AUTOMORPHISMS CONTROL SYSTEMS AND OBSERVABILITY

Victor Ayala Bravo

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Abstract. In this paper we consider the observability problem for a particular class of nonlinear systems on Lie groups that we called automorphisms control systems, in wich, every associated vector fields induces a one parameter groups of automorphisms of the state space. Such systems generalizes the so called bilinear systems wich are useful to approximate a large class of nonlinear systems and the result of this work characterize the observability of the direct product of automorphisms control systems wich is closely related with that sort of approximation.

Universidad Austral de Chile
Instituto de Matemática
Casilla 567 Valdivia, Chile
Professor Visitante do IMECC

Universidade Estadual de Campinas
Instituto de Matemática, Estatística e Ciência da Computação
Caixa Postal 6065
13.081 - Campinas - SP
BRASIL

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§1 – INTRODUCTION

A control system Σ is defined by the specification of the following data:

$$\dot{x} = F(x,u) \ , \ x \in M \ , \ u \in \Omega \ , \ h : M \to N,$$

where M and N are smooth manifold and h a differentiable function.

The family of associated vector fields of Σ is

$$D_\Sigma = \{x^u = F(\cdot\, , u) | u \in \Omega\},$$

Σ induces a "group" (G_Σ) and a "semi-group" (S_Σ) of local diffeomorphisms of M,

$$G_\Sigma = \{X^1_{t_1} \circ \ldots \circ X^k_{t_k} \mid t_j \in \mathbb{R}\, , \ X^j \in D_\Sigma\} \ ,$$

$$S_\Sigma = \{X^1_{t_1} \circ \ldots \circ X^k_{t_k} \mid t_j \geq 0\, , \ X^j \in D_\Sigma\}.$$

The orbit by Σ of $x \in M$ is the orbit by the action of G_Σ in x, i.e.

$$G_\Sigma(x) = \{\varphi(x) | \varphi \in G_\Sigma\}.$$

Σ is said to be transitive if for some $x \in M$, $G_\Sigma(x) = M$. This equivalence relation partitions M into smooth manifolds, [11].

Two elements $x, y \in M$ are called indistinguishable, written $x \sim y$, if

$$(h \circ \varphi)(x) = (h \circ \varphi)(y) \ , \ \forall \varphi \in S_\Sigma.$$

Indistinguishability is not equivalence relation unless some additional assumptions are made. For example, Σ analytic or forward complete are sufficient conditions to guarantee this property, [12].

Σ is said to be [locally] observable in $x \in M$ in case the equivalence class $C(x)$ of x by $\sim$ is trivial [in some neighbourhood of x], i.e., the family of applications $h \circ S_\Sigma$

separate x of any point of M [in some neighbourhood of x].

The basic questions related to this set-up is, whether a control system is [locally] observable or not and the negative case, to know under which conditions it is possible to construct a [weakly] minimal realization to Σ, i.e., to define a new system Σ' transitive, [locally] observable and a submersion $\pi : M \to M'$ such that for every $x \in M$,

$$(h \circ \varphi)(x) = (h' \circ \varphi')(\pi(x)) \ , \ \forall \varphi \in S_\Sigma,$$

where $\varphi' \in S_{\Sigma'}$ is determinated by the same control that define φ and $h = h' \circ \pi$.

Many authors have considered these problems. For instance in Hermann and Krener [9], Bastos [4], and Ayala [1] various co-distributions are defined and weakly minimal realizations are given for Σ; in Sussmann [12], Gauthier and Bornard [6], [7], San Martin and Ayala [3] are given minimal realizations for Σ studying regularity of equivalence relation. This work is related to particular analytic systems that we called automorphisms control systems (a.c.s.), i.e., non-linear systems where,

$M = G$, $N = G$, are Lie groups,

$u \in \Omega \subset \mathbb{R}^r$

$h \in \text{Hom}(G, G_*)$ is a continuous homomorphism and the dynamic F is given by

$$F(x,u) = X^0(x) + \Sigma_{i=1}^r u_i X^i(x)$$

with the additional condition: for every $i = 0, 1, \ldots, r$, X^i is an infinitesimal automorphism of G, i.e.,

$$X_t^i \in \text{Aut}(G), \ \forall t \in \mathbb{R} .$$

In particular, G_Σ (S_Σ) is a group (semi-group) of $\text{Aut}(G)$, the group of automorphisms of G.

We observe that a.c.s.'s generalizes the so-called bilinear systems (of theoretical and practical importance, [10]). In fact, a bilinear system is obtained when,

$$G = \mathbb{R}^m \ , \ G_s = \mathbb{R}^p \ , \ X^i \in M_m(\mathbb{R}) \text{ for } i = 0, 1, \ldots, r$$

$$\text{and} \quad h \in \text{Lin}(\mathbb{R}^m, \mathbb{R}^p).$$

In this case, $X_t^i = e^{tX^i} \in GL_m(\mathbb{R}), \ \forall t \in \mathbb{R} \ , \ i = 0, 1, \ldots, r.$

The specific interest in this paper. is to study observability of a.c.s.'s. Bilinear systems are also useful to approximate a large class of non-linear systems [5], [2] and the result of this work (Theorem 3.2) characterize the observability of the direct product of a.c.s.'s systems which is closely related with that sort of approximation.

§2 – OBSERVABILITY OF A.S.C.

Let Σ be an a.c.s. as defined in §1 and let $\mathcal{U}$ be the class of all piecewise constant control defined on $[0, +\infty)$ with values in Ω. We denote by 1 the identity of G.

We have the following basic fact:

Lema 2.1:

$C(1)$ is a G_Σ-invariant closed subgroup of G.

Proof:

1. Since h is a homomorphism then

$$C(1) = \{x \in G | S_\Sigma(x) \subset \text{Ker}(h)\}.$$

Since the vector fields are infinitesimal automorphims, we see that if $\varphi_t^u \in S_\Sigma$ and $x, y \in G$ then

$$\varphi_t^u(x \cdot y^{-1}) = \varphi_t^u(x) \cdot \varphi_t^u(y)^{-1}, \ \forall u \in \mathcal{U}.$$

In particular, since $\mathrm{Ker}(h)$ is a subgroup of G, it follows that

$$\varphi_t^u(y)^{-1} \in \mathrm{Ker}(h) \ , \ \text{for} \ y \sim 1.$$

Thus,

$$x, y \in C(1) \Longrightarrow S_\Sigma(x \cdot y^{-1}) \subset \mathrm{Ker}(h),$$

Consequently, $C(1)$ is a subgroup of G.

Let $(x_n) \subset C(1)$ be with $x_n \to x$. For every $y \in S_\Sigma(x)$, $\exists u \in \mathcal{U}$,

$$u = u_1 * \ldots * u_k \ , \ u_1, \ldots, u_k \in \Omega,$$

$\exists t_1, \ldots, t_k \geq 0$, $t =: (t_1, \ldots, t_k)$, satisfying

$$\varphi_t^u =: X_{t_1}^{u_1} \circ \ldots \circ X_{t_k}^{u_k} \in S_\Sigma \text{ and}$$

$$y = \varphi_t^u(x).$$

Clearly, by the continuity of φ_t^u it follows that

$$\varphi_t^u(x_n) \to y.$$

Now, for every $n \in \mathbb{N}$

$$\varphi_t^u(x_n) \subset S_\Sigma(x_n) \subset \mathrm{Ker}(h)$$

then,

$$y \in \mathrm{Ker}(h).$$

Thus, $x \in C(1)$ at hence this equivalence class is a closed subgroup.

The G_Σ-invariance follows by analytic continuity. In fact, for every $x \in C(1)$, for every $X^{u_1}, \ldots, X^{u_k} \in D_\Sigma$, $t_1, \ldots, t_k \geq 0$, we have:

$$h(X_{t_1}^{u_1} \circ \ldots \circ X_{t_k}^{u_k}(x)) = 1_{G_*}.$$

Since h is a continuous homomorphims h is analytic, [8], and the foregoing equality is true for $t_1, \ldots, t_k \in \mathbb{R}$. Then. G_Σ preserve $C(1)$. □

Proposition 2.2:

Let Σ be an a.c.s. on G. Then,

$\mathrm{Ker}(h)$ does not contain any nontrivial S_Σ-invariante subgroup.

Proof:

Let H be a not trivial S_Σ-invariant subgroup of $\mathrm{Ker}(h)$.

If $x \in H$ and $x \neq 1$ we have

$$S_\Sigma(x) \subset H \subset \mathrm{Ker}(h),$$

in particular $x \sim 1$ and therefore Σ is not observable.

Conversely suppose that Σ is not observable then, the indistiguishability class of $1 \in G$ is not trivial and the result follows from Lemma 2.1 □

Corollary 2.3:

Let Σ be an a.c.s. transitive on G. Then,

Σ is observable $\iff h$ isn't the zero homomorphism.

Proof:

Let $x \in C(1)$ be the transitivity of Σ, i.e., the equality $G_\Sigma(x) = G$, and the G_Σ-invariance of $C(1)$ shows that $G \subset \mathrm{Ker}(h)$. Thus $C(1)$ is trivial.

The converse is clear since (implicity) we suppose $G \neq \{1\}$ □

Corollary 2.4:

Let be Σ be an a.c.s. on G, then,

$$\Sigma/_{C(1)} =: (G/_{C(1)}, \Omega, \mathcal{U}, F^{*}, G_{s}, h')$$

is observable.

Proof:

We denote by $G \xrightarrow{\pi} G/_{C(1)}$, the projection of G on the homogeneous manifold $G/_{C(1)}$. This submersion induce vector fields

$$\tilde{X}^{i} = \pi_{*}X^{i}\ ,\quad i = 0, 1, \ldots, r.$$

In fact, $C(1)$ is G_{Σ}–invariant and each X^{i}_{t} with $t \in \mathbb{R}$ is an automorphism of G.

F^{*} is definid by

$$F^{*}(\pi(x), u) = \tilde{X}^{0}\ (\pi(x)) + \Sigma_{i=1}^{r} u_{i}\ \tilde{X}^{i}\ (\pi(x)),\quad u \in \Omega.$$

The output function h' of Σ^{*}, satisfies the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\pi} & G/_{C(1)} \\ h & & h' \\ & G_{s} & \end{array}$$

In fact,

$$x \sim y \implies y \cdot x^{-1} \sim 1\ ,$$

in particular, $h(x) = h(y)$ for every $y \in C(x)$.

Therefore, $\Sigma/_{C(1)}$ is well defined and the observability of this control systems follows from equality

$$C(C(1)) = \{C(1)\} \quad \square$$

Remark:

Is also clear that if Σ is transitive then, $\Sigma' = \Sigma/_{C(1)}$ is a minimal realization of Σ. This fact is a particular case of a minimal realization Theorem for analytic systems due to Sussmann, [12].

Example:

Let Σ um a.c.s. on $\mathbb{R}^m$ given by

$$F(x,u) = A^0 \cdot x + \Sigma_{i=1}^r u_i A^i \cdot x$$

where $A^i \in M_m(\mathbb{R})$, $i = 0,1,\ldots,r$, $\Omega \subset \mathbb{R}^r$, G_s an arbitrary Lie group and $h \in \text{Hom}(\mathbb{R}^m, G_s)$ a non-zero continuous homomorphism.

It is known that the Lie algebra generated by the fields $A^0, A^1, \ldots, A^r$ is integrated by a unique connected Lie subgroup H of $GL_m(\mathbb{R})$. Σ induces a right invariant system Σ_I on H and

$$G_{\Sigma_I}(\text{Id.}) = H.$$

In particular, each orbit of Σ is calculated by the action of H, i.e.,

$$G_\Sigma(x) = H(x).$$

Thus, the Corollary 2.3 implies that Σ restrict to $H(x)$ is observable, for every $x \in \mathbb{R}^m$.

$$\text{Let, } A^0 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Is clear that the Lie algebra generated by the fields A^0 and A^1 is the subalgebra of the anti-symmetric matrices in $M_3(\mathbb{R})$, therefore $H = SO(3)$ and then the orbits of Σ

are 2-dimensional spheres. It follows that this system restrict to these spheres in $\mathbb{R}^3$ is observable.

§3 – OBSERVABILITY OF THE DIRECT PRODUCT OF A.C.S.

We consider the following a.c.s.'s,

$$\Sigma_1 = (G_1, \Omega, \mathcal{U}, F_1, G_s, h_1),$$
$$\Sigma_2 = (G_1, \Omega, \mathcal{U}, F_2, G_s, h_2),$$

where, for every constant control $u \in \Omega \subset \mathbb{R}^r$

$$F_1(x, u) = X^0(x) + \Sigma_{i=1}^r u_i X^i(x),$$
$$h_1 \in \text{Hom.}(G_1, G_s),$$
$$F_2(y, u) = Y^0(y) + \Sigma_{i=1}^r u_i Y^i(y),$$
$$h_2 \in \text{Hom.}(G_2, G_s)$$

and $\mathcal{U}$ is the class of piecewise constant control with values in Ω.

Definition 3.1:

The direct product of Σ_1 and Σ_2 written $\Sigma_1 \bigoplus \Sigma_2$, is

$$\Sigma_1 \bigoplus \Sigma_2 = (G_1 \times G_2, \Omega, \mathcal{U}, F, G_s, h)$$

where,

$$F(x, y, u) = (F_1(x, u), F_2(y, u))$$
$$\text{and} \quad h \in \text{Hom.}(G_1 \times G_2, G_s)$$

is defined by

$$h(x,y) = h_1(x) \cdot h_2(y).$$

By the very if is evident that $\Sigma_1 \oplus \Sigma_2$ is an a.c.s.. Next, we characterize the observability of this system assuming that every component of the product is observable.

Theorem 3.2:

For $j = 1,2$ let Σ_j be an observable a.c.s. on G_j with output group G_s and output function h_j. Then,

$$\Sigma_1 \bigoplus \Sigma_2 \text{ is not observable } \iff$$

1. $\exists H_j \subset G_j$ a nontrivial G_{Σ_j}-invariant subgroup and a continuous isomorphism φ,

$$H_1 \xrightarrow{\varphi} H_2.$$

2. $\exists W_j \subset G_j$ a non trivial connected Lie subgroup such that $W_j \subset H_j \subset G_j$ and an isomorphism ψ

$$G_{\Sigma/w_1} \xrightarrow{\psi} G_{\Sigma/w_2}$$

with the following commuting property

$$\varphi \circ g = \psi(g) \circ \varphi\,, \;\; \forall g \in G_{\Sigma/w_1}.$$

3. $h_1/_{H_1} \cdot \left(h_2/_{H_2} \circ \varphi\right) = 1_{G_s}.$

Proof:

1. Let $C(1,1)$ be the indistinguishability class of $(1,1)$ by $\Sigma_1 \oplus \Sigma_2$. The Lemma 2.1 assevarate that $C(1,1)$ is a closed subgroup of $G_1 \times G_2$ and invariant by $G_{\Sigma_1 \bigoplus \Sigma_2}$.

In particular, this equivalence class is a Lie subgroup, [13].

By hypothesis $C(1,1)$ is not trivial and because every Σ_j is observable we have

$$C(1,1) \cap (G_1 \times \{1\}) = \{(1,1)\} = C(1,1) \cap (\{1\} \times G_2),$$

thus we can consider the canonic projections

$$C(1,1) \xrightarrow{\pi_j} G_j.$$

π_j is an injective homomorphism. In fact,

$$\pi_1(x_1, y_1) = \pi_1(x_2, y_2) \Longrightarrow x_1 = x_2 ,$$

since $(1, y_1 \cdot y_2^{-1}) \in C(1,1)$ it follows that $y_1 = y_2$. Analogous for π_2.

We define $H_j = \pi_j(C(1,1))$, and

$$H_1 \xrightarrow{\varphi} H_2 \text{ by :}$$

for $x \in H_1$, $\varphi(x)$ is the only element in H_2 such that

$$(x, \varphi(x)) \in C(1,1).$$

Otherwise, φ is determinated by the condition:

$$C(1,1) = \text{Graph}(\varphi).$$

φ is a group homomorfisms. In fact, let $x_1, x_2 \in H_1$ be then,

$$(x_1, \varphi(x_1)) \cdot (x_2, \varphi(x_2)) = (x_1 \cdot x_2, \varphi(x_1) \cdot \varphi(x_2)).$$

Since $C(1,1)$ is a group it follows that

$$\varphi(x_1 \cdot x_2) = \varphi(x_1) \cdot \varphi(x_2).$$

Clearly, φ is a groups isomorphism. To see the continuity of φ we observe that $\varphi = \pi_2 \circ \pi_1^{-1}$ and that for every $j = 1,2$, π_j is a homeomorphism on each connected component of H_j.

Let $g_1 \in G_{\Sigma_1}$ and $x \in H_1$, $\exists g_2 \in G_{\Sigma_2}$ such that

$$(g_1, g_2) \in G_{\Sigma_1 \oplus \Sigma_2}.$$

By the $G_{\Sigma_1 \oplus \Sigma_2}$-invariance of $C(1,1)$ we have

$$(g_1, g_2) \cdot (x, \varphi(x)) = (g_1 \cdot x,\ g_2 \cdot \varphi(x)) \in C(1,1),$$

in particular, $g_1 \cdot x \in H_1$. The G_{Σ_2}- invariance of H_2 follows from the epijectivity of φ.

2. We denote by $C(1,1)_0$ the connected component of $C(1,1)$ throught the identity. Then $C(1,1)_0$ is a Lie subgroup of $G_1 \times G_2$, [13].

For every $j = 1,2$ the projection

$$C(1,1)_0 \xrightarrow{\pi_j} G_j$$

induces a Lie algebra homomorphism

$$A.L.(C(1,1)_0) \xrightarrow{d\pi_j} A.L.(G_j)$$

in particular. $d\pi_j(A.L.(C(1,1)_0))$ is a subalgebra of $A.L.(G_j)$ which define a involutive and regular distribution on TG_j.

Let W_j be the integral connected Lie subgroup of this distributions, [13]. Then,

$$W_j \subset H_j \subset G_j.$$

W_j is connected and H_j is G_{Σ_j}-invariant thus, W_j is G_{Σ_j}-invariant. Consequently, we can to restrict each system Σ_j to the group W_j, in fact,

$$X_t \in \text{Aut.}(G_j) \Longrightarrow X_t/_{W_j} \in \text{Aut.}(W_j),$$

for each $X \in D_{\Sigma_j}$ and every $t \in \mathbb{R}$.

We denote by $\varphi_t^{j,u}$ the automorphism belong to G_{Σ_j} determinated by the constant control $u \in \Omega$ and $t \in \mathbb{R}$. On $\Omega \times \mathbb{R}$ we define the equivalence relation

$$\varphi_{t_1}^{j,u} \ \mathcal{R} \ \varphi_{t_2}^{j,v} \Longleftrightarrow \varphi_{t_1}^{j,u} = \varphi_{t_2}^{j,v}.$$

We considere the group of automorphisms

$$G_{\mathcal{R}(\Sigma_j)} = \text{gen.}\left\{\varphi_t^{j,u} | \overline{(u,t)} \in \Omega \times \mathbb{R}/\mathcal{R}\right\}.$$

Clearly,

$$G_{\Sigma_j} = G_{\mathcal{R}(\Sigma_j)}$$

therefore. we can suppose that $\mathcal{R}$ is a trivial relation on G_{Σ_j} and then the action of G_{Σ_j} on G_j is efective.

Now, we determinate the homomorphism ψ.

$$\text{Let} \quad G_{\Sigma_1/W_1} \xrightarrow{\psi} G_{\Sigma_2/W_2} \quad \text{be}$$

such that on the generators is define by

$$\psi(\varphi_t^{1,u}) = \varphi_t^{2,u} \quad , \quad (u,t) \in \Omega \times \mathbb{R}.$$

We claim that ψ is well define and injective. In fact,

$$\forall (g, \text{Id.}) \in G_{\Sigma_1 \oplus \Sigma_2} \ , \ \forall (x,y) \in C(1,1),$$

$$(g, \text{Id.}) \cdot (x,y) \in C(1,1).$$

Thus, $\quad \varphi(gx) = y = \varphi(x)$, in particular,

$$g \cdot x = x \ , \ \forall x \in W_1.$$

Since the action of $G_{\Sigma_1}/_{W_1}$ is efective on W_1 then, $g = \mathrm{Id}$, at hence

$$\varphi_t^{1,u} = \mathrm{Id}. \iff \varphi_t^{2,u} = \mathrm{Id}.$$

Therefore, $\forall (u, t_1), (v, t_2) \in \Omega \times I\!R$,

$$\varphi_{t_1}^{1,u} \quad = \quad \varphi_{t_2}^{1,v} \iff \varphi_{t_1}^{1,u} \circ (\varphi_{t_2}^{1,v})^{-1} = \mathrm{Id}.$$

$$\iff \varphi_{t_1}^{2,u} = \varphi_{t_2}^{2,v},$$

as claimed.

Since $C(1,1)$ is $G_{\Sigma_1/W_1 \bigoplus \Sigma_2/W_2}$–invariant we can consider the action

$$G_{\Sigma_1/W_1} \times C(1,1) \longrightarrow C(1,1)$$

$$(g, (x, \varphi(x))) \longrightarrow (g \cdot x, \psi(g) \circ \varphi(x))$$

Thus, for every $g \in G_{\Sigma_1/W_1}$ we have

$$\varphi \circ g = \psi(g) \circ \varphi.$$

3. Let $x \in H_1$ be then,

$$(x, \varphi(x)) \in C(1,1) \subset \mathrm{Ker}(h).$$

Consequently,

$$h_1/H_1 \cdot (h_2/H_2 \circ \varphi) = 1_{G_e}.$$

Conversely, let $T = \mathrm{Graph}(\varphi/W_1)$. By the hypothesis it follows that

$$\{(1,1)\} \overset{\subset}{\neq} T \subset C(1,1) \subset \mathrm{Ker}(h),$$

W_1 is a connected Lie groups, φ is a continuous isomorphism then, T is a connected subgroup at hence $S_{\Sigma_1 \oplus \Sigma_2}$-invariant. Therefore, Ker(h) contain a non trivial subgroup $S_{\Sigma_1 \oplus \Sigma_2}$-invariant and the Proposition 2.2 asserve that the direct product system is not observable. □

Remark:// Let Σ be an a.c.s. on G and $W \subset G$ a G_Σ-invariant Lie subgroup of G. We consider the representation

$$G_\Sigma \xrightarrow{\mu} \text{Aut.}(W)$$
$$\varphi_t^u \longrightarrow \mu(\varphi_t^u) = \varphi_t^u/W.$$

Clearly,

$$\mu(G_\Sigma) = G_{\Sigma/W}.$$

In particular,

$$G_{\Sigma/\text{Ker.}(\mu)} \simeq G_{\Sigma/W}.$$

If G_Σ does not contain normal subgroups then,

$$G_\Sigma \simeq G_{\Sigma/W}.$$

Thus, we have the following corollary.

Corollary 3.2:

In the conditions of the Theorem 3.2, if G_Σ does not contain normal subgroups then, that G_{Σ_1} and G_{Σ_2} are not isomorphisms is a necessary conditions for the observability of $\Sigma_1 \oplus \Sigma_2$. □

Suppose overthere that Σ_j is transitive on G_j then, we can state

Corollary 3.3:

$\Sigma_1 \oplus \Sigma_2$ is not observable $\Longleftrightarrow$

1. $G_1 \simeq_\varphi G_2$
2. $G_{\Sigma_1} \simeq_\psi G_{\Sigma_2}$
3. $h_1 \cdot (h_2 \circ \varphi) = 1_G$. $\square$

REFERENCES

[1] Ayala, V.
"Integrabilidade de Distribuições, Classes de Inobservabilidade e Realizações Fracamente Minimais".
Atas VI Escola de Geometria Diferencial., U.F. Ba., Bahia, Brasil, 1988, 67-73.

[2] Ayala, V.
"Sobre a Observabilidade de Sistemas de Controle". Tese de Doutorado. IMECC–UNICAMP, Brasil, 1988.

[3] Ayala, V. and San Martin, L.
"Minimal Realizations Under Controllability"
Submitted en Systems & Control Letters.

[4] Basto Gonçalves, J.
"Nonlinear Observability and Duality"
Systems & Control Letters 4 (1984), 97–101.

[5] Fliess, M.

"Fonctionnelles causales non linéaires et indeterminées non commutatives".

Bull. Soc. Math. France 109 (1981), 3-40.

[6] Gauthier, J.P. and Bonnard, G.

"Existence and uniqueness of minimal realizations in the C^∞ case".

Systems & Control Letters 1, (1982). 395-398.

[7] Gauthier, J.P. and Bonnard, G.

"Existence and uniqueness of minimal realizations for a class of C^∞ systems".

Siam J. Control and Optimization, Vol. 22, No. 4, July 1984.

[8] Hochild, G.

"Le structure des groupes de Lie". Dunod, Paris, 1968.

[9] Hermann, R. and Krener, A.

"Nonlinear Controllability and Observability".

IEEE Transactions on Automatic Control, Vol. AC-22, No. 5, October 1977.

[10] Mohler,

"Bilinear Control Processes".

Math. in Science and Engineering, Vol. 106.

Ac. Press New York and London 1973.

[11] Sussmann, H.

"Orbits of Families of Vector Fields and Integrability of Distributions"

Trans. Am. Math. Soc. Vol. 180, June 1973.

[12] Sussmann, H.

"Existence and uniqueness of minimal realizations of non-linear systems".

Mathematical Systems Theory 10, (1977), 263-284.

[13] Warner, F.

"Foundations of Differentiable Manifolds and Lie Groups".

Scott, Foreman and Company, Glenview, Illinois, 1971. □

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18/89 — **On Circular and Special Units of an Abelian Number Field** — *Trajano Nóbrega.*