ON THE VELOCITY WHICH APPEARS IN LORENTZ FORCE LAW: AN ILLUMINATING PUZZLE

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RELATÓRIO TÉCNICO Nº 31/90

Abstract. We discuss the meaning of the velocity which appears in Lorentz force law and many reasonable physical possibilities (velocity of the charge relative to an aether, to an inertial frame, to the magnet or wire which generates $\vec{B}$, etc). We show the correct meaning and then present a simple paradox: a magnetized needle placed near an infinite rectilinear charged wire seem by different inertial observers. The solution of this puzzle involves important electromagnetic principles.

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Setembro – 1990

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I - Introduction

Classical electromagnetism is composed of three distinct parts: (A) Maxwell's equations, (B) Constitutive relations depending on the medium (Ohm's law, $\vec{D} = \varepsilon\vec{E}$, etc), and (C) Lorentz force law. This last one states that a point charge q moving in an arbitrary electromagnetic field is acted on by a force

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B} \,. \tag{1}$$

In this equation $\vec{E} = \vec{E}(\vec{r}, t)$ is the electric field at a point $\vec{r}$ where the charge q is located, and $\vec{B} = \vec{B}(\vec{r}, t)$ is the magnetic induction at the same point.

The velocity $\vec{v}$ which appears in Eq. (1) is the instantaneous velocity of charge q. A fundamental question is: Velocity of q relative to what? Of course position, velocity and acceleration are not intrinsic properties of any body and a body A can have different velocities relative to different objects. What is the velocity of a man which is driving a car at 80 km/h? Relative to his own car it is zero, to the Earth it is 80 km/h, to another car which moves in the opposite direction at 60 km/h it is 140 km/h (neglecting relativistic corrections), to the Sun it is approximately 30 km/s, etc.

Physically there are many possible and meaningful possibilities: (A) The velocity of charge q relative to a fixed aether in space, or relative to an aether at rest in the frame of the "fixed stars" (like the aether of Maxwell and Fresnel[1]); (B) relative to the laboratory or to the Earth (or relative to an aether supposing that the Earth moving in space carries (drags) with it the surrounding aether); (C) relative to an inertial frame of reference; (D) relative to an arbitrary observer, not necessarily an inertial one; (E) relative to the macroscopic source of $\vec{B}$ (a magnet or a wire carrying a curre I) and (G) relative to the magnetic field. As a matter of fact in the development of electrodynamics many force laws were proposed with different quantities being relevant to them. In Weber's theory, for instance, which is the oldest of all these models, only the relative velocities and accelerations between interacting charges were important and the force had the same value for all observers[2-4]. In Clausius theory, on the other hand, it was introduced in the force law the velocities of the charges relative to an aether[5-6].

Curiously when most test - books introduce and present Lorentz force law in the first place, they don't state explicitly relative to what is the velocity of the charge q which appears in Eq. (1). Some examples: (I) Symon: "The force exerted by a magnetic field on a charged particle at a point $\vec{r}$ depends on the velocity $\vec{v}$ of the particle, and is given in terms of the magnetic induction $\vec{B}(\vec{r})$ by the equation $\vec{F} = \frac{q}{c}\vec{v} \times \vec{B}(\vec{r})$"[7]; Feynman: "We can write the force $\vec{F}$ on a charge q moving with a velocity $\vec{v}$ as [eq. (1)]." "The force on an electric charge depends not only on where it is, but also on how fast it is moving", [...], "It is possible [...] to write the magnetic force as $q\vec{v} \times \vec{B}$"[8]; Jackson: "Also essential for consideration of charged particle motion is the Lorentz force equation, [eq. (1)], that gives the force acting on a point charge q in the present of electromagnetic fields", "The total electromagnetic force on a charged particle is [eq. (1)]"[9]; Reitz and Milford: "For the purpose of defining the magnetic induction it is convenient to define $\vec{F}_m$, the magnetic force (frequently called the Lorentz force), as that part of the force exerted on a moving charge which is neither electrostatic nor mechanical. The magnetic induction, $\vec{B}$, is then defined as the vector which satisfies $\vec{F}_m = q\vec{v} \times \vec{B}$, for all velocities"[10]; Sears: "Force on a moving charge. [...] A positive charge q, moving with velocity v perpendicular to the direction of the induction, is found to experience a force F in the direction shown, perpendicular to its velocity v and to the induction B. The magnitude of this force is given by $F = qvB$.[...] Evidently, then, the force $\vec{F}$ on a charge q moving with velocity $\vec{v}$ in a magnetic field of flux density $\vec{B}$ is in vector notation, $\vec{F} = q\vec{v} \times \vec{B}$"[11]; Purcell: "We say that an electric current has associated with it a magnetic field which pervades the surrounding space. Some other current, or any moving charged particle which finds itself in this field experiences a force proportional to the strength of the magnetic field in that locality. The force is always perpendicular to the velocity, for a charged particle. The entire force on a particle carrying charge q is given by [eq. (1)] where $\vec{B}$ is the magnetic field. We shall take Eq. (1) as the definition of $\vec{B}$"[12]; Panofsky and Phillips: "According to Lorentz' electron theory, however, the only force which has physical significance is a resultant force which arises from the space-time forces acting on material charges and currents, namely, those obtained by averaging $\vec{F} = \rho(\vec{E} + \vec{u} \times \vec{B})$"[13]. We stop these examples here, although we could increase this list with many more books and authors. Reading all these passages a curious student could ask quite naturally and with complete reason: But velocity of the charge relative to what?

In our opinion this lack of a clear statement of the meaning of the velocity which appears in Eq. (1), when this equation is first presented, is the reason for the confusion of the students at this essential aspect of the theory. When we ask students who had taken courses of electromagnetism (undergraduate or graduate ones!) to explain the meaning of $\vec{v}$ in Eq. (1) we receive all sorts of answers from (A) to (G) above.

Of course when we study carefully any of the books cited above, especially the sections dealing with the special theory of relativity, we grasp the correct answer, letter (C): In Eq. (1) $\vec{v}$ is the velocity of the charge q relative to an inertial frame of reference. If this were stated more clearly when Lorentz force law is presented, the students could understand much more easily the interdependence and mutual transformation of the electric and magnetic fields. This interconnection of $\vec{E}$ and $\vec{B}$ only appears due to this meaning of $\vec{v}$. As a matter of fact, if $\vec{v}$ were, for instance, the velocity of q relative to the macroscopic source of $\vec{B}$ (let us suppose a magnet) then the magnetic force wouldn't change from an inertial frame S to another inertial frame S' moving with $\vec{V}$ relative to S. This is because in this case the magnetic component of Eq. (1) would read $\vec{F}_m = q(\vec{v}_q - \vec{v}_m) \times \vec{B}_m$, where $\vec{v}_q$ and $\vec{v}_m$ are the velocities of the charge and magnet, respectively, relative to an inertial observer. If in frame S, $\vec{v}_m = 0$, then $\vec{F} = q\vec{v}_q \times \vec{B}_m$. If $\vec{V} = \vec{v}_q$ then $\vec{v}\,'_q = 0$ and $\vec{v}\,'_m = -\vec{v}_q$, so that $\vec{F}\,' = q'(0 - (-\vec{v}_q)) \times \vec{B}\,'_m = \vec{F}$, because $\vec{B}\,'_m = \vec{B}_m$ and $q' = q$. This indicates clearly that the transformation properties of $\vec{E}$ and $\vec{B}$ into $\vec{E}\,'$ and $\vec{B}\,'$, and vice-versa arises (is necessary) because $\vec{v}$ in Lorentz force law has different values in different inertial frames.

II - The Paradox

Almost all books dealing with electromagnetism from the point of view of the special theory of relativity discuss the problem of a rectilinear infinite wire, charged or not, carrying a current I and exerting a force on an external charge q at rest or moving parallel to the wire. They analyse this situation in different frames of reference moving parallel to the wire and arrive at the famous transformation laws for the fields.

In this work we present another situation, slightly different from this one. It is quite simple, but extremely illuminating. In figure 1a is represented a rectilinear infinite wire along the Z axis charged uniformly with a linear charge density λ. It generates a radial electric field $\vec{E} = \lambda\hat{\rho}/(2\pi\varepsilon_0\rho)$. In this same inertial frame of reference S in which the charges of the wire are at rest, it is placed at rest a magnetized needle NS. Its axis $N-S$ is placed parallel to the wire and although free to turn in any direction it remains at rest (no one has ever observed an action of an electric field in a magnet). In figure 1b this same system is seem from an inertial frame S' moving with a constant velocity $\vec{v} = -v\hat{z}$ relative to S. As in S there is no magnetic field, the transformation laws (see any of the references listed above) give the fields in S' as:

$$\left.\begin{aligned}
\vec{E}\,'_\rho &= \gamma\vec{E}_\rho = \left(1-\frac{v^2}{c^2}\right)^{-1/2}\frac{\lambda\hat{\rho}}{2\pi\varepsilon_0\rho}\,, \\
\vec{E}\,'_\theta &= \vec{E}_\theta = 0\,, \\
\vec{E}\,'_z &= \vec{E}_z = 0\,, \\
\vec{B}\,' &= -\frac{\vec{v}}{c^2}\times\vec{E}\,' = \frac{v\mu_0\gamma\lambda}{2\pi\rho}\hat{\theta}\,, \\
\vec{B}\,'_\rho &= \vec{B}\,'_z = 0\,.
\end{aligned}\right\} \qquad (2)$$

All these results can also be calculated straight alway in S' using that in this frame there are charges moving to the right with $v\hat{z}$ (and then generating a magnetic field $\vec{B}\,'$), and the linear charge density is now $\lambda' = \gamma\lambda$.

The paradox appears because while in the frame S there was no torque in the magnetic needle, there is a magnetic field in S' which should turn the need so as to let it orthogonal to the wire (this was the famous discovery of Oersted of the deflection of a magnetic needle by a long current carrying wire!). Of course the needle cannot be parallel to the wire in S and orthogonal to it in S' as both are the same system only seem by different inertial frames. But what is the source (in S') of the opposite torque which will balance that of the magnetic field $\vec{B}$ '?

In figures 2a to 2d we replaced the magnetic needle by a single neutral circuit *abcd* where flows the constant current I. This is presented as an analogous system reduced to its essential aspects so that a solution for this puzzle can be more easily obtained. In these figures *ab* and *cd* are arcs of circle centered on the Z axis ($x = y = 0$, $\rho_{ab} = \rho_0 =$ constant for any θ ; $\rho_{cd} = \rho_1 =$ constant for any θ , $\rho_1 > \rho_0$). *bc* and *da* are the radial parts of the circuit. As a magnet or this circuit are neutral, the electric field will have no influence on them. But in S' there is a poloidal magnetic field which will act on the circuit *abcd* by Grasmann's force (sometimes known as Biot-Savart's law)

$$d\vec{F} = Id\vec{\ell} \times \vec{B} \,. \tag{3}$$

This expression can be obtained from Eq. (1) in this case of neutral currents. This is easily done taking Eq. (1) and adding the forces acting on the opposite charges of the neutral current element, q_+ and $q_- = -q_+$:

$$\begin{aligned} \vec{F} &= \vec{F}_+ + \vec{F}_- = (q_+\vec{E} + q_+\vec{v}_+ \times \vec{B}) + \\ &+ (q_-\vec{E} + q_-\vec{v}_- \times \vec{B}) = q_+(\vec{v}_+ - \vec{v}_-) \times \vec{B} = \\ &= Id\vec{\ell} \times \vec{B} \,. \end{aligned} \tag{4}$$

Usually we have $\vec{v}_+ = 0$ so that $Id\vec{\ell} = q_-\vec{v}_- = -q_+\vec{v}_-$. Anyway this is not necessary and, for instance, we can add a velocity $\vec{v}$ to the positive and negative charges of a wire (as when the wire itself is moving in space) that even so it will remain valid $Id\vec{\ell} = q_+\vec{V}_+ + q_-\vec{V}_- = q_+(\vec{v}_+ - \vec{v}_-)$ where now $\vec{V}_+(\vec{V}_-)$ is the total velocity of $q_+(q_-)$ composed of the drifting velocity $\vec{v}_+(\vec{v}_-)$ and of the velocity of the wire as a whole, $\vec{v}$: $\vec{V}_+ = \vec{v} + \vec{v}_+$ and $\vec{V}_- = \vec{v} + \vec{v}_-$.

From (2) and (3) we see immediately that there is no force in *ab* or *cd*, while the forces on *bc* or *da* are oppositely directed, as to generate a torque on the circuit. What is the source of the opposite torque which opposes this one?

Following the good example of Feynman and his famous paradoxical experiment[14], we will not give an answer to this problem. With his unsolved problem Feynman let most of us thinking about a deep problem in electromagnetic theory and gave us a golden opportunity to learn and understand important physical concepts. With this problem (simple to state, delicate to solve), we hope to give to students and teachers of physics a novel chance to improve their own feeling of what is meaningful in the natural world.

Acknowledgements - One of the authors (A. K. T. Assis) wishes to thank Fundação de Amparo à Pesquisa do Estado de São Paulo, FAPESP, and Conselho Nacional de Desenvolvimento Científico e Tecnológico, CNPq (Brazil), for financial support during the last years.

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Figure Captions

Figure 1 - (A) System seem from frame S: An infinite rectilinear charged wire at rest along the Z-axis generating a radial electric field $\vec{E}$. NS is a magnetized needle, also at rest in this frame, with its axis of symmetry parallel to the Z-axis.

(B) The same system as in Fig. 1a, but now seem from a frame S' which has a velocity $\vec{v} = -v\hat{z}$ relative to S. λ and $\vec{E}$ are now transformed to $\gamma\lambda$ and $\gamma\vec{E}$, and in addition to this there appears a poloidal magnetic field $\vec{B}$ which should exert a torque on the magnetic needle.

Figure 2 - (A) and (B): The same as in figure 1a, but now the magnetic needle was replaced by a neutral electrical circuit *abcd* where is flowing the constant current I. This circuit is in the XY plane while the charged line is along the Z-axis. *bc* and *da* are radial segments of current while *ab* and *cd* are segments of the circuit along arcs of circle centered on the Z-axis.

(C) and (D): The same as figs. 2 *a* and *b* seem from a frame S' which has a velocity $\vec{v} = -v\hat{z}$ relative to S. The torque due to the poloidal $\vec{B}$ field should act on the radial sides *bc* and *da*.

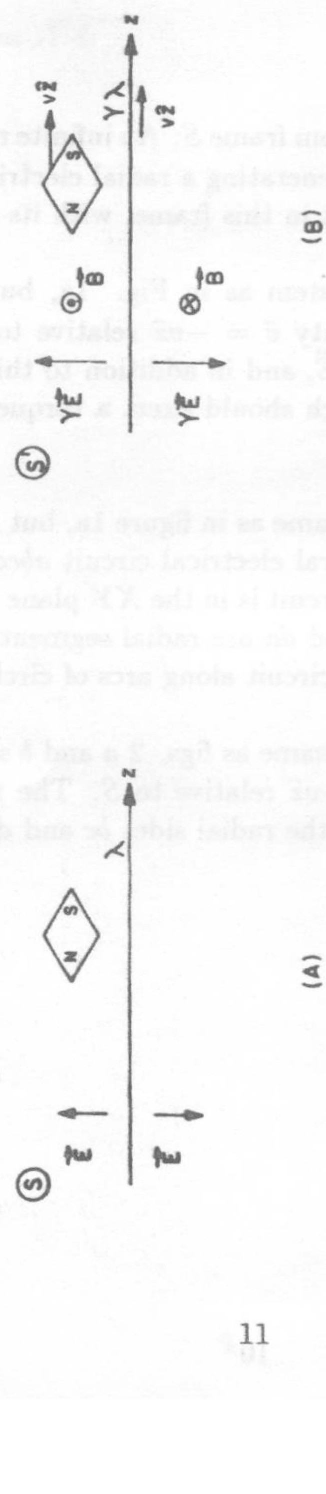

Figura 1

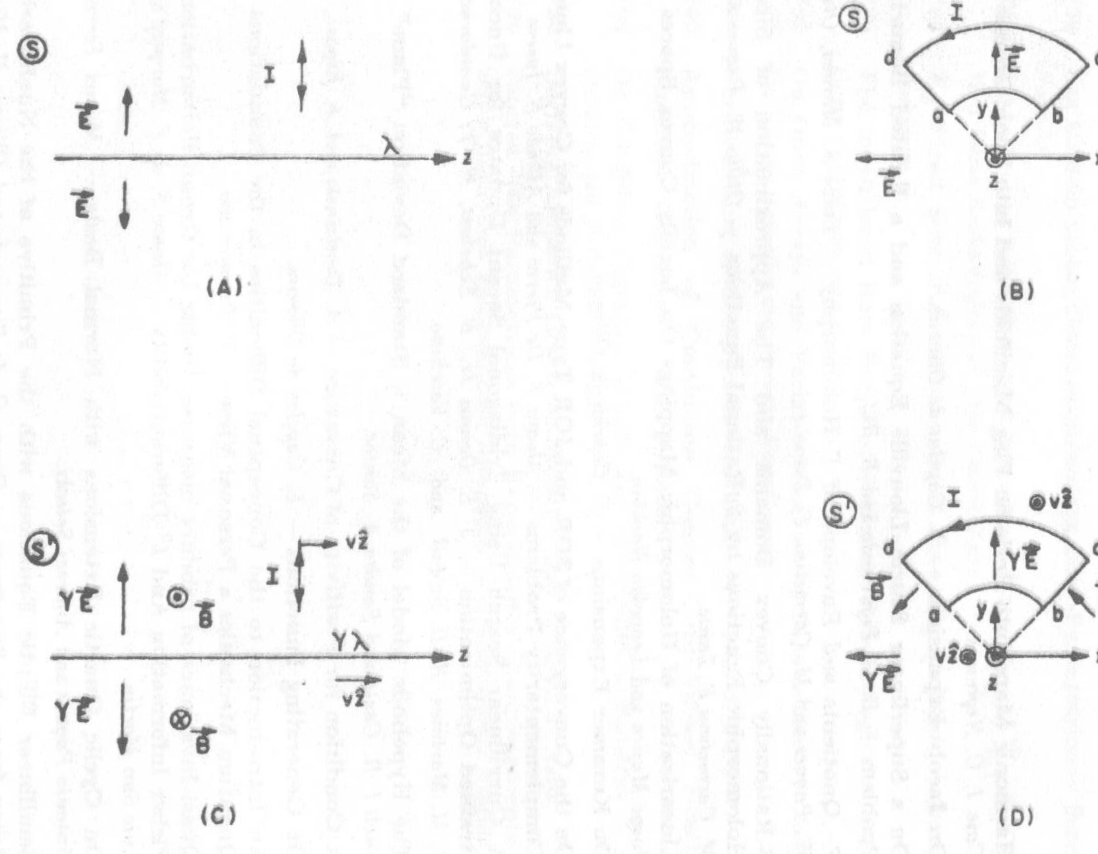

Figura 2

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