# A HOPF BIFURCATION THEOREM FOR EVOLUTION EQUATIONS OF HYPERBOLIC TYPE

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**ABSTRACT:** In this work we give a version of Hopf Bifurcation theorem in infinite dimension for semi–linear evolution equations. We show existence and uniqueness of a family of periodic solutions and we study its symmetry and stability properties. We do not assume analyticity nor compactness of the semi–group generated by the linear part; we also allow the parameter to perturb the unbounded part.

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# A HOPF BIFURCATION THEOREM FOR EVOLUTION EQUATIONS OF HYPERBOLIC TYPE

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**Abstract:** In this work we give a version of Hopf Bifurcation theorem in infinite dimension for semi linear evolution equations. We show existence and uniqueness of a family of periodic solutions and we study its symmetry and stability properties. We do not assume analyticity nor compactness of the semi-group generated by the linear part; we also allow the parameter to perturb the unbounded part.

## Introduction:

In the paper The Hopf Bifurcation Theorem in Infinite Dimensions [1], Crandall and Rabinowitz used the Liapunov-Schmidt method to study a semi-linear evolution equation where the semi group generated is holomorphic. To get their results, they introduced an unknown period $p$ as a new parameter in the equation and looked for solutions having a known period.

In this paper we will use the same technique for semi-linear evolution equations of the form

$$\frac{d}{ds} v(s) = A(r)v(s) + f(r, v(s)),$$

in a real Banach space $W$, where the perturbation appear also in the unbounded part $A(r)$ of the equation and the semi group $T(r,t)$, $t \geq 0$, generated by $A(r)$ is not necessarily analytic. We will show the existence and uniqueness of a bifurcating family of periodic solutions and we will study its simmetry and stability properties.

The main difference between Crandall and Rabinowitz [1] and our work, is that, we do not have analyticity assumptions; so we had an extra work to show the regularity of the periodic solutions. This problem was studied by Hale and De Oliveira [12] for functional equations. They used Fourier series to prove that every periodic solution must be $C^1$.

Our version of Hopf's bifurcation theorem can be applied to hyperbolic systems, such as, one dimensional wave equation and transmission line equation (which is a linearized version of the equations studied by Brayton and Miranker [13]). These applications will be treated in a subsequent paper.

Marsden and McCracken [9], devote one section to study Hopf's bifurcation for $PDE$ and present a theorem that holds for parabolic type equations. They assume that the flow must be smooth in $t$ and in the parametric $\mu$ of the system, for $t > 0$. Our hypothesis $HA4$) is analogue to the hypothesis (8.3) of Marsden and McCracken ([9] pg. 254), it requires a dichotomy of the space, such that the semi group $T(r,t)$ decays to zero exponentially in an invariant subspace of codimension two.

It is well known that the asymptotic behavior of a $C_0$ semi group $T(t)$, $t \geq 0$, is determined by its spectrum $\sigma(T(t))$. For instance, to say that $T(t)$ decays to zero exponentially in an invariant subspace of finite codimension is the same as to say that the essential spectral radius of $T(t)$ is less than one. For semi groups which are either compact or analytic, the spectrum of the infinitesimal generator $A$, $\sigma(A)$, gives all the important informations about $\sigma(T(t))$ (this takes care of retarded equations and parabolic systems), but, in general, since the continuous spectrum of $T(t)$ cannot be determined by the inspection of the spectrum of $A$, to apply our theorems, the asymptotic behavior of each problem has to be studied in particular. For difference equations and neutral $FDE$ that analysis has been carried out by D. Henry [3] and for hyperbolic systems by Lopes, Neves & Ribeiro [4]. In both works, it is shown basically, that the asymptotic behavior of the semi groups generated by those equations is determined by the spectrum of the corresponding infinitesimal generator.

For retarded functional equations and hyperbolic systems it is known that the spectrum $\sigma(A)$ of the operator $A$ is given by zeros of an entire function $h(\lambda)$ and it is also known that for these equations, the dimension of the range of the spectral projection, corresponding to a single characteristic root $\lambda_0$, is equal to the multiplicity of the spectral point $\lambda_0$, as a root of the characteristic equation $h(\lambda) = 0$, (see Levinger [6] and Lin & Neves [5]). These results can be used to compute the dimension of the finite invariant subspace on the dichotomy.

Another important question is that the semi group $T(r,t)$ is continuous in the strong operator topology (that is, $(r,t) \mapsto T(r,t)u$ is continuous for each $u$ fixed in Banach space). In general, the spectrum $\sigma(T(r,t))$ may present an "explosion" as a function of $r$. This implies the asymptotic behavior of $T(r,t)$ is very sensitive to perturbations of the parameter. This phenomenon has been studied for difference equations by Hale [10], and it is called parametric unstability, and also for hyperbolic systems by Lopes [8]. These papers give us conditions, under which, it is possible to estimate the "explosion" of the spectrum.

This paper is divided in two sections. The existence, uniqueness and simmetry properties are presented in section one. The analysis of the characterisitc exponents and linearized stability results appear in section two.

## Section 1. The Hopf Bifurcation Theorem

We will discuss the Hopf Bifurcation theorem in a real Banach space $W, \| \ \|$) for semi linear evolution equation of the form

$$\frac{d}{ds} v = A(r)v(s) + f(r, v(s)) \tag{1}$$

under the following assumptions on $A$ and $f$:

$HA$) For each $r \in (-r_0, r_0)$, $r_0 > 0$:

1. $A(r)$ is a closed densely defined linear operator on $W$, with domain $D(A(r)) = D$ independent on $r$.

2. $A(r)$ is strongly continuously differentiable on $D$ that is, $r \mapsto A(r)u$ for $u \in D$ has a continuous derivative.

3. $A(r)$ is infinitesimal generator of a strongly continuous semi-group $\{T(r,t) : t \geq 0\}$.

4. i) $W$ can be decomposed in a direct sum of $A(r)$–invariant subspaces, $W = X_r \oplus Y_r$, with strongly continuous differentiable projections $P(r)$ on $Y_r$ along $X_r$, $P(r) : W \to Y_r$, such that

$$\|T(r,t)(I - P(r))w\| \leq M \exp(-\gamma\, t)\|(I - P(r))w\|; \quad M \geq 1, \quad t \geq 0 \tag{2}$$

$$\gamma > 0, \qquad w \in W.$$

ii) $Y_r$ is a two (real) dimensional subspace and the matrix of the restriction of $A(r)$ to $Y_r$, with respect to a suitable basis $\{e_1(r), e_2(r)\}$, is

$$[A(r)] = \begin{bmatrix} \alpha(r) & \beta(r) \\ -\beta(r) & \alpha(r) \end{bmatrix} \tag{3}$$

where, $\alpha, \beta$ are continuously differentiable, $\alpha(0) = 0$, $\beta(0) \neq 0$ and $\alpha$ satisfies the Hopf's hypothesis:

$H\alpha$) $\alpha'(0) \neq 0$.

The requirements on $f$ in (1) are the following:

$Hf)$ The function $f : (-r_0, r_0) \times W \to W$ is $C^2$–function such that

$$f(r,0) = 0 \quad \text{and} \quad f_w(r,0) = 0 \quad \text{for} \quad |r| < r_0.$$

Before stating the main result, it may be helpful to remind some of the well known properties of $A(r)$, under the hypotheses $HA)$, (see Krein [7]):

i) $A(r)A^{-1}(s)$ is a bounded linear operator on $W$, continuous in $r$ and $s$ in the uniform operator topology.

ii) $A(r)A^{-1}(s)$ is strongly differentiable in $r$, with $A'(r)A^{-1}(s)$ bounded and strongly continuous in $r$ and $s$.

Furthermore, as a consequence of Trotter approximation theorem, the semi–group $T(r,t)$ generated by $A(r)$ is also strongly continuously differentiable in $r$ on $D$.

We will use in this paper $C_b(\mathbb{R}, W)$ to denote the space of continuous bounded functions from $\mathbb{R}$ into $W$ with the usual supremum norm $\|\cdot\|_\infty$ and $C_{w_0}(\mathbb{R}, W)$ to denote the subspace of $C_b(\mathbb{R}, W)$ of all $w_0$–periodic functions.

Our infinite dimensional version of the Hopf bifurcation theorem can now be stated:

**Theorem 1.1 (BIFURCATION).** Let $(HA)$, $(H\alpha)$ and $(Hf)$ be satisfied. Then there are positive real numbers $a_1 > 0$, $b_1 > 0$, $c_1 > 0$ and a continuously differentiable function

$$(p, r, v) : (-a_1, a_1) \to \mathbb{R} \times \mathbb{R} \times C_b(\mathbb{R}, W)$$

with the following properties:

i) For each $a \in (-a_1, a_1)$, $v(a)$ is a strong $\frac{2\pi}{\beta(0)}(1 + p(a))$–periodic solution of

$$\frac{d}{ds} v(a) = A(r(a))v(a) + f(r(a), v(a)).$$

ii) $p(0) = r(0) = \frac{dr}{da}(0) = \frac{dp}{da}(0) = 0, \quad v(0) = 0,$

$$v(a) \neq 0 \ \text{ if } \ 0 < |a| < a_1, \quad |p(a)| < c_1 \ \text{ and } \ \|v(a)\|_\infty < b_1.$$

iii) Except for translations on $s$, every $w_0$–periodic solution $w(s)$ of (1), with

$$|w_0 - \frac{2\pi}{\beta(0)}| < c_1 \ \text{ and } \ \|w\|_\infty < b_1$$

belongs to the family above.

**Proof.** We are looking for periodic solutions of the equation (1), but we do not know, at the first sight, the period of such solutions. So, we will introduce a new parameter $p$ relative to the unknown period. If $p$ is near zero and

$$s = (1 + p)t, \qquad w(t) = v((1 + p)t)$$

the equation (1) is equivalent to

$$\frac{d}{dt}w(t) = (1+p)[A(r)w(t) + f(r, w(t))] \tag{4}$$

and we will seek nontrivial $\frac{2\pi}{\beta(0)}$–pediodic solutions of (4).

The equation (4) can be viewed as a system

$$\frac{d}{dt}[(I - P(r))w] = (1+p)[A(r)(I-P(r))w + (I-P(r))f(r,w)] \tag{5}$$

$$\frac{d}{dt}(P(r)w) = (1+p)[A(r)(P(r)w) + P(r)f(r,w)]. \tag{6}$$

If $w(p, r, \cdot)$ is a $\frac{2\pi}{\beta(0)}$–periodic solution of this system, we have from equation (5) that

$$\begin{aligned}(I - P(r))w(p,r,t) &= T(r,(1+p)t)x_0 + \\ &+ (1+p)\int_0^t T(r,(1+p)(t-s))(I-P(r))f(r,w(p,r,s)ds,\end{aligned} \tag{7}$$

where

$$(I - P(r))w(p, r, \frac{2\pi}{\beta(0)}) = x_0$$

then defining

$$x_0(\cdot, r, \cdot) : (-1, \infty) \times C_{\frac{2\pi}{\beta(0)}}(I\!R, W) \to X_r; \quad |r| < r_0$$

by

$$\begin{aligned}&x_0(p, r, w(\cdot)) = \\ &= (I - T(r,(1+p)\frac{2\pi}{\beta(0)}))^{-1}(1+p)\int_0^{\frac{2\pi}{\beta(0)}} T(r,(1+p)(\frac{2\pi}{\beta(0)} - s))(I - P(r)) \\ &\quad f(r, w(s))ds\end{aligned} \tag{8}$$

we conclude that $x_0$ in (7) must satisfy

$$x_0 = x_0(p, r, w(p, r, \cdot)). \tag{9}$$

On the other hand, the equation (6) can be considered as a equation in $I\!R^2$, taking the coordinates with respect to the basis given in $HA4$), that is

$$\frac{d}{dt}[P(r)w] = (1+p)[A(r)][P(r)w] + (1+p)[P(r)f(r,w)], \quad \text{in} \quad I\!R^2,$$

where the brackets denote coordinates (or components).

For technical reasons, we will work with this equation in the following equivalent from:

$$\frac{d}{dt}[P(r)w] = [A(0)][P(r)w] + g(p, r, w) \tag{10}$$

where

$$g:(-1,\infty)\times(-r_0,r_0)\times W\to \mathbb{R}^2$$

is given by

$$(11)\qquad g(p,r,w)\doteq(1+p)([A(r)][P(r)w]+[P(r)f(r,w)])-[A(0)][P(r)w].$$

The following properties of $g$ can be easily verified:

$g_1$) $g(p,r,0)=0;\quad p>-1,\quad |r|<r_0$

$g_2$) $g$ and $g_w$ are continuously differentiables with $g_w(0,0,0)=0$.

For the sake of notations simplicity, we will identify, throughout the paper, elements of $Y_r$ with their coordinates in $\mathbb{R}^2$. in such way that the brackets can be eliminated.

From (10) and Fredholm's alternative (see Hale [11], pg. 263), we have

$$(12)\qquad Qg(p,r,w(p,r,\cdot))=0$$

where $Q$ is the linear projection from $C_{\frac{2\pi}{\beta(0)}}(\mathbb{R},\mathbb{R}^2)$ onto the subspace of $C_{\frac{2\pi}{\beta(0)}}(\mathbb{R},\mathbb{R}^2)$ spanned by the $\frac{2\pi}{\beta(0)}$-periodic solutions of the adjoint equation

$$(13)\qquad \dot z=-[A(0)]^t z=[A(0)]z\qquad (^t \text{ denotes transpose})$$

that is, a two dimensional subspace spanned by

$$\phi_1(t)=\begin{pmatrix}\cos(\beta(0)t)\\ -\mathrm{sen}(\beta(0)t)\end{pmatrix}\qquad \phi_2(t)=\begin{pmatrix}\mathrm{sen}(\beta(0)t)\\ \cos(\beta(0)t)\end{pmatrix}.$$

Furthernore, there is a unique $\frac{2\pi}{\beta(0)}$-periodic solution of (10), $Kg(p,r,w(p,r,\cdot))$ such that

$$QKg(p,r,w(p,r,\cdot))=0$$

and $K(I-Q)$ is a continuous linear operator taking $C_{\frac{2\pi}{\beta(0)}}(\mathbb{R},\mathbb{R}^2)$ into $C_{\frac{2\pi}{\beta(0)}}(\mathbb{R},\mathbb{R}^2)$. Therefore, except a translation on time, there exists $a\in\mathbb{R}$, such that

$$(14)\qquad P(r)w(p,r,t)=a\phi_1(t)+Kg(p,r,w(p,r,w(p,r,\cdot)))(t).$$

Now, using the equalities (7) and (14), we define the map

$$F:\mathbb{R}\times(-1,\infty)\times(-r_0,r_0)\times C_{\frac{2\pi}{\beta(0)}}(\mathbb{R},W)\to C_{\frac{2\pi}{\beta(0)}}(\mathbb{R},W)$$

by

$$\begin{aligned}(15)\qquad F(a,p,r,w)(t)&=w(t)-T(r,(1+p)t)x_0(p,r,w)-\\ &-(1+p)\int_0^t T(r,(1+p)(t-s))(I-P(r))f(r,w(s))ds-\\ &-a\phi_1(t)-K(I-Q)g(p,r,w)(t)\end{aligned}$$

where $x_0(p,r,w)$ and $g(p,r,w(s))$ are given respectively in (8) and (11). The Theorem 1.1 is a consequence of the following lemmas:

**Lemma 1.1.** Let $(HA), (H\alpha)$ and $(Hf)$ be satisfied. Then there exist constants $a_1 > 0$, $p_1 > 0$, $0 < r_1 \le r_0$ and a unique continuous function $w(a,p,r)$ defined for $|a| < a_1$, $|p| < p_1$ and $|r| < r_1$ with values in $C_{\frac{2\pi}{\beta(0)}}(\mathbb{R}, W)$ such that:

i) $F(a,p,r,w(a,p,r)) = 0$.

ii) $w(0,p,r) = 0$ and $w(a,p,r) \neq 0$ if $a \neq 0$.

iii) $w$ is continuously differentiable in the first variable $a$, with $\frac{\partial w}{\partial a}(0,0,0) = \phi_1(\cdot)$.

**Lemma 1.2.** Under the hypotheses of Lemma 1.1 and choosing the constants $a_1$, $p_1$ and $r_1$ smaller if necessary, the function $w(a,p,r)(t)$, given by Lemma 1.1, is continuously differentiable in all the variables. In particular $w(a,p,r,)$ is a strong $\frac{2\pi}{\beta(0)}$-periodic solution of

$$\frac{d}{dt}w = (1+p)[A(r)w + f(r,w)] - Qg(p,r,w) \tag{16}$$

**Lemma 1.3:** Under hypotheses of Lemma 1.1 and choosing $a_1$ still smaller if necessary, there exist continuously differentiable functions $p(a)$ and $r(a)$ defined for $|a| < a_1$ such that:

i) $Qg(p(a), r(a), w(a,p(a),r(a))) = 0$.

ii) $p(0) = r(0) = \frac{dr}{da}(0) = \frac{dp}{da}(0) = 0$.

In particular, setting

$$v(a)(s) = w(a,p(a),r(a))\left(\frac{s}{1+p(a)}\right)$$

we have that $v(a)$ is the family of $\frac{2\pi}{\beta(0)}$-periodic solutions satisfying Theorem 1.1.

We will finish this section proving the lemmas and the simmetry properties of the functions $w$, $p$ and $r$.

**Proof of Lemma 1.1.** Since $F$ is continuous, continuously differentiable in the variables $a$, $w$ and

$$F(0,0,0,0) = 0 \quad \text{and} \quad F_w(0,0,0,0) = I$$

the implicit function theorem implies the existence of constants $a_1$, $p_1$, $r_1$ and a function $w(a,p,r)$ defined for $|a| < a_1$, $|p| < p_1$, $|r| < r_1$, satisfying the conditions i) and iii) of the lemma. We also have that $F(0,p,r,0) = 0$, then $w(0,r,p) = 0$ follows from the uniqueness of the simplicit function theorem. Finally, since

$$QP(r)w(a,p,r)(t) = a\phi_1(t) \tag{17}$$

is nonzero if $a \neq 0$, we have $w(a,p,r) \neq 0$ if $a \neq 0$.

**Proof of Lemma 1.2.** First of all, we will prove that $w(a,r,p)(t)$ is continuously differentiable in $t$. We will show that $w(a,r,p)(t)$ can be uniformly approximated in the $C^1$-topology.

We will use, for simplicity, the notations

$$\begin{aligned} (I - P(r))w(a,p,r) &= x(a,p,r) \\ P(r)w(a,p,r) &= y(a,p,r). \end{aligned} \tag{18}$$

Of course, it suffices to show that $x(a,p,r)(t)$ is continuously differentiable in $t$. We define

$$S_{a,p,r} : C_{\frac{2\pi}{\beta(0)}}(\mathbb{R}, X_r) \to C_{\frac{2\pi}{\beta(0)}}(\mathbb{R}, X_r)$$

by

$$\begin{aligned} S_{a,p,r}(x)(t) &= T(r,(1+p)t)x_0(p\cdot r, x + y(a,p,r)) + \\ &+ \ (1+p)\int_0^t T(r,(1+p)(t-s))(I-P(r))f(r,x(s)+y(a,p,r)(s))ds \end{aligned} \tag{19}$$

where $x_0(p,r,x+y(a,p,r))$ is given in (8). The operator $S_{a,p,r}$ is well defined and, moreover, it is a uniform contraction in a neighborhood of zero for $a$, $p$, $r$ small enough. To be clearer, since $f$ satisfies

$$f(r,0) = 0 \quad \text{and} \quad f_w(r,0) = 0$$

and $y(a,p,r)$ is continuous with $y(0,p,r) = 0$, we can choose constants, that we call again $a_1$, $p_1$, $r_1$, and $\varepsilon_0$, such that, for $|a| < a_1$, $|p| < p$, $|r| < r_1$ and $||w_1|| \leq 2\varepsilon_0$, we have

$$\begin{aligned} ||(I - P(r))f(r,w_1)|| &\leq \frac{\gamma}{4M^3}\,||w_1|| \\ ||(I - P(r))f_w(r,w_1)\cdot w|| &\leq \frac{\gamma}{4M^3}||w|| \\ ||y(a,p,r)||_\infty &\leq \varepsilon_0 \end{aligned} \tag{20}$$

where $M$ and $\gamma$ are the constants given in (2). If $\overline{B_{\varepsilon_0}}$ denotes the closed ball of radius $\varepsilon_0$ in $C_{\frac{2\pi}{\beta(0)}}(\mathbb{R}, X_r)$, then a simple computation shows that

$$S_{a,p,r} : \overline{B_{\varepsilon_0}} \longrightarrow \overline{B_{\varepsilon_0}}, \quad |a| < a_1, \ \ |p| < p_1, \ \ |r| < r_1$$

and

$$||S_{a,p,r}(x_1) - S_{a,p,r}(x_2)||_\infty \leq \frac{1}{2}\,||x_1 - x_2||_\infty$$

for $x_1$ and $x_2$ in $\overline{B_{\varepsilon_0}}$. Therefore $x(a,p,r)$ is the unique fixed point of $S_{a,p,r}$ in $\overline{B_{\varepsilon_0}}$ and $x(a,p,r)$ can be approximated by the following successive approximation

$$x_0(a,p,r) = 0$$
$$x_{n+1}(a,p,r) = S_{a,p,r}(x_n(a,p,r)); \qquad n = 0,1,2,\dots$$

therefore, $x_n(a,p,r)(t)$ is continuous in all the variables and continuously differentiable in $t$ for $n = 0,1,2,\dots$. Moreover, since $\dot{x}_{n+1}(a,p,r)$ is a $\frac{2\pi}{\beta(0)}$-periodic solution of the linearized equation

$$\begin{aligned} \dot{v} &= (1+p)[A(r)v + (I - P(r))f_w(r, x_n(a,p,r) + y(a,p,r)) \cdot \\ &\quad \cdot (\dot{x}_n(a,p,r) + \dot{y}(a,p,r))], \end{aligned}$$

it is easy to see that $\dot{x}_{n+1}(a,p,r)$ satisfies:

$$\begin{aligned} (21)\quad & \dot{x}_{n+1}(a,p,r)(t) = \\ &= T(r,(1+p)t)\frac{\partial}{\partial w} x_0(p,r,x_n(a,p,r)+y(a,p,r)) \cdot (\dot{x}_n(a,p,r)+\dot{y}(a,p,r)) + \\ &+ (1+p)\int_0^t T(r,(1+p)(t-s))(I-P(r))f_w(r,x_n(a,p,r)(s)+y(a,p,r)(s)) \cdot \\ &\cdot (\dot{x}_n(a,p,r)(s)+\dot{y}(a,p,r)(s))ds, \end{aligned}$$

where $x_0(p,r,w(\cdot))$ is given in (8). Therefore if

$$K = \sup_{\substack{|r|<r_1 \\ \|w\|\le 2\varepsilon_0}} \|(I-P(r))\frac{\partial^2 f}{\partial w^2}(r,w)\|_{\mathcal{L}(X_r,\mathcal{L}(X_r))},$$

the mean value inequality implies

$$(22) \qquad \|(I-P(r))(f_w(r,w_1) - f_w(r,w_2)) \cdot w\| \le K\|w_1 - w_2\|\,\|w\|$$

for $w \in W$ and $|r| < r_1$, $\|w_i\| \le 2\varepsilon_0$, $i = 1,2$, then using (20), (21), (22) and some elementary computations, we can show that

$$\|\dot{x}_{n+1}(a,p,r)\|_\infty \le \frac{1}{2}\|\dot{x}_n(a,p,r)\|_\infty + \frac{1}{2}\|\dot{y}(a,p,r)\|_\infty \quad n = 0,1,2,\dots$$

therefore

$$\|\dot{x}_n(a,p,r)\|_\infty \le \|\dot{y}(a,p,r)\|_\infty; \qquad n = 0,1,2,\dots$$

and furthermore, for $n = 0,1,2,\dots$,

$$\begin{aligned} &\|\dot{x}_{n+1}(a,p,r)(t) - \dot{x}_n(a,p,r)(t)\| \le \\ &\le K_1\|x_n(a,p,r) - x_{n-1}(a,p,r)\|_\infty + \frac{1}{2}\|\dot{x}_n(a,p,r) - \dot{x}_{n-1}(a,p,r)\|_\infty \end{aligned}$$

where $K_1$ is a constant independent of $a$, $p$, $r$ and $t$, so we have

$$\begin{aligned} &\|\dot{x}_{n+1}(a,p,r) - \dot{x}_n(a,p,r)\|_\infty \le \\ &\le \frac{n}{2^{n-1}} K_1\|x_1(a,p,r)\|_\infty + \frac{1}{2^n}\|\dot{x}_1(a,p,r)\|_\infty \end{aligned}$$

and this inequality implies that $\dot{x}_n(a,p,r)(t)$ converges, as $n \to \infty$, uniformly in $a$, $p$, $r$ and $t$, if $|a| < a_1$, $|p| < p_1$, $|r| < r_1$ and $t \geq 0$, then $x(a,p,r)(t)$ is continuously differentiable in $t$ and

$$\dot{x}(a,p,r)(t) = \lim_{n\to\infty} \dot{x}_n(a,p,r)(t).$$

It remains to prove the differentiability in the variables $p$ and $r$, and this will be done via the well known. Fixed Point theorem depending on parameters. We know that $w(a,p,r)$ is fixed point of the operator

$$E(a,p,r,w)(t) = w(t) - F(a,p,r,w)(t)$$

with $F(a,p,r,w)$ given in (15). Then all we have to show is that $E(a,p,r,w)$, for $w$ in the fixed points set

$$\mathcal{F} = \{w(a,p,r) : |a| < a_1, \quad |r| < r_1, \quad |p| < p_1\}$$

has continuous derivatives in $p$ and $r$. Let $w(t)$ be in $\mathcal{F}$, then $w(t)$ is continuously differentiable, from first part, and therefore, to get the differentiability of $E(a,p,r,w(\cdot))$ in $p$ and $r$ it suffices to analyse the expression

$$\varphi(p,r,t) = \int_0^t T(r,(1+p)(t-s))(1+p)(I-P(r))f(r,w(s))ds$$

showing that:

i) $\varphi(p,r,t)$ belongs to the domain $D$ of $A(r)$,

ii) $\varphi(p,r,t)$ has continuous derivatives in $p$ and $r$.

The first one follows from

$$\varphi(p,r,t) = \int_0^{(1+p)t} T(r,s)(I-P(r))f(r,w(t-\frac{s}{1+p}))ds$$

that implies that $\varphi(p,r,t)$ is also continuously differentiable in $t$ and therefore belongs to the domain $D$. The second conclusion follows writting $\varphi(p,r,t)$ in the form

$$\begin{aligned}
&\varphi(p,r,t) = \\
&= \int_0^t T(r,(1+p)(t-s))(1+p)A(r)A^{-1}(r)(I-P(r))f(r,w(s))ds = \\
&= -\int_0^t \frac{d}{ds} T(r,(1+p)(t-s)A^{-1}(r)(I-P(r))f(r,w(s))ds
\end{aligned}$$

and integrating by parts, to get

$$\begin{aligned}
&\varphi(p,r,t) = \\
&= T(r,(1+p)t)A^{-1}(r)(I-P(r))f(r,w(0)) - A^{-1}(r)(I-P(r))f(r,w(t)) + \\
&+ \int_0^t T(r,(1+p)(t-s))A^{-1}(r)(I-P(r))f_w(r,w(s))\dot{w}(s)ds
\end{aligned}$$

that is continuously differentiable in $p$ and $r$.

**Proof of Lemma 1.3.** First of all, note that $Qg(p,r,w(a,p,r)(\cdot))$ is identically zero, for $a=0$, since $w(0,p,r)=0$ and $g(p,r,0)=0$. Then we define

$$(23) \qquad R(a,p,r)=\begin{cases} \frac{1}{a}Qg(p,r,w(a,p,r)(\cdot)); & \text{for } a\neq 0\\ Qg_w(p,r,0)\frac{\partial w}{\partial a}(0,p,r)(\cdot); & \text{for } a=0\end{cases}$$

and we will use the implicit function theorem in $R(a,p,r)=0$ to get uniques continuously differentiables functions $p(a)$ and $r(a)$ defined in a neighborhood of zero and assuming real values such that

$$R(a,p(a),r(a))=0$$
$$p(0)=r(0)=0$$

and

$$\frac{dr}{da}(0)=\frac{dp}{da}(0)=0.$$

We have $R(0,0,0)=0$ and

$$R(0,p,0)=pQ[A(0)]P(0)\frac{\partial}{\partial a}w(0,p,0)(\cdot)$$
$$R(0,0,r)=Q([A(r)]-[A(0)])P(r)\frac{\partial}{\partial a}w(0,0,r)(\cdot)$$

therefore, since $\frac{\partial}{\partial a}w(0,0,0)=\phi_1(\cdot)$, we have that

$$R_p(0,0,0)=Q[A(0)]\phi_1(\cdot)=-\beta(0)\phi_2(\cdot)$$
$$R_r(0,0,0,)=Q\frac{d}{dr}[A(r)]_{r=0}\phi_1(\cdot)$$
$$=\alpha'(0)\phi_1(\cdot)-\beta'(0)\phi_2(\cdot).$$

Consequentely, the Hopf's hypothesis $H\alpha)$ implies that $\frac{\partial R}{\partial(p,r)}(0,0,0)$ is an isomorphism and the remainder of the proof is straight–forward, since $\frac{\partial R}{\partial a}(0,0,0)=0$.

We will show now some symmetry properties of the bifurcating family of periodic solutions. In order to get these properties, we will change $a\ \phi_1(t)$ by $(a\cos\theta)\phi_1(t)+(a\,\mathrm{sen}\,\theta)\phi_2(t)$, $\theta\in I\!R$, in (15) obtaining the function $F$, depending on $\theta$, that is

$$F(a,p,r,w,\theta)(t)=w(t)-T(r,(1+p)t)x_0(p,r,w)-$$
$$-\quad(1+p)\int_0^t T(r,(1+p)(t-s))(I-P(r))f(r,w(s))ds-$$
$$-\quad a\phi_1(t-\frac{\theta}{\beta(0)})-K(I-Q)g(p,r,w)(t).$$

In the same way we did before, we get $w(a,p,r,\theta)$. In particular, $w(a,p,r)=w(a,p,r,0)$. Using the function $w(a,p,r,\theta)$, we introduce

$$R(a,p,r,\theta)=\begin{cases}\frac{1}{a}Qg(p,r,w(a,p,r,\theta))(\cdot); & \text{for } a\neq 0\\ Qg_w(p,r,0)\frac{\partial}{\partial a}w(0,p,r,\theta)(\cdot); & \text{for } a=0\end{cases}$$

to obtain the functions $p(a,\theta)$ and $r(a,\theta)$ by implicit function theorem. In particular $p(a) = p(a,0)$ and $r(a) = r(a,0)$.

The symmetry result is the following:

**Theorem 1.2.** (SYMMETRY). Under the hypotheses of Theorem 1.1, we have

a) $w(a,p,r,\theta) = w(-a,p,r,\theta+\pi)$,

b) $w(a,p,r,\theta)(t+\frac{\pi}{\beta(0)}) = w(-a,p,r,\theta)(t)$,

c) $p(a,\theta)$ and $r(a,\theta)$ are even functions of $a$.

**Proof.** We will use the local uniqueness of the implicit function theorem to get these properties. The first one is a consequence of the trivial equality

$$F(a,p,r,w,\theta) = F(-a,p,r,\theta,\ \theta+\pi).$$

The item b) is deduced from the equality

$$F(a,p,r,w(\frac{\pi}{\beta(0)}+\cdot),\theta+\pi)(t) = F(a,p,r,w,\theta)(t+\frac{\pi}{\beta(0)})$$

whose verification we omitted, but it can be done using only usual changes of variables.

The last part c) is derived from the relation

$$R(a,p,r,\theta)(t+\frac{\pi}{\beta(0)}) = -R(-a,p,r,\theta)(t)$$

that follows trivially from b).

## Section 2. Linearized Stability

First of all, we are going to review the concepts and some results, it may be helpful to locate the problem.

From the first section, we have that

$$w(a)(t) = w(a,p(a),r(a))(t); \qquad |a| < a_1, \qquad t \geq 0$$

are $\frac{2\pi}{\beta(0)}$-periodic functions satisfying

$$\dot{w}(a)(t) = (1+p(a))[A(r(a))w(a)(t) + f(r(a),w(a)(t))]. \tag{24}$$

Let $S(a,t,\tau)$, $t \geq \tau$, be the evolution operator of the linearized equation

$$\dot{u} = (1+p(a))[A(r(a))u + f_w(r(a),w(a)(t))u] \tag{25}$$

such that $u(t) = S(a,t,\tau)w_0$ is the solution of (25) with $u(\tau) = w_0$. The characteristic multipliers of the problem (25) are the nonzero eigenvalues of $S(a) =$

$S(a, \frac{2\pi}{\beta(0)}, 0)$. If $k(a)$ is such that $e^{k(a)\frac{2\pi}{\beta(0)}}$ is characteristic multiplier, then we say that $k(a)$ is a characteristic exponent of (25). Equivalently, $k(a)$ is a characteristic exponent of (25) if and only if the equation

$$\dot{u} = (1 + p(a))\,[A(r(a))u + f_w(r(a), w(a)t)u] - k(a)u \tag{26}$$

has a nontrivial $\frac{2\pi}{\beta(0)}$–periodic solution.

For $a = 0$, 1 is a characteristic multiplier with multiplicity 2 ($k = 0$ is a double characteristic exponent) with the corresponding eigenfunctions given, following the notations of section 1, by

$$\phi_1(t) \qquad \text{and} \qquad \phi_2(t).$$

Moreover, a simple differentiation of (24) show us that $k(a) = 0$ (or $2\pi\, in\, \frac{\beta(0)}{2\pi}$, $n = 0, \pm 1, \pm 2, \ldots$) should remain as characteristic exponents, for a in a neighborhood of zero.

Under our hypotheses, we have the following result of linearized stability (see Henry [2], pg. 158): If 1 is a simple eigenvalue of $S(a)$ and the remainder of the spectrum of $S(a)$ is strictly inside the unit circle, then the orbit $\{w(a)(t) : a \le t \le \frac{2\pi}{\beta(0)}\}$ is asymptotically stable (as a set) with asymptotic phase. Furthermore, if the spectrum of $S(a)$ contains points outside the unit circle, then the orbit $\{w(a)(t) : 0 \le t \le \frac{2\pi}{\beta(0)}\}$ is unstable (as a set). Therefore we will get the stability properties of the $\frac{2\pi}{\beta(0)}$–periodic solution $w(a)$, of (24), if we know the localization of the spectrum of $S(a)$, for $a \neq 0$ close to zero.

From the variation constants formula we know that

$$\begin{aligned} S(a,t,0)w &= T(r(a),(1+p(a))t)w + \\ &(1+p(a)) \int_0^t T(r(a),(1+p(a))(t-s)) f_w(r(a), w(a)(s)) S(a,s,0) w\, ds \end{aligned}$$

so, the Gronwall's inequality implies that

$$\{S(a,t,0)w : |a| < a_1, \quad 0 \le t \le \frac{2\pi}{\beta(0)}, \quad \|w\| = 1\}$$

is a bounded set of $W$. Therefore, since $f_w(0,0) = 0$, we have that

$$\|S(a) - T(a)\|_{\mathcal{L}(W)} \longrightarrow 0 \qquad \text{as} \qquad a \to 0$$

where

$$S(a) = S(a, \frac{2\pi}{\beta(0)}, 0) \qquad \text{and} \qquad T(a) = T(r(a), (1+p(a))\frac{2\pi}{\beta(0)}).$$

Therefore, for a sufficiently close to zero we conclude that the spectrum of $S(a)$ is contained in a circle with radius less than one with exception of two characteristic multipliers which correspond to small perturbation of the double multiplier 1 when $a = 0$. Since, we know that $k(a) = 0$ remains as characteristic exponent for $a \neq 0$, what we have to do is to look for a second small characteristic exponent which we will denote by $k^*(a)$, and to determine the sign of Re $k^*(a)$ (that will give us the expected stability properties of the solution $w(a)$). What we will really have is that $k^*(a)$ is real and the sign of $k^*(a)$ is determined by the signs of $\alpha'(0)$ and $ar'(a)$, for small $a$. The result of stability is the following:

**Theorem 2.1** (LINEARIZED STABILITY): Under the hypotheses of section 1. If $ar'(a)\alpha'(0) > 0$, $a \neq 0$, then $k^*(a) < 0$ and $w(a)$ is asymptotically stable. If $ar'(a)\alpha'(0) < 0$, $a \neq 0$, then $k^*(a) > 0$ and $w(a)$ is unstable.

To study the characteristic exponent $k^*(a)$ we have to analyse the equation (26), or more generally

$$\text{(27)} \qquad \begin{cases} \dot{v} = (1+p(a))\,[A(r(a))v + f_w(r(a), w(a)t)v] + h(t) \\ v(0) = v\left(\frac{2\pi}{\beta(0)}\right) \end{cases}$$

where $h$ is a $\frac{2\pi}{\beta(0)}$–periodic function. Adopting the same procedure used in the section 1 (equation (27)) to get the integrated form (15), that is, projecting the equation on the invariant subspaces $X_{r(a)}$ and $Y_{r(a)}$ and solving them separately as we did before, we obtain the following integrated form for the problem (27)

$$\text{(28)} \qquad \begin{cases} F_w(a)\cdot v - QP(r(a))v = L(a)h \\ Q[g_w(a)\cdot v + P(r(a))h] = 0 \end{cases}$$

where $F_w(a) = F_w(a, p(a), r(a), w(a))$, $g(a) = g(p(a), r(a), w(a)(\cdot))$ etc., and $L(a)$ is a bounded linear operator on $C_{\frac{2\pi}{\beta(0)}}(\mathbb{R}, W)$ given by

$$\begin{aligned} &(L(a)h)(t) = \\ &= T(r(a),(1+p(a))t)[I - T(r(a),(1+p(a))\frac{2\pi}{\beta(0)})]^{-1}\cdot \\ &\int_0^{\frac{2\pi}{\beta(0)}} T(r(a),(1+p(a))(\frac{2\pi}{\beta(0)} - s))\cdot(I - P(r(a))h(s)ds + \\ &+ \int_0^t T(r(a),(1+p(a))(t-s))(I - P(r(a))h(s)ds + [K(I-Q)P(r(a))h](t). \end{aligned}$$

Therefore $k(a)$ is a characteristic exponent of (25) if and only if, the problem

$$\text{(29)} \qquad \begin{cases} F_w(a)u - QP(r(a))u = L(a)k(a)u \\ Q[g_w(a)u - k(a)P(r(a))u] = 0 \end{cases}$$

has a nontrivial solution in $C_{\frac{2\pi}{\beta(0)}}(\mathbb{R}, W)$.

Observe that $k(a) = 0$ remains as characteristic exponent. The implicit function theorem shows that there exists a unique continuous $\frac{2\pi}{\beta(0)}$–periodic function $u^*(a)$, for a in neighborhood of zero, such that

$$\text{(30)} \qquad \begin{aligned} F_w(a)u^*(a) - \dot{\phi}_1(\cdot) = 0 \\ u^*(0)(t) = \dot{\phi}_1(t) \end{aligned}$$

therefore, from the first section, we also have that

$$au^*(a) = \dot{w}(a) \tag{31}$$

and

$$Qg_w(a)u^*(a) = 0 \tag{32}$$

that is, $u^*(a)$ is the eigenfunction corresponding to the characteristic exponent $k(a) = 0$ with

$$QP(r(a))u^*(a) = \dot{\phi}_1(\cdot) = -\beta(0)\phi_2(\cdot). \tag{33}$$

Since, $u^*(a)$ is a $\frac{2\pi}{\beta(0)}$-periodic solution of

$$\dot{v} = (1 + p(a))\left[A(r(a))v + f_w(r(a), w(a))v\right],$$

with the same proof of Lemma 1.2, we have that $u^*(a)(t)$ is continuously differentiable.

We also have $w(a)$ continuously differentiable with

$$QP(0)w'(0) = \phi_1(\cdot).$$

Then we can state the following lemmas, that, in particular, prove Theorem 2.1.

**Lemma 2.1.** There exists a unique continuous function $v^*(a, k, \eta)$ defined for $a$, $k$, $\eta$ (real) near zero, with values in $C_{\frac{2\pi}{\beta(0)}}(I\!R, W)$ such that:

i) $v^*(0, 0, 0) = \phi_1(\cdot)$

ii) $QP(r(a))v^*(a, r, \eta) = QP(r(a))w'(a)$

iii) $F_w(a)v^*(a, k, \eta) - QP(r(a))\dot{w}'(a) = -L(a)(kv^*(a, k, \eta) + \eta u^*(a))$

iv) $v^*(a, k, \eta)(t)$ is continuously differentiable in all the variables.

**Lemma 2.2.** There exist continuously differentiables functions $k^*(a)$ and $\eta^*(a)$ defined near $a = 0$ with values in $I\!R$, which satisfy:

i) $k^*(0) = \eta^*(0) = 0$

ii) $Qg_w(a)v^*(a) = k^*(a)QP(r(a))w'(a) + \eta^*(a)\dot{\phi}_1(\cdot)$ where $v^*(a) = v^*(a, k^*(a), \eta^*(a))$.

iii) If $k^*(a) \neq 0$, then $k^*(a)$ is a characteristic exponent of (25) (that is, $k^*(a)$ is the second small characteristic exponent).

**Lemma 2.3.** For $k^*(a)$ and $\eta^*(a)$ given in Lemma 2.2 we have that

i) $\|w'(a) - v^*(a)\|_\infty \leq c|ar'(a)|$

ii) $|k^*(a) + ar'(a)\alpha'(0)| \leq |ar'(a)|o(1)$, as $a \to 0$, in particular, $k^*(a)$ and $-ar'(a)\alpha'(0)$ have the same sign. Furthermore

iii) $|\eta^*(a) + \frac{ap'(a)}{1+p(a)} + ar'(a)\frac{\beta'(0)}{\beta(0)}| \leq |ar'(a)|o(1)$, as $a \to 0$.

**Proof of Lemma 2.1.** The map

$$\mathcal{F}(a,k,\eta,v) = F_w(a)v - QP(r(a))w'(a) + L(a)(kv + \eta u^*(a))$$

defined for $a$, $k$, $\eta$ near zero and $v \in C_{\frac{2\pi}{\beta(0)}}(I\!R, W)$ with values in $C_{\frac{2\pi}{\beta(0)}}(I\!R, W)$ is continuous and continuously differentiable in $v$, with

$$\mathcal{F}(0,0,0,\phi_1(\cdot)) = 0$$

and

$$\mathcal{F}_v(0,0,0,\phi_1(\cdot)) = I$$

therefore, the implicit function theorem gives $v^*(a,k,\eta)$ satisfying i), ii) and iii) of the lemma. The continuous differentiability of $v^*$ follows again using the same procedure adopted in Lemma 1.2.

**Proof of Lemma 2.2.** The existence of $k^*(a)$ and $\eta^*(a)$ satisfying i) and ii) is a consequence of the implicit function theorem, since $g_w(0) = 0$. Now, if $k^*(a) \neq 0$, then, using the Lemma 2.1 and (30), (32), (33), we obtain that

$$F_w(a)(v^*(a) + \frac{\eta^*(a)}{k^*(a)}\, u^*(a)) - (QP(r(a))w'(a) + \frac{\eta^*(a)}{k^*(a)}\, \phi_1(\cdot)) =$$
$$= \quad -L(a)k^*(a)(v^*(a) + \frac{\eta^*(a)}{k^*(a)}\, u^*(a))$$

and

$$Q[g_w(a)(v^*(a) + \frac{\eta^*(a)}{k^*(a)}\, u^*(a)) -$$
$$-k^*(a)P(r(a))(v^*(a) + \frac{\eta^*(a)}{k^*(a)}\, u^*(a))] = 0$$

therefore, from (29), we have $k^*(a)$ as a characteristic exponent with the corresponding eigenfunction

$$v^*(a) + \frac{\eta^*(a)}{k^*(a)}\, u^*(a).$$

**Proof of Lemma 2.3.** From the first section, we have that $w(a)$ is continuously differentiable, and $w'(a)$ is a $\frac{2\pi}{\beta(0)}$-periodic solution of

$$\dot{v} = (1 + p(a))\left[A(r(a))v + f_w(r(a), w(a))v\right] + h(a)$$

where

$$h(a) = \frac{p'(a)}{1 + p(a)}\, \dot{w}(a) + r'(a)(1 + p(a))(A'(r(a))w(a) + f_r(r(a), w(a))$$

therefore, (28) implies that

$$F_w(a)w'(a) - QP(r(a))w'(a) = L(a)h(a).$$

Since $\dot{w}(a) = au^*(a)$, we obtain using Lemmas 2.1 and 2.3, that

$$(34) \qquad F_w(a)(w'(a) - v^*(a)) = L(a)\,[k^*(a)v^*(a) + (\eta^*(a) + \frac{ap'(a)}{1+p(a)})u^*(a) + $$
$$+r'(a)(1+p(a))(A'(r(a))w(a) + f_r(r(a), w(a)))]$$

and looking at

$$A'(r(a))w(a) = A'(r(a))A^{-1}(r(a))A(r(a))w(a)$$

we can see that

$$A'(r(a))w(a) + f_r(r(a), w(a)) = aO(1) \quad \text{as} \quad a \to 0$$

because $A'(r(a))A^{-1}(r(a))$ is bounded and strongly continuous,

$$A(r(a))w(a) = \frac{au^*(a)}{1+p(a)} - f(r(a), w(a))$$

and $f(r(a), w(a)) = ao(1)$ as $a \to 0$, therefore

$$(35) \qquad F_w(a)(w'(a) - v^*(a)) = L(a)\,[k^*(a)v^*(a) + (\eta^*(a) + \frac{ap'(a)}{1+p(a)})u^*(a)] + $$
$$+ar'(a)O(1) \quad \text{as} \quad a \to 0.$$

On the other hand, differentiating the relation

$$Qg(a) = Qg(p(a), r(a), w(a)) = 0$$

gives

$$Qg_w(a)w'(a) = -p'(a)Qg_p(a) - r'(a)Qg_r(a) =$$
$$= \frac{-ap'(a)}{1+p(a)}\,\dot{\phi}_1(\cdot) - r'(a)(1+p(a))\frac{d}{dr}\,[A(r)]_{r=r(a)}a\phi_1(\cdot) -$$
$$- \quad -r'(a)Q((1+p(a))\frac{\partial}{\partial r}\left[P(r)f(r,w)\right]_{\substack{r=r(a)\\ w=w(a)}} +$$
$$+ \quad ((1+p(a))\,[A(r(a))] - [A(0)])\,\frac{d}{dr}\left[P(r)w\right]_{\substack{r=r(a)\\ w=w(a)}} \quad )$$

then, we have

$$(36)\; Qg_w(a)w'(a) =$$
$$= \frac{-ap'(a)}{1+p(a)}\,\dot{\phi}_1(\cdot) - ar'(a)(1+p(a))(\alpha'(r(a))\phi_1(\cdot) - \beta'(r(a))\phi_2(\cdot)) +$$
$$+ar'(a)o(1) \quad \text{as} \quad a \to 0.$$

and therefore, from Lemma 2.2, we obtain

$$(37)\qquad \begin{aligned} Qg_w(a)(w'(a)-v^*(a)) &= \\ &= -k^*(a)QP(r(a))w'(a) - \left(\eta^*(a)+\frac{ap'(a)}{1+p(a)}\right)\dot{\phi}_1(\cdot) - \\ &- ar'(a)(1+p(a))(\alpha'(r(a))\phi_1(\cdot)-\beta'(r(a))\phi_2(\cdot)) + \\ &+ ar'(a)o(1) \qquad \text{as} \qquad a\to 0. \end{aligned}$$

Since,

$$ar'(a)(1+p(a))(\alpha'(r(a))\phi_1(\cdot)-\beta'(r(a))\phi_2(\cdot)) = ar'(a)O(1)$$

we have, taking together (35) and (37), that

$$(38)\qquad \begin{aligned} &k^*(a)QP(r(a))w'(a) + \left(\eta^*(a)+\frac{ap'(a)}{1+p(a)}\right)\dot{\phi}_1(\cdot) + \\ &+ (F_w(a)+Qg_w(a))(w'(a)-v^*(a)) - \\ &- L(a)\left[k^*(a)v^*(a) + \left(\eta^*(a)+\frac{ap'(a)}{1+p(a)}\right)u^*(a)\right] = ar'(a)O(1). \end{aligned}$$

To estimate $\|w'(a)-v^*(a)\|_\infty$ we observe that $w'(a)-v^*(a)$ belongs to

$$V_a = \{v \in C_{\frac{2\pi}{\beta(0)}}(\mathbb{R},W) : QP(r(a))v = 0\}$$

and $C_{\frac{2\pi}{\beta(0)}}(\mathbb{R},W)$ is isomorphic to $\mathbb{R}^2 \times V_a$, so we can define the linear operator

$$K(a) : \mathbb{R}^2 \times V_a \longrightarrow C_{\frac{2\pi}{\beta(0)}}(\mathbb{R},W)$$

by

$$\begin{aligned} K(a)(k,\eta,v) &= kQP(r(a))w'(a) + \eta\dot{\phi}_1(\cdot) + \\ &+ (F_w(a) - Qg_w(a))v - L(a)[kv^*(a) + \eta u^*(a)] \end{aligned}$$

and, we have, $K(0) = I$ and $K(a)$ continuous, therefore, from (38), there is a positive constant $c$ such that

$$(39)\qquad |k^*(a)| + |\eta^*(a) + \frac{ap'(a)}{1+p(a)}| + \|w'(a)-v^*(a)\|_\infty \le c|ar'(a)|$$

that proves the item i) of the lemma, and implies

$$k(a)QP(r(a))w'(a) = k(a)\dot{\phi}_1(\cdot) + ar'(a)o(1)$$

and

$$Qg_w(a)(w'(a)-v^*(a)) = ar'(a)o(1) \qquad \text{as} \qquad a\to 0.$$

Now, using these two equalities and $\phi_2(\cdot) = \frac{-1}{\beta(0)}\dot{\phi}_1(\cdot)$ in (37), we obtain

$$\begin{aligned} &(k^*(a) + ar'(a)(1+p(a))\alpha'(r(a))\phi_1(t) + \\ &+ \left(\eta^*(a) + \frac{ap'(a)}{1+p(a)} + \frac{ar'(a)(1+p(a))\beta'(r(a))}{\beta(0)}\right)\dot{\phi}_1(t) = \\ &= ar'(a)o(1) \qquad \text{as} \qquad a\to 0 \end{aligned}$$

therefore, for $t = 0$, we have

$$|k^*(a) + ar'(a)(1 + p(a))\alpha'(r(a))| \leq |ar'(a)|o(1)$$

and

$$|\beta(0)(\eta^*(a) + \frac{ap'(a)}{1 + p(a)}) + ar'(a)(1 + p(a))\beta'(r(a))| \leq |ar'(a)|o(1) \quad \text{as} \quad a \to 0$$

and finally,

$$\begin{aligned} |k^*(a) + ar'(a)\alpha'(0)| &\leq |k^*(a) + ar'(a)(1 + p(a))\alpha'(r(a))| + \\ &+ |ar'(a)|\,|\alpha'(0) - (1 + p(a))\alpha'(r(a))| \leq |ar'(a)|o(1). \end{aligned}$$

In the same way,

$$\left|\eta^*(a) + \frac{ap'(a)}{1 + p(a)} + ar'(a)\frac{\beta'(0)}{\beta(0)}\right| \leq |ar'(a)|o(1)$$

and the proof is completed.

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