LOCAL MINIMIZERS OF A QUADRATIC FUNCTION WITH A SPHERICAL CONSTRAINT

José Mario Martínez

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Universidade Estadual de Campinas
Instituto de Matemática, Estatística e Ciência da Computação
Caixa Postal 6065
13.081 – Campinas – SP
BRASIL

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José Mario Martínez
Applied Mathematics Laboratory
IMECC–UNICAMP
Caixa Postal 6065
13.081 – Campinas. SP

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1. Introduction

We consider the problem

$$\text{Minimize} \quad q(s) \qquad (1.1)$$
$$s.t. \quad \|s\| = \Delta$$

where

$$q(s) = \frac{1}{2} s^T G s + g^T s, \qquad (1.2)$$

$s, b \in \mathbb{R}^n$, $G = G^T$, and $\|\cdot\| = \|\cdot\|_2$.

Global solutions of (1.1) are well understood, and may be obtained using the theory in [8, 12, 13]. Our main result concerns the existence and computation of local-nonglobal (LNG) solutions of (1.1).

Our interest in LNG solutions of (1.1) derives from the following observation:

Proposition 1.1. Assume that s^* is a global solution of the following nonlinear programming problem:

$$\text{(1.3)} \qquad \begin{array}{l} \text{Minimize } p(s) \\ s.t. \quad g_i(s) \le 0, \qquad i = 1, \ldots, m. \end{array}$$

where g_i, $i = 1, \ldots, m$, are continuous functions, and

$$g_1(s^*) = 0$$

$$g_i(s^*) < 0, \qquad i = 2, \ldots, m.$$

Then, s^* is a local solution of

$$\text{(1.4)} \qquad \begin{array}{l} \text{Minimize } p(s) \\ s.t. \quad g_1(s) = 0. \end{array}$$

This proposition generalizes an observation of [2]. It follows from Proposition 1.1 that, if $\bar{s}$ is a local solution of (1.4), and $g_i(\bar{s}) \le 0$, $i = 2, \ldots, m$, then $\bar{s}$ is a good "candidate" to minimizer of (1.3).

Now, Proposition 1.1 is a useful tool for solving (1.3) only if local minimizers of (1.4) are easy to find. We will show that this is the case when (1.4) has the form (1.1) - (1.2).

We would like to mention here a few situations where the identity between (1.4) and (1.1) occurs.

(a) Trust region methods for minimization with simple constraints:

Assume that we have the problem:

$$\text{(1.5)} \qquad \begin{array}{l} \text{Minimize } f(x) \\ s.t. \quad x \ge 0, \qquad x \in \mathbb{R}^n \end{array}$$

and we want to solve it using the trust-region approach (see [3, 4, 5, 8, 10, 12, 13]). At each iteration k of such an algorithm, we should solve a problem of type:

$$\text{Minimize} \quad \nabla f(x^k)^T s + \tfrac{1}{2} s^T B_k s$$

$$(1.6) \qquad s.t. \quad x^k + s \geq 0$$

$$\|s\|_p \leq \Delta_k$$

It is usually chosen $\| \cdot \|_p = \| \cdot \|_\infty$ (see [3, 4]) so that the feasible set of (1.6) is a box. However, the use of $\| \cdot \|_2$ instead of $\| \cdot \|_\infty$ in (1.6) has some advantages. In fact, the number of sets of possible active constraints of (1.6) if $\| \cdot \|_p = \| \cdot \|_\infty$ is 3^n, independently of the number of binding constraints of (1.5) at x^k. On the other hand, if $\| \cdot \|_p = \| \cdot \|_2$ this number goes from only 2 (if $x^k > 0$ and Δ_k is small), to $2^{n+1} - 1$ (if $x^k = 0$). Another reason for prefering $\| \cdot \|_2$ to $\| \cdot \|_\infty$ in (1.6) is the global information contained in the constraint $\|s\|_2 \leq \Delta_k$. In fact, a global minimizer $\bar{s}$ of the quadratic objective function subject to $\|s\|_2 \leq \Delta_k$ always exists, and so, if $x^k + \bar{s} \geq 0$, $\bar{s}$ is a global minimizer of (1.6). Of course, the same observation is true if $\| \cdot \|_p = \| \cdot \|_\infty$, but, in this case, to find a global minimizer with the sole constraint $\|s\|_\infty \leq \Delta_k$ is as difficult as the initial problem.

(b) Minimization of a general function on a sphere.

Assume that we want to solve:

$$\text{Minimize} \quad f(x)$$

$$(1.7) \qquad s.t. \quad \|x\| = \theta$$

An appealing idea is the solve (1.7) using a direct trust-region approach. This amounts to solve, at each iteration, a problem like this:

$$\text{Minimize} \quad \nabla f(x^k) s + \tfrac{1}{2} s^T B_k s$$

$$(1.8) \qquad s.t. \quad \|x^k + s\| = \theta$$

$$\|s\| \leq \Delta.$$

Like (1.6), it is easy to see that (1.8) is of the form (1.3), where (1.4) has the form (1.1).

(c) Trust–region algorithms for nonlinear programmig: Celis–Dennis–Tapia [1] introduced a trust–region algorithm for the equality constained nonlinear optimization problem:

$$\text{Minimize } f(x)$$

$$(1.9) \qquad s.t. \quad h(x) = 0, \qquad h : \mathbb{R}^n \to \mathbb{R}^m.$$

At each iteration of their algorithm they propose to solve a subproblem of the following type:

$$\text{Minimize } \nabla \ell(x^k)^T s + \tfrac{1}{2} s^T B_k s$$

$$(1.10) \qquad s.t. \quad \|s\| \leq \Delta$$

$$\|h'(x^k)s + h(x^k)\| \leq \theta$$

where $\ell(x)$ is an approximation of the Lagrangean function. A completely satisfactory algorithm for solving (1.10) is not known (see [2, 14, 15, 16]). However, problem (1.10) has the form (1.3) and so, Proposition 1.1 applies to this case. For $n = 2$, this observation helps to solve completely the problem (see [2]). For general n, the knowledge of local and global solutions of the quadratic function restricted to the sphere should be a considerable step towards the solution of (1.10).

This paper is organized as follows. The main result is proved in Section 2. There we essentially prove that problem (1.1) has at most one LNG solution, and we characterize this solution in terms of the eigensystem of G. The main theorem is a generalization of Lemma 3.4 of [2], but the proof in the n–dimensional case in much more involved. In Section 3 we suggest an algorithm for finding the LNG solution, which may be used jointly with the classical algorithms of Gay [8] and Moré–Sorensen [13] for finding global solutions. In Section 4 we address problem (1.10) under a different point of view than the usually taken. Instead of trying to find a global solution of (1.10), we find a feasible point for (1.10) which is a good approximation to the solution and is sufficiently good in the sense that satisfies the conditions which are enough to prove global convergence of the CDT algorithm (see [6]).

2. Local Minimizers of a Quadratic Function on a Sphere

We consider the problem

$$\text{Minimize } q(s)$$

$$(2.1) \qquad s.t. \quad \|s\| = \Delta$$

where

$$q(s) = \frac{1}{2}\, s^T G s + g^T s, \tag{2.2}$$

$s,\ b \in \mathbb{R}^n$. G symmetric.

Therefore (see, for instance, Golub–Van Loan [9])

$$G = QDQ^T \tag{2.3}$$

where $Q = (v_1, \ldots, v_n)$ is orthogonal. $D = \mathrm{Diag}(\lambda_1, \ldots \lambda_n)$, $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$.

Let $c = Q^T g$. Define

$$\varphi(\mu) = \sum_{i \in I} \frac{c_i^2}{(\lambda_i + \mu)^2} \tag{2.4}$$

where

$$I = \{i \in \{1, \ldots, n\} \mid c_i \neq 0\}. \tag{2.5}$$

Therefore,

$$\varphi'(\mu) = -2 \sum_{i \in I} \frac{c_i^2}{(\lambda_i + \mu)^3} \tag{2.6}$$

and

$$\varphi''(\mu) = 6 \sum_{i \in I} \frac{c_i^2}{(\lambda_i + \mu)^4}. \tag{2.7}$$

Using (2.3) – (2.7) it is easy to prove the following properties of φ.

Lemma 2.1. (a) φ is well-defined for all $\mu \in \Omega = \mathbb{R} - \{-\lambda_i,\ i \in I\}$. Moreover, $\varphi \in C^\infty(\Omega)$.

(b) $\varphi''(\mu) \geq 0$ for all $\mu \in \Omega$. Thus φ is convex on any interval contained in Ω.

(c) If $i \in I$ $(-\lambda_i \notin \Omega)$, we have:

$$\lim_{\mu \to -\lambda_i} \varphi(\mu) = \infty,$$

$$\lim_{\mu \to -\lambda_i^+} \varphi'(\mu) = -\infty, \quad \text{and}$$

$$\lim_{\mu \to -\lambda_i^-} \varphi'(\mu) = \infty.$$

(d) $\lim_{\mu \to \infty} \varphi(\mu) = \lim_{\mu \to -\infty} \varphi(\mu) = 0.$

(e) For all $\mu \in \Omega$, $\varphi(\mu) = \|-(G + \mu I)^{+} g\|^2$.

If s^* is a local minimizer of (2.1), we have, by standard Lagrange multiplier theory, that:

$$(G + \mu I)s^* + g = 0. \tag{2.8}$$

$$\|s^*\| = \Delta$$

for some $\mu \in \mathbb{R}$.

The following theorem proves that $\mu \in [-\lambda_2, \infty)$.

Theorem 2.1. If s^* is a local minimizer of (2.1), then (2.8) holds with $\mu \geq -\lambda_2$.

Proof. Define

$$\ell(s) = q(s) + \frac{\mu}{2}\|s\|^2, \tag{2.9}$$

the Lagrangean function associated to (2.1). Then,

$$\nabla \ell(x) = (G + \mu I)s + g, \tag{2.10}$$

$$\nabla^2 \ell(s) = G + \mu I \tag{2.11}$$

The necessary second order conditions for a local minimizer (see [11]) state that

$$w^T(G+\mu I)w \geq 0 \tag{2.12}$$

for all w in the tangent subspace to $\|s\| = \Delta$ at s^*.

Assume by contradiction that $\mu < -\lambda_2$. By (2.3), we have:

$$G+\mu I = Q(D+\mu I)Q^T = Q \begin{pmatrix} \lambda_1+\mu & & & \\ & \lambda_2+\mu & & \\ & & \ddots & \\ & & & \lambda_n+\mu \end{pmatrix} Q^T. \tag{2.13}$$

Now, consider the plane P spanned by v_1, v_2. If $v \in P$ we have $v = \alpha v_1 + \beta v_2$, and so, by (2.13),

$$v^T(G+\mu I)v = (\alpha v_1 + \beta v_2)^T(G+\mu I)(\alpha v_1 + \beta v_2) = \tag{2.14}$$

$$= \alpha^2 v_1^T(G+\mu I)v_1 + 2\alpha\beta v_1^T(G+\mu I)v_2 + \beta^2 v_2^T(G+\mu I)v_2.$$

But

$$v_1^T(G+\mu I)v_2 = v_1^T(v_1,\ldots,v_n)\begin{bmatrix} \lambda_1+\mu & & & \\ & \lambda_2+\mu & & \\ & & \ddots & \\ & & & \lambda_n+\mu \end{bmatrix}\begin{pmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{pmatrix} v_2 \tag{2.15}$$

$$= e_1^T \begin{bmatrix} \lambda_1+\mu & & & \\ & \lambda_2+\mu & & \\ & & \ddots & \\ & & & \lambda_n+\mu \end{bmatrix} e_2 = 0.$$

Moreover,

$$\alpha^2 v_1^T(G+\mu I)v_1 = \alpha^2 v_1^T(v_1,\ldots,v_n)\begin{bmatrix} \lambda_1+\mu & & & \\ & \lambda_2+\mu & & \\ & & \ddots & \\ & & & \lambda_n+\mu \end{bmatrix}\begin{pmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{pmatrix} v_1 \tag{2.16}$$

$$= \alpha^2 e_1^T \begin{pmatrix} \lambda_1 + \mu & & & \\ & \lambda_2 + \mu & & \\ & & \ddots & \\ & & & \lambda_n + \mu \end{pmatrix} e_1 = \alpha^2(\lambda_1 + \mu).$$

Analogously,

$$\beta^2 v_2^T (G + \mu I) v_2 = \beta^2(\lambda_2 + \mu). \tag{2.17}$$

Hence, by (2.14) – (2.17), we deduce that

$$v^T (G + \mu I) v < 0 \tag{2.18}$$

for all $v \in P$, $v \neq 0$.

Now, the dimension of the tangent hyperplane T at s^* is $n-1$, and the dimension of P is 2. Therefore, there exists some nonnull $w \in P \cap T$. Hence, (2.12) does not hold, and so, s^* is not a local minimizer. $\square$

Let us call S_1 the subspace spanned by $\{v_1, \dots, v_j\}$, where j is such that $\lambda_1 = \cdots = \lambda_j$.

The following theorem, due to Gay [8] and Moré–Sorensen [13] characterize solutions of (2.8) for $\mu \geq -\lambda_1$.

Theorem 2.2. If $\mu \geq -\lambda_1$ satisfies (2.8) then s^* is a global minimizer of (2.1). If $g \notin S_1^\perp$ or $\|\varphi(-\lambda_1)\| > \Delta^2$, there exists only one solution of (2.8) for $\mu \in [-\lambda_1, \infty)$ and, in this case $\mu > -\lambda_1$. However, if $g \in S_1^\perp$ and $\varphi(-\lambda_1) \leq \Delta^2$, (2.8) is satisfied with $\mu = -\lambda_1$ for any s^* such that $\|s^*\| = \Delta$ lying in the linear manifold

$$V = \{s \in \mathbb{R}^n \mid x = -(G - \lambda_1 I)^+ g + \sum_{i=1}^{j} \alpha_j v_j\}.$$

Proof. See Gay [8] and Moré–Sorensen [13].

By Theorems 2.1 and 2.2, it only remaians to characterize solutions of (2.8) for $\mu \in [-\lambda_2, -\lambda_1)$. This is done in the following theorem.

Theorem 2.3. (i) If $c_1 = 0$ $(g \perp S_1)$ there are no local minimizers of (2.1) with $\mu \in [-\lambda_2, -\lambda_1)$.

(ii) There exists at most one local minimizer of (2.1) with $\mu \in [-\lambda_2, -\lambda_1)$. For this minimizer, $\varphi(\mu)$ is well-defined and $\varphi'(\mu) \geq 0$.

(iii) If (s^*, μ) satisfies (2.8) for $\mu \in [-\lambda_2, -\lambda_1)$, and $\varphi'(\mu) > 0$, then s^* is a strict local minimizer of (2.1).

Proof. Let us first prove (i). If (s^*, μ) satifies (2.8) and $\mu \in (-\lambda_2, -\lambda_1)$, $G + \mu I$ is nonsingular, and so,

$$s^* = -(G + \mu I)^{-1} g. \tag{2.19}$$

Therefore,

$$s^* = -(v_1, \ldots, v_n)(D + \mu I)^{-1} c = -\sum_{i \in J} \frac{c_i}{(\lambda_i + \mu)} v_i$$

$$= -\sum_{\substack{i \in J \\ i \neq 1}} \frac{c_i}{(\lambda_i + \mu)} v_i.$$

Hence, v_1 is a tangent vector to the sphere at s^*. But

$$v_1^T (G + \mu I) v_1 = e_1^T (D + \mu I) e_1 = \lambda_1 + \mu < 0. \tag{2.20}$$

Therefore, the necessary condition (2.12) does not hold and so, s^* is not a local minimizer.

If (s^*, μ) satisfies (2.8) and $\mu = -\lambda_2$, s^* is a particular solution of the consistent system

$$(G - \lambda_2 I) s + g = 0.$$

Hence,

$$s^* = -(G - \lambda_2 I)^+ g + \sum_{\lambda_j = \lambda_2} \alpha_j v_j$$

$$= -\sum_{\substack{j=3 \\ \lambda_j \neq \lambda_2}}^{n} \frac{c_j}{\lambda_j - \lambda_2} v_j + \sum_{\lambda_j = \lambda_2} \alpha_j v_j$$

So, v_1 is also a tangent vector to the sphere at s^*. Now, as in (2.20), we have

$$v_1^T (G + \mu I) v_1 = v_1^T (G - \lambda_2 I) v_1 = \lambda_1 - \lambda_2 < 0.$$

Therefore, s^* is not a local minimizer.

The proof of (i) is complete. From now on, we assume $c_1 \neq 0$. Assume first that (x^*, μ) satisfies (2.8) and $\mu \in (-\lambda_2, -\lambda_1)$. Therefore, $G + \mu I$ is nonsingular, and

$$s^* = -(G+\mu I)^{-1} g = \frac{-c_1}{\lambda_1 + \mu} v_1 - \cdots - \frac{c_n}{\lambda_n + \mu} v_n \tag{2.21}$$

Since $c_1 \neq 0$, we deduce from (2.21) that the columns of W are a basis of the tangent hyperplane at s^*, where:

$$W = \left(\frac{c_2}{\lambda_2+\mu} v_1 - \frac{c_1}{\lambda_1+\mu} v_2, \frac{c_3}{\lambda_3+\mu} v_1 - \frac{c_1}{\lambda_1+\mu} v_3, \ldots, \frac{c_n}{\lambda_n+\mu} v_1 - \frac{c_1}{\lambda_1+\mu} v_n \right) \tag{2.22}$$

$$= (v_1, \ldots, v_n) \begin{pmatrix} \frac{c_2}{\lambda_2+\mu} & \frac{c_3}{\lambda_3+\mu} & \cdots & \frac{c_n}{\lambda_n+\mu} \\ \frac{-c_1}{\lambda_1+\mu} & 0 & \cdots & 0 \\ 0 & \frac{-c_1}{\lambda_1+\mu} & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \frac{-c_1}{\lambda_1+\mu} \end{pmatrix}.$$

Therefore,

$$B = B(\mu) = W^T (G + \mu I) W = \tag{2.23}$$

$$\begin{bmatrix} \frac{c_2^2(\lambda_1+\mu)}{(\lambda_2+\mu)^2} + \frac{c_1^2(\lambda_2+\mu)}{(\lambda_1+\mu)^2} & \frac{c_2c_3(\lambda_1+\mu)}{(\lambda_2+\mu)(\lambda_3+\mu)} & \cdots & \frac{c_2c_n(\lambda_1+\mu)}{(\lambda_2+\mu)(\lambda_n+\mu)} \\ \frac{c_2c_3(\lambda_1+\mu)}{(\lambda_2+\mu)(\lambda_3+\mu)} & \frac{c_3^2(\lambda_1+\mu)}{(\lambda_3+\mu)^2} + \frac{c_1^2(\lambda_3+\mu)}{(\lambda_1+\mu)^2} & \cdots & \frac{c_3c_n(\lambda_1+\mu)}{(\lambda_3+\mu)(\lambda_n+\mu)} \\ \vdots & \vdots & & \vdots \\ \frac{c_2c_n(\lambda_1+\mu)}{(\lambda_2+\mu)(\lambda_n+\mu)} & \frac{c_3c_n(\lambda_1+\mu)}{(\lambda_3+\mu)(\lambda_n+\mu)} & \cdots & \frac{c_n^2(\lambda_1+\mu)}{(\lambda_n+\mu)^2} + \frac{c_1^2(\lambda_n+\mu)}{(\lambda_1+\mu)^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{c_1^2(\lambda_2+\mu)}{(\lambda_1+\mu)^2} & & & 0 \\ & \frac{c_1^2(\lambda_3+\mu)}{(\lambda_1+\mu)^2} & & \\ & & \ddots & \\ 0 & & & \frac{c_1^2(\lambda_n+\mu)}{(\lambda_1+\mu)^2} \end{bmatrix} + \sigma u u^T$$

where

$$u = \begin{bmatrix} \frac{c_2}{\lambda_2+\mu} \\ \vdots \\ \frac{c_n}{\lambda_n+\mu} \end{bmatrix}, \quad \sigma = \lambda_1 + \mu.$$

Hence,

$$B = \hat{B}(I + \sigma \hat{B}^{-1} u u^T), \tag{2.24}$$

where

$$\hat{B} = \begin{bmatrix} \frac{c_1^2(\lambda_2+\mu)}{(\lambda_1+\mu)^2} & & 0 \\ & \ddots & \\ 0 & & \frac{c_1^2(\lambda_n+\mu)}{(\lambda_1+\mu)^2} \end{bmatrix}$$

Thus, (see [9])

$$\det(B) = \det(\hat{B})(1 + \sigma u^T \hat{B}^{-1} u). \tag{2.25}$$

Now,

$$\det(\hat{B}) = \frac{c_1^{n-2}(\lambda_2+\mu)\cdots(\lambda_n+\mu)}{(\lambda_1+\mu)^{2n-2}}, \tag{2.26}$$

and

$$(2.27)\qquad 1+\sigma u^T\widehat{B}^{-1}u = 1+(\lambda_1+\mu)\left(\frac{c_2}{\lambda_2+\mu},\dots,\frac{c_n}{\lambda_n+\mu}\right)\begin{pmatrix}\frac{(\lambda_1+\mu)^2c_2}{c_1^2(\lambda_2+\mu)^2}\\ \vdots\\ \frac{(\lambda_1+\mu)^2c_n}{c_1^2(\lambda_n+\mu)^2}\end{pmatrix}$$

$$= 1+\frac{c_2^2(\lambda_1+\mu)^3}{c_1^2(\lambda_2+\mu)^3}+\dots+\frac{c_n^2(\lambda_1+\mu)^3}{c_1^2(\lambda_n+\mu)^3}.$$

Hence, by (2.25) - (2.27),

$$(2.28)\qquad \det(B) = c_1^{2n-2}\frac{(\lambda_2+\mu)\cdots(\lambda_n+\mu)(\lambda_1+\mu)^3}{(\lambda_1+\mu)^{2n-2}}$$

$$\left[\frac{1}{(\lambda_1+\mu)^3}+\frac{c_2^2}{c_1^2(\lambda_2+\mu)^3}+\dots+\frac{c_n^2}{c_1^2(\lambda_n+\mu)^3}\right]=$$

$$=\frac{c_1^{2n-4}(\lambda_2+\mu)\cdots(\lambda_n+\mu)}{(\lambda_1+\mu)^{2n-5}}\left[\frac{c_1^2}{(\lambda_1+\mu)^3}+\dots+\frac{c_n^2}{(\lambda_n+\mu)^3}\right]=$$

$$=\frac{-c_1^{2n-4}(\lambda_2+\mu)\cdots(\lambda_n+\mu)}{2(\lambda_1+\mu)^{2n-5}}\left[\frac{-2c_1^2}{(\lambda_1+\mu)^3}+\dots+\frac{-2c_n^2}{(\lambda_n+\mu)^3}\right]$$

Thereore, by (2.6) and (2.28),

$$(2.29)\qquad \det(B) = \frac{-c_1^{2n-4}(\lambda_2+\mu)\cdots(\lambda_n+\mu)}{2(\lambda_1+\mu)^{2n-5}}\varphi'(\mu)$$

But, for all $\mu\in(-\lambda_2,-\lambda_1)$, $c_1\neq 0$,

$$(2.30)\qquad \frac{-c_1^{2n-4}(\lambda_2+\mu)\cdots(\lambda_n+\mu)}{2(\lambda_1+\mu)^{2n-5}}>0,$$

So, for all $\mu\in(-\lambda_2,-\lambda_1)$, $\det(B)$ has the same sign as $\varphi'(\mu)$. But (2.12) implies that $\det(B)\geq 0$, and so, that $\varphi'(\mu)\geq 0$. That is, $\varphi'(\mu)\geq 0$ is a necessary condition for s^* being a minimizer of (2.1) with $\mu\in(-\lambda_2,-\lambda_1)$.

Now, the equation $\varphi(\mu)=\Delta^2$ has at most two solutions in $(-\lambda_2,-\lambda_1)$. If it has one solution μ, it must be $\varphi'(\mu)=0$. If it has two solutions by Lemma 2.1 we necessarily have $\varphi'(\mu_1)<0$ and $\varphi'(\mu_2)>0$, if $\mu_1<\mu_2$. So, only the larger one defines a condidate to local minimizer.

Let us show now that, if $\mu\in(-\lambda_2,-\lambda_1)$ and $\varphi'(\mu)>0$, then μ is a strict local minimizer of (2.1).

Suppose, by contradiction, that μ does not define a local minimizer. Thus, $B = B(\mu)$ defined by (2.23) is not positive definite. But, since $c_1 \neq 0$ and $\lambda_1 \neq \lambda_2$ we easily see that $B(\hat{\mu})$ is positive definite if $\hat{\mu} \in (\mu, -\lambda_1)$ is close enough to $-\lambda_1$. Hence, there exists $\overline{\mu} \in (\mu, \hat{\mu})$ such that $B(\hat{\mu})$ has a zero eigenvalue. Therefore, $\det(B(\overline{\mu})) = 0$ and hence $\varphi'(\overline{\mu}) = 0$, contradicting convexity of φ. Finally, since $\varphi'(\mu) > 0$, $B(\mu)$ is nonsingular and hence the sufficient second order conditions for strict local minimizers (see [11]) also hold at s^*. Therefore, $\varphi'(\mu) > 0$ and $\mu \in (-\lambda_2, -\lambda_1)$ imply that s^*, defined by (2.8), is a strict local minimizer of (2.1).

Now assume that (2.8) holds with $\mu = -\lambda_2$. Define $\ell \geq 2$ by $\lambda_2 = \cdots = \lambda_\ell$. Therefore, the system

$$(G - \lambda_2 I)s + g = 0 \tag{2.31}$$

admits a linear manifold of solutions given by

$$s = -(G - \lambda_2 I)^+ g + \alpha_2 v_2 + \cdots + \alpha_\ell v_\ell, \tag{2.32}$$

with $\alpha_2, \ldots, \alpha_\ell \in I\!R$.

Moreover, $c_2 = \cdots = c_\ell = 0$, and

$$(G - \lambda_2 I)^+ g = \frac{c_1}{\lambda_1 - \lambda_2} v_1 + \frac{c_{\ell+1}}{\lambda_{\ell+1} - \lambda_2} v_{\ell+1} + \cdots + \frac{c_n}{\lambda_n - \lambda_2} v_n \tag{2.33}$$

Assume first that s^* has the form (2.32)–(2.33) with some $\alpha_i \neq 0$. Without loss of generality assume that $\alpha_2 \neq 0$. Consider the plane P spanned by $\{v_1, v_2\}$. If $w \in P$, then $w = \alpha v_1 + \beta v_2$, $\alpha, \beta \in I\!R$. Therefore,

$$w^T(G - \lambda_2 I)w = (\alpha v_1 + \beta v_2)^T (G - \lambda_2 I)(\alpha v_1 + \beta v_2) = \tag{2.34}$$

$$= \alpha^2 v_1^T (G - \lambda_2 I) v_1 + 2\alpha\beta v_1^T (G - \lambda_2 I) v_2 + \beta^2 v_2^T (G - \lambda_2 I) v_2 =$$

$$= \alpha^2 e_1^T (D - \lambda_2 I) e_1 + 2\alpha\beta e_1^T (D - \lambda_2 I) e_2 + \beta^2 e_2^T (D - \lambda_2 I) e_2 = \alpha^2(\lambda_1 - \lambda_2).$$

Hence $w^T(G - \lambda_2 I)w < 0$ unless w is a multiple of v_2.

Now consider the intersection of P with the tangent hyperplane at s^*. From the preceedings arguments, it follows that this intersection contains a vector w such that $w^T(G - \lambda_2 I)w < 0$ unless it is generated by v_2. But, if this is the case, v_2 is a tangent vector and so, by (2.32) - (2.33), $\alpha_2 = 0$. Hence, we proved that, if s^* is a local minimizer and has the from (2.32) - (2.33), then $\alpha_2 = \cdots = \alpha_\ell = 0$.

Therefore, it only remains to analyze the case where

$$s^* = -(G - \lambda_2 I)^+ g, \quad \|s^*\| = \Delta. \tag{2.35}$$

In this case, by (2.33), (2.35),

$$s^* = \frac{c_1}{\lambda_1 - \lambda_2} v_1 + \frac{c_{\ell+1}}{\lambda_{\ell+1} - \lambda_2} v_{\ell+1} + \cdots + \frac{c_n}{\lambda_n - \lambda_2} v_n. \tag{2.36}$$

Hence, using the convention $\frac{0}{0} = 0$, the columns of W given by (2.22) are a basis of the tangent hyperplane at x^*.

Therefore, the reasoning which lead to (2.23) - (2.30) may be repeated here. This means that, if s^* is a local minimizer, it must satisfy $\varphi'(-\lambda_2) \geq 0$ and, conversely, if $\varphi'(-\lambda_2) \gtrless 0$, s^* is a strict local minimizer. This completes the proof of Theorem 2.3. □

3. An Algorithm for Finding the Local Nonglobal Minimizer of the Problem

In this section we introduce an algorithm to calculate a local–nonglobal minimizer of problem (2.1).

We use the definition (2.4) for the function φ.

We assume that we have already found a global minimizer s_G of (2.1), with a corresponding Lagrange multiplier μ_G.

If $\mu_G = -\lambda_1$, that is, if $G + \mu_G I$ is singular, then $c_1 = 0$ and we know, from Theorem 2.3, that there are no local–nonglobal minimizers of (2.1). Therefore, we may assume that $\mu_G > -\lambda_1$.

The algorithm for computing a local–nonglobal minimizer of (2.1) is divided into two phases. In phase 1, we try to find μ such that $\varphi(\mu) \geq \Delta^2$ and $\varphi'(\mu) \geq 0$.

Algorithm 3.1 (Phase 1). Let μ_0 be such that $G + \mu_0 I$ is not positive semidefinite ($\mu_0 < -\lambda_1$). Let $\sigma_1, \sigma_2 \in (0, 1)$, $\sigma_1 < \sigma_2$ and let $\mu_0^U = \mu_G$.

Given μ_k, μ_k^U, the steps to obtain μ_{k+1}, μ_{k+1}^U are following:

Step 1. Compute $\varphi(\mu_k)$, $\varphi'(\mu_k)$.

Step 2. If $\varphi(\mu_k) \geq \Delta^2$ and $\varphi'(\mu_k) \geq 0$, stop Phase 1.

Step 3. Choose $\bar{\mu} \in [\mu_k + \sigma_1(\mu_k^U - \mu_k),\ \mu_k + \sigma_2(\mu_k^U - \mu_k)]$.

Step 4. If $G + \bar{\mu} I \geq 0$, define $\mu_{k+1} = \mu_k$, $\mu_{k+1}^U = \bar{\mu}$. Else, define $\mu_{k+1} = \bar{\mu}$, $\mu_{k+1}^U = \mu_k^U$.

Theorem 3.1. Algorithm 3.1 (Phase 1) stops at some finite k giving μ_k such that $\varphi(\mu_k) \geq \Delta^2$ and $\varphi'(\mu_k) \geq 0$, or it generates an infinite sequence such that $\lim_{k\to\infty} \mu_k = -\lambda_1$. Moreover, in this last case, $\lim_{k\to\infty} \mu_k^U = -\lambda_1$ and $c_1 = 0$.

Proof. If the sequence is not finite, it follows, from Step 3, that

$$\lim_{k\to\infty}(\mu_k^U - \mu_k) = 0. \tag{3.1}$$

But, by the definition of μ_k^U and μ_k, we have: $\mu_k^U \geq -\lambda_1$ and $\mu_k < -\lambda_1$. Therefore, by (3.1),

$$\lim_{k\to\infty}\mu_k^U = \lim_{k\to\infty}\mu_k = -\lambda_1.$$

Let us now prove that $c_1 = 0$ when the algorithm generates an infinite sequence. In fact, if $c_1 \neq 0$, we have, from Lemma 2.1, that:

$$\lim_{\mu\to-\lambda_1}\varphi(\mu) = \lim_{\mu\to-\lambda_1^-}\varphi'(\mu) = \infty.$$

Therefore, since $\lim_{k\to\infty}\mu_k = -\lambda_1$ and $\mu_k < -\lambda_1$ for all $k = 0, 1, 2, \ldots$, the conditions required at Step 2 should be satisfied for some finite k. This completes the proof. □

From Theorems 3.1 and 2.3, we know now that (2.1) has a local–nonglobal solution only if Algorithm 3.1 (Phase 1) stops at some finite k. So, let us assume that this is the case. Let us call $\mu_k^L = \mu_{k-1}$. Therefore, we have:

$$\varphi(\mu_k) \geq \Delta^2, \quad \varphi'(\mu_k) \geq 0. \tag{3.2}$$

and:

$$\text{Either} \quad \varphi(\mu_k^L) < \Delta^2 \quad \text{or} \quad \varphi'(\mu_k^L) < 0. \tag{3.3}$$

Algorithm 3.1 (Phase 2). Given μ_k^L, μ_k, the steps to obtain μ_{k+1}^L, μ_{k+1} are the following:

Step 1. If $\varphi(\mu_k) = \Delta^2$ and $\varphi'(\mu_k) \geq 0$, stop.

Step 2. Compute

$$\mu_k^N = \mu_k - \frac{(\varphi(\mu_k) - \Delta^2)}{\varphi'(\mu_k)}.$$

If $\mu_k^N < \mu_k^L$, stop.

Step 3. Choose $\tilde{\mu} \in [\mu_k^L + \sigma_1(\mu_k - \mu_k^L),\ \mu_k^L + \sigma_2(\mu_k - \mu_k^L)]$.

Step 4. If $\varphi(\tilde{\mu}) \geq \Delta^2$ and $\varphi'(\tilde{\mu}) \geq 0$, define $\mu_{k+1}^L = \mu_k^L,\ \mu_{k+1} = \tilde{\mu}$.
Else, define $\mu_{k+1}^L = \tilde{\mu},\ \mu_{k+1} = \mu_k$.

Theorem 3.2. Assume that, in Algorithm 3.1 (Phase 1), μ_0 is chosen in the interval $[-\lambda_2, -\lambda_1)$, and that Phase 1 stops at some finite k. Then:

(i) If Algorithm 3.1 (Phase 2) stops at Step 2, there is no local-nonglobal minimizer of (2.1).

(ii) If Algorithm 3.1 (Phase 2) defines an infinite sequence μ_k, then there exists $\mu^* \in (-\lambda_2, -\lambda_1)$ such that

$$\lim_{k\to\infty} \mu_k = \lim_{k\to\infty} \mu_k^L = \mu^*,$$

$$\varphi'(\mu^*) \geq 0, \quad \varphi(\mu^*) \geq \Delta^2, \quad \text{and}$$

$$\varphi(\mu^*) = \Delta^2 \quad \text{or} \quad \varphi'(\mu^*) = 0.$$

(iii) If $\varphi'(\mu^*) = 0$ and $\varphi(\mu^*) > \Delta^2$, there is no local-nonglobal minimizer of (2.1). If $\varphi'(\mu^*) > 0$ and $\varphi(\mu^*) = 0$ then μ^* defines the local-nonglobal minimizer of (2.1) ($s^* = -(G - \mu^* I)^{-1} g$).

Proof. Let us first prove (i). We proceed by contradiction. Assume that a local-nonglobal minimizer of (2.1) exists. Then, by Theorem 2.3, its Lagrange multiplier μ^* must satisfy:

$$\varphi(\mu^*) = \Delta^2, \quad \varphi'(\mu^*) \geq 0.$$

But, by construction, we have, for all k in Phase 2:

$$\varphi(\mu_k) \geq \Delta^2, \quad \varphi'(\mu_k) \geq 0.$$

and

$$\text{either} \quad \varphi(\mu_k) < \Delta^2 \quad \text{or} \quad \varphi'(\mu_k) < 0.$$

Hence, by Lemma 2.1, $\mu^* \in [\mu_k^L, \mu_k]$.

But, since φ is a convex function, $\mu_k^N \geq \mu^*$, contradicting the fact that $\mu_k^N < \mu_k^L$. Therefore (i) is proved.

Now, let us prove (ii). By Steps 3 and 4 of the algorithm, we see that

$$\lim_{k\to\infty} (\mu_k - \mu_k^L) = 0.$$

Therefore, there exists μ^* such that

$$\mu^* = \lim_{k\to\infty} \mu_k = \lim_{k\to\infty} \mu_k^L.$$

Hence,

$$\varphi(\mu^*) = \lim_{k\to\infty} \varphi(\mu_k) \geq \Delta^2 \tag{3.4}$$

$$\varphi'(\mu^*) = \lim_{k\to\infty} \varphi'(\mu_k) \geq 0. \tag{3.5}$$

We consider two possibilities:

(a) There exists a subsequence $\mu_{k_j}^L$, $j = 0, 1, 2, \ldots$ of (μ_k) such that

$$\varphi(\mu_{k_j}^L) < \Delta^2.$$

(b) For all μ_k^L generated in Phase 2 of Algorithm 2.1, we have $\varphi(\mu_k^L) \geq \Delta^2$.

Assume first that (a) holds. Then

$$\varphi(\mu^*) = \lim_{j\to\infty} \varphi(\mu_{k_j}^L) \leq \Delta^2. \tag{3.6}$$

Now, if (b) holds, it follows that $\varphi'(\mu_k^L) \leq 0$ for all μ_k^L. Hence,

$$\varphi'(\mu^*) = \lim_{j\to\infty} \varphi'(\mu_{k_j}^L) \leq 0. \tag{3.7}$$

Thus, (ii) follows from (3.4) – (3.7).

Finally, (iii) follows from Theorem 2.3 and the fact that $\varphi'(\mu^*) = 0$ is a sufficient condition for μ^* to be a local minimizer of φ. □

Remark. The efficiency of Algorithm 2.1 depends entirely of the way $\tilde{\mu}$ is chosen at Step 3 of Phase 1 and at Step 3 of Phase 2. Using the particular shape of φ it seems that its approximation by a rational function is recommended, as in the Hebden–Moré scheme (see [10, 12]) for finding global minimizers of 2.1.

4. Application to the Subproblem in the *CDT* Method

At each iteration of the *CDT* method for equality constrained minimization [1] a problem of the following type must be solved:

$$\text{Minimize } \tfrac{1}{c} s^T G s + g^T s$$

$$\text{(4.1)} \qquad s.t. \quad \|As - b\| \leq \theta,$$

$$\|s\| \leq \Delta$$

where G is as in (2.2), (2.3), $A \in \mathbb{R}^{m \times n}$ and the region defined by the two constraints is known to have nonvoid interior. A completely satisfactory algorithm for solving (4.1) is not known (see [2, 14, 15, 16]). However, El Alem [6] proved that global convergence of the *CDT* algorithm is obtained if a feasible point for (4.1) is found such that:

$$\text{(4.2)} \qquad Gs + g + \mu s + \rho A^T (As - b) = 0$$

with $\mu, \rho \geq 0$.

Here we propose an algorithm for obtaining a feasible point which satisfies (4.2) and is probably a good approximation to the solution of (4.1) in a finite number of steps.

Algorithm 4.1.

Set $\rho = 0$.

Step 1. Find s_1, a global solution, and $\{s_1, \ldots, s_q\}$ a set of global or local solutions of

$$\text{Minimize } \tfrac{1}{2} s^T G s + g^T s + \rho \|As - b\|^2$$

$$\text{(4.3)} \qquad s.t. \quad \|s\| \leq \Delta.$$

Step 2. If, for all $i = 1, \ldots, q$, $\|As_i - b\| > \theta$, go to Step 4.

Step 3. Choose

$$s = \text{Argmin}\,\{\frac{1}{2}s_i^T G s_i + g^T s_i \mid \|As_i - b\| \leq \theta\}. \tag{4.4}$$

Stop.

Step 4. $\rho \longleftarrow \max\{10, 10\rho\}$. Go to Step 1.

The following theorem states that the desired step 5 is obtained in a finite number of steps.

Theorem 4.1. After a finite number of steps, algorithm 4.1 obtains a feasible s satisfying (4.2).

Proof. Consider the following auxiliary problem:

$$\begin{aligned} &\text{Minimize } \|As - b\| \\ &s.t. \quad \|s\| \leq \Delta. \end{aligned} \tag{4.5}$$

Assume that, at any global minimizer $\bar{s}$ of (4.5), we have

$$\|A\bar{s} - b\| = m.$$

Since the feasible region of (4.1) has interior points, we must have $m < \theta$.

Now, let ρ_k a sequence of positive numbers such that $\rho_k \to \infty$, and let s_k be a global solution of

$$\begin{aligned} &\text{Minimize } \frac{1}{2}s^T G s + g^T s + \rho_k[\|As - b\|^2 - m^2] \\ &s.t. \quad \|s\| \leq \Delta. \end{aligned} \tag{4.6}$$

Using standard external penalization theory (see [7, 11]) we verify that any limit point of (s_k) is a global solution of:

$$\begin{aligned} &\text{Minimize } \frac{1}{2}s^T G s + g^T s \\ &s.t. \quad \|s\| \leq \Delta, \\ &\|As - b\|^2 = m^2. \end{aligned} \tag{4.7}$$

Therefore, for some k, the solution of (4.6) satifies

$$||As_k - b||^2 \leq m^2 + (\theta^2 - m^2) \leq \theta^2.$$

Since (4.6) is equivalent to (4.3), this reasoning applies to the successive solutions of (4.3) in Algorithm 4.1. Therefore, the algorithm stops.

But, if s is (local or global) solution of (4.3), we have, by standard (Lagrange multiplier theory, that (4.2) holds. Hence, the proof is complete. □

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