CLIFFORD ALGEBRAS AND THE HIDDEN GEOMETRICAL NATURE OF SPINORS

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Abstract: Many different definitions and representations of spinors are given in the literature, but there are no single reference explaining how they are related, which may explain why considerable confusion on the subject persists. Here and in following papers (II and III) we deal with three different definitions for spinors, (i) the covariant definition (E. Cartan) based on group theory representation, (ii) the ideal definition, based on real Clifford algebra ($\mathbb{R}_{p,q}$) methods, and (iii) the operator definition, where spinors are interpreted as particular elements an appropriated Clifford algebras (not necessarily elements of lateral ideals). By introducing the concept of spinorial metric on the space of algebraic spinors (i.e., elements of lateral ideals in appropriated Clifford algebras) we prove that for $p + q \leq 5$ that there exists an equivalence from the group theoretical point of view between covariant and algebraic spinors. Tho this end we study the Clifford and the twisted Clifford groups, a subject that is also necessary, e.g., for the construction of the Clifford bundle for Lorentzian manifolds with $(p,q) = (3,1)$. We give explicit construction of the representative of Pauli spinors in $\mathbb{R}_{3,0}$ and Dirac spinors, Majorana spinors, dotted and undotted two component spinors in $\mathbb{R}_{1,3}$, $\mathbb{R}_{3,1}$ and $\mathbb{R}_{4,1}$. The problem of the transformations laws of algebraic spinors is also treated in details and a satisfactory mathematical solution is presented. Our approach clears among others the geometrical meaning of spinors and shows that the usual claim that spinors are objects more fundamental than tensors is non-sequitur. Also our techniques permit, e.g., the construction of sets of Majorana or Dirac matrices in very time saving way.

In paper II we show how to obtain an almost elementary proof of Geroch's theorem without using the sophisticated techniques of algebraic topology. In paper III we study the problem of **algebraic spinor fields,** the Spinors and Clifford bundles and Dirac equations.

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Clifford Algebras and the Hidden Geometrical Nature of Spinors.

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Introduction

There appears in the literature three essentially different definitions of spinors: There are:

(I) The covariant definiton, (E. Cartan [1], R. Brauer and H. Weyl [2]), where a particular kind of a covariant spinor(c-spinor) is a set of complex variables defined by its transformations under a particular spin group.

(II) The ideal definition (C. Chevalley [3], M. Riez [4] and W. Graf [9]) where a particular kind of an algebraic spinor (e-spinor) is an element of a lateral ideal (defined by the idempotent e) in an appropriate Clifford algebra. (When e is primitive we write a-spinor, instead of e-spinor).

(III) The operator definition (D. Hestenes [5]) where a particular kind of operator spinor (o-spinor) is a Clifford number in an appropriate Clifford algebra $\mathbb{R}_{p,q}$ determining a set of tensors by bilinear mappings.[1]

The so-called pure spinors recently used by Caianiello [6] and Budinich and Trautman [7] are special cases of c-spinors (or e-spinors) and will be not analysed in this paper. From the point of view of this paper they are not so fundamental as it is usually thought.

The usual presentation of e-spinors as elements of lateral ideals in Clifford algebras as well as the introduction in this context of the groups $Spin_+(p,q)$, does not leave clear the relation between these objects and the c-spinors and the universal covering groups of some groups $SO_+(p,q)$ used in theoretical physics. The same is true in relation with o-spinors.

The main purpose of the present paper is to clear up the situation, and in the process we obtain very interesting results. In particular we are going to prove that

[1] for our notations see § 2

all the c-spinors used by physicists can be represented by appropriate e-spinors. From the explicit construction of the e-spinors (representing c-spinors) by the "idempotent method" (§ 2.3), we will see that e-spinors are nothing more than the sum of multivectors (or multiforms). This result is at variance with the usual claim [6,10,11,12] that spinors are more fundamental then tensors.

Also, from our approach the geometrical meaning of Pauli c-spinors, Weyl spinors (i.e., two component dotted and undotted c-spinor) and Dirac c-spinors become apparent. The geometrical meaning of these objects have already been discussed in the literature [11,12,13,14,15] using different approaches without a common geometrical basis.

To formulate our problem we start by remembering the kinds of c-spinors used by physicists,

(i) Pauli c-spinors – these are the vectors of a complex 2- dimensional space $\boldsymbol{C}^2$ equipped with the spinorial metric

$$\beta_p : \boldsymbol{C}^2 \times \boldsymbol{C}^2 \to \boldsymbol{C} \qquad \beta_p(\psi, \varphi) = \psi^* \varphi \tag{1}$$

$$\psi = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad \varphi = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad z_i, y_i \in \boldsymbol{C}, \quad i = 1,2 \quad \text{and} \quad \varphi^* = (\bar{z}_1, \bar{z}_2)$$

where in this text $\bar{z}$ always means the complex conjugated of $z \in \boldsymbol{C}$.

The spinorial metric is invariant under the action of the group $SU(2)$, i.e., if $u \in SU(2)$, then $\beta_p(u\psi, u\varphi) = \beta_p(\psi, \varphi)$. As it is well known, Pauli c-spinors carry the fundamental (irreducible) representation $D^{1/2}$ of $SU(2)$[16,17].

(ii) Weyl c-spinors – the objects have been introduced by Weyl [18] and called by van der Waerden [19] undotted and dotted two component spinors. We have the following definitions

Contravariant Undotted Spinors – these are the elements of a complex 2-dimensional space $\boldsymbol{C}^2$ equipped with the spinorial metric

$$\beta : \boldsymbol{C}^2 \times \boldsymbol{C}^2 \to \boldsymbol{C} \quad ; \quad \beta(\eta, \xi) = \eta^t \mathrm{C} \xi \tag{2}$$

$$\eta = \begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix}, \quad \mathrm{C} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The spinorial metric β is invariant under the action of the group $SL(2, \boldsymbol{C})$, i.e., if $\eta \mapsto u\eta$; $\quad \xi \mapsto u\xi$, then

$$\beta(\eta, \xi) = \beta(u\eta, u\xi) \leftrightarrow u^t \mathrm{C} u = \mathrm{C} \leftrightarrow u \in SL(2, \mathrm{C}) \tag{3}$$

Covariant Undotted Spinors – these are the elements of the dual space $\overset{\Delta}{\boldsymbol{C}}{}^2$, defined by

$$\overset{\Delta}{\boldsymbol{C}}{}^2 \ni \overset{\Delta}{\eta} : \boldsymbol{C}^2 \to \boldsymbol{C} \quad ; \quad \overset{\Delta}{\eta}(\xi) \equiv \overset{\Delta}{\eta}\,\xi = \beta(\eta, \xi) \tag{4}$$

It follows that

$$\overset{\Delta}{\eta} = \eta^t \mathrm{C} = (\eta_1, \eta_2) = (\eta^2, -\eta^1) \tag{5}$$

The transformation law of the covariant undotted spinors that leaves the spinorial metric invariant under $SL(2, \boldsymbol{C})$ is then

$$\overset{\Delta}{\eta} \longmapsto \overset{\Delta}{\eta}\, u^{-1} \quad , \quad u \in SL(2, \boldsymbol{C}) \tag{6}$$

Contravariant Dotted Spinors – these are the elements of the space $\dot{\boldsymbol{C}}^2 \equiv (\boldsymbol{C}^2)^*$, i.e., $\dot{\boldsymbol{C}}^2 \ni \dot{\eta} = (\eta^{\dot{1}}, \eta^{\dot{2}}) \equiv (\bar{\eta}^1, \bar{\eta}^2) = \eta^*, \eta \in \boldsymbol{C}^2$) equipped with the spinorial metric $\dot{\beta}$,

$$\dot{\beta} : \dot{\boldsymbol{C}}^2 \times \dot{\boldsymbol{C}}^2 \to \boldsymbol{C} \quad , \quad \dot{\beta}(\dot{\eta}, \dot{\xi}) = \dot{\eta} \mathrm{C} \dot{\xi}^t \tag{7}$$

and we have that

$$\dot{\beta}(\dot{\eta}, \dot{\xi}) = \dot{\beta}(\dot{\eta} u^*, \dot{\xi} u^*) \leftrightarrow u^* \mathrm{C} u^{*t} = \mathrm{C} \tag{8}$$

Covariant Dotted Spinors – these are the element of the dual space $\overset{\Delta}{\dot{\boldsymbol{C}}}{}^2$, defined by

$$\overset{\Delta}{\dot{\boldsymbol{C}}}{}^2 \ni \overset{\Delta}{\dot{\xi}} = \dot{\beta}(\ \ , \dot{\xi}) \quad ; \quad \dot{\eta}(\overset{\Delta}{\dot{\xi}}) \equiv \dot{\eta}\, \overset{\Delta}{\dot{\xi}} = \dot{\eta} \mathrm{C} \dot{\xi}^t \tag{9}$$

It follows that

$$\overset{\Delta}{\dot{\xi}} = \mathrm{C} \dot{\xi}^t = \begin{pmatrix} \xi_{\dot{1}} \\ \xi_{\dot{2}} \end{pmatrix} \equiv \begin{pmatrix} \xi^{\dot{2}} \\ -\xi^{\dot{1}} \end{pmatrix} \equiv \begin{pmatrix} \bar{\xi}^2 \\ -\bar{\xi}^1 \end{pmatrix} \tag{10}$$

It is clear that the laws of transformations of the dotted spinors under the action of $SL(2, \boldsymbol{C})$ are

$$\dot{\eta} \longmapsto \dot{\eta} u^* \quad ; \quad \overset{\Delta}{\dot{\eta}} \to (u^*)^{-1}\, \overset{\Delta}{\dot{\eta}} \tag{11}$$

The matrices u and $(u^*)^{-1}$ are the (non-equivalent) representations $D^{(1/2,0)}$ and $D^{(0,1/2)}$ of $SL(2, \boldsymbol{C})$.

(iii) **Dirac c-spinors** – these are the vectors of a complex 4-dimensions space $\boldsymbol{C}^4$ equipped with the spinorial metric [17,20]

$$\beta_d : \boldsymbol{C}^4 \times \boldsymbol{C}^4 \to \boldsymbol{C} \qquad \beta_d = (\psi_d, \phi_d) = \psi_d^t B \phi_d$$

where a Dirac c-spinor $\psi_d(\phi_d)$ is defined as

$$\boldsymbol{C}^2 \oplus \overset{\Delta}{\dot{\boldsymbol{C}}}{}^2 = \boldsymbol{C}^4 \ni \psi_d = \eta + \overset{\Delta}{\xi} = \begin{pmatrix} \eta^1 \\ \eta^2 \\ \xi_1 \\ \xi_2 \end{pmatrix} \tag{12}$$

In the canonical basis of $\boldsymbol{C}^4$ the matrix B is the representation of β_d and we have

$$B = \begin{pmatrix} \mathbf{C} & 0 \\ 0 & \mathbf{C} \end{pmatrix} \tag{13}$$

The spinorial metric β_d is invariant under $SL(2, \boldsymbol{C})$ in the following sense

$$\beta_d(\psi_d, \phi_d) = \beta_d(\rho(u)\psi_d, \rho(u)\phi_d)$$

$$\rho(u) = \begin{pmatrix} u & 0 \\ 0 & (u^*)_{-1} \end{pmatrix} \qquad u \in SL(2, \boldsymbol{C}) \tag{14}$$

The transformation law of the Dirac c-spinors are then

$$\psi_d \longmapsto \begin{bmatrix} u & 0 \\ 0 & (u^*)^{-1} \end{bmatrix} \psi_d \tag{15}$$

which means that Dirac c-spinors, as is well known, carry the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ of $SL(2, \boldsymbol{C})$.

(iv) **Standard Dirac c-spinors** – If $\rho(u)$ is a representation of $SL(2, \boldsymbol{C})$ then $S\rho(u)S^{-1}$, with $SS^{-1} = S^{-1}S = 1$ is also a representation. Under a similarity transformation the spinor $\psi_d \longmapsto S\psi_d$, which in general mixes the components of $\boldsymbol{C}^2$ with those of $\overset{\Delta}{\dot{\boldsymbol{C}}}{}^2$. A particular mixing is convenient in writting Dirac's equation. We define standard Dirac spinors as the objects ψ_s such that

$$\boldsymbol{C}^4 \ni \psi_s = \begin{pmatrix} \phi \\ \lambda \end{pmatrix} \tag{16}$$

where $\phi = \frac{1}{\sqrt{2}}(\xi + \overset{\Delta}{\dot{\eta}})$; $\lambda = \frac{1}{\sqrt{2}}(\xi - \overset{\Delta}{\dot{\eta}})$ where $\xi \in \boldsymbol{C}^2$ and $\overset{\Delta}{\dot{\eta}} \in \overset{\Delta}{\dot{\boldsymbol{C}}}{}^2$ and the sums in ϕ and λ are in the sense of sums of complex numbers for each component. It is well known that ψ_d and ψ_s are related by a unitary transformation

$(S^{-1} = S^*)$ which leave unchanged the bilinear covariant constructed from ψ_d and ψ_d^* [17,20].

We now ask the main question to which this paper is adressed: to which Clifford algebras are the c-spinors described in (i), (ii), (iii) and (iv) above to be associated?

We are going to give an original answer to the above question by intoducing a natural scalar product (see § 3) in certain lateral ideals of certain *real* Clifford algebras that "mimic" what has been described in (i), (ii), (iii) and (iv) above. To this end in section 2 we give the main properties of Clifford algebras over the reals [3,4,21,22,23,24,25,26,27]. The material presented fixes our notation and is the minimum necessary to permit the formulation of our ideas in a rigorous way.

In section 3 we define the a-spinors as elements of minimal lateral ideals and the e-spinors are the elements of lateral ideals (*not necessarily minimal*) in real Clifford algebras. The a-spinors or e-spinors of each one of the Clifford algebras *studied* in this paper has a natural F-linear space structure over one of the following fields $F = \mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$, respectively the real, complex and quaterniom fields (§ 2).

We introduce for each a-spinor space $I = \mathbb{R}_{p,q}e$ a *natural scalar product* (spinorial metric) i.e., a non-degenerated bilinear mapping $\Gamma : I \times I \to Fe$, where F is the natural scalar field associated with the vector structure of $I \subset \mathbb{R}_{p,q}$.

Our approach to the natural scalar product shows for $p + q \leq 5$, the groups $Spin_+(p,q)$ are the groups that leave the spinorial metric invariant. Thus our approach to the scalar product is different from the one discussed by Lounesto[8] and as we shall see offers a solution for the main question formulated above.

In § 4 we analyse in detail the special cases $SU(2) \simeq Spin(3,0)$ and $SL(2,\mathbb{C}) \simeq Spin_+(1,3)$ and identify respectively the ideals that contain the objects corresponding to Pauli c-spinors in $\mathbb{R}_{3,0}$ and the Weyl c-spinors and Dirac c-spinors in $\mathbb{R}_{1,3}$ (the space-time algebra) and $\mathbb{R}_{3,1}$ (the Majorana algebra). Our identifications are all based on explicit proofs that the representative space of a-spinors (or e-spinors) of each one of the c-spinors mentioned above carry the correct representation of the corresponding *spin* group (according to the theory of group representation). We show also that the original Dirac algebra $\mathbb{C}(4)$ must be identified for physical reasons with the real Clifford algebra $\mathbb{R}_{4,1}$.

Now, it is well known that physical theories use spinor fields. Indeed, there are theories which use c-spinor fields [28,29] and theories that use e-spinor fields [9,30,31,32,33]. c-spinor fields are sections of the so-called Spinor bundle and the e-spinor fields are elements of the Clifford bundle. These two bundles are of very different nature. In particular the existence of the Spinor bundle imposes several constraints on the base manifold of the bundle (which is taken as a Lorentzian manifold modeling space-time [10,34,35,36]), which are different from the constraints imposed for the existence of the Clifford bundle [9,37].

Now, the construction of the Clifford bundle for a base space-time of signature $(3,1)$ needs the use of the tilted Clifford group [10,22,27] which are introduced in § 3 together with the $Pin(p,q)$, $(Pin^{\square}(p,q))$ and $Spin(p,q)$ and $(Spin^{\square}(p,q))$ groups.

The use of the Clifford bundle in references [9,30,31,32,33] show explicitly some problems with the "transformation law" of e-spinors. This problem is completed solved from the mathematical point of view in § 3.5 and in [39].

In another publication (called II) we study the structure of the Spinor bundle, the Clifford bundle and a new bundle which we call the Spinor-Clifford-bundle. In II we show how to write Dirac's equation on the Clifford bundles $\mathcal{C}(\mathbb{R}_{1,3})$ and $\mathcal{C}(R_{3,1})$. This motivates the operator definition of spinor given by Hestenes for $\mathbb{R}_{1,3}$ and generalized by Dimakis [38] for all real Clifford algebras.

We call also the attention of the reader that in II[37] we give a new proof Geroch's theorem that require only the explicit construction of the Weyl algebraic spinor and the spinorial metric within $\mathbb{R}_{1,3}$ and elementary facts about associated bundles and the bundle-reduction process. This is to be compared with the original Geroch's proof which uses the full algebraic topology machiney.

For methodological reasons the definition of o-spinors is presented in another publication, called III[39].

Finally in § 5 we present our conclusions.

2. Some General Features About Clifford Algebras

Let V be a vector space of finite dimension n over the field F and let Q be a non-degenerate quadratic form on V. The Clifford algebra $C(V,Q) = T(V)/I_Q$ where $T(V)$ is the tensor algebra of $V(T(V) = \sum_{i=1}^{\infty} T^i(V)$; $T^{(0)}(V) = F$; $T^1(V) = V$; $T^r(V) = \otimes^r V)$ and I_Q is the bilateral ideal generated by the elements of the form $x \otimes x - Q(x)1$, $x \in V$. The signature of Q is arbitrary. The Clifford algebra so constructed is an associative algebra with unit. The space V is naturally imbedded in $C(V,Q)$.

$$V \overset{i}{\hookrightarrow} T(V) \overset{j}{\to} T(V)/I_Q = C(V,Q), \quad i_Q = j \circ i; \quad \text{and} \quad V \equiv I_Q(V) \subset C(V,Q).$$

Let $C^+(V,Q)$ (respectively $C^-(V,Q)$) be the j-image of $\sum_{i=0}^{\infty} T^{2i}(V)$ (respectively $\sum_{i=0}^{\infty} T^{2i+1}(V)$) in $C(V,Q)$. The elements of $C^+(V,Q)$ form a subalgebra of $C(V,Q)$ called the even subalgebra of $C(V,Q)$.

$C(V,Q)$ has the following universal property: "If A is an associative F-algebra with unit then all linear mappings $\phi: V \to A$ such that $(\phi(x))^2 = Q(x)1$, $\forall x \in V$ can be extended in a unique way to a homomorphism $\phi: C(V,Q) \to A$"

In $C(V,Q)$ there exist three linear mappings which are quite natural. They are extensions of the mappings:

(a) **Main Involution** - an automorphism $^{\square}: C(V,Q) \to C(V,Q)$ extension of $\alpha: V \to T(V)/I_Q$, $\quad \alpha(x) = -i_Q(x) = -x, \forall x \in V$.

(b) **Reversion** - an antiautomorphism $^{*}: \quad C(V,Q) \to C(V,Q)$ extension of $^{t}: T^{r}(V) \to T^{r}(V)$, $\quad T^{r}(V) \ni x = x_{i_1} \otimes \cdots \otimes x_{i_r} \to x^{t} = x_{i_r} \otimes \cdots \otimes x_{i_1}$.

(c) **Conjugation** - $\tilde{}: C(V,Q) \to C(V,Q)$, defined by the composition of the automorphism $^{\square}$ with the antiautomorphism *, i.e., if $x \in C(V,Q)$, then $\tilde{x} = (x^{*})^{\square}$.

$C(V,Q)$ can be described through its generators, i.e., if $\{e_i\}, i = 1,2,\ldots,n$ is a Q-orthonormal basis of V, then $C(V,Q)$ is generated by 1 and the $e_i^{\prime}s$ subject to the conditions $e_ie_i = Q(e_i)1$ and $e_ie_j + e_je_i = 0, i \neq j, \quad i,j = 1,2,\ldots,n$. If V is a n-dimensional real vector space then we can choose a basis $\{e_i\}$ for V such that $Q(e_i) = \pm 1$.

2.2. The Real Clifford Algebras $\mathbb{R}_{p,q}$

Let $\mathbb{R}^{p,q}$ be a real vector space of dimension $p+q=n$ equipped with a metric $g: \mathbb{R}^{p,q} \times \mathbb{R}^{p,q} \to R$. Let $\{e_i\}$ be the canonical basis of $\mathbb{R}^{p,q}$ such that

$$g(e_i,e_j) = g_{ij} = g(e_j,e_i) = g_{ji} = \begin{cases} +1 & i = j = 1,2,\ldots,p \\ -1 & i = j = p+1,\ldots,p+q = n \\ 0 & i \neq j \end{cases}$$

The Clifford algebra $\mathbb{R}_{p,q} = C(\mathbb{R}^{p,q}, Q)$; $p+q=n$, in the Clifford algebra over the real field $\mathbb{R}$, generated by 1 and the $\{e_i\}$, $i = 1,\ldots,n$ such that $Q(e_i) = g(e_i,e_i)$. $\mathbb{R}_{p,q}$ is obviously of dimension 2^n and it is the direct sum of the vector spaces $\mathbb{R}^k_{p,q}$ of dimensions $\binom{n}{k}$, $0 \leq k \leq n$. The canonical basis for $\mathbb{R}^k_{p,q}$ are the elements $e_A = e_{\alpha_1}\cdots e_{\alpha_k}$, $1 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq n$. The element $e_J = e_1 \cdots e_n \in \mathbb{R}^n_{p,q}$ commutes (n-odd) or anti-commutes (n-even) with all vectors $e_1,\ldots,e_n$ in $\mathbb{R}^1_{p,q} = \mathbb{R}^{p,q}$. The center of $\mathbb{R}_{p,q}$ is $\mathbb{R}^0_{p,q} = \mathbb{R}$ if n is even and it is the direct sum $\mathbb{R}^0_{p,q} \oplus \mathbb{R}^n_{p,q}$ if n is odd. [23,24] All Clifford algebras are semi-simple. If $p+q=n$ is even $\mathbb{R}_{p,q}$ is a simple algebra and if $p+q=n$ is odd we have the following possibilities:

(a) $\mathbb{R}_{p,q}$ is simple $\leftrightarrow e_J^2 = -1 \leftrightarrow p-q \neq 1 \pmod 4 \leftrightarrow$ center $\mathbb{R}_{p,q}$ is isomorphic to C.

(b) $\mathbb{R}_{p,q}$ is not simple $\leftrightarrow e_J^2 = +1 \leftrightarrow p-q = 1 \pmod 4) \leftrightarrow$ center $\mathbb{R}_{p,q}$ is isomorphic to $\mathbb{R}^0_{p,q} \oplus \mathbb{R}^n_{p,q}$.

From the fact that all semi-simple algebras are the direct sum of two simple algebras [23] and from

Weddenburn's Theorem: "If A is a simple algebra then A is equivalent to $F(m)$, where F is a division algebra and m and F are unique (modulo isomorphisms)" we obtain from the point of view of representation theory $\mathbb{R}_{p,q} \simeq F(m)$ or $\mathbb{R}_{p,q} \simeq F(m) \oplus F(m)$ where $F(m)$ is the matrix algebra of dimension $m \times m$ (for some m) with coefficients in $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

Table I (where $[n/2]$ means the integral part of $(n/2)$ presents the representation of $\mathbb{R}_{p,a}$, as a matrix algebra [23,25].

$p-q \pmod 8$	0	1	2	3	4	5	6	7
$\mathbb{R}_{p,q}$	$\mathbb{R}(2^{[n/2]})$	$\mathbb{R}(2^{[n/2]}) \oplus \mathbb{R}(2^{[n/2]})$	$\mathbb{R}(2^{[n/2]})$	$\mathbb{C}(2^{[n/2]})$	$\mathbb{H}(2^{[n/2-1]})$	$\mathbb{H}(2^{[n/2]-1}) \oplus \mathbb{H}(2^{[n/2]-1})$	$\mathbb{H}(2^{[n/2-1]})$	$\mathbb{C}(2^{[n/2]})$

Table I - Representation of the real Clifford algebra $\mathbb{R}_{p,q}$ as a matrix algebra

2.3. Minimal Lateral Ideals of $\mathbb{R}_{p,q}$

The minimal left ideals of a semi-simple algebra A are of the type Ae, where $e(e^2 = e)$ is a primitive idempotent of A. A idempotent is primitive if it cannot be written as a sum of two non zero orthogonal idempotents, i.e., $e \neq \hat{e} + \check{e}$, where $\hat{e}^2 = \hat{e}$, $\check{e}^2 = \check{e}$ and $\hat{e}\check{e} = \check{e}\hat{e} = 0$[23]. Recall that when $p + q = n$ is even $\mathbb{R}_{p,q} \simeq F(m)$. (Table I). We also have the

Theorem. The maximum number of pairwise orthogonal idempotents in $F(m)$ is m[24].

The decomposition of $\mathbb{R}_{p,q}$ into minimal ideals is then characterized by a *spectral set* $\{e_{pq,i}\}$ of idempotent elements of $\mathbb{R}_{p,q}$ such that

(a) $$\sum e_{pq,i} = 1$$

(b) $$e_{pq,i}\, e_{pq,j} = \delta_{ij}\, e_{pq,i}$$

(c) rank of $e_{pq,i}$ is minimal $\neq 0$, i.e., $e_{pq,i}$ is primitive.

where rank of $e_{pq,i}$ is defined as the rank of the $\oplus \Lambda^d(R^{p,q})$-morphism $e_{pq,i}$: $\psi \to \psi\, e_{pq,i}$, where $\oplus \Lambda^d(\mathbb{R}^{p,q})$ is the exterior algebra of $\mathbb{R}^{p,q}$. Then $\mathbb{R}_{p,q} = \sum I^i_{p,q}$, $I^i_{p,q} = \mathbb{R}_{p,q} e_{pq,i}$ and $\psi \in I^i_{p,q} \subset \mathbb{R}_{p,q}$ is such that $\psi\, e_{pq,i} = \psi$. Conversely, any element $\psi \in I^i_{p,q}$ can be characterized by an idempotent $e_{pq,i}$ of minimal rank $\neq 0$ with $\psi\, e_{pq,i} = \psi$.

We have the

Theorem [8]: A minimal left ideal of $\mathbb{R}_{p,q}$ is of the type $I_{p,q} = \mathbb{R}_{p,q} e_{pq}$ where $e_{pq} = 1/2(1 + e_{\alpha_1}) \cdots 1/2(1 + e_{\alpha_k})$ is a primitive idempotent of $\mathbb{R}_{p,q}$ and where $e_{\alpha_1}, \ldots, e_{\alpha_k}$ is a set of commuting elements of the canonical basis of $\mathbb{R}_{p,q}$ such that $(e_{\alpha_i})^2 = 1$, $i = 1, \ldots, k$ that generates a group of order $k = q - r_{q-p}$ and r_i are the Radon-Hurwitz numbers, defined by the recurrence formula $r_{i+8} = r_i + 4$ and

i	0	1	2	3	4	5	6	7
r_i	0	1	2	2	3	3	3	3

Table II – Radon-Hurwitz number

If we have a linear mapping $L_a : \mathbb{R}_{p,q} \to \mathbb{R}_{p,q} \ni L_a(x)$, $\forall x \in \mathbb{R}_{p,q}$ and where $a \in \mathbb{R}_{p,q}$, the since $I_{p,q}$ is invariant under left multiplication with arbitrary elements of $\mathbb{R}_{p,q}$, we can consider $L_a|_{I_{p,q}} : I_{p,q} \to I_{p,q}$. We have the

Theorem: If $p + q = n$ is even or odd with $p - q \neq 1 \pmod 4$ then

$$\mathbb{R}_{p,q} \simeq \mathcal{L}_F(I_{p,q}) \simeq F(m) \tag{17}$$

where $F \simeq \mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$, $\mathcal{L}_F(I_{p,q})$ is the algebra of linear transformations in $I_{p,q}$ over the field F, $m = dim_F(I_{p,q})$ and $F \simeq eF(m)e$, e being the representation of e_{pq} in $F(m)$. If $p + q = n$ is odd, with $p - q = 1 \pmod 4$, then $\mathbb{R}_{p,q} \simeq \mathcal{L}_F(I_{p,q}) \simeq F(m) \oplus F(m)$, $m = dim_F(I_{p,q})$ and $e_{pq}\mathbb{R}_{p,q}e_{pq} \simeq \mathbb{R} \oplus \mathbb{R}$ or $\mathbb{H} \oplus \mathbb{H}$.

With the above isomorphisms we can identify the minimal left ideals $\mathbb{R}_{p,q}$ with the column matrices of $F(m)$.

Now, with the ideas introduced above it is a simple exercise to find a primitive idempotent of $\mathbb{R}_{p,q}$. We have the following algorithm. We first give a look in Table I and find to which matrix algebra our particular $\mathbb{R}_{p,q}$ is isomorphic. Let $\mathbb{R}_{p,q} \simeq F(m)$ for a particular F and m. [2] Next we take from the canonical basis $\{e_A\}$ of $\mathbb{R}_{p,q}$

$$e_A = e_{\beta_1} \cdots e_{\beta_k}, \quad 1 \leq \beta_1 \leq \cdots \leq \beta_k \leq n, \quad p+q=n$$

a element $e_{\alpha_1} \in \{e_A\}$ such that $e_{\alpha_1}^2 = 1$. We then construct the idempotent $e_{pq} = 1/2(1+e_{\alpha_1})$ and calculate $dim_F(I_{p,q})$. If $dim_F(I_{p,q}) = m$ then e_{pq} is primitive. If $dim_F(I_{p,q}) \neq m$ then choose [3] $\{e_A\} \ni e_{\alpha_2} | e_{\alpha_2}^2 = 1$ and construct the idempotent $e'_{pq} = 1/2(1+e_{\alpha_1})1/2(1+e_{\alpha_2})$ and the calculate $dim_F(I'_{p,q})$ where $I'_{p,q} = \mathbb{R}_{p,q}e'_{pq}$. If $dim_F(I'_{p,q}) = m$, then e_{pq} is primitive. Otherwise repit the procedure. According to the theorem above the process is finite.

We will discuss the problem of the *equivalence of* representations of $\mathbb{R}_{p,q}$ when we take the minimal left ideals (instead of some vector space isomorphic to them) as representation modules of $\mathbb{R}_{p,q}$, after the introduction of the concept of the Clifford groups (§ 3).

3. Algebraic Spinors, Spin Group, Spinorial Representation and Spinorial Metric.

We continue to use the notation of § 2, but here V refers always to a real vector space.

3.1. The Group $C^*(V,Q)$.

The elements $u \in C(V,Q)$ such that there exists $u^{-1} \in C(V,Q)$, $uu^{-1} = u^{-1}u = 1$ constitute a non abelian group which we denote by $C^*(V,Q)$. When $(V,Q) \equiv \mathbb{R}^{p,q}$ we denote $C^*(V,Q)$ by $\mathbb{R}^*_{p,q}$.

$C^*(V,Q)$ acts naturally on $C(V,Q)$ as an algebra automorphism through its adjoint representation Ad

$$\begin{aligned} Ad: \ & C^*(V,Q) \to Aut(C(V,Q)) \\ & Ad_u : C(V,Q) \to C(V,Q) \\ & \qquad Ad_u(x) = uxu^{-1}, \quad u \in C^*(V,Q); \quad x \in C(V,Q) \end{aligned} \tag{18}$$

For what follows it is also important to consider the so-called twisted adjoint

[2] We are supposing $\mathbb{R}_{p,q}$ is simple. The procedure is also straightforward when $\mathbb{R}_{p,q}$ is semi-simple.

[3] All elements e_{α_i} are actual commuting elements as stated in the last theorem.

representation of $C^*(V,Q)$,

$$Ad_u^{\square} : C(V,Q) \to C(V,Q) \qquad Ad_u^{\square} x = u^{\square} x u^{-1} \tag{19}$$

Observe that $Ad_u^{\square} = Ad_u$ for $u \in C^*(V,Q) \cap C^+(V,Q)$.

3.2. The Clifford and the Twisted Clifford Groups

We define the *Clifford group* $\Gamma(V,Q)$ of $C(V,Q)$ by

$$\Gamma(V,Q) = \{u \in C^*(V,Q), Ad_u(V) = V\} \tag{20}$$

We define the special Clifford group $\Gamma^+(V,Q)$ of $C(V,Q)$ by

$$\Gamma^+(V,Q) = \Gamma(V,Q) \cap C^+(V,Q) \tag{21}$$

We introduce the "norm mapping" $N : C(V,Q) \to C(V,Q)$ by

$$N(x) = x^* x \tag{22}$$

We have the

Proposition: The following mappings are homomorphisms

$$N|_{\Gamma(V,Q)} : \Gamma(V,Q) \to \mathbb{R}^*, \quad \dim(V) \text{ is even} \qquad \mathbb{R}^* = \mathbb{R} - \{0\}$$

$$N|_{\Gamma^+}(V,q) : \Gamma^+(V,Q) \to \mathbb{R}^*, \dim(V) \text{ is odd} \tag{23}$$

The *reduced Clifford group* is defined by

$$\Gamma_0(V,Q) = \{u \in \Gamma(V,Q) | N(u) = +1 \text{ and } \dim(V) \text{ is even}\} \tag{24}$$

The *reduced special Clifford group* is defined by

$$\Gamma_0^+(V,Q) = \{u \in \Gamma^+(V,Q) | N(u) = +1 \text{ and } \dim(V) \text{ is odd}\} \tag{25}$$

Γ_{0+}^+ denotes the component of Γ_0^+ containing the identity. (26)

We now define *the twisted Clifford group* $\Gamma^{\square}(V,Q)$ of $C(V,Q)$ by

$$\Gamma^{\square}(V,Q) = \{u \in C^*(V,Q), \ Ad_u^{\square}(V) = V\} \tag{27}$$

The *special twisted Clifford group* $\Gamma^{\square\,+}(V,Q)$ is defined by

$$\Gamma^{\square\,+}(V,Q) = \Gamma^{\square}(V,Q) \cap C^+(V,Q) \tag{28}$$

When $(V,Q) \equiv I\!R^{p,q}$ we use the notations $\Gamma(p,q)$ and $\Gamma^{\square}(p,q)$ for $\Gamma(V,Q)$ and $\Gamma^{\square}(V,Q)$, etc ...

It is clear from eq.(20) [eq.(27)] that $\Gamma^{\square}(V,Q) \subset \Gamma(V,Q)$.

Let us call σ the restriction of Ad to $\Gamma(V,Q)$. Since when $u \in \Gamma(V,Q)$, the linear morphism $\sigma(u) : V \to V$ satisfies $Q(\sigma(u)x) = Q(uxu^{-1}) = uxu^{-1}uxu^{-1} = ux^2u^{-1} = Q(x)$ it follows that we have the following group morphism

$$Ad|_{\Gamma(V,Q)} \equiv \sigma : \Gamma(V,Q) \to O(V,Q) \tag{29}$$

It is very important to observe that when $dim(V)$ is odd the morphism σ is not *onto*, its image being the special orthogonal group $SO(V,Q)$.

We then have [10,21,22,23]

Proposition:

$$\begin{aligned} \sigma(\Gamma(V,Q)) &= O(Q,V), \quad \text{if} \quad dim(V) \quad \text{is even} \\ \sigma(\Gamma^{+}(V,Q)) &= SO(V,Q), \quad \text{if} \quad dim(V) \quad \text{is odd} \end{aligned} \tag{30}$$

For what follows we need the result [23].

Proposition: The kernel of the mapping

$$Ad^{\square} : \Gamma^{\square} \to Aut(V) \quad \text{is} \quad R^{*} \tag{31}$$

Consider now the "*twisted norm mapping*" $N^{\square} : C(V,Q) \to C(V,Q)$

$$N^{\square}(x) = \hat{\tilde{x}}x \tag{32}$$

If $v \in V \hookrightarrow C(V,Q)$, $N(v) = (v^{*})^{\square}v = -v^2 = -Q(v)$. This means that $N(v)$ coincides with the "square of $v \in (V,-Q)$". In the particular case when $(V,Q) \equiv I\!R^{p,q}$, then $(V,-Q) \equiv I\!R^{q,p}$, which is a real vector space of dimension $n = p+q$ with a metric of signature (q,p).

$N^{\square}$ has important properties when restricted to $\Gamma^{\square}(V,Q)$. We have:

Proposition [24]:

$$\begin{aligned} N^{\square}|_{\Gamma^{\square}}(V,Q) : \Gamma^{\square}(V,Q) \to I\!R^{*} \quad &\text{is a homomorphism} \\ \text{and for} \quad \Gamma^{\square}(V,Q) \ni u \quad &\text{it is} N^{\square}(u^{\square}) = N^{\square}(u). \end{aligned} \tag{33}$$

More important is the

Proposition: Let $u \in \Gamma^{\square}(V,Q)$ and $Ad^{\square}|_{\Gamma(V,Q)} \equiv \sigma^{\square}$. Then the linear morphism $\sigma^{\square}(u) : V \to V$ satisfies $N(\sigma^{\square}(u)v) = N(v) = -Q(v)$.

$$\text{It follows that} \quad \sigma^{\square} \subset O(V,-Q) \tag{34}$$

The proof of Proposition (34) is trivial. It is important to observe that as $O(V,Q) \simeq O(V,-Q)$ we have also that $\sigma^{\square} \subset O(V,-Q)$.

We close this § 2.3 with the following observation: It is easily to show that if $v \in V$ and $Q(v) \neq 0$, then $Ad_v(V) = V$ and $Ad_v^{\square}(V) = V$. Therefore we can show [10] that the elements of both $\Gamma(V,Q)$ and $\Gamma^{\square}(V,Q)$ can be written as a product of a finite number of vectors v_i, $Q(v_i) \neq 0$. We have, e.g.,

$$\Gamma^{\square}(V,Q) = \{v_1 \cdots v_r \in C(V,Q), \quad v_i = 1,\ldots,r, \quad Q(v_i) \neq 0\} \tag{35}$$

3.3. The groups $Pin(p,q), Spin(p,q), Pin^{\square}(p,q)$ and $Spin^{\square}(p,q)$

We define the $Pin(p,q)$ group when $p+q=n$ is even as the subgroup of $\Gamma(p,q)$ such that

$$Pin(p,q) = \{u \in \Gamma(p,q) | N(u) = \pm 1\} \tag{36}$$

It is clear that $Pin(p,q) \simeq \Gamma(p,q)/\mathbb{R}^*_+$, where $\mathbb{R}^*_+$ are the non-negative real numbers.

We also define the $Spin(p,q)$ group when $p+q=n$ is *even* or *odd* by

$$Spin(p,q) = \{u \in \Gamma^+(p,q) | N(u) = \pm 1\} \tag{37}$$

It is clear that $Spin(p,q) \simeq \Gamma^+(p,q)/\mathbb{R}^*_+$.

We define the $Spin_+(p,q)$ group when $p+q=n$ is *even* or *odd* by

$$Spin_+(p,q) = \{u \in \Gamma^+(p,q) | N(u) = +1\} \tag{38}$$

Comparing eq.(38) with eq.(25) we see that $Spin_+(p,q) = \Gamma_0^+(p,q)$ when $p+q=n$ is *odd*. It is clear that $Spin_+(p,q) = Pin_+(p,q)$ where $Pin_+(p,q)$ is the component of $Pin(p,q)$ containing the identity.

The following theorem holds true [22,23].

Theorem:

$$Ad|_{Pin(p,q)} : Pin(p,q) \to O(p,q) \quad \text{is onto with kernel}$$

$$Z_2 \quad \text{when} \quad p+q=n \quad \text{is} \in \text{even}$$

$$Ad|_{Spin(p,q)} : Spin(p,q) \to SO(p,q) \quad \text{is onto with kernel}$$

$$Z_2 \quad \text{when} \quad p+q=n \quad \text{is} \quad \textit{even} \quad \text{or} \quad \textit{odd} \tag{39}$$

Theorem (39) means that $O(p,q) \simeq Pin(p,q)/Z_2$ when $(p+q)$ is even and $SO(p,q) \simeq Spin(p,q)/Z_2$ when $(p+q)$ is odd or even. It is also clear from the above considerations that $SO_+(p,q) \simeq \frac{Spin_+(p,q)}{Z_2}$, where $SO_+(p,q)$ and $Spin_-(p,q)$ are respectively the components of $SO(p,q)$ and $Spin(p,q)$ connected with the identity.

The above groups are used for the case when $p+q=n$ is even, e.g., by Bugajska [11] in her study of the spinor structure of space time. Also related definitions are given by Crumeyrolle [10].

For the construction of the Clifford bundle (in III) when $p+q=3+1$ we need also the following definitions

$$Pin^{\square}(V,Q) = \{u \in \Gamma^{\square}(V,Q) | N^{\square}(u) = \pm 1\} \tag{40}$$

$$Spin^{\square}(V,Q) = \{u \in \Gamma^{\square\,+}(V,Q) | N(u) = \pm 1\} \tag{41}$$

$$Spin^{\square}_{+}(V,Q) = \{u \in \Gamma^{\square\,+}(V,Q) | N(u) = +1\} \tag{42}$$

$Spin^{\square}_{+}(V,Q)$ is the connected component of $Spin^{\square}(p,q)$.

We saw above that $Pin(p,q)$ is a covering of $O(p,q)$ only when $p+q$ is *even*. For the groups $Pin^{\square}(p,q)$ and $Spin^{\square}(p,q)$ we have [10].

Theorem:

$$\begin{aligned} &Ad^{\square}|_{Pin^{\square}(p,q)} : Pin^{\square}(p,q) \to O(q,p) \quad \text{is onto with kernel } Z_2 \\ &Ad^{\square}|_{Spin^{\square}(p,q)} : Spin^{\square}(p,q) \to SO(q,p) \text{is onto with kernel } Z_2 \end{aligned} \tag{43}$$

It follows that the following exact sequences are valid for all (p,q).

$$\begin{aligned} &O \to Z_2 \to Pin^{\square}(p,q) \overset{Ad^{\square}}{\to} O(q,p) \to 1 \\ &O \to Z_2 \to Spin^{\square}(p,q) \overset{Ad^{\square}}{\to} O(q,p) \to 1 \\ &O \to Z_2 \to Spin^{\square}_{+}(p,q) \overset{Ad^{\square}}{\to} SO_{+}(p,q) \end{aligned} \tag{44}$$

We now give the relation between the various groups defined above

Proposition: When

$$p + q = n \quad \text{is even} \quad \Gamma^{\square}(p,q) = \Gamma(p,q)$$

and when

$$p + q = n \quad \text{is odd} \quad \Gamma^{+}(p,q) \subset \Gamma^{\square}(p,q) \subset \Gamma(p,q) \qquad (45)$$

Proposition: When

$$p + q = n \quad \text{is even} \quad Pin(p,q) = Pin^{\square}(p,q)$$

and when

$$p + q = n \quad \text{is even or odd} \quad Spin(p,q) = Spin^{\square}(p,q) \qquad (46)$$

The proofs of Propositions (45) and (46) are trivial.
We resume the above results in Tables III and IV.

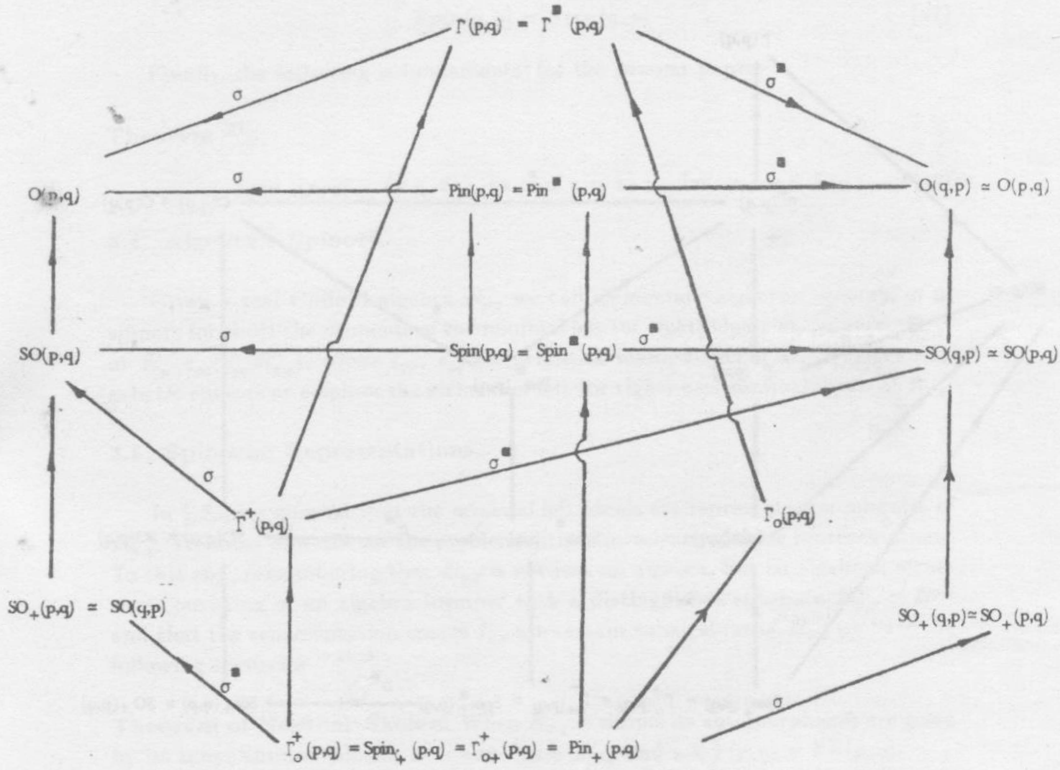

Table III: $p - q = n$ is even and $n > 2$

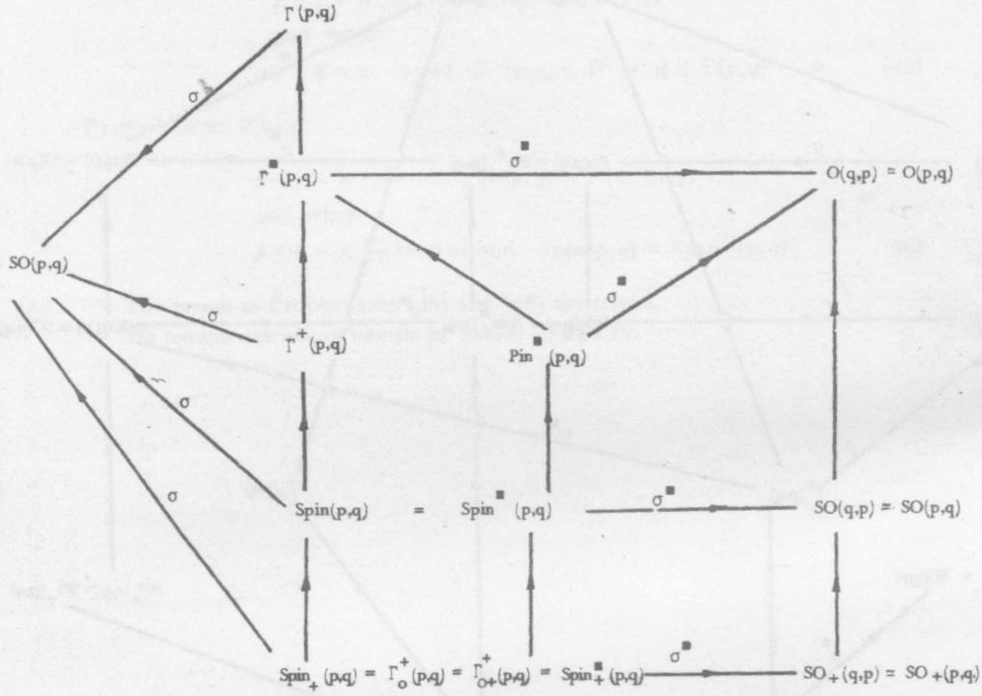

Table IV - $p - q = n$ is odd

Since $\mathbb{R}^+_{p,q} \simeq \mathbb{R}^+_{q,p}$ it follows that

$$Spin(p,q) \simeq Spin(q,p) \tag{47}$$

Finally, the following is fundamental for the present paper:

Theorem [21]:

$$Spin_+(p,q) = \{u \in \mathbb{R}^+_{p,q} | \tilde{u}u = u^{\sim}u = 1\} \; ; \quad \text{for} \quad p+q \leq 5. \tag{48}$$

3.4. Algebraic Spinors.

Given a real Clifford algebra $\mathbb{R}_{p,q}$ we call elementary algebraic spinors, or a-spinors for short the elements of the minimal left (or right) ideals $\mathbb{R}_{p,q}e_{p,q}$ $(e_{pq}\mathbb{R}_{p,q})$ or $\mathbb{R}^+_{p,q}e'_{pq}(e'_{pq}\mathbb{R}^-_{p,q})$, where e_{pq}, e'_{pq} are primitive idempotents of $\mathbb{R}_{p,q}$. We call algebraic spinors or e-spinor the element of left (or right) *non* minimal ideals of $\mathbb{R}_{p,q}$.

3.5. Spinorial Representations.

In § 2.3 we showed that the minimal left ideals are representation modules of $\mathbb{R}_{p,q}$. We must now discuss the problem of the equivalence of these representations. To this end, remembering that $\mathbb{R}_{p,q}$ is not just an algebra, but an algebraic structure consisting of an algebra together with a distinguished subspace $\mathbb{R}^1_{p,q} \simeq \mathbb{R}^{p,q}$ and that the representation spaces $I_{p,q}$ are certain subalgebras of $\mathbb{R}_{p,q}$ we have the following theorems [9,22,23].

Theorem of Noether-Skolen: When $\mathbb{R}_{p,q}$ is simple its automorphisms are given by its inner automorphisms $x \to uxu^{-1}, x \in \mathbb{R}_{p,q}$ and $u \in \Gamma(p,q) = \Gamma^{\square}(p,q)$.

Theorem: When $\mathbb{R}_{p,q}$ is simple all their finite-dimensional irreducible representations are equivalent under inner automorphisms.

In view of the above theorems we define that two representations $I_{p,q}$ and $I'_{p,q}$ of $\mathbb{R}_{p,q}$ are equivalent if $I'_{pq} = uI_{pq}u^{-1}$ for some $u \in \Gamma(p,q)$.

Now, if we consider the definition of the group $Spin^+(p,q)$ we see that the ideals $I_{p,q}$ can be made into *spinorial* representation of $SO_+(p,q)$ (in the sense of group theory)by postulating $I_{p,q} \to uI_{p,q}$, for $u \in Spin_+(p,q)$. This is exactly the idea behind the introduction of the spinorial metric (§ 3.6) which is necessary in order to "mimic" the results in (i), (ii), (iii) and (iv) of § 1.

The transformation $\psi \longmapsto u\psi, \psi \in I_{p,q}$ corresponds to the usual transformation of c-spinors, but the use of this transformation involving other Clifford numbers would contradict the fact that the $I'_{p,q}$s are substructures of $\mathbb{R}_{p,q}$.

Observe that when $\mathbb{R}_{p,q}$ is semi-simple and e_{pq} is a primitive idempotent then $\mathbb{R}_{p,q} e_{pq} \mathbb{R}_{p,q}$ is a bilateral ideal and $1e1 = e \neq 0$. It follows that $\mathbb{R}_{p,q} e_{pq} \mathbb{R}_{p,q} = \mathbb{R}_{p,q}$, from where we can write

$$\mathbb{R}_{p,q} = I(p,q)({}^*I_{p,q}) \tag{49}$$

where

$$I_{p,q} = \mathbb{R}_{p,q} e_{pq} \quad \text{and} \quad e_{pq} \mathbb{R}_{p,q} = {}^* I_{p,q}$$

The meaning of eq.(48) is then that $\forall x \in \mathbb{R}_{p,q}$ can be written as sum of elements of the tensor product [4] of the spinor spaces $I_{p,q}$ and ${}^*I_{p,q}$, i.e., any $x \in \mathbb{R}_{p,q}$ can be considered as a rank-two spinor.

This decomposition of antisymmetric tensors are the ones generally, presented in textbooks of theoretical physics and group theory and which gave birthy to the believe that spinors are more fundamental than tensors. However from the results of § 2.3 we know that $\forall x \in \mathbb{R}_{p,q}$ can be written as

$$x = \psi_1 + \psi_2 + \cdots + \psi_n \tag{50}$$

where $\psi_i \in I'_{p,q}$, $I'_{p,q} = \mathbb{R}_{p,q} e_{pq,i}$, $\sum_{i=1}^n e_{pq,i} = 1$.

From eq.(50) it is clear that e-spinors can be written as sum of antisymmetric tensors (for explicit examples see § 4).

Now, let $\{e_{pq,i}\}$ and $\{e'_{pq,i}\}$ be two set of primitive idempotent $\sum_{i=1}^n e_{pq,i} = 1$, $\sum_{i=1}^n e'_{pq,i} = 1$ and $e'_{pq,i} = u e_{pq} u^{-1}$, $u \in \Gamma(p,q)$.

Given $x \in \mathbb{R}_{p,q}$ we have

$$x = \sum_{i=1}^n x e_{pq,i} = \sum_{i=1}^n x e'_{pq,i} = \psi_1 + \cdots + \psi_n = \psi'_1 + \cdots + \psi'_n$$

and then

$$\psi'_i = u \psi_i u^{-1} \tag{51}$$

On the other side if a given $x \in \mathbb{R}_{p,q}$ can be written as

$$x = \psi {}^*\varphi, \quad \psi \in I_{p,q}, \quad {}^*\varphi \in {}^*I_{p,q} \tag{52}$$

we can write

$$\begin{aligned}
x &= \psi {}^*\varphi = \sum \psi_\alpha e_{pq} e_{pq} {}^*\varphi_\alpha \quad ; \quad \psi_\alpha, {}^*\varphi \in \mathbb{R}_{p,q} \\
&= \sum \psi_\alpha (u^{-1} e'_{pq} u)(u^{-1} e'_{pq} u) {}^*\varphi_\alpha \\
&= \sum (u^{-1} \psi'_\alpha e'_{pq}) u u^{-1} (e'_{pq} {}^*\varphi'_\alpha u) \quad ; \quad \psi'_\alpha = u \psi_\alpha u^{-1} \\
&= u^{-1} [\sum (\psi'_\alpha e'_{pq}) u u^{-1} (e'_{pq} {}^*\varphi'_\alpha)] u
\end{aligned} \tag{53}$$

[4] remember the definition of the Clifford product

In this case the factor $uu^{-1} = 1$ can be eliminated or retaining without affecting the result. Then, from the usual decomposition of antisymmetric tensors as tensor products of spinors we infer that we can choose as transformation laws for the spinors:

$$\begin{aligned} &\psi \to \psi' = u\psi && \text{(a)} \\ \text{or} & && \\ &\psi \to \psi' = u\psi u^{-1} && \text{(b)} \end{aligned} \tag{54}$$

Eq.(54.b) is what results from the "sum decomposition" of x into spinors.

Observe that if $\psi \in I_{p,q}$ then $u\psi \in I_{p,q}$ but $u\psi u^{-1} \notin I_{p,q}$, in general.

In physical theories which use spinor fields the observables are usually associated with bilinear functions of spinors (i.e., the observables are tensors) and the problem with (54.a) and (54.b) does not occur. There are a paper in the literature[40] saying that the transformation (54.a) can be directly observed. However the arguments given by the authors are not very strong - we will come back to this particular point into another paper.

3.6. Scalar Product of ϵ-Spinors. The Spinorial Metric

In § 2.3 we saw that $\mathbb{R}_{p,q}$ is simple, a minimal left ideal $I_{p,q}$ of $\mathbb{R}_{p,q}$ is of the form $I_{p,q} = \mathbb{R}_{p,q}e_{pq}$ where e_{pq} is a primitive idempotent of $\mathbb{R}_{p,q}$ and $F \simeq e_{pq}\mathbb{R}_{p,q}e_{pq}$ with $F = \mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$, depending of $p - q = 0, 1, 2 (mod\ 8)$, $p - q = 3, 7 (mod\ 8)$ or $p - q = 4, 5, 6 (mod\ 8)$ respectively (Table I). We can then define a right action F in $I_{p,q}$, $I_{p,q} \times F \to I_{p,q}$ by $I_{p,q} \times F \ni (\psi, \alpha) \to \psi\alpha \in I_{p,q}$. In this way $I_{p,q}$ has a natural linear vector space structure over the field F, whose elements are the natural "scalars" of the vector space $I_{p,q}$.

These remarks suggest us to seach for a *"natural scalar product"* on $I_{p,q}$, i.e., a non-degenetared bilinear mapping $\Gamma : I_{p,q} \times I_{p,q} \to F$. To this end we observe that if f and g are F-endomorphisms in $\mathbb{R}_{p,q}$ then we can define a bilinear mapping Γ in $\mathbb{R}_{p,q}$ using f and g. We simply take $\Gamma(\psi, \varphi) = f(\psi)g(\varphi)$, $\psi, \varphi \in \mathbb{R}_{p,q}$. Considering that $I_{p,q} = \mathbb{R}_{p,q}e_{pq}$ has a natural structure of vector space over F we can take the restriction of Γ to $I_{p,q}$, and ask the following question:

For $\psi, \varphi \in I_{p,q}$ when does $\Gamma(\psi, \varphi) \in F$?

As we saw in § 2.1 we have three natural isomorphisms defined in $\mathbb{R}_{p,q}$, the main involution, the reversion and the conjugation, denoted respectively by $^{\square}$, * and $^{\sim}$. Combining these isomorphisms with the identity mapping we can define the following bilinear mappings

$$\Gamma_i : I_{p,q} \times I_{p,q} \to \mathbb{R}_{p,q}, \quad i = 1, 2, 3$$

$$
\begin{aligned}
\Gamma_1(\psi,\varphi) &= \psi^{\Box}\varphi \\
\Gamma_2(\psi,\varphi) &= \psi^{\cdot}\varphi \\
\Gamma_3(\psi,\varphi) &= \tilde{\psi}\varphi, \quad \forall\psi,\varphi \in I
\end{aligned}
\tag{55}
$$

As already observed in § 2.1 the main involution is an automorphisms whereas the reversion and conjugation are antiautomorphisms.' An automorphism (anti-auotmorphism) transforms an element of a minimal left ideal in an element of a minimal left ideal (minimal right ideal).

To see the validity of these statements it is enough to observe that the image of a primitive idempotent under a isomorphism is a primitive idempotent and that if $\psi \in I_{p,q} = \mathbb{R}_{p,q}e_{pq}$ then $\psi = xe_{pq}$ with $x \in \mathbb{R}_{p,q}$ and

$$
\begin{aligned}
\psi^{\Box} &= (xe_{pq})^{\Box} = x^{\Box}e^{\Box}_{pq} \Rightarrow \psi^{\Box} \in I'_{p,q} = \mathbb{R}_{p,q}e^{\Box}_{pq} \\
\psi^{\cdot} &= (xe_{pq})^{\cdot} = e^{\cdot}_{pq}x^{\cdot} \Rightarrow \psi^{\cdot} \in {}^{\cdot}I_{p,q} = e^{\cdot}_{pq}\mathbb{R}_{p,q} \\
\tilde{\psi} &= (xe_{pq})^{\sim} = \tilde{e}_{pq}\tilde{x} \Rightarrow \tilde{\psi} \in \tilde{I}_{p,q} = \tilde{e}_{pq}\mathbb{R}_{p,q}
\end{aligned}
\tag{56}
$$

Using the isomorphimsm $\mathbb{R}_{p,q} \simeq \mathcal{L}_F(I_{p,q}) \simeq F(m)$, $m = dim_F(I_{p,q})$ (when $\mathbb{R}_{p,q}$ is simple, cf. § 2.3) we identify the elements of the minimal left ideals of $\mathbb{R}_{p,q}$ with the column matrices of $F(m)$. Then. if $\psi \in I_{p,q}$ has a representation as a column matrix of $F(m)$ then $\psi^{\cdot}$ and $\tilde{\psi}$ have representations as row matrices of $F(m)$, and we get that $\psi^{\cdot}\varphi$ and $\tilde{\psi}\varphi$ are elements of F.

We identify the scalars of the vector structure of $I_{p,q}$ with multiples of

$$
e_{pq} = \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}
\tag{57}
$$

i.e., as matrices in $F(m)$ multiples of the matrix in eq.(57). Sometimes it may be convenient to choose the 1 in $e_{p,q}$ in another line [see eq. (81)]. Through isomorphisms of $\mathbb{R}_{p,q}$ (multiplication by a convenient invertible element $u \in \mathbb{R}_{p,q}$) we can transport $\psi^{\cdot}\varphi$ or $\tilde{\psi}\varphi$ to the position (1,1) in the matrix representation of these operations. We then conclude that the *natural scalar products* in $I_{p,q}$ are

$$
\beta_i = I_{p,q} \times I_{p,q} \to F \quad , \quad i = 1, 2
\tag{58}
$$

$\beta_1(\psi,\varphi) = u'\psi^{\cdot}\varphi$ and $\beta_2(\psi,\varphi) = u\tilde{\psi}\varphi$, $\forall\psi,\varphi \in I_{p,q}$ and $u, u' \in \mathbb{R}_{p,q}$ are convenient invertible elements.

Lounesto[8] obtains the scalar products in eq.(58) using similar arguments and immediately proceeds to the classification of the group of automorphisms of these scalar products, i.e., the homomorphisms of right F-modules, $I_{p,q} \to I_{p,q}$, $\psi \to s\psi$, $s \in \mathbb{R}_{p,q}$ which preserve the products in eq.(58). Observe that from $\beta_1(s\psi, s\varphi) = \beta_1(\psi, \varphi)$ we get $s^* s = 1$ and from $\beta_2(s\psi, s\varphi) = \beta_2(\psi, \varphi)$ we get $\tilde{s}s = 1$ $(\psi, \varphi \in I_{p,q})$. Lounesto calls $G_1 = \{s \in \mathbb{R}_{p,q}: s^* s = 1\}$, $G_2 = \{s \in \mathbb{R}_{p,q}, \tilde{s}s = 1\}$.

So in Lounesto's paper there does not appear clearly any relationship between the groups $Spin(p,q)$ and the groups G_1 and G_2 with the consequence that we do not have a clear basis to mimic within the Clifford algebras $\mathbb{R}_{p,q}$ (for appropriate p and q) the results described in (i), (ii), (iii) and (iv) of § 1. We can mimic these results within some Clifford algebras by introducing the concept of spinorial metric.

Observe that since $Spin(p,q) \subset \mathbb{R}^+_{p,q}$ it seems interesting to define a scalar product in an ideal $I^+_{p,q} = \mathbb{R}^+_{p,q} e_{pq}$. The reason is that such a scalar product is now *unique*, since if $s \in \mathbb{R}^+_{p,q}$, then $s^* = \tilde{s}$. This unique scalar product will be called in what follows the spinorial metric

$$\beta : I^+_{p,q} \times I^+_{p,q} \to F \tag{59}$$

define by $\beta(\psi, \varphi) = u\tilde{\psi}\varphi$. We see that $G = \{s \in \mathbb{R}^+_{p,q} | \tilde{s}s = 1\}$ is the group of automorphisms of the spinorial metric just defined and $G \subset G_1$, $G \subset G_2$.

We now recall Theorem (48) of § 3 which says that for $p + q \leq 5$

$$Spin_+(p,q) = \{u \in \mathbb{R}^+_{p,q} | \tilde{u}u = u^* u = 1\}$$

With the result we get a new interpretation of the groups $Spin_+(p,q)$ for $p + q \leq 5$, namely, these are the groups that leave the Spinorial metric of eq.(59) invariant. But even more important is the fact that now we know the way to mimic within appropriate Clifford algebras (i), (ii), (iii) and (iv) of § 1 and thus we can make a representation within Clifford algebras of the Pauli c-spinors, undotted and dotted bidimensional c-spinors and Dirac c-spinors. This is done in § 4, and in II.

4. Representation of Pauli c-Spinors, Undotted and Dotted Two-Dimensional Spinors and Dirac c-Spinors by Appropriated Algebraic Spinors

4.1. Pauli a-spinors and the Group $SU(2)$

The algebra $\mathbb{R}_{3,0}$ (Pauli-algebra) is isomorphic to $\mathbf{C}(2)$ (see Table I). $\mathbb{R}_{3,0}$ is generated by 1 and $\sigma_i, i = 1,2,3$ subject to the condition $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}, \delta_{ij} = diag(+1,+1,+1)$. It is trivial to verify that $e_{30} = 1/2(1 + \sigma_3)$ is a primitive idempotent of $\mathbb{R}_{3,0}$. Now, consider $x \in \mathbb{R}_{3,0}$.

$$x = a_0 + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 + a_4\sigma_1\sigma_2 + a_5\sigma_1\sigma_3 + a_6\sigma_2\sigma_3 + a_7\sigma_1\sigma_2\sigma_3 \quad (60)$$

$$a_i \in \mathbb{R}, \quad i = 0, \ldots, 7$$

The elements $\varphi \in I_{3,0} \equiv I_p = \mathbb{R}_{3,0}e_{30} = \mathbb{R}^+_{3,0}e_{30}$ (Pauli a-spinors) are of the form

$$\varphi = [(a_0 + a_3) - (a_4 + a_7)\sigma_1\sigma_2\sigma_3]e_{30} + [(a_1 + a_5) - (a_2 + a_6)\sigma_1\sigma_2\sigma_3]\sigma_1 e_{30} \quad (61)$$

It is then immediate that $e_{30}\mathbb{R}_{3,0}e_{30} \simeq \mathbf{C}$ has basis $\{1, \sigma_1\sigma_2\sigma_3\}e_{30}$ and the spinorial basis is $\alpha = \{e_{30}, \sigma_1 e_{30}\}$. We now show that the elements of I_p are the representatives of Pauli c-spinors $|(i\ )$ of § 1].

Using the isomorphism $\mathbb{R}_{3,0} \overset{f}{\simeq} \mathcal{L}_{\mathbf{C}}(I_p)$ (§ 2.3), $f(x)\psi = x\psi, u \in \mathbb{R}_{3,0}, \psi \in I_p$ we obtain the representation of $x \in \mathbb{R}_{3,0}$ in the α-basis through the following algorithm: [5] Put

$$e_{30} = |1>, \quad \sigma_1 e_{3,0} = |2>; \quad <1| = e_{30}^*, \quad <2| = (\sigma_1 e_{30})^* = e_{30}\sigma_1 \quad (62)$$

Then

$$1 = \sum_i |i><i|, \quad i = 1,2; \quad <i|j> = \delta_{ij}e_{30}$$

$$x = x\sum |i><i| \Rightarrow x|i> = \sum x_{ji}\,|j>; \quad x_{ji} = <j|x|i>$$

$$e_{30} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \quad \sigma_1 e_{30} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (63)$$

We also have the following matrix representations for $x, x^\square, x^*$ and $\tilde{x} \in \mathbb{R}_{3,0}$.

$$\mathbf{C}(2) \ni x = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix}; \quad x^\square = \begin{bmatrix} \overline{z_4} & -\overline{z_3} \\ -\overline{z_2} & \overline{z_1} \end{bmatrix};$$

$$x^* = \begin{bmatrix} \overline{z_1} & \overline{z_3} \\ \overline{z_2} & \overline{z_4} \end{bmatrix}; \quad \tilde{x} = \begin{bmatrix} z_4 & -z_2 \\ -z_3 & z_1 \end{bmatrix} \quad (64)$$

[5] If $x \in \mathbb{R}_{3,0}$ we use the same letter for $f(x) \in \mathbf{C}(2) \simeq \mathcal{L}_{\mathbf{C}}(I_p)$. This causes no confusion.

From eq.(64) we see that the main antiautomorphism * corresponds in the Pauli-algebra to the operation * in matrix algebra.

We now define the spinorial metric

$$\beta : I_p \times I_p \to C; \beta(\psi,\varphi) = 2\langle \psi^* \varphi \rangle_0 = \bar{\psi}^1 \varphi^1 + \bar{\psi}^2 \varphi^2 \tag{65}$$

where $\langle\ \rangle_0$ means the scalar part of the Pauli-number.

The representation of β in the α-basis is then

$$[\beta]_\alpha = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{1}_2 \tag{66}$$

Also $\beta(\psi,\varphi) = \beta(u\psi, u\varphi) \Longleftrightarrow u^* u = 1 \Longleftrightarrow u \in U(2)$.

Now, if $x \in \mathbb{R}^+_{3,0} \simeq \mathbb{R}_{0,2} \simeq \mathbb{H}$, we have the following representation for x in the α-basis

$$x = \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix} \quad \text{and} \quad \hat{x} = x^* = \begin{bmatrix} \bar{z} & \bar{w} \\ -w & z \end{bmatrix} \tag{67}$$

Then $N^\square(x) = \hat{x}x = det\, x \cdot \mathbb{1} \Longrightarrow N^\square(x) = 1 \Longleftrightarrow det\, x = 1$. So, the elements $u \in \mathbb{R}^+_{3,0}$ such that $\beta(u\psi, u\varphi) = \beta(\psi,\varphi), \psi, \varphi \in I_p$ satisfy $\hat{u}u = 1$ and $det\, u = -1$, which means that $u \in SU(2) \simeq Spin_+(3,0)$. Our statement that Pauli c-spinors are represented by the elements of $I_p = \mathbb{R}^+_{3,0} e_{30}$ (Pauli a-spinors) is then proved.

4.2. Representative of Weyl c-Spinors and Dirac c-Spinors within $\mathbb{R}_{1,3}$ and $SL(2,\mathbf{C})$

The algebra $\mathbb{R}_{1,3}$ is generated by 1 and the vectors e_μ such that

$$e_\mu e_\nu + e_\nu e_\mu = 2\eta_{\mu\nu} \quad \eta_{\mu\nu} = diag(+1,-1,-1,-1); \quad \mu,\nu = 0,1,2,3.$$

Consider the isomorphism $\mathbb{R}^+_{3,0} \overset{f}{\simeq} \mathbb{R}_{1,3}$, where f is the linear extension of $f(\sigma_i) = e_i e_0$ and $\sigma_i \in \mathbb{R}^{3,0}$ as in § 4.1. Since $e_{30} = 1/2(1+\sigma_3)$ is a primitive idempotent of $\mathbb{R}_{3,0}$, $f(e_{30}) = 1/2(1 + e_3 e_0)$ is a primitive idempotent of $\mathbb{R}^+_{1,3}$. Also, since $\mathbb{R}_{1,3} \simeq \mathbb{H}(2)$, $e_{13} = f(e_{30}) = e$ is also a primitive idempotent of $\mathbb{R}_{1,3}$ since

$$\dim_{\mathbb{R}} \mathbb{R}_{1,3} e = 2^4/2 \quad \text{and} \quad dim_{\mathbb{H}} \mathbb{R}_{1,3} e = 2$$

$I_D = \mathbb{R}_{1,3} e$ is a bidimensional quaternionic space and $\psi_D \in I_D$ is a representation of the Dirac spinors as we shall prove. Let $a \in \mathbb{R}_{1,3} e$

$$\begin{aligned} a &= s + (a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3) \\ &+ (a_{01} e_0 e_1 + a_{02} e_0 e_2 + a_{03} e_0 e_3 + a_{12} e_1 e_2 + a_{13} e_1 e_3 + a_{23} e_2 e_3) \\ &+ (a_{012} e_0 e_1 e_2 + a_{013} e_0 e_1 e_3 + a_{023} e_0 e_2 e_3 + a_{123} e_1 e_2 e_3) \\ &+ p e_0 e_1 e_2 e_3 \end{aligned} \tag{68}$$

Then $\psi_D \in \mathbb{R}_{1,3}e$ is such that

$$\psi_D = [x_0 + x_1e_1 + x_2e_2 + x_3e_1e_2]e + [y_0 - y_1e_1 + y_2e_2 + y_3e_1e_2]e_1e_0e \quad (69)$$

where $e\mathbb{R}_{1,3}e \simeq \mathbb{H}$ has basis $\{1, e_1, e_2, e_1e_2\}e$.

Contravariant Undotted a-Spinors

Consider the minimal left ideal $I = \mathbb{R}^+_{1,3}e$. Then $\eta \in I$ can be written as

$$\eta = [x_0 + x_3\mathrm{i}]e + [y_0 + y_3\mathrm{i}]e_0e_1 \quad (70)$$

where $e\mathbb{R}^+_{1,3}e \simeq \mathbb{C}$ has basis $\{1, \mathrm{i}\}e$, where

$$\mathrm{i} = e_0e_1e_2e_3 \quad ; \quad x_0, x_3, y_0, y_3 \in \mathbb{R}$$

We write

$$\eta = \eta^1e + \eta^2e_1e_0e = \eta^1s_1 + \eta^2s_2 \simeq \begin{bmatrix} \eta^1 & 0 \\ \eta^2 & 0 \end{bmatrix} \eta^i \in \mathbb{C}e \quad , \quad i = 1, 2 \quad (71)$$

Covariant undotted a-spinors

Consider the space $\tilde{I} = (\mathbb{R}^+_{1,3}e)^{\sim}$

$$\tilde{I} \ni \tilde{\eta} = \tilde{e}\eta^1 - \tilde{e}e_1e_0\eta^2 \simeq \begin{pmatrix} 0 & 0 \\ -\eta^2 & \eta^1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \eta_1 & \eta_2 \end{pmatrix}$$

We can identify the covariant undotted spinors as the elements

$$\overset{\Delta}{\eta} = e_1e_0\tilde{\eta} = (\eta^1e + \eta^2ee_1e_0)\sigma_3\sigma_1 \simeq \begin{pmatrix} -\eta^2 & \eta^1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \eta_1 & \eta_2 \\ 0 & 0 \end{pmatrix} = \eta^t\boldsymbol{C}$$

$$\sigma_3 = e_3e_0 \quad , \quad \sigma_1 = e_1e_0 \quad , \quad C = \sigma_3\sigma_1 \quad (72)$$

Note that we can define the spinorial metric as

$$\beta : I \times I \to \boldsymbol{C} \; ; \; \beta(\eta, \xi) = 2\langle\overset{\Delta}{\eta}\,\xi\rangle_0 = 2\,\langle\eta^t\,C\,\xi\,\rangle_0 \quad (73)$$

Contravariant dotted spinors

Consider the ideal $\dot{I} = {}^*I = e_0\tilde{I}e_0$

$$\begin{aligned} \dot{I} \ni \dot{\eta} &= e_0(\tilde{e}\eta^1 - \tilde{e}e_1e_0\eta^2)e_0 \\ &= e\bar{\eta}^1 + ee_1e_0\bar{\eta}^2 \simeq \begin{pmatrix} \bar{\eta}^1 & \bar{\eta}^2 \\ 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} \eta^{\dot{1}} & \eta^{\dot{2}} \\ 0 & 0 \end{pmatrix} \end{aligned} \quad (74)$$

Covariant dottet spinors

Consider the ideal $\tilde{\dot{I}} = -e_0 I e_0$

$$\begin{aligned} \tilde{\dot{I}} \ni \tilde{\dot{\eta}} &= -\bar{\eta}^1 \tilde{e} + \bar{\eta}^2 e_1 e_0 \tilde{e} \\ &\simeq \begin{pmatrix} 0 & \bar{\eta}^2 \\ 0 & -\bar{\eta}^1 \end{pmatrix} = \begin{pmatrix} 0 & \eta_{\dot{1}} \\ 0 & \eta_{\dot{2}} \end{pmatrix} \end{aligned}$$

We can identify the covariant dotted spinors as the elements

$$\begin{aligned} \overset{\Delta}{\dot{\eta}} = \tilde{\dot{\eta}}\, \sigma_1 &= -\tilde{\dot{\eta}}\, \sigma_3 \sigma_1 = -(\bar{\eta}^1 \tilde{e} + \bar{\eta}^2 e_1 e_0 \tilde{e}) \sigma_3 \sigma_1 \\ &= \sigma_3 \sigma_1 (\bar{\eta}^1 e + \bar{\eta}^2 \tilde{e} e_1 e_0) = \sigma_3 \sigma_1 (\bar{\eta}^1 e + \bar{\eta}^2 e_1 e_0 e) \\ &\simeq C \dot{\eta}^t = \begin{pmatrix} \eta^{\dot{2}} & -\eta^{\dot{1}} \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{75}$$

The spinorial metric in this case is defined by

$$\dot{\beta} : \dot{I} \times \dot{I} \to \boldsymbol{C} \quad ; \quad \dot{\beta}(\dot{\eta}, \dot{\xi}) = 2\langle \dot{\eta} \sigma_3 \sigma_1 \dot{\xi}^t \rangle_0 = 2\langle \dot{\eta}\ \overset{\Delta}{\dot{\xi}} \rangle_0 \tag{76}$$

In the spinorial basis $\alpha = \{e, \dot{e}_1 e_0 e\}$ of I we have the following representation for $\sigma_i = e_i e_0, \quad i = 1, 2, 3$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad ; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{77}$$

Observe now that we can write from eqs.(68, 70, 75) that

$$\psi_D = \varphi + e_0\ \overset{\Delta}{\dot{\chi}} \tag{78}$$

Now, let $x \in \mathbb{R}^+_{1,3}$. Then we have that if $x \in \boldsymbol{C}(2)$ is the representative of $x \in \mathbb{R}^+_{1,3}$ then

$$x = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \Rightarrow e_0 x e_0 = \begin{pmatrix} \bar{z}_4 & -\bar{z}_3 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} = (x^*)^{-1} \tag{79}$$

From eqs. (78) and (79) we have that for $u \in Spin_+(1,3)$ that $\psi_D \to \psi'_D = u\psi_D$ and

$$u\psi_D = u\varphi + u e_0\ \overset{\Delta}{\dot{\chi}} = u\varphi - e_0 [(u^*)^{-1}\ \overset{\Delta}{\dot{\chi}}] = \varphi' + e_0\ \overset{\Delta'}{\dot{\chi}} \tag{80}$$

Eq. (80) shows that I_D is the carrier space of the representation $D^{(1/2,0)} \oplus D^{(0,1/2)}$ of $Spin_+(1,3) \simeq SL(2, \boldsymbol{C})$ as defined in (iii) of § 1.

Observe also that from eq. (69) we can write

$$\psi_D = \psi_1 e_0 e + \psi_2 e_1 e + \psi_3 e + \psi_4 e_0 e_1 e \tag{81}$$

with $\psi_i \in e\mathbb{R}_{1,3}e \simeq \mathbb{C}$ with basis $\{1, i\}e_2e_1$.

A complex spinorial basis for I_D is then $\alpha_D = \{e_0e, e_1e, e, e_0e_1e\}$.

Consider now the injection

$$\begin{aligned}\gamma: \mathbb{R}_{1,3} &\to \mathcal{L}_{\mathbb{C}}(I_D)r \\ x &\longmapsto \gamma(x): I_D \to I_D \\ &\qquad \psi_D \longmapsto u\psi_D\end{aligned}$$

We get the following representation for $e_\mu, \mu = 0, 1, 2, 3$ in the α_D-basis

$$\gamma(e_0) = \gamma_0 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix} \quad \gamma(e_i) = \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad , \quad i = 1, 2, 3 \tag{82}$$

We can also mimic the spinorial metric in $\mathbb{C}(4)$ [(iii) of § 1] defining

$$\beta_D: I_D \times I_D \to \mathbb{C} \quad , \quad \beta_D(\psi, \varphi) = 2\langle \psi^* e_3e_1\varphi\rangle_0 \tag{83}$$

4.3. Representations of Weyl and Dirac c-Spinors in $\mathbb{R}_{3,1}$

$\mathbb{R}_{3,1} \simeq \mathbb{R}(4)$ (see Table I), the Majorana algebra is generated by 1 and the vectors e_μ such that $\bar{e}_\mu\bar{e}_\nu + \bar{e}_\nu\bar{e}_\mu = -2\eta_{\mu\nu}$; $\eta_{\mu\nu} = diag(+1, -1, -1, -1), \mu, \nu = 0, 1, 2, 3$. We can easily verify that $\bar{e} = 1/2(1 + \overline{e_3}\ \overline{e_0})$ is a primitive idempotent of $\mathbb{R}^+_{3,1} \simeq \mathbb{R}_{3,0} \simeq \mathbb{R}^-_{1,3}$. Then each $\varphi \in I = \mathbb{R}^-_{3,1}\bar{e}$ can be written as

$$\varphi = \varphi_1\bar{e} + \varphi_2\bar{e}_1\bar{e}_0\bar{e} \tag{84}$$

where $\varphi_1, \varphi_2 \in \bar{e}\mathbb{R}^+_{3,1}\bar{e} \simeq \mathbb{C}$ has basis $\{1, \bar{i}\}\bar{e}$, where $\bar{i} = -\overline{e_0}\ \bar{e}_1\bar{e}_2\bar{e}_3 = -\sigma_1\sigma_2\sigma_3$, $\sigma_i = \bar{e}_i\bar{e}_0, \quad i = 1, 2, 3$.

It follows that the structure of the Weyl spinors is equal in the $\mathbb{R}_{1,3}$ algebra. What we want now is to represent the Dirac spinors inside $\mathbb{R}_{3,1}$. Observe that unlike the case of $\mathbb{R}_{1,3}$. $\bar{e} = 1/2(1 + \bar{e}_3\bar{e}_0)$ is not a primitive idempotent of $\mathbb{R}_{3,1}$.

However, for each $x \in \mathbb{R}_{3,1}$ we can write

$$\mathbb{R}_{1,3} \ni x = x^+ + \bar{e}_0y^+ \quad ; \quad x^+, y^+ \in \mathbb{R}^+_{3,1} \tag{85}$$

Also if $u \in \mathbb{R}^-_{3,1}$ and if $u \in \mathbb{C}(2)$ is the representative of u in the canonical spinorial basis then

$$u = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \Rightarrow \bar{e}_0u\bar{e}_0 = \begin{pmatrix} -\bar{z}_4 & \bar{z}_3 \\ \bar{z}_2 & -\bar{z}_1 \end{pmatrix} = -(u^*)^{-1} \tag{86}$$

It follows that the objects of the non-minimal ideal $\bar{I}_D = \mathbb{R}_{1,3}\bar{e}$, $\psi_D = \varphi + \bar{e}_0\ \overset{\Delta}{\dot{\chi}}$ transforms under the action of $u \in Spin_+(3,1) \simeq Spin_+(1,3) \simeq SL(2, \mathbb{C})$ as

$$\bar{\psi}_D \to u\bar{\psi}_D = u\varphi + u\bar{e}_0\ \overset{\Delta}{\dot{\chi}} = u\varphi + \bar{e}_0[(u^*)^{-1}\ \overset{\Delta}{\dot{\chi}}] \tag{87}$$

From eq. (87) it is follows that the non-minimal ideal $\overline{I}_D$ is the carrier space of the representation $D^{(1/2,0)} \oplus D^{(0,1/2)}$ of the group $SL(2, \mathbf{C})$. $\overline{\psi}_D$ is an c-spinor according to the definition given in § 3.4.

This example shows that when working with Clifford algebras we cannot restrict the representation of the c-spinors used by physicist *only* to elements of minimal lateral ideals.

Majorana Spinors.

The elements of the minimal left ideals of $\mathbb{R}_{3,1}$ are the Majorana spinors. They can be constructed by the standard procedure used above.

Since $\hat{e} = 1/4(1 + \bar{e}_3\bar{e}_0)(1 + \bar{e}_2)$ is a primitive idempotent of $\mathbb{R}_{3,1}$ we have

$$I_M = \mathbb{R}_{3,1}\hat{e} \ni \psi_M = \psi_1\bar{e}_1 \overset{\Delta}{e} + \psi_2\bar{e} + \psi_3\bar{e}_0\bar{e}_1 \overset{\Delta}{e} + \psi_4\bar{e}_0 \overset{\Delta}{e} \tag{88}$$

where $\psi_i \in \hat{e}\mathbb{R}_{3,1}\hat{e} \simeq \mathbb{R}$, $i = 1, 2, 3, 4$.

It is interesting to compare eqs. (88) with eq. (81) which express ψ_M in I_M and ψ_D in I_D.

Consider now the isomorphism

$$\begin{aligned} \gamma : \mathbb{R}_{3,1} &\rightarrow \mathcal{L}_{\mathbb{R}}(I_M) \\ x &\longmapsto \gamma(x) : I_M \rightarrow I_M \\ &\qquad \psi_M \longmapsto \psi_M \end{aligned}$$

We get the following representation for $\bar{e}_\mu, \mu = 0, 1, 2, 3$ in the $\alpha_M = \{\bar{e}_1, \hat{e}, \bar{e}_1, \bar{e}_0\bar{e}_1\hat{e}, \bar{e}_0\hat{e}\}$ basis $[\gamma(e_\mu) \equiv \gamma_\mu]$

$$\gamma_0 = \begin{pmatrix} 0 & -\mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}; \ \gamma_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}; \ \gamma_2 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}; \ \gamma_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \tag{89}$$

Eqs. (89) shows a set of what is usually called in the literature Majorana matrices and our presentation shows how easily it is to find a set of Majorana matrices with the techniques of this paper.

4.4. Representations of Dirac c-Spinors within the $\mathbb{R}_{4,1}$ Algebra

From Table I we see that $\mathbb{R}_{4,1}$, $\mathbb{R}_{2,3}$ and $\mathbb{R}_{0,5}$ are isomorphic to the algebra $\mathbf{C}(4)$ which is the usual Dirac algebra of physicists. In order to identity the algebra that carries the physical interpretation associated with space-time ($\mathbb{R}^{1,3}$) we proceed as follows. Let $E_A, A = 0, 1, 2, 3, 4$ be an orthonormal basis for $\mathbb{R}^{p,q}$

with $p + q = 5$. The volume element is $E_J = E_0E_1E_2E_3E_4$ and we get $E_J^2 = -1$ for $q = 1, 3, 5$. Now define

$$e_\mu = E_\mu E_4 \tag{90}$$

and impose that e_μ is an orthonormal basis for $\mathbb{R}^{1,3}$, i.e.,

$$e_0^2 = -E_0^2E_4^2 = +1 \quad , \quad e_k^2 = -E_k^2E_4^2 = -1 \quad , \quad k = 1, 2, 3 \tag{91}$$

Eq. (91) is satisfied when $p = 4, q = 1$, i.e., $E_4^2 = E_k^2 = -E_0^2 = 1$ and we conclude that the real Clifford algebra associated with space-time ($\mathbb{R}^{1,3}$) and isomorphic to $\mathbb{C}(4)$ is $\mathbb{R}_{4,1}$.

Eq. (91) shows that $\mathbb{R}_{1,3} \overset{g}{\simeq} \mathbb{R}_{4,1}^+$ where g is the linear extension of $g(e_\mu) = E_\mu E_4$, $\mu = 0, 1, 2, 3$. We already saw in § 4.2 that $f(e_{30})$ is a primitive idempotent of $\mathbb{R}_{1,3}$ and we have that $g(f(e_{30}))$ is a primitive idempotent of $\mathbb{R}_{4,1}^+$. Then $\bar{I}_D = \mathbb{R}_{4,1}^+ g(f(e_{30}))$ is a minimal ideal $\mathbb{R}_{4,1}^+$ which is a 4-dimensional vector-space over the complex field and its elements, the Dirac a-spinors, are representations in $\mathbb{R}_{4,1}$ of Dirac c-spinors.

4.5. Representation of the Standard Dirac c-Spinors within the $\mathbb{R}_{1,3}$ Algebra

It is obvious that $\check{e} = 1/2(1 + e_0)$ is a primitive idempotent of $\mathbb{R}_{1,3}$. Also we can easily verify that for any $x \in \mathbb{R}_{1,3}$ there exists $y \in \mathbb{R}_{1,3}^+$ such that $x\,\check{e} = y\,\check{e}$. It follows that $\check{I}_D = \mathbb{R}_{1,3}^+\,\check{e}$ is a minimal left ideal of $\mathbb{R}_{1,3}$. The elements $\check{\psi}_D \in \check{I}_D$ can be written in the form

$$\check{\psi} = \psi_1\,\check{e} + \psi_2 e_3 e_1\,\check{e} + \psi_3 e_3\,\check{e} + \psi_4 e_1\,\check{e} \tag{92}$$

$\psi_i \in \check{e}\,\mathbb{R}_{1,3}\,\check{e} \simeq \mathbb{C}$ with basis $\{1, e_2e_1\}\,\check{e}$.

The $\check{\psi}'\,s$ are the representative of standard Dirac c-spinors, which are the kind of c-spinors that appear in the usual form of the Dirac equation[17]. The isomorphism

$$\begin{aligned}\gamma:\ \mathbb{R}_{1,3} &\to \mathcal{L}_{\mathbb{C}}(\check{I}_D)\\ x &\longmapsto \gamma(x) : \check{I}_D \to \check{I}_D\\ &\qquad\qquad \check{\psi} \longmapsto \check{\psi}\end{aligned}$$

gives through the technique already introduced in § 4.1 the following representation for e_μ, $\mu = 0, 1, 2, 3$ and $e_5 = e_0e_1e_2e_3$ in $\check{\alpha}_D = \{\check{e}, e_3e_1\,\check{e}, e_3\,\check{e}, e_1\,\check{e}\}$, a complex spinorial basis for $\check{I}_D$.

Putting $\gamma(\check{\alpha}_D) = \{|1>, |2>, |3>, |4>\}$ and $\gamma(e_\mu) = \gamma_\mu, \gamma(e_5) = \gamma_5$,

$$i = |\delta_{i1}|; \quad \gamma_0 = \begin{bmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{bmatrix} \; ; \; \gamma_k = \begin{bmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{bmatrix} \; ; \; \gamma_5 = \begin{bmatrix} 0 & i\mathbb{1}_2 \\ i\mathbb{1}_2 & 0 \end{bmatrix} \tag{93}$$

where $|\delta_{i1}|$ is the 4×4 matrix with one in the i-line of the first column, all other elements being zero. Also $i = \sqrt{-1}$. The set of γ-matrices in (93) is usually known as the standard representation of Dirac-matrices[17].

Observe now that the idempotents $\check{e} = 1/2(1 + e_0)$ and $e = 1/2(1 + e_3e_0)$ are related by

$$e = u\,\check{e}\,u^{-1} \; ; \; u = (1 + e_3) \tag{94}$$

Since $u = (1 + e_3) \notin \Gamma(1,3)$, the ideals I_D and $\check{I}_D$ are not equivalent (module) representations of $\mathbb{R}_{1,3}$, although from the point of view of group theory both ideals are carrier spaces of the representation $D^{(1/2,0)} \oplus D^{(0,1/2)}$ of $SL(2,\mathbb{C})$. This point is important for paper number II of this series. There we will need also the following results which are trivially established

$$|1> = \gamma_0|1>; \quad i|1> = \gamma_2\gamma_1|1>; \quad |2> = -\gamma_5\gamma_2|1>; \quad |3> = \gamma_3|1>; \quad |4> = \gamma_1|1> \tag{95}$$

5. Conclusions

Hestenes[41] said about the theory of spinors: "I have not met anyone who was not dissatisfied with his first reading on the subject".

Well, the reasons for such statement are in our view due to three main facts,

(A) The usual representation of c-spinors such as introduced in (i), (ii), (iii) and (iv) of § 1 does not emphasize the geometrical meaning of these objects.

(B) There are not clear connection between the abstract concepts of c-spinor and the more abstract concepts of a-spinors, e-spinors and o-spinors as element of particular Clifford algebras.

(C) The represenation of a-spinor (or e-spinor) fields as sections of some Clifford bundles (over space-time) and the problem of the "transformation law" of spinors.

As to (A) we think that the situation has been partially clarified with the presentation by Hestenes of the geometrical meaning of Pauli-spinors and of Dirac-spinors[13,14] and also by Penrose and Rindler[12] of the quasi geometrical representation of the undotted and dotted two components spinors.

As to (B) we think that the present paper shows in a clear way how to obtain relations between all c-spinors used by physicists and a-spinors and e-spinors. The problem of o-spinors as already said in the introduction will be treated in another publication (III)[39].

From our approach, § 3.5 and § 4 it is clear that $\dot{a}$-spinors and e-spinors can be though as elements of the exterior algebra of the vector space $V = \mathbb{R}^{p,q}$. It follows that the usual claim that spinors are more fundamental then tensors is **non-sequitur**.

It is very important to emphasize that all our a-spinors or (e-spinors) are elements of *real Clifford algebras*. Other approaches to the subject of algebraic spinors like, e.g., Crumeyrolle [10], Bugajska [11] and Salingrados and Wene[26] complexific $\mathbb{R}_{1,3}$ or $\mathbb{R}_{3,1}$ (the complexification being isomorphic to $\mathbb{R}_{4,1} \simeq \mathbb{C}(4)$) introducing unnecessary complications. The reason for such a complexification is the need to use a de Witt[10,11] basis for the spinor-space, since these authors at the time of their writtings did not seen aware of the idempotent method used in this paper.

Another "need" for complexification comes according to the view of Salingrados and Wene[26] from the fact that $\mathbb{R}_{1,3}$ has only two idempotents and the formulation of quantum electrodynamics, as is well known needs four idempotents (there called projection operators). This "difficulty" can be easily solved following Hestenes[41] simply by introducing a single operator that belongs to the dual space of $\mathbb{R}_{1,3}$ (here considered as a vector space over $\mathbb{R}$).

We must say that we cannot properly discuss here on some distinctive features of the different representations of Dirac a-spinor (or e-spinor) fields and related Dirac's like equations over Lorentzian manifolds. The interested reader is invited to see our publications, II and III.

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