ON THE SIMPLE QUOTIENTS OF TOURNAMENTS THAT ADMIT EXACTLY ONE HAMILTONIAN CYCLE

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RELATÓRIO TÉCNICO Nº 38/89

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Outubro – 1989

On the Simple Quotients of Tournaments that Admit Exactly one Hamiltonian Cycle

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Introduction

In 1960, P. Camion [3] gave some characterizations for the tournaments admitting a unique hamiltonian cycle. In 1969, R.J. Douglas [5] described a structural characterization of these tournaments. In this paper we give a presentation of these tournaments, which we call *Douglas tournaments*, by relating them to the class of the normal tournaments by means of the following result (Theorems 3.10 and 3.11):

Theorem: Let H_n be a hamiltonian tournament with $cc(H_n) = k \geq 3$. H_n is a Douglas tournament if and only if:

1.1) H_n has as simple quotient a tournament Q_m $(m \geq 5)$ such that:

a) Q_m is normal;

b) the subtournament of the poles in Q_m is transitive;

c) the poles of Q_m are all of class 1;

d) between two poles x_i and x'_j of Q_m of class 1, the following rules of adjacencies hold $x_i \to x'_j$ implies $j \leq i+1$.

1.2) H_n can be constructed from Q_m by replacing all the vertices of Q_m but the vertices $a_2, \ldots, a_{k-1}$ of its characteristic cycle A_k, by some transitive tournament.

2) H_n is the composition of a singleton and two transitive tournaments with a 3-cycle.

AMS Subject Classification (1980) - 05C20.

1. Notations and Preliminary Results

A *tournament* T_n is a digraph in which every pair of vertices is joined by exactly one arc. We will often identify T_n or a cycle C with their vertex sets $V(T_n)$ or $V(C)$.

If the arc joining the vertices v_i and v_j is directed from v_i to v_j, v_i is said to be a *predecessor* of v_j and v_j a *successor* of v_i and we prefer write $v_i \rightarrow v_j$ instead of (v_i, v_j). Moreover $A \rightarrow B$ denotes that the vertices of the subtournament A are all predecessors of all the vertices of the subtournament B.

Let T_n be a tournament of order n. A cycle C of T_n is *coned* by a vertex v (i.e. v cones C) if there exists a vertex $v \in V(T_n) - V(C)$ such that either $v \rightarrow V(C)$ or $V(C) \rightarrow v$.

Let H_n be a hamiltonian tournament. If no vertex of $H_n - C$ cones C , C is said to be *non-coned* in H_n and the vertices of $H_n - C$ are called *poles* associated to C.

A non-coned cycle C of H_n is *minimal* if each cycle C' such that $V(C') \subset V(C)$ is coned by at least one vertex of H_n. A minimal cycle is *characteristic* if its possesses the shortest lenght of the minimal cycles and its length is the *cyclic characteristic* $cc(H_n)$ of H_n. Then the difference $n - cc(H_n)$ is called *cyclic difference* $cd(H_n)$ of H_n.

A vertex v of H_n is called a *neutral vertex* of H_n if $H_n - v$ is hamiltonian, and the number of neutral vertices of H_n is denoted by $\nu(H_n)$. We recall that between $\nu(H_n)$ and $cd(H_n)$, the inequality $\nu(H_n) \geq cd(H_n)$ holds (see [2]).

Definition 1.1: The tournament A_n $(n \geq 4)$, with vertex set $V(A_n) = \{a_1, a_2, \ldots, a_n\}$ and arc set $E(A_n) = \{a_i \rightarrow a_j / j < i - 1 \text{ or } j = i + 1\}$ is called the *bineutral tournament of order n*.

Remark: It is known that A_n is the unique hamiltonian tournament that has exactly 2 neutral vertices. Moreover, for $n \geq 5$, $\{a_1, a_2, a_{n-1}, a_n\}$ is its maximal transitive subtournament.

The vertices of a subtournament S of T are *equivalent* if each vertex, not in $V(S)$, cones $V(S)$. If the vertices of T_n can be partitioned into disjoint subtournaments $S^{(1)}, S^{(2)}, \ldots, S^{(n)}$ of equivalent vertices and R_m denotes the tournament of the m vertices $w_1, w_2, \ldots, w_n$ in which $w_i \rightarrow w_j$ if and only if $S^{(i)} \rightarrow S^{(j)}$, then $T_n = R_m(S^{(1)}, S^{(2)}, \ldots, S^{(m)})$ is the *composition* of the m *components* $S^{(1)}, S^{(2)}, \ldots, S^{(m)}$ with the *quotient* R_m.

A tournament T_n is *simple* if $T_n = R_m(S^{(1)}, S^{(2)}, \ldots, S^{(m)})$ implies that either $m = 1$

or $m = n$.

We also recall the following results about the quotient tournaments:

Proposition 1.2: The quotient tournament R_m is isomorphic to a subtournament of T_n.

Proposition 1.3: Every tournament T_n, with $n \geq 2$, admits exactly one simple quotient tournament.

Proposition 1.4: A tournament H_n is hamiltonian if and only if every one of its quotient tournament is hamiltonian.

Proposition 1.5: Let H_n be a hamiltonian tournament and Q_m one of its quotient tournaments, then $cc(H_n) = cc(Q_m)$.

Now we recall the definitions and the structural characterization of normal tournaments (see Nos. 3 and 6 of [4]).

Definition 1.6: A hamiltonian tournament H_n is said to be *normal* if it contains only one minimal cycle (the characteristic one) or equivalently if $cd(H_n) = \nu(H_n)$.

Proposition 1.7: Let H_n be a normal tournament, then its characteristic cycle C_k is either the 3-cycle A_3 $(k = 3)$ or a bineutral tournament $A_k (k \geq 4)$.

Proposition 1.8: Let H_n be a normal tournament with cyclic characteristic k $(k \geq 3)$ and let A_k be its characteristic cycle. A pole z, associated to A_k, must have the following adjacencies with respect to A_k:

1) $(a_{i+1}, a_{i+2}, \ldots, a_k) \to z \to (a_1, a_2, \ldots, a_i)$ $i = 1, 2, \ldots, k-1$;

2) $(a_i, a_{i+2}, a_{i+3}, \ldots, a_k) \to z \to (a_1, a_2, \ldots, a_{i-1}, a_{i+1})$ $i = 1, 2, \ldots, k-1$.

Definition 1.9: The pole z is called *pole of kind i and class* 1 *or class* 2 and denoted by x_i or y_i if its adjacencies are given by the previous conditions 1) or 2) respectively.

Theorem 1.10: A tournament H_n is normal if, and only if, the following conditions are true:

a) H_n allows a bineutral tournament A_k $(k \geq 4)$ or the 3-cycle A_3 as a minimal tournament;

b) the poles associated to A_k are of the $(k-1)$ kinds of class 1 $(x_1, x_2, \ldots, x_{k-1})$ and of the $(k-1)$ kinds of class 2 $(y_1, y_2, \ldots, y_{k-1})$, considered in Definition 1.9;

c) the subtournament P_{n-k} of the poles associated to A_k is not hamiltonian;

d) P_{n-k} allows the following transitive composition:

$$P_{n-k} = Tr^*_{k+1}(T^{(0)}, T^{(1)}, \dots, T^{(k)}) \text{ (that is, } T^{(i)} \rightarrow T^{(j)} \iff i > j)$$

where:

$T^{(0)}$	is formed by poles of type	x_1
$T^{(1)}$	is formed by poles of type	x_1, x_2, y_1
$T^{(2)}$	is formed by poles of type	x_1, x_2, x_3, y_2
....		
$T^{(i)}$	is formed by poles of type	$x_{i-1}, x_i, x_{i+1}, y_i$
....		
$T^{(k-2)}$	is formed by poles of type	$x_{k-3}, x_{k-2}, x_{k-1}, y_{k-2}$
$T^{(k-1)}$	is formed by poles of type	$x_{k-2}, x_{k-1}, y_{k-1}$
$T^{(k)}$	is formed by poles of type	x_{k-1},

and each component different from $T^{(0)}$ and $T^{(k)}$ can be empty or not.

2. The $\mathcal{D}$ Tournaments

Definition 2.1: We call a ***tournament of Douglas*** every tournament that admits exactly one hamiltonian cycle. We denote by D_n one such tournament of order n, and by $\mathcal{D}$ the class of all these tournaments. We denote by $v_1 \Longrightarrow v_2 \Longrightarrow \cdots \Longrightarrow v_n \Longrightarrow v_1$ the hamiltonian cycle of D_n.

We now prove the following propositions:

Proposition 2.2: Every component $S^{(i)}$ of a D_n tournament is transitive.

Proof: In fact, if it were not transitive then we would have at least two distinct hamiltonian paths in $S^{(i)}$, hence two hamiltonian cycles in D_n. □

Proposition 2.3: Every hamiltonian subtournament H_m of a tournament D_n is a D_m tournament.

Proof: If $m = n - 1$ this is true, for otherwise every hamiltonian cycle of D_m would be extended to a hamiltonian cycle of D_n.

If $m < n - 1$, two different possibilities must be considered:

1) There exists a chain of Hamiltonian subtournaments $H'_{m+1}, H'_{m+2}, \dots, H'_{n-1}$ such that $V(H_m) \subset V(H'_{m+1}) \subset \cdots \subset V(H'_{n-1}) \subset V(H_n)$. Then step by step, it follows that $H'_{n-1} \in \mathcal{D}, H'_{n-2} \in \mathcal{D}, \dots, H_m \in \mathcal{D}$.

2) There exists a hamiltonian subtournament H'_s $(m \leq s < n-1)$, such that between

H_m and H'_s there is a chain of hamiltonian subtournaments as in 1), whereas, for each subtournament T'_{s+1} such that $V(H'_s) \subset V(T'_{s+1})$, T'_{s+1} is not hamiltonian, i.e. H'_s is coned by all the vertices $V(H_n) - V(H'_s) = \{v_1, v_2, \ldots, v_{n-s}\}$. Then H_n must be the composition $H_n = H'_{n-s+1}(\{v_1\}, \{v_2\}, \ldots, \{v_{n-s}\}, H'_s)$, which is a contradiction to Proposition 2.2. □

Corollary 2.4: If $D_n \in \mathcal{D}$ is the composition $D_n = R_m(S^{(1)}, S^{(2)}, \ldots, S^{(m)})$, then $R_m \in \mathcal{D}$.

Proof: Since R_m is isomorphic to a subtournament of D_n, R_m belongs to $\mathcal{D}$ by Proposition 2.3. □

Corollary 2.5: A hamiltonian tournament T_n $(n \geq 4)$ is a D_n tournament if and only if for every quadruple of different vertices a, b, c, d such that the pairs (a, b) and (c, d) are separated in a cycle of T_n, then the same pairs are separated in every other cycle of T_n containing them.

Proof: If T_n is a D_n tournament then the result is obvious, since the induced ordering is the same in all the hamiltonian subtournaments of D_n.

On the other hand, we consider a hamiltonian tournament T_n which is not a D_n tournament, and let C and C' be two of its hamiltonian cycles. There exist two consecutive vertices a and c in C and non consecutive in C'. Hence there exist a vertex b in the path in C' joining a to c, and a vertex d in the path in C' from c to a. Therefore the pairs (a, b) and $(c. d)$ are separated in C' and not in C. □

3. Relations between $\mathcal{D}$ and Normal Tournaments

We now characterize the tournaments of Douglas in terms of normal tournaments.

Theorem 3.1: Every characteristic cycle C_k of a D_n tournament (with $cc(D_n) = k \geq 4$) is bineutral.

Proof: We shall prove by absurd. So let's assume there exist 3 neutral vertices in C_k, namely x, y, z, then there exist three poles called x', y', z' coning $C_k - x$, $C_k - y$, $C_k - z$ (see Lemma 15, [2]), respectively. If, for example, $x' \to y', x \to x'$ and $y \to y'$, we obtain $x \to x' \to (C_k - x)$, $y \to y' \to (C_k - y)$.

We have to consider two different cases:

(i) The hamiltonian path from x to y in C_k contains some other vertices, let's say u

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