EQUIVARIANT HARMONIC MAPS INTO FLAG MANIFOLDS

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RELATÓRIO TÉCNICO Nº 46/87

ABSTRACT. This paper is about harmonic maps from closed Riemann surfaces into flag manifolds. We construct some new examples of hamonic maps $\phi : T^2 = S^1 \times S^1 \longrightarrow F(n)$ which are not holomorphic with respect to any almost complex structure on $F(n)$.

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Outubro – 1987

§1. INTRODUCTION

This note studies the problem of understanding harmonic maps from closed Riemann surfaces into flag manifolds.

In the late 60's Chern [6] and Calabi [5] published several papers on minimal immersions into spheres or more generally real projective spaces, which are in the spirit of this investigation. Then Simons [18], Lawson [14], Hsiang and Lawson [13] published important papers in this direction.

The problem was reexamined by physicîsts Glaser and Stora [11] and Din-Zakrzewski [8]. They called the attention that the right problem should be to look at harmonic maps into complex projective spaces. Inspired by these ideas Eells and Wood [10] gave a complete classification for harmonic maps from $\mathbb{C}P^1$ to $\mathbb{C}P^n$, and some important partial results for the higher genus cases in terms of holomorphic data. A number of related results have appeared including Burstall and Wood [4], Chern and Wolfson [7], Uhlenbeck [19]. These authors have studied harmonic maps into other homogeneous symmetric spaces like Lie groups and Grassmannians.

On the other hand, very little is known about the homogeneous (non-symmetric) case. One reason that we want to understand the non-symmetric case, beside its own intrinsic interest, is that the finite dimensional flags model in finite dimensions the geometry of the Loop group; i.e., maps from S^1 to a compact Lie group G. In a well-known paper Atiyah calls attention to the identification of holomorphic maps into the loop group and instantons [2].

Secondly the study of the critical points of a functional on a space of maps is difficult to treat in general. The energy functional whose critical points are the harmonic maps is more tractable from the point of view of computations than for example, the Yang-Mills functional, but appears to share some of their important properties. See Wolfson's paper [21] for more details.

If one wants to understand the problem of harmonic maps into a non-symmetric space it is natural to start by understanding harmonic maps from closed Riemann surfaces to flag manifolds. These are by definition the quotient of a compact Lie group by any maximal torus. Some of the first work in this problem was done by Guest [12] using a entirely different approach.

In §2 we state some basic facts about maps into flag manifolds.

In §3 we describe the harmonic and holomorphic equations in terms of projection operators and derive a topological restriction for a totaly isotropic map $\phi\ T^2 : \mathbb{C}P^1 \longrightarrow F(n)$ to be holomorphic with respect to a non-integrable almost complex structure on $F(n)$.

In §4 we construct some new examples of harmonic maps $\phi : T^2 = S^1 \times S^1 \longrightarrow F(n)$ which are not holomorphic with respect to any almost complex structure on $F(n)$. These examples are obtained by studying some equivariant harmonic maps with respect to an S^1 action. This method reduces a partial differential equation (harmonic map equation) to an ordinary differential equation of second order and therefore we expect in general lots of solutions. The approach is based on [20].

We will consider throughout this paper $F(n)$ equipped with the normal Killing form metric.

The contents of this note are part of my doctoral thesis [15]. I want to express my gratitude to my thesis advisor Prof. Karen Uhlenbeck for her deep advise, criticism and encouragement.

§2. SOME BASIC FACTS ABOUT MAPS INTO FLAG MANIFOLDS

A flag manifold is a homogeneous space $G/_T$ where G is a compact Lie group and T is any maximal torus. We denote by $F(n)$ the flag with $G = U(n)$ and $T = \underbrace{U(1) \ \ldots \ U(1)}_{n\text{-times}}$.

The Killing form of $U(n)$ is a positive definite inner product $\langle\,,\,\rangle$ on the Lie algebra $\mu(n)$, and one has the decomposition $\mu(n) = p \oplus \underbrace{\mu(1) \oplus \ldots \oplus \mu(1)}_{n\text{-times}}$. Note that there are X, Y, Z in p such that $[X,[Y,Z]] \notin p$, hence according to Cartan's theorem $F(n)$ cannot be a symmetric space.

We have that $p = \sum_{s\in S} E_s$ where $S \subseteq \mathfrak{h}^*$ is the set of roots and E_s is the root-space corresponding to $s \in S$. We have $p \otimes_{\mathbb{R}} \mathbb{C} = \sum_{s\in\Delta'} E_s$, where Δ' is the subset of complementary roots.

An T-invariant almost complex structure on $F(n)$ corresponds to an T-invariant endomorphism J of p with $J^2 = -I$. Such endomorphisms correspond to some decomposition $S = S^+ \in S^-$ with the property $S^- = \{-\alpha\ ;\ \alpha \in S^+\}$, where by the decomposition $p \otimes_{\mathbb{R}} \mathbb{C} = p_{(1,0)} \oplus p_{(0,1)}$ into (1,0) and (0,1) parts is given by:

$$p \otimes_{\mathbb{R}} \mathbb{C} = \Big(\sum_{s\in S^+} E_s\Big) \oplus \Big(\sum_{s\in S^-} E_s\Big)\,.$$

The almost complex structure is integrable precisely when S^+ is the set of positive roots with respect to a choice of fundamental Weyl chamber D in $\mu(1) \oplus \ldots \oplus \mu(1)$.

A general (T-invariant) almost complex structure is specified by whether or not is agrees with J on each $E_s \oplus E_{-s}$, so there are $2^{|S^+|}$ possibilities. From these only $n!$ = order of the Weyl group of $U(n)$ are integrable.

If we put on $F(n)$ the metric induced from the Killing form metric on $U(n)$ and consider $F(n)$ equipped with any integrable almost complex structure we see that $F(n)$ with the induced normal Killing form metric is not Kähler.

Throughout all this note we will consider $F(n)$ equipped with the Killing form metric.

Let $\underline{\mathbb{C}}^n$ denote the trivial holomorphic vector bundle $M^2 \times \mathbb{C}^n$ over M^2.

We use extrinsic differential geometry and think of $\phi : M^2 \longrightarrow F(n)$ as a map or as a subbundle of $\underline{\mathbb{C}}^n$ via the pull-back of tautologously defined vector bundles on $F(n)$. Note that we also think of $F(n)$ as the set of n-tuples $(L_1,\dots,L_n)$. Here L_i is a 1-dimensional subspace of $\underline{\mathbb{C}}^n$, L_i is perpendicular to L_j if $i \neq j$ and $L_1 \oplus \dots \oplus L_n = \mathbb{C}^n$. Then the tautologously defined vector bundles on $F(n)$ have as fibres over a flag $(L_1,\dots,L_n)$ the vector spaces $L_1,\dots,L_n$ respectively.

As usual we identify a smooth map $\phi : M^2 \longrightarrow \mathbb{C}P^{n-1}$ with a subbundle $\underline{\phi}$ of $\underline{\mathbb{C}}^n$ of rank one which has fibre at $x \in M$ given by: $\underline{\phi}_x = T_{\phi(x)}$ where T is the tautological line bundle over $\mathbb{C}P^{n-1}$; i.e., $\phi = \phi^*(T)$. Any subbundle $\underline{\phi}$ of $\mathbb{C}^n$ inherits a metric denoted by $\langle\ ,\ \rangle_\phi$ and connection denoted by D_ϕ, from the flat metric and connection ∂ on $\underline{\mathbb{C}}^n$.

Explicitly: $\langle V,W\rangle_\phi = \langle V,W\rangle$, $\forall V,W \in \underline{\phi}_x$, $x \in M$ and $(D_\phi)_Z W = \pi_\phi(\partial_Z W)$, $W \in \Gamma(\underline{\phi})$, $Z \in T(M)^{(1,0)}$. Here $\pi_\phi : \mathbb{C}^n \longrightarrow \phi$ denotes the Hermitian projection in the subbundle $\underline{\phi}$.

Note that we always describe $F(n)$ in terms of the natural embedding $F(n) \longrightarrow \mathbb{C}P^{n-1} \times \dots \times \mathbb{C}P^{n-1}$. So $\phi : M^2 \longrightarrow F(n)$ is described as $\phi = (\pi_1,\dots,\pi_n)$ where $\pi_i : M^2 \longrightarrow \mathbb{C}P^{n-1}$ and $\pi_i\pi_j = \delta_{ij}\pi_i$.

Let us now put these facts above in a more algebraic fashion.

Let G be a connected compact Lie group, g its Lie algebra and $T \subseteq G$ be a maximal torus with Lie algebra $\mathfrak{h}$. Since T is compact, the set $Ad_g(T) = \{Ad_g(t);\ t \in T\}$ is compact. Then there exists an $Ad_g(T)$-invariant inner product $\langle\ ,\ \rangle$ on g and writing k for $\mathfrak{h}^\perp$ in g and π, $I-\pi$ for orthogonal projections from g onto $\mathfrak{h}$, k with kernels $k,\mathfrak{h}$ respectively, we have $Ad_g(T)(k) \subseteq k$, in particular $[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h}$, $[\mathfrak{h},k] \subseteq k$.

Now consider the fibration $T \dashrightarrow G \longrightarrow G/_T$. This defines a principal T-bundle on which G operates transitively on the left. This principal T-bundle carries a natural left-invariant connection as well other additional structures. On the one hand, the $Ad_g(T)$-invariant inner product $\langle\ ,\ \rangle$ on g and its restriction to k extend uniquely to a left G-invariant and right T-invariant Riemannian metric on $G/_T$. On the other hand, we also have two distinguished 1-forms on G with values in g, namely the left-invariant Maurer-Cartan form $g^{-1}dg$ and the right-invariant.

Maurer-Cartan from $dg.g^{-1}$. The natural left-invariant connection mentioned above can now be described in several equivalent ways:

(a) left translations of G_g, $g \in G$ to $g = G_e$ and takes the vertical spaces to $\mathfrak{h}$ and the horizontal spaces to k.

(b) The connection form is the vertical part of the left-invariant Maurer-Cartan form on G : $A = \pi(g^{-1}dg)$.

Now let $g : M^2 \longrightarrow U(n)$. g can be thought as: $g = (X_1, \ldots, X_n)$ where X_i is a matrix with n-rows and one column and $g.g^* = I$. We introduce the orthogonal projections $\pi_1(g) = X_1.X_1^*, \ldots, {}_n(g) = X_n X_n^*$. Hence we must have $\pi_1 + \ldots + \pi_n = I$ and $\pi_i\pi_j = 0$ if $i \neq j$ and $\pi_i^2 = \pi_i$ since $g^*g = I$ and $X_i^*.X_j = \delta_{ij}$.

Alternatively we can think of π_i as: $\pi_i(g) = gE_ig^*$ where

$$E_i = {}_i\begin{bmatrix} 0 & & \overset{i}{} & & \\ & \ddots & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 0 \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix}$$ Gauge transformations act on g as:

$$(X_1, \ldots, X_n) \longrightarrow (X_1 h_1, \ldots, X_n h_n) \quad \text{where} \quad \begin{bmatrix} h_1 & & \\ & \ddots & \\ & & h_n \end{bmatrix} \in U(n).$$

The gauge potential is:

$$A_\mu = \begin{bmatrix} X_1^* \partial_\mu (X_1) \\ \cdot \\ \cdot \\ X_n^* \partial_\mu (X_n) \end{bmatrix}$$ and the covariant derivative is:

$$D_\mu (X_1,\ldots,X_n) = (\pi_1(\partial_\mu(X_1)),\ldots,\pi_n(\partial_\mu(X_n)))$$

where $\mu = \partial/_{\partial z}$ or $\partial/_{\partial \bar{z}}$. By composing g with $\pi : U(N) \longrightarrow U(n)/_T = F(n)$ we can think of $\phi = \pi \circ g : M^2 \longrightarrow F(n)$ as $\phi = (\pi_1,\ldots,\pi_n)$. Then each such ϕ determines the tautologously defined vector bundles $\pi_1,\ldots,\pi_n$ over M^2. We study the second fundamental forms of these tautological bundles $\pi_1,\ldots,\pi_n$ in $\mathbb{C}^n$.

Let $\frac{\partial \pi_i}{\partial x} = \partial_{\frac{\partial}{\partial x}} \pi_i$ be the covariant derivative of π_i with respect to x. We can prove:

2.1. PROPOSITION.

a) $\pi_i(\frac{\partial \pi_i}{\partial x})\pi_i = 0$

b) $\frac{\partial \pi_i}{\partial x} = -\frac{\partial}{\partial x}(\pi_i^\perp)$

c) $\pi_i(\frac{\partial}{\partial x}(\pi_i^\perp))\pi_i = 0$

d) $\pi_i^\perp(\frac{\partial \pi_i}{\partial x})\pi_i^\perp = 0$

e) $\frac{\partial \pi_i}{\partial x} = A_x^i + (A_x^i)^*$ where $A_x^i = \pi_i^\perp(\frac{\partial \pi_i}{\partial x})\pi_i$

f) $\pi_i A_x^i \pi_i = A_x^i$ and $\pi_i((A_x^i)^*)\pi_i^\perp = (A_x^i)^*$

g) $\pi_i(A_x^i) = 0$

h) $\pi_i^\perp(A_x^i) = 0$

PROOF. We have that $\frac{\partial\pi_i}{\partial x} = \frac{\partial}{\partial x}(\pi_i.\pi_i) = \frac{\partial\pi_i}{\partial x}.\pi_i + \pi_i.\frac{\partial\pi_i}{\partial x}$ then $(\frac{\partial\pi_i}{\partial x})\pi_i = \frac{\partial\pi_i}{\partial x}\pi_i^2 + \pi_i\frac{\partial\pi_i}{\partial x}\pi_i$, hence we have proved a).

Now $0 = \frac{\partial I}{\partial x} = \frac{\partial}{\partial x}(\pi_i + \pi_i^{\perp}) = \frac{\partial\pi_i}{\partial x} + \frac{\partial\pi_i^{\perp}}{\partial x}$ hence follow b) and c). But $(\pi_i^{\perp})^2 = \pi_i^{\perp}$, then if we proceed as we did in the proof of a) we get that $\pi_i^{\perp}(\frac{\partial\pi_i^{\perp}}{\partial x})\pi_i^{\perp} = 0$, now by using b) we have d).

On the other hand, $\frac{\partial\pi_i}{\partial x} = I\frac{\partial\pi_i}{\partial x}I = (\pi_i + \pi_i^{\perp})\frac{\partial\pi_i}{\partial x}(\pi_i + \pi_i^{\perp})$, by using have: $\frac{\partial\pi_i}{\partial x} = A_x^i + (A_x^i)^* = \pi_i^{\perp}(\frac{\partial\pi_i}{\partial x})\pi_i + \pi_i(\frac{\partial\pi_i}{\partial x})\pi_i^{\perp}$. Now f), g) and h) follow from the fact that $(\pi_i^{\perp})^2 = \pi_i^{\perp}$ and $\pi_i^{\perp}.\pi_i = \pi_i^{\perp}\pi_i = 0$.

Therefore A_x^i is the projection of $\frac{\partial\pi_i}{\partial x}$ on $\pi_i^{\perp}$.

We call the partial second fundamental forms of $\phi = (\pi_1,\ldots,\pi_n) : M^2 \longrightarrow F(n)$ the maps: $A_x^{ij} = \pi_i(A_x^j) = \pi_i(\pi_j^{\perp}\frac{\partial\pi_j}{\partial x}\pi_j) = \pi_i\frac{\partial\pi_j}{\partial x}\pi_j = \pi_i\frac{\partial\pi_j}{\partial x}$ if $i\neq j$.

Note that $A_x^{ij} \in Hom(\pi_j,\pi_i)$ and $\sum_i A_x^{ij}$ is the second fundamental form of the span of π_i.

Now if we think in M^2 as a complex 1-dimensional manifold, then we define:

$$\frac{\partial\pi_i}{\partial z} = \frac{1}{2}(\frac{\partial\pi_i}{\partial x} - \sqrt{-1}\frac{\partial\pi_i}{\partial y}) \text{ and } \frac{\partial\pi_i}{\partial\bar{z}} = \frac{1}{2}(\frac{\partial\pi_i}{\partial x} + \sqrt{-1}\frac{\partial\pi_i}{\partial y}).$$

We also define:

$$A_z^j = \pi_j^{\perp}\frac{\partial\pi_j}{\partial z} = \sum_{i(\neq j)} A_z^{ij}, \quad A_{\bar{z}}^j = \sum_{i(\neq j)} A_{\bar{z}}^{ij}$$

and $A_{\bar{z}}^{ij} = \pi_i \frac{\partial\pi_j}{\partial z}$ and $A_{z}^{ij} = \pi_i \frac{\partial\pi_j}{\partial\bar{z}}$.

The following formula will be very useful.

2.2. PROPOSITION. $\sum_{i,j=1}^{n} \left(\frac{\partial}{\partial z}(A_{\bar{z}}^{ij}) - \frac{\partial}{\partial\bar{z}}(A_{z}^{ij})\right) = 2 \sum_{\substack{i,j=1\\(i\neq j)}}^{n} [A^{ij}, A^{ji}]$

$+ \sum_{\substack{k,i,j=1\\(i\neq j)}}^{n} [A_{z}^{ij}, A_{\bar{z}}^{jk} + A_{\bar{z}}^{ki}]$.

PROOF. See [15] or [16].

Now consider $\mu = z$ or $\bar{z}$ and

$$A_\mu = \begin{bmatrix} 0 & A_\mu^{12} & \cdots & A_\mu^{1n} \\ A_\mu^{21} & 0 & \cdots & A_\mu^{2n} \\ \vdots & \vdots & & \vdots \\ A_\mu^{n1} & A_\mu^{n2} & \cdots & 0 \end{bmatrix}$$

We can rewrite the formula above as:

2.3. PROPOSITION. $\frac{\partial}{\partial z}(A_{\bar{z}}) - \frac{\partial}{\partial\bar{z}}(A_z) = [A_z, A_{\bar{z}}] + [A_z, A_{\bar{z}}]_h$

§3. HARMONIC AND HOLOMORPHIC MAPS INTO FLAG MANIFOLDS

We now study the energy integral in terms of projection operators and write down the Euler-Lagrange equations for our variational problem.

3.1. DEFINITION. Given a smooth map $\phi = (\pi_1,\ldots,\pi_n) : M^2 \longrightarrow F(n) = U(n)/_T$ where $\phi_i = \phi E_i \phi^*$, we define the energy of ϕ as:

$$E(\phi) = \frac{1}{2}\sum_{i=1}^{n}\int_{M^2} \left(\left|\frac{\partial\pi_i}{\partial x}\right|^2 + \left|\frac{\partial\pi_i}{\partial y}\right|^2\right) dxdy .$$

We will prove next some formulas that come from the conservation laws associated with the invariance of E under the action U(n) according to Nöether's theorem.

We call $q : M^2 \longrightarrow \mu(n)$ a angular momentum map. Given $\phi = (\pi_1,\ldots,\pi_n) : M^2 \longrightarrow F(n)$ let $[\pi_i, q] = \pi_i q - q\pi_i$. The map q gives rise naturally to a variation of ϕ, $\delta\phi(q): M^2 \longrightarrow F(n)$ given by:

$$\delta\phi(q)(x) = \left(\frac{d}{dt}\Big|_{t=0} - \exp ade^{-tq(x)}\pi_1,\ldots,\frac{d}{dt}\Big|_{t=0} \exp ad^{-tq(x)}.\pi_n(x)\right)$$

$= ([\pi_1(x), q(x)],\ldots,[\pi_n(x), q(x)])$. Then we can compute the first variation of the enery for the map ϕ.

3.2. PROPOSITION. Let $\phi = (\pi_1,\ldots,\pi_n) : M^2 \longrightarrow F(n)$ be a smooth map. Then:

$$(\delta E)(\delta\phi(q)) = -\sum_{i=1}^{n} \langle\langle [\pi_i, \Delta\pi_i], q \rangle\rangle \quad \text{where } \langle\langle\, ,\rangle\rangle \text{ is the } L^2\text{-}$$

Hilbert inner product on $C^o(M^2, F(n))$.

PROOF. $(\delta E)(\delta\phi(q)) = \sum_{i=1}^{n}\int_{M^2} \left(\left\langle\frac{\partial\pi_i}{\partial x}, \frac{\partial}{\partial x}(\delta\pi_i(q))\right\rangle + \right.$

$\left. + \left\langle\frac{\partial\pi_i}{\partial y}, \frac{\partial}{\partial y}(\delta\pi_i(q))\right\rangle\right) dxdy.$

But $\partial(M^2)$ is empty then if we integrate by parts we have:

$$(\delta E)(\delta\phi(q)) = \sum_{i=1}^{n} \int_{M^2} \langle -\Delta\pi_i, \delta\pi_i(q)\rangle\, dxdy = \sum_{i=1}^{n} \langle\langle -\Delta\pi_i, [\pi_i, q]\rangle\rangle .$$

But by using the cyclic property of trace we can easily see that $\langle\langle A,[B,C]\rangle\rangle = \langle\langle [B^*,A],C\rangle\rangle$. The finally we can prove:

$$(\delta E)(\delta\phi(q)) = -\sum_{i=1}^{n} \langle\langle [\pi_i, \Delta\pi_i], q\rangle\rangle .$$

We known that $\phi = (\pi_1,\dots,\pi_n) : M^2 \longrightarrow F(n)$ is harmonic if and only if it is a critical point of the energy integral, i. e., for any variation $\delta\phi(q)$ of ϕ we have $(\delta E)(\delta\phi(q)) = 0$. Therefore $\phi = (\pi_1,\dots,\pi_n) : M^2 \longrightarrow F(n)$ is harmonic if and only if $\sum_{i=1}^{n} [\pi_i, \Delta\pi_i] = 0$ if and only if $\frac{\partial}{\partial z}(A_{\bar z}) + \frac{\partial}{\partial \bar z}(A_z) = 0$. Then we prove:

$$A_\mu = \begin{bmatrix} 0 & A_\mu^{12} & \dots & A_\mu^{1n} \\ A_\mu^{21} & 0 & & A_\mu^{2n} \\ \vdots & \vdots & & \vdots \\ A_\mu^{n1} & \dots & \dots & 0 \end{bmatrix}$$

Then $tr(A_\mu) = tr(A_\mu^2) = 0$.

PROOF. See [15] or [16].

We have seen that the energy of a map $\phi = (\pi_1,\dots,\pi_n) : M^2 \longrightarrow F(n)$ is given by:

$$E(\phi) = \frac{1}{2} \sum_{i=1}^{n} \int_{M^2} \left(\left\langle \frac{\partial\pi_i}{\partial z}, \frac{\partial\pi_i}{\partial z} \right\rangle + \left\langle \frac{\partial\pi_i}{\partial \bar z}, \frac{\partial\pi_i}{\partial \bar z} \right\rangle \right) V_g =$$

$$= \sum_{i=1}^{n} \int_{M^2} \left|\frac{\partial \pi_i}{\partial \mu}\right|^2 V_g = \sum_{i,j=1}^{n} \int_{M^2} |A_\mu^{ij}|^2 V_g = - \sum_{i,j=1}^{n} \int_{M^2} \mathrm{tr}(A_\mu^{ij} A_{\bar{\mu}}^{ji}) V_g$$

where $\mu = z$ or $\bar{z}$ and A_μ^{ij} are the partial second fundamental forms associated to ϕ.

Now let $[1,n] = \{x \in \mathbb{Z} ; 1 \leq x \leq n\}$. Consider $D = \{(i,i) ; 1 \leq i \leq n\}$ and S^+ to be a partition of $([1,n] \times [1,n] - D)$ containing $\frac{(n^2-n)}{2}$ elements such that if $(i,j) \in S^+$ then $(j,i) \notin S^+$. We denote S^- the complement of S^+ in $[1,n] \times [1,n] - D$. We call S^+ a positive system in $1,n$.

Let E^o and $\bar{E}$ denote the ∂ and $\bar{\partial}$-energy respectively, defined by:

$$E^o_{S^+}(\phi) = \sum_{(i,j)\in S^+} \int_{M^2} |A_z^{ij}|^2 V_g \quad \text{and}$$

$$\bar{E}_{S^+}(\phi) = \sum_{(i,j)\in S^-} \int_{M^2} |A_z^{ij}|^2 V_g = \sum_{(i,j)\in S^+} \int_{M^2} |A_{\bar{z}}^{ij}|^2 V_g$$

where S^+ is a positive system in $[1,n]$. Therefore $\phi = (\pi_1, \ldots, \pi_n) : M^2 \longrightarrow F(n)$ is holomorphic with respect to the almost complex structure determined by S^+ if and only if $\bar{E}_{S^+}(\phi) = \sum_{(i,j)\in S^+} \int_{M^2} |A_{\bar{z}}^{ij}|^2 V_g = 0$ if and only if $A_{\bar{z}}^{ij} = 0$; $\forall (i,j) \in S^+$.

3.4. DEFINITION. Let $\phi = (\pi_1, \ldots, \pi_n) : M^2 \longrightarrow F(n)$ be a harmonic map. ϕ is called totally isotropic if $[A_z, A_{\bar{z}}]_p = 0$, where $[A_z, A_{\bar{z}}]_p$ denotes the off diagonal part of the n×n-matrix $[A_z, A_{\bar{z}}]$.

We can prove:

3.5. THEOREM. Let $\phi = (\pi_1,\dots,\pi_n) : M^2 \longrightarrow F(n)$ be a totally isotropic map. Then $A_\mu^{ij} \in \mathrm{Hom}(\pi_j,\pi_i) \equiv \pi_j^* \otimes \pi_i$ is a holomorphic section of the line bundle $\pi_j^* \otimes \pi_i$ over M^2 when the total space of such bundle has a suitable complex structure.

PROOF. See [15] or [16].

We now will prove a interesting result for harmonic maps $\phi = (\pi_1,\dots,\pi_n) : T^2 = S^1 \times S^1 \longrightarrow F(n)$ which consists of a purely topological restriction for ϕ to be holomorphic with respect to some non-integrable almost complex structure on $F(n)$.

3.6. COROLLARY. Let $\phi = (\pi_1,\dots,\pi_n) : T^2 \longrightarrow F(n)$ be a totally isotropic map, holomorphic with respect to some non-integrable almost complex structure on $F(n)$ but not with respect to any integrable one. Then $C_1[\pi_1] = \dots = C_1[\pi_n] = 0$, where $C_1[\pi_i]$ denotes the first Chern number of π_i.

PROOF. We give the proof for $n=3$, and for arbitrary n the proof is entirely similar.

Without loss of generality we can assume say, that $A_z^{12} \neq 0$, $A_z^{31} \neq 0$ and $A_z^{23} \neq 0$ otherwise ϕ would be holomorphic with respect to some integrable almost complex structure.

But according to 3.5 Theorem $T(T^2)^* \otimes \mathrm{Hom}(\pi_j,\pi_i)$ has a holomorphic section if and only if $-C_1[T^2] + C_1[\pi_i] - C_1[\pi_1] =$ $= C_1[\pi_i] - C_1[\pi_j] \geq 0$.

Therefore $C_1[\pi_1] \geq C_1[\pi_2]$, $C_1[\pi_2] \geq C_1[\pi_3]$ and $C_1[\pi_3] \geq$ $\geq C_1[\pi_1]$ i.e., $C_1[\pi_1] = C_1[\pi_2] = C_1[\pi_3]$. But $\pi_1 + \pi_2 + \pi_3$ is equal to the trivial bundle over T^2, hence $C_1[\pi_1] + C_1[\pi_2] +$ $+ C_1[\pi_3] = 3C_1[\pi_1] = 0$. Therefore $C_1[\pi_1] = C_1[\pi_2] = C_1[\pi_3] = 0$.

§4. EQUIVARIANT HARMONIC MAPS INTO FLAG MANIFOLDS

In this paragraph we will study harmonic maps which are equivariant with respect to an S^1 action from $T^2 = S^1 \times S^1$ to $F(n)$.

Often interesting examples of solutions to non-linear problems are found by examining an equivariant case. The assumption of equivariance under a continuous group action whose orbits have co-dimension one in the domain manifold reduces a partial differential equation to an ordinary differential equation, then we can use the theorem of existence and uniqueness of solutions of ordinary differential equations.

Now let us recall some useful facts from the general theory of equivariant harmonic maps and the relationship between harmonic maps and minimal immersions.

Let (M^n, g) be a Riemannian manifold and $Iso(M)$ its full isometry group. Myers and Steenrod showed that $Iso(M)$ is naturally a Lie group which acts differentiably on M.

A Lie subgroup G of $Iso(M)$ is called an isometry group of M, and the co-dimension of the maximal dimensional orbits is defined to be the cohomogeneity of G. The cohomogeneity of $Iso(M)$ is called the cohomogeneity of M.

Let G be a compact, connected group of isometries of M. An immersion $f : N \to M$ is called G-invariant if there is a smooth action of G on N such that $g.f = f.g$, $\forall g \in G$.

The "submanifold" f is said to be minimal it its mean curvature vector field vanishes identically.

4.1. DEFINITION. By an equivariant variation of a G-invariant submanifold $f : N \to M$ we mean a differentiable variation $f_t : N \to M$, $-\varepsilon < t < \varepsilon$, $f_o = f$, through submanifolds such that $g.f_t = f_t.g$ for all $g \in G$ and all t. We recall the following useful result proved by Hsiang and Lawson [13].

4.2. THEOREM. Let N be a compact manifold and $f : N \longrightarrow M$ be a G-invariant submanifold of M. Then $f : N \longrightarrow M$ is minimal if and only if the volume of N is stationary with respect to all compactly supported equivariant variations.

PROOF. See [13].

Now let us recall the close relationship between harmonic maps and minimal immersions. We start by stating the following easy consequence of Riemann-Roch's theorem, namely:

4.3. PROPOSITION. If $\phi : (\mathbb{C}P^1, g) \longrightarrow (N^n, h)$ is harmonic then ϕ is conformal.

We also have:

4.4. PROPOSITION. If $\phi : (M,g) \longrightarrow (N,h)$ is a nonconstant, harmonic and conformal, then it is a minimal branched immersion; i. e., it is a conformal minimal immersion except at isolated points where $d\phi \equiv 0$, around the image of these points there are normal coordinates in which ϕ is of the form

$\phi^1(z) = C\mathrm{Re}(z^k) + (|z^k|)$, $\phi^2(z)\ C\,\mathrm{Im}(z^k) + \sigma(|z^k|), \ldots, \phi^\alpha(z) = \sigma(|z^k|)$, for all $\alpha \geq 3$.

PROOF. See [9].

Therefore every nonconstant harmonic map from $\mathbb{C}P^1$ is a minimal branched immersion.

Now let us study the differential equations found in [20], adapted to our non-symmetric case.

Consider $\rho : S^1 \longrightarrow U(n)$ given by $\rho(\exp(\sqrt{-1}\,\theta)) = \exp(A\theta)$ where A is some fixed matrix in $\mu(n)$ and we also assume $\exp(2\pi A) = I$.

Let e be a basis of $\mathbb{R}$ and consider $(d\rho)(1)(e) = A \in \mu(n)$.

Assume further that the set of equivariant harmonic maps

$$F_\rho = \{\phi \in C^\infty(S^1 \times \mathbb{R}, F(n));\ \phi(\exp(\sqrt{-1}\,\theta), t =$$

$$= \rho(\exp(\sqrt{-1}\,\theta))f(t) \text{ where } f(t) = (f_1(t),\dots,f_n(t)),$$

for all $\exp(\sqrt{-1}\,\theta) \in S^1\}$ is non-empty. Note that $U(n)$ acts on $F(n)$ by conjugation:

$$U(n) \times F(n) \longrightarrow F(n)$$

$$(A\,,X) \longrightarrow A\,XA^{-1}.$$

Let $\phi = (\pi_1,\dots,\pi_n) : S^1 \times \mathbb{R} \longrightarrow F(n)$ given by:

$(\exp(\sqrt{-1}\,\theta), t) = (\pi_1(\theta,t),\dots,\pi_n(\theta,t)) = \exp(A\theta).(f_1(t),\dots,f_n(t)) =$ $= \exp(A\theta).f(t)$, where f_i's are projection operators and $A \in \mu(n)$. Note that $\pi_i(\theta,t) = \exp(A\theta).f_i(t).\exp(-A\theta)$. We have:

4.5. LEMMA. Let $\phi(\theta,t) = (\pi_1(\theta,t),\dots,\pi_n(\theta,t)) : S^1 \times \mathbb{R} \longrightarrow F(n)$ be an equivariant harmonic map. Then:

$$E(\phi(\theta,t)) = \frac{1}{2}\sum_{i=1}^{n}\int_{S^1\times\mathbb{R}} \{\langle \frac{\partial f_i}{\partial t}, \frac{\partial f_i}{\partial t}\rangle + \langle [A,f_i(t)],[A,f_i(t)]\rangle\}\,dt$$

PROOF. We know that $\|d\pi_i\|^2 = \left\|\frac{\partial \pi_i}{\partial t}\right\|^2 + \left\|\frac{\partial \pi_i}{\partial \theta}\right\|^2$. But $\frac{\partial \pi_i}{\partial t} =$

$= \exp(A\theta)\,\frac{\partial f_i}{\partial t}\,\exp(-A\theta)$ and $\frac{\partial \pi_i}{\partial \theta} = A\,\exp(A\theta)f_i(t)\exp(-A\theta)\,-$

$-\exp(A\theta)f_i(t).A\,\exp(-A\theta) = \exp(A\theta)\,Af_i(t)\,\exp(-A\theta)\,-$

$-\,\exp(A\theta)f_i(t)A\,\exp(-A\theta) = \exp(A\theta)[A,f_i(t)]\exp(-A\theta)$. Therefore

$$\left\|\frac{\partial \pi_i}{\partial t}\right\|^2 = tr(\exp(A\theta)\,\frac{\partial f_i}{\partial t}\,\exp(-A\theta)\,\exp(A\theta).(\frac{\partial f_i}{\partial t})^*\,\exp(-A\theta)) =$$

$= tr(\frac{\partial f_i}{\partial t}\cdot(\frac{\partial f_i}{\partial t})^*) = \left\|\frac{\partial f_i}{\partial t}\right\|^2$. In a similar way, by using again the cyclic property of the trace we can see that $\left\|\frac{\partial\pi_i}{\partial\theta}\right\|^2 = \|[A,f_i]\|^2$.

Now we prove the harmonic map equations for our equivariant case.

4.6. PROPOSITION. The Euler-Lagrange equations for an equivariant harmonic map into $F(n)$ are:

$$\sum_{i=1}^{n} [f_i(t), f_i''(t) + [A,[A,f_i(t)]]] = 0 .$$

Hence the partial differential equation becomes a second order ordinary differential equation.

PROOF. By above we have:

$$E(\phi) = \frac{1}{2}\sum_{i=1}^{n}\int_{S^1\times\mathbb{R}} \{\langle\frac{\partial f_i}{\partial t},\frac{\partial f_i}{\partial t}\rangle + \langle[A,f_i],[A,f_i]\rangle\}\,dt \text{ . Hence}$$

$$(\delta E)(\delta\phi) = \mathrm{Re}\{\sum_{i=1}^{n}\int_{S^1\times\mathbb{R}} \langle\frac{\partial f_i}{\partial t},\frac{\partial}{\partial t}(\delta f_i)\rangle + \langle[A,f_i],[A,\delta f_i]\rangle\,dt\}$$

$$\underset{\text{parts}}{\overset{\text{by}}{=\!=\!=}} \mathrm{Re}\{\sum_{i=1}^{n}\int_{S^1\times\mathbb{R}} \langle -f_i'',\delta f_i\rangle + \langle -[A,[A,f_i]],\delta f_i\rangle\,dt\} =$$

$$= \mathrm{Re}\ -\{\sum_{i=1}^{n}\langle\!\langle [f_i,f_i''] + [f_i,[A,[A,f_i]]],\ q\rangle\!\rangle^2\} =$$

$$= \mathrm{Re}\{\sum_{i=1}^{n}\langle\!\langle f_i, f_i'' + [A,[A,f_i]], q\rangle\!\rangle^2\}. \text{ Therefore}$$

$\phi = \exp(A\theta)(f_1(t),\ldots,f_n(t))\exp(-A\theta) : S^1\times\mathbb{R} \longrightarrow F(n)$ is an equivariant harmonic map if and only if $\sum_{i=1}^{n}[f_i, f_i'' + [A,[A,f_i]]] = 0$

by the fundamental lemma of the calculus of variations.

Now by studying special cases of the general ordinary differential equation found in 4.6 Proposition we will construct new examples of harmonic maps $\phi = (\pi_1, \ldots, \pi_n) : T^2 = S^1 \times S^1 \longrightarrow F(n)$ which are not holomorphic with respect to any almost complex structure on $F(n)$ where $F(n)$ is equipped with the Killing form metric.

Consider a local chart $U \subseteq \mathbb{R}^2$ for a Riemann surface M^2.

Now consider B_1 and B_2 in $\mu(n)$ such that $[B_1, B_2] = 0$. Then we can define locally the following map:

$$U \xrightarrow{\ \tilde{\phi}\ } U(n)$$

$$(x,y) \longrightarrow \exp(B_1 x + B_2 y) \ .$$

We have seen that $\tilde{\phi}$ induces a map $\phi = (\pi_1, \ldots, \pi_n) : U \longrightarrow F(n)$ given by:

$$\pi_i = \tilde{\phi} E_i \tilde{\phi}^* = \exp(B_1 x + B_2 y) . E_i . \exp(-B_1 x - B_2 y)$$

where

$$E_i = \ _i\!\begin{bmatrix} 0 & & & \overset{i}{} & & & \\ & \ddots & & & & 0 & \\ & & 0 & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & 0 & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}$$

We can prove:

4.7. PROPOSITION. Let $\phi = (\pi_1, \ldots, \pi_n) : U \longrightarrow F(n)$ given by:

$\pi_i = \exp(B_1x + B_2y)E_i \exp(-B_1x - B_2y)$ where B_1, B_2 are in $\mu(n)$ and $[B_1, B_2] = 0$.

Then: $A_x = \sum_{\substack{i,j \\ i\neq j}} \exp(B_1x + B_2y)E_iB_1E_j \exp(-B_1x - B_2y)$

and

$$A_y = \sum_{\substack{i,j \\ i\neq j}} \exp(B_1x + B_2y)E_iB_2E_j \exp(-B_1x - B_2y)$$

PROOF. We will prove the expression for A_x and the one for A_y is proved similarly.

$$A_x^{ji} = \pi_j \frac{\partial \pi_i}{\partial x} = \pi_j(B_1 \exp(B_1x + B_2y)E_i \exp(-B_1x - B_2y) -$$

$$- \exp(B_1x + B_2y)E_iB_1 \exp(-B_1x - B_2y)).$$ But since $B_1.B_2 = B_2.B_1$ we have:

$$A_x^{ji} = (\exp(B_1x + B_2y)E_j \exp(-B_1x - B_2y))(B_1.\exp(B_1x + B_2y).$$

$$.E_i \exp(-B_1x - B_2y) - \exp(B_1x + B_2y)E_iB_1 \exp(-B_1x - B_2y)) =$$

$$= \exp(B_1x + B_2y).E_jB_1E_i \exp(-B_1x - B_2y).$$

Now by using the remarks after the proof of 3.2 Proposition we can find the Euler-Lagrange equations for an equivariant harmonic map.

4.8. PROPOSITION. Let $\phi = (\pi_1, \ldots, \pi_n) : U \longrightarrow F(n)$ be a smooth map such that $\pi_i = \exp(B_1x + B_2y)E_i \exp(-B_1x - B_2y)$ where B_1 and B_2 are in $\mu(n)$ and $[B_1, B_2] = 0$. Then ϕ is harmonic if and only if $[B_1, \text{diag } B_1] + [B_2, \text{diag } B_2] = 0$, where diag B_i denotes the diagonal part of B_i, $i=1,2$.

PROOF. According to 3.2 Proposition is harmonic if and only if $\frac{\partial}{\partial x}(A_x) + \frac{\partial}{\partial y}(A_y) = 0$. Hence let us compute $\frac{\partial}{\partial x}(A_x)$ and $\frac{\partial}{\partial y}(A_y)$.

$\frac{\partial}{\partial x}(A_x) = B_1 \exp(B_1x + B_2y)E_iB_1E_j \exp(-B_1x - B_2y) - \exp(B_1x +$

$+ B_2y)E_iB_1E_j \exp(-B_1x - B_2y) = \exp(B_1x + B_2y)[B_1, \sum_{\substack{i,j \\ i\neq j}} E_iB_1E_j].$

$.\exp(-B_1x - B_2y) = \exp(B_1x + B_2y).[B_1,\text{diag } B_1]\exp(-B_1x - B_2y)$.

Similarly we prove that

$\frac{\partial}{\partial y}(A_y) = \exp(B_1x + B_2y)[B_2,\text{diag } B_2]\exp(-B_1x - B_2y)$. Therefore

$\frac{\partial}{\partial x}(A_x) + \frac{\partial}{\partial y}(A_y) = 0$ if and only if $[B_1,\text{diag } B_1] + [B_2,\text{diag } B_2] = 0$.

Now let us study $f : \mathbb{R} \longrightarrow U(n)$ where $f(x) = \exp(Bx)$ and

$$B = \begin{bmatrix} 0 & \alpha\sqrt{-1} & 0 & 0 & \dots & 0 \\ \alpha\sqrt{-1} & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \beta\sqrt{-1} & \dots & 0 \\ 0 & 0 & \beta\sqrt{-1} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in \mu(n) \ , \quad n \geq 4$$

and α and β are non-zero real numbers. Then $\exp(Bx) = I + Bx +$

$+ \frac{(Bx)^2}{2!} + \dots + \frac{(Bx)^n}{n!} + \dots$. But

$$(Bx)^2 = \begin{bmatrix} -\alpha^2x^2 & 0 & 0 & 0 & \dots & 0 \\ 0 & -\alpha^2x^2 & 0 & 0 & \dots & 0 \\ 0 & 0 & -\beta^2x^2 & 0 & \dots & 0 \\ 0 & 0 & 0 & -\beta^2x^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \text{ and so on} \dots$$

Therefore $f(x) = \exp(Bx) =$

$$= \begin{bmatrix} \cos\alpha x & 0 & 0 & 0 & & 0 \\ 0 & \cos\alpha x & 0 & 0 & \dots & 0 \\ 0 & 0 & \cos\beta x & 0 & \dots & 0 \\ 0 & 0 & 0 & \cos\beta x & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} +$$

$$+ \sqrt{1} \begin{bmatrix} 0 & \sin\alpha x & 0 & 0 & \dots & 0 \\ \sin\alpha x & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \sin\beta x & \dots & 0 \\ 0 & 0 & \sin\beta x & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Now let us consider for example:

$$B_1 = \begin{bmatrix} 0 & \alpha\sqrt{-1} & 0 & 0 & \dots & 0 \\ \alpha\sqrt{-1} & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \beta\sqrt{-1} & 0 & \\ 0 & 0 & \beta\sqrt{-1} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

and

$$B_2 = \begin{bmatrix} 0 & \beta\sqrt{-1} & 0 & 0 & \dots & 0 \\ \beta\sqrt{-1} & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \alpha\sqrt{-1} & 0 & \\ 0 & 0 & \alpha\sqrt{-1} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

where α and β are non-zero real numbers such that $\alpha/_\beta$ or $\beta/_\alpha \in \mathbb{Z}$. Then B_1 and $B_2 \in \mu(n)$ and $[B_1, B_2] = 0$.

Now let us consider $\phi : \mathbb{R}^2 \longrightarrow U(n)$ given by:

$(x,y) \longrightarrow \exp(B_1 x + B_2 x)$. Since $\alpha/_\beta$ or $\beta/_\alpha \in \mathbb{Z}$ there is such that $\alpha\gamma = 2\pi\beta n_1$ or $\alpha\beta = 2\pi\alpha n_2$ where n_1, n_2 are arbitrary integers. Then $\tilde{\phi}$ induces:

$$\phi : \frac{\mathbb{R}^2}{(\mathbb{Z} \otimes \mathbb{Z})} \longrightarrow F(n) \quad \text{given by:}$$

$\phi = (x + \gamma n, y + \gamma m) = \tilde{\phi}(x,y)(E_1, \dots, E_m)\tilde{\phi}(x,y) = \exp(B_1 x + B_2 y)$. $.(E_1, \dots, E_n)\exp(-B_1 x - B_2 y)$. But $\operatorname{diag} B_1 = \operatorname{diag} B_2 = 0$. Then according to 4.8 Proposition $\phi : T^2 = S^1 \times S^1 \longrightarrow F(n)$, $n \geq 4$ is harmonic. Therefore we have proved the following result:

4.9. THEOREM. Let $\phi = (\pi_1, \dots, \pi_n) : T^2 \longrightarrow F(n)$ where $\pi_i = \exp(B_1 x + B_2 y).E_i.\exp(-B_1 x - B_2 y)$ where $B_1, B_2 \in \mu(n)$, $n \geq 4$ are as above. Then ϕ is harmonic but is not holomorphic with respect to any almost complex structure on $F(n)$.

PROOF. By above we have seen that ϕ is harmonic. On the other hand, $A_z^{12} = A_x^{12} + \sqrt{-1}\ A_y^{12}$ and $A_z^{21} = A_x^{21} + \sqrt{-1}\ A_y^{21}$ are both non-zero according to 4.7 Proposition. Therefore ϕ is not holomorphic with respect to any almost complex structure on $F(n)$.

Clearly the same method produces different examples of harmonic maps $\phi : T^2 \longrightarrow F(n)$ that are not holomorphic with respect to any almost complex structure on $F(n)$. It is a intersting problem to classify all harmonic maps from T^2 to $F(n)$.

Finally, we want to point out that we cannot expect to use this equivariant harmonic map method developed above to produce harmonic maps $\phi = (\pi_1,\dots,\pi_n) : \mathbb{C}P^1 \longrightarrow F(n)$, because in general $tr(A_z^2) \neq 0$ which contradicts 3.3 Theorem. In fact: $tr(A_z^2) =$ $= tr(A_x^2 + A_y^2) + 2\sqrt{-1}\ tr(A_x.A_y)$. But $tr(A_x^2) = tr(B_1^2)$ and $tr(A_y^2) =$ $= tr(B_2^2)$ hence $tr(A_x^2 + A_y^2) \neq 0$ in general. This fact provides another piece of evidence to our conjecture that any harmonic map $\phi : \mathbb{C}P^1 \longrightarrow F(n)$ must be holomorphic with respect to some almost complex structure on $F(n)$. See [15] or [16] for more details.

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