau_k]$, where the fixed point set is $F_k = \text{Fix}\,(T) \cup (\cup_0^{k-1} V(n, j))$. Futhermore, the normal bundle η_k to F_k is $\eta \oplus 1_C^k$, where η is the normal bundle to the Fix (T) on M^n, and $1_C^k = \oplus_0^{k-1} 1_C^{k-j}$, with 1_C^{k-j} the trivial complex bundle over $V(n, j)$.

Lemma 5.05. If η is the normal bundle to the fixed point set of $[M^n, T]$, then $[V(n, k)] = [CP(\eta \oplus 1_C^{k+1})] + \sum_{j=0}^{k-1} [CP(k-j)][V(n, j)]$. □

Lemma 5.06. Let λ_n be the canonic complex line bundle over $CP(n)$. Then

$$\overline{J}(\lambda_n) = 1 + \sum_{i=0}^{\infty} [V(n+1, i)]\theta^{n+i+1}.$$

proof. Let T_0 be a semi free S^1-action on $CP(n+1)$ defined by T_0: $[z_0, z_1, \ldots, z_n] \mapsto [sz_0, z_1, \ldots, z_n]$, where s belongs to S^1. The normal bundle to the fixed point set is $\lambda \oplus 1_C^{(n+1)}$, where λ is the canonic complex line bundle to $CP(n)$ and $1_C^{(n+1)}$ is the trivial complex bundle to $CP(0)$. Now, we are going to consider the manifolds with semi free S^1-action $[V(n+1, k), \tau_k]$, where $[V(n+1, 0), \tau_0] = [CP(n+1), T_0]$. The fixed point set of $[V(n+1, k), \tau_k]$ is $[\nu \oplus 1_C^k \to F] + [1_C^k \to V(n+1, 0)] + [1_C^{k-1} \to V(n+1, 1)] + \ldots + [1_C \to V(n+1, k-1)]$, where $\nu \to F$ is the bundle $(1_C^{(n+1)} \to CP(0)) \cup (\lambda \to CP(n))$.

Taking $k = 2$, we see that the fixed point set of $[V(n+1, 2), \tau_2]$ is $[\nu \oplus 1_C^2 \to F] + [1_C^2 \to V(n+1, 0)] + [1_C \to V(n+1, 1)]$. Since $\overline{J}[V(n+1, 2), \tau_2] = [V(n+1, 2)]\theta^{n+3} +$ terms with power bigger than $n+3$, and $\overline{J}[V(n+1, 2), \tau_2] = \overline{J}[\nu \oplus 1_C^2 \to F] + \overline{J}[1_C^2 \to V(n+1, 0)] + \overline{J}[1_C \to$

$V(n+1,\ 1)]$, we get $[V(n+1,\ 2)]\theta^{n+2}+$ terms with power bigger than $n+2 = \overline{J}[\nu \to F] + [V(n+1,\ 0)]\theta^{n+1} + [V(n+1,\ 1)]\theta^{n+2}$. Therefore, $\overline{J}[\nu \to F] = [V(n+1,0)]\theta^{n+1} + [V(n+1,1)]\theta^{n+2} + [V(n+1,\ 2)]\theta^{n+3}+$ terms with power bigger than $n+3$.

On the other hand, $\overline{J}[\nu \to F] = \overline{J}[1_C^{n+1} \to CP(0)] + \overline{J}(\lambda_n)$. Hence, $\overline{J}(\lambda_n) = 1 + [V(n+1,\ 0)]\theta^{n+1} + [V(n+1,\ 1)]\theta^{n+2} + [V(n+1,\ 2)]\theta^{n+3}+$ terms with power bigger than $n+3$. Finally, in general we have the result, that is, $\overline{J}(\lambda_n) = 1 + \sum_{i=0}^{\infty}[V(n+1,\ i)]\theta^{n+i+1}$. □

6. Algebraic interpretation of the geometric construction $\mathcal{K}$.

As in [1], in this section we are going to express $SF_*(S^1)$ as a direct sum of certain submodules.

Let $[CP(n),\ T_0]$ be in $SF_{2n}(S^1)$, where $T_0 : [z_0,\ldots,z_n] \mapsto [sz_0,\ z_1,\ldots,z_n]$, $s \in S^1$. The fixed point set of this semi free S^1-action is $(1_C^n \to CP(0)) \cup (\lambda \to CP(n-1))$. Therefore, one can write $[CP(n),\ T_0] = \alpha_n + \alpha_1^n$, where $\alpha_n = [CP(n-1), \lambda]$. Thus, $\alpha_n = [CP(n),\ T_0] + \alpha_1^n$. Since $M_*(S^1)$ is a polynomial algebra over N_* generated by the elements α_n in $M_{2n}(S^1)$, for $n \geq 1$, we have that the elements $[CP(n), T_0]$, for $n \geq 2$, together $\alpha_1 = i$ in $M_2(S^1)$, constitute another system of polynomial generators for $M_*(S^1)$. We are going to denote by Q the polynomial subalgebra of $SF_*(S^1)$ generated by $[CP(n),\ T_0]$; and by Q_{2m} the intersection $Q \cap SF_{2m}(S^1)$. Thus, $M_*(S^1) = Q[i]$. By (5.01), $SF_{2m}(S^1)$ consist of polynomials P in i over Q for which $\overline{J}(P)$ belongs to the ideal generated by θ^m.

Lemma 6.01. Given an element y in $M_{2m}(S^1)$, there is an unique polynomial F in $N_*[i]$, with no constant terms, such that $y + F$ belongs to $SF_{2m}(S^1)$.

proof. Since $M_*(S^1) = Q[i]$, we have that $y = q_0 + q_1 i + q_2 i^2 + \ldots + q_{m-1} i^{m-1} + q_m i^m$, where q_j is in $Q_{2(m-j)}$. Thus, since $\overline{J}(i) = 1$ and $\overline{J}$ is stable, we have that $\overline{J}(y) = \overline{J}(q_0) + \overline{J}(q_1 i) + \overline{J}(q_2 i^2) + \ldots + \overline{J}(q_{m-1} i^{m-1}) + \overline{J}(q_m i^m) = \overline{J}(q_0) + \overline{J}(q_1) + \ldots + \overline{J}(q_{m-1}) + \overline{J}(q_m)$. Next, since q_j belongs to $Q_{2(m-j)} = Q \cap SF_{2m}(S^1)$, then $\overline{J}(q_j) = [W_{2j}^{2(m-j)}]\theta^{m-j} + [W_{2j}^{2(m-j)+2}]\theta^{m-j+1} + \ldots + [W_{2j}^{2(m-1)}]\theta^{m-1} + [W_{2j}^{2m}]\theta^m +$ terms with power bigger than m, where $[W_{2j}^{2(m-j)}]$ is in $N_{2(m-j)}$. Thus, $\overline{J}(y) = [W_{2m}^0] + ([W_{2m}^2] + [W_{2(m-1)}^2])\theta + \cdots + (\sum_{k=0}^{j}[W_{2(m-k)}^{2j}])\theta^j + \ldots + (\sum_{k=0}^{m-1}[W_{2(m-k)}^{2(m-1)}])\theta^{m-1} +$

$(\sum_{k=0}^{m}[W^{2m}_{2(m-k)}])\theta^k+$ terms with power bigger than m. Let $F = [W^0_{2m}]i^m + ([W^2_{2m}] + [W^2_{2(m-1)}])i^{m-1} + \ldots + (\sum_{k=0}^{j}[W^{2j}_{2(m-k)}])i^{m-j} + \ldots + (\sum_{k=0}^{m-1}[W^{2(m-1)}_{2(m-k)}])i$ be in $N_*[i]$. Then, we have $\overline{J}(y+F) = (\sum_{k=0}^{m}[W^{2m}_{2(m-k)}])\theta^m+$ terms with power bigger than m, i. e., $\overline{J}(y+F)$ is in the ideal generated by θ^m. Therefore, by (5.01), $y+F$ belongs to $SF_{2m}(S^1)$. □

Let $\varepsilon : Q \to N_*$ be the aumentation homomorphism. Thus, we have the following lemma.

Lemma 6.02. Let q_j be in $SF_{2(m-j)}(S^1)$, for $0 \leq j \leq m$. Then $\mathcal{K}q_j$ is in $SF_{2(m-j+1)}(S^1)$, where we denote by $\mathcal{K}q_j$ the element $i(q_j + [W^{2(m-j)}_j])$, and $[W^{2(m-j)}_j] = \varepsilon(q_j)$

proof. Since i_{q_j} is in $M_{2(m-j+1)}(S^1)$, there is an unique element F in $N_*[i]$, with no constant term, such that $iq_j + F$ belongs to $S_{2(m-j+1)}(S^1)$.

Now, we are going to verify that $F = [W^{2(m-j)}_{2j}]\theta^{m-j+1}+$ terms with power bigger than $m-j+1$. We know that $\overline{J}(q_j) = [W^{2(m-j)}_{2j}]\theta^{m-j} + [W^{2(m-j+1)}_{2j}]\theta^{m-j+1}+$ terms with power bigger than $m-j+1$. Thus, $\overline{J}(iq_j + [W^{2(m-j)}_{2j}]i) = [W^{2(m-j+1)}_{2j}]\theta^{m-j+1}+$ terms with power bigger than $m-j+1$. Therefore, $\overline{J}(\mathcal{K}q_j)$ is in the ideal generated by θ^{m-j+1}, and this fact imply that $\mathcal{K}q_j$ belongs to $SF_{2(m-j+1)}(S^1)$. □

Remark 6.03. Denoting by $\mathcal{K}^2q_j$ the element $i\mathcal{K}q_j$, by (6.02), we have that $\mathcal{K}^2q_j$ is in $SF_{2(m-j+2)}(S^1)$. Thus, successively, $\mathcal{K}^nq_j = i\mathcal{K}^{n-1}q_j$ belongs to $SF_{2(m-j+n)}(S^1)$, and $\mathcal{K}^nq_j = i^nq_j + [W^{2(m-j)}_{2j}]i^n$.

Lemma 6.04. Let q_j be in $SF_{2(m-j)}(S^1) \cap Q$, $j = 0, \ldots, m$. If $y = q_0 + q_1i + q_2i^2 + \ldots + q_{m-1}i^{m-1} + q_mi^m$ belongs to $SF_{2m}(S^1)$, then $y = q_0 + \mathcal{K}q_1 + \mathcal{K}^2q_2 + \ldots + \mathcal{K}^mq_m$ and $\varepsilon(q_j) = 0$.

proof. We can write $y = q_0+\mathcal{K}q_1+\mathcal{K}^2q_2+\ldots+\mathcal{K}^{m-1}q_{m-1}+\mathcal{K}^mq_m+[W^{2(m-1)}_2]i+[W^{2(m-2)}_4]i^2+\ldots+[W^2_{2(m-1)}]i^{m-1}+[W^0_{2m}]i^m$, where $[W^{2(m-j)}_{2j}] = \varepsilon(q_j)$. Since y is in $SF_{2m}(S^1)$, and q_0, $\mathcal{K}q_1$, $\mathcal{K}^2q_2, \ldots, \mathcal{K}^mq_m$ belong to $SF_{2m}(S^1)$, we must have

$(\sum_{j=1}^{m}[W_{2j}^{2(m-j)}]i^j)$ in $SF_{2m}(S^1)$. Next, since $\overline{J}([W_{2j}^{2(m-j)}]i^j) = [W_{2j}^{2(m-j)}]\theta^{m-j}$ and $m-j < m$, we conclude that $W_{2j}^{2(m-j)}$, $j = 1, \ldots, m$, are boundary manifolds. Thus, $(\sum_{j=1}^{m}[W_{2j}^{2(m-j)}]i^j) = 0$. Therefore, $y = q_0 + \mathcal{K}q_1 + \mathcal{K}^2 q_2 + \ldots + \mathcal{K}^m q_m$. □

Denoting by $Q^+_{2(m-j)} = ker(Q_{2(m-j)} \to N_{2(m-j)})$, we have the following theorems.

Theorem 6.05. $SF_{2m}(S^1)$ is the direct sum of Q_{2m} and $\mathcal{K}^m Q^+_{2(m-n)}$, for $m \geq n > 0$; and $\mathcal{K}^n$ embed $Q^+_{2(m-n)}$ in $SF_{2m}(S^1)$. □

Theorem 6.06. The N_*- module $SF_*(S^1)$ is the direct sum of Q and N_*-submodules $\mathcal{K}^n Q^+$, for $n > 0$; and $\mathcal{K}^n$ embed Q^+ in $SF_*(S^1)$, where $Q^+ = ker(Q \to N_*)$. □

7. $\mathbb{Z}_p$-actions, p an odd prime.

We denote by $[M^n, T]$ a closed manifold M^n together a p-periodic diffeomorphism T, p an odd prime. We have the following bordism groups: the bordism group of free $\mathbb{Z}_p$-actions $N_*(\mathbb{Z}_p)$, the bordism group of semi free $\mathbb{Z}_p$-actions $SF_*(\mathbb{Z}_p)$, and the bordism group of semi free $\mathbb{Z}_p$-actions on manifolds with boundary which is free on the boundary $M_*(\mathbb{Z}_p)$.

Theorem 7.01. We have the exact sequence

$$\ldots \to N_n(\mathbb{Z}_p) \to SF_n(\mathbb{Z}_p) \xrightarrow{j_*} M_n(\mathbb{Z}_p) \to N_{n-1}(\mathbb{Z}_p) \to \ldots$$

where $M_n(\mathbb{Z}_p) = \oplus N_{n-2k}(BU(k_1) \times \ldots \times BU(k_{(p-1)/2}))$. (see [4], 38.3.) □

Consider the homomorphism $\overline{\varepsilon} : N_*(\mathbb{Z}_p) \to N_*$ defined by $\overline{\varepsilon}[M^n, T] = [M^n/T]$. Let $\widetilde{N}_*(\mathbb{Z}_p)$ the reduced group, i. e., $\widetilde{N}_*(\mathbb{Z}_p) = ker\overline{\varepsilon}$.

Theorem 7.02. The sequence

$$0 \to N_n \xrightarrow{i_*} SF_n(\mathbb{Z}_p) \xrightarrow{j_*} M_n(\mathbb{Z}_p) \xrightarrow{\partial} \widetilde{N}_{n-1}(\mathbb{Z}_p) \to 0$$

is exact. The homomorphism i_* is defined by $i_*[M^n] = [M^n \times \mathbb{Z}_p,\ 1 \times \sigma]$, where σ is the p-periodic map which permuts the elements of $\mathbb{Z}_p$. □

Theorem 7.03. $\widetilde{N}_*(\mathbb{Z}_p)$ is a N_*-module generated by the elements $\{[S^{2k-1}, \rho]\}$, where $\rho = \dfrac{2\pi i}{p}$. □

The N_*-modules $SF_*(\mathbb{Z}_p)$ and $M_*(\mathbb{Z}_p)$ are graduated rings with multiplication induced by the cartesian product $[M_0,\ T_0][M_1,\ T_1] = [M_0 \times M_1,\ T_0 \times T_1]$.

Denoting by I the image of i_*, then I is the ideal of $SF_*(\mathbb{Z}_p)$ generated by $[\mathbb{Z}_p,\ \sigma]$, since j_* is a ring homomorphism. Therefore $\widehat{SF}_*(\mathbb{Z}_p) = SF_*(\mathbb{Z}_p)/I$ is a ring and we have the following theorem.

Theorem 7.04. The sequence

$$0 \to \widehat{SF}_*(\mathbb{Z}_p) \xrightarrow{j_*} M_*(\mathbb{Z}_p) \xrightarrow{\partial} \widetilde{N}_*(\mathbb{Z}_p) \to 0$$

is exact. □

Consider the set $\{\overline{\alpha}_{2k-1} : k = 1,\ 2, \ldots\}$ of generators of $\widetilde{N}_*(\mathbb{Z}_p)$, where $\overline{\alpha}_{2k-1} = [S^{2k-1}, \rho]$ and $\rho = \exp(2\pi i/p)$. There are closed manifolds M^{4k}, $k = 1,\ 2, \ldots$, where $\overline{\beta}_{2k-1} = p\overline{\alpha}_{2k-1} + [M^4]\overline{\alpha}_{2k-5} + [M^8]\overline{\alpha}_{2k-9} + \ldots = 0$, for $k = 1,\ 2, \ldots$, in $\widetilde{N}_*(\mathbb{Z}_p)$. Thus, $\widetilde{N}_*(\mathbb{Z}_p)$ is isomorphic as a N_*-module to the quotient of the N_*-free module generated by the elements $\overline{\alpha}_1,\ \overline{\alpha}_3,\ \overline{\alpha}_5, \ldots$, by the submodule generated by $\overline{\beta}_1,\ \overline{\beta}_3,\ \overline{\beta}_5, \ldots$

Now, we consider the following diagram

$$\begin{array}{ccccccc}
0 & \to & SF_*(S^1) & \xrightarrow{j} & M_*(S^1) & \to & N_*(S^1) & \to 0 \\
 & & \downarrow r_p & & \downarrow r'_p & & \downarrow r''_p & \\
0 & \to & \widehat{SF}_*(\mathbb{Z}_p) & \xrightarrow{j_*} & M_*(\mathbb{Z}_p) & \to & \widetilde{N}_*(\mathbb{Z}_p) & \to 0
\end{array}$$

where r_p is the homomorphism sending the S^1-action $[M, S]$ to the restriction $\mathbb{Z}_p$- action.

Since $N_*(S^1)$ is a N_*-free module generated by $\alpha_{2k-1} = [S^{2k-1}, S]$, where S is the S^1-action on S^{2k-1} given by $s(t,\ (z_0, \ldots, z_{2k-1})) = (tz_0,\ tz_1, \ldots, tz_{2k-1})$, $t \in S^1$; and $M_*(S^1)$ is a polynomial algebra generated by $\lambda_i : [\lambda \to CP(i)]$, we are going to define a N_*-submodule B of $M_*(S^1)$

in the following way: the kernel of r''_p is the N_*-free module generated by $\beta_k = p\alpha_{2k-1} + [M^4]\alpha_{2k-5} + [M^8]\alpha_{2k-9} + \ldots$, where $r''_p(\beta_k) = \overline{\beta}_{2k-1} = 0$ in $\widetilde{N}_*(\mathbb{Z}_p)$. Let $\widehat{\beta}_k$ be defined by: $\widehat{\beta}_k = p\lambda_0^k + [M^4]\lambda_0^{k-2} + [M^8]\lambda_0^{k-4} + \ldots$ in $M_*(S^1)$, and let B be the N_*-free module generated by $\widehat{\beta}_1, \widehat{\beta}_3, \widehat{\beta}_5, \ldots$, which is a submodule of $M_*(S^1)$. Since $\overline{J}(\widehat{\beta}_k) = 1 + [M^4]\theta^2 + [M^8]\theta^4 + \ldots$, where $\overline{J}$ is the Boardman homomorphism, and denoting by x_n a basic n-dimensional element of N_*, we have that

$$\begin{aligned}
\overline{J}(x_{2k-2}\widehat{\beta}_1) &= x_{2k-2}\theta^{k-1} + \ldots \\
\overline{J}(x_{2k-6}\widehat{\beta}_3) &= x_{2k-6}\theta^{k-3} + x_{2k-6}[M^4]\theta^{k-1} + \ldots \\
\overline{J}(x_{2k-10}\widehat{\beta}_5) &= x_{2k-10}\theta^{k-5} + x_{2k-10}[M^4]\theta^{k-3} + \ldots \\
&\vdots \\
\overline{J}(x_4\widehat{\beta}_{k-2}) &= x_4\theta^2 + x_4[M^4]\theta^4 + \ldots
\end{aligned}$$

So, any combination of the elements above has power of $\theta \leq k-1$. Therefore, it follows that there isn't element in $SF_{2k}(S^1)$ with image nozero in B. Hence, we have that $j(SF_*(S^1)) \cap B = (0)$.

Theorem 7.05. The homomorphism $r_p : SF_*(S^1) \to \widehat{SF}_*(\mathbb{Z}_p)$ is 1-1.

proof. we have the following commutative diagram

$$\begin{array}{ccccccc}
0 & \to & SF_*(S^1) & \xrightarrow{j} & M_*(S^1) & \xrightarrow{\cong} & \oplus N_k(BU(r)) \\
 & & \downarrow r_p & & \downarrow r'_p & & \downarrow r \\
0 & \to & SF_*(\mathbb{Z}_p) & \xrightarrow{j_*} & M_*(\mathbb{Z}_p) & \xrightarrow{\cong} & \oplus N_*(BU(r_1) \times \ldots \times BU(r_k))
\end{array}$$

where r is given by inclusion on the first factor. Therefore, r is 1 - 1. This imply that r'_p is 1 - 1,and finally r_p is 1 - 1, since $r'_p \circ j = j_* \circ r_p$ and j, j_* are 1 - 1. □

Remark. 7.06. For $p = 3$, we have that r'_p is an isomorphism because $M_*(S^1) \simeq \oplus N_*(BU(r))$, $M_*(\mathbb{Z}_3) \simeq \oplus N_*(BU(r))$ and the map r is an isomorphism. Thus, since $B \cap \overline{J}(SF_*(S^1)) = (0)$, we have that $j_*(\widehat{SF}_*(\mathbb{Z}_3)) \simeq j(SF_*(S^1)) \oplus B$, and $\widehat{SF}_*(\mathbb{Z}_3) \simeq j(SF_*(S^1)) \oplus B$, since j_* is 1 - 1. Therefore, $SF_*(\mathbb{Z}_3) \simeq N_* \oplus j(SF_*(S^1)) \oplus B$.

We are going to denote by $(\mathbb{Z}_p)^k_n$ the set of classes in the bordism group N_n which are represented by a n-manifold which is the fixed point set of a closed $(n+k)$-manifold with a semi-free $\mathbb{Z}_p$-action. We have that $(\mathbb{Z}_p)^k_n$ is a subgroup of N_n, $(\mathbb{Z}_0)^0_n \simeq N_n$ and $(\mathbb{Z}_p)^k_* = \oplus_{n\geq 0}(\mathbb{Z}_p)^k_n$ is an ideal of N_*.

Theorem 7.07. $(\mathbb{Z}_p)^2_n \simeq N_n$.

proof. Let $\widehat{\beta}_k = p\lambda_0^k + [M^4]\lambda_0^{k-2} + [M^8]\lambda_0^{k-4} + \ldots$ be in $M_*(\mathbb{Z}_p)$. Since $\partial(\widehat{\beta_k})$ in $\widetilde{N}_*(\mathbb{Z}_p)$ is a boundary, we have that $\widehat{\beta}_k$ belongs to the image of j_*. In particular, $\widehat{\beta}_1 = p\lambda_0$ belongs to the image of j_*. Therefore, $[x_n]p\lambda_0$ belongs to the image of j_*, where x_n is a n-dimensional generator, $n \neq 2^r - 1$,of N_*. Thus, there is a semi free $\mathbb{Z}_p$-action $[M^{n+2}, T]$ with fixed point set $[x_n] + \ldots + [x_n]$ (p-times), where p is an odd prime. Since $[x_n] + \ldots + [x_n] = [x_n]$, we have that $[x_n]$ is in $(\mathbb{Z}_p)^2_n$. Hence, $(\mathbb{Z}_p)^2_n \simeq N_n$. □

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