EMBEDDINGS OF FRECHET SPACES

IN UNIFORM FRECHET ALGEBRAS

Jorge Mujica

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Abstract. It is shown that every Fréchet space is topologically isomorphic to a complemented subspace of a uniform Fréchet algebra.

Universidade Estadual de Campinas
Instituto de Matemática, Estatística e Ciência da Computação
IMECC – UNICAMP
Caixa Postal 6065
13.081 – Campinas – SP
BRASIL

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Introduction

A well known result of Milne [10] asserts that every Banach space is isometrically isomorphic to a complemented subspace of a uniform Banach algebra. Since Enflo [6] has constructed a Banach space without the approximation property, Milne's theorem implies the existence of a uniform Banach algebra without the approximation property.

In this note we first present Milne's proof from the point of view of infinite dimensional holomorphy, and next follow the same approach to extend Milne's theorem to the realm of Fréchet spaces.

I am grateful to Aleksander Pelczynski for observing that Milne's theorem is closely connected with a result in [13]. That observation ultimately led to the proof of Milne's theorem presented here. I am also grateful to João B. Prolla for some helpful comments.

1. Notation and Terminology

Throughout this note $C(X)$ denotes the locally multiplicatively convex algebra of all complex-valued continuous functions on a topological space X, with the compact-open topology. A uniform Banach algebra (resp. uniform Fréchet algebra) is a closed subalgebra of $C(X)$ which contains the constants and separates the points of X, for a compact Hausdorff space X (resp. hemicompact k-space X).

If U is an open subset of a complex locally convex space E, then $H(U)$ denotes the locally multiplicatively convex algebra of all complex-valued holomorphic functious on U, with the compact-open topology, whereas $H^\infty(U)$ denotes the Banach algebra of all bounded members of $H(U)$, with the sup norm.

If E is a real topological vector space, then the Cartesian product $E \times E$, with the operations

$$(x, y) + (s, t) = (x + s, y + t),$$
$$(\alpha + \beta i)(x, y) = (\alpha x - \beta y, \alpha y + \beta x),$$

is a complex topological vector space, called the complexification of E, and denoted by $E_{\mathbb{C}}$. If E is a real Banach space, then $E_{\mathbb{C}}$ is a complex Banach space for the norm

$$\|(x, y)\| = \sup_\theta \max\{\|x \cos\theta - y \sin\theta\|, \|y \cos\theta + x \sin\theta\|\}.$$

Furthermore, the mapping $x \to (x, 0)$ embeds E isometrically into $E_{\mathbb{C}}$, and the projection $(x, y) \to (x, 0)$ has norm one.

We refer to [9] of [14] for the terminology from topological vector spaces, to [7] for the terminology from topological algebras, and to [5] or [12] for the terminology from infinite dimensional complex analysis.

2. Embeddings of Banach Spaces in Uniform Banach Algebras

2.1 Milne's Theorem [10]. Every Banach space is isometrically isomorphic to a 1-complemented subspace of a uniform Banach algebra.

Proof. Consider first the case of a complex Banach space E. If $U_{E'}$ denotes the open unit ball of the dual E', then the bidual E'' is isometrically isomorphic to a 1–complemented subspace of $H^\infty(U_{E'})$. Indeed, if $S : E'' \hookrightarrow H^\infty(U_{E'})$ is the inclusion mapping, and $T : H^\infty(U_{E'}) \to E''$ is defined by $Tf = Df(0)$ for every $f \in H^\infty(U_{E'})$, then $ToS(x'') = x''$ for every $x'' \in E'', S$ is an isometry and $\|T\| = 1$. If A is the subalgebra of $H^\infty(U_{E'})$ generated by $S(E)$, then $Tf \in E$ for every $f \in A$. Indeed, each $f \in A$ can be written as a sum $f = \sum_{m=0}^{N} f_m$, where

$$f_m(x') = \sum_{j=1}^{jm} c_{mj}(x'(x_{mj}))^m,$$

with $c_{mj} \in \mathbb{C}$ and $x_{mj} \in E$. Since $Df(0) = f_1$, it follows that $Tf \in E$, as asserted. Whence it follows that $Tf \in E$ for every $f \in \overline{A}$, the closure of A in $H^\infty(U_{E'})$. Thus we have a commutative diagram

$$\begin{array}{ccccc} E'' & \overset{S}{\hookrightarrow} & H^\infty(U_{E'}) & \overset{T}{\longrightarrow} & E'' \\ \uparrow & & \uparrow & & \uparrow \\ E & \hookrightarrow & \overline{A} & \longrightarrow & E \end{array}$$

and E is isometrically isomorphic to a 1–complemented subspace of $\overline{A}$. Since $\overline{A}$ may be regarded also as a closed subalgebra of $C(\overline{U}_{E'}, \sigma(E', E))$, the proof is complete in the case of a complex Banach space E.

Finally, if E is a real Banach space, then E is isometrically isomorphic to 1–complemented subspace of its complexification $E_{\mathbb{C}}$. Thus the desired conclusion follows from the first part of the proof.

3. Embeddings of Frechet Spaces in Uniform Frechet Algebras

3.1 Theorem. Every Fréchet space E is topologically isomorphic to a complemented subspace of a uniform Fréchet algebra A. If E is, in addition. Montel, Schwartz or nuclear, then we can take an A with the same property.

Proof. Consider first the case of a complex Fréchet space E, and let E'_c denote the dual of E, with the topology of uniform convergence on the compact subsets of E. Then E'_c is a hemicompact k-space (see [11]), and $H(E'_c)$ is a uniform Fréchet algebra. Since $(E'_c)'_c = E$, the method of proof of Theorem 2.1 shows that E is topologically isomorphic to a complemented subspace of $H(E'_c)$ (see [1] or [5]). If E is a Fréchet–Montel space, then $H(E'_c)$ is a Fréchet–Montel space as well, by a result of Dineen [4]. If E is a Fréchet–Schwartz space, then $H(E'_c)$ is a Fréchet–Schwartz space as well, by a result of Colombeau and Perrot [3]. And if E is a Fréchet–nuclear space, then $H(E'_c)$ is a Fréchet–nuclear space as well, by a result of Boland [2]. This completes the proof in the case of a complex Fréchet space E.

Finally, if E is a real Fréchet space, then E is topologically isomorphic to a complemented subspace of its complexification $E_{\mathbb{C}}$. Thus the desired conclusions follow easily from the first part of the proof.

Since Hogbe–Nlend [8] has constructed a Fréchet–Schwartz space without the approximation property, Theorem 3.1 implies the existence of a uniform Fréchet–Schwartz algebra without the approximation property.

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Instituto de Matemática
Universidade Estadual de Campinas
Caixa Postal 6065
13081 Campinas, SP, Brazil

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