REPRESENTABLE OPERATORS AND THE DUNFORD-PETTIS THEOREM

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Abstract. The aim of this article is to present a modern approach to the Dunford-Pettis theorem about the representation of linear, continuous operators on L_1 with values in a Banach space by densities; as will be seen this result is nothing else but a Radon-Nikodým theorem. Moreover, a systematic investigation of these "representable" operators will be given, as well as some remarks about the rôle of the Radon-Nikodým property of Banach spaces in the theory of operator ideals. It is not possible to give all proofs: if not otherwise stated the reader may find the missing proofs in [6], in particular in section 3 and appendices B - D.

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The aim of this article is to present a modern approach to the Dunford-Pettis theorem about the representation of linear, continuous operators on L_1 with values in a Banach space by densities; as will be seen this result is nothing else but a Radon-Nikodým theorem. Moreover, a systematic investigation of these "representable" operators will be given, as well as some remarks about the rôle of the Radon-Nikodým property of Banach spaces in the theory of operator ideals. It is not possible to give all proofs: if not otherwise stated the reader may find the missing proofs in [6], in particular in section 3 and appendices B - D.

Contents:

Notation:

Let E, F, G be normed spaces
B_E: closed unit ball of E
$\mathcal{L}(E,F) := \{T : E \to F |$ linear, continuous $\}$
$E' := \mathcal{L}(E, \mathbb{K})$ (topological) dual of E
$\langle x, x' \rangle := x'(x)$ (duality bracket) whenever $x \in E$ and $x' \in E'$
$\kappa_E : E \hookrightarrow E''$ the canonical embedding into the bidual
$\mathcal{B}(E, F; G) := \{\Phi : E \times F \to G \mid$ bilinear, continuous $\}$
$E \stackrel{1}{=} F$: E and F are metrically isomorphic
$(\Omega, \Sigma, \mu) = (\Omega, \mu)$ measure space with the σ-algebra Σ being μ-complete.

1. The projective tensor product

1.1 Let E and F be vector spaces (over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ of scalars), then the algebraic tensor product $E \otimes F$ has the following universal property: For every vector space G and every *bilinear* mapping $\Phi : E \times F \to G$ there is a unique *linear* operator $\Phi^L : E \otimes F \to G$ such that $\Phi(x, y) = \Phi^L(x \otimes y)$

$$\begin{tikzcd} E \times F \arrow[rr, "\Phi"] \arrow[d] & & G \\ E \otimes F \arrow[urr, "\Phi^L"'] & & \end{tikzcd}$$

The operator Φ^L is called the *linearization* of Φ. If E and F are normed spaces then there is a unique norm $\pi(\,\cdot\,; E, F)$ on $E \otimes F$ such that for all normed spaces G and bilinear $\Phi : E \times F \to G$ the following holds:

(a) Φ is continuous if and only if its linearization Φ^L is continuous.
(b) In this case: $\|\Phi\| = \|\Phi^L\|$

where $\|\Phi\| := \sup \{\|\Phi(x, y)\|_G \mid x \in B_E, y \in B_F\}$. Notation: $E \otimes_\pi F$ for $E \otimes F$ equipped with this norm and $E \tilde{\otimes}_\pi F$ for the completion. The universal property says:

$$\mathcal{B}(E, F; G) \stackrel{1}{=} \mathcal{L}(E \otimes_\pi F, G).$$

Since every operator $T \in \mathcal{L}(E, F')$ defines a unique bilinear continuous form $\varphi_T \in \mathcal{B}(E, F; \mathbb{K})$ by

$$\varphi_T(x, y) := \langle Tx, y \rangle_{F', F}$$

and $T \rightsquigarrow \varphi_T$ is also an onto isometry $\mathcal{L}(E, F') \to \mathcal{B}(E, F; \mathbb{K})$ one has in particular

$$(E \otimes_\pi F)' = \mathcal{B}(E, F; \mathbb{K}) = \mathcal{L}(E, F').$$

1.2. The norm π on $E \otimes F$ can be calculated as follows:

$$\pi(z; E, F) = \inf \left\{ \sum_{n=1}^{N} \|x_n\| \, \|y_n\| \;\middle|\; z = \sum_{n=1}^{N} x_n \otimes y_n; N \in \mathbb{N}, \right.$$
$$\left. x_n \in E,\ y_n \in F \right\}.$$

It is a non-trivial result (originally due to Grothendieck) that for every $z \in E\tilde{\otimes}_\pi F$ there is an absolutely convergent series $\sum(x_n \otimes y_n)$ in $E\tilde{\otimes}_\pi F$ with limit z. Moreover,

$$\pi(z; E, F) = \inf\left\{\sum_{n=1}^{\infty} ||x_n||\, ||y_n|| \;\middle|\; z = \sum_{n=1}^{\infty} x_n \otimes y_n\right\}.$$

for every $z \in E\tilde{\otimes}_\pi F$. Note, that $\pi(x \otimes y; E, F) = ||x||\, ||y||$.

1.3. If $T_i \in \mathcal{L}(E_i, F_i)$ then there is a unique linear map $S : E_1 \otimes E_2 \to F_1 \otimes F_2$ satisfying

$$S(x_1 \otimes x_2) = T_1 x_1 \otimes T_2 x_2;$$

this map is denoted by $T_1 \otimes T_2$. It is straightforward to see that

$$||T_1 \otimes T_2 : E_1 \otimes_\pi E_2 \to F_1 \otimes_\pi F_2|| = ||T_1||\, ||T_2||.$$

Moreover, the formula given in 1.2. for π shows easily that if $Q_i : E_i \xrightarrow{1} F_i$ are metric surjections (i.e. F_i has the quotient norm with respect to Q_i) then

$$Q_i \otimes Q_2 : E_1 \otimes_\pi E_2 \to F_1 \otimes_\pi F_2$$

is a metric surjection as well. This is why the norm π is called the *projective norm* on the tensor product $E \otimes F$.

1.4. If μ is a measure and E a Banach space then $L_1(\mu) \otimes E$ is the subspace of (classes of) those Bochner integrable functions in $L_1(\mu; E)$ which have finite dimensional range (μ-almost everywhere). The following result will be crucial.

PROPOSITION: *For all $\tilde{f} \in L_1(\mu) \otimes E$ one has*

$$\pi(\tilde{f}; L_1(\mu), F) = \int_\Omega ||f(\omega)||_E \mu(d\omega).$$

This means that $L_1(\mu) \otimes_\pi E$ is an isometric subspace of $L_1(\mu, E)$. Since it is even dense in the Banach space $L_1(\mu, E)$ it follows that

$$L_1(\mu, E) \stackrel{1}{=} L_1(\mu)\tilde{\otimes}_\pi E.$$

See also Alencar's article [2] in this volume for a proof of these facts.

2. The weak version of the Dunford-Pettis theorem

2.1. If the Radon-Nikodým theorem holds for a measure μ (these measures are called *localizable*) it says that $L_1(\mu)' \overset{1}{=} L_\infty(\mu)$, the latter being the space of classes of measurable functions which are locally μ-almost everywhere equal to a bounded function. In other words: if $\varphi : L_1(\mu) \to I\!\!K$ is linear and continuous then there is a unique $\tilde{g} \in L_\infty(\mu)$ such that

$$\langle \varphi, \tilde{f} \rangle = \int f(\omega) g(\omega) \mu(d\omega)$$

for all $\tilde{f} \in L_1(\mu)$. Recalling from 1.1. and 1.4. the relations

$$\mathcal{L}(L_1(\mu), E') = (L_1(\mu) \otimes_\pi E)' = (L_1(\mu, E))'$$

one sees that representing operators $L_1 \to E'$ by "densities" might be nothing else than determining the dual of $L_1(\mu, E)$.

2.2. Let $\mathcal{M}_\mu(E', E)$ be the vector space of all functions $g : \Omega \to E'$ such that

(a) g is $\sigma(E', E)$-measurable, i.e. $\langle g(\cdot), x \rangle_{E', E}$ is measurable for all $x \in E$

(b) $\sup \{ ||g(\omega)||_{E'} \mid \omega \in \Omega \} < \infty$.

One can verify that for each $g \in \mathcal{M}_\mu(E', E)$ and each $\tilde{f} \in L_1(\mu, E)$ the function

$$\langle g(\cdot), f(\cdot) \rangle_{E', E}$$

is integrable; now it is obvious that φ_g

$$\langle \varphi_g, \tilde{f} \rangle := \int \langle g(\omega), f(\omega) \rangle \mu(d\omega)$$

defines a linear continuous functional on $L_1(\mu, E)$. It is the aim to show that the mapping

$$\begin{array}{rcl} \mathcal{M}_\mu(E', E) & \longrightarrow & (L_1(\mu, E))' = \mathcal{L}(L_1(\mu), E') \\ g & \rightsquigarrow & \varphi_g \end{array}$$

with an absolutely convergent series in $L_1(\mu)\widetilde{\otimes}_\pi E$ (see 1.2.). This implies that it is enough to find a $g \in \mathcal{M}_\mu(E', E)$ satisfying

$$\langle \varphi, \widetilde{h} \otimes x \rangle = \int \langle g(\omega), x \rangle h(\omega) \mu(d\omega)$$

for all $x \in E$ and $\widetilde{h} \in L_1(\mu)$. Fix $x \in E$; then φ_x defined by

$$\langle \varphi_x, \widetilde{h} \rangle := \langle \varphi, \widetilde{h} \otimes x \rangle$$

is linear and satisfies

$$|\langle \varphi_x, \widetilde{h} \rangle| \leq \|\varphi\| \, \pi(\widetilde{h} \otimes x; L_1, E) = \|\varphi\| \, \|\widetilde{h}\| \, \|x\|.$$

It follows that $\varphi_x \in L_1(\mu)'$ and $\|\varphi_x\| \leq \|\varphi\| \, \|x\|$; therefore there is a unique $\widetilde{g}_x \in L_\infty(\mu)$ (with norm $\|\varphi_x\|$) representing φ_x. Using a lifting λ_∞ one chooses the representative $\lambda_\infty(\widetilde{g}_x) \in \mathcal{L}_\infty(\mu)$ of the class $\widetilde{g}_x$ and sees that

$$\langle \varphi, \widetilde{h} \otimes x \rangle = \langle \varphi_x, \widetilde{h} \rangle = \int \lambda_\infty(\widetilde{g}_x)(\omega) \, h(\omega) \mu(d\omega).$$

It is obvious that $x \rightsquigarrow \varphi_x$ is linear; since L_1' and L_∞ are linearly isometric, the linearity of λ_∞ implies that for each $\omega \in \Omega$ the mapping

$$g(\omega) : E \to I\!K; \ \langle g(\omega), x \rangle := \lambda_\infty(\widetilde{g}_x)(\omega)$$

is linear. Moreover, by the very properties of λ_∞

$$|\langle g(\omega), x \rangle| = |\lambda_\infty(\widetilde{g}_x)(\omega)| \leq \|\widetilde{g}_x\| = \|\varphi_x\| \leq \|\varphi\| \, \|x\|$$

which means that $g(\omega) \in E'$ and $\|g(\omega)\|_{E'} \leq \|\varphi\|$. It is clear that this function $g : \Omega \to E'$ is a function as wanted. □

2.3. For $T \in \mathcal{L}(L_1(\mu), E') \stackrel{1}{=} (L_1(\mu, E))'$ (see 2.1.) this representation gives

$$\langle T\widetilde{h}, x \rangle = \langle \varphi_T, \widetilde{h} \otimes x \rangle = \int \langle g(\omega), h(\omega) x \rangle \mu(d\omega)$$

hence the

DUNFORD-PETTIS THEOREM (weak version): *Let μ be a strictly localizable measure, E a Banach space and $T \in \mathcal{L}(L_1(\mu), E')$. Then there is a $g \in \mathcal{M}_\mu(E', E)$ with $||g(\omega)||_{E'} \leq ||T||$ for all $\omega \in \Omega$ such that*

$$\langle T\tilde{h}, x\rangle = \int_\Omega \langle g(\omega), x\rangle \, h(\omega)\mu(d\omega)$$

for all $\tilde{h} \in L_1(\mu)$.

The function g will be called a *$\sigma(E', E)$-density of T.*

2.4. Let $\langle E_1, E_2\rangle$ be a dual system (think at $\langle E', E\rangle$ or $\langle E, E'\rangle$) then a function $f : \Omega \to E_1$ is called *$\sigma(E_1, E_2)$-Pettis integrable* if

(a) For all $x_2 \in E_2$ the function $\langle f(\cdot), x_2\rangle$ is integrable
(b) For each integrable A there is $x_A \in E_1$ such that for all $x_2 \in E_2$

$$\langle x_A, x_2\rangle = \int_A \langle f(\omega), x_2\rangle \mu(d\omega)$$

In this case the element x_A is unique and x_Ω is called the $\sigma(E_1, E_2)$-Pettis integral of f:

$$x_\Omega = \int_\Omega f(\omega)\mu(d\omega) \quad (\sigma(E_1, E_2)\text{−Pettis integral})$$

It is immediate to see that each Bochner integrable function $f : \Omega \to E$ is $\sigma(E, E')$-Pettis integrable with the same integral.
Using this notation, the Dunford-Pettis theorem reads

$$T\tilde{h} = \int_\Omega h(\omega)g(\omega)\mu(d\omega) \quad (\sigma(E', E)\text{−Pettis integral})$$

for all $\tilde{h} \in L_1(\mu)$. Under which circumstances this integral can be interpreted as a Bochner integral?

3. Representable operators

3.1. The $\sigma(E', E)$-density $g : \Omega \to E'$ in the weak version of the Dunford-Pettis theorem is uniformly bounded and $\sigma(E', E)$-measurable. Pettis' measurability criterion says that a function g is *μ-measurable* (i.e. g is the pointwise limit of a sequence of μ-step functions) if it is $\sigma(E', E)$-measurable and has separable range almost everywhere (i.e. there is a norm-separable subspace F such that $g(\omega) \in F$ for μ-almost all $\omega \in \Omega$). Assuming now that μ is a finite measure and the density has separable range a.e., then for each $h \in \mathcal{L}_1(\mu)$ the product $h \cdot g$ is μ-measurable and $||h(\omega)g(\omega)|| \leq c|h(\omega)|$, hence $h \cdot g$ is Bochner integrable and the integral in the Dunford-Pettis theorem is a Bochner integral.

3.2. This phenomenon will be studied now systematically. For this suppose, from now on, that *μ is a finite measure.*

DEFINITION: *Let E be a Banach space. An operator $T \in \mathcal{L}(L_1(\mu), E)$ is called* Riesz representable (*in short*: representable) *if there is bounded μ-measurable function $g : \Omega \to E$ such that*

$$T\tilde{f} = \int_\Omega fgd\mu \quad \text{(Bochner integral)}$$

for all $\tilde{f} \in L_1(\mu)$.

The function g is called a *Riesz density* for T. The very properties of Bochner integrals imply that if g is a Riesz density of $T : L_1 \to E$ and $S : E \to F$, then $S \circ g$ is a Riesz density of $S \circ T$.
If g is μ-measurable and bounded one denotes

$$||g||_\infty := || \; ||g(\cdot)||_E \; ||_{L_\infty} = \text{ess sup} \, ||f(\cdot)||_E \, .$$

One can show, that if g is a Riesz density of $T \in \mathcal{L}(L_1(\mu), E)$ then $||T|| = ||g||_\infty$.

REMARK:

(1) *If g_1 and g_2 are Riesz densities for T then $g_1 = g_2$ μ-a.e..*
(2) *$g(\omega) \in \overline{T(L_1)}$ μ-a.e..*

Proof: For (1) note that $g_1 - g_2$ is a Riesz density for the operator $T - T = 0$ hence $||g_1 - g_2|| = ||0|| = 0$. For the second statement consider

$$L_1(\mu) \xrightarrow{T} E \xrightarrow{Q} E/\overline{T(L_1)}$$

then $Q \circ g$ is a Riesz density of $Q \circ T = 0$, hence $Q \circ g = 0$ μ-a.e. by (1); this is the statement (2). □

3.3. The arguments in 3.1. show that the weak version of the Dunford-Pettis theorem gives the

PROPOSITION: *If E is a Banach space with separable dual E' then every operator $T : L_1(\mu) \to E'$ is Riesz representable.*

A famous result of Davis-Figiel-Johnson-Pelczyński [4] states that every weakly compact operator T (i.e. the image of the unit ball is weakly relatively compact) factors through a reflexive space; clearly, if T has separable range then this reflexive space can be choosen to be separable as well. So if $T \in \mathcal{L}(L_1, E)$ has separable range and is weakly compact it factors

$$\begin{array}{ccc} L_1 & \xrightarrow{T} & E \\ \mathrm{U} & & \mathrm{V} \\ G & & G \text{ separable} \end{array}$$

– and a Riesz density of U gives one for $T = V \circ U$:

COROLLARY: *Every weakly compact operator $T : L_1(\mu) \to E$ with separable range is Riesz representable.*

3.4. Before continuing, it is worthwhile to see an operator which is *not* representable – and some of the structural consequences of this fact. Take the space c_0 of zero-sequences and the operator

$$\begin{array}{rcl} \mathcal{F} : L_1([0, 2\pi]) & \longrightarrow & c_0 \\ \hat{f} & \rightsquigarrow & (\hat{f}(n))_{n \in \mathbb{N}} \end{array}$$

of Fourier-coefficients:

$$\hat{f}(n) := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(t) \exp(int) dt;$$

$(\hat{f}(n)) \in c_0$ by the Riemann-Lebesgue lemma.
If g were a Riesz density of $\mathcal{F}$ or of $\kappa_{c_0} \circ \mathcal{F} : L_1 \to c_0 \hookrightarrow \ell_\infty$, then (looking at components) one would obtain that

$$g(t) = \left(\frac{1}{\sqrt{2\pi}} \exp{(int)} \right)_{n \in \mathbb{N}} \in \ell_\infty$$

almost everywhere; g (as a measurable function) is separably-valued a.e. but the function on the right side is not! It follows that the following three operators are *not* representable:

(1) $\mathcal{F} : L_1[0,2\pi] \to c_0$
(2) $\kappa_{c_0} \circ \mathcal{F} : L_1[0,2\pi] \to \ell_\infty$
(3) $id : L_1[0,2\pi] \to L_1[0,2\pi]$ (identity map).

Not that $g(t) = ((2\pi)^{-1/2} \exp(int))_{n\in\mathbb{N}}$ defines a $\sigma(\ell_\infty, \ell_1)$-density of $\kappa_{c_0} \circ \mathcal{F}$ – with values in $\ell_\infty \setminus c_0$.

COROLLARY: *The Banach spaces c_0 and $L_1[0,2\pi]$ are not isomorphic to the dual of any Banach space.*

Proof: Since c_0 and $L_1[0,2\pi]$ are separable this is an immediate consequence of proposition 3.3. and the existence of non-representable operators. □

The result about $L_1[0,2\pi]$ is an old celebrated result of Gelfand.

3.5. The following lemma will lead immediately to the strong version of the Dunford-Pettis theorem.

LEMMA: *Let $\mathcal{A}$ be an operator ideal and E a Banach space. If every operator $T \in \mathcal{A}(L_1(\mu), E)$ with separable range is representable then every operator $T \in \mathcal{L}(L_1(\mu), E)$ is representable.*

Sketch of the proof: The following, non-obvious fact will be used: *representable operators throw weakly compact sets into compact sets.*
For the lemma it is enough to show that every $T : L_1(\mu) \to E$ has separable range. Since the step functions are dense in $L_1(\mu)$ and compact sets are separable it suffices to show that

$$\{T(\tilde{\chi}_A) \mid A \subset \Omega \ \mu\text{−integrable}\}$$

is relatively compact = relatively sequentially compact. So take a sequence of integrable $A_n \subset \Omega$ and consider the σ-algebra $\mathcal{A}_0$ generated by

$\{A_n \mid n \in \mathbb{N}\}$. Then T_0 defined by

$$T_0(\tilde{f}) := T(\mathbf{E}(\tilde{f} \mid \mathcal{A}_0))$$

(conditional expectation) is an operator in $\mathcal{A}$, and has a separable range (since the $\mathcal{A}_0$-measurable, integrable functions form a separable subspace of L_1): by assumption, T_0 is separable. The unit ball of L_∞ is weakly compact in L_1 hence the set $\{\tilde{\chi}_{A_n}\}$ is weakly compact and $\{T_0(\tilde{\chi}_{A_n}) = T(\tilde{\chi}_{A_n})\}$ is therefore relatively compact, which implies that $(T(\tilde{\chi}_{A_n}))$ has a convergent subsequence. □

3.6. Applying this lemma to the operator ideal $\mathcal{W}$ of weakly compact operators, the corollary 3.3 gives the

DUNFORD-PETTIS THEOREM (strong version): *Let μ be a finite measure and E a Banach space. Then every weakly compact operator $T : L_1(\mu) \to F$ is Riesz representable.*

One can show that every operator $L_1(\mu) \to \ell_1$ is representable – even that each representable operator factors through ℓ_1 (Lewis-Stegall theorem). So any surjective $T : L_1[0,1] \to \ell_1$ shows that representable operators need not be weakly compact.

3.7. An interesting structural consequence is the following

PROPOSITION: *$L_1(\mu)$ has no reflexive, infinite dimensional, complemented subspaces (μ a finite measure).*

Proof: Let $F \subset L_1(\mu)$ be a complemented, reflexive subspace and P a projection. Then P is weakly compact since the unit ball of F is weakly compact and hence representable. Therefore $B_F = P(B_F)$ is even compact (see the fact mentioned at the beginning of the proof of 3.5) and therefore the Banach space F is finite dimensional. □

4. An application to operator-valued measures

4.1. Let $\mathcal{A}$ be a σ-algebra of subsets of a set Ω, μ a (non-negative) finite measure on $\mathcal{A}$ and $M : \mathcal{A} \to \mathcal{L}(E,F)$ a σ-additive function (=: vector measure) where E and F are Banach spaces. Note the special cases $\mathcal{L}(E,\mathbb{K}) = E'$ and $\mathcal{L}(\mathbb{K},F) = F$.

PROPOSITION: *There is an operator $T : L_1(\mu) \to \mathcal{L}(E,F)$ such that*

$$M(A) = T(\widetilde{\chi}_A)$$

for all $A \in \mathcal{A}$ if and only if for all $x \in E$ and $y' \in F$ the signed measure $\langle M(\cdot)x, y'\rangle_{F,F'}$ is μ-absolutely continuous with density in $\mathcal{L}_\infty(\mu)$.

Proof: The condition is clearly necessary. Conversely, define

$$T(\sum_{n=1}^{N} \lambda_n \widetilde{\chi}_{A_n}) := \sum_{n=1}^{N} \lambda_n M(A_n)$$

which – as usual – is a good definition. If $g_{x,y'} \in \mathcal{L}_\infty(\mu)$ is a Radon-Nikodým density of $\langle M(\cdot)x, y'\rangle$ with respect to μ one obtains for each step function $h \in \mathcal{L}_1(\mu)$

$$\langle T(\widetilde{h})x, y'\rangle = \int_\Omega h(\omega) g_{x,y'}(\omega)\mu(d\omega)$$

and hence

$$|\langle T(\widetilde{h})x, y'\rangle| \leq ||\widetilde{h}||_{L_1} \quad ||\widetilde{g}_{x,y'}||_{L_\infty}.$$

It follows from the uniform boundedness principle that

$$\{T(\widetilde{h}) \in \mathcal{L}(E,F) \mid h \text{ step function}, \ ||\widetilde{h}||_{L_1} \leq 1\}$$

is uniformly bounded in $\mathcal{L}(E,F)$. This implies that T is continuous and hence extendable to a continuous, linear operator $L_1(\mu) \to \mathcal{L}(E,F)$. □

4.2. So the search for "densities" g such that

$$M(A) = \int_A g(\omega)\mu(d\omega) \in \mathcal{L}(E,F)$$

(as a certain Pettis integral or Bochner integral) is reduced to the representability of operators. Using

$$(E \otimes_\pi F)' = (E\widetilde{\otimes}_\pi F)' = \mathcal{L}(E,F')$$

and the weak and strong Dunford-Pettis theorem one obtains the

THEOREM:

(1) *Let μ be a strictly localizable measure on a set Ω, E and F normed spaces and $T : L_1(\mu) \to \mathcal{L}(E, F')$ a linear and continuous operator, then there is a function $D : \Omega \to \mathcal{L}(E, F')$ such that*

(a) *$\langle D(\cdot)x, y\rangle$ is μ-measurable for all $x \in E$ and $y \in F$*

(b) *For all $\tilde{h} \in L_1(\mu)$*

$$T\tilde{h} = \int \tilde{h}(\omega)D(\omega)\mu(d\omega)$$

as a $\sigma(\mathcal{L}(E,F'), E\tilde{\otimes}_\pi F)$-Pettis integral; in particular: the integral is a $\sigma(\mathcal{L}(E,F'), E \otimes F)$-Pettis integral

(c) *$\|D(\omega)\| \leq \|T\|$ for all $\omega \in \Omega$.*

(2) *If μ is finite and $\mathcal{L}(E, F')$ separable or reflexive, then D is μ-measurable and the integral in* (1) (b) *is a Bochner integral.*

Just some examples for the Bochner integrability of the density – in other words, when T is a representable operator (E, F Banach spaces):

(1) If every operator $E \to F'$ is compact, E' and F' are separable and one of these spaces has the approximation property, then

$$\mathcal{L}(E, F') = \mathcal{K}(E, F') = E'\tilde{\otimes}_\varepsilon F'$$

(ε the injective tensor product, $\mathcal{K}$ the ideal of compact operators) is separable.

(2) $\mathcal{L}(E, F')$ is reflexive if and only both spaces E and F are reflexive and $\mathcal{L}(E, F') = \mathcal{K}(E, F')$.

Pitt's theorem states that

$$\mathcal{L}(\ell_p, \ell_q) = \mathcal{K}(\ell_p, \ell_q)$$

whenever $1 \leq q < p < \infty$ – hence every $T : L_1(\mu) \to \mathcal{L}(\ell_p, \ell_q)$ is representable. Note that $\mathcal{L}(\ell_2, \ell_2) = (l_2\tilde{\otimes}_\pi l_2)'$ is neither separable nor reflexive.

4.3. Using the embedding $\mathcal{L}(E, F) \subset \mathcal{L}(E, F'')$ one obtains for operators $T : L_1(\mu) \to \mathcal{L}(E, F)$ a $\sigma(\mathcal{L}(E, F''), E\tilde{\otimes}_\pi F')$-density. The example with the

Fourier transform in 3.4. (put $E := I\!K,\ F := c_0, F'' = \ell_\infty$) shows that in general, the density cannot be chosen to have values in $\mathcal{L}(E, F)$.

5. The Radon Nikodým property of Banach spaces

5.1. A Banach space E has the *Radon-Nikodým property* if, for each finite measure μ, each operator $T : L_1(\mu) \to E$ is representable; actually it is enough to check this for the Lebesgue measure on $[0,1]$ (a nice direct proof of this fact was recently given by Botelho [3]). Taking into account the relation between operators and vector-valued measures established in 4.1. one can show that for these spaces a Radon-Nikodým theorem for certain vector-valued measures holds – with Bochner integrable "derivative". I do not want to go into the details of this measure-theoretic aspect and refer the reader to the monography of Diestel and Uhl [7], [8] and Alencar [2].

5.2 In the scalar case the Radon-Nikodým theorem means $L_1' = L_\infty$ – and, as a consequence, $L_p' = L_{p'}$ for $1 \le p < \infty$.

PROPOSITION: *Let E be a Banach space and $1 \le p < \infty$. Then*

$$L_p(\mu, E)' = L_{p'}(\mu, E')$$

holds isometrically for all finite measures μ (or only the Lebesgue measure on $[0,1]$) if and only if E' has the Radon-Nikodým property.

5.3. From section 3 the following can easily be derived:

PROPOSITION:

(1) *Separable dual spaces have the Radon-Nikodým property.*

(2) *Reflexive spaces have the Radon-Nikodým property.*

(3) *E has the Radon-Nikodým property if and only if every separable closed subspace has it.*

Proof: (1) is a reformulation of proposition 3.3., (2) a direct consequence of the strong Dunford-Pettis theorem. The last statement follows from lemma 3.5 applied to the ideal $\mathcal{A} = \mathcal{L}$ of all operators. □

The spaces $c_0, \ell_\infty, C[0,1], L_1[0,1]$ do not have the Radon-Nikodým property (by the example in 3.4.); also $\mathcal{L}(\ell_2, \ell_2)$ does not have it since it contains c_0. In [7], p. 217-219 the reader can find a long list of equivalent formulations of the property – and a list of spaces which have or which do not have it. In the rest of this survey I want to point at the importance of the Radon-Nikodým property for the theory of operator ideals.

6. Nuclear Operators

6.1. An operator $T : E \to F$ is called *nuclear* if there are $x'_n \in E'$ and $y_n \in F$ with $\sum_{n=1}^{\infty} \|x'_n\| \, \|y_n\| < \infty$ and

$$Tx = \sum_{n=1}^{\infty} \langle y'_n, x \rangle y_n$$

for all $x \in E$. Denote by $x' \underline{\otimes} y$ the one-dimensional operator

$$(x' \underline{\otimes} y)\ (x) := \langle x', x \rangle y$$

for all $x \in E$; it follows that $\|x' \underline{\otimes} y\| = \|x'\| \, \|y\|$. With this notation $T \in \mathcal{L}(E, F)$ is nuclear if and only of it has a representation

$$T = \sum_{n=1}^{\infty} x'_n \underline{\otimes} y_n$$

the series being absolutely convergent in $\mathcal{L}(E, F)$. Taking

$$\mathrm{N}(T) := \inf\{ \sum_{n=1}^{\infty} \|x'_n\| \, \|y_n\| \;\Big|\; T = \sum_{n=1}^{\infty} x'_n \underline{\otimes} y_n \}$$

one obtains that the space $\mathcal{N}(E, F)$ of all nuclear operators with the norm N is a Banach space. It is easy to see that $(\mathcal{N}, \mathrm{N})$ is a Banach operator ideal. The bilinear map

$$\begin{array}{ccc} E' \times F & \to & \mathcal{N}(E, F) \\ (x', y) & \rightsquigarrow & x' \underline{\otimes} y \end{array}$$

is continuous, hence its linearization extends to the completion of the π-tensor product

$$J : E' \tilde{\otimes}_\pi F \to \mathcal{N}(E, F).$$

The represenation of elements is $E'\tilde{\otimes}_\pi F$ given in 1.2. shows that π is onto and even a metric surjection; if follows that

$$J(\sum_{n=1}^{\infty} x'_n \otimes y_n) = \sum_{n=1}^{\infty} x'_n \underline{\otimes} y_n.$$

It is a most important, but unpleasant fact, that J is not injective in general! This is why the different notation $\otimes$ and $\underline{\otimes}$ for tensors and operators was used. Clearly, J restricted to $E' \otimes F$ is injective, but not an isomorphism (into) of normed spaces. J is injective if E' or F has approximation property. Fortunately, most of the usual spaces are known to have the approximation property, but $\mathcal{L}(l_2, l_2)$ does not!

Nuclear operators are very important in Analysis in particular for questions where eigenvalues are involved or when would like to use the trace of operators in infinite dimensional Banach spaces; I cannot give details. However, it is usually quite difficult to decide whether a given operator is nuclear or not! This is why one needs general theorems in this direction.

6.2. If $T : L_1(\mu) \to E$ (μ a finite measure) is nuclear then there are $\tilde{g}_n \in L'_1 = L_\infty$ and $y_n \in E$ with $\sum \|\tilde{g}_n\|_\infty \, \|x_n\| < \infty$ and

$$T\tilde{h} = \sum_{n=1}^{\infty} \int h g_n d\mu \cdot x_n = \int h(\omega) \sum_{n=1}^{\infty} g_n(\omega) x_n \, \mu(d\omega)$$

for each $\tilde{h} \in L_1$. Since $\|\tilde{g}_n\|_1 \leq \mu(\Omega)\|\tilde{g}_n\|_\infty$ it follows that

$$g(\omega) := \sum_{n=1}^{\infty} g_n(\omega) x_n$$

defines (a.e.) a Bochner integrable functions which clearly is a Riesz density for the operator T.

REMARK: *Nuclear operators $L_1(\mu) \to E$ (μ a finite measure) are representable.*

One may interprete this reasoning in another way:

$$\Phi : \mathcal{N}(L_1(\mu), E) = L_\infty(\mu)\tilde{\otimes}_\pi E \hookrightarrow L_1(\mu)\tilde{\otimes}_\pi E = L_1(\mu, E)$$

and $\Phi(T)$ is just the Riesz density of T. (Note that $L_\infty(\mu)$ has the approximation property which implies that the map in the middle is injective, see [6], p.66).

Now suppose that the operator $T : L_1 \to E$ is representable with a Riesz density $\tilde{g} \in L_1(\mu, E) = L_1(\mu)\tilde{\otimes}_\pi E$ (the function g is even bounded!). Looking at

$$L_1(\mu)\tilde{\otimes}_\pi E \hookrightarrow L_1(\mu)''\tilde{\otimes}_\pi E = L_\infty(\mu)'\tilde{\otimes}_\infty E = \mathcal{N}(L_\infty(\mu), E)$$

one sees that $\tilde{g}$ corresponds to a certain nuclear operator $L_\infty(\mu) \to E$. Denoting by $I : L_\infty(\mu) \hookrightarrow L_1(\mu)$ the canonical embedding and using an expression $\tilde{g} = \sum \tilde{g}_n \otimes x_n \in L_1(\mu)\tilde{\otimes}_\pi E$ one gets for $\tilde{h} \in L_\infty(\mu)$

$$(T \circ I)(h) = \int h(\omega)g(\omega)\mu(d\omega) = \sum_{n=1}^{\infty} \int h(\omega)g_n(\omega)\mu(d\omega)x_n =$$

$$\sum_{n=1}^{\infty} \langle \tilde{h}, \tilde{g}_n \rangle_{L_\infty, L_1} \cdot x_n$$

which means $T \circ I = \sum_{n=1}^{\infty} \tilde{g}_n \otimes x_n$ and $T \circ I$ is nuclear: T being representable implies that $T \circ I : L_\infty \to E$ is nuclear. The converse of this is also true:

THEOREM (Grothendieck): *Let μ be a finite measure and E a Banach space. Then $T : L_1(\mu) \to E$ is representable if and only if $T \circ I : L_\infty(\mu) \to E$ is nuclear.*

There are two important additional details for this result:
(1) If μ is a Borel-Radon measure on a compact space K then I may be replaced by the canonical map $I : C(K) \to L_1(K, \mu)$.
(2) $||T|| = ||\tilde{g}||_\infty$ and $\mathrm{N}(T \circ I) = ||\tilde{g}||_1 = \int_\Omega ||g(\omega)||_E \mu(d\omega)$ for the Riesz density g of T. (Here I is either $L_\infty \to L_1$ or $C(K) \to L_1$)

6.3. This nice result has powerful consequences for the theory of operator ideals. Just one example. An operator $T \in \mathcal{L}(E, F)$ is called *integral* if it admits a factorization

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "R"] & F \arrow[r, hook, "\kappa_F"] & F'' \\ L_\infty(\mu) \arrow[rr, "I"] & & L_1(\mu) \arrow[u, "S"] \end{tikzcd}$$

for some finite measure μ. It is called *Pietsch intgral* if the factorization can be choosen such that S takes values in F instead of F''. Notation: $\mathcal{PI}(E,F)$ and $\mathcal{I}(E,F)$. Since F' is complemented is F''' one has

$$\mathcal{PI}(E,F') = \mathcal{I}(E,F')$$

but in general integral operators are not Pietsch integral: Alencar [1] observed that there is a Banach space E (necessarily without the approximation property) and an operator $T \in \mathcal{I}(E,F)$ such that T' even is nuclear, but T is *not* Pietsch integral. On the other hand, it is easy to see that every nuclear operator is Pietsch integral.

THEOREM: *A Banach space F has the Radon-Nikodým property if and only if $\mathcal{PI}(E,F) = \mathcal{N}(E,F)$ for all Banach spaces E.*

Proof: Assume F to have the Radon-Nikodým property and $T \in \mathcal{PI}(E,F)$ then

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ {\scriptstyle R} & \cdot/\cdot & {\scriptstyle S} \\ L_\infty & \xrightarrow{I} & L_1 \end{array}$$

S is representable hence Grothendieck's theorem 6.2. gives that $S \circ T$ is nuclear and therefore also $T = S \circ I \circ R$. Conversely, take $T \in \mathcal{L}(L_1(\mu), F)$ then the operator U

$$U : L_\infty(\mu) \xhookrightarrow{I} L_1(\mu) \xrightarrow{T} F.$$

is Pietsch integral hence nuclear: again Grothendieck's theorem gives that T is representable. Since μ and T were arbitrary, F has the Radon-Nikodým property. □

6.4. One can show that $\mathcal{I}(E,F') = (E \otimes_\varepsilon F)'$ holds – this was Grothendieck's original definition of integral operators (see also Alencar [2]). When is the natural map

$$J : E'\tilde{\otimes}_\pi F' \longrightarrow \mathcal{N}(E,F') \hookrightarrow \mathcal{I}(E,F') = (E \otimes_\varepsilon F)'$$

surjective? Using $\mathcal{PI}(E,F') = \mathcal{I}(E,F')$ the above theorem shows that this is true if F' has the Radon-Nikodým property; J is injective if F' has the approximation property.

COROLLARY: *Let F' have the approximation property. F' has the Radon-Nikodým property if property if and only if*

$$E'\tilde{\otimes}_\pi F' = (E \otimes_\varepsilon F)'$$

for all Banach spaces E.

It follows from 5.3. that $E'\tilde{\otimes}_\pi F' = (E \otimes_\varepsilon F)'$ holds of F' is separable or reflexive (and has the approximation property). This duality relation between the injective and projective tensornorms is an important tool in Functional Analysis.

7. Historical Note

The study of representable operators and vector-valued Radon-Nikodým theorems goes back to the work of Dunford, Pettis and Phillips in the late thirties. Later on, in his thesis 1955, Grothendieck gave a powerful push to the theory (and even extended it to spaces more general than Banach spaces, see also Defant [5]) when he studied, what he called "propriété de Phillips". The important monography of Diestel and Uhl, 1977, presents nearly all about the topic what was known at that time (see also [8]). The present survey offers no new results – only the presentation is different from the former ones.

Bibliography

[1] Alencar, R.: *Multilinear mappings of nuclear and integral type*; Proc. Amer. Math. Soc. 44(1985)33-38

[2] Alencar, R.: *Function spaces and tensor products*; 34º Seminário Brasileiro de Análise, this volume

[3] Botelho. G.: *Aspectos analíticos e geométricos da propriedade de Radon-Nikodým em espaços de Banach*; Tese de mestrado, Unicamp, 1989

[4] Davis, W. J. – T. Figiel – W. B. Johnson – A. Pelczyński: *Factoring weakly compact operators*; J. Funct. Analysis 17(1974)311-327

[5] Defant, A: *The local Radon-Nikodým property for duals of locally convex spaces*; Bull. Soc. Roy. Sci. Liège 53(1984)233-246

[6] Defant, A. – K. Floret: *Tensor norms and operator ideals*; to appear in North-Holland Math. Studies

[7] Diestel, J. – J. J. Uhl: *Vector measures*; Math. Surveys 15, Amer. Math. Soc., 1977

[8] Diestel, J. – J. J. Uhl: *Progress in vector measures 1977-1983*; in: Lecture Notes Math. 1033(1983)144-192

[9] Floret, K.: *Der Satz von Dunford-Pettis und die Darstellung von Massen mit Werten in lokalkonvexen Räumen*; Math. Ann. 208(1974)203-212

[10] Floret, K.: *Maß-und Integrationstheorie*; Teubner, 1981

[11] Grothendieck, A.: *Produits tensoriels topologiques et espaces nucléaires*; Memoirs Amer. Math. Soc. 16, 1955

[12] Jonescu-Tulcea, A. – C. Jonescu-Tulcea: *Topics in the theory of lifting*; Springer, Ergebn. Math. Grenzgeb. 48, 1961

[13] Pietsch, A.: *Operator ideals*; Deutscher Verlag der Wiss., 1978 and North-Holland, 1980

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