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On energy-critical nonlinear Schrödinger systems

Um estudo sobre sistemas não lineares de Schrödinger com energia crítica

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RESUMO

O principal objetivo desta tese é estudar sistemas acoplados de equações de Schrödinger não linear no caso em que o espaço de energia $\dot{H}^1(\mathbb{R}^d)$ é crítico em relação ao *scaling*. O estudo é dividido em duas partes. Na primeira estudamos um sistema com não linearidades cúbicas, onde o espaço de energia crítica é o $\dot{H}^1(\mathbb{R}^4)$. Nosso principal objetivo é mostrar *blow-up* em tempo finito para soluções cujo o dado inicial é radialmente simétrico. Começamos aplicando o método do ponto fixo para mostrar boa colocação local do problema de Cauchy associado. Em seguida, provamos existência de soluções *ground state*. Para tal, utilizamos o método de concentração e compacidade para encontrar uma solução para um problema de minimização restrito deduzido a partir de uma desigualdade crítica do tipo Sobolev. Por fim, para obter o resultado de *blow-up* em tempo finito, utilizamos uma versão modificada do método de convexidade.

A segunda parte trata de um sistema com não linearidades gerais com crescimento do tipo quadrático, onde o espaço de energia é $\dot{H}^1(\mathbb{R}^6)$. Aqui o principal objetivo é provar um resultado de *scattering* e boa colocação global. Iniciamos provando boa colocação local, onde também utilizamos o método do ponto fixo, entretanto, provamos o resultado com o dado inicial no espaço de Sobolev não homogêneo $H^1(\mathbb{R}^6)$ e, em seguida, mostramos um resultado de estabilidade que nos permite trabalhar com o dado inicial no espaço de Sobolev homogêneo $\dot{H}^1(\mathbb{R}^6)$. Para provar a existência global, utilizamos o método de concentração-compacidade e rigidez, que consiste em admitir que o resultado é falso e provar a existência de uma solução particular, chamada de solução crítica. Em seguida, provamos que tal solução não pode existir, chegando em uma contradição.

Palavras-chave: Sistema de equações de Schrödinger; Energia crítica; Soluções *ground state*; *Blow-up*; Boa colocação; Scattering;

ABSTRACT

The main goal of this thesis is to study coupled systems of nonlinear Schrödinger equations in the case where the energy space $\dot{H}^1(\mathbb{R}^d)$ is critical with respect to scaling. The study is divided into two parts. In the first one, we study a system with cubic nonlinearities, where the critical energy space is $\dot{H}^1(\mathbb{R}^4)$. Our main objective is to show blow-up in finite time for solutions whose initial data is radially symmetric. We start by applying the fixed point method to show the local well-posedness of the associated Cauchy problem. Next, we prove the existence of ground state solutions. To this end, we use the concentration-compactness method to find a solution of a restricted minimization problem deduced from a critical Sobolev-type inequality. Finally, to obtain the blow-up in finite time result, we use a modified version of the convexity method.

The second part of the work deals with a system with general nonlinearities with quadratic growth. In contrast, the critical energy space is $\dot{H}^1(\mathbb{R}^6)$. Our main goal is to prove a scattering result and global well-posedness. We start by proving local well-posedness, where we use the fixed point method. However, we prove the result with the initial data in the inhomogeneous Sobolev space $H^1(\mathbb{R}^6)$ and then we show a stability result that allows us to work with the initial data in the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^6)$. To show the global existence, we use the method that consists of admitting that the result is false and proving the existence of a particular solution, called the critical solution. Then, we prove that such type of solutions cannot exist, arriving at a contradiction.

Keywords: Schrödinger systems; Energy critical; Well-posedness; Ground state solution; Blow-up; Scattering.

LIST OF SYMBOLS

$\mathbb{N}$	the set of natural number.
$\mathbb{R}$	the set of real numbers.
$\mathbb{R}_+$	the set of nonnegative real numbers.
$\mathbb{C}$	the set of complex numbers.
$\mathbb{R}^d$	the d-dimensional Euclidean space.
$\mathbb{C}^d$	the d-dimensional complex space.
$\mathrm{Re}(z)$	the real part of the complex number z.
$\mathrm{Im}(z)$	the imaginary part of the complex number z.
$\bar{z}$	conjugate of a complex number z.
$\vert \cdot \vert$	the euclidean norm $\sqrt{x_1^2 + ... + x_d^2}$, where $x \in \mathbb{R}^d$.
$\Vert \cdot \Vert_X$	norm in the space X.
$\partial_{x_i} = \partial_i$	the partial derivative, $\frac{\partial}{\partial x_i}$, with respect to the x_i variable.
u_{x_i}	$= \frac{\partial u}{\partial x_i} = \partial_i u$.
u_t	$= \frac{\partial u}{\partial t} = \partial_t u$.
∇u	the gradient vector $(u_{x_1}, ..., u_{x_d})$.
Δu	the Laplacian operator $\sum_{i=1}^{d} u_{x_i x_i}$.

B_r	$= \{x \in \mathbb{R}^n; \lvert x\rvert < r\}$.
$\int f$	$= \int_{\mathbb{R}^d} f(x)dx$.
$\mathcal{C}(X)$	the set of continuous functions in X.
$\mathcal{C}^k(X)$	the set of functions with continuous derivatives of order $k \geqslant 0$ in X.
$\mathcal{C}_0^\infty(X)$	the set of all functions in the class C^∞ with compact support in X.
$L^p(X)$	the Lebesgue space of all p-integrable functions.
$H^{s,p}(X)$	$(s \in \mathbb{R}, 1 \leqslant p \leqslant \infty)$ Sobolev spaces.
$H^s(X)$	$= H^{s,2}(X)$.
$\dot{H}^{s,p}$	the homogeneous (generalized) Sobolev space.
$\dot{H}^s(X)$	$= \dot{H}^{s,2}(X)$.
$\mathcal{S}'(X)$	the space of Schwatrz functions in X.
$\mathcal{F}(f) = \hat{f}$	the Fourier transform.
$\mathcal{F}^{-1}(f) = \check{f}$	the inverse Fourier transform.
$e^{it(a\Delta+b)}u_0$	$= \left(e^{-it(a\lvert\xi\rvert^2-b)}\hat{u}_0\right)^\vee$ with a and b constants.
$X \hookrightarrow Y$	continuous inclusion of space X into space Y.
$\mathbf{A} = A^l$	the product $A \times ... \times A$ (l times).
$\mathbf{v}$	the vector $(v_1, ..., v_l)$.
$\lvert\mathbf{u}\rvert$	the vector $(\lvert u_1\rvert, ..., \lvert u_l\rvert)$.
$(\delta_\lambda f)(x)$	the dilation by $1/\lambda$, that is, $(\delta_\lambda f)(x) = f(x/\lambda)$.
C	a constant that may change from one line to the next.

Contents

CHAPTER 1

INTRODUCTION

In this work we will study to distinct nonlinear systems of Schrödinger equations. The first one is the following cubic-type system,

$$\begin{cases} iu_t + \Delta u - u + \left(\dfrac{1}{9}|u|^2 + 2|w|^2\right) u + \dfrac{1}{3}\bar{u}^2 w = 0, \\ i\sigma w_t + \Delta w - \mu w + (9|w|^2 + 2|u|^2)w + \dfrac{1}{9}u^3 = 0, \end{cases} \tag{1.1}$$

where $u = u(t,x)$ and $w = w(t,x)$ are complex valued functions with $(t,x) \in \mathbb{R} \times \mathbb{R}^4$, Δ represents the standard Laplacian operator and $\sigma, \mu > 0$. This model describes the interaction between an optical beam and its third harmonic in a material with Kerr-type nonlinear response. For a more detailed explanation of the model, the reader can check (SAMMUT; BURYAK; KIVSHAR, 1998).

The second one is a l-component nonlinear Schrödinger system with quadratic-growth nonlinearities. Precisely, we will show a scattering result and global well-posedness to the following Cauchy problem

$$\begin{cases} i\alpha_k \partial_t u_k + \gamma_k \Delta u_k = -f_k(u_1, ..., u_l), \\ (u_1(0,x), ..., u_l(0,x)) = (u_{10}, ..., u_{l0}), \quad k = 1, ..., l, \end{cases} \tag{1.2}$$

where $u_1, ..., u_l$ are complex-valued functions on the variables $(t,x) \in \mathbb{R} \times \mathbb{R}^6$, $\alpha_k, \gamma_k > 0$, are real constants and the nonlinearities $f_k : \mathbb{C}^l \longrightarrow \mathbb{C}$ satisfy a quadratic-type growth.

The main goal of this work is to study nonlinear systems of Schrödinger equations in the energy-critical case. This term comes from the fact that not only the class of solutions, but also the energy, are left invariant under the transformation

$$f(t,x) \mapsto f_\lambda(t,x) := \lambda^{\frac{d-2}{2}} f(\lambda^2 t, \lambda x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \tag{1.3}$$

called scaling symmetry. This defines a notion of criticality. Precisely, a quick computation shows that $\dot{H}^1(\mathbb{R}^d)$ is the critical (scaling invariant) Sobolev space if $p := (d+2)/(d-2)$, where p denotes the power of the nonlinearities. Therefore, the critical dimensions for systems (1.1) and (1.2) are, respectively, $d = 4$ and $d = 6$.

The first one is devoted to study system (1.1). From a mathematical point of view, system (1.1) has been studied in several cases. In (PAVA; PASTOR, 2009), the authors established local and global well-posedness for the associated Cauchy problem with periodic initial data in dimension one. Also in one space dimension, (PASTOR, 2010) is concerned with nonlinear and spectral stability of periodic traveling wave solutions. The author proved the existence of two smooth curves of periodic solutions depending on the cnoidal type functions and a stability result under perturbations having the same minimal wavelength and zero mean over their fundamental period. For the multidimensional case, (OLIVEIRA; PASTOR, 2021) proved the existence and stability of ground state solutions, the local and global well-posedness and established several criteria for blow-up in finite time in the energy space $H^1(\mathbb{R}^d)$. In (RAMADAN; STEFANOV, 2024), the authors studied solitary waves for (1.1). They constructed the waves in largest possible parameter space and provided a complete classification of their stability. In (COLIN; WATANABE, 2023), it was proved the existence of stable standing wave solutions as well as the correspondence between minimizers and ground state solutions. In the three dimensional case, (ARDILA; DINH; FORCELLA, 2021) studied the asymptotic dynamics for solutions to (1.1). They provided sharp threshold criteria leading to global well-posedness and scattering of solutions, as well as formation of singularities in finite time for symmetric initial data. Also, in (ZHANG; DUAN, 2023), it was proved existence results for normalized ground state solutions in the L^2-subcritical case and L^2-supercritical cases and established the nonexistence of normalized ground state solutions in the L^2-critical case and a new blow-up criterion which is related to normalized solutions.

Our goal is to study the system in the energy space $H^1(\mathbb{R}^d)$. This terminology comes from the fact that, at least in a formal level, the system conserves energy and mass, respectively given by,

$$E(u,w) := \frac{1}{2}\int(|\nabla u|^2+|\nabla w|^2+|u|^2+\mu|w|^2)-\int\left(\frac{1}{36}|u|^4+\frac{9}{4}|w|^4+|u|^2|w|^2+\frac{1}{9}\mathrm{Re}(\bar{u}^3 w)\right) \tag{1.4}$$

and

$$M(u,w) := \int(|u|^2+3\sigma|w|^2). \tag{1.5}$$

Our main goal is to prove existence of blow-up solutions for system (1.1). To do this, we will use the ideas presented in (NOGUERA; PASTOR, 2022).

First, we establish the local well-posedness for the Cauchy problem associated to (1.1). We set the space

$$Y(I) := (\mathcal{C} \cap L_t^\infty H_x^1) \cap L_t^4 H_x^{1,8/3}, \tag{1.6}$$

for a time interval $I = [-T, T]$ with $T > 0$. The result is the following.

Theorem 1.1. *For any $u_0, w_0 \in H^1(\mathbb{R}^4)$, there exists $T(u_0, w_0) > 0$, such that the system (1.1) admits a unique solution $(u, w) \in Y(I) \times Y(I)$, with $I = [-T(u_0, w_0), T(u_0, w_0)]$. In addition, the following blow-up alternative holds: There exist times $T_*, T^* \in (0, \infty)$such that the solution can be extended to $(-T_*, T^*)$ and if $T^* < \infty$, then*

$$\lim_{t \to T^*} (\|\nabla u(t)\|_{L_t^q L_x^r} + \|\nabla w(t)\|_{L_t^q L_x^r}) = \infty,$$

for any pair (q, r) satisfying $2 \leqslant q, r \leqslant \infty$, $\dfrac{2}{q} = 2 - \dfrac{4}{r}$. A similar result holds if $T_* < \infty$.

We establish the local well-posedness using the fixed point method to find solutions of the equivalent integral equations

$$\begin{cases} u(t) = U(t)u_0 + i\displaystyle\int_0^t U(t-s)F(u(s), w(s))ds, \\ w(t) = W(t)w_0 + i\displaystyle\int_0^t W(t-s)G(u(s), w(s))ds, \end{cases} \tag{1.7}$$

where $U(t) = e^{it(\Delta+1)}$, $W(t) = e^{it(a\Delta+b)}$, are the corresponding unitary groups associated to the linear part of (1.1), with $a = 1/\sigma$, $b = \mu/\sigma$, and

$$F(u, w) = \left(\frac{1}{9}|u|^2 + 2|w|^2\right) u + \frac{1}{3}\bar{u}^2 w \quad \text{and} \quad G(u, w) = a(9|u|^2 + 2|w|^2)w + \frac{1}{9}au^3, \tag{1.8}$$

are the nonlinearities. This will be addressed in Section 3.1

Next, in Section 3.2, we study a special class of solutions called *ground states* which are defined as follows. Recall that standing waves are solutions of (1.1) of the form

$$u(t, x) = e^{i\omega t}P(x), \quad w(t, x) = e^{3i\omega t}Q(x), \tag{1.9}$$

where P and Q are real functions with fast decay at infinity. Using (1.9) in (1.1), one can see that (P, Q) must satisfy

$$\begin{cases} \Delta P - (\omega + 1)P + \left(\dfrac{1}{9}P^2 + 2Q^2\right) P + \dfrac{1}{3}P^2 Q = 0, \\ \Delta Q - (\mu + 3\sigma\omega)Q + (9Q^2 + 2P^2)Q + \dfrac{1}{9}P^3 = 0. \end{cases} \tag{1.10}$$

It is known from (OLIVEIRA; PASTOR, 2021), Lemma 2.2, that if $(P, Q) \in H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d)$ is a solution to (1.10) then the identity

$$(d-4)\int (|\nabla P|^2 + |\nabla Q|^2)dx + d(\omega + 1)\int P^2 dx + d(\mu + 3\sigma\omega)\int Q^2 dx = 0. \tag{1.11}$$

is satisfied for all t. Thus, if $d = 4$ then

$$(\omega + 1)\int P^2 dx + (\mu + 3\sigma\omega)\int Q^2 dx = 0,$$

which implies that the system has non-trivial solution only if $\omega = -1$ and $\mu = 3\sigma$. In these conditions, the system (1.10) reduces to

$$\begin{cases} \Delta P + \left(\frac{1}{9}P^2 + 2Q^2\right)P + \frac{1}{3}P^2 Q = 0, \\ \Delta Q + (9Q^2 + 2P^2)Q + \frac{1}{9}P^3 = 0, \end{cases} \tag{1.12}$$

and the corresponding action functional can be written as

$$S(P, Q) = \frac{1}{2}K(P, Q) - N(P, Q), \tag{1.13}$$

where

$$K(P, Q) = \int \left(|\nabla P|^2 + |\nabla Q|^2\right) dx, \quad N(P, Q) = \int \left(\frac{1}{36}P^4 + \frac{9}{4}Q^4 + P^2Q^2 + \frac{1}{9}P^3 Q\right) dx. \tag{1.14}$$

Precisely, we have the definition

Definition 1.2. *We say that*

(i) A pair of functions $(P, Q) \in \dot{H}^1(\mathbb{R}^4) \times \dot{H}^1(\mathbb{R}^4)$ *is a weak solution to* (1.12)*, if for all* $(f, g) \in \dot{H}^1(\mathbb{R}^4) \times \dot{H}^1(\mathbb{R}^4)$,

$$\begin{aligned} \int \nabla P \cdot \nabla f dx &= \int \left(\frac{1}{9}P^3 + 2Q^2 P + \frac{1}{3}P^2 Q\right) f dx, \\ \int \nabla Q \cdot \nabla g dx &= \int \left(9Q^3 + 2P^2 Q + \frac{1}{9}P^3\right) g dx. \end{aligned} \tag{1.15}$$

(ii) A solution $(P_0, Q_0) \in \dot{H}^1(\mathbb{R}^4) \times \dot{H}^1(\mathbb{R}^4)$ *is a ground state of* (1.12) *if*

$$S(P_0, Q_0) = \inf\{S(P, Q);\ (P, Q) \in \mathcal{C}\}$$

where $\mathcal{C}$ *denotes the set of all non-trivial solutions of* (1.12)*. The set of all ground states of* (1.12) *will be denote by* $\mathcal{G}$*.*

The main result of Section 3.2 is the following.

Theorem 1.3. *There exists a ground state solution* (P_0, Q_0) *for system* (1.12)*, i.e.,* $\mathcal{G}$ *is non-empty.*

For this purpose, we shall use the concentration-compactness method, introduced in (LIONS, 1985). We proceed in three steps. First, we deduce a critical Sobolev-type inequality corresponding to our system and derive a minimization problem. Additionally, we establish a localized version of the Sobolev inequality, which will be useful to our purpose. Later, we prove a result inspired in the limit case lemma presented in (LIONS, 1985), which is called concentration-compactness lemma II. Finally, the third step is to prove that the minimization problem has a minimizer, implying the existence of a ground state solution. Finally, we establish an optimal constant for the minimization problem.

Finally, in Section 3.3, we prove the main result of this part,

Theorem 1.4. *Suppose* $(u_0, w_0) \in H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4)$ *and let* (u, w) *be the corresponding solution of* (1.1) *defined in the maximal time interval of existence* I. *If* (u_0, w_0) *is a pair of radially symmetric functions satisfying*

$$E(u_0, w_0) < \mathcal{E}(P, Q) \tag{1.16}$$

$$K(u_0, w_0) > K(P, Q), \tag{1.17}$$

where (P, Q) *is any ground state solution, and* $\mathcal{E}$ *is the energy defined in* (3.25), *then the time interval* I *is finite.*

As usual we use the convexity method to obtain this kind of result, which consist in deriving a contradiction by working with the virial identity

$$\mathcal{V}(t) = \int \phi(x)|u(t,x)|^2 dx + \int \phi(x)\sigma^2|w(t,x)|^2 dx,$$

where $\phi \in C_0^\infty(\mathbb{R}^4)$, and its derivative

$$\mathcal{V}'(t) = 2\mathrm{Im}\int \nabla\phi(\nabla u\bar{u} + \sigma\nabla w\bar{w})dx - 4\int \phi\mathrm{Im}\left(\frac{1}{2}\bar{u}f(u,w) + \frac{\sigma}{2}\bar{w}f(u,w)\right)dx, \tag{1.18}$$

with $f(u,w) = \left(\frac{1}{9}|u|^2 + 2|w|^2\right)u + \frac{1}{3}\bar{u}^2 w$ and $g(u,w) = (9|w|^2 + 2|u|^2)w + \frac{1}{9}u^3$.

Notice that the second term in (1.18) does not vanishes necessarily , which brings some difficulties in order to apply the method. To avoid this problem, we used a modification of the method presented in (INUI; KISHIMOTO; NISHIMURA, 2020), which consists in working with radially symmetric solutions and the function

$$\mathcal{R}(t) = 2\mathrm{Im}\int_{\mathbb{R}^4} \nabla\phi(\nabla u\bar{u} + \sigma\nabla w\bar{w})dx \tag{1.19}$$

instead of the usual $\mathcal{V}$.

In the second part of this work, we will be focused on system (1.2). In this case we will assume that the nonlinearities f_k, $k = 1, \ldots, l$ satisfies the following hypothesis

(H1)

$$f_k(\mathbf{0}) = 0, \quad k = 1, \ldots, l.$$

(**H2**) For all $\mathbf{z}, \mathbf{z}' \in \mathbb{C}^l$

$$\left|\frac{\partial}{\partial z_m}[f_k(\mathbf{z}) - f_k(\mathbf{z}')]\right| + \left|\frac{\partial}{\partial \bar{z}_m}[f_k(\mathbf{z}) - f_k(\mathbf{z}')]\right| \lesssim \sum_{j=1}^{l} |z_j - z_j'|, \quad k, m = 1, ..., l.$$

(**H3**) There exists a function $F : \mathbb{C}^l \longrightarrow \mathbb{C}$, such that

$$f_k(\mathbf{z}) = \frac{\partial F}{\partial \bar{z}_k}(\mathbf{z}) + \overline{\frac{\partial F}{\partial z_k}}(\mathbf{z}), \quad k = 1, ..., l.$$

(**H4**) For all $\theta \in \mathbb{R}$ and $\mathbf{z} \in \mathbb{C}^l$,

$$\mathrm{Re}F\left(e^{i\frac{\alpha_1}{\gamma_1}\theta} z_1, ..., e^{i\frac{\alpha_l}{\gamma_l}\theta} z_l\right) = \mathrm{Re}F(\mathbf{z}).$$

(**H5**) The function F is homogeneous of degree 3, that is, for all $\mathbf{z} \in \mathbb{C}^l$ and $\lambda > 0$, it holds

$$F(\lambda \mathbf{z}) = \lambda^3 F(\mathbf{z}).$$

(**H6**) It holds

$$\left|\mathrm{Re}\int_{\mathbb{R}^d} F(\mathbf{u})dx\right| \leqslant \int_{\mathbb{R}^d} F(\,|\mathbf{u}|\,)dx.$$

(**H7**) The function F is real-valued in $\mathbb{R}^l$, that is, if $(y_1, ..., y_l) \in \mathbb{R}^l$ then

$$F(y_1, ..., y_l) \in \mathbb{R}.$$

Moreover, the functions f_k are nonnegative on the positive cone on $\mathbb{R}^l$, that is, for $y_i \geqslant 0$, $i = 1, ..., l$

$$f_k(y_1, ..., y_l) \geqslant 0.$$

(**H8**) The function F may be written as a sum $F_1 + ... + F_m$, where F_s, $s = 1, ..., m$, is super-modular on $\mathbb{R}^d_+$, $1 \leqslant d \leqslant l$, and vanishes on hyperplanes, that is, for any $i, j \in \{1, ..., d\}$, $i \neq j$ and $k, h > 0$, we have

$$F_s(y + he_i + ke_j) + F_s(y) \geqslant F_s(y + he_i) + F_s(y + ke_j), \quad y \in \mathbb{R}^d_+,$$

and $F_s(y_1, ..., y_d) = 0$ if $y_j = 0$ for some $j \in \{1, ..., d\}$.

Remark 1.5. *The following system is an example satisfying the conditions (H1)-(H8)*

$$\begin{cases} i\partial_t u_1 + \Delta u_1 = -2\bar{u}_1 u_2, \\ i\partial_t u_2 + \kappa\Delta u_2 = -u_1^2, \end{cases} \tag{1.20}$$

where $F(z_1, z_2) = \bar{z}_1^2 z_2$. Other models with quadratic-type growth nonlinearities satisfying (H1)-(H8) can be found in (KIVSHAR et al., 2000), (NOGUERA; PASTOR, 2022) and (PASTOR, 2019)

Hypothesis (**H3**) – (**H5**) guarantee that the system (1.2) conserves both mass and energy given, respectively, by

$$Q(\mathbf{u}(t)) := \sum_{k=1}^{l} \frac{\alpha_k^2}{\gamma_k} \|u_k(t)\|_{L^2}^2, \tag{1.21}$$

and

$$E(\mathbf{u}(t)) := \sum_{k=1}^{l} \gamma_k \|\nabla u_k\|_{L^2}^2 - 2\mathrm{Re} \int_{\mathbb{R}^d} F(\mathbf{u}(t))dx, \tag{1.22}$$

that is, provided there exists a solution to the system, then

$$Q(\mathbf{u}(t)) = Q(\mathbf{u}_0) \quad \text{and} \quad E(\mathbf{u}(t)) = E(\mathbf{u}_0).$$

We also denote the kinetic energy and potential energy, respectively, by

$$K(\mathbf{u}) = \sum_{k=1}^{l} \gamma_k \|\nabla u_k\|_{L^2}^2 \quad \text{and} \quad P(\mathbf{u}) = \mathrm{Re} \int_{\mathbb{R}^d} F(\mathbf{u}(t))dx.$$

Thus, with this notation, the energy becomes $E(\mathbf{u}) = K(\mathbf{u}) - 2P(\mathbf{u})$.

This kind of system has been studied in (NOGUERA; PASTOR, 2021), where the local and global well-posedness was proved on $L^2(\mathbb{R}^d)$ and $H^1(\mathbb{R}^d)$, $1 \leqslant d \leqslant 6$, existence and stability/instability of ground state solutions, and the dichotomy global existence versus blow-up in finite time, in the cases $1 \leqslant d \leqslant 5$. In (NOGUERA; PASTOR, 2022) was treated the H^1 critical, that is, when $d = 6$. The authors proved existence of ground state solutions and conditions to a radial solution blow-up in finite time. On both works, the hypothesis (H4) was replaced by

(**H4***) There are positive constants $\sigma_1, ..., \sigma_l$ such that for any $\mathbf{z} \in \mathbb{C}^l$

$$\mathrm{Im} \sum_{k=1}^{l} \sigma_k f_k(\mathbf{z}) \bar{z}_k = 0.$$

Recall that, a ground state solution in $\mathbb{R}^6$ is a solution to the elliptic system

$$-\gamma_k \Delta \psi_k = f_k(\boldsymbol{\psi}), \quad k = 1, .., l, \tag{1.23}$$

where ψ_k are real-valued functions with decay to zero at infinity. Under our hypothesis (see (NOGUERA; PASTOR, 2022), Theorem 3.3), the set of ground state solutions of (1.23), denoted by $\mathcal{G}_6$ is nonempty if $d = 6$. Besides that, we have the following Gagliardo-Nirenberg inequality (see (NOGUERA; PASTOR, 2022), Corollary 4.12),

$$P(\mathbf{u}) \leqslant C_6 K(\mathbf{u})^{3/2}, \tag{1.24}$$

for all functions $\mathbf{u} \in \mathcal{D} := \{\boldsymbol{\psi} \in \dot{\mathbf{H}}^1(\mathbb{R}^6); P(\boldsymbol{\psi}) > 0\}$, with optimal constant C_6 given by

$$C_6 := \frac{1}{3^{3/2}} \frac{1}{E(\boldsymbol{\psi})^{1/2}} = \frac{1}{3} \frac{1}{K(\boldsymbol{\psi})^{1/2}}. \tag{1.25}$$

where $\boldsymbol{\psi}$ is a ground state solution to (1.23) (see (NOGUERA; PASTOR, 2022), Corollary 3.14).

Turning back to our problem, we start with the following definitions.

Definition 1.6. *(Solution) By a solution to the system* (1.2) *we will understand a function* $\mathbf{u} : I \times \mathbb{R}^6$*, defined on a non-empty time interval* $I \subset \mathbb{R}$*, with* $0 \in I$*, if it lies in the class* $\mathbf{C}_t^0\dot{\mathbf{H}}_x^1(K \times \mathbb{R}^6) \cap \mathbf{L}_{t,x}^4(K \times \mathbb{R}^6)$ *for all compact interval* $K \subset I$*, and satisfy the Duhamel formula*

$$\begin{cases} u_k(t) = U_k(t)u_{k0} + i\displaystyle\int_0^t U_k(t-s)\frac{1}{\alpha_k}f_k(\mathbf{u})ds, \\ (u_1(0,x), ..., u_l(0,x)) = (u_{10}, ..., u_{l0}) := \mathbf{u}_0, \end{cases} \tag{1.26}$$

where $U_k(t)$ *denotes the corresponding unitary group defined by* $U_k(t) = e^{it\frac{\gamma_k}{\alpha_k}\Delta}$, $k = 1, ..., l$*, and* $t \in I$*. The interval* I *is said to be the lifespan of* $\mathbf{u}$*. We say that* $\mathbf{u}$ *is a maximal solution if the solution cannot be extended to an interval* $J \supset I$ *strictly larger then* I*. We say that the solution is global if* $I = \mathbb{R}$*.*

Definition 1.7. *(Scattering size). Let* $\mathbf{u}$ *be a solution of* (1.2)*. The scattering size of* $\mathbf{u}$ *on a time interval* I *is defined as*

$$S_I(\mathbf{u}) := \sum_{k=1}^{l}\int_I\int_{\mathbb{R}^6}|u_k(t,x)|^4dxdt.$$

Definition 1.8. *(Blow-up) We say that a solution* $\mathbf{u}$ *of* (1.2) *blows-up forward in time if there exists* $t_1 \in I$ *such that*

$$S_{[t_1,\sup I)}(\mathbf{u}) = \infty,$$

and $\mathbf{u}$ *blows-up backward in time, if there exists* $t_2 \in I$ *such that*

$$S_{(\inf I, t_2]}(\mathbf{u}) = \infty.$$

We say that $\mathbf{u}$ *blows-up in finite time, if it blows-up both forward and backward in time.*

The local theory for (1.2) will be treated in Chapter 4, Section 4.1. We summarize the results in the following theorem.

Theorem 1.9. *Given* $\mathbf{u}_0 \in \dot{\mathbf{H}}_x^1(\mathbb{R}^6)$*, there exists a unique maximal-lifespan solution* $\mathbf{u} : I \times \mathbb{R}^6 \to \mathbb{C}$ *to* (1.2) *with initial data* $\mathbf{u}(0) = \mathbf{u}_0$*. This solution has the following properties:*

- *(Local existence)* I *is an open neighborhood of* 0*.*
- *(Blow-up criterion) If* $\sup(I)$ *is finite, then* $\mathbf{u}$ *blows-up forward in time; if* $\inf(I)$ *is finite, then* $\mathbf{u}$ *blows-up backward in time.*

- *(Small data global existenece) If* $\|\nabla \mathbf{u}_0\|_{\mathbf{L}^2}$ *is sufficiently small, then* $\mathbf{u}$ *is a global solution which does not blow-up either forward or backward in time. Indeed, in this case,* $S_{\mathbb{R}}(\mathbf{u}) \lesssim \|\nabla \mathbf{u}_0\|_{\mathbf{L}^2}^4$.

The main result of this part of the work is the following.

Theorem 1.10. *(Spacetime bounds). Let* $d = 6$ *and assume (H1)-(H8). Consider* $\mathbf{u}_0 \in \dot{\mathbf{H}}_x^1$ *and* $\mathbf{u} : I \times \mathbb{R}^6 \longrightarrow \mathbb{C}^l$ *the corresponding solution to* (1.2)*. Let* $\boldsymbol{\psi} \in \mathcal{G}_6$ *be a ground state. If*

$$E(\mathbf{u}_0) < E(\boldsymbol{\psi}), \tag{1.27}$$

and

$$K(\mathbf{u}_0) < K(\boldsymbol{\psi}), \tag{1.28}$$

then

$$S_I(\mathbf{u}) < \infty.$$

Corollary 1.11. *(Global well-posedness and Scattering). Let* $\mathbf{u}$ *be a maximal solution to* (1.2) *on the time interval* I*. Assume also* (1.27) *and* (1.28)*. Then* $I = \mathbb{R}$ *and*

$$S_{\mathbb{R}}(\mathbf{u}) < \infty. \tag{1.29}$$

In particular, the solution scatters, that is, there exist asymptotic states $\mathbf{u}^{\pm} \in \dot{\mathbf{H}}_x^1$ *such that*

$$\|\mathbf{u}(t) - \mathbf{U}(t)\mathbf{u}^{\pm}\|_{\dot{\mathbf{H}}_x^1} \to 0 \quad as \quad t \to \pm\infty,$$

where $\mathbf{U}(t) = (U_1(t), ..., U_l(t))$.

As we will see, the result in Theorem 1.10 is sharp, in the sense that if we reverse inequality (1.28) then the corresponding solution blows-up in finite time. Precisely, we have the following result.

Theorem 1.12. *Let* $\mathbf{u}_0 \in \dot{\mathbf{H}}^1$ *and let* $\mathbf{u}$ *be the corresponding solution of* (1.2) *defined in the maximal time interval of existence* I*. Assume that*

$$E(\mathbf{u}_0) < E(\boldsymbol{\psi}), \tag{1.30}$$

and

$$K(\mathbf{u}_0) > K(\boldsymbol{\psi}). \tag{1.31}$$

Then, if $x\mathbf{u}_0 \in \mathbf{L}^2(\mathbb{R}^6)$ *or* $\mathbf{u}_0 \in \dot{\mathbf{H}}^1$ *is radially symmetric we have that* I *is finite.*

Remark 1.13. *In the radial case, Theorem 1.12 was proved in Theorem 4.1 (ii) of (NOGUERA; PASTOR, 2022).*

To prove Theorem 1.10, we follow the ideas presented in (KILLIP; VISAN, 2010), which basically consists in assuming that the conclusion of Theorem 1.10 is false and construct a special type of solution, called critical solution, which we will prove that cannot exist. This method is called concentration/compactness and rigidity argument and was first introduced by (KENIG; MERLE, 2006). This will be organized as follows.

In Section 4.1 we will prove the local well-posedness in $\dot{\mathbf{H}}^1_x$ by using the approach presented in (KILLIP; VISAN, 2013).We start proving local well-posedness assuming that the initial data belongs to the inhomogeneous Sobolev space $\mathbf{H}^1_x(\mathbb{R}^6)$, using the usual method of contraction presented in (CAZENAVE, 2003). The next step is to present some stability results which allows us to prove continuous dependence of the solution $\mathbf{u}$ upon the initial data $\mathbf{u}_0$ in the critical space $\dot{\mathbf{H}}^1_x$. This allows us to treat the initial data in the homogeneous Sobolev space $\dot{\mathbf{H}}^1_x$, since every function in $\dot{\mathbf{H}}^1_x$ can be well approximated by $\mathbf{H}^1_x$ functions. At the end of the section, we will prove a standard blow-up result.

In Section 4.2, we will prove the existence of critical solutions. We will see that such solutions have many properties, one of then is almost periodicity modulo symmetries, which we define as follows.

Definition 1.14. *(Almost periodicity modulo symmetries). A solution $\mathbf{u}$ to (1.2) on a time interval I is said to be almost periodic modulo symmetries if there exist functions $N : I \longrightarrow \mathbb{R}^+$, $x : I \longrightarrow \mathbb{R}^6$ and $C : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, such that for all $t \in I$ and $\eta > 0$:*

$$\sum_{k=1}^{l} \int_{|x-x(t)|\geqslant C(\eta)/N(t)} \gamma_k |\nabla u_k(t,x)|^2 dx \leqslant \eta$$

and

$$\sum_{k=1}^{l} \int_{|\xi|\geqslant C(\eta)N(t)} \gamma_k |\xi|^2 |\hat{u}_k(t,\xi)|^2 d\xi \leqslant \eta.$$

N is called scale frequency function of the solution $\mathbf{u}$, x is the spacial center function and C is the compactness modulus function.

Remark 1.15. *We know that a family of functions $\mathcal{F} \subset \dot{H}^1(\mathbb{R}^6)$, is compact if, and only if, $\mathcal{F}$ is bounded in $\dot{H}^1(\mathbb{R}^6)$ and, for all $\eta > 0$, there exists a compactness modulus function $C(\eta) > 0$, such that*

$$\int_{|x|\geqslant C(\eta)/N(t)} |\nabla f(x)|^2 dx + \int_{|\xi|\geqslant C(\eta)N(t)} |\xi|^2 |\hat{f}(\xi)|^2 d\xi \leqslant \eta$$

for all functions $f \in \mathcal{F}$. See Appendix A for more details. In particular, by Sobolev embedding, every compact set in $\dot{H}^1_x(\mathbb{R}^6)$ is compact in $L^3_x(\mathbb{R}^6)$. Therefore, any solution $\mathbf{u} : I \times \mathbb{R}^6 \to \mathbb{C}$ to (1.2) that is almost periodic modulo symmetries must also satisfy

$$\sum_{k=1}^{l} \int_{|x-x(t)|\geqslant C(\eta)/N(t)} |u_k(t,x)|^3 \lesssim \eta,$$

for all $t \in I$ and $\eta > 0$.

Remark 1.16. *Another consequence of compactness modulo symmetries is the existence of a function* $c : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$\int_{|x-x(t)|\leqslant c(\eta)/N(t)} |\nabla \mathbf{u}(t,x)|^2 dx + \int_{|\xi|\leqslant c(\eta)N(t)} |\xi|^2 |\hat{\mathbf{u}}|^2 d\xi \leqslant \eta,$$

for all $t \in I$ *and* $\eta > 0$. *See the Appendix A for more details.*

The main result of the section is.

Theorem 1.17. *(Reduction to almost periodic solutions). Suppose that Theorem 1.10 fails. Then there exists a maximal solution* $\mathbf{u}_c : I_c \times \mathbb{R}^6 \longrightarrow \mathbb{C}^l$ *to* (1.2) *such that*

$$\sup_{t\in I_c} K(\mathbf{u}_c) < K(\boldsymbol{\psi}),$$

$\mathbf{u}_c$ *is almost periodic modulo symmetries and* $\mathbf{u}_c$ *blows-up in time. Moreover,* $\mathbf{u}_c$ *has minimum kinetic energy among all solutions that blows-up in time, that is,*

$$\sup_{t\in I} K(\mathbf{u}(t)) \geqslant \sup_{t\in I_c} K(\mathbf{u}_c),$$

for all maximal solutions $\mathbf{u}$ *that blows-up at least in one direction.*

To guarantee the existence of such kind of solution, we will need an auxiliary result, called Palais-Smale property. In order to show that our system satisfies such property, we will use the nonlinear profile decomposition and stability theory. All these tools will be discussed in Chapter 2.

From this, in Section 4.3, we will see that it is possible to classify the solutions $\mathbf{u}_c$ to (1.2), founded in Theorem 1.17, with more refined properties, according to different kinds of scale functions $N(t)$. Such type of classification was studied in (KILLIP; TAO; VISAN, 2009) and (KILLIP; VISAN, 2010). The result states the following.

Proposition 1.18. *(The enemies). Suppose that Theorem 1.10 fails. Then there exists a maximal solution* $\mathbf{u}_c : I_c \times \mathbb{R}^6 \longrightarrow \mathbb{C}^l$*, which is almost periodic modulo symmetries and satisfy*

$$S_{I_c}(\mathbf{u}_c) = \infty \quad \textit{and} \quad \sup_{t\in I_c} K(\mathbf{u}_c(t)) < K(\boldsymbol{\psi}). \tag{1.32}$$

Moreover, the time interval I_c *and the scale function* $N(t)$ *satisfy one of the three following scenarios:*

(i) We have $|\inf I_c| < \infty$ *or* $|\sup I_c| < \infty$*;*

(ii) We have $I_c = \mathbb{R}$ *and*

$$N(t) = 1, \quad \forall t \in \mathbb{R};$$

(iii) We have $I_c = \mathbb{R}$ and

$$\inf_{t\in\mathbb{R}} N(t) \geqslant 1, \quad \textit{and} \quad \limsup_{t\to\infty} N(t) = \infty.$$

Remark 1.19. *From the literature (see (KILLIP; VISAN, 2010)) the three scenarios are known, respectively, as Finite-time blow-up, Soliton-like solution and Low-to-high frequency cascade.*

Finally, to conclude the proof of Theorem 1.10, we will show that the critical solution $\mathbf{u}_c$ cannot satisfy any one of these conditions, that is, we will exclude case by case the possibilities, arriving to a contradiction. This is the motivation to call the three scenarios "the enemies".

In Section 4.4, we will exclude the first case, showing that the $\mathbf{L}^2$-norm of $\mathbf{u}_c(t)$ converges to zero when t goes to infinty. Since the mass of the system is conserved, this implies that $\mathbf{u}_c$ is identically zero.

For the remaining cases, we will need to show that the solution $\mathbf{u}_c$ has some negative regularity that is, the solution is in a Sobolev space of negative index, and this is done in two steps. First, we show that the solution belongs to $\mathbf{L}_t^\infty(\mathbf{L}_x^p)$, this guarantee that the function decays at infinity faster than a function in $\mathbf{u} \in \mathbf{L}_t^\infty(\dot{\mathbf{H}}_x^1)$. The second step is to improve the decay previously established to $\mathbf{L}^2$ spaces. This will be done in Section 4.5.

Finally, in Section 4.6, we will use the negative regularity to deduce some compactness properties of $\mathbf{u_c}$ in $\mathbf{L}^2$, then, we show that $\mathbf{u}_c$ has zero momentum, and finally, use a virial identity to exclude the Soliton case. Next, in Section 4.7, we use the negative regularity joint with the conservation of mass to exclude the low-to-high cascade frequency. Last but not least, in Section 4.8, we show the scattering result of Corollary 1.11 and the blow-up result of Theorem 1.12.

CHAPTER 2

NOTATION AND PRELIMINARY RESULTS

Throughout the work we will use the standard notation in PDEs. Indeed, C will represent a generic constant which may vary from inequality to inequality. If a and b are positive constants, we denote $a \lesssim b$ whenever $a \leqslant Cb$ for some constant $C > 0$, similar for the case $a \gtrsim b$. We write $X\pm$ for any quantity of the form $X \pm \epsilon$ for any small $\epsilon > 0$. Given a subset A, we denote by $\mathbf{A}$ the product $A \times ... \times A$ (l-times). In particular, if A is a Banach space, then $\mathbf{A}$ also is with the usual norm given by the sum. For a complex number $z \in \mathbb{C}$, $\operatorname{Re} z$ and $\operatorname{Im} z$ represents its real and imaginary parts. Also, $\bar{z}$ denotes the complex conjugate of z. We set $\left|\mathbf{z}\right|$ for the vector $(|z_1|, ..., |z_l|)$. This is not to be confused with $|\mathbf{z}| = \sqrt{|z_1|^2 + ... + |z_l|^2}$ which denotes the usual norm of the vector $\mathbf{z} \in \mathbb{C}^l$.

We denote the stantard Sobolev, the homogeneous Sobolev and the Lebesgue spaces by $H^{s,p} = H^{s,p}(\mathbb{R}^d)$, $\dot{H}^{s,p} = \dot{H}^{s,p}(\mathbb{R}^d)$ and $L^p = L^p(\mathbb{R}^d)$, respectively, with its usual norms. We denote $H^s = H^{s,2}$ and $\dot{H}^s = \dot{H}^{s,2}$. Given a time interval I, the mixed Lebesgue space $L_t^p L_x^q(I \times \mathbb{R}^d)$ is denoted by $L_t^p L_x^q$ and will be endowed with the norm

$$\|f\|_{L_t^p L_x^q} = \left(\int_I \left(\int_{\mathbb{R}^d} |f(t,x)|^q dx \right)^{p/q} dt \right)^{1/p},$$

with the obvious modification if either $p = \infty$ or $q = \infty$.

A pair (q, r) is called admissible with $2 \leqslant q, r \leqslant \infty$ if $\dfrac{2}{q} = \dfrac{d}{2} - \dfrac{d}{r}$. For a fixed space time slab $I \times \mathbb{R}^d$, we set the Strichartz norm by

$$\|u\|_{\mathcal{S}^0(I)} := \sup_{(q,r) \text{ admissible}} \|u\|_{L_t^q L_x^r(I \times \mathbb{R}^d)} \quad \text{and} \quad \|u\|_{\mathcal{S}^1(I)} := \|\nabla u\|_{\mathcal{S}^0(I)}. \tag{2.1}$$

We define the Fourier transform on $\mathbb{R}^d$ by

$$\hat{f}(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx.$$

For $s \in \mathbb{R}$, we define the fractional differentiation/integral operator

$$\widehat{|\nabla|^s f}(\xi) := |\xi|^s \hat{f}(\xi),$$

which defines the homogeneous Sobolev norm

$$\|f\|_{\dot{H}^s_x(\mathbb{R}^d)} := \||\nabla|^s f\|_{L^2_x(\mathbb{R}^d)}.$$

If no confusion is caused, we denote $\int_{\mathbb{R}^d} f(x)dx$ simply by $\int f$. We start with the results that will be used throughout the work.

Theorem 2.1. *(Strichartz estimates, (CAZENAVE, 2003) Theorem 2.3.3) The following inequalities hold.*

(i) If (q, r) is an admissible pair. Then, for all $f \in L^2(\mathbb{R}^d)$.

$$\|e^{it\Delta} f\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|f\|_{L^2_x(\mathbb{R}^d)}.$$

(ii) Let I be a time interval and $t_0 \in \bar{I}$. If (q_1, r_1) and (q_2, r_2) are two admissible pairs, then

$$\left\| \int_{t_0}^t e^{i(t-s)\Delta} f(\cdot, s)ds \right\|_{L^{q_1}_t L^{r_1}_x(I\times\mathbb{R}^d)} \lesssim \|f\|_{L^{q_2'}_t L^{r_2'}_x(I\times\mathbb{R}^d)}$$

and

$$\left\| \int_a^b e^{i(t-s)\Delta} f(\cdot, s)ds \right\|_{L^{q_1}_t L^{r_1}_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|f\|_{L^{q_2'}_t L^{r_2'}_x([a,b]\times\mathbb{R}^d)}.$$

Proposition 2.2. *Let (q, r) be an admissible pair. Given $u_0 \in L^2_x(\mathbb{R}^d)$ and $\epsilon > 0$ then there exist $T > 0$ such that*

$$\left(\int_0^T \|e^{it\Delta} u_0\|^r_{L^q_x} dt\right)^{1/r} < \epsilon. \tag{2.2}$$

In addition, there exist $\delta > 0$ such that if $\|v_0 - u_0\|_{L^2_x} < \delta$, then

$$\left(\int_0^T \|e^{it\Delta} v_0\|^r_{L^q_x} dt\right)^{1/r} < \epsilon,$$

where $e^{it\Delta}$ is the unitary group associated to the linear part of the Schrödinger equation.

Proof. See (LINARES; PONCE, 2015), page 100. □

Remark 2.3. *The above results still holds if we replace $e^{it\Delta}$ with $U(t)$, $W(t)$ and $U_k(t)$, $k = 1, ..., l$. defined, respectively, on the pages 12 and 17.*

2.1 Preliminaries: First part

2.1.1 Measures

We denote the set of all continuous bounded functions and all continuous function with compact support on X, respectively, by $\mathcal{C}_b(X)$ and $\mathcal{C}_c(X)$. When it comes to Radon measures, if X is a locally compact Hausdorff space, we shall denote by $\mathcal{M}_+(X)$ the Banach space of all non-negative measures, by $\mathcal{M}_+^b(X)$ the space of all bound (or finite) measures and $\mathcal{M}_+^1$ the space of all probability measures. Given two measures ν and μ, we write $\nu \ll \mu$ if the measure ν is absolutely continuous with respect to the measure μ. For all $\mu \in \mathcal{M}_+^b(X)$, $\|\mu\| := \mu(X)$ is called the mass of μ.

Let us introduce some convergence notions of measures.

Definition 2.4. *(i) A sequence $(\mu_m) \subset \mathcal{M}_+$ is said to converge vaguely to μ in $\mathcal{M}_+(X)$, and denoted by $\mu_m \overset{*}{\rightharpoonup} \mu$, if $\int_X f d\mu_m \to \int_X f d\mu$ for all $f \in \mathcal{C}_c(X)$.*

(ii) A sequence $(\mu_m) \subset \mathcal{M}_+^b(X)$ is said to converge weakly to μ, in $\mathcal{M}_+^b(X)$, and denoted by $\mu_m \rightharpoonup \mu$, if $\int_X f d\mu_m \to \int_X f d\mu$, for all $f \in \mathcal{C}_b(X)$.

(iii) A sequence $(\mu_m) \subset \mathcal{M}_+^b$ is said to be uniformly tight if, for every $\epsilon > 0$, there exists a compact subset $K_\epsilon \subset X$ such that $\mu_m(X \backslash K_\epsilon) \leqslant \epsilon$ for all m. We also say that a set $\mathcal{H} \subset \mathcal{M}_+(X)$ is vaguely bounded if $\sup_{\mu \in \mathcal{H}} \left| \int_X f d\mu \right| < \infty$ for all $f \in \mathcal{C}_c(X)$.

To finish this section, we state a result that guarantees the existence of vaguely convergent sequences. The proof can be found in Theorems 30.6 and 31.2 in (BAUER, 2001).

Lemma 2.5. *Let X be a locally compact Hausdorff space. Then*

(i) Every vaguely bounded sequence in $\mathcal{M}_+(X)$ contains a vaguely convergent subsequence;

(ii) If $\mu_m \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}_+(X)$ and $(\|\mu_m\|)$ is bounded, then μ is finite.

2.1.2 Some estimates

Now, we presents an adapted version of the generalized Brezis-Lieb Lemma (see (BRÉZIS; LIEB, 1983), Theorem 2). Let $f : \mathbb{R}^l \to \mathbb{R}$ be a continuous function satisfying $f(0, ..., 0) = 0$, for all, $a, b \in \mathbb{R}^l$, and $\epsilon > 0$

$$|f(a + b) - f(b)| \leqslant \epsilon\zeta(a) + \psi_\epsilon(b), \tag{2.3}$$

where ζ and ψ_ϵ are non-negative functions.

Lemma 2.6. *Let $v_m = u_m - u$ be a sequence of measurable functions from $\mathbb{R}^d \to \mathbb{R}^l$ such that*

(i) $v_m \to 0$ *a. e.;*

(ii) $f(u) \in L^1(\mathbb{R}^d)$*;*

(iii) $\int \zeta(v_m)(x)dx \leqslant M < \infty$, *for some constant* M, *independent of* m*;*

(iv) $\int \psi_\epsilon(u)(x)dx < \infty$, *for any* $\epsilon > 0$.

Then, as $m \to \infty$,

$$\int |f(u_m) - f(v_m) - f(u)|dx \to 0.$$

Lemma 2.7. *Let $I \subset \mathbb{R}$ be an open interval with $0 \in I$, $a \in \mathbb{R}$, $b > 0$ and $q > 1$. Define $\gamma = (bq)^{-1/(q-1)}$ and $f(r) = a - r + br^q$, for $r > 0$. Let $G(t)$ be a nonnegative continuous function such that $f \circ G \geqslant 0$ in I. Assume that $a < \left(1 - \frac{1}{q}\right)\gamma$, we have*

(i) If $G(0) < \gamma$ *then* $G(t) < \gamma$, *for all* $t \in I$*;*

(ii) If $G(0) > \gamma$ *then* $G(t) > \gamma$, *for all* $t \in I$.

Proof. See Lemma 3.1 in (PASTOR, 2015). □

2.2 Preliminaries: Second part

2.2.1 Some estimates

Lemma 2.8. *(Acausal Gronwall's inequality). Given η, C, γ, $\gamma' > 0$, let $\{x_k\}_{k\geqslant 0}$ be a bounded nonnegative sequence obeying*

$$x_k \leqslant C2^{-\gamma k} + \eta \sum_{l<k} 2^{-\gamma|k-l|}x_l + \eta \sum_{l\geqslant k} 2^{-\gamma'|k-l|}x_l,$$

for all $k \geqslant 0$. If, $\eta \leqslant \frac{1}{4}\min\{1 - 2^{-\gamma}, 1 - 2^{-\gamma'}, 1 - 2^{\rho-\gamma}\}$ for some $0 < \rho < \gamma$, then

$$x_k \leqslant (4C + \|x\|_{\ell^\infty})2^{-\rho\gamma}.$$

Proof. See (KILLIP; VISAN, 2011, Lemma 5.3). □

From now on we will assume that the hypothesis (H1)-(H8) hold. Furthermore, for $1 \leqslant p \leqslant \infty$ we denote by p' its Hölder's conjugate, that is, $1/p + 1/p' = 1$. We start with the following dispersive estimate.

Lemma 2.9. *If $2 \leqslant p \leqslant \infty$ and $t \neq 0$, then, for $k = 1, ..., l$,*

$$\|U_k(t)f\|_{L^p_x(\mathbb{R}^d)} \lesssim |t|^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|f\|_{L^{p'}_x(\mathbb{R}^d)}, \quad \forall f \in L^{p'}_x(\mathbb{R}^d).$$

Proof. See Proposition 2.2.3 in (CAZENAVE, 2003). □

Next lemma has an important role in the proof of Palais-Smale condition.

Lemma 2.10. *Given $\phi \in \dot{H}^1_x(\mathbb{R}^d)$, $R > 0$ and $T > 0$*

$$\|\nabla e^{it\Delta}\phi\|^3_{L^2_{t,x}([-T,T]\times\{|x|\leqslant R\})} \lesssim T^{\frac{2}{d+2}}R^{\frac{3d+2}{2(d+2)}}\|e^{it\Delta}\phi\|_{L^{2(d+2)/(d-2)}_{t,x}}\|\nabla\phi\|^2_{L^2_x}.$$

Proof. See (KILLIP; VISAN, 2010), Lemma 2.5. □

Next results are some consequences of our assumptions on the nonlinearities f_k, $k = 1, ..., l$.

Lemma 2.11. *Suppose that the nonlinearities f_k obey (H1) and (H2). Then*

(i) For all $\mathbf{z}, \mathbf{z}' \in \mathbb{C}^l$, we have

$$|f_k(\mathbf{z}) - f_k(\mathbf{z}')| \lesssim \sum_{m=1}^{l}\sum_{j=1}^{l}(|z_j| + |z'_j|)|z_m - z'_m|, \quad k = 1, ..., l.$$

In particular,

$$|f_k(\mathbf{z})| \lesssim \sum_{j=1}^{l}|z_j|^2, \quad k = 1, ..., l.$$

(ii) Let $\mathbf{u}$ and $\mathbf{u}'$ be complex-valued functions defined on $\mathbb{R}^d$. Then

$$|\nabla[f_k(\mathbf{u}) - f_k(\mathbf{u}')]| \lesssim \sum_{m=1}^{l}\sum_{j=1}^{l}|u_j||\nabla(u_m - u'_m)| + \sum_{m=1}^{l}\sum_{j=1}^{l}|u_j - u'_j||\nabla u'_m|.$$

(iii) Let $1 < p, q, r < \infty$ be such that $\dfrac{1}{r} = \dfrac{1}{p} + \dfrac{1}{q}$ and $s \in (0, 1)$. Then, for $k = 1, ..., l$,

$$\|\nabla f_k(\mathbf{u})\|_{\mathbf{L}^r} \lesssim \|\mathbf{u}\|_{\mathbf{L}^p_x}\|\nabla\mathbf{u}\|_{\mathbf{L}^q_x}$$

and

$$\|f_k(\mathbf{u})\|_{\mathbf{H}^{s,r}} \lesssim \|\mathbf{u}\|_{\mathbf{L}^p_x}\|\mathbf{u}\|_{\mathbf{H}^{s,q}}. \tag{2.4}$$

Proof. For (i) and (ii) see Lemma 2.2, Corolário 2.3 and Lemma 2.4 in (NOGUERA; PASTOR, 2021). Part (iii) is a consequence of (H2) and Leibniz's rule (see Proposition 5.1 in (TAYLOR, 2000) and Corollary 2.5 in (NOGUERA; PASTOR, 2021)). □

Lemma 2.12. *Assume that f_k satisfy (H2) for all $k = 1, ..., l$. For all $J \in \mathbb{N}$ we have*

$$\left|\nabla\left[\sum_{j=1}^{J} f_k(\mathbf{u}_j) - f_k\left(\sum_{j=1}^{J}\mathbf{u}_j\right)\right]\right| \lesssim \sum_{j\neq i}|\nabla \mathbf{u}_j||\mathbf{u}_i|, \quad k = 1, ..., l.$$

Proof. Let $\mathbf{u}_j = (u_{j1}, ..., u_{jl})$ and $f_k(\mathbf{z}) = f_k(z_1, ..., z_l)$. By the chain rule

$$\frac{\partial f_k}{\partial x_j}(\mathbf{u}) = \sum_{m=1}^{l}\left(\frac{\partial f_k}{\partial z_m}(\mathbf{u})\frac{\partial u_m}{\partial x_j} + \frac{\partial f_k}{\partial \bar{z}_m}(\mathbf{u})\frac{\partial \bar{u}_m}{\partial x_j}\right),$$

and we get the following

$$\frac{\partial f_k}{\partial x_i}\left(\sum_{j=1}^{J}\mathbf{u}_j\right) = \sum_{m=1}^{l}\left[\frac{\partial f_k}{\partial z_m}\left(\sum_{j=1}^{J}\mathbf{u}_j\right)\frac{\partial}{\partial x_i}\left(\sum_{n=1}^{J}u_{nm}\right) + \frac{\partial f_k}{\partial \bar{z}_m}\left(\sum_{j=1}^{J}\mathbf{u}_j\right)\frac{\partial}{\partial x_i}\left(\sum_{n=1}^{J}\bar{u}_{nm}\right)\right]$$

Therefore, by the triangle inequality,

$$\begin{aligned}\left|\sum_{j=1}^{J}\frac{\partial f_k}{\partial x_i}(\mathbf{u}_j) - \frac{\partial f_k}{\partial x_i}\left(\sum_{j=1}^{J}\mathbf{u}_j\right)\right| &\leqslant \left|\sum_{m=1}^{l}\left[\sum_{n=1}^{J}\frac{\partial f_k}{\partial z_m}(\mathbf{u}_n)\frac{\partial u_{nm}}{\partial x_i} - \frac{\partial f_k}{\partial z_m}\left(\sum_{j=1}^{J}\mathbf{u}_j\right)\left(\sum_{n=1}^{J}\frac{\partial u_{nm}}{\partial x_i}\right)\right]\right| \\ &+ \left|\sum_{m=1}^{l}\left[\sum_{n=1}^{J}\frac{\partial f_k}{\partial \bar{z}_m}(\mathbf{u}_n)\frac{\partial \bar{u}_{nm}}{\partial x_i} - \frac{\partial f_k}{\partial \bar{z}_m}\left(\sum_{j=1}^{J}\mathbf{u}_j\right)\left(\sum_{n=1}^{J}\frac{\partial \bar{u}_{nm}}{\partial x_i}\right)\right]\right|.\end{aligned} \tag{2.5}$$

Now, notice that for each m, we get

$$\begin{aligned}&\left|\sum_{n=1}^{J}\frac{\partial f_k}{\partial z_m}(\mathbf{u}_n)\frac{\partial u_{nm}}{\partial x_i} - \frac{\partial f_k}{\partial z_m}\left(\sum_{j=1}^{J}\mathbf{u}_j\right)\left[\sum_{n=1}^{J}\frac{\partial u_{nm}}{\partial x_i}\right]\right| \\ &\qquad = \left|\sum_{n=1}^{J}\left[\left(\frac{\partial f_k}{\partial z_m}(\mathbf{u}_n) - \frac{\partial f_k}{\partial z_m}\left(\sum_{j=1}^{J}\mathbf{u}_j\right)\right)\frac{\partial u_{nm}}{\partial x_i}\right]\right| \\ &\qquad \leqslant \sum_{n=1}^{l}\left|\frac{\partial f_k}{\partial z_m}(\mathbf{u}_n) - \frac{\partial f_k}{\partial z_m}\left(\sum_{j=1}^{J}\mathbf{u}_j\right)\right|\left|\frac{\partial u_{nm}}{\partial x_i}\right| \\ &\qquad \lesssim \sum_{n=1}^{J}\left|\mathbf{u}_n - \sum_{j=1}^{j}\mathbf{u}_j\right|\left|\frac{\partial u_{nm}}{\partial x_i}\right| \\ &\qquad \lesssim \sum_{n\neq j}|\mathbf{u}_j|\left|\frac{\partial \mathbf{u}_n}{\partial x_i}\right|,\end{aligned}$$

where we used (H2) in the second last inequality. In the same way,

$$\left|\sum_{n=1}^{J}\frac{\partial f_k}{\partial \bar{z}_m}(\mathbf{u}_n)\frac{\partial \bar{u}_{nm}}{\partial x_i} - \frac{\partial f_k}{\partial \bar{z}_m}\left(\sum_{j=1}^{J}\mathbf{u}_j\right)\left[\sum_{n=1}^{J}\frac{\partial \bar{u}_{nm}}{\partial x_i}\right]\right| \lesssim \sum_{n\neq j}|\mathbf{u}_j|\left|\frac{\partial \bar{\mathbf{u}}_n}{\partial x_i}\right|.$$

Then, for each $i = 1, .., l$,

$$\left|\sum_{j=1}^{J}\frac{\partial f_k}{\partial x_i}(\mathbf{u}_j) - \frac{\partial f_k}{\partial x_i}\left(\sum_{j=1}^{J}\mathbf{u}_j\right)\right| \lesssim \sum_{n\neq j}|\mathbf{u}_j|\left|\frac{\partial \mathbf{u}}{\partial x_i}\right|,$$

which implies

$$\left|\nabla\left[\sum_{j=1}^{J} f_k(\mathbf{u}_j) - f_k\left(\sum_{j=1}^{J}\mathbf{u}_j\right)\right]\right| \lesssim \sum_{n\neq j}|\mathbf{u}_j||\nabla \mathbf{u}_n|.$$

□

Lemma 2.13. *Supose that (H3) and (H4) hold, then f_k satisfies the Gauge condition, that is, for all $\theta \in \mathbb{R}$,*

$$f_k\left(e^{i\frac{\alpha_1}{\gamma_1}\theta}u_1, \dots, e^{i\frac{\alpha_l}{\gamma_l}\theta}u_l\right) = e^{i\frac{\alpha_k}{\gamma_k}\theta} f_k(\mathbf{u}), \quad k = 1, \dots, l.$$

Proof. See Lema 2.8 in (NOGUERA; PASTOR, 2021). □

Next, we show some properties of the potential function F.

Lemma 2.14. *Assume that (H1)-(H5) hold.*

(i) For all $\mathbf{z} \in \mathbb{C}^l$,

$$|\mathrm{Re}F(\mathbf{z})| \lesssim \sum_{j=1}^{l}|z_j|^3.$$

(ii) We have

$$\mathrm{Re}\sum_{k=1}^{l} f_k(\mathbf{u})\nabla\bar{u}_k = \mathrm{Re}[\nabla F(\mathbf{u})]$$

and

$$\mathrm{Re}\sum_{k=1}^{l} f_k(\mathbf{u})\bar{u}_k = \mathrm{Re}[3F(\mathbf{u})].$$

(iii) The potential function vanishes at zero, that is, $F(\mathbf{0}) = 0$.

Proof. For (i) and (ii) see Lemmas 2.10 and 2.11, respectively, in (NOGUERA; PASTOR, 2021). Finally, (iii) is consequence of (H5). □

Lemma 2.15. *(Refined Fatou's Lemma) Supose that $\{f_n\} \subset L^p_x(\mathbb{R}^6)$ is such that $\limsup \|f_n\|_{L^p} < \infty$. If $f_n \to f$ almost everywhere, then*

$$\int_{\mathbb{R}^6} ||f_n|^p - |f_n - f|^p - |f|^p|\, dx \to 0.$$

In particular, $\|f_n\|^p_{L^p} - \|f_n - f\|^p_{L^p} \to \|f\|^p_{L^p}$.

Proof. The proof can be found in (BRÉZIS; LIEB, 1983). □

Lemma 2.16. *(Gagliardo-Nirenberg's Inequality) Let $1 < q < p \leqslant \infty$ and $s > 0$ be such that*

$$\frac{1}{p} = \frac{1}{q} - \frac{\theta s}{d},$$

for some $0 < \theta < 1$. Then for all $f \in \dot{H}^{s,q}_x(\mathbb{R}^d)$ we have

$$\|f\|_{L^p_x(\mathbb{R}^d)} \lesssim_{d,p,q,s} \|f\|^{1-\theta}_{L^q_x(\mathbb{R}^d)}\|f\|^{\theta}_{\dot{H}^{s,q}_x(\mathbb{R}^d)}.$$

Proof. See (TAO, 2006), Apendix A. □

Lemma 2.17. *(Young's inequality) Let $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, $1 \leqslant p, q \leqslant \infty$ with $\frac{1}{p} + \frac{1}{q} \geqslant 1$. Then, $f * g \in L^r(\mathbb{R}^d)$, where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. Morover*

$$\|f * g\|_{L^r} \leqslant \|f\|_{L^p}\|g\|_{L^q}.$$

Proof. See (LINARES; PONCE, 2015), Section 2.1.1. □

2.2.2 Littlewood-Paley theory

Let $\varphi(\xi)$ be a radial bump function with support on the ball $\{\xi \in \mathbb{R}^6 : |\xi| \leqslant 11/10\}$ and equal to 1 on the ball $\{\xi \in \mathbb{R}^6 : |\xi| \leqslant 1\}$. For each dyadic number $N > 0$, that is, $N = 2^j$ for $j \in \mathbb{Z}$, we set the Fourier multipliers

$$\begin{aligned}
\widehat{P_{\leqslant N} f}(\xi) &:= \varphi(\xi/N)\hat{f}(\xi), \\
\widehat{P_{> N} f}(\xi) &:= (1 - \varphi(\xi/N))\hat{f}(\xi), \\
\widehat{P_N f}(\xi) &:= (\varphi(\xi/N) - \varphi(2\xi/N))\hat{f}(\xi),
\end{aligned}$$

and in a similar way, $P_{<N}$ and $P_{\geqslant N}$. Note, in particular, the telescoping identities

$$P_{\leqslant N} f = \sum_{M \leqslant N} P_M f; \quad P_{>N} f = \sum_{M>N} P_M f; \quad f = \sum_M P_M f.$$

Moreover, for $M < N$, we set

$$P_{M<\cdot\leqslant N} := P_{\leqslant N} - P_{\leqslant M} = \sum_{M<N'\leqslant N} P_{N'},$$

Since Littlewood-Paley operators are Fourier multipliers, they comute with the propagator $U_k(t)$ and the operator $i\alpha_k\partial_t + \gamma_k\Delta$. Besides that, using Fourier trasnform properties, up to a constant, $P_{\leqslant N}$ is a convolution operator, as the following

$$\begin{aligned}
P_{\leqslant N} f(x) &= \frac{1}{(2\pi)^d}\int \varphi\left(\frac{\xi}{N}\right) e^{ix\cdot\xi}\hat{f}(\xi)d\xi \\
&= \frac{1}{(2\pi)^d}\int N^d\check{\varphi}(Ny)f(x-y)dy \\
&= \frac{1}{(2\pi)^d}[N^d\check{\varphi}(N\cdot)] * f.
\end{aligned}$$

Using Lemma 2.17 with $r = p$ and $q = 1$, we obtain

$$\|P_{\leqslant N} f\|_{L^p} = C\|[N^d\check{\varphi}(N\cdot)] * f\|_{L^p} \lesssim \|\check{\varphi}\|_{L^1}\|f\|_{L^p} \lesssim \|f\|_{L^p}, \tag{2.6}$$

where $C > 0$ does not depend on N. Next, we enunciate some estimates that will be useful in our analysis.

Remark 2.18. *Let f and g be functions and N be a dyadic number. Then*

$$P_N(g_{\leqslant \frac{N}{10}} f) = P_N(g_{\leqslant \frac{N}{10}} f_{> \frac{N}{10}}). \tag{2.7}$$

Indeed, note that by definition of Littlewood-Paley projections we have

$$\widehat{P_N f(\xi)} = \psi_N(\xi)\widehat{f}(\xi),$$

where $\psi_N(\xi) := \psi_1\left(\frac{\xi}{N}\right) = \varphi\left(\frac{\xi}{N}\right) - \varphi\left(\frac{2\xi}{N}\right)$ *and* $\psi_1\left(\frac{\xi}{N}\right) = 0$ *if* $\frac{11}{10}N \leqslant |\xi| \leqslant \frac{N}{2}$. *Now, observe that*

$$P_N(g_{\leqslant \frac{N}{10}} f) = P_N(g_{\leqslant \frac{N}{10}} f_{> \frac{N}{10}}) + P_N(g_{\leqslant \frac{N}{10}} f_{\leqslant \frac{N}{10}}) := P_N(g_{\leqslant \frac{N}{10}} f_{> \frac{N}{10}}) + I.$$

Then, we get (2.7)*if we show that I vanishes. More precisely,*

$$\begin{aligned}
\widehat{I}(\xi) &= \psi_N(\xi)\widehat{P_{\leqslant \frac{N}{10}} g} * \widehat{P_{\frac{N}{10}} f}(\xi) \\
&= \psi_N(\xi)\int \widehat{P_{\leqslant \frac{N}{10}} g}(\xi - \xi_1)\widehat{P_{\frac{N}{10}} f}(\xi_1)d\xi_1 \\
&= \psi_N(\xi)\int \varphi\left(10\frac{(\xi-\xi_1)}{N}\right)\varphi\left(10\frac{\xi_1}{N}\right)\widehat{g}(\xi-\xi_1)\widehat{f}(\xi_1)d\xi_1 \\
&= 0,
\end{aligned}$$

since $\left|10\frac{(\xi-\xi_1)}{N}\right| \geqslant \frac{10}{N}(|\xi| - |\xi_1|) \geqslant \frac{10}{N}\left(\frac{11}{10}N - \frac{11}{100}N\right) \geqslant \frac{11}{10}.$

Lemma 2.19. *(Bernstein's estimates). For $s \geqslant 0$ and $1 \leqslant p \leqslant q \leqslant \infty$:*

$$\begin{gathered}
\|P_{\geqslant N} f\|_{L^p_x} \lesssim N^{-s}\||\nabla|^s P_{\geqslant N} f\|_{L^p_x}, \\
\|P_{\leqslant N} f\|_{L^p_x} \lesssim N^{-s}\||\nabla|^s P_{\leqslant N} f\|_{L^p_x}, \\
\||\nabla|^{\pm s} P_N f\|_{L^p_x} \sim N^{\pm s}\|P_N f\|_{L^p_x}, \\
\|P_{\leqslant N} f\|_{L^q_x} \lesssim N^{\frac{6}{p}-\frac{6}{q}}\|P_{\leqslant N} f\|_{L^p_x}, \\
\|P_N f\|_{L^q_x} \lesssim N^{\frac{6}{p}-\frac{6}{q}}\|P_N f\|_{L^p_x},
\end{gathered}$$

Proof. See (TAO, 2006), page 333. □

We also need the following vector version of the nonlinear Bernstein's estimate.

Lemma 2.20. *Let $g : \mathbb{C}^l \longrightarrow \mathbb{C}$ be a Hölder continuous function of order 1, then*

$$\|P_N g(\mathbf{u})\|_{L^p_x} \lesssim N^{-1}\|\nabla \mathbf{u}\|_{L^p_x},$$

for any $1 \leqslant p < \infty$ and $\mathbf{u} \in \dot{\mathbf{H}}^{1,p}$.

Proof. See (KILLIP; VISAN, 2013), Lemma A.13. □

2.2.3 Linear profile decomposition

In this section we follow the ideas presented in (KOCH; TATARU; VISAN, 2014), with suitable adaptations to our case, in order to establish the linear profile decomposition corresponding to the Schrödinger propagator $U_k(t)$, $k = 1, ..., l$, for bounded sequences in $\dot{H}^1$. Such type of decomposition was first obtained by (KERAANI, 2001), relying on an improved Sobolev inequality proved by (GERARD Y. MEYER, 1997). We start with the following estimate that is a refinement of the Strichartz estimates which shows that linear solutions with non-trivial spacetime norm must concentrate on at least one frequency annulus.

Lemma 2.21. *(Refined Strichartz estimate). For all $h \in \dot{H}^1(\mathbb{R}^6)$ we have*

$$\|U_k(t)h\|_{L^4_{t,x}} \lesssim \|h\|^{1/2}_{\dot{H}^1(\mathbb{R}^6)} \sup_{N\in 2^{\mathbb{Z}}} \|U_k(t)P_N h\|^{1/2}_{L^4_{t,x}}, \quad k = 1, ..., l.$$

Proof. See (KOCH; TATARU; VISAN, 2014), Lemma 3.1, page 239. □

With this result at hand we may prove the inverse Strichartz's inequality, which goes one step further than the last lemma, and shows that linear solutions with non-trivial spacetime norm contain a bubble of concentration around some point in spacetime. In this sense, we introduce the notation $\mathbf{U}(t)\mathbf{u} = (U_1(t)u_1, ..., U_l(t)u_l)$.

Lemma 2.22. *Let $(\mathbf{h}_m) \subset \dot{\mathbf{H}}^1(\mathbb{R}^6)$. Suppose that*

$$\lim_{m\to\infty} \|\mathbf{h}_m\|_{\dot{\mathbf{H}}^1} = A < \infty \quad \text{and} \quad \lim_{m\to\infty} \|\mathbf{U}(t)\mathbf{h}_m\|_{\mathbf{L}^4_{t,x}} = \epsilon > 0.$$

Then, there is a subsequence in m, $\boldsymbol{\phi} \in \dot{\mathbf{H}}^1$, $(\lambda_m) \subset (0, \infty)$, and $(t_m, x_m) \subset \mathbb{R} \times \mathbb{R}^6$ such that

$$\lambda_m^2[\mathbf{U}(t)\mathbf{h}_m](\lambda_m x + x_m) \rightharpoonup \boldsymbol{\phi}(x), \quad \text{weakly in } \dot{\mathbf{H}}^1, \tag{2.8}$$

$$\liminf_{m\to\infty} \left\{\|\mathbf{h}_m\|^2_{\dot{\mathbf{H}}^1} - \|\mathbf{h}_m - \boldsymbol{\phi}_m\|^2_{\dot{\mathbf{H}}^1}\right\} = \|\boldsymbol{\phi}\|^2_{\dot{\mathbf{H}}^1} \gtrsim \epsilon^{12} A^{-10}, \tag{2.9}$$

$$\liminf_{m\to\infty} \left\{\|\mathbf{U}(t)\mathbf{h}_m\|^4_{\mathbf{L}^4_{t,x}} - \|\mathbf{U}(t)(\mathbf{h}_m - \boldsymbol{\phi}_m)\|^4_{\mathbf{L}^4_{t,x}}\right\} \gtrsim \epsilon^{24} A^{-20}, \tag{2.10}$$

$$\liminf_{m\to\infty} \left\{\|\mathbf{h}_m\|^3_{\mathbf{L}^3_{t,x}} - \|\mathbf{h}_m - \boldsymbol{\phi}_m\|^3_{\mathbf{L}^3_{t,x}} - \|\mathbf{U}(-\lambda_m^{-2} t_m)\boldsymbol{\phi}\|^3_{\mathbf{L}^3_{t,x}}\right\} = 0, \tag{2.11}$$

where,

$$\boldsymbol{\phi}_m := \lambda_m^{-2}[\mathbf{U}(\lambda_m^{-2} t_m)\boldsymbol{\phi}]\left(\frac{x - x_m}{\lambda_m}\right). \tag{2.12}$$

Proof. We start noticing that, up to a subsequence, we may assume

$$\|\mathbf{h}_m\|_{\dot{\mathbf{H}}^1_x} \leqslant 2A \quad \text{and} \quad \|\mathbf{U}(t)\mathbf{h}_m\|_{\mathbf{L}^4_{t,x}} \geqslant \frac{\epsilon}{2}.$$

Then, by Lemma 2.21, we see that for each m, there is $N_m \in 2^{\mathbb{Z}}$ such that

$$\|\mathbf{U}(t)P_{N_m}\mathbf{h}_m\|_{\mathbf{L}^4_{t,x}} \gtrsim \epsilon^2 A^{-1}.$$

On the other hand, by Strichartz and nonlinear Bernstein's inequality, we obtain

$$\|\mathbf{U}(t)P_{N_m}\mathbf{h}_m\|_{\mathbf{L}^4_{t,x}} \lesssim \|P_{N_m}\mathbf{h}_m\|_{\mathbf{L}^2_x} \lesssim N_m^{-1}A.$$

By interpolation,

$$\begin{aligned}\epsilon^2 A^{-1} &\lesssim \|\mathbf{U}(t)P_{N_m}\mathbf{h}_m\|_{\mathbf{L}^4_{t,x}} \\ &\lesssim \|\mathbf{U}(t)P_{N_m}\mathbf{h}_m\|^{2/3}_{\mathbf{L}^{8/3}_{t,x}}\|\mathbf{U}(t)P_{N_m}\mathbf{h}_m\|^{1/3}_{\mathbf{L}^{\infty}_{t,x}} \\ &\lesssim N_m^{-2/3}A^{2/3}\|\mathbf{U}(t)P_{N_m}\mathbf{h}_m\|^{1/3}_{\mathbf{L}^{\infty}_{t,x}}.\end{aligned}$$

Therefore

$$N_m^{-2}\|\mathbf{U}(t)P_{N_m}\mathbf{h}_m\|_{\mathbf{L}^{\infty}_{t,x}} \gtrsim A\left(\frac{\epsilon}{A}\right)^6.$$

Then, there is a subsequence $(t_m, x_m) \in \mathbb{R}\times\mathbb{R}^6$ such that

$$N_m^{-2}\left|[\mathbf{U}(t_m)P_{N_m}\mathbf{h}_n](x_m)\right| \gtrsim A\left(\frac{\epsilon}{A}\right)^6. \tag{2.13}$$

We define now the special scale $\lambda_m = N_m^{-1}$. It remains to find the profile $\boldsymbol{\phi}$ and show that it satisfies (2.8) trough (2.10). Indeed, setting

$$\mathbf{g}_m(x) := \lambda_m[\mathbf{U}(t_m)\mathbf{h}_m](\lambda_m x + x_m),$$

a change of variables gives us

$$\|\mathbf{g}_m\|_{\dot{\mathbf{H}}^1_x} = \|\mathbf{h}_m\|_{\dot{\mathbf{H}}^1_x} \lesssim A.$$

Hence, up to a subsequence, we may choose $\boldsymbol{\phi}$ such that $\mathbf{g}_m \rightharpoonup \boldsymbol{\phi}$ weakly in $\dot{\mathbf{H}}^1_x$. This proves (2.8). Note that the assymptotic decoupling statement in (2.9) follows since $\dot{\mathbf{H}}^1_x$ is a Hilbert space. To prove the lower bound in (2.9), we consider $\check{\psi} := P_1\delta_0$ to denote the convolution kernel associated with P_1. Observe that, by definition, $\psi(\xi) = (P_1\delta_0)\hat{}= (\varphi(\xi)-\varphi(2\xi))\,\hat{\delta}_0 = \varphi(\xi)-\varphi(2\xi)$. Therefore, using Plancherel theorem,

$$\begin{aligned}\left\langle U_k(t_m)h_{km}, \lambda_m^{-4}\check{\psi}\left(\frac{x-x_m}{\lambda_m}\right)\right\rangle_{L^2_x} &= \int U_k(t_m)h_{km}(x)\overline{\lambda_m^{-4}\check{\psi}\left(\frac{x-x_m}{\lambda_m}\right)}dx \\ &= \int (U_k(t_m)h_{km})\hat{}(\xi)\left[\overline{\lambda_m^{-4}\check{\psi}\left(\frac{x-x_m}{\lambda_m}\right)}\right]\hat{}(\xi)d\xi \\ &= \int (U_k(t_m)h_{km})\hat{}(\xi)\lambda_m^{-4}\lambda_m^6 e^{ix_m\xi}\psi(\lambda_m\xi)d\xi \\ &= N_m^{-2}\int (U_k(t_m)h_{km})\hat{}(\xi)e^{ix_m\xi}\left[\varphi\left(\frac{\xi}{N_m}\right)-\varphi\left(\frac{2\xi}{N_m}\right)\right]d\xi \\ &= N_m^{-2}[U_k(t_m)P_{N_m}h_{km}](x_m).\end{aligned}$$

Thus, using change of variables and (2.13),

$$\begin{aligned}|\langle\phi_k,\check{\psi}\rangle_{L^2_x}| = |\lim_{m\to\infty}\langle g_{km},\check{\psi}\rangle_{L^2_x}| &= \left|\lim_{m\to\infty}\left\langle U_k(t_m)h_{km}, \lambda_m^{-4}\check{\psi}\left(\frac{x-x_m}{\lambda_m}\right)\right\rangle_{L^2_x}\right| \\ &= \lim_{m\to\infty} N_m^{-2}\left|[U_k(t_m)P_{N_m}h_{km}](x_m)\right| \\ &\gtrsim A\left(\frac{\epsilon}{A}\right)^6.\end{aligned} \tag{2.14}$$

On the other hand, by Hölder's inequality and Sobolev's embedding,

$$|\langle \phi_k, \check{\psi} \rangle_{L^2_x}| \lesssim \|\phi_k\|_{L^6_x} \|\check{\psi}\|_{L^{6/5}_x} \lesssim \|\phi_k\|_{\dot{H}^1_x}.$$

Putting both inequalities together and summing over k, we arrive at the lower bound of (2.9).

We now show (2.10). We start with decoupling for the $\mathbf{L}^4_{t,x}$ norm. Observe that, for $k = 1, ..., l$,

$$(i\alpha_k \partial_t)^{1/2} U_k(t) = (-\gamma_k \Delta)^{1/2} U_k(t)$$

as can be checked in (KOCH; TATARU; VISAN, 2014, page 242). Then by Hölder's inequality, on any compact set $K \subset \mathbb{R} \times \mathbb{R}^6$, we obtain, for $k = 1, ..., l$,

$$\|U_k(t) g_{km}\|_{H^{1/2}_{t,x}(K)} \lesssim \|\langle -\gamma_k \Delta \rangle^{1/2} g_{km}\|_{L^2_{t,x}(K)} \lesssim A.$$

Using this together with Rellich-Kondrashov theorem, up to a subsequence, we have

$$\mathbf{U}(t)\mathbf{g}_m \to \mathbf{U}(t)\boldsymbol{\phi} \quad \text{strongly in} \quad \mathbf{L}^2_{t,x}(K),$$

This is because $\mathbf{g}_m \rightharpoonup \boldsymbol{\phi}$ weakly in $\dot{\mathbf{H}}^1_x$, implies that $\mathbf{U}(t)\mathbf{g}_m$ converge to $\mathbf{U}(t)\boldsymbol{\phi}$ in the distribution sense on $\mathbb{R} \times \mathbb{R}^6$. Passing to another subsequence, we get that $\mathbf{U}(t)\mathbf{g}_m \to \mathbf{U}(t)\boldsymbol{\phi}$ almost everywhere on K. Finally, by a diagonal argument and, again, passing to a subsequence if necessary,

$$\mathbf{U}(t)\mathbf{g}_m \to \mathbf{U}(t)\boldsymbol{\phi} \quad \text{a.e. in } \mathbb{R} \times \mathbb{R}^6.$$

By Lemma 2.15 and a change of variables

$$\lim_{m\to\infty} \left[\|\mathbf{U}(t)\mathbf{h}_m\|^4_{\mathbf{L}^4_{t,x}} - \|\mathbf{U}(t)(\mathbf{h}_m - \boldsymbol{\phi}_m)\|^4_{\mathbf{L}^4_{t,x}} \right] = \|\mathbf{U}(t)\boldsymbol{\phi}\|^4_{\mathbf{L}^4_{t,x}}.$$

Thus, (2.10) will be proved provided we show that

$$\|\mathbf{U}(t)\boldsymbol{\phi}\|_{\mathbf{L}^4_{t,x}} \gtrsim \epsilon \left(\frac{\epsilon}{A}\right)^5. \tag{2.15}$$

To this end, we use (2.14), Mikhlin multiplier Theorem and Bernstein's estimate to get

$$\begin{aligned} A\left(\frac{\epsilon}{A}\right)^6 \lesssim \left|\langle \boldsymbol{\phi}, \check{\boldsymbol{\psi}} \rangle_{\mathbf{L}^2_x}\right| &= \left|\langle \mathbf{U}(t)\boldsymbol{\phi}, \mathbf{U}(-t)\check{\boldsymbol{\psi}} \rangle_{\mathbf{L}^2_x}\right| \\ &\lesssim \|\mathbf{U}(t)\boldsymbol{\phi}\|_{\mathbf{L}^4_x} \|\mathbf{U}(t)\check{\boldsymbol{\psi}}\|_{\mathbf{L}^{4/3}_x} \\ &\lesssim \|\mathbf{U}(t)\boldsymbol{\phi}\|_{\mathbf{L}^4_x} \end{aligned}$$

uniformly on $|t| \leqslant 1$. Then

$$\begin{aligned} \|\mathbf{U}(t)\boldsymbol{\phi}\|_{\mathbf{L}^4_{t,x}} &= \left[\int \|\mathbf{U}(t)\boldsymbol{\phi}\|^4_{\mathbf{L}^4_x} dt\right]^{1/4} \\ &\gtrsim \left[\int_0^1 \|\mathbf{U}(t)\boldsymbol{\phi}\|^4_{\mathbf{L}^4_x} dt\right]^{1/4} \\ &\gtrsim A\left(\frac{\epsilon}{A}\right)^6 = \epsilon\left(\frac{\epsilon}{A}\right)^5, \end{aligned}$$

showing (2.10). Finally, to show (2.11), we observe that, passing to a subsequence if necessary, we may assume that $\lambda_m^{-2} t_m \to t_0 \in [-\infty, \infty]$. If $|t_0| = \infty$, we approximate $\boldsymbol{\phi}$ on $\dot{\mathbf{H}}_x^1$ by Schwartz functions and use the fact that by Lemma 2.9

$$\|U_k(-t_m \lambda_m^{-2})\psi_k\|_{L_x^3} \to 0, \quad m \to \infty$$

for any $\psi_k \in \mathcal{S}(\mathbb{R}^6)$, $k = 1, ..., l$. Now, if $t_0 \in (-\infty, \infty)$, then (2.8) may be taken as $\lambda_m^2 \mathbf{h}_m(\lambda_m x + x_m) \rightharpoonup \mathbf{U}(t_0)\boldsymbol{\phi}(x)$ weakly in $\dot{\mathbf{H}}_x^1$. Using Rellich-Kondrashov Theorem and Lemma 2.15 as before, we get the desired. □

Definition 2.23. *(Symmetry group) For any position $x_0 \in \mathbb{R}^6$ and scale parameter $\lambda > 0$, we set the unitary transformation $\mathbf{g}_{x_0,\lambda} : \dot{\mathbf{H}}^1(\mathbb{R}^6) \to \dot{\mathbf{H}}^1(\mathbb{R}^6)$ by*

$$[\mathbf{g}_{x_0,\lambda}\mathbf{f}](x) := \lambda^{-2}\mathbf{f}\left(\lambda^{-1}(x - x_0)\right).$$

Let $\mathbf{G}$ be the collection of such transformations. For a function $\mathbf{u} : I \times \mathbb{R}^6 \to \mathbb{C}$, we define

$$[T_{\mathbf{g}_{x_0,\lambda}}\mathbf{u}](t, x) := \lambda^{-2}\mathbf{u}\left(\lambda^{-2}t, \lambda^{-1}(x - x_0)\right).$$

Thus, if $\mathbf{u}$ is a solution to (1.2), then, for $\mathbf{g} \in \mathbf{G}$, $T_{\mathbf{g}}\mathbf{u}$ is also a solution to (1.2) with initial data $\mathbf{g}\mathbf{u}_0$.

Remark 2.24. *In order to simplify the formulas along the work, we will use the following notations for $\lambda_n^j > 0$ and $x_n^j \in \mathbb{R}^6$,*

$$(\mathbf{g}_n^j\mathbf{u})(x) := (\lambda_n^j)^{-2}\mathbf{u}\left(\frac{x - x_n^j}{\lambda_n^j}\right) \quad \textit{and} \quad [(\mathbf{g}_n^j)^{-1}\mathbf{u}](x) := (\lambda_n^j)^2\mathbf{u}(\lambda_n^j x + x_n^j).$$

Note that $\nabla\mathbf{g}_n^j\mathbf{u}(x) = (\lambda_n^j)^{-3}\nabla\mathbf{u}\left(\frac{x - x_n^j}{\lambda_n^j}\right)$. Then,

$$\begin{aligned}
\|\mathbf{g}_n^j\mathbf{u}\|_{\dot{\mathbf{H}}_x^1} &= \left(\int |\nabla\mathbf{g}_n^j\mathbf{u}(x)|^2 dx\right)^{1/2} \\
&= \left(\int \left|(\lambda_n^j)^{-3}\nabla\mathbf{u}\left(\frac{x - x_n^j}{\lambda}\right)\right|^2 dx\right)^{1/2} \\
&= \left(\int (\lambda_n^j)^{-6}(\lambda_n^j)^6|\nabla\mathbf{u}(y)|^2 dy\right)^{1/2} \\
&= \left(\int |\nabla\mathbf{u}(y)|^2 dy\right)^{1/2} \\
&= \|\mathbf{u}\|_{\dot{\mathbf{H}}_x^1}.
\end{aligned} \tag{2.16}$$

In the same way, it is possible to show that $\|\mathbf{u}\|_{\dot{\mathbf{H}}_x^1} = \|(\mathbf{g}_n^j)^{-1}\mathbf{u}\|_{\dot{\mathbf{H}}_x^1}$ and $\langle\mathbf{g}_n^j\mathbf{u}_1, \mathbf{u}_2\rangle_{\dot{\mathbf{H}}_x^1} = \langle\mathbf{u}_1, (\mathbf{g}_n^j)^{-1}\mathbf{u}_2\rangle_{\dot{\mathbf{H}}_x^1}$ for all $\mathbf{u}_1, \mathbf{u}_2 \in \dot{\mathbf{H}}_x^1$. Besides that, we also use the notation

$$\boldsymbol{\phi}_n^j(x) := (\lambda_n^j)^{-2}[\mathbf{U}(t_n^j)\boldsymbol{\phi}^j]\left(\frac{x - x_n^j}{\lambda_n^j}\right) = [\mathbf{g}_n^j\mathbf{U}(t_n^j)\boldsymbol{\phi}^j](x).$$

Remark 2.25. *Notice that if $f \in C_0^\infty(\mathbb{R}^d)$ and $\{(t^n, x^n)\} \subset \mathbb{R} \times \mathbb{R}^d$ is a sequence. Then, for $k = 1, ..., l$, we have $U_k(t^n) f(x + x^n) \rightharpoonup 0$ weakly in $\dot{H}_x^1$ as $n \to \infty$ whenever $|t^n| \to \infty$ or $|x^n| \to \infty$. Indeed, we need to show that for all $u \in (\dot{H}_x^1)^*$*

$$\int \nabla u \cdot \nabla U_k(t^n) f(x + x^n) \to 0.$$

Since $|x^n| \to \infty$, there is no loss of generality in assuming that $|x_1^n| \to \infty$ as $n \to \infty$. Using Fourier transform and the change of variables $x_1^n \xi = \eta$, we have

$$\begin{aligned}\int \nabla u \cdot \nabla U_k(t^n) f(x + x^n) &= -\int |\xi| \hat{u} |\xi| e^{-it^n|\xi|^2} e^{ix^n\cdot\xi} \hat{f}(\xi) d\xi \\ &= -\int \frac{|\eta|^2}{|x_1^n|^2} \hat{u}\left(\frac{\eta}{x_1^n}\right) e^{-it^n \frac{|\eta|^2}{|x_1^n|^2}} e^{ix^n \cdot \frac{\eta}{x_1^n}} \hat{f}\left(\frac{\eta}{x_1^n}\right) \frac{d\eta}{|x_1^n|^d} \to 0,\end{aligned}$$

as $n \to \infty$. The case when $|t_n| \to \infty$ is treated in the same way with a change of variables $\eta = \sqrt{t_n}\xi$.

Theorem 2.26. *(Linear profiile decomposition). Let $\{\mathbf{u}_n\}$ be a sequence of bounded functions in $\dot{\mathbf{H}}^1(\mathbb{R}^6)$. Passing to be a subsequence if necessary, there is $J^* \in \{0, 1, ...\} \cup \{\infty\}$, functions $\{\boldsymbol{\phi}^j\}_{j=1}^{J^*} \subset \dot{\mathbf{H}}^1(\mathbb{R}^6)$, symmetry group elements $\mathbf{g}_n^j \in \mathbf{G}$, and $\{(t_n^j, x_n^j)\} \subset \mathbb{R} \times \mathbb{R}^6$ such that for all $0 \leqslant J \leqslant J^*$ finite, we have the decomposition*

$$\mathbf{u}_n = \sum_{j=1}^{J} [\mathbf{g}_n^j \mathbf{U}(t_n^j) \boldsymbol{\phi}^j] + \mathbf{w}_n^J \tag{2.17}$$

with the following properties:

$$\lim_{J \to J^*} \limsup_{n \to \infty} \|\mathbf{U}(t)\mathbf{w}_n^J\|_{\mathbf{L}_{t,x}^4} = 0, \tag{2.18}$$

$$\lim_{n \to \infty} \left[\|\nabla \mathbf{u}_n\|_{\mathbf{L}_x^2}^2 - \sum_{j=1}^{J} \|\nabla \boldsymbol{\phi}^j\|_{\mathbf{L}_x^2}^2 - \|\nabla \mathbf{w}_n^J\|_{\mathbf{L}_x^2}^2 \right] = 0, \tag{2.19}$$

$$\mathbf{U}(-t_n^J) \left[(\mathbf{g}_n^J)^{-1} \mathbf{w}_n^J \right] \rightharpoonup 0 \quad \text{in } \dot{\mathbf{H}}^1(\mathbb{R}^6). \tag{2.20}$$

Moreover, if $j \neq k$, then

$$\frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + \frac{|x_n^j - x_n^k|^2}{\lambda_n^j \lambda_n^k} + \frac{|t_n^j (\lambda_n^j)^2 - t_n^k (\lambda_n^k)^2|}{\lambda_n^j \lambda_n^k} \to \infty \quad \text{as } n \to \infty. \tag{2.21}$$

Proof. We follow the ideias presented in (KOCH; TATARU; VISAN, 2014, Chapter 4, page 246). We proceed inductively. To start, we define $\mathbf{w}_n^0 := \mathbf{u}_n$. Now, suppose that we have the decomposition up to level $J \geqslant 0$ obeying the hypothesis (2.19) and (2.20). Then, up to a subsequence, we define

$$A_J := \lim_{n \to \infty} \|\mathbf{w}_n^J\|_{\dot{\mathbf{H}}_x^1} \quad \text{and} \quad \epsilon_J := \lim_{n \to \infty} \|\mathbf{U}(t)\mathbf{w}_n^J\|_{\mathbf{L}_{t,x}^4}.$$

If $\epsilon_J = 0$, it is enough to choose $J^* = J$. Otherwise, we apply Proposition 2.22 in $\mathbf{w}_n^J$. Therefore, passing to a subsequence in n, we find $\boldsymbol{\phi}^{J+1} \in \dot{\mathbf{H}}_x^1$, $\{\lambda_n^{J+1}\} \subset (0,\infty)$, and $\{(t_n^{J+1}, x_n^{J+1})\} \subset \mathbb{R} \times \mathbb{R}^6$, where we denote the time parameters given in Proposition 2.22 as the following: $t_n^{J+1} = -\lambda_n^{-2} t_n$.

According to Proposition 2.22, the profile $\boldsymbol{\phi}^{J+1}$ is setting as a weak limit, namely,

$$\boldsymbol{\phi}^{J+1} = \lim_{n\to\infty} (\mathbf{g}_n^{J+1})^{-1}[\mathbf{U}(t_n^{J+1}(\lambda_n^{J+1})^2)\mathbf{w}_n^J] = \lim_{n\to\infty} \mathbf{U}(-t_n^{J+1})[(\mathbf{g}_n^{J+1})^{-1}\mathbf{w}_n^J].$$

Define $\boldsymbol{\phi}_n^{J+1} := \mathbf{g}_n^{J+1}\mathbf{U}(t_n^{J+1})\boldsymbol{\phi}^{J+1}$. Now, setting $\mathbf{w}_n^{J+1} := \mathbf{w}_n^J - \boldsymbol{\phi}_n^{J+1}$, by definition of $\boldsymbol{\phi}^{J+1}$, we obtain

$$\mathbf{U}(t_n^{J+1})(\mathbf{g}_n^{J+1})^{-1}\mathbf{w}_n^{J+1} \rightharpoonup 0, \quad \text{weakly in } \dot{\mathbf{H}}_x^1.$$

This proves (2.20) up to level $J+1$. Moreover, by Proposition 2.22, we also have

$$\lim_{n\to\infty} \left\{ \|\mathbf{w}_n^J\|_{\dot{\mathbf{H}}_x^1}^2 - \|\mathbf{w}_n^{J+1}\|_{\dot{\mathbf{H}}_x^1}^2 - \|\boldsymbol{\phi}^{J+1}\|_{\dot{\mathbf{H}}_x^1}^2 \right\} = 0.$$

Combining with the induction hypothesis, we get (2.19) up to level $J+1$.

Passing to a new subsequence, and using, again, Proposition 2.22, we get

$$\begin{aligned} A_{J+1}^2 &= \lim_{n\to\infty} \|\mathbf{w}_n^{J+1}\|_{\dot{\mathbf{H}}_x^1}^2 \leqslant A_J^2 \left[1 - C\left(\frac{\epsilon_J}{A_J}\right)^{12} \right] \leqslant A_J^2, \\ \epsilon_{J+1}^4 &= \lim_{n\to\infty} \|\mathbf{U}(t)\mathbf{w}_n^{J+1}\|_{\mathbf{L}_{t,x}^4}^4 \leqslant \epsilon_J^4 \left[1 - C\left(\frac{\epsilon_J}{A_J}\right)^{20} \right]. \end{aligned} \tag{2.22}$$

If $\epsilon_{J+1} = 0$, we stop and put $J^* = J+1$. In this case, (2.18) is immediately verified. If $\epsilon_{J+1} > 0$, we proceed with the induction process. If the algorithm does not finishes in finitely many steps, we choose $J^* = \infty$. In this case, (2.22) implies that $\epsilon_J \to 0$ if $J \to \infty$ and then, (2.18) hold.

Next, we show the orthogonality condition (2.21). Suppose that such a condition is false for some pair (j,k). Without loss of generality, we may assume that $j < k$ and (2.21) is true to all pairs (j,m), with $j < m < k$. Passing to a subsequence, we may assume that

$$\frac{\lambda_n^j}{\lambda_n^k} \to \lambda_0 \in (0,\infty), \quad \frac{x_n^j - x_n^k}{\sqrt{\lambda_n^j \lambda_n^k}} \to x_0 \quad \text{and} \quad \frac{t_n^j(\lambda_n^j)^2 - t_n^k(\lambda_n^k)^2}{\lambda_n^j \lambda_n^k} \to t_0. \tag{2.23}$$

From the inductive relation

$$\mathbf{w}_n^{k-1} = \mathbf{w}_n^j - \sum_{m=j+1}^{k-1} \boldsymbol{\phi}_n^m$$

and from the definition of $\boldsymbol{\phi}^k$, we get

$$\begin{aligned} \boldsymbol{\phi}^k &= \lim_{n\to\infty} \mathbf{U}(-t_n^k)[(\mathbf{g}_n^k)^{-1}\mathbf{w}_n^{k-1}] \\ &= \lim_{n\to\infty} \mathbf{U}(-t_n^k)[(\mathbf{g}_n^k)^{-1}\mathbf{w}_n^j] - \sum_{m=j+1}^{k-1} \lim_{n\to\infty} \mathbf{U}(-t_n^k)[(\mathbf{g}_n^k)^{-1}\boldsymbol{\phi}_n^m]. \end{aligned} \tag{2.24}$$

To reach the contradiction, we prove that these weak limits are all zero, which contradicts the nontriviality of $\boldsymbol{\phi}^k$.

We write,

$$\begin{aligned}\mathbf{U}(-t_n^k)[(\mathbf{g}_n^k)^{-1}\mathbf{w}_n^j] &= \mathbf{U}(-t_n^k)(\mathbf{g}_n^k)^{-1}\mathbf{g}_n^j\mathbf{U}(t_n^j)[\mathbf{U}(-t_n^j)(\mathbf{g}_n^j)^{-1}\mathbf{w}_n^j] \\ &= (\mathbf{g}_n^k)^{-1}\mathbf{g}_n^j\mathbf{U}\left(t_n^j - t_n^k\frac{(\lambda_n^k)^2}{(\lambda_n^j)^2}\right)[\mathbf{U}(-t_n^j)(\mathbf{g}_n^j)^{-1}\mathbf{w}_n^j].\end{aligned}$$

Note that by (2.23)

$$t_n^j - t_n^k\frac{(\lambda_n^k)^2}{(\lambda_n^j)^2} = \frac{t_n^j(\lambda_n^j)^2 - t_n^k(\lambda_n^k)^2}{\lambda_n^j\lambda_n^k}\cdot\frac{\lambda_n^k}{\lambda_n^j} \to \frac{t_0}{\lambda_0}. \tag{2.25}$$

Using (2.25), (2.20), together with the following facts: The adjoints of the unitary operators $(\mathbf{g}_n^k)^{-1}\mathbf{g}_n^j$ converge strongly and, if $\mathbf{f}_n \rightharpoonup 0$ in $\dot{\mathbf{H}}_x^1$, then $\mathbf{U}(t_n)f_n \rightharpoonup 0$ in $\dot{\mathbf{H}}_x^1$, we get that the first term on the right-hand side of (2.24) vanishes.

To complete the proof, it remains to show that the second term on the right-hand side of (2.24) vanishes. To this end, consider $j < m < k$ and write

$$\mathbf{U}(-t_n^k)(\mathbf{g}_n^k)^{-1}\boldsymbol{\phi}_n^m = (\mathbf{g}_n^k)^{-1}\mathbf{g}_n^j\mathbf{U}\left(t_n^j - t_n^k\frac{(\lambda_n^k)^2}{(\lambda_n^j)^2}\right)[\mathbf{U}(-t_n^j)(\mathbf{g}_n^j)^{-1}\boldsymbol{\phi}_n^m].$$

Arguing as before, it suffices to show that

$$\mathbf{U}(-t_n^j)(\mathbf{g}_n^j)^{-1}\boldsymbol{\phi}_n^m = \mathbf{U}(-t_n^j)(\mathbf{g}_n^j)^{-1}\mathbf{g}_n^m\mathbf{U}(t_n^m)\boldsymbol{\phi}^m \rightharpoonup 0 \quad \text{in } \dot{\mathbf{H}}_x^1.$$

By density, this reduces to show that

$$I_n := \mathbf{U}(-t_n^j)(\mathbf{g}_n^j)^{-1}\mathbf{g}_n^m\mathbf{U}(t_n^m)\boldsymbol{\phi} \rightharpoonup 0 \quad \text{in } \dot{\mathbf{H}}_x^1, \tag{2.26}$$

for all $\boldsymbol{\phi} \in \mathbf{C}_c^\infty(\mathbb{R}^d)$. We may rewrite I_n as the following

$$I_n = \left(\frac{\lambda_n^j}{\lambda_n^m}\right)^2\left[\mathbf{U}\left(t_n^m - t_n^j\left(\frac{\lambda_n^j}{\lambda_n^m}\right)^2\right)\boldsymbol{\phi}\right]\left(\frac{\lambda_n^j x + x_n^j - x_n^m}{\lambda_n^m}\right).$$

Recalling that (2.21) holds for the pair (j, m), we first show that (2.26) holds when the scaling parameters are not comparable, that is,

$$\lim_{n\to\infty}\left(\frac{\lambda_n^j}{\lambda_n^m} + \frac{\lambda_n^m}{\lambda_n^j}\right) = \infty. \tag{2.27}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}|\langle I_n, \boldsymbol{\psi}\rangle_{\dot{\mathbf{H}}_x^1}| &\lesssim \min\left\{\|\Delta I_n\|_{\mathbf{L}_x^2}\|\boldsymbol{\psi}\|_{\mathbf{L}_x^2}, \|I_n\|_{\mathbf{L}_x^2}\|\Delta\boldsymbol{\psi}\|_{\mathbf{L}_x^2}\right\} \\ &\lesssim \min\left\{\frac{\lambda_n^j}{\lambda_n^m}\|\Delta\boldsymbol{\phi}\|_{\mathbf{L}_x^2}\|\boldsymbol{\psi}\|_{\mathbf{L}_x^2}, \frac{\lambda_n^m}{\lambda_n^j}\|\boldsymbol{\phi}\|_{\mathbf{L}_x^2}\|\Delta\boldsymbol{\psi}\|_{\mathbf{L}_x^2}\right\},\end{aligned}$$

which converge to zero if $n \to \infty$, for all $\boldsymbol{\psi} \in \mathbf{C}_c^\infty(\mathbb{R}^6)$. Therefore, if (2.27) holds, we get (2.26).

From now on, we assume that

$$\lim_{n\to\infty} \frac{\lambda_n^j}{\lambda_n^m} = \lambda_1 \in (0,\infty).$$

Suppose now that the time parameters diverge, that is,

$$\lim_{n\to\infty} \frac{|t_n^j(\lambda_n^j)^2 - t_n^m(\lambda_n^m)^2|}{\lambda_n^j \lambda_n^m} = \infty.$$

Thus, we also have

$$\left| t_n^m - t_n^j \left(\frac{\lambda_n^j}{\lambda_n^m}\right)^2 \right| = \frac{|t_n^j(\lambda_n^j)^2 - t_n^m(\lambda_n^m)^2|}{\lambda_n^j \lambda_n^m} \cdot \frac{\lambda_n^j}{\lambda_n^m} \to \infty, \quad \text{as } n \to \infty.$$

Under this condition, (2.26) follows from

$$\lambda_1^2 \left[\mathbf{U}\left(t_n^m - t_n^j \left(\frac{\lambda_n^j}{\lambda_n^m}\right)^2\right) \boldsymbol{\phi}\right] \left(\lambda_1 x + \frac{x_n^j - x_n^m}{\lambda_n^m}\right) \rightharpoonup 0 \quad \text{in } \dot{\mathbf{H}}_x^1,$$

which is a direct consequence of Remark 2.25.

Finally, if we have

$$\frac{\lambda_n^j}{\lambda_n^m} \to \lambda_1 \in (0,\infty), \quad \frac{|t_n^j(\lambda_n^j)^2 - t_n^m(\lambda_n^m)^2|}{\lambda_n^j \lambda_n^m} \to t_1, \quad \text{but} \quad \frac{|x_n^j - x_n^m|^2}{\lambda_n^j \lambda_n^m} \to \infty. \tag{2.28}$$

Then, we should have $t_n^m - t_n^j(\lambda_n^j)^2/(\lambda_n^m)^2 \to \lambda_1 t_1$. Then, it suffices to show that

$$\lambda_1^2 \mathbf{U}(t_1\lambda_1)\boldsymbol{\phi}(\lambda_1 x + y_n) \rightharpoonup 0 \quad \text{in } \dot{\mathbf{H}}_x^1, \tag{2.29}$$

where

$$y_n := \frac{x_n^j - x_n^m}{\lambda_n^m} = \frac{x_n^j - x_n^m}{\sqrt{\lambda_n^j \lambda_n^m}} \sqrt{\frac{\lambda_n^j}{\lambda_n^m}} \to \infty \quad \text{as}\, n \to \infty.$$

and this follows from Remark 2.25.

Finally, we prove the last statement of the theorem, with respect to behavior of t_n^j. For each j, passing to a subsequence, we may assume that $t_n^j \to t^j \in [-\infty,\infty]$. Using a diagonal argument, we may assume that such limit exists for all $j \geqslant 1$.

Fixing $j \geqslant 1$, if $t^j = \pm\infty$, there is nothing to show. Suppose that $t^j \in (-\infty,\infty)$. Then, since we change $\boldsymbol{\phi}^j$ by $\mathbf{U}(t^j)\boldsymbol{\phi}^j$, we may redefine $t_n^j \equiv 0$. Indeed, we may incorporate the errors into $\mathbf{w}_n^J$, namely,

$$\lim_{n\to\infty} \|\mathbf{g}_n^j \mathbf{U}(t_n^j)\boldsymbol{\phi}^j - \mathbf{g}_n^j \mathbf{U}(t^j)\boldsymbol{\phi}^j\|_{\dot{\mathbf{H}}_x^1} = 0,$$

which follows from the strong convergence of the linear propagator, finishing the proof. □

2.2.4 Asymptotic decoupling

We start defining the operators T_n^j by

$$(T_n^j u)(t,x) := (\lambda_n^j)^{-2} u\left(\frac{t}{(\lambda_n^j)^2} + t_n^j, \frac{x - x_n^j}{\lambda_n^j}\right),$$

where $(\lambda_n^j) \subset (0,\infty)$ and $(t_n^j, x_n^j) \subset \mathbb{R} \times \mathbb{R}^d$. We have the following result

Lemma 2.27. *Suppose that the parameters associated to j,k are orthogonal in the sense of (2.21). Then, for each $\psi^j, \psi^k \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$,*

$$\|T_n^j\psi^j T_n^k\psi^k\|_{L_{t,x}^{\frac{d+2}{d-2}}} + \|T_n^j\psi^j \nabla(T_n^k\psi^k)\|_{L_{t,x}^{\frac{d+2}{d-1}}} + \|\nabla(T_n^j\psi^j)\nabla(T_n^k\psi^k)\|_{L_{t,x}^{\frac{d+2}{d}}}$$

converges to zero as $n \to \infty$.

Proof. See (KOCH; TATARU; VISAN, 2014, Lemma 7.1, page 261). □

2.2.5 Coercivity lemmas

Lemma 2.28. *Let $I \subset \mathbb{R}$ be an open interval with $0 \in I$, $a \in \mathbb{R}$, $b > 0$ and $q > 1$. Define $\gamma = (bq)^{-1/(q-1)}$ and $f(r) = a - r + br^q$, for $r > 0$. Let $G(t)$ be a nonnegative continuous function such that $f \circ G \geqslant 0$ in I. Assume that $a < (1-\delta)\left(1 - \frac{1}{q}\right)\gamma$, for some $\delta > 0$ sufficiently small, we have*

(i) If $G(0) < \gamma$ then there exists $\delta_1 = \delta_1(\delta) > 0$ such that $G(t) < (1-\delta_1)\gamma$, for all $t \in I$;

(ii) If $G(0) > \gamma$ then there exists $\delta_2 = \delta_2(\delta)$ such that $G(t) > (1+\delta_2)\gamma$, for all $t \in I$.

Proof. See Corollary 3.2 in (PASTOR, 2015). □

The next lemmas reproduce observations of (KENIG; MERLE, 2006). We include details for the sake of completeness.

Lemma 2.29. *(Coercivity I). Assume that $\mathbf{u}_0 \in \dot{\mathbf{H}}^1(\mathbb{R}^6)$ and let $\mathbf{u}$ be a solution of (1.2) with maximal existence interval I. Let $\boldsymbol{\psi} \in \mathcal{G}_6$ be a ground state. Suppose that*

$$E(\mathbf{u}_0) < (1-\tilde{\delta})E(\boldsymbol{\psi}).$$

(i) If

$$K(\mathbf{u}_0) < K(\boldsymbol{\psi}),$$

then there exists $\tilde{\delta}_1 = \tilde{\delta}_1(\tilde{\delta})$ such that

$$K(\mathbf{u}(t)) < (1-\tilde{\delta}_1)K(\boldsymbol{\psi}),$$

for all $t \in I$.

(ii) If

$$K(\mathbf{u}_0) > K(\boldsymbol{\psi}),$$

then there exists $\tilde{\delta}_2 = \tilde{\delta}_2(\tilde{\delta})$ *such that*

$$K(\mathbf{u}(t)) > (1 + \tilde{\delta}_2)K(\boldsymbol{\psi}),$$

for all $t \in I$.

Proof. From the conservation of the energy and (1.24), we deduce

$$K(\mathbf{u}(t)) \leqslant E(\mathbf{u}_0) + 2C_6 K(\mathbf{u}(t))^{3/2}, \quad \forall t \in I. \tag{2.30}$$

Let $G(t) = K(\mathbf{u}(t))$, $a = E(\mathbf{u}_0)$, $b = 2C_6$ and $q = 3/2$ in Lemma 2.28. By (2.30) we see that $f \circ G \geqslant 0$ on I. Besides that, (1.25) gives us

$$\gamma = (bq)^{-\frac{1}{q-1}} = (3C_6)^{-2} = K(\boldsymbol{\psi}).$$

By Lemma 2.28 we get the result. □

Lemma 2.30. *(Energy coercivity). Under hypothesis of Lemma 2.29 we have*

(i) If

$$K(\mathbf{u}_0) < K(\boldsymbol{\psi}),$$

then there exists $\delta' = \delta'(\tilde{\delta}) > 0$ *such that*

$$K(\mathbf{u}(t)) - 3P(\mathbf{u}(t)) \geqslant \delta' K(\mathbf{u}(t)),$$

for all $t \in I$.

(ii) If

$$K(\mathbf{u}_0) > K(\boldsymbol{\psi}),$$

then there exists $\delta'' = \delta''(\tilde{\delta}) > 0$ *such that*

$$K(\mathbf{u}(t)) - 3P(\mathbf{u}(t)) \leqslant -\delta'' K(\mathbf{u}(t)),$$

for all $t \in I$.

Proof. We will just show item (i). The second one is proved in an analogous way. Using (1.24), Lemma 2.28 and Lemma 2.29, we deduce

$$1 - 3\frac{P(\mathbf{u}(t))}{K(\mathbf{u}(t))} \geqslant 1 - 3C_6 K(\mathbf{u}(t))^{1/2} = 1 - \left[\frac{K(\mathbf{u}(t))}{K(\boldsymbol{\psi})}\right]^{1/2} \geqslant 1 - (1 - \tilde{\delta}_1)^{1/2} =: \delta'.$$

Multiplying both sides by $K(\mathbf{u}(t))$ we get the result. □

Lemma 2.31. *(Energy trapping). Let* $\mathbf{u}$ *be a solution of* (1.2) *with maximal existence interval* I *and initial data* $\mathbf{u}_0$*. If* $E(\mathbf{u}_0) \leqslant (1-\delta)E(\boldsymbol{\psi})$ *and* $K(\mathbf{u}_0) \leqslant (1-\delta')K(\boldsymbol{\psi})$*, then*

$$K(\mathbf{u}(t)) \sim E(\mathbf{u}(t)), \quad \forall t \in I. \tag{2.31}$$

Proof. By (1.24) and $E(\mathbf{u}_0) \leqslant (1-\delta)E(\boldsymbol{\psi})$ we obtain

$$\begin{aligned} E(\mathbf{u}(t)) &\leqslant K(\mathbf{u}(t)) + |P(\mathbf{u}(t))| \\ &\leqslant K(\mathbf{u}(t)) + C_6|K(\mathbf{u}(t))|^{3/2} \\ &\leqslant \left(1 + C_6[(1-\delta')K(\boldsymbol{\psi})^{1/2}]\right) K(\mathbf{u}(t)). \end{aligned}$$

On the other hand,

$$\begin{aligned} E(\mathbf{u}(t)) &\geqslant \frac{1}{3}K(\mathbf{u}(t)) + \frac{2}{3}[K(\mathbf{u}(t)) - 3P(\mathbf{u}(t))] \\ &\geqslant \frac{1}{3}K(\mathbf{u}(t)) + \frac{2}{3}\delta' K(\mathbf{u}(t)) \\ &= \frac{1}{3}(1 + 2\delta')K(\mathbf{u}(t)). \end{aligned}$$

Combining both inequalities, we get the result. □

2.2.6 Virial identities

In this section we present some virial identities that will be useful in our analyses. Originally, these kind of identity was introduced in (GLASSEY, 1973) in the context of the wave equation.

Proposition 2.32. *Assume that* $\mathbf{u}_0 \in \mathbf{H}^1(\mathbb{R}^6)$ *and* $x\mathbf{u}_0 \in \mathbf{L}^2(\mathbb{R}^6)$*. Define*

$$V(t) = \sum_{k=1}^{l} \frac{\alpha_k^2}{\gamma_k}\|xu_k(t)\|_{L^2}^2 = \sum_{k=1}^{l} \frac{\alpha_k^2}{\gamma_k}\int |x|^2|u_k(t,x)|^2 dx. \tag{2.32}$$

Then,

$$V'(t) = 4\sum_{k=1}^{l} \alpha_k \mathrm{Im}\int \nabla u_k \cdot x\bar{u}_k dx \tag{2.33}$$

and

$$V''(t) = 12E(\mathbf{u}_0) - 4K(\mathbf{u}), \tag{2.34}$$

for all $t \in I$*.*

Proof. See Proposition 5.3 in (NOGUERA; PASTOR, 2021). □

Proposition 2.33. *Assume that* $\mathbf{u}_0 \in \mathbf{H}^1(\mathbb{R}^6)$ *and let* $\mathbf{u}$ *be the corresponding solution of* (1.2)*. Let* $\varphi \in C_0^\infty(\mathbb{R}^6)$ *and define*

$$M(t) = \frac{1}{2}\int \varphi(x)\left(\sum_{k=1}^{l}\frac{\alpha_k^2}{\gamma_k}|u_k|^2\right) dx.$$

Then,

$$M'(t) = \sum_{k=1}^{l} \alpha_k \mathrm{Im} \int \nabla\varphi \cdot \nabla u_k \bar{u}_k dx,$$

and

$$\begin{aligned} M''(t) = 2 \sum_{1\leqslant m,j\leqslant 6} \mathrm{Re} \int \frac{\partial^2\varphi}{\partial x_m \partial x_j} \left[\sum_{k=1}^{l} \gamma_k \partial_{x_j} \bar{u}_k \partial_{x_m} u_k \right] dx \\ - \frac{1}{2} \int \Delta^2\varphi \left(\sum_{k=1}^{l} \gamma_k |u_k|^2 \right) dx - \mathrm{Re} \int \Delta\varphi F(\mathbf{u}) dx. \end{aligned} \tag{2.35}$$

Proof. See Theorem 5.7 in (NOGUERA; PASTOR, 2021). □

CHAPTER 3

BLOW-UP OF THE RADIALLY SYMMETRIC SOLUTIONS FOR A CUBIC NLS TYPE SYSTEM IN DIMENSION 4

In this chapter we will study the following cubic-type system

$$\begin{cases} iu_t + \Delta u - u + \left(\frac{1}{9}|u|^2 + 2|w|^2\right)u + \frac{1}{3}\bar{u}^2 w = 0, \\ i\sigma w_t + \Delta w - \mu w + (9|w|^2 + 2|u|^2)w + \frac{1}{9}u^3 = 0. \end{cases} \tag{3.1}$$

We are going to show local well-posedness to the Cauchy problem associated, existence of ground state solutions and blow-up for radially symmetric initial data.

3.1 Local well-posedness

As mentioned before, this section is devoted to prove the local well-posedness to the Cauchy problem associated to (1.1) in $H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4)$. We work in the space $Y(I)$, defined in (1.6), in which the norm is given by

$$\|f\|_Y = \|f\|_{L_t^\infty H_x^1} + \|f\|_{L_t^4 H_x^{1,\frac{8}{3}}}.$$

Before proceeding to the main result, notice that, using Hölder and Sobolev's inequalities, we have

$$\|fgh\|_{L_x^{\frac{8}{5}}} \lesssim \|f\|_{L_x^8}\|g\|_{L_x^8}\|h\|_{L_x^{\frac{8}{3}}} \lesssim \|f\|_{H_x^{1,\frac{8}{3}}}\|g\|_{H_x^{1,\frac{8}{3}}}\|h\|_{H_x^{1,\frac{8}{3}}}. \tag{3.2}$$

and

$$\|fgh\|_{L_t^{4/3}L_x^{\frac{8}{5}}} \lesssim \|f\|_{L_t^4 H_x^{1,\frac{8}{3}}}\|g\|_{L_t^4 H_x^{1,\frac{8}{3}}}\|h\|_{L_t^4 H_x^{1,\frac{8}{3}}}. \tag{3.3}$$

In a similar way, exchanging fgh for the product $fg\nabla h$ we obtain

$$\|fg\nabla h\|_{L_t^{\frac{4}{3}}L_x^{\frac{8}{5}}} \lesssim \|f\|_{L_t^4 L_x^{\frac{8}{3}}}\|g\|_{L_t^4 L_x^{\frac{8}{3}}}\|\nabla h\|_{L_t^4 L_x^{\frac{8}{3}}} \lesssim \|f\|_{L_t^4 H_x^{1,\frac{8}{3}}}\|g\|_{L_t^4 H_x^{1,\frac{8}{3}}}\|h\|_{L_t^4 H_x^{1,\frac{8}{3}}}. \tag{3.4}$$

Proof of Theorem 1.1. We start with some estimates for the nonlinearities F and G defined in (1.8). A direct calculation shows that

$$|F(u,w) - F(u',w')| \lesssim (|u|^2 + |u'|^2 + |w|^2 + |w'|^2)(|u-u'| + |w-w'|),$$

and

$$|G(u,w) - G(u',w')| \lesssim (|u|^2 + |u'|^2 + |w|^2 + |w'|^2)(|u-u'| + |w-w'|).$$

Then, using (3.2), we get

$$\begin{aligned}\|(|u|^2 + |u'|^2 + |w|^2 + |w'|^2)(|u-u'|)\|_{L_x^{\frac{8}{5}}} \\ \lesssim \left(\|u\|^2_{H_x^{1,\frac{8}{3}}} + \|u'\|^2_{H_x^{1,\frac{8}{3}}} + \|w\|^2_{H_x^{1,\frac{8}{3}}} + \|w'\|^2_{H_x^{1,\frac{8}{3}}}\right)\|u-u'\|_{H_x^{1,\frac{8}{3}}}\end{aligned}$$

and

$$\begin{aligned}\|(|u|^2 + |u'|^2 + |w|^2 + |w'|^2)(|u-u'|)\|_{L_x^{\frac{8}{5}}} \\ \lesssim \left(\|u\|^2_{H_x^{1,\frac{8}{3}}} + \|u'\|^2_{H_x^{1,\frac{8}{3}}} + \|w\|^2_{H_x^{1,\frac{8}{3}}} + \|w'\|^2_{H_x^{1,\frac{8}{3}}}\right)\|w-w'\|_{H_x^{1,\frac{8}{3}}}.\end{aligned}$$

Hence,

$$\begin{aligned}\|F(u,w) - F(u',w')\|_{L_x^{\frac{8}{5}}} \\ \lesssim \left(\|u\|^2_{H_x^{1,\frac{8}{3}}} + \|u'\|^2_{H_x^{1,\frac{8}{3}}} + \|w\|^2_{H_x^{1,\frac{8}{3}}} + \|w'\|^2_{H_x^{1,\frac{8}{3}}}\right)\left(\|u-u'\|_{L_x^{\frac{8}{3}}} + \|w-w'\|_{L_x^{\frac{8}{3}}}\right),\end{aligned}$$

and using (3.3), we get

$$\begin{aligned}\|F(u,w) - F(u',w')\|_{L_t^{\frac{4}{3}}L_x^{\frac{8}{5}}} \lesssim \\ \left(\|u\|^2_{L_t^4 H_x^{1,\frac{8}{3}}} + \|u'\|^2_{L_t^4 H_x^{1,\frac{8}{3}}} + \|w\|^2_{L_t^4 H_x^{1,\frac{8}{3}}} + \|w'\|^2_{L_t^4 H_x^{1,\frac{8}{3}}}\right)\left(\|u-u'\|_{L_t^4 H_x^{1,\frac{8}{3}}} + \|w-w'\|_{L_t^4 H_x^{1,\frac{8}{3}}}\right).\end{aligned} \tag{3.5}$$

In a similar way, we get

$$\begin{aligned}\|G(u,w) - G(u',w')\|_{L_t^{\frac{4}{3}}L_x^{\frac{8}{5}}} \lesssim \\ \left(\|u\|^2_{L_t^4 H_x^{1,\frac{8}{3}}} + \|u'\|^2_{L_t^4 H_x^{1,\frac{8}{3}}} + \|w\|^2_{L_t^4 H_x^{1,\frac{8}{3}}} + \|w'\|^2_{L_t^4 H_x^{1,\frac{8}{3}}}\right)\left(\|u-u'\|_{L_t^4 H_x^{1,\frac{8}{3}}} + \|w-w'\|_{L_t^4 H_x^{1,\frac{8}{3}}}\right).\end{aligned} \tag{3.6}$$

Once more, a direct calculation give us

$$|\nabla[F(u,w) - F(u',w')]| \lesssim (|u|^2 + |u'|^2 + |w|^2 + |w'|^2)(|\nabla[u-u'] + \nabla[w-w'])$$

and

$$|\nabla[G(u,w)-G(u',w')]| \lesssim (|u|^2+|u'|^2+|w|^2+|w'|^2)(|\nabla[u-u']+\nabla[w-w']).$$

By the same argument used previously, we obtain

$$\|\nabla[F(u,w)-F(u',w')]\|_{L_t^{\frac{4}{3}}L_x^{\frac{8}{5}}} \lesssim \left(\|u\|^2_{L_t^4H_x^{1,\frac{8}{3}}}+\|u'\|^2_{L_t^4H_x^{1,\frac{8}{3}}}+\|w\|^2_{L_t^4H_x^{1,\frac{8}{3}}}+\|w'\|^2_{L_t^4H_x^{1,\frac{8}{3}}}\right)\left(\|u-u'\|_{L_t^4H_x^{1,\frac{8}{3}}}+\|w-w'\|_{L_t^4H_x^{1,\frac{8}{3}}}\right) \tag{3.7}$$

and

$$\|\nabla[G(u,w)-G(u',w')]\|_{L_t^{\frac{4}{3}}L_x^{\frac{8}{5}}} \lesssim \left(\|u\|^2_{L_t^4H_x^{1,\frac{8}{3}}}+\|u'\|^2_{L_t^4H_x^{1,\frac{8}{3}}}+\|w\|^2_{L_t^4H_x^{1,\frac{8}{3}}}+\|w'\|^2_{L_t^4H_x^{1,\frac{8}{3}}}\right)\left(\|u-u'\|_{L_t^4H_x^{1,\frac{8}{3}}}+\|w-w'\|_{L_t^4H_x^{1,\frac{8}{3}}}\right). \tag{3.8}$$

By Strichartz's inequality, (3.5), (3.6), (3.7) and (3.8), we have

$$\left\|\int_0^t U(t-\tau)[F(u,w)-F(u',w')]d\tau\right\|_{L_t^4H_x^{1,\frac{8}{3}}} \lesssim \|F(u,w)-F(u',w')\|_{L_t^{\frac{4}{3}}L_x^{\frac{8}{5}}}+\|\nabla[F(u,w)-F(u',w')]\|_{L_t^{\frac{4}{3}}L_x^{\frac{8}{5}}} \lesssim \left(\|u\|^2_{L_t^4H_x^{1,\frac{8}{3}}}+\|u'\|^2_{L_t^4H_x^{1,\frac{8}{3}}}+\|w\|^2_{L_t^4H_x^{1,\frac{8}{3}}}+\|w'\|^2_{L_t^4H_x^{1,\frac{8}{3}}}\right)\left(\|u-u'\|_{L_t^4H_x^{1,\frac{8}{3}}}+\|w-w'\|_{L_t^4H_x^{1,\frac{8}{3}}}\right). \tag{3.9}$$

Similarly

$$\left\|\int_0^t W(t-\tau)[G(u,w)-G(u',w')]d\tau\right\|_{L_t^4H_x^{1,8/3}} \lesssim \left(\|u\|^2_{L_t^4H_x^{1,\frac{8}{3}}}+\|u'\|^2_{L_t^4H_x^{1,\frac{8}{3}}}+\|w\|^2_{L_t^4H_x^{1,\frac{8}{3}}}+\|w'\|^2_{L_t^4H_x^{1,\frac{8}{3}}}\right)\left(\|u-u'\|_{L_t^4H_x^{1,\frac{8}{3}}}+\|w-w'\|_{L_t^4H_x^{1,\frac{8}{3}}}\right). \tag{3.10}$$

Finally, taking $(u',w')=(0,0)$ in (3.9) and (3.10) we have

$$\left\|\int_0^t U(t-\tau)F(u,w)d\tau\right\|_{L_t^4H_x^{1,8/3}} \lesssim \left(\|u\|_{L_t^4H_x^{1,\frac{8}{3}}}+\|w\|_{L_t^4H_x^{1,\frac{8}{3}}}\right)^3 \tag{3.11}$$

and

$$\left\|\int_0^t W(t-\tau)G(u,w)d\tau\right\|_{L_t^4H_x^{1,8/3}} \lesssim \left(\|u\|_{L_t^4H_x^{1,\frac{8}{3}}}+\|w\|_{L_t^4H_x^{1,\frac{8}{3}}}\right)^3. \tag{3.12}$$

Moreover, combining Strichartz's inequality, (3.5), (3.6), (3.7) and (3.8), we obtain

$$\begin{aligned}&\left\|\int_0^t U(t-\tau)[F(u,w)-F(u',w')]d\tau\right\|_{L_t^\infty H_x^1} \lesssim \\ &\|F(u,w)-F(u',w')\|_{L_t^{\frac{4}{3}}L_x^{\frac{8}{5}}} + \|\nabla[F(u,w)-F(u',w')]\|_{L_t^{\frac{4}{3}}L_x^{\frac{8}{5}}} \lesssim \\ &\left(\|u\|^2_{L_t^4 H_x^{1,\frac{8}{3}}} + \|u'\|^2_{L_t^4 H_x^{1,\frac{8}{3}}} + \|w\|^2_{L_t^4 H_x^{1,\frac{8}{3}}} + \|w'\|^2_{L_t^4 H_x^{1,\frac{8}{3}}}\right)\left(\|u-u'\|_{L_t^4 H_x^{1,\frac{8}{3}}} + \|w-w'\|_{L_t^4 H_x^{1,\frac{8}{3}}}\right)\end{aligned} \tag{3.13}$$

and, similarly

$$\begin{aligned}&\left\|\int_0^t W(t-\tau)[G(u,w)-G(u',w')]d\tau\right\|_{L_t^\infty H_x^1} \lesssim \\ &\left(\|u\|^2_{L_t^4 H_x^{1,\frac{8}{3}}} + \|u'\|^2_{L_t^4 H_x^{1,\frac{8}{3}}} + \|w\|^2_{L_t^4 H_x^{1,\frac{8}{3}}} + \|w'\|^2_{L_t^4 H_x^{1,\frac{8}{3}}}\right)\left(\|u-u'\|_{L_t^4 H_x^{1,\frac{8}{3}}} + \|w-w'\|_{L_t^4 H_x^{1,\frac{8}{3}}}\right).\end{aligned} \tag{3.14}$$

Taking $(u',w')=(0,0)$, we get

$$\left\|\int_0^t U(t-\tau)F(u,w)d\tau\right\|_{L_t^\infty H_x^1} \lesssim \left(\|u\|_{L_t^4 H_x^{1,\frac{8}{3}}} + \|w\|_{L_t^4 H_x^{1,\frac{8}{3}}}\right)^3, \tag{3.15}$$

and

$$\left\|\int_0^t W(t-\tau)G(u,w)d\tau\right\|_{L_t^\infty H_x^1} \lesssim \left(\|u\|_{L_t^4 H_x^{1,\frac{8}{3}}} + \|w\|_{L_t^4 H_x^{1,\frac{8}{3}}}\right)^3. \tag{3.16}$$

Existence and uniqueness: Define the operator $\mathcal{H}(u,w) = (H_1(u,w), H_2(u,w))$ where,

$$\begin{aligned}H_1(u(t),w(t)) &= U(t)u_0 + i\int_0^t U(t-s)F(u(s),w(s))ds,\\ H_2(u(t),w(t)) &= W(t)w_0 + i\int_0^t W(t-s)G(u(s),w(s))ds.\end{aligned}$$

By the Strichartz inequality, (2.2) and (3.11), we get that for any $\epsilon > 0$ fixed there exists $T > 0$ such that

$$\begin{aligned}\|H_1(u,w)\|_{L_t^4 H_x^{1,8/3}} &\lesssim \|U(t)u_0\|_{L_t^4 H_x^{1,8/3}} + \left\|\int_0^T U(t-s)F(u,w)ds\right\|_{L_t^4 H_x^{1,8/3}}\\ &\lesssim \epsilon + \left(\|u\|_{L_t^4 H_x^{1,\frac{8}{3}}} + \|w\|_{L_t^4 H_x^{1,\frac{8}{3}}}\right)^3,\end{aligned} \tag{3.17}$$

and similarly, using (3.12),

$$\begin{aligned}\|H_2(u,w)\|_{L_t^4 H_x^{1,8/3}} &\lesssim \|W(t)w_0\|_{L_t^4 H_x^{1,8/3}} + \left\|\int_0^T W(t-s)G(u,w)ds\right\|_{L_t^4 H_x^{1,8/3}}\\ &\lesssim \epsilon + \left(\|u\|_{L_t^4 H_x^{1,\frac{8}{3}}} + \|w\|_{L_t^4 H_x^{1,\frac{8}{3}}}\right)^3.\end{aligned} \tag{3.18}$$

On the other hand, by (3.15), we have

$$\begin{aligned}\sup_{t\in[0,T]}\|H_1(u(t),w(t))-U(t)u_0\|_{H^1_x}&=\left\|\int_0^t U(t-\tau)F(u,w)d\tau\right\|_{L^\infty_t H^1_x}\\&\lesssim\left(\|u\|_{L^4_tH^{1,\frac{8}{3}}_x}+\|w\|_{L^4_tH^{1,\frac{8}{3}}_x}\right)^3,\end{aligned}\tag{3.19}$$

and, by (3.16),

$$\begin{aligned}\sup_{t\in[0,T]}\|H_2(u(t),w(t))-W(t)w_0\|_{H^1_x}&=\left\|\int_0^t W(t-\tau)G(u,w)d\tau\right\|_{L^\infty_t H^1_x}\\&\lesssim\left(\|u\|_{L^4_tH^{1,\frac{8}{3}}_x}+\|w\|_{L^4_tH^{1,\frac{8}{3}}_x}\right)^3.\end{aligned}\tag{3.20}$$

Now, set the norms

$$\begin{aligned}\|v\|_1&:=\|v(t)-U(t)u_0\|_{L^\infty_tH^1_x}+\|v\|_{L^4_tH^{1,\frac{8}{3}}_x},\\\|v\|_2&:=\|v(t)-W(t)w_0\|_{L^\infty_tH^1_x}+\|v\|_{L^4_tH^{1,\frac{8}{3}}_x},\end{aligned}$$

and consider the ball

$$\bar{B}(T,a)=\{(v,h)\in Y\times Y;\ \|(v,h)\|_T:=\|v\|_1+\|h\|_2<a\}.$$

Then, using (3.17), (3.18), (3.19) and (3.20) we have for all $\epsilon>0$ there exists $T>0$ such that for $(u,w)\in\bar{B}(T,a)$

$$\begin{aligned}\|H_1(u,w)\|_1&=\|H_1(u,w)-U(t)u_0\|_{L^\infty_tH^1_x}+\|H_1(u,w)\|_{L^4_tH^{1,\frac{8}{3}}_x}\\&\lesssim\epsilon+\left(\|u\|_{L^4_tH^{1,\frac{8}{3}}_x}+\|w\|_{L^4_tH^{1,\frac{8}{3}}_x}\right)^3\\&\lesssim\epsilon+a^3\end{aligned}\tag{3.21}$$

and

$$\|H_2(u,w)\|_2\lesssim\epsilon+a^3.\tag{3.22}$$

Choosing $a=2\varepsilon$ we have

$$\|\mathcal{H}(u,w)\|_T=\|H_1(u,w)\|_1+\|H_2(u,w)\|_2\lesssim\left(\frac{1}{2}+a^2\right)a.$$

Now, choosing $\varepsilon>0$ such that $a^2<\dfrac{1}{2}$ we have that $\mathcal{H}$ is well defined on $\bar{B}(T,a)$. It remains to show that $\mathcal{H}$ is a contraction in $\bar{B}(T,a)$. Indeed, take $(u,w),(u',w')\in\bar{B}(T,a)$. By (3.9), (3.10), (3.13) and (3.14), we have

$$\begin{aligned}\|\mathcal{H}(u,w)-\mathcal{H}(u',w')\|_T&=\|H_1(u,w)-H_1(u',w')\|_1+\|H_2(u,w)-H_2(u',w')\|_2\\&=\|\int_0^T U(t-\tau)[F(u,w)-F(u',w')]\|_1+\|\int_0^T W(t-\tau)[G(u,w)-G(u',w')]\|_2\\&\lesssim a^2(\|u-u'\|_{L^4_tH^{1,\frac{8}{3}}_x}+\|w-w'\|_{L^4_tH^{1,\frac{8}{3}}_x})\\&\lesssim a^2\left(\|u-u'\|_1+\|w-w'\|_2\right)\\&\lesssim a^2\|(u,w)-(u',w')\|_T.\end{aligned}$$

But, we choose ε such that $a^2 < \frac{1}{2}$, so $2a^2 < 1$ hence $\mathcal{H}$ is a contraction. By the fixed point theorem, there exists a unique solution on $\bar{B}(T,a)$.

The blow-up alternative can be done as in Theorem 4.5.1 of (CAZENAVE, 2003). We will omit the details. □

3.2 Existence of ground state solution

This section is devoted to prove the existence of ground state solutions. As mentioned before, we will follow the ideas in (NOGUERA; PASTOR, 2022). We start with the deduction of a critical Sobolev-type inequality.

3.2.1 Critical Sobolev-type inequality and localized version

The first result states that the function N must be positive for a pair (P,Q) of non-trivial solutions to (1.12).

Lemma 3.1. *Let $\mathcal{N} := \{(P,Q) \in \dot{H}^1(\mathbb{R}^4) \times \dot{H}^1(\mathbb{R}^4);\ N(P,Q) > 0\}$. Then $\mathcal{C} \subset \mathcal{N}$, where $\mathcal{C}$ denotes the set of all non-trivial solutions of (1.12) and N is defined in (1.14).*

Proof. Let $(P,Q) \in \mathcal{C}$. Taking $(f,g) = (P,Q)$ in (1.15) we have

$$\int |\nabla P|^2 = \int \frac{1}{9}P^4 + 2Q^2P^2 + \frac{1}{3}P^3Q$$

and

$$\int |\nabla Q|^2 = \int 9Q^4 + 2Q^2P^2 + \frac{1}{9}P^3Q.$$

By summing both equations, we get

$$K(P,Q) = \int \frac{1}{9}P^4 + 9Q^4 + 4Q^2P^2 + \frac{4}{9}P^3Q = 4N(P,Q). \tag{3.23}$$

Since (P,Q) are non-trivial, it follows that $N(P,Q) > 0$ and $(P,Q) \in \mathcal{N}$ as desired. □

Let us introduce the functional

$$J(P,Q) := \frac{K(P,Q)^2}{N(P,Q)}, \quad (P,Q) \in \mathcal{N}. \tag{3.24}$$

Remark 3.2. *(i) The energy functional for system (1.12) is*

$$\mathcal{E}(P,Q) := \frac{1}{2}K(P,Q) - N(P,Q),\ (P,Q) \in \dot{H}^1(\mathbb{R}^4) \times \dot{H}^1(\mathbb{R}^4). \tag{3.25}$$

Then, if (P,Q) is a non-trivial solution, we have $\mathcal{E}(P,Q) = S(P,Q)$.

(ii) Observe that, using (3.23),

$$S(P,Q) = \frac{1}{2}K(P,Q) - N(P,Q) = N(P,Q).$$

Moreover,

$$J(P,Q) = \frac{K(P,Q)^2}{N(P,Q)} = 16N(P,Q) = 16S(P,Q).$$

Hence, a non-trivial solution of (1.12) is a ground state if, and only if, it has least energy $\mathcal{E}$ among all solutions if, and only if, it minimizes J.

From now on, we will assume that u and w are real-valued functions. We start noticing that if we apply Hölder's inequality with $p = 4$ and $q = \frac{4}{3}$, in view of $ab \lesssim (a^p + b^q)$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int u^3 w \leqslant \int |u|^3|w| \leqslant \|u^3\|_{L^{\frac{4}{3}}}\|w\|_{L^4} = \|u\|^3_{L^4}\|w\|_{L^4} \lesssim ((\|u\|^3_{L^4})^{\frac{4}{3}} + \|w\|^4_{L^4}) \lesssim (\|u\|^4_{L^4} + \|w\|^4_{L^4}) \tag{3.26}$$

and $u^2w^2 \leqslant |u|^2|w|^2 \lesssim (|u|^2)^2 + (|w|^2)^2 = |u|^4 + |w|^4$, thus

$$\int u^2w^2 \lesssim \int |u|^4 + |w|^4 = \|u\|^4_{L^4} + \|w\|^4_{L^4}. \tag{3.27}$$

Therefore, we have $N(u,w) \lesssim \|u\|^4_{L^4} + \|w\|^4_{L^4}$. Using Sobolev's inequality

$$\|f\|^4_{L^4} \lesssim \|\nabla f\|^4_{L^2}$$

we get,

$$\begin{aligned} N(u,w) &\lesssim \|u\|^4_{L^4} + \|w\|^4_{L^4} \\ &\lesssim \|\nabla u\|^4_{L^2} + \|\nabla w\|^4_{L^2} \\ &\lesssim \|\nabla u\|^4_{L^2} + \|\nabla w\|^4_{L^2} + 2(\|\nabla u\|^2_{L^2}\|\nabla w\|^2_{L^2}) \\ &\lesssim K(u,w)^2. \end{aligned} \tag{3.28}$$

Hence, if $(u,w) \in \mathcal{N}$ then

$$N(u,w) \lesssim K(u,w)^2. \tag{3.29}$$

Consequently, there exists a positive constant C such that

$$\frac{1}{C} \leqslant J(u,w), \quad \forall (u,w) \in \mathcal{N}, \tag{3.30}$$

that is, the functional J is bounded from below by a positive constant. The best constant we can place in (3.30) is given by

$$C_4^{-1} = \inf\{J(u,w);\ (u,w) \in \mathcal{N}\}, \tag{3.31}$$

where the subscript 4 in C_4 is motivated by the dimension $d = 4$. To show that this infimum is attained, we will consider the normalized version of the problem as follows

$$I = \inf\{K(u, w);\ (u, w) \in \mathcal{N},\ \ N(u, w) = 1\}. \tag{3.32}$$

Since we assumed that N can take negative values outside the origin, we should slightly modify our problem. For this end, we are considering the problem in $\mathcal{N}$ instead of $\dot{H}^1(\mathbb{R}^4) \times \dot{H}^1(\mathbb{R}^4)$.

Definition 3.3. *A minimizing sequence for (3.31) is a sequence (u_m, w_m) in $\mathcal{N}$ such that $J(u_m, w_m) \to C_4^{-1}$. In the same way, a minimizing sequence for (3.32) is a sequence (u_m, w_m) in $\mathcal{N}$ such that $N(u_m, w_m) = 1$, for all m and $K(u_m, w_m) \to I$.*

Next, notice that $\partial_{x_i}|u| = \dfrac{u}{|u|}\partial_{x_i} u$ thus

$$|\nabla |u||^2 = \sum_{i=1}^{d} \left| \frac{u}{|u|}\partial_{x_i} u \right|^2 \leqslant \sum_{i=1}^{d} |\partial_{x_i} u|^2 \leqslant |\nabla u|^2.$$

In the same way, $|\nabla |w||^2 \leqslant |\nabla w|^2$ and then $K(|u|, |w|) \leqslant K(u, w)$. Moreover,

$$N(u, w) \leqslant |N(u, w)| \leqslant \int \frac{1}{36}|u|^4 + \frac{9}{4}|w|^4 + |u|^2|w|^2 + \frac{1}{9}|u|^3|w| = N(|u|, |w|).$$

Hence, $J(|u|, |w|) \leqslant J(u, w)$, that is, if (u_m, w_m) is a minimizing sequence for (3.31) (or (3.32)) then so is $(|u_m|, |w_m|)$. In particular, there is no loss of generality in assuming that minimizing sequences are always non-negative.

Remark 3.4. *Observe that $C_4 = I^{-2}$. Indeed, denote $A := \{(u, w) \in \mathcal{N},\ N(u, w) = 1\}$. Then, for any $(u, w) \in A$ we have $J(u, w) = K(u, w)^2$ and, hence, $C_4^{-1} \leqslant K(u, w)^2$ or, equivalently, $C_4^{-1/2} \leqslant K(u, w)$, for all $(u, w) \in A$, that is, $C_4^{-1/2}$ is a lower bound to the set $\{K(u, w);\ (u, w) \in \mathcal{N},\ N(u, w) = 1\}$. Therefore, $C_4^{-1/2} \leqslant I$, i.e., $I^{-2} \leqslant C_4$. On the other hand, since N and K are homogeneous of degree 4 and 2, respectively, we have that $J(\lambda(u, w)) = J(u, w)$, for any $\lambda > 0$. Now, given $\epsilon > 0$, let $(u, w) \in \mathcal{N}$ be such that $J(u, w) < C_4^{-1} + \epsilon$ and set $(\tilde{u}, \tilde{w}) := N(u, w)^{-1/4}(u, w)$. Then $N(\tilde{u}, \tilde{w}) = 1$, $J(\tilde{u}, \tilde{w}) = J(u, w)$ and*

$$I^2 \leqslant K(\tilde{u}, \tilde{w})^2 = J(\tilde{u}, \tilde{w}) = J(u, w) < C_4^{-1} + \epsilon.$$

Hence, $C_4 \leqslant I^{-2}$ and then $C_4 = I^{-2}$. Therefore, (3.29) becomes

$$N(u, w) \leqslant I^{-2} K(u, w)^2, \quad \forall (u, w) \in \mathcal{N}. \tag{3.33}$$

In addition, if (u, w) is a minimizer for (3.32), then $K(u, w) = I$ and $N(u, w) = 1$, so

$$J(u, w) = \frac{K(u, w)^2}{N(u, w)} = I^2 = C_4^{-1}.$$

Thus, (u, w) is also a minimizer for (3.31).

Before proceeding, it is convenient to set the function

$$\varphi(u,w) := \frac{1}{36}u^4 + \frac{9}{4}w^4 + u^2w^2 + \frac{1}{9}u^3w.$$

Notice that φ is homogeneous of degree 4. Also, for $R > 0$ and $y \in \mathbb{R}^4$, the function $u^{R,y} := R^{-1}u(R^{-1}(x-y))$, satisfies

$$\begin{aligned} N(u^{R,y}, w^{R,y}) &= N(R^{-1}u(R^{-1}(x-y)), R^{-1}w(R^{-1}(x-y))) \\ &= R^{-4}\int \varphi(u(R^{-1}(x-y)), w(R^{-1}(x-y)))dx \\ &= \int \varphi(u(z), w(z))dz \\ &= N(u,w), \end{aligned}$$

where we used the change of variables $z = R^{-1}(x-y)$. In the same way, since K is homogeneous of degree 2, it follows that

$$\begin{aligned} K(u^{R,y}, w^{R,y}) &= K(R^{-1}u(R^{-1}(x-y)), R^{-1}w(R^{-1}(x-y))) \\ &= R^{-2}\int |\nabla(u(R^{-1}(x-y))|^2 + |\nabla w(R^{-1}(x-y))|^2 dx \\ &= \int |\nabla u(z)|^2 + |\nabla w(z)|^2 dz \\ &= K(u,w). \end{aligned}$$

Thus, the functionals K and N are invariant under the transformation

$$(u,w) \mapsto (u^{R,y}, w^{R,y}) = (R^{-1}u(R^{-1}(x-y)), R^{-1}w(R^{-1}(x-y))). \tag{3.34}$$

As mentioned before, to finish this part we will set some results about a localized version of Sobolev's inequality. We will start with a useful tool to achieve the goal. The result was essentially proved in (FLUCHER; MüLLER, 1999), Lemma 8.

Lemma 3.5. *For all $\delta > 0$, there exists a constant $C(\delta) > 0$ with the following property: if $r/R < C(\delta)$ and $x \in \mathbb{R}^4$, then there is a cut-off function $\chi_R^r \in H^{1,\infty}(\mathbb{R}^4)$ such that $\chi_R^r = 1$ on $B(x,r)$, $\chi_R^r = 0$ outside $B(x,R)$ and*

$$K(\chi_R^r u, \chi_R^r w) \leqslant \int_{B(x,R)} \left(|\nabla u|^2 + |\nabla w|^2\right) dy + \delta K(u,w), \tag{3.35}$$

and

$$K((1-\chi_R^r)u, (1-\chi_R^r)w) \leqslant \int_{\mathbb{R}^4\setminus B(x,r)} \left(|\nabla u|^2 + |\nabla w|^2\right) dy + \delta K(u,w), \tag{3.36}$$

for any $(u,w) \in \dot{H}^1(\mathbb{R}^4) \times \dot{H}^1(\mathbb{R}^4)$.

Proof. There is no loss of generality in assuming $x = 0$. Define the function

$$\chi_R^r(y) := \begin{cases} 1, & |y| \leqslant r, \\ \dfrac{\log(|y|/R)}{\log(r/R)}, & r \leqslant |y| \leqslant R, \\ 0, & |y| \geqslant R. \end{cases}$$

Note that for $r < |y| < R$ and $1 \leqslant i \leqslant 4$, we have

$$\frac{\partial \chi_R^r}{\partial y_i} = \frac{1}{\log(r/R)} \frac{y_i/(R|y|)}{|y|/R} = \frac{y_i}{\log(r/R)|y|^2}.$$

Since $R > r$ and $\log(r/R) = -\log(R/r)$, we have

$$|\nabla \chi_R^r| = \frac{|y|}{\log(R/r)|y|^2} = \frac{1}{\log(R/r)|y|}.$$

Hence

$$|\nabla \chi_R^r|^4 = \frac{1}{\log(R/r)^4 |y|^4}.$$

Besides that, $\nabla \chi_R^r = 0$ for $|y| < r$. Therefore, using polar coordinates we have

$$\begin{aligned} \int_{B(0,R)} |\nabla \chi_R^r|^4 dy &= \frac{1}{\log(R/r)^4} \int_{\{r \leqslant |y| \leqslant R\}} \frac{1}{|y|^4} dy \\ &= \frac{1}{\log(R/r)^4} \int_r^R \left(\int_{\partial B(0,t)} \frac{1}{|y|^4} dS(y) \right) dt \\ &= \frac{1}{\log(R/r)^4} \int_r^R \alpha_4 \frac{t^3}{t^4} dt \\ &= \frac{\alpha_4}{\log(R/r)^4} \int_r^R \frac{1}{t} dt \\ &= \frac{\alpha_4}{\log(R/r)^3}. \end{aligned}$$

where α_4 is the measure of the unit sphere in $\mathbb{R}^4$. Now, using Young's inequality, we have

$$\begin{aligned} |\nabla(\chi_R^r u)|^2 &= |u \nabla \chi_R^r + \chi_R^r \nabla u|^2 \\ &= |u|^2 |\nabla \chi_R^r|^2 + 2(u \nabla \chi_R^r)(\chi_R^r \nabla u) + |\chi_R^r|^2 |\nabla u|^2 \\ &\leqslant |u|^2 |\nabla \chi_R^r|^2 + \frac{1}{\epsilon} |u \nabla \chi_R^r|^2 + \epsilon |\chi_R^r \nabla u|^2 + |\chi_R^r|^2 |\nabla u|^2 \\ &= (1+\epsilon) |\chi_R^r|^2 |\nabla u|^2 + \left(1 + \frac{1}{\epsilon}\right) |u|^2 |\nabla \chi_R^r|^2. \end{aligned}$$

In addition, Hölder's inequality with $p = q = 2$ implies

$$\int_{B(0,R)} |u|^2 |\nabla \chi_R^r|^2 dy \leqslant C \|u\|_{L^4}^2 \left(\int_{B(0,R)} |\nabla \chi_R^r|^4 dy \right)^{1/2}.$$

Hence, using the above estimates and Sobolev's inequality

$$\|u\|_{L^4}^2 \leqslant C \|\nabla u\|_{L^2}^2$$

we have

$$\begin{aligned}\int_{B(0,R)} |\nabla[\chi_R^r u]|^2 dy &\leqslant \int_{B(0,R)} (1+\epsilon)|\chi_R^r|^2|\nabla u|^2 + \left(1+\frac{1}{\epsilon}\right)\int_{B(0,R)} |u|^2|\nabla\chi_R^r|^2. \\ &\leqslant \int_{B(0,R)} (1+\epsilon)|\chi_R^r|^2|\nabla u|^2 + \left(1+\frac{1}{\epsilon}\right) C\|u\|_{L^4}^2 \left(\int_{B(0,R)} |\nabla\chi_R^r|^4 dy\right)^{1/2} \\ &\leqslant (1+\epsilon)\int_{B(0,R)} |\chi_R^r|^2|\nabla u|^2 dy + \left(1+\frac{1}{\epsilon}\right)\frac{C\alpha_4^{1/2}}{(\log(R/r))^{3/2}}\int_{\mathbb{R}^4} |\nabla u|^2 dy.\end{aligned}$$

Similarly for w,

$$\int_{B(0,R)} |\nabla[\chi_R^r w]|^2 dy \leqslant \int_{B(0,R)} (1+\epsilon)|\chi_R^r|^2|\nabla w|^2 + \left(1+\frac{1}{\epsilon}\right)\frac{C\alpha_4^{1/2}}{\log(R/r)^{3/2}}\int_{\mathbb{R}^4} |\nabla w|^2 dy.$$

Summing the above estimates, we get

$$K(\chi_R^r u, \chi_R^r w) \leqslant \int_{B(0,R)} |\nabla u|^2 + |\nabla w|^2 dy + \left[\epsilon + \left(1+\frac{1}{\epsilon}\right)\frac{\xi^2}{\log(R/r)^{3/2}}\right] K(u,w),$$

where $\xi = \sqrt{C}\alpha_4^{1/4}$. Taking $\epsilon = \sqrt{\delta+1}-1$ and

$$C(\delta) := exp\left[-\left(\frac{\xi}{\sqrt{\delta+1}-1}\right)^{4/3}\right],$$

we have that if $r/R \leqslant C(\delta)$ then

$$\left[\epsilon + \left(1+\frac{1}{\epsilon}\right)\frac{\xi^2}{\log(R/r)^{3/2}}\right] \leqslant \delta$$

and (3.35) follows. To show (3.36), observe that

$$\begin{aligned}&\int_{\mathbb{R}^4\backslash B(0,r)} |\nabla[(1-\chi_R^r)u]|^2 dy \leqslant \\ &\qquad (1+\epsilon)\int_{\mathbb{R}^4\backslash B(0,r)} |1-\chi_R^r|^2|\nabla u|^2 dy + \left(1+\frac{1}{\epsilon}\right)\int_{\mathbb{R}^4\backslash B(0,r)} |u|^2|\nabla(1-\chi_R^r)|^2 dy,\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^4\backslash B(0,r)} |\nabla[(1-\chi_R^r)w]|^2 dy \leqslant \\ &\qquad (1+\epsilon)\int_{\mathbb{R}^4\backslash B(0,r)} |1-\chi_R^r|^2|\nabla w|^2 dy + \left(1+\frac{1}{\epsilon}\right)\int_{\mathbb{R}^4\backslash B(0,r)} |w|^2|\nabla(1-\chi_R^r|^2 dy.\end{aligned}$$

Thus, since $|\nabla(1-\chi_R^r)|^2 = |\nabla\chi_R^r|^2$ and $\chi_R^r = 0$ outside $B(0,R)$, then (3.36) follows as in (3.35) thereby, finishing the proof. □

With this at hand, we can state the localized version of the Sobolev inequality as follows.

Corollary 3.6. *Let $(u, w) \in \dot{H}^1(\mathbb{R}^4) \times \dot{H}^1(\mathbb{R}^4)$ with $u, w > 0$. Fix $\delta > 0$ and $r/R \leqslant C(\delta)$ with $C(\delta)$ as in Lemma 3.5. Then*

$$\int_{B(x,R)} \varphi(u,w)dy \leqslant I^{-2}\left[\int_{B(x,R)} |\nabla u|^2 + |\nabla w|^2 dy + \delta K(u,w)\right]^2, \tag{3.37}$$

$$\int_{\mathbb{R}^4\backslash B(x,R)} \varphi(u,w)dy \leqslant I^{-2}\left[\int_{\mathbb{R}^4\backslash B(x,R)} |\nabla u|^2 + |\nabla w|^2 dy + (2\delta+\delta^2)K(u,w)\right]^2. \tag{3.38}$$

Proof. There is no loss of generality in assuming $x = 0$. Observe that $\chi_R^r = 1$ on $B(0,r)$ and $supp(\chi_R^r) = \overline{B(0,R)}$. Then, (3.33) and (3.35) give us

$$\begin{aligned}\int_{B(0,R)} \varphi(u,w)dx &\leqslant \int_{\mathbb{R}^4} \varphi(\chi_R^r u, \chi_R^r w)dx \\ &\leqslant I^{-2}K(\chi_R^r u, \chi_R^r w)^2 \\ &\leqslant I^{-2}\left[\int |\nabla u|^2 + |\nabla w|^2 dx + \delta K(u,w)\right]^2,\end{aligned}$$

which is exactly (3.37). For (3.38), we use the function $(1-\chi_R^r)\chi_{R_2}^{R_1}$ where $r < R < R_1 < R_2$ and $R_1/R_2 \leqslant C(\delta)$. Naturally, $(1-\chi_R^r)\chi_{R_2}^{R_1} = 1$ on $B(0,R_1)\backslash B(0,R)$ and we have

$$\begin{aligned}\int_{B(0,R_1)\backslash B(0,R)} \varphi(u,w)dx &= \int_{B(0,R_1)\backslash B(0,R)} \varphi\left((1-\chi_R^r)\chi_{R_2}^{R_1}u, (1-\chi_R^r)\chi_{R_2}^{R_1}w\right)dx \\ &\leqslant \int_{B(0,R_1)} \varphi\left((1-\chi_R^r)\chi_{R_2}^{R_1}u, (1-\chi_R^r)\chi_{R_2}^{R_1}w\right)dx \\ &\leqslant I^{-2}\left[\int_{B(0,R_2)} |\nabla[(1-\chi_R^r)u]|^2 + |\nabla[(1-\chi_R^r)w]|^2 dx + \delta K((1-\chi_R^r)u, 1-\chi_R^r)w)\right]^2,\end{aligned}$$

where, in the last inequality, we use (3.37). Taking R_1 and R_2 as large as we want such that $\frac{R_1}{R_2} \leqslant C(\delta)$, the last inequality gives us

$$\int_{\mathbb{R}^4\backslash B(0,R)} \varphi(u,w)dx \leqslant I^{-2}[K((1-\chi_R^r)u, 1-\chi_R^r)w) + \delta K((1-\chi_R^r)u, 1-\chi_R^r)w]^2.$$

Finally, using (3.36), we get

$$\int_{\mathbb{R}^4\backslash B(0,R)} \varphi(u,w)dx \leqslant I^{-2}\left[\int_{\mathbb{R}^4\backslash B(0,R)} |\nabla u|^2 + |\nabla w|^2 dy + \delta K(u,w)) + \delta(1+\delta)K(u,w)\right]^2,$$

as desired. □

3.2.2 Concentration-compactness method

We start with a result, that is called concentrarion-compactness lemma I, which is a slightly modification of the Lemma presented in (LIONS, 1984).

Lemma 3.7. *[Concentrarion-compactness lemma I]. Suppose that (ν_m) is a sequence in $\mathcal{M}_+^1(\mathbb{R}^4)$. Then, there is a subsequence, still denoted by (ν_m), such that one of the following conditions holds:*

(i) (Vanishing) For all $R > 0$ it holds

$$\lim_{m\to\infty}\left(\sup_{x\in\mathbb{R}^4}\nu_m(B(x,R))\right) = 0.$$

(ii) (Dichotomy) There is a number $\lambda \in (0,1)$ such that for all $\epsilon > 0$ there exists $R > 0$ and a sequence (x_m) with the following property: given $R' > R$

$$\nu_m(B(x_m,R)) \geqslant \lambda - \epsilon,$$

$$\nu_m(\mathbb{R}^4\backslash B(x_m,R')) \geqslant 1 - \lambda - \epsilon,$$

for m sufficiently large.

(iii) (Compactness) There exists a sequence $(x_m) \subset \mathbb{R}^4$ such that for each $\epsilon > 0$ there is a radius $R > 0$ with the property

$$\nu_m(B(x_m,R)) \geqslant 1 - \epsilon,$$

for all m.

Proof. One can see the proof in (FLUCHER; MüLLER, 1999) Lemma 23. □

To achieve our goal and find a minimizer for the minimization problem (3.32), we will built a suitable sequence of probability Radon measures and then, as a consequence of Lemma 3.7, up to a subsequence, it will satisfy one of the three conditions above. From that, we will avoid vanishing and dichotomy implying in the compactness of the sequence. Hence, we shall get a vague convergence in $\mathcal{M}_+^b(\mathbb{R}^4)$. Such convergence allows us to use, what is called concentration-compactness lemma II, which was inspired in the limit case lemma in (LIONS, 1985), roughly speaking, this guarantee dilation invariance for the minimization problem.

Lemma 3.8. *[Concentration-compactness lemma II] Let $(u_m, w_m) \subset \dot{H}^1(\mathbb{R}^4) \times \dot{H}^1(\mathbb{R}^4)$ be a sequence such that $u_m, w_m \geqslant 0$ and*

$$\begin{cases} (u_m, w_m) \rightharpoonup (u,w), & in \quad \dot{H}^1(\mathbb{R}^4)\times\dot{H}^1(\mathbb{R}^4), \\ \mu_m := (|\nabla u_m|^2 + |\nabla w_m|^2)dx \overset{*}{\rightharpoonup} \mu, & in \quad \mathcal{M}_+^b(\mathbb{R}^4) \\ \nu_m := \varphi(u_m, w_m)dx \overset{*}{\rightharpoonup} \nu, & in \quad \mathcal{M}_+^b(\mathbb{R}^4). \end{cases} \tag{3.39}$$

Then,

(i) There exists an at most countable set J, a family of distinct points $\{x_j \in \mathbb{R}^4;\, j \in J\}$, and a family of non-negative numbers $\{a_j;\, j \in J\}$ such that

$$\nu = \varphi(u,w)dx + \sum_{j\in J} a_j \delta_{x_j}. \tag{3.40}$$

(ii) Moreover, we have

$$\mu \geqslant \left(|\nabla u|^2 + |\nabla w|^2\right) dx + \sum_{j\in J} b_j \delta_{x_j} \tag{3.41}$$

fore some family $\{b_j;\, j \in J\}, b_j > 0$, such that

$$a_j \leqslant I^{-2} b_j^2, \quad \forall j \in J. \tag{3.42}$$

In particular, $\sum_{j\in J} a_j^{1/2} < \infty$.

Remark 3.9. *Since $u_m, w_m \geqslant 0$, then $\varphi(u_m,w_m) \geqslant 0$. Thus, ν_m is indeed a positive measure. In addition, the weak converge of $(u_m,w_m) \rightharpoonup (u,w)$ implies that, up to a subsequence, we have $(u_m,w_m) \to (u,w)$ a.e. in $\mathbb{R}^4$. Hence $u,w \geqslant 0$.*

Proof. **Step 1.** Assume that $(u,w) = (0,0)$.

Let $\xi \in C_0^\infty(\mathbb{R}^4)$. From the weak convergence of (ν_m) and the homogeneity of φ,

$$\begin{aligned} \int |\xi|^4 d\nu &= \lim_{m\to\infty} \int |\xi|^4 \varphi(u_m,w_m)dx \\ &= \lim_{m\to\infty} \int \varphi(|\xi|u_m, |\xi|w_m)dx \\ &\leqslant I^{-2} \liminf_{m\to\infty} K(\xi u_m, \xi w_m)^2. \end{aligned} \tag{3.43}$$

Since $(u_m,w_m) \rightharpoonup (0,0)$ in $\dot{H}^1(\mathbb{R}^4) \times \dot{H}^1(\mathbb{R}^4)$, we know that (see Theorem 8.6 of (LIEB; LOSS, 2001)), for all $M \subset \mathbb{R}^4$ with finite measure, by Lemma 2.5, we have

$$\chi_M(u_m,w_m) \to (0,0), \tag{3.44}$$

strongly in $L^2(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$. Then, taking $M = \operatorname{supp}|\nabla\xi|$ and using the triangular inequality, we obtain

$$\begin{aligned} &\left|\left(\|\nabla(\xi u_m)\|_{L^2}^2 + \|\nabla(\xi w_m)\|_{L^2}^2\right)^{1/2} - \left(\|\xi\nabla(u_m)\|_{L^2}^2 + \|\xi\nabla(w_m)\|_{L^2}^2\right)^{1/2}\right| \\ &\quad\leqslant \left(\|\nabla(\xi u_m) - \xi\nabla u_m\|_{L^2}^2 + (\|\nabla(\xi w_m) - \xi\nabla w_m\|_{L^2}^2\right)^{1/2} \\ &\quad= \left(\|u_m\nabla\xi\|_{L^2}^2 + \|w_m\nabla\xi\|_{L^2}^2\right)^{1/2} \\ &\quad\lesssim \left(\int |\chi_M u_m|^2 + |\chi_M w_m|^2\right)^{1/2} \to 0, \quad m\to\infty. \end{aligned}$$

Combining with the vague convergence of (μ_m), we get

$$\begin{aligned}\liminf_{m\to\infty} K(\xi u_m, \xi w_m)^2 &= \liminf_{m\to\infty}\left(\int |\xi|^2(|\nabla u_m|^2 + |\nabla w_m|^2)dx\right)^2\\ &= \liminf_{m\to\infty}\left(\int |\xi|^2 d\mu_m\right)^2\\ &= \left(\int |\xi|^2 d\mu\right)^2.\end{aligned}$$

Then, from (3.43), we deduce

$$\int |\xi|^4 d\nu \leqslant I^{-2}\left(\int |\xi|^2 d\mu\right)^2, \quad \xi \in C_0^\infty(\mathbb{R}^4). \tag{3.45}$$

We claim that (3.45) implies that

$$\nu(E) \leqslant I^{-2}\mu(E)^2, \quad \forall E \in \mathcal{B}(\mathbb{R}^4). \tag{3.46}$$

Indeed, let $U \subset \mathbb{R}^4$ be an open set and take a compact set $K \subset U$. By C^∞ Urysohn's lemma (see (FOLLAND, 1999), Lemma 8.18), there exists $g \in C_0^\infty(\mathbb{R}^4)$ obeying $0 \leqslant g \leqslant 1$, $g = 1$ on K and $\operatorname{supp}(g) \subset U$. Thus, by (3.45)

$$\nu(K) = \int_K g^4 d\nu \leqslant \int g^4 d\nu \leqslant I^{-2}\left(\int g^2 d\mu\right)^2 \leqslant I^{-2}\left(\int_{\operatorname{supp}(g)} g^2 d\mu\right)^2 \leqslant I^{-2}\left(\int_U d\mu\right)^2.$$

Thus, $\nu(K) \leqslant I^{-2}\mu(U)^2$, for all $K \subset U$ compact. Since ν is a Radon measure, by its inner regularity, we have

$$\nu(U) \leqslant I^{-2}\mu(U)^2, \quad \forall U \subset \mathbb{R}^4, \quad U \text{ open}. \tag{3.47}$$

Now, if $E \in \mathcal{B}(\mathbb{R}^4)$, where $\mathcal{B}$ denotes the Borel σ-algebra, and U is an open set $E \subset U$, then from (3.47), we get $\nu(E) \leqslant \nu(U) \leqslant I^{-2}\mu(U)^2$. Since μ is a Radon measure, we can use its outer regularity to get

$$\nu(E) \leqslant I^{-2}\mu(E)^2, \quad \forall E \in \mathcal{B}(\mathbb{R}^4).$$

Now, consider $D = \{x \in \mathbb{R}^4;\ \mu(\{x\}) > 0\}$. We may write $D = \bigcup_{k=1}^{\infty} D_k$, where $D_k = \{x \in \mathbb{R}^4;\ \mu(\{x\}) > 1/k\}$. Since μ is a finite measure, then D_k is finite for all k. Indeed, assume that exists k_0 such that D_{k_0} has infinitely many elements, i.e., $D_{k_0} = \{x_j, j \in \mathbb{N}\}$. Then $\mu(D_{k_0}) = \sum_{j\in\mathbb{N}} \mu(\{x_j\}) > \sum_{j\in\mathbb{N}} 1/k_0 = \infty$, which contradicts the fact that μ is finite. Hence, D_k is finite for all $k \in \mathbb{N}$ and the set D is at most countable. Thus we write $D = \{x_j;\ j \in J\}$, with $J \subset \mathbb{N}$.

Set $b_j = \mu(\{x_j\})$, $j \in J$, then for any $E \in \mathcal{B}(\mathbb{R}^4)$, we have

$$\sum_{j\in J} b_j \delta_{x_j}(E) = \sum_{\substack{j\in J\\ x_j\in E}} b_j = \sum_{\substack{j\in J\\ x_j\in E}} \mu(\{x_j\}) \leqslant \mu(E), \tag{3.48}$$

where $\delta_{x_j}(E) = 1$ if $x_j \in E$ and $\delta_{x_j}(E) = 0$ if not. From (3.47) we have (3.41) for the case $(u, w) = (0, 0)$.

Now, observe that from (3.46) we have $\nu \ll \mu$ and, by Radon-Nikodym theorem (see (EVANS; GARIEPY, 1992), Section 1.6) there is a non-negative function $h \in L^1(\mathbb{R}^4, \mu)$ such that

$$\nu(E) = \int_E h(x)d\mu(x), \quad \forall E \in \mathcal{B}(\mathbb{R}^4). \tag{3.49}$$

Moreover, h satisfies

$$h(x) = \lim_{r\to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))}, \quad \mu \text{ a.e. } x \in \mathbb{R}^4. \tag{3.50}$$

Using (3.50) and (3.46), we get $0 \leqslant h(x) \leqslant I^{-2}\mu(\{x\})$. Thus, $h(x) = 0$, μ a.e. on $\mathbb{R}^4 \backslash D$. In particular, we can rewrite the integral (3.49) as

$$\int_E h(x)d\mu(x) = \sum_{\substack{j\in J \\ x_j \in E}} h(x_j)\mu(\{x_j\}). \tag{3.51}$$

Setting $a_j = \nu(\{x_j\})$, $j \in J$, we have from (3.49) and (3.51) that in fact $a_j = h(x_j)b_j$, $\forall j \in J$. Then, for all $E \in \mathcal{B}(\mathbb{R}^4)$, we have

$$\nu(E) = \sum_{\substack{j\in J \\ x_j \in E}} h(x_j)\mu(\{x_j\}) = \sum_{\substack{j\in J \\ x_j \in E}} a_j = \sum_{j\in J} a_j\delta_{x_j}(E),$$

which establishes (3.40) for $(u, w) = (0, 0)$. Finally, inequality (3.42) follows immediately from the definitions of a_j e b_j and (3.46). Note also that by taking $E = \mathbb{R}^4$ in (3.48) we deduce that $\sum_{j\in J} b_j$ is convergent. Hence, the convergence of the series $\sum_{j\in J} a_j^{1/2}$ follows from (3.46).

Step 2. Case $(u, w) \neq (0, 0)$.

Since $u_m, w_m \geqslant 0$, then we have $\varphi(u, w) \geqslant 0$ thus $\varphi(u, w)dx$ defines a positive measure.

Claim The measures

$$\mu - (|\nabla u|^2 + |\nabla w|^2)dx \quad \text{and} \quad \nu - \varphi(u, w)dx \tag{3.52}$$

are non-negative.

Indeed, set $(y_m, z_m) = (u_m - u, w_m - w)$ and consider the sequence of measures

$$\tilde{\mu}_m := (|\nabla y_m|^2 + |\nabla z_m|^2)dx \quad \text{and} \quad \tilde{\nu}_m := \varphi(|y_m|, |z_m|)dx.$$

Since $(y_m, z_m) \rightharpoonup (0, 0)$ in $\dot{H}^1(\mathbb{R}^4) \times \dot{H}^1(\mathbb{R}^4)$, the sequence $(K(y_m, z_m))$ is uniformly bounded. Hence, since

$$\left|\int f d\tilde{\mu}_m\right| \leqslant \|f\|_{L^\infty} K(y_m, z_m), \quad f \in C_0^\infty(\mathbb{R}^4),$$

we have that $(\tilde{\mu}_m)$ is vaguely bounded on $\mathcal{M}_+^b(\mathbb{R}^4)$. Therefore, Lemma 2.5 gives us a subsequence, still denoted by $(\tilde{\mu}_m)$, and $\tilde{\mu} \in \mathcal{M}_+^b(\mathbb{R}^4)$ obeying

$$\tilde{\mu}_m \overset{*}{\rightharpoonup} \tilde{\mu}, \quad \text{in} \quad \mathcal{M}_+^b(\mathbb{R}^4). \tag{3.53}$$

Now, if

$$\mu_m \overset{*}{\rightharpoonup} \tilde{\mu} + (|\nabla u|^2 + |\nabla w|^2)dx, \quad \text{in} \quad \mathcal{M}_+^b(\mathbb{R}^4), \tag{3.54}$$

then by uniqueness of the vague limit,

$$\mu = \tilde{\mu} + (|\nabla u|^2 + |\nabla w|^2)dx$$

and, by the finiteness of all involved measures, we may conclude that $\mu - (|\nabla u|^2 + |\nabla w|^2)dx$ is non-negative.

Now, we turn our attention to establish (3.54). Since $\partial_{x_i} y_m \rightharpoonup 0$ and $\partial_{x_i} z_m \rightharpoonup 0$ in $L^2(\mathbb{R}^4)$ and $f\partial_{x_i} u, f\partial_{x_i} w \in L^2(\mathbb{R}^4)$ for each $f \in C_c(\mathbb{R}^4)$, then

$$\begin{aligned} \lim_{m\to\infty} \int f \nabla y_m \cdot \nabla u dx = 0, \\ \lim_{m\to\infty} \int f \nabla z_m \cdot \nabla w dx = 0. \end{aligned} \tag{3.55}$$

Thus,

$$\begin{aligned} 0 &\leqslant \left| \int f d\mu_m - \int f \left[d\tilde{\mu} + (|\nabla u|^2 + |\nabla w|^2)dx \right] \right| \\ &= \left| \int f(|\nabla u_m|^2 + |\nabla w_m|^2)dx - \int f \left[d\tilde{\mu} + (|\nabla u|^2 + |\nabla w|^2)dx \right] \right| \\ &= \left| \int f \left[|\nabla y_m|^2 + 2\nabla y_m \cdot \nabla u + |\nabla u|^2 + |\nabla z_m|^2 + 2\nabla z_m \cdot \nabla w + |\nabla w|^2 \right] dx \right. \\ &\quad \left. - \int f d\tilde{\mu} - \int f(|\nabla u|^2 + |\nabla w|^2)dx \right| \\ &\leqslant \left| \int f d\tilde{\mu}_m - \int f d\tilde{\mu} \right| + 2\left[\left| \int f \nabla y_m \cdot \nabla u dx \right| + \left| \int f \nabla z_m \cdot \nabla w dx \right| \right]. \end{aligned}$$

Since the first and the second terms goes to zero by (3.53) and (3.55), respectively, then (3.54) holds.

Now, let us show that $(\tilde{\nu}_m)$ is vaguely bounded in $\mathcal{M}_+^b(\mathbb{R}^4)$. As seen before, $(K(y_m, z_m))$ is uniformly bounded. Then, (3.29), implies

$$\left| \int f d\tilde{\nu}_m \right| \leqslant \|f\|_{L^\infty} \int \varphi(|y_m|, |z_m|)dx = CN(|y_m|, |z_m|) \leqslant CK(|y_m|, |z_m|)^2 < M,$$

for some constant M. Again, from Lemma 2.5, we obtain a subsequece, still denoted by $(\tilde{\nu}_m)$, such that

$$\tilde{\nu}_m \overset{*}{\rightharpoonup} \tilde{\nu}, \quad \text{in} \quad \mathcal{M}_+^b(\mathbb{R}^4), \tag{3.56}$$

Now, observe that if

$$\nu_m \overset{*}{\rightharpoonup} \tilde{\nu} + \varphi(u, w)dx, \quad \text{in} \quad \mathcal{M}_+^b(\mathbb{R}^4), \tag{3.57}$$

holds, then $\nu = \tilde{\nu} + \varphi(u, w)dx$ and, hence, $\nu - \varphi(u, w)dx$ is non-negative. So, let us prove (3.57).

We know that $\varphi(u, w) \leqslant C(|u|^4 + |w|^4)$. Then, we are able to use Brezis-Lieb's Lemma 2.6 with $\varphi(|u|, |w|)$ instead of $G(x)$ in the following way: We first assume that $(y_m, z_m) \to 0$ a.e. in $\mathbb{R}^4$ (see Remark 3.9). Then, by Sobolev's inequality $\|f\|_{L^4}^2 \leqslant \|\nabla f\|_{L^2}^2$, we have $(u, w) \in L^4(\mathbb{R}^4) \times L^4(\mathbb{R}^4)$. Then, $\varphi(|u|, |w|) \in L^1(\mathbb{R}^4)$. Moreover, we have that (y_m, z_m) is uniformly bounded in $L^4(\mathbb{R}^4)$. Thus,

$$|\varphi(|a_1 + b_1|, |a_2 + b_2|) - \varphi(|b_1|, |b_2|)| \leqslant \epsilon\phi(a_1, a_2) + \psi_\epsilon(b_1, b_2),$$

where $\phi(a_1, a_2) = |a_1|^4 + |a_2|^4$ and $\psi_\epsilon(b_1, b_2) \leqslant C_\epsilon(|b_1|^4 + |b_2|^4)$ with $\epsilon > 0$. Also,

$$\int \phi(y_m, z_m)dx \leqslant M \quad \text{and} \quad \int \psi_\epsilon(u, w)dx < \infty,$$

for M indenpendent of ϵ and m. The Brezis-Lieb Lemma gives us

$$\lim_{m\to\infty} \int |\varphi(|u_m|, |w_m|) - \varphi(|y_m|, |z_m|) - \varphi(|u|, |w|)|dx = 0. \tag{3.58}$$

Hence, for all $g \in C_c(\mathbb{R}^4)$,

$$\begin{aligned}
0 &\leqslant \left| \int g d\nu_m - \int g[d\tilde{\nu} + \varphi(u, w)dx] \right| \\
&= \left| \int g\varphi(u_m, w_m)dx - \int g\varphi(|y_m|, |z_m|)dx + \int g\varphi(|y_m|, |z_m|)dx - \int g[d\tilde{\nu} + \varphi(u, w)]dx \right| \\
&\leqslant \|g\|_{L^\infty} \int |\varphi(|u_m|, |w_m| - \varphi(|y_m|, |z_m|) - \varphi(|u|, |w|)|dx + \left| \int g d\tilde{\nu}_m - \int f\tilde{\nu} \right|.
\end{aligned}$$

The first term vanishes by taking the limit and using (3.58). The second term goes to zero by the vague convergence of $(\tilde{\nu}_m)$. So (3.57) holds and, consequently, $\nu - \varphi(u, w)dx$ is non-negative, which finish the claim. Therfore,

$$\begin{cases} (|\nabla y_m|^2 + |\nabla z_m|^2)dx \overset{*}{\rightharpoonup} \mu - (|\nabla u|^2 + |\nabla w|^2)dx, & \text{in } \mathcal{M}_+^b(\mathbb{R}^4), \\ \varphi(|y_m|, |z_m|)dx \overset{*}{\rightharpoonup} \nu - \varphi(u, w)dx, & \text{in } \mathcal{M}_+^b(\mathbb{R}^4), \end{cases}$$

and we complete the proof of the lemma after applying Step 1. We notice that Step 1 still holds even if we do not have $u_m, w_m \geqslant 0$; in that case we change (ν_m) in (3.39) by $\nu_m := \varphi(|u_m|, |w_m|)dx$. □

Before proving Theorem 1.3, we will establish an adapted version of Lemma 1.7.4 in (CAZENAVE, 2003), which will help us to avoid the vanishing property.

Lemma 3.10. *Let* $(u_m, w_m) \subset L^3(\mathbb{R}^4) \times L^3(\mathbb{R}^4)$ *be such that,* $u_m, w_m \geqslant 0$ *and* $\int \varphi(u_m, w_m)dx = 1$*, for any* $m \in \mathbb{N}$*. Let* $Q_m(R)$ *be the concentration function of* $\varphi(u_m, w_m)$*, that is,*

$$Q_m(R) := \sup_{y \in \mathbb{R}^4} \int_{B(y,R)} \varphi(u_m, w_m)dx, \quad R > 0.$$

Then, for each m *there is* $y = y(m, R)$ *such that*

$$Q_m(R) = \int_{B(y,R)} \varphi(u_m, w_m)dx.$$

Proof. Fix $m \in \mathbb{N}$. By the definition of Q_m, for any $R > 0$ there is (y_i) in $\mathbb{R}^4$ such that

$$Q_m(R) = \lim_{i \to \infty} \int_{B(y_i,R)} \varphi(u_m, w_m)dx > 0.$$

Hence, there exists i_0 such that if $i > i_0$ then $\int_{B(y_i,R)} \varphi(u_m, w_m)dx \geqslant \epsilon$, where $\epsilon > 0$.

Let us show that (y_i) is bounded. If not, there is a subsequence, still denoted by (y_i), such that $B(y_j, R) \cap B(y_i, R) = \varnothing$, $\forall i \neq j$. Thus

$$1 = \int \varphi(u_m, w_m)dx \geqslant \sum_{i \geqslant i_0} \int_{B(y_i,R)} \varphi(u_m, w_m)dx = \infty,$$

which is an absurd. Therefore (y_j) has a convergent subsequence (y_{j_k}) with limit $y = y(m, R)$. Applying the dominated convergence theorem, we get

$$Q_m(R) = \lim_{j_k \to \infty} \int_{B(y_{j_k},R)} \varphi(u_m, w_m)dx = \int_{B(y,R)} \varphi(u_m, w_m)dx,$$

which finish the proof. □

3.2.3 Proof of Theorem 1.3

Following the strategy in (NOGUERA; PASTOR, 2022), before proceeding to the proof of Theorem 1.3, we first state the following result.

Theorem 3.11. *Suppose that* (u_m, w_m) *is a minimizing sequence for* (3.32) *with* $u_m, w_m \geqslant 0$*. Then, up to translation and dilation* (u_m, w_m) *is relatively compact in* $\mathcal{N}$*, that is, there exist a subsequence* (u_{m_j}, w_{m_j}) *and sequences* $(R_j) \subset \mathbb{R}$*,* $(y_j) \subset \mathbb{R}^4$ *such that the pair* (v_j, z_j) *given by*

$$v_j := R_j^{-1} u_{m_j}(R_j^{-1}(x - y_j)), \quad z_j := R_j t^{-1} w_{m_j}(R_j^{-1}(x - y_j)),$$

strongly converges in $\mathcal{N}$ *to some* (v, z)*, which minimizes* (3.32)*.*

Proof. The proof will proceed in 6 steps. We start taking (u_m, w_m) in $\mathcal{N}$ a minimizing sequence for (3.32) with $u_m, w_m \geqslant 0$. Then,

$$\lim_{m\to\infty} K(u_m, w_m) = I, \quad N(u_m, w_m) = \int \varphi(u_m, w_m)dx = 1, \forall m. \tag{3.59}$$

Step 1. There exist sequences (R_m) in $\mathbb{R}$ and (y_m) in $\mathbb{R}^4$ such that

$$v_m := R^{-1}u_m(R_m^{-1}(x - y_m)), \quad z_m := R^{-1}w_m(R_m^{-1}(x - y_m)) \tag{3.60}$$

satisfies

$$\sup_{y\in\mathbb{R}^4} \int_{B(y,1)} \varphi(v_m, z_m)dx = \int_{B(0,1)} \varphi(v_m, z_m)dx = \frac{1}{2}. \tag{3.61}$$

To show that, let us take $R > 0, s \in \mathbb{R}^4$ and consider the following scaling

$$v_m^{R,s} := R^{-1}u_m(R^{-1}(x - s)), \quad z_m^{R,s} := R^{-1}w_m(R^{-1}(x - s)).$$

From (3.34) we get $K(v_m^{R,s}, z_m^{R,s}) = K(u_m, w_m)$ and $N(v_m^{R,s}, z_m^{R,s}) = N(u_m, w_m) = 1$. Let us consider the concentration function corresponding to $\varphi(v_m, z_m)$ given by

$$Q_m^{R,s}(t) = \sup_{y\in\mathbb{R}^4} \int_{B(y,t)} \varphi(v_m^{R,s}(x), z_m^{R,s}(x))dx.$$

A change of variables give us $Q_m(t/R) = Q_m^{R,s}(t)$ for all $t \geqslant 0$ and $s \in \mathbb{R}^4$, where Q_m is defined as in Lemma 3.10. In particular, for all m, Q_m is a non-decreasing function with $Q_m(0) = 0$, $Q_m(1/R) = Q_m^{R,s}(1)$ and $Q_m(t) \to 1$ as $t \to \infty$. Therefore,

$$\lim_{R\to 0^+} Q_m^{R,s}(1) = \lim_{R\to 0^+} Q_m(1/R) = 1.$$

Consequently, for any m we can find $R_m > 0$ obeying

$$Q_m^{R_m,s}(1) = Q_m(1/R_m) = \frac{1}{2}, \quad \forall s \in \mathbb{R}^4, \tag{3.62}$$

i.e.,

$$\sup_{y\in\mathbb{R}^4} \int_{B(y,1)} \varphi(v_m^{R_m,s}, z_m^{R_m,s})dx = Q_m^{R_m,s}(1) = \frac{1}{2}, \quad \forall s \in \mathbb{R}^4. \tag{3.63}$$

On the other hand, since $\int \varphi(v_m^{R_m,s}, z_m^{R_m,s})dx = 1$ and $v_m^{R_m,s}, z_m^{R_m,s} \geqslant 0$, Lemma 3.10 gives us $y_m \in \mathbb{R}^4$ obeying

$$\begin{aligned}\sup_{y\in\mathbb{R}^4} \int_{B(y,1)} \varphi(v_m^{R_m,s}(x), z_m^{R_m,s}(x))dx &= \int_{B(y_m,1)} \varphi(v_m^{R_m,s}(x), z_m^{R_m,s}(x))dx \\ &= \int_{B(0,1)} \varphi(R_m^{-1}u_m(R_m^{-1}(r + y_m - s)), R_m^{-1}w_m(R_m^{-1}(r + y_m - s)))dr,\end{aligned}$$

where we used the change of variables $x = r + y_m$. Choosing $s = 2y_m$ and using (3.63), we get

$$\begin{aligned}
&\int_{B(0,1)} \varphi(R_m^{-1}u_m(R_m^{-1}(r-y_m), R_m^{-1}w_m(R_m^{-1}(r-y_m))dr \\
&\qquad = \sup_{y\in\mathbb{R}^4}\int_{B(y,1)} \varphi(R_m^{-1}u_m(R_m^{-1}(x-2y_m)), R_m^{-1}w_m(R_m^{-1}(x-2y_m)))dx \\
&\qquad = Q_m^{R_m,2y_m}(1) \\
&\qquad = \frac{1}{2},
\end{aligned}$$

which is the second equality in (3.61). For the first one, observe

$$\begin{aligned}
\sup_{y\in\mathbb{R}^4}\int_{B(y,1)} \varphi(v_m, z_m)dx &= \sup_{y\in\mathbb{R}^4}\int_{B(y,1)} \varphi(R_m^{-1}u_m(R_m^{-1}(x-y_m)), R_m^{-1}w_m(R_m^{-1}(x-y_m)))dx \\
&= \sup_{y\in\mathbb{R}^4}\int_{B(y,1)} \varphi(v_m^{R_m,y_m}, z_m^{R_m,y_m})dx \\
&= \frac{1}{2},
\end{aligned}$$

where in the last equality we used (3.63).

Next, from (3.34) and Step 1, (v_m, z_m) is also a minimizing sequence for (3.32) with $v_m, z_m \geqslant 0$, that is,

$$\lim_{m\to\infty} K(v_m, z_m) = I, \quad N(v_m, z_m) = \int \varphi(v_m, z_m)dx = 1, \forall m \in \mathbb{N}. \tag{3.64}$$

Particularly, (v_m, z_m) is uniformly bounded in $\mathcal{N}$. Thus, there exist $(v, z) \in \dot{H}^1(\mathbb{R}^4) \times \dot{H}^1(\mathbb{R}^4)$ such that, up to a subsequence,

$$(v_m, z_m) \rightharpoonup (v, z) \quad \text{in } \dot{H}^1(\mathbb{R}^4) \times \dot{H}^1(\mathbb{R}^4). \tag{3.65}$$

Let us show that $(v_m, z_m) \to (v, z)$ in $\mathcal{N}$ and (v, z) is a minimizer for (3.32). Indeed, from Remark (3.9) we have $v, z \geqslant 0$. Set the sequence of measure

$$\mu_m := (|\nabla v_m|^2 + |\nabla z_m|^2)dx, \quad \nu_m := \varphi(v_m, z_m)dx. \tag{3.66}$$

The identity in (3.64) give us that (ν_m) is a probability sequence of measures for all m. Thus, by Lemma 3.7, up to a subsequence, occurs one of the following cases: vanishing, dichotomy or compactness. Let us exclude the vanishing and dichotomy cases.

Step 2 Vanishing does not occur.

Indeed, in view of (3.61) it follows that for $R = 1$

$$\lim_{m\to\infty}\sup_{y\in\mathbb{R}^4} \nu_m(B(y,1)) \geqslant \frac{1}{2}.$$

Step 3. Dichotomy does not occur.

Suppose the opposite. Then, there is $\lambda \in (0,1)$ such that for all $\epsilon > 0$, there exist $R > 0$ and a sequence (x_m) in $\mathbb{R}^4$ such that given $R' > R$ and m sufficiently large,

$$\nu_m(B(x_m, R) \geqslant \lambda - \epsilon, \quad \nu_m(\mathbb{R}^4 \backslash B(x_m, R')) \geqslant 1 - \lambda - \epsilon. \tag{3.67}$$

Thus, for m sufficiently large, fixing $\delta > 0$, Corollary 3.6 yields that choosing ρ satisfying $R < \rho < R'$ with $\rho/R' \leqslant C(\delta)$ and $R/\rho \leqslant C(\delta)$ then

$$\int_{B(x_m,R)} \varphi(v_m, z_m)dx \leqslant I^{-2}\left[\int_{B(x_m,\rho)} |\nabla v_m|^2 + |\nabla z_m|^2 dx + \delta K(v_m, z_m)\right]^2$$

and

$$\int_{\mathbb{R}^4\backslash B(x_m,R')} \varphi(v_m, z_m)dy \leqslant I^{-2}\left[\int_{\mathbb{R}^4\backslash B(x_m,\rho)} |\nabla v_m|^2 + |\nabla z_m|^2 dy + (2\delta + \delta^2)K(u, w)\right]^2.$$

Combining both inequalities with (3.67), we get

$$I\left[(\lambda - \epsilon)^{1/2} + (1 - \lambda - \epsilon)^{1/2}\right] \leqslant K(v_m, z_m) + (3\delta + \delta^2)K(v_m, z_m). \tag{3.68}$$

From (3.64) the right-hand side of (3.68) is bounded by $K(v_m, z_m) + (3\delta + \delta^2)M$, where $M > 0$ does not depend on m. Therefore, taking $\delta, \epsilon \to 0$ and $m \to \infty$ leads to

$$I[\lambda^{1/2} + (1 - \lambda)^{1/2}] \leqslant I, \tag{3.69}$$

that is, $\lambda^{1/2} + (1 - \lambda)^{1/2} \leqslant 1$. But this contradicts the fact that if $\lambda \in (0,1)$ then $\lambda^{1/2} + (1 - \lambda)^{1/2} > 1$. Hence, dichotomy does not occurs.

Thereby, Lemma 3.7 implies that compactness occurs, that is, there is a sequence (x_m) in $\mathbb{R}^4$ such that for all $\epsilon > 0$ there is a radius $R > 0$ such that

$$\nu_m(B(x_m, R)) \geqslant 1 - \epsilon, \quad \forall m. \tag{3.70}$$

Step 4. The sequence (ν_m) is uniformly tight.

Indeed, we first show that $B(x_m, R) \cap B(0,1) \neq \varnothing$, for all m. Otherwise, there is m_0 such that $B(x_{m_0}, R) \cap B(0,1) = \varnothing$. Taking $\epsilon \in (0, 1/2)$ in (3.70) leads us to

$$\int_{B(x_{m_0},R)} \varphi(v_{m_0}, w_{m_0})dx > \frac{1}{2}.$$

Combining with (3.61) we have

$$\int \varphi(v_{m_0}, w_{m_0})dx \geqslant \int_{B(x_{m_0},R)} \varphi(v_{m_0}, w_{m_0})dx + \int_{B(0,1)} \varphi(v_{m_0}, w_{m_0})dx > \frac{1}{2} + \frac{1}{2} = 1,$$

which is a contradiction with (3.64). Hence, the claim follows.

Now, since $B(x_m, R) \subset B(0, 2R + 1)$, for all m, (3.70) give us

$$\nu_m(B(0, 2R + 1)) \geqslant 1 - \epsilon, \quad \forall m.$$

Then, because (ν_m) is a sequence of probability measures,

$$\nu_m\left(\mathbb{R}^4\backslash\overline{B(0,2R+1)}\right) = 1-\nu_m(B(0,2R+1)) \leqslant \epsilon, \quad \forall m.$$

that is, (ν_m) uniformly tight.

Step 5. Up to a subsequence, (ν_m) weakly converge to $\nu \in \mathcal{M}_+^1(\mathbb{R}^4)$.

In fact, note that for each $f \in \mathcal{C}_c(\mathbb{R}^4)$,

$$\left|\int f d\nu_m\right| \leqslant \|f\|_{L^\infty}\nu_m(\mathbb{R}^4) = \|f\|_{L^\infty} < \infty.$$

Hence by Lemma 2.5, there is $\nu \in \mathcal{M}_+^b(\mathbb{R}^4)$ such that, up to a subsequence, $\nu_m \rightharpoonup \nu$ weakly in $\mathcal{M}_+^b(\mathbb{R}^4)$, that is,

$$\int f d\nu_m \to \int f d\nu, \quad \forall f \in \mathcal{C}_b(\mathbb{R}^4). \tag{3.71}$$

In particular, taking $f \equiv 1$, we have

$$\nu(\mathbb{R}^4) = \lim_{m\to\infty}\nu_m(\mathbb{R}^4) = 1, \tag{3.72}$$

which implies that $\nu \in \mathcal{M}_+^1(\mathbb{R}^4)$.

Now, since $K(v_m, z_m)$ is uniformly bounded, then (μ_m) is vaguely bounded. Therefore, up to a subsequence, there is $\mu \in \mathcal{M}_+^b(\mathbb{R}^4)$ obeying

$$\mu_m \overset{*}{\rightharpoonup} \mu \quad \text{in} \quad \mathcal{M}_+^b(\mathbb{R}^4). \tag{3.73}$$

Thus, with (3.65), (3.71) and (3.73) in hand, we can use Lemma 3.8 to get

$$\mu \geqslant (|\nabla v|^2 + |\nabla z|^2)dx + \sum_{j\in J} b_j\delta_{x_j}, \quad \nu = \varphi(v,z)dx + \sum_{j\in J} a_j\delta_{x_j} \tag{3.74}$$

for a family $\{x_j \in \mathbb{R}^4;\, j \in J\}$ with J at most countable and $a_j, b_j \geqslant 0$ satisfying

$$a_j \leqslant I^{-2}b_j^2, \quad \forall j \in J \tag{3.75}$$

with $\sum_{j\in J} a_j^{1/2}$ convergent. Hence, (3.33), (3.72) and (3.75) lead us to

$$\begin{aligned} I = \liminf_{m\to\infty}\mu_m(\mathbb{R}^4) &\geqslant \mu(\mathbb{R}^4) \\ &\geqslant K(v,z) + \sum_{j\in J} b_j \\ &\geqslant I\left[N(v,z)^{1/2} + \sum_{j\in J} a_j^{1/2}\right] \\ &\geqslant I\left[N(v,z) + \sum_{j\in J} a_j\right]^{1/2} \\ &= I[\nu(\mathbb{R}^4)]^{1/2} \\ &= I, \end{aligned} \tag{3.76}$$

where we have used that $\lambda \mapsto \lambda^{1/2}$ is a strictly concave function. Then, for all the inequalities above to be in fact equalities, it is necessary that at most one of the terms $N(v,z)$ or a_j, $j \in J$ must be different from zero.

Step 6. $a_j = 0$ for all $j \in J$.

Suppose that there exist $j_0 \in J$ such that $a_{j_0} \neq 0$. Then, by the above discussion, (3.72) and the decomposition (3.74) it follows that $\nu = a_{j_0}\delta_{x_{j_0}}$, and hence

$$1 = \nu(\mathbb{R}^4) = a_{j_0}. \tag{3.77}$$

The condition (3.61) gives us

$$\frac{1}{2} \geqslant \int_{B(x_{j_0},1)} \varphi(v_m, z_m)dx = \nu_m(B(x_{j_0},1)), \quad \forall m,$$

which leads to

$$\frac{1}{2} \geqslant \lim_{m\to\infty} \nu_m(B(x_{j_0},1)) = \nu(B(x_{j_0},1)) = \int_{B(x_{j_0},1)} d\nu = a_{j_0},$$

where the first equality is a consequence of weak convergence (3.71). But, this contradicts (3.77) which finish this step.

With this in hand, we must be in the case $\nu = \varphi(u,v)dx$ and from (3.72), we obtain

$$N(v,z) = \int \varphi(v,z)dx = 1, \tag{3.78}$$

which means that $(v,z) \in \mathcal{N}$.

To show that (v,z) is a minimizer for (3.32), it remains to guarantee that $K(v,z) = I$. Indeed, from the definition of I and (3.78) it follows that $I \leqslant K(v,z)$. On the other hand, the lower semi-continuity of the weak convergence (3.65), gives

$$K(v,z) \leqslant \liminf K(v_m,z_m) = I.$$

Then $K(v,z) = I$ and $(v_m,z_m) \to (v,z)$ strongly in $\mathcal{N}$, completing the proof. □

Corollary 3.12. *There is $(v,z) \in \mathcal{N}$ satisfying $N(v,z) = 1$ and $K(v,z) = C_4^{-1/2}$, where C_4 is the best constant in the critical Sobolev-type inequality (3.29).*

We are now in position to prove Theorem 1.3.

Proof of Theorem 1.3. We start applying Theorem 3.11 to get a minimizer of (3.32), which will be denoted by (v,z). By Lagrange's multiplier theorem, there is a constant Λ such that for any pair $(f,g) \in \dot{H}^1(\mathbb{R}^4) \times \dot{H}^1(\mathbb{R}^4)$ it holds

$$\begin{aligned} 2\int \nabla v \cdot \nabla f dx &= \Lambda \int \left(\frac{1}{9}v^3 + 2z^2 v + \frac{1}{3}v^2 z\right) f dx,\\ 2\int \nabla z \cdot \nabla g dx &= \Lambda \int \left(9z^3 + 2v^2 z + \frac{1}{9}v^3\right) g dx. \end{aligned} \tag{3.79}$$

Taking $f = v$ and $g = z$ we see that $\Lambda \neq 0$. Then, setting $(P_0, Q_0) := \left(\frac{\Lambda}{2}\right)^{\frac{1}{2}}(v, z)$ we deduce that (P_0, Q_0) is non-trivial. Let us show that (P_0, Q_0) is indeed a ground state solution for (1.12). Note that

$$\begin{aligned}\int \nabla P_0 \cdot \nabla f dx &= \left(\frac{\Lambda}{2}\right)^{\frac{1}{2}} \int \nabla v \cdot \nabla f dx \\ &= \int \left(\frac{\Lambda}{2}\right)^{\frac{3}{2}} \left(\frac{1}{9}v^3 + 2z^2 v + \frac{1}{3}v^2 z\right) f dx \\ &= \int \left(\frac{1}{9}P_0^3 + 2Q_0^2 P_0 + \frac{1}{3}P_0^2 Q_0\right) f dx\end{aligned}$$

and

$$\begin{aligned}\int \nabla Q_0 \cdot \nabla f dx &= \left(\frac{\Lambda}{2}\right)^{\frac{1}{2}} \int \nabla z \cdot \nabla g dx \\ &= \int \left(\frac{\Lambda}{2}\right)^{\frac{3}{2}} \left(9z^3 + 2v^2 z + \frac{1}{9}v^3\right) g dx \\ &= \int \left(9Q_0^3 + 2P_0^2 Q_0 + \frac{1}{9}P_0^3\right) g dx.\end{aligned}$$

therefore (P_0, Q_0) is a solution of (1.12). Also, from Remark 3.2 we have $J(P_0, Q_0) = 4^{-2} S(P_0, Q_0)$. Then,

$$J(P_0, Q_0) = \frac{K(P_0, Q_0)^2}{N(P_0, Q_0)} = \frac{K\left(\left(\frac{\Lambda}{2}\right)^{\frac{1}{2}}(v, z)\right)^2}{N\left(\left(\frac{\Lambda}{2}\right)^{\frac{1}{2}}(v, z)\right)} = \frac{\left(\frac{\Lambda}{2}\right)^2 K(v, z)^2}{\left(\frac{\Lambda}{2}\right)^2 N(v, z)} = J(v, z),$$

and consequently, (P_0, Q_0) minimizes J and then minimizes the action functional S. Hence (P_0, Q_0) is a ground state solution of (1.12). □

Corollary 3.13. *The inequality*

$$N(u, w) \leqslant C_4^{opt} K(u, w)^2, \tag{3.80}$$

holds for all $(u, w) \in \mathcal{N}$*, with the optimal constant given by*

$$C_4^{opt} = \frac{1}{16\mathcal{E}(P, Q)}, \tag{3.81}$$

where (P, Q) *is any ground state solution of* (1.12).

Proof. We have seen in Remark 3.4 that (3.80) holds with

$$C_4^{-1} = (C_4^{opt})^{-1} = \inf\{J(u, w); (u, w) \in \mathcal{N}\}.$$

Now, if (P, Q) is a ground state of (1.12), then Remark 3.2 leads to

$$C_4^{-1} = J(P, Q) = 16S(P, Q) = 16\mathcal{E}(P, Q),$$

which is the desired. □

3.3 Blow-up

As mentioned before, the goal of this section is to establish some blow-up results. We start considering for each $\phi \in C_0^\infty(\mathbb{R}^4)$ the funcion

$$\mathcal{V}(t) = \int \phi(x)(|u|^2 + 3\sigma|w|^2)dx.$$

Now, setting $\mathcal{U}(t) = \int \phi|u|^2 dx$, we have that

$$\begin{aligned}
\mathcal{U}'(t) &= 2\mathrm{Re}\int \phi\bar{u}u_t dx \\
&= 2\mathrm{Re}\int \phi\bar{u}(i\Delta u - iu + if(u,w))dx \\
&= 2\mathrm{Re}\int i\phi\bar{u}\Delta u dx - 2\mathrm{Re}\int i\phi|u|^2 dx + 2\mathrm{Re}\int i\phi\bar{u}f(u,w)dx \\
&= 2\mathrm{Re}\int -i(\nabla\phi\bar{u} + \phi\nabla\bar{u})\nabla u dx - 2\mathrm{Re}\int i\phi|u|^2 dx + 2\mathrm{Re}\int i\phi\bar{u}f(u,w)dx \\
&= 2\mathrm{Im}\int \bar{u}\nabla\phi\nabla u dx + 2\mathrm{Im}\int \phi|\nabla u|^2 dx + 2\mathrm{Im}\int \phi|u|^2 dx - 2\mathrm{Im}\int \phi\bar{u}f(u,w)dx,
\end{aligned}$$

where $f(u,w) = \left(\frac{1}{9}|u|^2 + 2|w|^2\right)u + \frac{1}{3}\bar{u}^2 w$. Similarly, for $\mathcal{W}(t) = \int \phi 3\sigma|w|^2 dx$, we obtain

$$\mathcal{W}'(t) = 2\mathrm{Im}\int 3\bar{w}\nabla\phi\nabla w dx - 2\mathrm{Im}\int 3\bar{w}g(u,w)dx.$$

where $g(u,w) = (9|w|^2 + 2|u|^2)w + \frac{1}{9}u^3$. Now, if $\mathcal{V}(t) = \mathcal{U}(t) + \mathcal{W}(t)$, we have

$$\mathcal{V}'(t) = 2\mathrm{Im}\int \nabla\phi(\bar{u}\nabla u + 3\bar{w}\nabla w)dx - 2\mathrm{Im}\int \phi\bar{u}f(u,w) + 3\bar{w}g(u,w))dx. \tag{3.82}$$

As mentioned before, since the second term in (3.82) does not necessarily vanishes, we will follow the ideas presented in (INUI; KISHIMOTO; NISHIMURA, 2020) and work with radially symmetric solutions and the function

$$\mathcal{R}(t) = 2\mathrm{Im}\int \nabla\phi(\bar{u}\nabla u + 3\bar{w}\nabla w)dx \tag{3.83}$$

instead of $\mathcal{V}$. Following the strategy presented in (KAVIAN, 1987, Lemma 2.9), we have

$$\begin{aligned}
\mathcal{R}'(t) =& 4\sum_{1\leqslant m,j\leqslant 4} \mathrm{Re}\int \frac{\partial^2\phi}{\partial x_m \partial x_j}(\partial_{x_j}\bar{u}\partial_{x_m}u + \partial_{x_j}\bar{w}\partial_{x_m}w)dx \\
&- \int \Delta^2\phi(|u|^2 + |w|^2)dx - 2\mathrm{Re}\int \Delta\phi H(u,w)dx,
\end{aligned} \tag{3.84}$$

where $H(u,w) := \bar{u}f(u,w) + \bar{w}g(u,w)$. See also Proposition 2.33. This last identity is known as localized virial identity. Now, observe that if u_0, w_0 are radially symmetric, so are the respective solutions u, w. Besides, if we also take ϕ to be radially symmetric, we can write $\phi(x) = \phi(|x|)$, $u(x) = (|x|)$ and $w(x) = w(|x|)$. Then, for $r = |x|$, we have

$$\frac{\partial \phi}{\partial x_m} = \phi'(r)\frac{x_m}{r} \quad \text{and} \quad \frac{\partial^2 \phi}{\partial x_m \partial x_j} = \phi''(r)\frac{x_m x_j}{r^2} + \phi'(r)\frac{\delta_{mj}}{r} - \phi'(r)\frac{x_m x_j}{r^3},$$

where δ_{mj} is the Kroenecker delta. Multiplying the second derivative by $(x_m x_j)/r^2$ and summing in m and j, we obtain

$$\begin{aligned}\sum_{1\leqslant m,j\leqslant 4} \frac{\partial^2 \phi}{\partial x_m \partial x_j} \cdot \frac{x_m x_j}{r^2} &= \phi''(r) \sum_{1\leqslant m,j\leqslant 4} \frac{x_m^2 x_j^2}{r^4} + \phi'(r) \sum_{1\leqslant m,j\leqslant 4} \frac{x_m x_j}{r^3}\delta_{mj} - \phi'(r) \sum_{1\leqslant m,j\leqslant 4} \frac{x_m^2 x_j^2}{r^5} \\ &= \phi''(r).\end{aligned}$$

Now, since $\dfrac{\partial}{\partial x_m} u = u'(r)\dfrac{x_m}{r}$, then $|\nabla u|^2 = |u'(r)|^2$, whence

$$\frac{\partial}{\partial x_j}\bar{u}\frac{\partial}{\partial x_m}u = |\nabla u|^2 \frac{x_m x_j}{r^2}.$$

Therefore, gathering all the above information leads to

$$\sum_{1\leqslant m,j\leqslant 4} \mathrm{Re}\frac{\partial^2 \phi}{\partial x_m \partial x_j}(\partial_{x_j}\bar{u}\partial_{x_m}u + \partial_{x_j}\bar{w}\partial_{x_m}w) = \phi''(r)\left(|\nabla u|^2 + |\nabla w|^2\right).$$

Doing the same for w and replacing in (3.84), we may rewrite $\mathcal{R}'$ as

$$\mathcal{R}'(t) = 4\int \phi''(|\nabla u|^2 + |\nabla w|^2)dx - \int \Delta^2\phi(|u|^2 + |w|^2)dx - 2\mathrm{Re}\int \Delta\phi H(u,w)dx \quad (3.85)$$

Let us introduce the functional

$$\mathcal{P}(u,w) = \int \left(\frac{1}{36}|u|^4 + \frac{9}{4}|w|^4 + |u|^2|w|^2 + \frac{1}{9}\mathrm{Re}(\bar{u}^3 w)\right) dx.$$

Observe that

$$H(u,w) = \bar{u}f(u,w) + \bar{w}g(u,w) = \frac{1}{9}|u|^4 + 9|w|^4 + 4|u|^2|w|^2 + \frac{3}{9}\bar{u}^3 w + \frac{1}{9}u^3\bar{w},$$

and consequently

$$\mathrm{Re}\int H(u,w)dx = \int \left(\frac{1}{9}|u|^4 + 9|w|^4 + 4|u|^2|w|^2 + \frac{4}{9}\mathrm{Re}(\bar{u}^3 w)\right) dx = \frac{1}{4}\mathcal{P}(u,w).$$

Now, we define the "Pohozaev" functional by

$$\tau(u(t),w(t)) = K(u(t),w(t)) - 4\mathcal{P}(u(t),w(t)) \quad (3.86)$$

Using the definitions of the energy (1.4) and the mass (1.5) we may rewrite

$$\tau(u,w) = 4E(u,w) - K(u,w) - 2M(u,w). \quad (3.87)$$

Our first result reads as follow.

Theorem 3.14. *Assume that* $(u_0, w_0) \in H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4)$ *and let* (u, w) *be the corresponding solution of* (1.1) *defined in the maximal time interval of existence* I. *If*

$$E(u_0, w_0) < \mathcal{E}(P, Q) \tag{3.88}$$

and

$$K(u_0, w_0) > K(P, Q), \tag{3.89}$$

where (P, Q) *is a ground state and* $\mathcal{E}$ *is the energy in* (3.25), *then there exists* $\delta > 0$, *such that* $\tau(u(t), w(t)) \leqslant -\delta < 0$, *for all* $t \in I$.

Proof. Notice that from the definition of the energy (3.25) and (3.23) we have

$$K(P, Q) = 4\mathcal{E}(P, Q). \tag{3.90}$$

Moreover, using (3.80) we get $|\mathcal{P}(u, w)| \leqslant N(|u|, |w|) \leqslant C_4^{opt} K(|u|, |w|)^2 \leqslant C_4^{opt} K(u, w)^2$. Thus, by conservation of the energy

$$\begin{aligned} K(u, w) &= 2E(u_0, w_0) - M(u, w) + 2\mathcal{P}(u, w) \\ &\leqslant 2E(u_0, w_0) + 2|\mathcal{P}(u, w)| \\ &\leqslant 2E(u_0, w_0) + 2C_4^{opt} K(u, w)^2. \end{aligned} \tag{3.91}$$

Therefore, taking $a = 2E(u_0, w_0)$, $b = 2C_4^{opt}$ and $q = 2$ in Lemma 2.7, we have $\gamma = (4C_4^{opt})^{-1}$ and $f(r) = 2E(u_0, w_0) - r + 2C_4^{opt} r^2$, for $r > 0$. Also, setting $G(t) = K(u(t), w(t))$, it follows from (3.91) that

$$f \circ G(t) = 2E(u_0, w_0) - K(u(t), w(t)) + 2C_4^{opt} K(u(t), w(t))^2 \geqslant 0.$$

Thus,

$$a < \left(1 - \frac{1}{q}\right)\gamma \Leftrightarrow E(u_0, w_0) < \frac{1}{16}(C_4^{opt})^{-1} = \mathcal{E}(P, Q),$$

and

$$G(0) > \gamma \Leftrightarrow K(u_0, w_0) > \frac{1}{4C_4^{opt}} = 4\mathcal{E}(P, Q) = K(P, Q).$$

Therefore, applying Lemma 2.28 we get

$$K(u(t), w(t)) > K(P, Q), \quad \forall t \in I. \tag{3.92}$$

The hypothesis (3.88) together with the energy conservation gives us

$$4E(u(t), w(t)) = 4E(u_0, w_0) < 4\mathcal{E}(P, Q) = K(P, Q) < K(u(t), w(t)),$$

and as a consequence of (3.87)

$$\tau(u(t), w(t)) < 0, \quad t \in I. \tag{3.93}$$

Now, let us show that there is $\theta > 0$, such that

$$\tau(u(t), w(t)) < -\theta K(u(t), w(t)), \forall t \in I. \tag{3.94}$$

Indeed, if $E(u_0, w_0) \leqslant 0$, then we can take $\theta = 1$, and by (3.87) we have the desired estimate. On the other hand, suppose that $E(u_0, w_0) > 0$ and (3.94) does not hold. Thus, there exist sequences $(t_m) \subset I$ and $\theta_m \to 0$ obeying

$$-\theta_m \frac{1}{4} K(u(t_m), w(t_m)) \leqslant \tau(u(t_m), w(t_m)) < 0.$$

Which implies

$$\begin{aligned} E(u(t_m), w(t_m)) &= \frac{1}{4}\tau(u(t_m), w(t_m)) + \frac{1}{4}K(u(t_m), w(t_m)) + \frac{1}{2}M(u(t_m), w(t_m)) \\ &\geqslant (1-\theta_m)\frac{1}{4}K(u(t_m), w(t_m)). \end{aligned}$$

Again, the energy conservation, (3.90) and (3.92) lead to

$$\begin{aligned} E(u_0, w_0) = E(u(t_m), w(t_m)) &\geqslant (1-\theta_m)\frac{1}{4}K(u(t_m), w(t_m)) \\ &> (1-\theta_m)\frac{1}{4}K(P, Q) \\ &\geqslant (1-\theta_m)\mathcal{E}(P, Q). \end{aligned}$$

Taking $m \to \infty$ we arrive at a contradiction with (3.88). Hence, the result follows from (3.92) and (3.94). □

Lemma 3.15. *For $x \in \mathbb{R}^4$, we set $r = |x|$. Given a constant $c > 0$, define*

$$\chi(r) = \begin{cases} r^2, & 0 \leqslant r \leqslant 1, \\ c, & r \geqslant 3. \end{cases} \tag{3.95}$$

Assume also that $\chi''(r) \leqslant 2$ and $0 \leqslant \chi'(r) \leqslant 2r$, $\forall r \geqslant 0$. Let $\chi_R(r) = R^2\chi(r/R)$. Then

(i) If $r \leqslant R$,

$$\Delta\chi_R(r) = 8 \quad \textit{and} \quad \Delta^2\chi_R(r) = 0. \tag{3.96}$$

(i) If $r \geqslant R$,

$$\Delta\chi_R(r) \leqslant C \quad \textit{and} \quad |\Delta^2\chi_R(r)| \leqslant \frac{C}{R^2}, \tag{3.97}$$

where C is a constant independent of R.

Proof. (i) Since $r \leqslant R$ then $\chi_R(r) = r^2$. Hence,

$$\partial_{x_i}\chi_R(r) = \partial_{x_i}(|x|^2) = 2x_i \quad \Rightarrow \quad \partial^2_{x_i}\chi_R(r) = 2.$$

Thus, $\Delta\chi_R(r) = 8$ and $\Delta^2\chi_R(r) = 0$.

(ii) A straightforward calculation leads to

$$\frac{\partial^k \chi_R(r)}{\partial r^k} = \frac{\chi^{(k)}(r/R)}{R^{k-2}}.$$

So, for $k = 0, 1, ...$ we have

$$\left|\frac{\partial^k \chi_R(r)}{\partial r^k}\right| \leqslant \frac{C}{R^{k-2}}. \tag{3.98}$$

On the other hand,

$$\partial_{x_i}\chi_R(r) = R^2 \partial_{x_i}\chi(|x|/R) = R\frac{x_i}{|x|} \cdot \chi'(r/R)$$

and

$$\partial^2_{x_i}\chi_R(r) = R\left[\frac{|x|^2 - x_i^2}{|x|^3} \cdot \chi'(r/R) + \frac{1}{R}\frac{x_i^2}{|x|^2} \cdot \chi''(r/R)\right].$$

Therefore,

$$\Delta\chi_R(r) = \frac{3}{r}\frac{\partial\chi_R(r)}{\partial r} + \frac{\partial^2\chi_R(r)}{\partial r^2}.$$

Again, a straightforward calculation leads to

$$\Delta^2\chi_R(r) = \frac{\partial^4\chi_R(r)}{\partial r^4} + \frac{6}{r}\frac{\partial^3\chi_R(r)}{\partial r^3} - \frac{3}{r^2}\frac{\partial^2\chi_R(r)}{\partial r^2} + \frac{3}{r^3}\frac{\partial\chi_R(r)}{\partial r}.$$

Hence, using (3.98) and the fact that $1/r \leqslant 1/R$, allow us to obtain

$$\Delta\chi_R(r) \leqslant C \quad \text{and} \quad |\Delta^2\chi_R(r)| \leqslant \frac{C}{R^2}.$$

□

Now, we are in position to prove Theorem 1.4.

Proof of Theorem 1.4. Consider $I = (T_*, T^*)$. Let us focus in the case $T^* < \infty$, for T_* the argument follows similarly. Suppose by contradiction that $T^* = \infty$. Taking $\phi(x) = \chi_R(|x|)$, defined as in Lemma 3.15, in (3.83) and (3.85) we obtain

$$\mathcal{R}(t) = 2\text{Im}\int \nabla\chi_R(\nabla u\bar{u} + \sigma\nabla w\bar{w})dx$$

and

$$\begin{aligned}\mathcal{R}'(t) &= 8\tau(u, w) + 4\int(\chi_R'' - 2)(|\nabla u|^2 + |\nabla w|^2)dx - \int \Delta^2\chi_R(|u|^2 + |w|^2)dx \\ &\quad - 2\text{Re}\int(\Delta\chi_R - 8)H(u, w)dx \\ &=: 8\tau(u, w) + \mathcal{R}_1(t) + \mathcal{R}_2(t) + \mathcal{R}_3(t).\end{aligned}$$

As in Lemma 3.15, $\chi_R'' \leqslant 2$, for any $r \geqslant 0$, so $\mathcal{R}_1 \leqslant 0$. Now, using conservation of the mass and (3.96), we get

$$\mathcal{R}_2(t) \leqslant \int |\Delta^2\chi_R|(|u|^2 + |w|^2)dx \leqslant CR^{-2}\int_{\{|x|\geqslant R\}}(|u|^2 + |w|^2)dx \leqslant CR^{-2}M(u_0, w_0).$$

Besides, since $|\mathrm{Re}(z)| \leqslant |z|$ for any $z \in \mathbb{C}$, then (3.26), (3.27) and (3.96) give us

$$\begin{aligned}\mathcal{R}_3 &= -\mathrm{Re}\int_{\{|x|\geqslant R\}} (\Delta\chi_R - 8)H(u,w)dx\\ &\leqslant C\int_{\{|x|\geqslant R\}} |\mathrm{Re}H(u,w)|dx\\ &\leqslant C\int_{\{|x|\geqslant R\}} |u|^4 + |w|^4 dx\\ &= C(\|u\|^4_{L^4(|x|\geqslant R)} + \|w\|^4_{L^4(|x|\geqslant R)}).\end{aligned}$$

We know from the literature (see (OGAWA; TSUTSUMI, 1991), equation 3.7), that for $f \in H^1(\mathbb{R}^4)$ radially symmetric, it holds the radial Gagliardo-Nirenberg inequality

$$\int_{\{|x|\geqslant R\}} |f|^4 \leqslant CR^{-3}\|f\|^3_{L^2(|x|\geqslant R)}\|\nabla f\|_{L^2(|x|\geqslant R)}.$$

Then, by Young's inequality, for $\epsilon > 0$, we obtain

$$\begin{aligned}\mathcal{R}_3 &= C(\|u\|^4_{L^4(|x|\geqslant R)} + \|w\|^4_{L^4(|x|\geqslant R)})\\ &\leqslant CR^{-3}(\|u\|^3_{L^2(|x|\geqslant R)}\|\nabla u\|_{L^2(|x|\geqslant R)} + \|w\|^3_{L^2(|x|\geqslant R)}\|\nabla w\|_{L^2(|x|\geqslant R)})\\ &\leqslant C_\epsilon R^{-6}(\|u\|^6_{L^2(|x|\geqslant R)} + \|w\|^6_{L^2(|x|\geqslant R)}) + \epsilon K(u,w)\\ &\leqslant C_\epsilon R^{-6}M(u_0,w_0)^3 + \epsilon K(u,w),\end{aligned}$$

where C_ϵ depends on μ and ϵ.

Now, from (3.87) we have

$$\epsilon K(u,w) \leqslant -\epsilon\tau(u,w) + 4\epsilon E(u_0,w_0),$$

then, gathering all above estimates we obtain,

$$\mathcal{R}'(t) \leqslant (8-\epsilon)\tau(u,w) + CR^{-2}M(u_0,w_0) + C_\epsilon R^{-6}M(u_0,w_0)^3 + 4\epsilon E(u_0,w_0), \quad \epsilon > 0. \tag{3.99}$$

Therefore, for $\epsilon \in (0,1)$, Lemma 3.14 and the energy conservation lead to

$$\mathcal{R}'(t) \leqslant -(8-\epsilon)\delta + CR^{-2}M(u_0,w_0) + C_\epsilon R^{-6}M(u_0,w_0)^3 + 4\epsilon E(u_0,w_0). \tag{3.100}$$

Hence, fixing R as large as necessary and ϵ as small as necessary, we get $\mathcal{R}'(t) \leqslant -2\delta$. Integrating in $[0,t)$ we obtain

$$\mathcal{R}(t) \leqslant -2\delta t + \mathcal{R}(0). \tag{3.101}$$

On the other hand, by Hölder's inequality,

$$\begin{aligned}|\mathcal{R}(t)| &\leqslant 2R\int |\chi'(|x|/R)|(|\nabla u||u| + \sigma|\nabla w||w|)dx\\ &\leqslant CR(\|u\|_{L^2}\|\nabla u\|_{L^2} + \|w\|_{L^2}\|\nabla w\|_{L^2})\\ &\leqslant CRM(u_0,w_0)^{1/2}K(u,w)^{1/2}.\end{aligned} \tag{3.102}$$

Taking T_0 sufficiently large such that $\mathcal{R}(0)/\delta < T_0$, by (3.101) we get

$$\mathcal{R}(t) \leqslant -\delta t < 0, \quad t \geqslant T_0. \tag{3.103}$$

Consequently, (3.102) and (3.103) imply that

$$\delta t \leqslant -\mathcal{R}(t) = |\mathcal{R}(t)| \leqslant CRM(u_0, w_0)^{1/2} K(u, w)^{1/2},$$

that is, for some positive constant C_0,

$$K(u(t), w(t)) \geqslant C_0 t^2, \quad t \geqslant T_0. \tag{3.104}$$

Now, since ϵ can be chosen arbitrarily small, from (3.99) and (3.87), we deduce that

$$\mathcal{R}'(t) \leqslant 32E(u_0, w_0) - 8K(u, w) + CR^{-2}M(u_0, w_0) + CR^{-6}M(u_0, w_0)^3, \tag{3.105}$$

where we have used the energy conservation once again. Notice that in the above inequality, several terms are independent of t. Thus, we may choose $T_1 > T_0$, so that

$$C_0 4T_1^2 \geqslant 32E(u_0, w_0) + CR^{-2}M(u_0, w_0) + CR^{-6}M(u_0, w_0)^3.$$

Then, from (3.104) and (3.105) we arrive at

$$\mathcal{R}'(t) \leqslant -4K(u(t), w(t)), \quad t > T_1.$$

Hence, integrating in $[T_1, t)$, we get

$$\mathcal{R}(t) \leqslant -4 \int_{T_1}^{t} K(u(s), w(s)) ds,$$

and combining with (3.102), leads to

$$4 \int_{T_1}^{t} K(u(s), w(s)) ds \leqslant -\mathcal{R}(t) \leqslant |\mathcal{R}(t)| \leqslant CRM(u_0, w_0)^{1/2} K(u(t), w(t))^{1/2}. \tag{3.106}$$

Setting $\eta(t) := \displaystyle\int_{T_1}^{t} K(u(s), w(s)) ds$ and $A := \dfrac{16}{C^2 R^2 M(u_0, w_0)}$, we may write

$$A \leqslant \frac{\eta'(t)}{\eta^2(t)},$$

taking $T' > T_1$ and integrating over $[T', t)$, we get

$$A(t - T') \leqslant \int_{T'}^{t} \frac{\eta'(s)}{\eta^2(s)} ds = \frac{1}{\eta(T')} - \frac{1}{\eta(t)} \leqslant \frac{1}{\eta(T')},$$

that is,

$$0 < \eta(T') \leqslant \frac{1}{A(t - T')}.$$

Hence, making $t \to \infty$ we derive a contradiction. Therefore, $T^* < \infty$, and the proof is complete. □

CHAPTER 4

SCATTERING FOR A QUADRATIC TYPE NLS SYSTEM IN DIMENSION 6

4.1 Local theory in $\dot{\mathbf{H}}_x^1$

In this section we will prove the local well-posedness of (1.2) in $\dot{\mathbf{H}}_x^1$. We will use the approach presented in (KILLIP; VISAN, 2013). The first step is to prove the local well-posedness assuming that the initial data belongs to the inhomogeneous Sobolev space $\mathbf{H}_x^1(\mathbb{R}^6)$, using the usual method of contraction, presented in (CAZENAVE, 2003). Next step is to present a stability lemma, which allows us to show uniform continuous dependence of the solution $\mathbf{u}$ to the initial data $\mathbf{u}_0$. This result allows us to work with the initial data in the homogeneous Sobolev space $\dot{\mathbf{H}}_x^1$, since every function in $\dot{\mathbf{H}}_x^1$ can be well approximated by $\mathbf{H}_x^1$ functions. At the end of the section,we show a standard blow-up result. We start with the following result.

Theorem 4.1. *(Standard local well-posedness). Suppose the hypothesis (H1) and (H2) hold. Let $\mathbf{u}_0 \in \mathbf{H}_x^1(\mathbb{R})$. Let there exists $\eta_0 > 0$ such that if $0 < \eta \leqslant \eta_0$ and I is a compact interval containing zero such that*

$$\|\mathbf{U}(t)\mathbf{u}_0\|_{\mathbf{L}_t^4\dot{\mathbf{H}}_x^{1,\frac{12}{5}}(I\times\mathbb{R}^6)} \leqslant \eta, \tag{4.1}$$

then there exists a unique solution $\mathbf{u}$ to (1.2) on $I \times \mathbb{R}^6$. Moreover, we have the bounds

$$\|\mathbf{u}\|_{\mathbf{L}_t^4\dot{\mathbf{H}}_x^{1,\frac{12}{5}}} \leqslant 2\eta \tag{4.2}$$

$$\|\nabla\mathbf{u}\|_{\mathcal{S}^0(I\times\mathbb{R}^6)} \lesssim \|\nabla\mathbf{u}_0\|_{\mathbf{L}_x^2} + \eta^2 \tag{4.3}$$

$$\|\mathbf{u}\|_{\mathcal{S}^0(I\times\mathbb{R}^6)} \lesssim \|\mathbf{u}_0\|_{\mathbf{L}^2_x}. \tag{4.4}$$

Proof. As mentioned before, we will use the contraction mapping argument. Define the solution map $\boldsymbol{\phi}(\mathbf{u})(t) = (\phi_1(\mathbf{u})(t), ..., \phi_l(\mathbf{u})(t))$, with

$$\phi_k(\mathbf{u})(t) := U_k(t)u_{k0} - i\int_0^t \frac{1}{\alpha_k}U_k(t-s)f_k(\mathbf{u}(s))ds,$$

on the set $B_1 \cap B_2$ where

$$B_1 := \left\{\mathbf{u} \in \mathbf{L}^\infty_t\mathbf{H}^1_x(I\times\mathbb{R}^6);\ \|\mathbf{u}\|_{\mathbf{L}^\infty_t\mathbf{H}^1_x(I\times\mathbb{R}^6)} \lesssim 2\|\mathbf{u}_0\|_{\mathbf{H}^1_x} + (2\eta)^2\right\}$$
$$B_2 := \left\{\mathbf{u} \in \mathbf{L}^4_t\mathbf{H}^{1,\frac{12}{5}}_x(I\times\mathbb{R}^6);\ \|\mathbf{u}\|_{\mathbf{L}^4_t\dot{\mathbf{H}}^{1,\frac{12}{5}}_x(I\times\mathbb{R}^6)} \leqslant 2\eta \text{ and } \|\mathbf{u}\|_{\mathbf{L}^4_t L^{\frac{12}{5}}_x(I\times\mathbb{R}^6)} \lesssim \|\mathbf{u}_0\|_{\mathbf{L}^2_x}\right\},$$

with the metric

$$d(\mathbf{u},\mathbf{v}) := \|\mathbf{u}-\mathbf{v}\|_{\mathbf{L}^4_t L^{\frac{12}{5}}_x(I\times\mathbb{R}^6)}.$$

Note that, with the metric d, both B_1 and B_2 are closed, and therefore, complete (see (CAZENAVE, 2003, Theorem 4.4.1)). Using the Strichartz inequality, Lemma 2.11- (iii), and Sobolev's embedding, we get that for $\mathbf{u} \in B_1 \cap B_2$, $k = 1, ..., l$

$$\begin{aligned}
\|\phi_k(\mathbf{u})\|_{\mathbf{L}^\infty_t\mathbf{H}^1_x(I\times\mathbb{R}^6)} &\lesssim \|\mathbf{u}_0\|_{\mathbf{H}^1_x} + \|f_k(\mathbf{u})\|_{\mathbf{L}^2_t\mathbf{H}^{1,\frac{3}{2}}_x(I\times\mathbb{R}^6)}\\
&\lesssim \|\mathbf{u}_0\|_{\mathbf{H}^1_x} + \|\mathbf{u}\|_{\mathbf{L}^4_t\mathbf{H}^{1,\frac{12}{5}}}\|\mathbf{u}\|_{\mathbf{L}^4_{t,x}(I\times\mathbb{R}^6)}\\
&\lesssim \|\mathbf{u}_0\|_{\mathbf{H}^1_x} + (2\eta + \|\mathbf{u}_0\|_{\mathbf{L}^2_x})\|\mathbf{u}\|_{\mathbf{L}^4_t\dot{\mathbf{H}}^{1,\frac{12}{5}}(I\times\mathbb{R}^6)}\\
&\lesssim \|\mathbf{u}_0\|_{\mathbf{H}^1_x} + (2\eta + \|\mathbf{u}_0\|_{\mathbf{L}^2_x})(2\eta).
\end{aligned}$$

Thus,

$$\|\boldsymbol{\phi}(\mathbf{u})\|_{\mathbf{L}^\infty_t\mathbf{H}^1_x(I\times\mathbb{R}^6)} = \sum_{k=1}^{l}\|\phi_k(\mathbf{u})\|_{\mathbf{L}^\infty_t\mathbf{H}^1_x(I\times\mathbb{R}^6)} \lesssim \|\mathbf{u}_0\|_{\mathbf{H}^1_x} + (2\eta + \|\mathbf{u}_0\|_{\mathbf{L}^2_x})(2\eta)$$

similarly,

$$\|\boldsymbol{\phi}(\mathbf{u})\|_{\mathbf{L}^4_t\mathbf{L}^{\frac{12}{5}}_x(I\times\mathbb{R}^6)} \lesssim \|\mathbf{u}_0\|_{\mathbf{L}^2_x}(1+2\eta).$$

Arguing as above and using (4.1), we obtain

$$\|\boldsymbol{\phi}(\mathbf{u})\|_{\mathbf{L}^4_t\dot{\mathbf{H}}^{1,\frac{12}{5}}_x} \lesssim \eta + (2\eta)^2.$$

Therefore, taking η_0 small enough such that $0 < \eta \leqslant \eta_0$, the functional $\boldsymbol{\phi}$ maps the set $B_1 \cap B_2$ to itself. Now, repeating the above computations and using Lemma 2.11 item (i), allow us to obtain

$$\|\boldsymbol{\phi}(\mathbf{u}) - \boldsymbol{\phi}(\mathbf{v})\|_{\mathbf{L}^4_t\mathbf{L}^{\frac{12}{5}}_x(I\times\mathbb{R}^6)} \lesssim (2\eta)\|\mathbf{u}-\mathbf{v}\|_{\mathbf{L}^4_t\mathbf{L}^{\frac{12}{5}}_x(I\times\mathbb{R}^6)}.$$

Then, $\boldsymbol{\phi} : B_1 \cap B_2 \to B_1 \cap B_2$ is a contraction, provided η_0 is small enough. The fixed point theorem guarantees the existence of a unique function $\mathbf{u} \in B_1 \cap B_2$ satisfying $\boldsymbol{\phi}(\mathbf{u}) = \mathbf{u}$. In addition, $\boldsymbol{\phi}$ maps into $\mathbf{C}^0_t\mathbf{H}^1_x$. So, we may conclude that $\mathbf{u}$ is indeed a solution to (1.2). □

Corollary 4.2. *There is $\eta_0 > 0$ such that for all $\mathbf{u}_0 \in \mathbf{H}^1(\mathbb{R}^6)$ satisfying $\|\nabla \mathbf{u}_0\|_{\mathbf{L}^2} \lesssim \delta_0$, then the solution given in of Theorem 1.9 extends globally.*

Proof. Note that by the Strichartz inequality,

$$\|\nabla \mathbf{U}(t)\mathbf{u}_0\|_{\mathbf{L}_t^4 \mathbf{L}_x^{\frac{12}{5}}(\mathbb{R}\times\mathbb{R}^6)} \lesssim \|\nabla \mathbf{u}_0\|_{\mathbf{L}_x^2}.$$

Thus, under the hypothesis, (4.1) holds with $I = \mathbb{R}$. □

As a next step, we prove a stability result that, as we mentioned before, will allow us to remove the restriction on the initial data. More precisely, we remove the condition that the initial data belongs to the inhomogeneous Sobolev space $\mathbf{H}_x^1$. This result is also important to prove the Palais-Smale condition and existence of a critical solution. In (KOCH; TATARU; VISAN, 2014) can be found a more general result, the one we will state next is a short version, which is enough for our purpose.

Lemma 4.3. *Let I be a compact interval containing 0 and $\mathbf{v} : I \times \mathbb{R}^6 \to \mathbb{C}^l$ be an approximate solution of (1.2) in the sense that*

$$i\alpha_k \partial_t v_k + \gamma_k \Delta v_k = -f_k(\mathbf{v}) + e_k$$

for some function $\mathbf{e} = (e_1, ..., e_l)$. Assume also that

$$\|\mathbf{v}\|_{\mathbf{L}_t^\infty \dot{\mathbf{H}}_x^1} \leqslant E, \tag{4.5}$$

$$S_I(\mathbf{v}) \leqslant L, \tag{4.6}$$

where E, L are positive constants and $S_I(\mathbf{v})$ is defined on page 17. Let $\mathbf{u}_0 \in \dot{\mathbf{H}}^1$. Assume that

$$\|\mathbf{u}_0 - \mathbf{v}(0)\|_{\dot{\mathbf{H}}_x^1} \leqslant \epsilon, \tag{4.7}$$

$$\|\nabla \mathbf{e}\|_{\mathbf{L}_{t,x}^{8/5}} \leqslant \epsilon, \tag{4.8}$$

for some $0 < \epsilon < \epsilon_1$, where ϵ_1 is a constant depending on E and L. Then, there exists a unique solution $\mathbf{u} : I \times \mathbb{R}^6 \to \mathbb{C}^l$ to (1.2) with initial data $\mathbf{u}(0, x) = \mathbf{u}_0$ such that

$$S_I(\mathbf{v} - \mathbf{u}) \leqslant C(E, L)\epsilon, \tag{4.9}$$

$$\|\nabla(\mathbf{u} - \mathbf{v})\|_{\mathcal{S}^0(I)} \leqslant C(E, L)\epsilon, \tag{4.10}$$

$$\|\nabla \mathbf{u}\|_{\mathcal{S}^0(I)} \leqslant C(E, L). \tag{4.11}$$

Proof. We will follow the ideas presented in (KOCH; TATARU; VISAN, 2014). First, we prove the result under the additional hypothesis that $\mathbf{u}_0 \in \mathbf{L}_x^2$ (and consequently $\mathbf{u}_0 \in \mathbf{H}_x^1$). This allows us to use Theorem 4.1 to ensure the existence of $\mathbf{u}$. We will remove such assumption later. Also, there is no loss of generality in assuming that $\mathbf{u}$ is defined on the interval I.

We start assuming that

$$\|\nabla \mathbf{v}\|_{\mathbf{L}_t^4\mathbf{L}_x^{12/5}(I\times\mathbb{R}^6)} \leqslant \delta \tag{4.12}$$

for some $\delta > 0$ small enough depending on E. Without loss of generality, we may assume that $0 = \inf I$. Now, Let $\mathbf{w} = \mathbf{u} - \mathbf{v}$. Thus, $\mathbf{w}$ solves the following system

$$(i\alpha_k\partial_t + \gamma_k\Delta)w_k + e_k = f_k(\mathbf{v}) - f_k(\mathbf{u}).$$

Set $A(t) = \sum_{k=1}^{l} A_k(t)$, where

$$A_k(t) := \|\nabla[(i\alpha_k\partial_t + \gamma_k\Delta)w_k + e_k]\|_{L_t^2L_x^{3/2}([0,t]\times\mathbb{R}^6)}.$$

Using the integral equation for w_k, $k = 1, ..., l$, and taking the gradient, we deduce

$$\nabla w_k(t) = U_k(t)\nabla w(0) + i\int_0^t U_k(t-s)\nabla[f_k(\mathbf{v}) - f_k(\mathbf{u}) - e_k]ds.$$

Then, by Sobolev embbeding, Strichartz's inequality, (4.7) and (4.8), we have for $k = 1, ..., l$,

$$\begin{aligned}\|w_k\|_{L^4_{t,x}([0,t]\times\mathbb{R}^6)} &\lesssim \|\nabla w_k\|_{L_t^4L_x^{12/5}((0,t]\times\mathbb{R}^6)}\\ &\lesssim \|w_k(0)\|_{\dot{H}^1_x} + A_k(t) + \|\nabla e_k\|_{L^{8/5}_{t,x}([0,t]\times\mathbb{R}^6)}\\ &\lesssim A_k(t) + \epsilon.\end{aligned}$$

Therefore

$$\|\mathbf{w}\|_{\mathbf{L}^4_{t,x}([0,t]\times\mathbb{R}^6)} \lesssim \|\nabla\mathbf{w}\|_{\mathbf{L}_t^4L_x^{12/5}([0,t]\times\mathbb{R}^6)} \lesssim A(t) + \epsilon. \tag{4.13}$$

On the other hand, by Lemma 2.11 (ii), (4.12) and (4.8), we obtain

$$\begin{aligned}A_k(t) &= \|\nabla[f_k(\mathbf{u}) - f_k(\mathbf{v})]\|_{L_t^2L_x^{3/2}}\\ &\lesssim \|\mathbf{u}\|_{\mathbf{L}^4_{t,x}}\|\nabla\mathbf{w}\|_{\mathbf{L}_t^4\mathbf{L}_x^{12/5}} + \|\mathbf{w}\|_{\mathbf{L}^4_{t,x}}\|\nabla\mathbf{v}\|_{\mathbf{L}_t^4\mathbf{L}_x^{12/5}}\\ &\lesssim \left[\|\mathbf{w}\|_{\mathbf{L}^4_{t,x}} + \|\mathbf{v}\|_{\mathbf{L}^4_{t,x}}\right]\|\nabla\mathbf{w}\|_{\mathbf{L}_t^4\mathbf{L}_x^{12/5}} + \|\mathbf{w}\|_{\mathbf{L}^4_{t,x}}\|\nabla\mathbf{v}\|_{\mathbf{L}_t^4\mathbf{L}_x^{12/5}}\\ &\lesssim [A(t)+\epsilon+\delta][A(t)+\epsilon] + [A(t)+\epsilon]\delta,\end{aligned} \tag{4.14}$$

where all space time norms are taken in $[0, t] \times \mathbb{R}^6$. Summing over k and taking $\delta > 0$ small enough, we obtain

$$0 \leqslant CA(t)^2 - A(t) + C\epsilon^2. \tag{4.15}$$

Now, observe that if we take $h(x) = Cx^2 - x + C\epsilon^2$, we have a parabola facing upwards and roots given by

$$x = \frac{1 \pm \sqrt{1-4C^2\epsilon^2}}{2C}.$$

Moreover, notice that $\dfrac{1 \pm \sqrt{1-4C^2\epsilon^2}}{2C} < \epsilon$ if, and only if, $\epsilon < \dfrac{1}{2C}$. Then, by (4.15) we get $h(A(t)) \geqslant 0$ and, since $A(t) \geqslant 0$ for all $t \in I$ and $A(0) = 0$, by continuity we should have

$$A(t) \leqslant \frac{1-\sqrt{1-4C^2\epsilon^2}}{2C}.$$

for all $t \in I$. Hence, if ϵ is sufficiently small, we deduce

$$A(t) \lesssim \epsilon, \quad \forall t \in I,\ k = 1, ..., l, \tag{4.16}$$

for $0 < \epsilon < \epsilon_1$. This, together with (4.13) gives us

$$S_I(\mathbf{v} - \mathbf{u}) \lesssim \epsilon. \tag{4.17}$$

Now, to obtain (4.10) using Strichartz's inequality, (4.7), (4.8) and (4.16), we get for $k = 1, ..., l$

$$\|\nabla w_k\|_{\mathcal{S}^0(I)} \lesssim \|u_k(0) - v_k(0)\|_{\dot{H}^1_x} + A_k(t) + \|\nabla e_k\|_{L^2_t L^{3/2}_x(I\times\mathbb{R}^6)} \lesssim \epsilon. \tag{4.18}$$

Combining with (4.17) and (4.18), we get the desired result under (4.12).

Furthermore, to obtain (4.11), observe that by (4.12) and (4.13), for $k = 1, ..., l$,

$$\|\nabla u_k\|_{L^4_t L^{12/5}_x(I\times\mathbb{R}^6)} \lesssim \|\nabla w_k\|_{L^4_t L^{12/5}_x(I\times\mathbb{R}^6)} + \|\nabla v_k\|_{L^4_t L^{12/5}_x(I\times\mathbb{R}^6)} \lesssim \epsilon + \delta.$$

After combine this together with Strichartz's inequality, Sobolev's embedding and (4.5), we deduce

$$\begin{aligned} \|\nabla u_k\|_{L^\infty_t L^2_x(I\times\mathbb{R}^6)} &\lesssim \|v_k(t_0)\|_{\dot{H}^1_x} + \|u_k(0) - v_k(t_0)\|_{\dot{H}^1_x} + \|\nabla u_k\|^2_{L^4_t L^{12/5}_x(I\times\mathbb{R}^6)} \\ &\lesssim E + \epsilon + [\epsilon + \delta]^2 \lesssim E, \end{aligned} \tag{4.19}$$

provided $\delta, \epsilon \leqslant \epsilon_0 = \epsilon_0(E)$.

Next we will remove the assumption (4.12). First, we note that (4.6) implies $\nabla \mathbf{u} \in L^\infty_t L^2_x(I \times \mathbb{R}^6)$. Indeed, subdividing I into $N_0 \sim (1 + \dfrac{L}{\eta})^4$ subintervals I_j such that on each I_j we have

$$\|\mathbf{v}\|_{\mathbf{L}^4_{t,x}(I_j\times\mathbb{R}^6)} \leqslant \eta,$$

and using Strichartz's inequality, Lemma 2.11 and (4.5), we may estimate, for $k = 1, ..., l$,

$$\begin{aligned} \|\nabla v_k\|_{\mathcal{S}^0(I_j)} &\lesssim \|v_k\|_{L^\infty_t \dot{H}^1_x(I\times\mathbb{R}^6)} + \|\nabla f_k(\mathbf{v})\|_{L^2_{t,x}(I_j\times\mathbb{R}^6)} + \|\nabla e_k\|_{L^2_t L^{3/2}_x(I\times\mathbb{R}^6)} \\ &\lesssim E + \|\mathbf{v}\|_{L^4_{t,x}(I_j\times\mathbb{R}^6)} \|\nabla \mathbf{v}\|_{\mathcal{S}^0(I_j\times\mathbb{R}^6)} + \epsilon \\ &\lesssim E + \eta \|\nabla \mathbf{v}\|_{L^4_{t,x}(I_j\times\mathbb{R}^6)} + \epsilon. \end{aligned}$$

Thus, taking η small enough, we get

$$\|\nabla v_k\|_{\mathcal{S}^0(I_j)} \lesssim E + \epsilon, \quad k = 1, ..., l.$$

Summing this bound over all intervals I_j, allow us to obtain

$$\|\nabla v_k\|_{\mathcal{S}^0(I\times\mathbb{R}^6)} \leqslant C(E,L), \quad k=1,...,l.$$

Now, we may subdivide I into N_1 subintervals $J_i=[t_i,t_{i+1}]$ such that on each J_i we get

$$\|\nabla \mathbf{v}\|_{\mathbf{L}^2_t\mathbf{L}^{12/5}_x(J_i\times\mathbb{R}^6)} \leqslant \delta, \quad k=1,...,l,$$

where δ is as in (4.12). Choosing ϵ_1 small enough, depending on ϵ_0 and N_1, the same argument above, implies that for each i and $0<\epsilon<\epsilon_1$,

$$\begin{aligned} S_{J_i}(\mathbf{v}-\mathbf{u}) &\leqslant C(i)\epsilon,\\ \|\nabla(\mathbf{v}-\mathbf{u})\|_{\mathcal{S}^0(J_i)} &\leqslant C(i)\epsilon,\\ \|\nabla\mathbf{u}\|_{\mathcal{S}^0(J_i)} &\leqslant C(i)E,\\ A(t_i) &\leqslant C(i)\epsilon, \end{aligned}$$

since (4.7) holds when 0 is replaced by t_i. We show this using an inductive argument. By Strichartz's inequality, we have, for $k=1,...,l$,

$$\begin{aligned} \|u_k(t_{i+1})-v_k(t_{i+1})\|_{\dot H^1_x} &\lesssim \|u_k(0)-v_k(t_0)\|_{\dot H^1_x} + \|\nabla e_k\|_{L^2_tL^{3/2}_x} + A_k(t_{i+1})\\ &\lesssim \epsilon + \sum_{j=0}^{i} C(j)\epsilon. \end{aligned}$$

where we take $t\in[0,t_{i+1}]$. Choosing ϵ_1 small enough depending on ϵ_0 and E, we can continue the inductive argument.

It remains to remove the additional hypothesis that $\mathbf{u}_0\in\mathbf{L}^2_x$. We use the usual limiting argument to this end. Let us approximate $\mathbf{u}_0\in\dot{\mathbf{H}}^1_x$ by a sequence $\{\mathbf{u}_0^n\}\subset\mathbf{H}^1_x$, that is, for any $\epsilon>0$, there exists $n_0>0$ such that for $n>n_0$,

$$\|\mathbf{u}_0-\mathbf{u}_0^n\|_{\dot{\mathbf{H}}^1_x} \leqslant \epsilon. \tag{4.20}$$

Observe that we can find an interval I_n such that

$$\|\mathbf{U}(t)\mathbf{u}_0^n\|_{\mathbf{L}^4_t\dot{\mathbf{H}}^{1,\frac{12}{5}}_x(I_n\times\mathbb{R}^6)} = \|\mathbf{U}(t)\nabla\mathbf{u}_0^n\|_{\mathbf{L}^4_t\mathbf{L}^{\frac{12}{5}}_x(I_n\times\mathbb{R}^6)} \leqslant \eta,$$

for some $0<\eta\leqslant\eta_0$. Then, by Theorem 4.1, we can find a sequence of solutions $\mathbf{u}^n: I_n\times\mathbb{R}^6\to\mathbb{C}$ to (1.2) in $\mathbf{H}^1_x$ with initial data $\mathbf{u}^n(0)=\mathbf{u}_0^n$, such that, for all $n>n_0$,

$$\|\mathbf{u}^n\|_{\mathbf{L}^\infty_t\dot{\mathbf{H}}^1_x(I_n\times\mathbb{R}^6)} \lesssim \|\nabla\mathbf{u}^n\|_{\mathcal{S}^0(I_n)} \lesssim \|\nabla\mathbf{u}_0^n\|_{\mathbf{L}^2_x}+\eta^2 \leqslant E.$$

and

$$\|\mathbf{u}^n\|_{\mathbf{L}^4_{t,x}(I_n\times\mathbb{R}^6)} \lesssim \|\mathbf{u}^n\|_{\mathbf{L}^4_t\dot{\mathbf{H}}^{1,\frac{12}{5}}_x(I_n\times\mathbb{R}^6)} \lesssim \|\nabla\mathbf{u}^n\|_{\mathcal{S}^0(I_n)} \lesssim \|\nabla\mathbf{u}_0^n\|_{\mathbf{L}^2_x}+\eta^2 \leqslant L.$$

for some $\eta,E,L>0$. Together with (4.20), we can apply the above result for $\mathbf{e}=0$ and $\mathbf{v}:=\mathbf{u}^m$, for $m>n_0$. Thus, we get a solution $\mathbf{w}_n: I_m\times\mathbb{R}^6\to\mathbb{C}$, with initial data $\mathbf{w}_n(0)=\mathbf{u}_0^n$ such that

$$\|\mathbf{v}-\mathbf{w}_n\|_{\dot{\mathbf{H}}^1_x} < \epsilon. \tag{4.21}$$

But, by uniquiness of solution, we must have $\mathbf{w}_n = \mathbf{u}^n$ and $I_n = I_m$ for $n, m > n_0$. Therefore, all solutions $\mathbf{u}^n$ with $n > n_0$ are defined in the same interval, namely, $\tilde{I}$. Rewriting (4.21) with $\mathbf{v} := \mathbf{u}^m$ and $\mathbf{w}_n = \mathbf{u}^n$ we get that the sequence $(\mathbf{u}^n)$ is Cauchy in $\dot{\mathbf{H}}^1_x$. Then, $\mathbf{u}_n$ converge to a solution $\mathbf{u} : \tilde{I} \times \mathbb{R}^6 \to \mathbb{C}$ with initial data $\mathbf{u}_0$ that obeys $\nabla \mathbf{u} \in \mathbf{L}^\infty_t \mathbf{L}^2_x(\tilde{I})$. This completes the proof. □

Now we are in a position to prove local well-posedness in the energy critical norm.

Corollary 4.4. *(Local well-posedness) Let $\mathbf{u}_0 \in \dot{\mathbf{H}}^1_x$. Then, there exists a compact time interval I containing 0 and a unique solution $\mathbf{u}$ to (1.2) with initial data $\mathbf{u}_0 = \mathbf{u}(0)$.*

Proof. Let $\mathbf{u}_0 \in \dot{\mathbf{H}}^1_x$. Since $\dot{\mathbf{H}}^1_x$ functions can be approximated by $\mathbf{H}^1_x$ functions, we may found a sequence $(\mathbf{u}^n_0) \subset \mathbf{H}^1_x$ such that, for any $\epsilon > 0$, there exists $n_0 > 0$ such that for $n > n_0$,

$$\|\mathbf{u}_0 - \mathbf{u}^n_0\|_{\dot{\mathbf{H}}^1_x} \leqslant \epsilon. \tag{4.22}$$

Now, by Theorem 4.1, given $\eta_0 > 0$, we can find a sequence of solutions to (1.2), $\mathbf{u}^n \subset \mathbf{H}^1_x$ with initial data $\mathbf{u}^n(0) = \mathbf{u}^n_0$, such that

$$\|\mathbf{u}^n\|_{\mathbf{L}^4_{t,x}} \lesssim \|\mathbf{u}^n\|_{\mathbf{L}^4_t \dot{\mathbf{H}}^{1,\frac{12}{5}}_x} \lesssim \|\nabla \mathbf{u}^n\|_{\mathcal{S}^0} \lesssim \|\nabla \mathbf{u}^n_0\|_{\mathbf{L}^2_x} + \eta^2 \leqslant L.$$

for some $0 < \eta \leqslant \eta_0$ and $L > 0$. Also, arguing as before, we have that all $\mathbf{u}^n$ is defined in the time interval I and is a Cauchy sequence in energy space $\dot{\mathbf{H}}^1_x$, and therefore,

$$\|\mathbf{u}^n\|_{\mathbf{L}^\infty_t \dot{\mathbf{H}}^1_x} \leqslant E.$$

Then, by Lemma 4.3, there exists a solution $\mathbf{u} : I \times \mathbb{R}^6 \to \mathbb{C}$, to (1.2) with initial data $\mathbf{u}(0) = \mathbf{u}_0$. □

We finish this section showing a standard blow-up criterion for solutions of (1.2).

Theorem 4.5. *(Standard blow-up criterion). Let $\mathbf{u}_0 \in \dot{\mathbf{H}}^1_x$ and $\mathbf{u}$ be the corresponding solution to (1.2) on $[0, T_0] \times \mathbb{R}^6$ such that*

$$\|\mathbf{u}\|_{\mathbf{L}^4_{t,x}([0,T_0]\times\mathbb{R}^6)} < \infty. \tag{4.23}$$

Then, there exists $\delta = \delta(\mathbf{u}_0)$ such that the solution $\mathbf{u}$ extends to a solution to (1.2) on the interval $[0, T_0 + \delta]$.

Proof. We follow the ideas presented in (TAO; VISAN, 2005). Let us denote the norm in (4.23) by M. The first step is to establish an $\dot{\mathbf{H}}^1$ bound on $\mathbf{u}$. To do this, we start subdviding $[0, T_0]$ into $N \sim \left(1 + \dfrac{M}{a}\right)^4$ subintervals J_i such that

$$\|\mathbf{u}\|_{\mathbf{L}^4_{t,x}(J_i\times\mathbb{R}^6)} < a, \tag{4.24}$$

where a is a small positive constant. By the Strichartz inequality, Lemma 2.11, we have for $k = 1, ..., l$,

$$\begin{aligned}\|u_k\|_{L_t^\infty \dot{H}_x^1(J_i\times\mathbb{R}^6)} &\lesssim \|u_k(t_i)\|_{\dot{H}_x^1(\mathbb{R}^6)} + \|\nabla f_k(\mathbf{u})\|_{\mathbf{L}_t^2\mathbf{L}_x^{\frac{3}{2}}} \\ &\lesssim \|u_k(t_i)\|_{\dot{H}_x^1(\mathbb{R}^6)} + \|u_k\|_{L^4_{t,x}(J_i\times\mathbb{R}^6)}\|\nabla u_k\|_{\mathcal{S}^0(J_i)} \\ &\lesssim \|u_k(t_i)\|_{\dot{H}_x^1(\mathbb{R}^6)} + a\|\nabla u_k\|_{\mathcal{S}^0(J_i)},\end{aligned}$$

for each interval J_i and any $t_i \in J_i$. If a is sufficiently small, we conclude

$$\|\mathbf{u}\|_{\mathbf{L}_t^\infty \dot{\mathbf{H}}_x^1(J_i\times\mathbb{R}^6)} \lesssim \|\mathbf{u}(t_i)\|_{\dot{\mathbf{H}}_x^1}.$$

Thus, inductively we may obtain a bound of the form

$$\|\mathbf{u}\|_{\mathbf{L}_t^\infty \dot{\mathbf{H}}_x^1([0,T_0]\times\mathbb{R}^6)} \leqslant C(\|\mathbf{u}(t_i)\|_{\dot{\mathbf{H}}_x^1}, M, a),$$

which implies

$$\|\mathbf{u}\|_{\mathbf{L}_t^4\dot{\mathbf{H}}_x^{1,\frac{12}{5}}([0,T_0]\times\mathbb{R}^6)} \leqslant C(\|\mathbf{u}(t_i)\|_{\dot{\mathbf{H}}_x^1}, M, a). \tag{4.25}$$

Now, let $0 \leqslant \tau < T_0$. By the Strichartz inequality, Lemma 2.11 and the Sobolev embedding, we have for $k = 1, ..., l$,

$$\begin{aligned}\|u_k - U(t-\tau)u_k(\tau)\|_{L_t^4\dot{H}_x^{1,\frac{12}{5}}([\tau,T_0]\times\mathbb{R}^6)} &\lesssim \|\nabla f_k(\mathbf{u})\|_{\mathbf{L}_t^2\mathbf{L}_x^{\frac{3}{2}}([\tau,T_0]\times\mathbb{R}^6)} \\ &\lesssim \|\mathbf{u}\|^2_{\mathbf{L}_t^4\dot{\mathbf{H}}_x^{1,\frac{12}{5}}([\tau,T_0]\times\mathbb{R}^6)}.\end{aligned} \tag{4.26}$$

Thus, by triangle inequality,

$$\|U_k(t-\tau)u_k(\tau)\|_{L_t^4\dot{H}_x^{1,\frac{12}{5}}([\tau,T_0]\times\mathbb{R}^6)} \lesssim \|\mathbf{u}\|^2_{\mathbf{L}_t^4\dot{\mathbf{H}}_x^{1,\frac{12}{5}}([\tau,T_0]\times\mathbb{R}^6)} + \|\mathbf{u}\|_{\mathbf{L}_t^4\dot{\mathbf{H}}_x^{1,\frac{12}{5}}([\tau,T_0]\times\mathbb{R}^6)}.$$

Let η_0 be as in Theorem 4.1. By (4.25), taking τ sufficiently close to T_0, we obtain

$$\|U_k(t-\tau)u_k(\tau)\|_{L_t^4\dot{H}_x^{1,\frac{12}{5}}([\tau,T_0]\times\mathbb{R}^6)} \leqslant \frac{\eta_0}{2}.$$

While from Strichartz's inequality we have

$$\|U_k(t-\tau)u_k(\tau)\|_{L_t^4\dot{H}_x^{1,\frac{12}{5}}(\mathbb{R}\times\mathbb{R}^6)} < \infty.$$

By the monotone convergence theorem, we deduce that there exists $\delta = \delta(\mathbf{u}_0) > 0$ such that

$$\|U_k(t-\tau)u_k(\tau)\|_{L_t^4\dot{H}_x^{1,\frac{12}{5}}([0,T_0+\delta]\times\mathbb{R}^6)} \leqslant \eta_0.$$

Again, by Theorem 4.1, there exists a unique solution to (1.2) with initial data $\mathbf{v}(\tau)$ at time $t = \tau$ which belongs to $\mathbf{C}\dot{\mathbf{H}}_x^1([\tau, T_0 + \delta] \times \mathbb{R}^6)$. By using the uniqueness of solution, we see that $\mathbf{u} = \mathbf{v}$ on $[\tau, T_0] \times \mathbb{R}^6$ and so, $\mathbf{v}$ is an extension of $\mathbf{u}$ to $[0, T_0 + \delta] \times \mathbb{R}^6$. □

Remark 4.6. *Note that in the contrapositive, this lemma asserts that if a solution* $\mathbf{u}$ *connot be continued beyond a time* $T_*, T^* > 0$ *, that is,* $\mathbf{u}$ *has maximal lifespan* $I = (-T_*, T^*)$ *and both* $T_*, T^* < \infty$*, then the* $L^4_{t,x}$*-norm must blow-up at that time* T_*, T^**, i.e.,*

$$S_I(\mathbf{u}) = \infty.$$

4.2 Existence of a critical solution

In this section we prove Theorem 1.17. We first define, for any $0 \leqslant K_0 \leqslant K(\boldsymbol{\psi})$, the function

$$L(K_0) := \sup\left\{S_I(\mathbf{u});\, \mathbf{u} : I \times \mathbb{R}^6 \to \mathbb{C}^l \text{ is a solution to (1.2) s.t. } \sup_{t\in I} K(\mathbf{u}(t)) \leqslant K_0\right\}.$$

Therefore, $L : [0, K(\boldsymbol{\psi})] \longrightarrow [0,\infty]$ is a nondecreasing function and since the ground state $\boldsymbol{\psi}$ is time independent, thus $S_{\mathbb{R}}(\boldsymbol{\psi}) = \infty$, and hence $L(K(\boldsymbol{\psi})) = \infty$. Let us show that L is continuous. Indeed, let $X = L^{-1}(a,b) = \{E \in (0, K(\boldsymbol{\psi}))|\, L(E) \in (a,b)\}$. We show that X^C is closed. Consider $(E_n) \subset X^C$ such that $E_n \to E$ when $n \to \infty$. Without loss of generality, we may assume that E_n is such that $L(E_n) \geqslant b$ for all n. The case when $L(E_n) \leqslant a$ is treated in a similar way. Thus, there exists a sequence $(\mathbf{v}_n)$ of solutions that obeys $\sup_{t\in I_n} K(\mathbf{v}_n) \leqslant E_n$,, such that $S_{I_n}(\mathbf{v}_n) \geqslant b - \dfrac{1}{n}$. Let $\epsilon > 0$ and $\mathbf{u}_0^n \in \dot{\mathbf{H}}^1$ be such that $\|\mathbf{u}_0^n - \mathbf{v}_n(0)\|_{\mathbf{L}_t^\infty \dot{\mathbf{H}}_x^1} < \epsilon$. By Lemma 4.3, there exists $\mathbf{u}_n^\epsilon$ solution with initial data $\mathbf{u}_n^\epsilon(0) = \mathbf{u}_0^n$ such that

$$\|\mathbf{v}_n - \mathbf{u}_n^\epsilon\|_{\mathbf{L}_t^\infty \dot{\mathbf{H}}^1} + S_{I_n}(\mathbf{v}_n - \mathbf{u}_n^\epsilon) \leqslant \epsilon. \tag{4.27}$$

Notice that from (4.27), we have for each n,

$$\|\mathbf{u}_n^\epsilon\|_{\mathbf{L}_t^\infty \dot{\mathbf{H}}^1} - \|\mathbf{v}_n\|_{\mathbf{L}_t^\infty \dot{\mathbf{H}}^1} \leqslant \left|\|\mathbf{u}_n^\epsilon\|_{\mathbf{L}_t^\infty \dot{\mathbf{H}}^1} - \|\mathbf{v}_n\|_{\mathbf{L}_t^\infty \dot{\mathbf{H}}^1}\right| \leqslant \|\mathbf{u}_n^\epsilon - \mathbf{v}_n\|_{\mathbf{L}_t^\infty \dot{\mathbf{H}}^1} < \epsilon.$$

Then

$$\|\mathbf{u}_n^\epsilon\|_{\mathbf{L}_t^\infty \dot{\mathbf{H}}^1} < \|\mathbf{v}_n\|_{\mathbf{L}_t^\infty \dot{\mathbf{H}}^1} + \epsilon \leqslant E_n + \epsilon,$$

which implies

$$\limsup_{n\to\infty,\epsilon\to 0} \|\mathbf{u}_n^\epsilon\|_{\mathbf{L}_t^\infty \dot{\mathbf{H}}^1} \leqslant E. \tag{4.28}$$

Thus, passing to a subsequence if necessary, we are able to get a famlily of solutions $\mathcal{F} = \{\mathbf{u}_n^\epsilon\}_{n\in\mathbb{N}}$ such that $\sup_{t\in I} K(\mathbf{u}_n^\epsilon) \leqslant E$. Furthermore, by (4.27),

$$S_{I_n}(\mathbf{v}_n) - S_{I_n}(\mathbf{u}_n^\epsilon) \leqslant |S_I(\mathbf{v}_n) - S_{I_n}(\mathbf{u}_n^\epsilon)| \leqslant S_{I_n}(\mathbf{v}_n - \mathbf{u}_n) \leqslant \epsilon,$$

where, $S_{I_n}(\mathbf{u}_n^\epsilon) \geqslant S_{I_n}(\mathbf{v}_n) - \dfrac{1}{n} \geqslant b - \dfrac{1}{n} - \epsilon$ and, therefore,

$$\sup\{S_{I_n}(\mathbf{u}_n^\epsilon);\, \mathbf{u}_n \in \mathcal{F}\} \geqslant b. \tag{4.29}$$

By arbitrariness of $\epsilon > 0$, combining (4.28) and (4.29), we conclude $L(E) \geqslant b$, that is, $E \in X^C$ which shows that X^C is closed. Consequently X is open and L is continuous. Hence, there must exist a critical energy, denoted by K_c, such that

$$L(K_0) \begin{cases} < \infty, & K_0 < K_c \\ = \infty, & K_0 \geqslant K_c. \end{cases} \tag{4.30}$$

In particular, if $\mathbf{u} : I \times \mathbb{R}^6 \longrightarrow \mathbb{C}^l$ is a maximal solution such that $\sup(K(\mathbf{u})) \leqslant K_c$, then $\mathbf{u}$ is global and

$$S_{\mathbb{R}}(\mathbf{u}) \leqslant L\left(\sup_{t\in\mathbb{R}} K(\mathbf{u}(t))\right) < \infty.$$

The next result is essential to reach the goal of this section. The proof is addapted to the one presented in (KILLIP; VISAN, 2010).

Theorem 4.7. *(Palais-Smale condition) Let $\mathbf{u}_n : I_n \times \mathbb{R}^6 \longrightarrow \mathbb{C}^l$ be a sequence of solutions to (1.2) such that*

$$\limsup_{n\to\infty}\left(\sup_{t\in I_n} \|\nabla \mathbf{u}_n(t)\|^2_{\mathbf{L}^2_x}\right) = K_c \tag{4.31}$$

and let $(t_n) \subset I_n$ be a sequence of times obeying

$$\lim_{n\to\infty} S_{\geqslant t_n}(\mathbf{u}_n) = \lim_{n\to\infty} S_{\leqslant t_n}(\mathbf{u}_n) = \infty.$$

Then, the sequence $\mathbf{u}_n(t_n)$ has a subsequence that converges in $\dot{\mathbf{H}}^1(\mathbb{R}^6)$ modulo symmetries.

Proof. We follow the ideas presented in (KILLIP; VISAN, 2010). Without loss of generality, we may assume that $t_n = 0$, for all n, by time-translation symmetry. Thus,

$$\lim_{n\to\infty} S_{\geqslant 0}(\mathbf{u}_n) = \lim_{n\to\infty} S_{\leqslant 0}(\mathbf{u}_n) = \infty. \tag{4.32}$$

By (4.31), the sequence $\mathbf{u}_n(0)$ is bounded in $\dot{\mathbf{H}}^1(\mathbb{R}^6)$. Therefore, up to a subsequence, we may apply Theorem 2.26 to get the following decomposition

$$\mathbf{u}_n(0) = \sum_{j=1}^{J} g_n^j \mathbf{U}(t_n^j)\boldsymbol{\phi}^j + \mathbf{w}_n^J,$$

where, for simplicity, we denote by $(g_n^j\mathbf{u})(x) := (\lambda_n^j)^{-1/2}\mathbf{u}\left(\dfrac{x - x_n^j}{\lambda_n^j}\right)$.

Refining the subsequence once to each j and using a diagonal argument, for each j, we may assume that $(t_n^j)_{n\geqslant 1}$ converge to some $t^j \in [-\infty, \infty]$. Thus, if $t^j \in (-\infty, \infty)$, since $U_k(0) = \mathrm{Id}$, for $k = 1, ..., l$, changing $\boldsymbol{\phi}^j$ by $\mathbf{U}(t^j)\boldsymbol{\phi}^j$, we may assume that $t^j = 0$. Besides that,

$$\sum_{j=1}^{J} g_n^j \mathbf{U}(t_n^j)\boldsymbol{\phi}^j + \mathbf{w}_n^J = \sum_{j=1}^{J} g_n^j[\mathbf{U}(t_n^j)\boldsymbol{\phi}^j + \boldsymbol{\phi}^j - \boldsymbol{\phi}^j] + \mathbf{w}_n^J = \sum_{j=1}^{J} g_n^j\boldsymbol{\phi}^j + \tilde{\mathbf{w}}^J,$$

where $\tilde{\mathbf{w}}^J = \sum_{j=1}^{J} g_n^j[\mathbf{U}(t_n^j)\boldsymbol{\phi}^j - \boldsymbol{\phi}^j] + \mathbf{w}^J$. Therefore, we may consider that $t_n^j \equiv 0$. Hence, either $t_n^j \equiv 0$ or $t_n^j \to \pm\infty$.

Now, we set the nonlinear profiles $\mathbf{v}^j : I^j \times \mathbb{R}^6 \longrightarrow \mathbb{C}^l$ associated to $\boldsymbol{\phi}^j$ and t_n^j by the following

- If $t_n^j \equiv 0$, then $\mathbf{v}^j$ is the maximal solution to (1.2) with initial data $\mathbf{v}^j(0) = \boldsymbol{\phi}^j$.
- If $t_n^j \to \infty$, then $\mathbf{v}^j$ is the maximal solution to (1.2) which scatters forward in time to $\mathbf{U}(t)\boldsymbol{\phi}^j$.
- If $t_n^j \to -\infty$, then $\mathbf{v}^j$ is the maximal solution to (1.2) which scatters backward in time to $\mathbf{U}(t)\boldsymbol{\phi}^j$.

Now, for each $j, n \geqslant 1$, consider $\mathbf{v}_n^j : I_n^j \times \mathbb{R}^6 \to \mathbb{C}^l$ given by

$$\mathbf{v}_n^j(t) := T_n^j \left[\mathbf{v}^j(\cdot + t_n^j)\right](t),$$

where T_n^j is defined as in Lemma 2.27 and $I_n^j := \{t \in \mathbb{R};\ (\lambda_n^j)^{-2}t + t_n^j \in I^j\}$. This way, each $\mathbf{v}_n^j$ is a maximal solution to (1.2) with initial data $\mathbf{v}_n^j(0) = g_n^j \mathbf{v}^j(t_n^j)$ defined on the interval $I_n^j = (-T_{n,j}^-, T_{n,j}^+)$, where $-\infty \leqslant -T_{n,j}^- < 0 < T_{n,j}^+ \leqslant \infty$.

Notice that for each $J \geqslant 1$, using (2.19) and (4.31)

$$\sum_{j=1}^{J} \|\nabla \boldsymbol{\phi}^j\|_{\mathbf{L}_x^2}^2 \leqslant \lim_{n\to\infty} \left[\sum_{j=1}^{J} \|\nabla \boldsymbol{\phi}^j\|_{\mathbf{L}_x^2}^2 + \|\nabla \mathbf{w}_n^J\|_{\mathbf{L}_x^2}^2\right] = \lim_{n\to\infty} \|\nabla \mathbf{u}_n\|_{\mathbf{L}_x^2}^2 \leqslant K_c. \tag{4.33}$$

Since this holds to every$J \geqslant 1$, the series is convergent. Hence, there exists $J_0 \geqslant 1$ such that

$$\|\nabla \boldsymbol{\phi}^j\|_{\mathbf{L}_x^2} \leqslant \eta_0, \quad \forall j \geqslant J_0,$$

where η_0 is the threshold in Corollary 4.2. Then, for every $n \geqslant 1$ and $j \geqslant J_0$, the solutions $\mathbf{v}_n^j$ are global and

$$\sup_{t\in\mathbb{R}} \|\nabla \mathbf{v}_n^j(t)\|_{\mathbf{L}_x^2}^2 + S_{\mathbb{R}}(\mathbf{v}_n^j(t)) \leqslant \|\nabla \boldsymbol{\phi}^j\|_{\mathbf{L}_x^2}^2. \tag{4.34}$$

Claim 1:(At least a bad profile) There exists $1 \leqslant j_0 < J_0$ such that

$$\limsup_{n\to\infty} S_{[0,T_{n,j_0}^+)}(\mathbf{v}_n^{j_0}) = \infty.$$

Indeed, suppose by contradiction that for $1 \leqslant j < J_0$,

$$\limsup_{n\to\infty} S_{[0,T_{n,j_0}^+)}(\mathbf{v}_n^{j}) < \infty. \tag{4.35}$$

In particular, this implies that $T_{n,j}^+ = \infty$, $1 \leqslant j < J_0$ and for all n large enough. Thus, subdividing $[0, \infty)$ into subintervals obeying $S_I(\mathbf{v}_n^j) < \delta$, applying Strichartz's inequality in each subinterval and summing, we deduce

$$\limsup_{n\to\infty} \|\mathbf{v}_n^j\|_{\mathcal{S}^1([0,\infty))} < \infty, \quad \text{for all } 1 \leqslant j < J_0. \tag{4.36}$$

Combining (4.34) with (4.35) and using (2.19) and (4.31), we have for n sufficiently large,

$$\begin{aligned}\sum_{j\geqslant 1} S_{[0,\infty)}(\mathbf{v}_n^j) &\lesssim \sum_{j=1}^{J_0-1} S_{[0,\infty)}(\mathbf{v}_n^j) + \sum_{j\geqslant J_0} S_{[0,\infty)}(\mathbf{v}_n^j) \\ &\lesssim 1 + \sum_{j\geqslant J_0} \|\nabla \boldsymbol{\phi}^j\|_{\mathbf{L}_x^2}^2 \\ &\lesssim 1 + K_c.\end{aligned} \tag{4.37}$$

Now, define the approximation

$$\mathbf{u}_n^J(t) := \sum_{j=1}^{J} \mathbf{v}_n^j(t) + \mathbf{U}(t)\mathbf{w}_n^J. \tag{4.38}$$

Note that,

$$\begin{aligned}\|\mathbf{u}_n^J(0) - \mathbf{u}_n(0)\|_{\dot{\mathbf{H}}^1} &\lesssim \left\| \sum_{j=1}^{J} g_n^j \mathbf{v}^j(t_n^j) - g_n^j \mathbf{U}(t_n^j)\boldsymbol{\phi}^j \right\|_{\dot{\mathbf{H}}^1} \\ &\lesssim \sum_{j=1}^{J} \|\mathbf{v}^j(t_n^j) - \mathbf{U}(t_n^j)\boldsymbol{\phi}^j\|_{\dot{\mathbf{H}}^1}.\end{aligned}$$

Consenquently, by the choice of $\mathbf{v}^j$,

$$\limsup_{n\to\infty} \|\mathbf{u}_n^J(0) - \mathbf{u}_n(0)\|_{\dot{\mathbf{H}}^1} = 0.$$

We show now that $\mathbf{u}_n^J$ does not blow-up foward in time. First, let us introduce the notation

$$\|\mathbf{u}\mathbf{v}\|_{\mathbf{L}_x^p}^p := \sum_{k=1}^{l} \|u_k v_k\|_{L_x^p}^p.$$

Now, note that

$$\limsup_{n\to\infty} \|\mathbf{v}_n^j \mathbf{v}_n^i\|_{L_{t,x}^2} = 0. \tag{4.39}$$

Indeed, recall that by (4.34) and (4.36), $\mathbf{v}_n^j \in \mathcal{S}^1([0,\infty))$ (see (2.1)) for any $j \geqslant 1$ and n large enough. Combining this with the Strichartz inequality one can see that

$$\|\mathbf{v}^j\|_{\dot{\mathbf{X}}^1([0,\infty)\times\mathbb{R}^6])} = \|\mathbf{v}_n^j\|_{\dot{\mathbf{X}}^1([0,\infty)\times\mathbb{R}^6])} \lesssim 1,$$

where $\dot{\mathbf{X}}^1 := \mathbf{L}_{t,x}^4 \cap L_t^{\frac{8}{3}}\dot{\mathbf{H}}_x^{1,\frac{8}{3}}$. Then, we may approximate $\mathbf{v}_n^j$ in $\dot{\mathbf{X}}^1$ by C_0^∞ functions, that is, given $\epsilon > 0$, there exists $\boldsymbol{\psi}_\epsilon^j \in C_0^\infty(\mathbb{R}\times\mathbb{R}^6)$ such that

$$\|\mathbf{v}_n^j - T_n^j \boldsymbol{\psi}_\epsilon^j\|_{\dot{\mathbf{X}}^1(\mathbb{R}\times\mathbb{R}^6)} < \epsilon. \tag{4.40}$$

Moreover, if $j \neq i$ and $\epsilon > 0$, using (4.36) and Lemma 2.27 we obtain for n sufficiently large,

$$\begin{aligned}
&\|\mathbf{v}_n^j \mathbf{v}_n^i\|_{\mathbf{L}_{t,x}^2([0,\infty)\times\mathbb{R}^6)} \\
&\quad\leqslant \|\mathbf{v}_n^j(\mathbf{v}_n^i - T_n^i\boldsymbol{\psi}_\epsilon^i)\|_{\mathbf{L}_{t,x}^2([0,\infty)\times\mathbb{R}^6)} + \|(\mathbf{v}_n^j - T_n^j\boldsymbol{\psi}_\epsilon^j)T_n^i\boldsymbol{\psi}_\epsilon^i\|_{\mathbf{L}_{t,x}^2([0,\infty)\times\mathbb{R}^6)} \\
&\qquad + \|T_n^j\boldsymbol{\psi}_\epsilon^j T_n^i\boldsymbol{\psi}_\epsilon^i\|_{\mathbf{L}_{t,x}^2([0,\infty)\times\mathbb{R}^6)} \\
&\quad\lesssim \|\mathbf{v}_n^j\|_{\dot{\mathbf{X}}^1(\mathbb{R})}\|\mathbf{v}_n^i - T_n^i\boldsymbol{\psi}_\epsilon^i\|_{\dot{\mathbf{X}}^1(\mathbb{R})} + \|\mathbf{v}_n^j - T_n^j\boldsymbol{\psi}_\epsilon^j\|_{\dot{\mathbf{X}}^1(\mathbb{R})}\|\boldsymbol{\psi}_\epsilon^i\|_{\mathcal{S}^1(\mathbb{R})} \\
&\qquad + \|T_n^j\boldsymbol{\psi}_\epsilon^j T_n^i\boldsymbol{\psi}_\epsilon^i\|_{\mathbf{L}_{t,x}^2([0,\infty)\times\mathbb{R}^6)} \\
&\quad\lesssim \epsilon,
\end{aligned} \tag{4.41}$$

and (4.39) holds. Also, notice that for a_j, $j = 1, ..., J$, we have

$$\left(\sum_{j=1}^{J} a_j\right)^4 = \sum_{i\neq k}\sum_{j=0}^{4} a_i^j a_k^{4-j} \lesssim \sum_{j=1}^{J} a_j^4 + C_J \sum_{i\neq k} a_j^2 a_k^2.$$

Then

$$\begin{aligned} S_{[0,\infty)}\left(\sum_{j=1}^{J}\mathbf{v}_n^j\right) = \left\|\sum_{j=1}^{J}\mathbf{v}_n^j\right\|_{\mathbf{L}_x^4}^4 &\lesssim \int \sum_{k=1}^{l}\left(\sum_{j=1}^{J}|v_{nk}^j|^4 + C_J\sum_{i\neq j}|v_{nk}^i v_{nk}^j|^2\right) \\ &= \sum_{j=1}^{J} S_{[0,\infty)}(\mathbf{v}_n^j) + C_j \sum_{i\neq j}\|\mathbf{v}_n^i\mathbf{v}_n^j\|_{\mathbf{L}_x^2}^2 \end{aligned} \tag{4.42}$$

Therefore, by (4.42), (2.18), (4.37) and (4.39)

$$\begin{aligned} \lim_{J\to\infty}\limsup_{n\to\infty} S_{[0,\infty)}(\mathbf{u}_n^J) &\lesssim \lim_{J\to\infty}\limsup_{n\to\infty}\left(S_{[0,\infty)}\left(\sum_{j=1}^{J}\mathbf{v}_n^j\right) + S_{[0,\infty)}(\mathbf{U}(t)\mathbf{w}_n^J)\right) \\ &\lesssim \lim_{J\to\infty}\limsup_{n\to\infty}\left(\sum_{j=1}^{J} S_{[0,\infty)}(\mathbf{v}_n^j) + C_J\sum_{j\neq i}\|\mathbf{v}_n^j\mathbf{v}_n^i\|_{L_{t,x}^2}^2\right) \\ &\lesssim 1 + K_c. \end{aligned} \tag{4.43}$$

Using the same argument that was used to obtain (4.36) from (4.35), we deduce

$$\lim_{J\to\infty}\limsup_{n\to\infty}\|\nabla\mathbf{u}_n^J\|_{\mathcal{S}^0([0,\infty))} \leqslant C < \infty, \tag{4.44}$$

where C depends only on K_c. In order to apply Lemma 4.3, we need to show that, for any $k = 1, ..., l$

$$\lim_{J\to\infty}\limsup_{n\to\infty}\left\|\nabla\left[(i\alpha_k\partial_t + \gamma_k\Delta)u_{kn}^J - f_k(\mathbf{u}_n^J)\right]\right\|_{L_{t,x}^{8/5}([0,\infty)\times\mathbb{R}^6)} = 0,$$

which, by definition of $\mathbf{u}_n^J$, we deduce

$$\begin{aligned} [i\alpha_k\partial_t + \gamma_k\Delta]u_{kn}^J - f_k(\mathbf{u}_n^J) &= \sum_{j=1}^{J} f_k(\mathbf{v}_n^j) - f_k(\mathbf{u}_n^J) \\ &= \sum_{j=1}^{J} f_k(\mathbf{v}_n^j) - f_k\left(\sum_{j=1}^{J}\mathbf{v}_n^j\right) + f_k(\mathbf{u}_n^J - \mathbf{U}(t)\mathbf{w}_n^J) - f_k(\mathbf{u}_n^J). \end{aligned} \tag{4.45}$$

Therefore, by triangle inequality, this is equivalent to show that

$$\lim_{J\to\infty}\limsup_{n\to\infty}\left\|\nabla\left[\sum_{j=1}^{J} f_k(\mathbf{v}_n^j) - f_k\left(\sum_{j=1}^{J}\mathbf{v}_n^j\right)\right]\right\|_{L_{t,x}^{8/5}([0,\infty)\times\mathbb{R}^6)} = 0 \tag{4.46}$$

and

$$\lim_{J\to\infty}\limsup_{n\to\infty}\|\nabla[f_k(\mathbf{u}_n^J - \mathbf{U}(t)\mathbf{w}_n^J) - f_k(\mathbf{u}_n^J)]\|_{L_{t,x}^{8/5}([0,\infty)\times\mathbb{R}^6)} = 0. \tag{4.47}$$

Let us start with (4.46). Note that, by Lemma 2.12 we may write

$$\left|\nabla\left[\sum_{j=1}^{J} f_k(\mathbf{v}_n^j) - f_k\left(\sum_{j=1}^{J}\mathbf{v}_n^j\right)\right]\right| \lesssim \sum_{j\neq i}|\nabla\mathbf{v}_n^j||\mathbf{v}_n^i|, \quad k=1,\dots,l.$$

Let us show that for $j\neq i$,

$$\limsup_{n\to\infty}\left\|\mathbf{v}_n^j\nabla\mathbf{v}_n^i\right\|_{\mathbf{L}_{t,x}^{8/5}([0,\infty)\times\mathbb{R}^6)} = 0.$$

Indeed, recall that we can approximate $\mathbf{v}_n^j$ and $\nabla\mathbf{v}_n^j$ in $\dot{\mathbf{X}}^1$ by C_0^∞ functions, that is, given $\epsilon>0$, there exists $\boldsymbol{\psi}_\epsilon^j\in C_c^\infty(\mathbb{R}\times\mathbb{R}^6)$ such that

$$\|\mathbf{v}_n^j - T_n^j\boldsymbol{\psi}_\epsilon^j\|_{\dot{\mathbf{X}}^1(\mathbb{R})} + \|\nabla\mathbf{v}_n^i - \nabla(T_n^i\boldsymbol{\psi}_\epsilon^i)\|_{\dot{\mathbf{X}}^1(\mathbb{R})} < \epsilon. \tag{4.48}$$

Hence, if $j\neq i$ and $\epsilon>0$, using (4.36) and Lemma 2.27 we deduce for n large enough that

$$\begin{aligned}
&\|\mathbf{v}_n^j\nabla\mathbf{v}_n^i\|_{\mathbf{L}_{t,x}^{8/5}([0,\infty)\times\mathbb{R}^6)}\\
&\quad\leqslant \|\mathbf{v}_n^j(\nabla\mathbf{v}_n^i - \nabla(T_n^i\boldsymbol{\psi}_\epsilon^i))\|_{\mathbf{L}_{t,x}^{8/5}([0,\infty)\times\mathbb{R}^6)} + \|(\mathbf{v}_n^j - T_n^j\boldsymbol{\psi}_\epsilon^j)\nabla(T_n^i\boldsymbol{\psi}_\epsilon^i)\|_{\mathbf{L}_{t,x}^{8/5}([0,\infty)\times\mathbb{R}^6)}\\
&\qquad + \|T_n^j\boldsymbol{\psi}_\epsilon^j\nabla(T_n^i\boldsymbol{\psi}_\epsilon^i)\|_{\mathbf{L}_{t,x}^{8/5}([0,\infty)\times\mathbb{R}^6)}\\
&\quad\lesssim \|\mathbf{v}_n^j\|_{\dot{\mathbf{X}}^1(\mathbb{R})}\|(\nabla\mathbf{v}_n^i - \nabla(T_n^i\boldsymbol{\psi}_\epsilon^i)\|_{\dot{\mathbf{X}}^1(\mathbb{R})} + \|\mathbf{v}_n^j - T_n^j\boldsymbol{\psi}_\epsilon^j\|_{\dot{\mathbf{X}}^1(\mathbb{R})}\|\nabla\boldsymbol{\psi}_\epsilon^i\|_{\dot{\mathbf{X}}^1(\mathbb{R})}\\
&\qquad + \|T_n^j\boldsymbol{\psi}_\epsilon^j\nabla(T_n^i\boldsymbol{\psi}_\epsilon^i)\|_{\mathbf{L}_{t,x}^{8/5}([0,\infty)\times\mathbb{R}^6)}\\
&\quad\lesssim \epsilon.
\end{aligned} \tag{4.49}$$

Then,

$$\limsup_{n\to\infty}\left\|\nabla\left[\sum_{j=1}^{J} f_k(\mathbf{v}_n^j) - f_k\left(\sum_{j=1}^{J}\mathbf{v}_n^j\right)\right]\right\|_{L_{t,x}^{8/5}([0,\infty))} \lesssim \limsup_{n\to\infty}\sum_{i\neq j}\left\|\mathbf{v}_n^j\nabla\mathbf{v}_n^i\right\|_{\mathbf{L}_{t,x}^{8/5}([0,\infty)\times\mathbb{R}^6)} = 0, \tag{4.50}$$

which proves (4.46). Now, consider (4.47). Henceforth, unless otherwise is said, the norms are taken on $[0,\infty)\times\mathbb{R}^6$. Combining Hölder's inequality, (4.38) and Lemma 2.11,

$$\left\|\nabla f_k\left(\mathbf{u}_n^J - \mathbf{U}(t)\mathbf{w}_n^J\right) - \nabla f_k(\mathbf{u}_n^J)\right\|_{\mathbf{L}_{t,x}^{8/5}} \leqslant \left\|\left(\sum_{j=1}^{J}\mathbf{v}_n^J\right)\nabla\mathbf{U}(t)\mathbf{w}_n^J\right\|_{\mathbf{L}_{t,x}^{8/5}} + \|\mathbf{U}(t)\mathbf{w}_n^J\|_{\mathbf{L}_{t,x}^4}\|\nabla\mathbf{u}_n^J\|_{\mathbf{L}_{t,x}^{8/3}}.$$

When we take the limit in time, the second term vanishes by (2.18) and (4.44). Hence, is enough to show

$$\lim_{J\to\infty}\limsup_{n\to\infty}\left\|\left(\sum_{j=1}^{J}\mathbf{v}_n^J\right)\nabla\mathbf{U}(t)\mathbf{w}_n^J\right\|_{\mathbf{L}_{t,x}^{8/5}} = 0. \tag{4.51}$$

Indeed, let $\eta>0$. By (4.37) there exists $J'=J'(\eta)\geqslant 1$ such that

$$\sum_{j\geqslant J'} S_{\geqslant 0}(\mathbf{v}_n^j) \leqslant \eta.$$

Using Hölder's inequality and an argument as in (4.43), we deduce

$$\limsup_{n\to\infty}\left\|\left(\sum_{j=J'}^{J}\mathbf{v}_n^j\right)\nabla\mathbf{U}(t)\mathbf{w}_n^J\right\|^4_{\mathbf{L}^{8/5}_{t,x}} \lesssim \limsup_{n\to\infty}\left(\sum_{j\geqslant J'} S_{[0,\infty)}(\mathbf{v}_n^j)\right)\|\nabla\mathbf{U}(t)(\mathbf{w}_n^J)\|^4_{\mathbf{L}^{8/5}_{t,x}}$$
$$\lesssim \eta.$$

Since $\eta>0$ is arbitrary, to show (4.51) it suffices to obtain

$$\lim_{J\to\infty}\limsup_{n\to\infty}\|\mathbf{v}_n^j\nabla\mathbf{U}(t)\mathbf{w}_n^J\|_{\mathbf{L}^{8/5}_{t,x}}=0,\quad 1\leqslant j\leqslant J'. \tag{4.52}$$

Fix $1\leqslant j\leqslant J'$, by a change of variables

$$\|\mathbf{v}_n^j\nabla\mathbf{U}(t)\mathbf{w}_n^J\|_{\mathbf{L}^{8/5}_{t,x}}=\|\mathbf{v}^j\nabla\tilde{\mathbf{w}}_n^J\|_{\mathbf{L}^{8/5}_{t,x}},$$

where $\tilde{\mathbf{w}}_n^J:=[(T_n^j)^{-1}\mathbf{U}(t)\mathbf{w}_n^J](\cdot-t_n^j)$. Note that,

$$S_{\mathbb{R}}(\tilde{\mathbf{w}}_n^J)=S_{\mathbb{R}}(\mathbf{U}(t)\mathbf{w}_n^J)\quad\text{and}\quad\|\nabla\tilde{\mathbf{w}}_n^J\|_{\mathbf{L}^{8/3}_{t,x}}=\|\nabla\mathbf{U}(t)\mathbf{w}_n^J\|_{\mathbf{L}^{8/3}_{t,x}}. \tag{4.53}$$

Again, using Hölder's inequality

$$\|\mathbf{v}^j\nabla\tilde{\mathbf{w}}_n^J\|_{\mathbf{L}^{8/5}_{t,x}}\lesssim\|\mathbf{v}^j\|_{\mathbf{L}^{8}_{t,x}}\|\nabla\tilde{\mathbf{w}}_n^J\|_{\mathbf{L}^{2}_{t,x}}.$$

By a density argument, we may assume $\mathbf{v}^j\in\mathbf{C}_0^\infty(\mathbb{R}\times\mathbb{R}^6)$. Therefore, it is sufficient to show that

$$\lim_{J\to\infty}\limsup_{n\to\infty}\|\nabla\tilde{\mathbf{w}}_n^J\|_{\mathbf{L}^{2}_{t,x}(K)}=0,$$

for all compact $K\subset\mathbb{R}\times\mathbb{R}^6$. However, this is a consequence of Lemma 2.10, (4.53) and (2.18). Hence, (4.47) follows.

Now we are in position to apply Lemma 4.3. Using (4.43), we deduce, for n sufficiently large,

$$S_{[0,\infty)}(\mathbf{u}_n)\lesssim 1+K_c,$$

which contradicts (4.32). This argument finish the proof of Claim 1.

Now, rearranging the index if necessary, we may assume that there exists $1\leqslant J_1<J_0$ such that

$$\begin{cases}\limsup\limits_{n\to\infty}S_{[0,T^+_{n,j})}(\mathbf{v}_j^n)=\infty, & \text{para } 1\leqslant j\leqslant J_1\\ \limsup\limits_{n\to\infty}S_{[0,\infty)}(\mathbf{v}_j^n)<\infty, & \text{para } j>J_1.\end{cases} \tag{4.54}$$

We can guarantee, up to a subsequence in n, that $S_{[0,T^+_{n,1})}(\mathbf{v}_n^1)\to\infty$.

For each $m,n\geqslant 1$, we set an integer $j=j(m,n)\in\{1,\ldots,J_1\}$ and an interval K_n^m of the form $[0,\tau]$ by

$$\sum_{1\leqslant j\leqslant J_1}S_{K_n^m}(\mathbf{v}_n^j)=S_{K_n^m}(v_n^{j(m,n)})=m. \tag{4.55}$$

By the pigeonhole principle, there exists a $1 \leqslant j_1 \leqslant J_1$ such that, for infinitely many m and n, we have $j(m,n) = j_1$. Note that the infinite set of n that this holds depends on m. Rearranging the index, we may assume that $j_1 = 1$. Moreover, by definition of the critical energy,

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \sup_{t\in K_n^m} \|\nabla \mathbf{v}_n^1(t)\|_{\mathbf{L}_x^2}^2 \geqslant K_c. \tag{4.56}$$

On the other hand, in view of (4.55), all $\mathbf{v}_n^j$ have finite scattering size in K_n^m for each $m \geqslant 1$. Then, by the same argument used in Claim 1, we see that, for n and J large enough, $\mathbf{u}_n^J$ is a good approximation for $\mathbf{u}_n$ in each interval K_n^m. Precisely, we have the following

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|\mathbf{u}_n^J - \mathbf{u}_n\|_{\mathbf{L}_t^\infty \dot{\mathbf{H}}^1(K_n^m \times \mathbb{R}^6)} = 0, \quad \forall m \geqslant 1. \tag{4.57}$$

Claim 2. For all $J \geqslant 1$, $m \geqslant 1$,

$$\limsup_{n\to\infty} \sup_{t\in K_n^m} \left| \|\nabla \mathbf{u}_n^J(t)\|_{\mathbf{L}_x^2}^2 - \sum_{j=1}^J \|\nabla \mathbf{v}_n^j(t)\|_{\mathbf{L}_x^2}^2 - \|\nabla \mathbf{w}_n^J\|_{\mathbf{L}_x^2}^2 \right| = 0.$$

Indeed, fix $J \geqslant 1$ and $m \geqslant 1$. Then, for all $t \in K_n^m$, by (4.38),

$$\begin{aligned}
\|\nabla \mathbf{u}_n^J(t)\|_{\mathbf{L}_x^2}^2 &= \langle \nabla \mathbf{u}_n^J(t), \nabla \mathbf{u}_n^J(t) \rangle \\
&= \sum_{j=1}^J \|\nabla \mathbf{v}_n^j(t)\|_{\mathbf{L}_x^2}^2 + \|\nabla \mathbf{w}_n^J(t)\|_{\mathbf{L}_x^2}^2 + \sum_{j\neq i} \langle \nabla \mathbf{v}_n^j(t), \nabla \mathbf{v}_n^i(t) \rangle \\
&\quad + \sum_{j=1}^J \left(\langle \nabla \mathbf{U}(t)\mathbf{w}_n^J, \nabla \mathbf{v}_n^j(t) \rangle + \langle \nabla \mathbf{v}_n^j(t), \nabla \mathbf{U}(t)\mathbf{w}_n^J \rangle \right).
\end{aligned}$$

Thus, to prove the claim, it is enough to show that for all sequence $(t_n) \subset K_n^m$

$$\langle \nabla \mathbf{v}_n^j(t_n), \nabla \mathbf{v}_n^i(t_n) \rangle \to 0 \text{ as } n \to \infty,\ i \neq j,\ 1 \leqslant i, j \leqslant J \tag{4.58}$$

and

$$\langle \nabla \mathbf{U}(t_n)\mathbf{w}_n^J, \nabla \mathbf{v}_n^j(t_n) \rangle \to 0 \text{ as } n \to \infty,\ 1 \leqslant j \leqslant J. \tag{4.59}$$

We just show the second case, which depends on (2.20). The first one is treated in the same way using (2.21). After performing a change of variables

$$\langle \nabla \mathbf{U}(t_n)\mathbf{w}_n^J, \nabla \mathbf{v}_n^j(t_n) \rangle = \left\langle \nabla \mathbf{U}(t_n(\lambda_n^j)^{-2})[(g_n^j)^{-1}\mathbf{w}_n^J], \nabla \mathbf{v}^j \left(\frac{t_n}{(\lambda_n^j)^2} + t_n^j \right) \right\rangle. \tag{4.60}$$

Since $t_n \in K_n^m \subset [0, T_{n,j}^+)$ for all $1 \leqslant j \leqslant J_1$, then $t_n(\lambda_n^j)^{-2} + t_n^j \in I^j$, for all $j \geqslant 1$, where I^j is the maximal interval of existence of $\mathbf{v}^j$. By (4.54), for $j > J_1$, $I^j = \mathbb{R}$. Refining the sequence once for each j and using again the diagonalisation argument, we may assume $t_n(\lambda_n^j)^{-2} + t_n^j$ converges for all j. Now, we fix $1 \leqslant j \leqslant J$.

Case 1: If $t_n(\lambda_n^j)^{-2} + t_n^j$ converges to some point τ^j in the interior of I^j, then by continuity of the flow, $\mathbf{v}^j\left(t_n(\lambda_n^j)^{-2} + t_n^j\right)$ converges to $\mathbf{v}^j(\tau^j)$ in $\dot{\mathbf{H}}^1(\mathbb{R}^6)$. On the other hand, using (4.33), we deduce

$$\limsup_{n\to\infty} \left\|\mathbf{U}(t_n(\lambda_n^j)^{-2})[(g_n^j)^{-1}\mathbf{w}_n^J]\right\|_{\dot{\mathbf{H}}^1(\mathbb{R}^6)} = \limsup_{n\to\infty} \|\mathbf{w}_n^J\|_{\dot{\mathbf{H}}^1(\mathbb{R}^6)} \leqslant K_c. \tag{4.61}$$

Combining with (4.60), we get

$$\begin{aligned}\lim_{n\to\infty}\langle\nabla\mathbf{U}(t_n)\mathbf{w}_n^J, \nabla\mathbf{v}_n^j(t_n)\rangle &= \lim_{n\to\infty}\left\langle\nabla\mathbf{U}\left(-\frac{t_n}{(\lambda_n^j)^2}\right)[(g_n^j)^{-1}\mathbf{w}_n^J], \nabla\mathbf{v}^j\left(\frac{t_n}{(\lambda_n^j)^2}+t_n^j\right)\right\rangle\\ &= \lim_{n\to\infty}\left\langle\mathbf{U}\left(-\frac{t_n}{(\lambda_n^j)^2}-t_n^j\right)\nabla\mathbf{U}\left(-\frac{t_n}{(\lambda_n^j)^2}\right)[(g_n^j)^{-1}\mathbf{w}_n^J], \mathbf{U}\left(-\frac{t_n}{(\lambda_n^j)^2}-t_n^j\right)\nabla\mathbf{v}^j(\tau^j)\right\rangle\\ &= \lim_{n\to\infty}\left\langle\nabla\mathbf{U}(-t_n^j)[(g_n^j)^{-1}\mathbf{w}_n^J], \nabla\mathbf{U}(-\tau^j)\mathbf{v}^j(\tau^j)\right\rangle.\end{aligned} \tag{4.62}$$

Using (2.20), we obtain (4.59).

Case 2: Consider now that $t_n(\lambda_n^j)^{-2} + t_n^j$ converges to $\sup I_j$. Then, we should have $\sup I^j = \infty$ and, consequently, $\mathbf{v}^j$ scatters foward in time. This holds if $t_n^j \to \infty$ when $n\to\infty$. Otherwise, suppose it does not hold. Then

$$\limsup_{n\to\infty} S_{[0,t_n]}(\mathbf{v}_n^j) = \limsup_{n\to\infty} S_{[t_n^j, t_n(\lambda_n^j)^{-2}+t_n^j]}(\mathbf{v}^j) = \infty,$$

which contradicts $t_n \in K_n^m$. Hence, it must exist $\boldsymbol{\psi}^j \in \dot{\mathbf{H}}^1$ such that

$$\lim_{n\to\infty}\left\|\mathbf{v}^j\left(t_n(\lambda_n^j)^{-2}+t_n^j\right) - \mathbf{U}(t_n(\lambda_n^j)^{-2}+t_n^j)\boldsymbol{\psi}^j\right\|_{\dot{\mathbf{H}}^1} = 0.$$

Doing the same as in (4.62), we arrive at

$$\lim_{n\to\infty}\langle\nabla\mathbf{U}(t_n)\mathbf{w}_n^J, \nabla\mathbf{v}_n^j(t_n)\rangle = \lim_{n\to\infty}\langle\nabla\mathbf{U}(-t_n^j)[(g_n^j)^{-1}\mathbf{w}_n^J], \nabla\boldsymbol{\psi}^j\rangle,$$

which, again by (2.20), implies (4.59).

Case 3: Now let us focus on the case that $t_n(\lambda_n^j)^{-2} + t_n^j$ converges to $\inf I^j$. Since $t_n(\lambda_n^j)^{-2} \geqslant 0$ and $\inf I^j < \infty$, for all $j \geqslant 1$, we see that t_n^j cannot converge to ∞. Moreover, if $t_n^j \equiv 0$, then $\inf I^j < 0$. Since $t_n(\lambda_n^j)^{-2} \geqslant 0$, then t_n^j cannot be identically zero. So, $t_n^j \to -\infty$ which leads to $\inf I^j = -\infty$ and $\mathbf{v}^j$ scatters backwards in time to $\mathbf{U}(t)\boldsymbol{\phi}^j$. Therefore,

$$\lim_{n\to\infty}\left\|\mathbf{v}^j\left(t_n(\lambda_n^j)^{-2}+t_n^j\right) - \mathbf{U}(t_n(\lambda_n^j)^{-2}+t_n^j)\boldsymbol{\phi}^j\right\|_{\dot{\mathbf{H}}^1} = 0.$$

Repeating the argument in (4.62), we have

$$\lim_{n\to\infty}\langle\nabla\mathbf{U}(t_n)\mathbf{w}_n^J, \nabla\mathbf{v}_n^j(t_n)\rangle = \lim_{n\to\infty}\langle\nabla\mathbf{U}(-t_n^j)[(g_n^j)^{-1}\mathbf{w}_n^J], \nabla\boldsymbol{\phi}^j\rangle,$$

which, again by (2.20), implies (4.59). Proving Claim 2.

Finally, by (4.31), (4.57) and Claim 2,

$$K_c \geqslant \limsup_{n\to\infty} \sup_{t_n\in K_n^m} \|\nabla \mathbf{u}_n^J(t)\|_{\mathbf{L}^2_{t,x}}^2 = \lim_{J\to\infty} \limsup_{n\to\infty} \left[\sup_{t\in K_n^m} \sum_{j=1}^{J} \|\nabla \mathbf{v}_n^j(t)\|_{\mathbf{L}^2_x}^2 + \|\nabla \mathbf{w}_n^J\|_{\mathbf{L}^2_x}^2 \right].$$

By (4.56), this implies $J_1 = 1$, $\mathbf{v}_n^j \equiv 0$ for all $j \geqslant 2$, and $\mathbf{w}_n := \mathbf{w}_n^1$ converges to zero strongly in $\dot{\mathbf{H}}^1$, that is,

$$\mathbf{u}_n(0) = g_n \mathbf{U}(\tau_n)\boldsymbol{\phi} + \mathbf{w}_n \tag{4.63}$$

for some $g_n \in G$, $\tau_n \in \mathbb{R}$ and functions $\boldsymbol{\phi}, \mathbf{w}_n \in \dot{\mathbf{H}}^1$. Moreover, the sequence τ_n obeys either $\tau_n \equiv 0$ or $\tau_n \to \pm\infty$.

If $\tau_n \equiv 0$, then we have that $\mathbf{u}_n(0)$ converges modulo symmetry to $\boldsymbol{\phi}$, which is the desired in this case.

To finish the proof, we show that this is the only possible case. Indeed, suppose whithout loss of generality that $\tau_n \to \infty$. The case $\tau_n \to -\infty$ is analogous. Thus, by Strichartz's inequality $S_{\mathbb{R}}(\mathbf{U}(t))\boldsymbol{\phi} < \infty$. Therefore,

$$\lim_{n\to\infty} S_{\geqslant 0}\left(\mathbf{U}(t)\mathbf{U}(\tau_n)\boldsymbol{\phi}\right) = 0.$$

Since linear solutions and scattering size are preserved by the action of g_n, this leads to

$$\lim_{n\to\infty} S_{\geqslant 0}\left(\mathbf{U}(t) g_n \mathbf{U}(\tau_n)\boldsymbol{\phi}\right) = 0.$$

Together with (4.63) and the fact that $\mathbf{w}_n \to 0$ in $\dot{\mathbf{H}}^1$, we deduce that

$$\lim_{n\to\infty} S_{\geqslant 0}\left(\mathbf{U}(t)\mathbf{u}_n(0)\right) = 0.$$

Applying Lemma 4.3, we deduce

$$\lim_{n\to\infty} S_{\geqslant 0}(\mathbf{u}_n) = 0,$$

which contradicts (4.32). This finishes the proof of Theorem 4.7. □

Now we have the necessary tools to prove Theorem 1.17.

Proof of Theorem 1.17. Suppose that Theorem 1.10 fails. Since $L(K(\boldsymbol{\psi})) = \infty$, by definition of critical energy K_c we must have $K_c \leqslant K(\boldsymbol{\psi})$. Therefore, we may choose a sequence of functions $\mathbf{u}_n : I_n \times \mathbb{R}^6 \longrightarrow \mathbb{C}^l$, with I_n compact, obeying

$$\sup_{n\geqslant 1} \sup_{t\in I_n} K(\mathbf{u}_n(t)) = K_c \quad \text{and} \quad \lim_{n\to\infty} S_{I_n}(\mathbf{u}_n) = \infty. \tag{4.64}$$

Let $t_n \in I_n$ be such that $S_{\geqslant t_n}(\mathbf{u}_n) = S_{\leqslant t_n}(\mathbf{u}_n) = \frac{1}{2} S_{I_n}(\mathbf{u}_n)$. Then,

$$\lim_{n\to\infty} S_{\geqslant t_n}(\mathbf{u}_n) = \lim_{n\to\infty} S_{\leqslant t_n}(\mathbf{u}_n) = \infty. \tag{4.65}$$

By time-translation symmetry, there is no loss of generality in assuming $t_n = 0$. Using Palais-Smale condition, we can find a function $\mathbf{u}_{c,0} \in \dot{\mathbf{H}}^1$ and $g_n \in G$ such that $g_n \mathbf{u}_n \to \mathbf{u}_{c,0}$ strongly in $\dot{\mathbf{H}}^1$, that is,

$$\lim_{n\to\infty} \|T_{g_n}\mathbf{u}_n(0) - \mathbf{u}_{c,0}\|_{\dot{\mathbf{H}}^1} = 0.$$

Let $\mathbf{u}_c : I_c \times \mathbb{R}^6 \longrightarrow \mathbb{C}^l$ be the maximal solution corresponding to the initial data $\mathbf{u}_{c,0}$. By Lemma 4.3, we have $I_c \subset \liminf I_n$ and

$$\lim_{n\to\infty} \|T_{g_n}\mathbf{u}_n - \mathbf{u}_c\|_{\mathbf{L}_t^\infty \dot{\mathbf{H}}^1(K\times\mathbb{R}^6)} = 0, \quad \forall K \subset I_c \text{ compact}.$$

Then, by (4.64),

$$\sup_{t\in I_c} K(\mathbf{u}_c) \leqslant K_c. \tag{4.66}$$

Now, suppose that $\mathbf{u}_c$ does not blow-up forward in time. Then $[0,\infty) \subset I_c$ and $S_{\geqslant 0}(\mathbf{u}_c) < \infty$. Invoking again Lemma 4.3, we obtain

$$S_{\geqslant 0}(\mathbf{u}_n) = S_{\geqslant 0}(T_{g_n}\mathbf{u}_n) < \infty,$$

for n large enough, which contradicts (4.65). A similar argument is used to the negative blow-up case. Hence $\mathbf{u}_c$ blows-up in finite time. Now, by Theorem 4.7,

$$\sup_{t\in I_c} K(\mathbf{u}_c(t)) \geqslant K_c. \tag{4.67}$$

Hence, by (4.66),

$$\sup_{t\in I_c} K(\mathbf{u}_c(t)) = K_c. \tag{4.68}$$

It remains to show that $\mathbf{u}_c$ is almost periodic modulo symmetries. For this purpose, consider a sequence of times $t_n \in I_c$. Since $\mathbf{u}_c$ blows-up in time, we have

$$S_{\geqslant t_n}(\mathbf{u}_c) = S_{\leqslant t_n}(\mathbf{u}_c) = \infty.$$

By Palais-Smale, there exists a sequence $\mathbf{u}_c(t_n)$ that converges in $\dot{\mathbf{H}}^1$ modulo symmetries. This implies that the orbit $\mathcal{F}_c := \{T_{g_n}\mathbf{u}_c(t_n);\, t_n \in I_c\}$ is pre-compact in $\dot{\mathbf{H}}^1$ modulo symmetries. Hence, by definition of T_{g_n}, it follows that $\mathbf{u}_c$ is almost periodic modulo symmetries. This completes the proof. □

4.3 The enemies

This section is devoted to prove Proposition 1.18. Since the proof does not rely on the nonlinearities, for the question of completness, we shall present here a slightly modified version of the proof given in (KILLIP; VISAN, 2010). We also use some ideas presented in (KILLIP; TAO; VISAN, 2009). Some of the tools is given in the Appendix A.1.

To begin with, we note that the existence of an almost periodic modulo symmetries solution $\mathbf{v} : J \times \mathbb{R}^6 \to \mathbb{C}^l$, with minimal kinetic energy is guaranteed by Theorem 1.17 in the last section. We denote the symmetry parameters of $\mathbf{v}$ by $N_{\mathbf{v}}(t)$ and $x_{\mathbf{v}}(t)$. It remains to construct a solution $\mathbf{u}_c : I_c \times \mathbb{R}^6 \to \mathbb{C}^l$ such that its frequency function $N(t)$ satisfies one of the following conditions: Finite-time blow-up, soliton or low-to-high frequency cascade. The construction of $\mathbf{u}_c$ is made by taking subsequential limits of normalizations of $\mathbf{v}$ at $t_0 \in J$, given by (A.5). This is an almost periodic solution and has symmetry parameters given by (A.6).

Using the definition of almost periodicity, given a sequence $t_n \in J$ we may get a subsequence such that $\mathbf{v}^{t_n}(0)$ converges to some $\mathbf{u}_0 \in \dot{\mathbf{H}}^1_x$. Moreover, if we denote by $\mathbf{u}$ the maximal solution with $\mathbf{u}(0) = \mathbf{u}_0$, then $\mathbf{u}$ is almost periodic modulo symmetries with the same compact modulus function as $\mathbf{v}$. Once we have the solution, we set the following quantities for $T > 0$,

$$osc(T) := \inf_{t_0 \in J} \frac{\sup\{N_{\mathbf{v}}(t);\ t \in J \text{ and } |t - t_0| \leqslant T N_{\mathbf{v}}(t_0)^{-2}\}}{\inf\{N_{\mathbf{v}}(t);\ t \in J \text{ and } |t - t_0| \leqslant T N_{\mathbf{v}}(t_0)^{-2}\}}$$

and

$$a(t_0) := \frac{N(t_0)}{\sup\{N(t);\ t \in J \text{ and } t \leqslant t_0\}} + \frac{N(t_0)}{\sup\{N(t);\ t \in J \text{ and } t \geqslant t_0\}}.$$

Then, to complete the proof, we divide in three scenarios. The first one is when oscilation is finite, that is,

(i) $\lim\limits_{T \to \infty} osc(T) < \infty$, which allow us to extract a soliton-like solution.

Here, we choose a sequence t_n such that

$$\limsup_{n \to \infty} \frac{\sup\{N(t);\ t \in J \text{ and } |t - t_n| \leqslant T N(t_n)^{-2}\}}{\inf\{N(t);\ t \in J \text{ and } |t - t_n| \leqslant T N(t_n)^{-2}\}} < \infty.$$

Then, we may find a number $A = A_{\mathbf{v}}$ and two sequences, $t_n \in J$ and $T_n \to \infty$, obeying

$$\frac{\sup\{N_{\mathbf{v}};\ |t - t_n| \leqslant T_n N_{\mathbf{v}}(t_n)^{-2}\}}{\inf\{N_{\mathbf{v}};\ |t - t_n| \leqslant T_n N_{\mathbf{v}}(t_n)^{-2}\}} < A,$$

for all n. Together with Remark A.5, we get

$$[t_n - T_n N_{\mathbf{v}}^{-2}, t_n + T_n N_{\mathbf{v}}^{-2}] \subset J$$

and

$$N_{\mathbf{v}}(t) \sim N_{\mathbf{v}}(t_n),$$

for all t in this interval. Now, define the normalizations $\mathbf{v}^{[t_n]}$ of $\mathbf{v}$ at times t_n. Then, $\mathbf{v}^{[t_n]}$ is a maximal solution with lifespan

$$J_n := \{s \in \mathbb{R};\ t_n + N_{\mathbf{v}}(t_n)^{-2} s \in J\} \supset [-T_n, T_n].$$

It is also an almost periodic solution modulo symmetries with compactness modulus function C and frequency scale function

$$N_{\mathbf{v}^{[t_n]}}(s) := \frac{1}{N_{\mathbf{v}}(t_n)} N_{\mathbf{v}}\left(t_n + N_{\mathbf{v}}(t_n)^{-2}s\right).$$

Particularly, we see that if $s \in [-T_n, T_n]$ then

$$N_{\mathbf{v}^{[t_n]}} \sim 1. \tag{4.69}$$

Lemma A.4 now implies, up to a subsequence, that $\mathbf{v}^{[t_n]}$ converge locally uniformly (see Appendix, Definition A.1) to an almost periodic modulo symmetries solution $\mathbf{u}$, with maximal interval of existence I containing the origin and energy $E(\mathbf{v})$. As $T_n \to \infty$, Lemma A.1 and (4.69) yield that $N_{\mathbf{u}}$ obeys

$$0 < \inf_{t \in I} N_{\mathbf{u}}(t) \leqslant \sup_{t \in I} N_{\mathbf{u}}(t) < \infty.$$

By Corollary A.6, I could not have finite endpoints, therefore I must be $\mathbb{R}$. Also, we can normalize $N \equiv 1$ by modifying C by a bounded quantity. Hence, $\mathbf{u}$ satisfies the conditions to be a soliton.

The other two scenarios happen when $osc(T)$ is unbounded. In this cases we work with $a(t_0)$, for $t_0 \in J$ to distinguish them. The second case is the following

(ii) $\lim_{T\to\infty} osc(T) = \infty$ and $\inf_{t_0 \in J} a(t_0) = 0$.

Since $a(t_0) = 0$, we may choose sequences $t_n^- < t_n < t_n^+$ from J such that $a(t_n) \to 0$, $N_{\mathbf{v}}(t_n^-)/N_{\mathbf{v}}(t_n^+) \to \infty$ and $N_{\mathbf{v}}(t_n^+)/N_{\mathbf{v}}(T_n^-) \to \infty$. Then, we choose times $t_n' \in (t_n^-, t_n^+)$ such that

$$N_{\mathbf{v}}(t_n') \leqslant 2\inf\{N(t);\ t \in [t_n^-, t_n^+]\}. \tag{4.70}$$

In this way, we have $N(t_n') \leqslant 2N(t_n)$, which allow us to deduce that

$$\frac{N_{\mathbf{v}}(t_n^-)}{N_{\mathbf{v}}(t_n')} \to \infty \quad \text{and} \quad \frac{N_{\mathbf{v}}(t_n^+)}{N_{\mathbf{v}}(t_n')} \to \infty. \tag{4.71}$$

Now, let us denote by $\mathbf{u}$ the subsequential limit of $\mathbf{v}^{[t_n']}$ and let I be its maximal interval of existence. If I is bounded, then $\mathbf{u}$ is a finite-time blow-up solution in the sense of Proposition 1.18. Thus it remains to consider the case $I = \mathbb{R}$.

Let $s_n^{\pm} := (t_n^{\pm})N_{\mathbf{v}}(t_n')^2$. From (4.71) we see that $N_{\mathbf{u}}(s_n^{\pm}) \to \infty$ and then, since $\mathbf{u}$ is a global solution, $s_n^{\pm} \to \infty$. Combining with (4.70) we have that $N_{\mathbf{u}}(t)$ is uniformly bounded from below in $t \in \mathbb{R}$. Rescaling $\mathbf{u}$, we may conclude that $N_{\mathbf{u}}(t) \geqslant 1$ for all $t \in \mathbb{R}$.

It follows from $osc(T) \to \infty$, that $N_{\mathbf{v}}(t)$ must show significant oscilation in neighborhoods of t_n', which also happens to $\mathbf{u}$. Combining this with the lower bound on $N_{\mathbf{u}}$, one can see that $\limsup_{|t|\to\infty} N_{\mathbf{u}}(t) = \infty$. Then, up to a time-translation, we have constructed a low-to-high cascade in the sense of Proposition 1.18.

The last case is when $a(t_0)$ is strictly positive, or

(iii) $\lim_{T\to\infty} osc(T) = \infty$ and $\inf_{t_0 \in J} a(t_0) = 2\epsilon > 0.$

Let us call $t_0 \in J$ *future-spreading* if $N(t) \leqslant \epsilon^{-1}N(t_0)$, for any $t \geqslant t_0$ and *past-spreading* if $N(t) \leqslant \epsilon^{-1}N(t_0)$ for any $t \leqslant t_0$. Thus, by hypothesis, every t_0 is past- or future-spreading.

Notice that J must be infinite in backward or forward time direction, since a single time is past- or future-spreading, respectively. Also, recall that finite-time blow-up is accompanied by $N_{\mathbf{v}} \to \infty$ as t approaches the blow-up time. Next we will show that either all sufficiently late times are future-spreading or all sufficiently early times are past-spreading. Otherwise, it would be possible to find a interval large enough such that it starts with a future-spreading time and it ends with a past-spreading time. This would be absurd, as it contradicts the divergence of $osc(T)$. We will focus only in the case where $t \geqslant t_0$ are future-spreading. The past-spreading case is analogous since we have time-reversal symmetry.

Take T obeying $osc(T) > 2\epsilon^{-1}$. Let us construct an increasing sequence $\{t_n\}_{n=0}^{\infty}$ such that

$$0 \leqslant t_{n+1} - t_n \leqslant 8TN(t_n)^{-2} \quad \text{and} \quad N(t_{n+1}) \leqslant \frac{1}{2}N(t_n). \tag{4.72}$$

Given t_n, set $t_n' := t_n + 4TN(t_n)^{-2}$. If $2N(t_n') \leqslant N(t_n)$ we choose $t_{n+1} = t_n'$ and the properties above follows. If $2N(t_n') > N(t_n)$, then

$$J_n := [t_n' - TN(t_n')^{-2}, t_n' + TN(t_n')^{-2}] \subset [t_n, t_n + 8TN(t_n)^{-2}].$$

As t_n is future-spreading, we may ensure that $N(t) \leqslant \epsilon^{-1}N(t_n)$ on J_n, however, by the choice of T, we can find $t_{n+1} \in J_n$ obeying $2N(t_{n+1}) \leqslant N(t_n)$.

Since we have a sequence of times satisfying (4.72), then any subsequential limit $\mathbf{u}$ of $\mathbf{v}^{[t_n]}$ is a finite-time blow-up solution. Indeed, setting $s_n := (t_0 - t_n)N(t_n)^{-2}$ we may notice that $N_{\mathbf{v}^{[t_n]}} \geqslant 2^n$. However,

$$|s_n| = N(t_n)^2 \sum_{k=0}^{n-1} [t_{k+1} - t_k] \leqslant 8T \sum_{k=0}^{n-1} \frac{N(t_n)^2}{N(t_k)^2} \leqslant 8T \sum_{k=0}^{n-1} 2^{-(n-k)} \leqslant 8T.$$

Thus, s_n is bounded and, therefore, the solution $\mathbf{u}$ must blow-up at some time $-8T \leqslant t < 0$.

This completes the proof of Proposition 1.18.

4.4 Finite-time blow-up

In this section we shall start the process of eliminating of our “enemies”. We start avoiding the finite-time blow-up solution.

Theorem 4.8. *There is no critical solution, in the sense of Theorem 1.17, for the system (1.2) which blows-up in finite time.*

Proof. Suppose that there exists a maximal finite-time blowing-up solution, namely, $\mathbf{u}_c : I_c \times \mathbb{R}^6 \longrightarrow \mathbb{C}^l$. There is no loss of generality in assuming that $\sup I_c < \infty$. Then

$$\liminf_{t \nearrow \sup I_c} N(t) = \infty. \tag{4.73}$$

Indeed, if (4.73) does not occur, we may choose $(t_n) \subset I_c$ converging to $\sup I_c$, and set $\mathbf{v}_n : I_n \times \mathbb{R}^6 \longrightarrow \mathbb{C}^l$ given by

$$\mathbf{v}_n(t,x) = \frac{1}{N(t_n)^2}\mathbf{u}_c\left(t_n + \frac{t}{N(t_n)^2}\,,\, x(t_n) + \frac{x}{N(t_n)}\right),$$

where $I_n := \{t_n + N(t_n)^{-2}t : \quad t \in I_c\}$. Theorem 1.17 tell us that $\mathbf{u}_c$ is almost periodic modulo symmetries, which implies that $\{\mathbf{v}_n(t,x)\}_{n\in\mathbb{N}}$ is also a solution for the system. Besides that, combining with Remark 1.15, we have that $\{\mathbf{v}_n(0)\} \subset \dot{\mathbf{H}}^1(\mathbb{R}^6)$ is pre-compact in $\dot{\mathbf{H}}^1_x$. Hence, after passing to a subsequence if necessary, there exists $\mathbf{v}_0$ such that

$$\lim_{n\to\infty} \|\mathbf{v}_n(0) - \mathbf{v}_0\|_{\dot{\mathbf{H}}^1} = 0. \tag{4.74}$$

Suppose that $\mathbf{v}_0 = 0$. Then, since $\|\nabla \mathbf{v}_n(0)\|_{\mathbf{L}^2} = \|\nabla \mathbf{u}_c(t_n)\|_{\mathbf{L}^2}$, by (4.74) we have that $\|\nabla \mathbf{u}_c(t_n)\|_{\mathbf{L}^2} \to 0$ as $n \to \infty$, that is, $K(\mathbf{u}_c(t_n)) \to 0$, as $n \to \infty$. By Lemma 2.31, $E(\mathbf{u}_c(t_n)) \sim K(\mathbf{u}_c(t_n))$. Taking $n \to \infty$, we get $E(\mathbf{u}_c(t_n)) \to 0$. By conservation of the energy, this leads to $E(\mathbf{u}_c) = 0$, which is a contradiction, since $\mathbf{u}_c \neq 0$. Then, $\mathbf{v}_0 \neq 0$. Let $\mathbf{v} : I \times \mathbb{R}^6 \to \mathbb{C}^l$ be the maximal solution to (1.2) with initial data $\mathbf{v}_0 = \mathbf{v}(0)$, where $I := (-T_*, T^*)$ satisfies $-\infty \leqslant -T_* < 0 < T^* \leqslant \infty$. By the well-posedness, for each compact interval $J \subset I$, we have $S_J(\mathbf{v}) < \infty$. This shows that $\mathbf{u}_c$ is well-posed with finite scatterting size on the interval $\{t_n + t(N(t_n))^{-2},\, t \in J\}$. However, as $t_n \nearrow \sup I_c$ and $\liminf_{n\to\infty} N(t_n) = \liminf_{t\nearrow \sup I_c} N(t) < \infty$, that is, $\mathbf{u}_c$ has finite scattering size beyond $\sup I_c$, which contradicts the existence of $t_1 \in I_c$ such that $S_{[t_1, \sup I_c)}(\mathbf{u}_c) = \infty$. Hence, (4.73) must hold.

Consider $\mathbf{u}_c = (u_{c1}, ..., u_{cl})$. Let $\eta \in (0,1)$ and $t \in I_c$. By Hölder's inequality and Sobolev's embedding, for $k = 1, ..., l$ and $R > 0$,

$$\begin{aligned}\int_{|x|<R} |u_{ck}|^2 dx &\leqslant \int_{|x-x(t)|\leqslant \eta R} |u_{ck}|^2 dx + \int_{|x|\leqslant R,\, |x-x(t)|>\eta R} |u_{ck}|^2 dx \\ &\lesssim \eta^2 R^2 \|u_{ck}\|^2_{L^3_x} + R^2 \left(\int_{|x-x(t)|>\eta R} |u_{ck}|^3 dx\right)^{2/3} \\ &\lesssim \eta^2 R^2 K(\boldsymbol{\psi}) + R^2 \left(\int_{|x-x(t)|>\eta R} |u_{ck}|^3 dx\right)^{2/3} \\ &\lesssim \eta^2 R^2 K(\boldsymbol{\psi}) + R^2 \eta^{2/3},\end{aligned}$$

where we used (4.73), almost periodicity modulo symmetries and Remark 1.15 in the last inequality. Then, letting $\eta \to 0$, we see that

$$\limsup_{t\to \sup I_c} \int_{|x|\leqslant R} |u_{ck}|^2 dx = 0, \quad \forall R > 0,\ k = 1, ..., l. \tag{4.75}$$

Now, consider

$$\phi(r) = \begin{cases} 1, & r \leqslant 1, \\ 0, & r \geqslant 2, \end{cases}$$

and

$$V_R(t) := \int \left(\sum_{k=1}^{l} \frac{\alpha_k^2}{\gamma_k} |u_{ck}|^2 \right) \phi\left(\frac{|x|}{R} \right) dx.$$

By (4.75),

$$\limsup_{t \to \sup I_c} V_R(t) = 0, \quad \forall R > 0. \tag{4.76}$$

Using Hardy's inequality (see (TAO, 2006, Lemma A.2)) and (1.32), we get

$$\begin{aligned} |V_R'(t)| &= 2 \left| \sum_{k=1}^{l} \alpha_k \mathrm{Im} \int \nabla \phi_R \cdot \nabla u_{ck} \bar{u}_{ck} dx \right| \\ &\lesssim \|\nabla \mathbf{u}_c\|_{\mathbf{L}_x^2} \left\| \frac{\mathbf{u}_c}{|x|} \right\|_{\mathbf{L}_x^2} \\ &\lesssim [K(\mathbf{u}_c)]^2 \\ &< [K(\boldsymbol{\psi})]^2. \end{aligned}$$

By the fundamental theorem of calculus,

$$V_R(t_1) \lesssim V_R(t_2) + |t_1 - t_2| [K(\boldsymbol{\psi})|]^2, \quad \forall t_1, t_2 \in I_c,\ R > 0.$$

Taking $t_2 \to \sup I_c$ and using (4.76), we see

$$V_R(t_1) \lesssim |\sup I_c - t_1| [K(\boldsymbol{\psi})]^2, \quad \forall t_1 \in I_c.$$

Invoking the conservation of mass and making $R \to \infty$,

$$Q(\mathbf{u}_{c0}) = Q(\mathbf{u}_c(t_1)) \lesssim |\sup I_c - t_1| [K(\boldsymbol{\psi})|], \quad \forall t_1 \in I_c.$$

Letting $t_1 \nearrow \sup I_c$, give us $\mathbf{u}_{c0} \equiv 0$. By uniquess of solution, it follows that $\mathbf{u}_c \equiv 0$, which contradicts (1.32). □

4.5 Negative Regularity

Before proceeding to exclusion of next two "enemies", we must prove that the critical solution has some negative regularity. We dedicate this section for this purpose. We begin by stating the main result of this section.

Theorem 4.9. *(Negative Regularity). Let* $\mathbf{u}$ *be a global solution to* (1.2) *that is almost periodic modulo symmetries. Suppose also that*

$$\sup_{t \in \mathbb{R}} \|\nabla \mathbf{u}\|_{\mathbf{L}^2} < \infty \tag{4.77}$$

and

$$\inf_{t\in\mathbb{R}} N(t) \geqslant 1. \tag{4.78}$$

Then $\mathbf{u} \in \mathbf{L}_t^\infty \dot{\mathbf{H}}_x^{-\epsilon}$ *for some* $\epsilon > 0$*. In particular,* $\mathbf{u} \in \mathbf{L}_t^\infty \mathbf{L}_x^2$*.*

To prove this theorem we will follow the strategy presented in (KILLIP; VISAN, 2010). The proof will be done in two steps. The first one is to prove that our solution lies in $\mathbf{L}_t^\infty \mathbf{L}_x^p$, for $2 < p < 3$. The second is to use the "double Duhamel trick" to improve regularity to $\mathbf{u} \in \mathbf{L}_t^\infty \dot{\mathbf{H}}_x^{1-s}$ for some $s > 0$. Having disposed of this two preliminary steps, we may derive Theorem 4.9. Before proceeding, we need to set some usefull tools. The first one is the following Duhamel's formula.

Lemma 4.10. *Let* $\mathbf{u}$ *be an almost periodic solution to* (1.2) *with maximal interval of existence* I*. Then, for all* $t \in I$*,*

$$\begin{aligned} u_k(t) &= \lim_{T \nearrow \sup I} i \int_t^T U_k(t-s) f_k(\mathbf{u}(s)) ds \\ &= -\lim_{T \searrow \inf I} i \int_T^t U_k(t-s) f_k(\mathbf{u}(s)) ds \end{aligned} \tag{4.79}$$

as weak limits in $\dot{\mathbf{H}}_x^1(\mathbb{R}^6)$*.*

Proof. The proof can be found in (TAO; VISAN; ZHANG, 2008), Section 6. □

Remark 4.11. *Assume that* $\mathbf{u}$ *obeys the hypotheses of Theorem 4.9. Consider* $\eta > 0$ *a small constant that will be chosen later. By Remark 1.16 combined with* (4.78)*, there is* $N_0 = N_0(\eta)$ *such that*

$$\|\nabla(P_{\leqslant N_0}\mathbf{u})\|_{\mathbf{L}_t^\infty \mathbf{L}_x^2} \leqslant \eta, \quad \forall \eta > 0. \tag{4.80}$$

Remark 4.12. *Define, for frequencies* $N \leqslant 10N_0$*,*

$$A(N) := N^{-1/2} \|P_N \mathbf{u}(t)\|_{\mathbf{L}_t^\infty \mathbf{L}_x^4(\mathbb{R}\times\mathbb{R}^6)}. \tag{4.81}$$

By Bernstein's inequality, Sobolev's embedding $H^1(\mathbb{R}^6) \hookrightarrow L^3(\mathbb{R}^6)$ *and* (4.77) *we see*

$$\begin{aligned} A(N) &= N^{-1/2} \|P_N \mathbf{u}(t)\|_{\mathbf{L}_t^\infty \mathbf{L}_x^4} \\ &\lesssim N^{-\frac{1}{2}} N^{\left(\frac{6}{3}-\frac{6}{4}\right)} \|P_N \mathbf{u}\|_{\mathbf{L}_t^\infty \mathbf{L}_x^3} \\ &\lesssim \|P_N \mathbf{u}\|_{\mathbf{L}_t^\infty \mathbf{L}_x^3} \\ &\lesssim \|\nabla \mathbf{u}\|_{\mathbf{L}_t^\infty \mathbf{L}_x^2}, \end{aligned}$$

which implies that $A(N)$ *is well defined.*

The next result is a recurrence formula to $A(N)$.

Lemma 4.13. *For all $N \leqslant 10N_0$,*

$$A(N) \lesssim \left(\frac{N}{N_0}\right)^{1/2} + \eta \sum_{\frac{N}{10} \leqslant N_1 \leqslant N_0} \left(\frac{N}{N_1}\right)^{1/2} A(N_1) + \eta \sum_{N_1 < \frac{N}{10}} \left(\frac{N_1}{N}\right)^{1/2} A(N_1), \tag{4.82}$$

where $A(N)$ is given by (4.81).

Proof. We first fix N such that $N \leqslant 10N_0$. By time-translation symmetry, it is sufficient to show that

$$N^{-1/2}\|P_N \mathbf{u}(0)\|_{\mathbf{L}^4_x} \lesssim \left(\frac{N}{N_0}\right)^{1/2} + \eta \sum_{\frac{n}{10} \leqslant N_1 \leqslant N_0} \left(\frac{N}{N_1}\right)^{1/2} A(N_1) + \eta \sum_{N_1 < \frac{N}{10}} \left(\frac{N_1}{N}\right)^{1/2} A(N_1). \tag{4.83}$$

By Duhamel's formula (4.79) and triangle inequality, we have

$$\begin{aligned} N^{-1/2}\|P_N u_k(0)\|_{L^4_x} \leqslant N^{-1/2} &\left\| \int_0^{N^{-2}} U_k(t) \frac{1}{\alpha_k} P_N f_k(\mathbf{u}(t)) dt \right\|_{L^4_x} \\ &+ N^{-1/2} \left\| \int_{N^{-2}}^{\infty} U_k(t) \frac{1}{\alpha_k} P_N f_k(\mathbf{u}(t)) dt \right\|_{L^4_x}. \end{aligned} \tag{4.84}$$

For the first term on the right-hand side of the last inequality, using Lemma 2.19, we may estimate

$$\begin{aligned} N^{-1/2} \left\| \int_0^{N^{-2}} U_k(t) \frac{1}{\alpha_k} P_N f_k(\mathbf{u}(t)) dt \right\|_{L^4_x} &\lesssim N^{-1/2} N^{3/2} \left\| \int_0^{N^{-2}} U_k(t) \frac{1}{\alpha_k} P_N f_k(\mathbf{u}(t)) dt \right\|_{L^2_x} \\ &\lesssim N \|P_N f_k(\mathbf{u}(t))\|_{L^\infty_t L^2_x} \left(\int_0^{N^{-2}} 1 dt \right) \\ &\lesssim N^{-1} N^{3/2} \|P_N f_k(\mathbf{u}(t))\|_{L^\infty_t L^{4/3}_x} \\ &= N^{1/2} \|P_N f_k(\mathbf{u}(t))\|_{L^\infty_t L^{4/3}_x}. \end{aligned} \tag{4.85}$$

Next, for the second term in the right-hand side of (4.84) we may apply Lemma 2.9 to get

$$\begin{aligned} N^{-1/2} \left\| \int_{N^{-2}}^{\infty} U_k(t) \frac{1}{\alpha_k} P_N f_k(\mathbf{u}(t)) dt \right\|_{L^4_x} &\lesssim N^{-1/2} \|P_N f_k(\mathbf{u}(t))\|_{L^\infty_t L^{4/3}_x} \left(\int_{N^{-2}}^{\infty} |t|^{-3/2} dt \right) \\ &= N^{1/2} \|P_N f_k(\mathbf{u}(t))\|_{L^\infty_t L^{4/3}_x}. \end{aligned} \tag{4.86}$$

From (4.85) and (4.86), we conclude

$$N^{-1/2}\|P_N \mathbf{u}(0)\|_{\mathbf{L}^4_x} \lesssim N^{1/2} \|P_N f_k(\mathbf{u}(t))\|_{L^\infty_t L^{4/3}_x}.$$

Hence, to obtain (4.83) we need to estimate $N^{1/2}\|P_N f_k(\mathbf{u}_c(t))\|_{L^\infty_t L^{4/3}_x}$. Notice that

$$\begin{aligned} f_k(\mathbf{u}_c) &= f_k(\mathbf{u}_c) - f_k(P_{\leqslant N_0} \mathbf{u}_c) + f_k(P_{\leqslant N_0} \mathbf{u}_c) \\ &=: g_k(\mathbf{u}_c) + f_k(P_{\leqslant N_0} \mathbf{u}_c). \end{aligned}$$

Then,

$$\begin{aligned} N^{1/2}\|P_N f_k(\mathbf{u}_c)\|_{L_t^\infty L_x^{4/3}} &\leqslant N^{1/2}\|P_N g_k(\mathbf{u}_c)\|_{L_t^\infty L_x^{4/3}} + N^{1/2}\|P_{\leqslant N_0} f_k(\mathbf{u}_c)\|_{L_t^\infty L_x^{4/3}} \\ &=: \mathcal{I} + \mathcal{J}. \end{aligned} \tag{4.87}$$

By Lemma 2.11 and the decomposition of the solution

$$\begin{aligned} |g_k(\mathbf{u})| &= |f_k(\mathbf{u}) - f_k(P_{\leqslant N_0}\mathbf{u})| \\ &= |f_k(P_{\leqslant N_0}\mathbf{u} + P_{>N_0}\mathbf{u}) - f_k(P_{\leqslant N_0}\mathbf{u})| \\ &\leqslant C\sum_{m=1}^{l}\sum_{j=1}^{l}\left(|P_{\leqslant N_0}u_j + P_{>N_0}u_j| + |P_{\leqslant N_0}u_j|\right)|P_{>N_0}u_m| \\ &\leqslant C\sum_{m=1}^{l}\sum_{j=1}^{l}|P_{\leqslant N_0}u_j||P_{>N_0}u_m| + C\sum_{m=1}^{l}|P_{>N_0}u_m|^2. \end{aligned}$$

Furthermore, using Hölder's inequality and Lemma 2.19, the first term in the right-hand side of (4.87) can be bounded as follows

$$\begin{aligned} \mathcal{I} &\lesssim N^{1/2}\|g_k(\mathbf{u})\|_{L_t^\infty L_x^{4/3}} \\ &\lesssim N^{1/2}\left[\sum_{m=1}^{l}\sum_{j=1}^{l}\||P_{\leqslant N_0}u_j||P_{>N_0}u_m|\|_{L_t^\infty L_x^{4/3}} + \sum_{m=1}^{l}\||P_{>N_0}u_m|^2\|_{L_t^\infty L_x^{4/3}}\right] \\ &\lesssim N^{1/2}\left[\sum_{m=1}^{l}\sum_{j=1}^{l}\|u_j\|_{L_t^\infty L_x^3}\|P_{>N_0}u_m\|_{L_t^\infty L_x^{12/5}} + \sum_{m=1}^{l}\|u_m\|_{L_t^\infty L_x^3}\|P_{>N_0}u_m\|_{L_t^\infty L_x^{12/5}}\right] \\ &\lesssim N^{1/2}N_0^{-1/2}\left[\sum_{m=1}^{l}\sum_{j=1}^{l}\|u_j\|_{L_t^\infty L_x^3}\||\nabla|^{1/2}u_m\|_{L_t^\infty L_x^{12/5}} + \sum_{m=1}^{l}\|u_m\|_{L_t^\infty L_x^3}\||\nabla|^{1/2}u_m\|_{L_t^\infty L_x^{12/5}}\right]. \end{aligned}$$

Using the embbedings $\dot{H}_x^1(\mathbb{R}^6) \hookrightarrow \dot{H}_x^{\frac{1}{2},\frac{12}{5}}(\mathbb{R}^6)$, see (BERGH; LöFSTRöM, 1976, Theorem 6.5.1, page 153), and $\dot{H}_x^1(\mathbb{R}^6) \hookrightarrow L_x^3(\mathbb{R}^6)$, see (TAO, 2006, page 335, A.11), and (4.77), the last inequality gives us

$$\mathcal{I} \lesssim N^{1/2}N_0^{-1/2} = \left(\frac{N}{N_0}\right)^{1/2}.$$

To estimate the second term in the right-hand side of (4.87), the fundamental theorem of calculus allows us to write

$$\begin{aligned} f_k(\mathbf{z}) - f_k(\mathbf{z}') = \sum_{m=1}^{l}(z_m - z'_m)\int_0^1 \frac{\partial f_k}{\partial z_m}(\mathbf{z}' + \theta(\mathbf{z}-\mathbf{z}'))d\theta \quad + \\ \sum_{m=1}^{l}\overline{(z_m - z'_m)}\int_0^1 \frac{\partial f_k}{\partial \bar{z}_m}(\mathbf{z}' + \theta(\mathbf{z}-\mathbf{z}'))d\theta. \end{aligned}$$

Taking $\mathbf{z} = P_{\frac{N}{10}\leqslant\cdot\leqslant N_0}\mathbf{u}$ and $\mathbf{z}' = P_{\leqslant N_0}\mathbf{u}$, we arrive at

$$f_k(P_{\leqslant N_0}\mathbf{u}) = f_k\left(P_{\frac{N}{10}\leqslant\cdot\leqslant N_0}\mathbf{u}\right) + \sum_{m=1}^{l} P_{<\frac{N}{10}}u_m \int_0^1 \frac{\partial f_k}{\partial z_m}\left(P_{\frac{N}{10}\leqslant\cdot\leqslant N_0}\mathbf{u} + \theta P_{<\frac{N}{10}}\mathbf{u}\right) d\theta$$
$$+ \sum_{m=1}^{l} \overline{P_{<\frac{N}{10}}u_m} \int_0^1 \frac{\partial f_k}{\partial \bar{z}_m}\left(P_{\frac{N}{10}\leqslant\cdot\leqslant N_0}\mathbf{u}{+}\theta P_{<\frac{N}{10}}\mathbf{u}\right) d\theta,$$

which implies

$$\begin{aligned}\mathcal{J} &\lesssim N^{1/2}\left\|P_N f_k\left(P_{\frac{N}{10}\leqslant\cdot\leqslant N_0}\mathbf{u}\right)\right\|_{L_t^\infty L_x^{4/3}}\\ &\quad+ N^{1/2}\sum_{m=1}^{l}\left\|P_N\left[P_{<\frac{N}{10}}u_m\int_0^1\frac{\partial f_k}{\partial z_m}\left(P_{\frac{N}{10}\leqslant\cdot\leqslant N_0}\mathbf{u}+\theta P_{<\frac{N}{10}}\mathbf{u}\right)d\theta\right]\right\|_{L_t^\infty L_x^{4/3}}\\ &\quad+ N^{1/2}\sum_{m=1}^{l}\left\|P_N\left[\overline{P_{<\frac{N}{10}}u_m}\int_0^1\frac{\partial f_k}{\partial \bar{z}_m}\left(P_{\frac{N}{10}\leqslant\cdot\leqslant N_0}\mathbf{u}+\theta P_{<\frac{N}{10}}\mathbf{u}\right)d\theta\right]\right\|_{L_t^\infty L_x^{4/3}}\\ &=: \mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3.\end{aligned} \tag{4.88}$$

At first, we shall work with $\mathcal{J}_2$ and $\mathcal{J}_3$. Effectively, it suffices to estimate $\mathcal{J}_2$, because $\mathcal{J}_3$ can be treated in an analogous way.

From $(H2)$, we have for $k = 1, ..., l$ that the complex derivatives of the nonlinearities f_k are Hölder continuous of order 1, hence Lemma 2.20 gives us, for $m = 1, ..., l$,

$$\begin{aligned}\left\|P_{>\frac{N}{10}}\frac{\partial f_k}{\partial z_m}(\mathbf{u})\right\|_{L_t^\infty L_x^2} &\lesssim \sum_{M>\frac{N}{10}}\left\|P_M\frac{\partial f_k}{\partial z_m}(\mathbf{u})\right\|_{L_t^\infty L_x^2}\\ &\lesssim \sum_{M>\frac{N}{10}} M^{-1}\|\nabla\mathbf{u}\|_{\mathbf{L}_t^\infty\mathbf{L}_x^2}\\ &\lesssim N^{-1}\|\nabla\mathbf{u}\|_{\mathbf{L}_t^\infty\mathbf{L}_x^2},\end{aligned}$$

since $\displaystyle\sum_{M>\frac{N}{10}} M^{-1} = \sum_{j>0} 2^{-j}\frac{10}{N} \lesssim N^{-1}$. Applying Hölder's inequality, Remark 2.18, (4.80) and (4.81),

$$\begin{aligned}\mathcal{J}_2 &= N^{1/2}\sum_{m=1}^{l}\left\|P_N\left[P_{<\frac{N}{10}}u_m\int_0^1\frac{\partial f_k}{\partial z_m}\left(P_{\frac{N}{10}\leqslant\cdot\leqslant N_0}\mathbf{u}+\theta P_{<\frac{N}{10}}\mathbf{u}\right)d\theta\right]\right\|_{L_t^\infty L_x^{4/3}}\\ &\lesssim N^{1/2}\sum_{m=1}^{l}\|P_{<\frac{N}{10}}u_m\|_{L_t^\infty L_x^4}\left\|P_{>\frac{N}{10}}\int_0^1\frac{\partial f_k}{\partial z_m}\left(P_{\frac{N}{10}\leqslant\cdot\leqslant N_0}\mathbf{u}+\theta P_{<\frac{N}{10}}\mathbf{u}\right)d\theta\right\|_{L_t^\infty L_x^2}\\ &\lesssim N^{1/2}\sum_{m=1}^{l}\|P_{<\frac{N}{10}}u_m\|_{L_t^\infty L_x^4}N^{-1}\|\nabla P_{<N_0}\mathbf{u}\|_{\mathbf{L}_t^\infty\mathbf{L}_x^2}\\ &\lesssim \eta N^{-1/2}\|P_{<\frac{N}{10}}\mathbf{u}\|_{\mathbf{L}_t^\infty\mathbf{L}_x^4}\\ &\lesssim \eta N^{-1/2}\sum_{N_1<\frac{N}{10}}\|P_{N_1}\mathbf{u}\|_{\mathbf{L}_t^\infty\mathbf{L}_x^4}\\ &= \eta\sum_{N_1<\frac{N}{10}}\left(\frac{N_1}{N}\right)^{1/2}A(N_1).\end{aligned}$$

Finally, we can estimate $\mathcal{J}_1$ in (4.88). By Lemma 2.11,

$$|f_k(\mathbf{z})| \lesssim \sum_{m=1}^{l} \bar{z}_m z_m,$$

so, using Lemma 2.11 and Hölder's inequality,

$$\begin{aligned}
\left\| P_N f_k \left(P_{\frac{N}{10} \leqslant \cdot \leqslant N_0} \mathbf{u} \right) \right\|_{L_t^\infty L_x^{4/3}} &\lesssim \left\| f_k \left(P_{\frac{N}{10} \leqslant \cdot \leqslant N_0} \mathbf{u} \right) \right\|_{L_t^\infty L_x^{4/3}} \\
&\lesssim \sum_{m=1}^{l} \| \overline{(P_{\frac{N}{10} \leqslant \cdot \leqslant N_0} u_m)} (P_{\frac{N}{10} \leqslant \cdot \leqslant N_0} u_m) \|_{L_t^\infty L_x^{4/3}} \\
&\lesssim \sum_{\frac{N}{10} \leqslant N_1, N_2 \leqslant N_0} \left[\sum_{m=1}^{l} \| \overline{(P_{N_1} u_m)} (P_{N_2} u_m) \|_{L_t^\infty L_x^{4/3}} \right] \\
&\lesssim \sum_{\frac{N}{10} \leqslant N_1 \leqslant N_2 \leqslant N_0} \left[\sum_{m=1}^{l} \| \overline{(P_{N_1} u_m)} \|_{L_t^\infty L_x^4} \| (P_{N_2} u_m) \|_{L_t^\infty L_x^2} \right] \\
&\quad + \sum_{\frac{N}{10} \leqslant N_2 \leqslant N_1 \leqslant N_0} \left[\sum_{m=1}^{l} \| \overline{(P_{N_1} u_m)} \|_{L_t^\infty L_x^2} \| (P_{N_2} u_m) \|_{L_t^\infty L_x^4} \right].
\end{aligned} \tag{4.89}$$

Therefore, using Lemma 2.19 and (4.80),

$$\begin{aligned}
\left\| P_N f_k \left(P_{\frac{N}{10} \leqslant \cdot \leqslant N_0} \mathbf{u} \right) \right\|_{L_t^\infty L_x^{4/3}} &\lesssim \eta \sum_{\frac{N}{10} \leqslant N_1 \leqslant N_2 \leqslant N_0} \sum_{m=1}^{l} N_2^{-1} \| P_{N_1} u_m \|_{L_t^\infty L_x^4} \\
&\quad + \eta \sum_{\frac{N}{10} \leqslant N_2 \leqslant N_1 \leqslant N_0} \sum_{m=1}^{l} N_1^{-1} \| P_{N_2} u_m \|_{L_t^\infty L_x^4} \\
&\lesssim \eta \sum_{\frac{N}{10} \leqslant N_1 \leqslant N_2 \leqslant N_0} \sum_{m=1}^{l} \left(\frac{N_1}{N_2} \right) N_1^{-1} \| P_{N_1} u_m \|_{L_t^\infty L_x^4} \\
&\quad + \eta \sum_{\frac{N}{10} \leqslant N_2 \leqslant N_1 \leqslant N_0} \sum_{m=1}^{l} \left(\frac{N_2}{N_1} \right) N_2^{-1} \| P_{N_2} u_m \|_{L_t^\infty L_x^4} \\
&\lesssim \eta \sum_{\frac{N}{10} \leqslant N_1 \leqslant N_0} N_1^{-1/2} A(N_1) \\
&\quad + \eta \sum_{\frac{N}{10} \leqslant N_2 \leqslant N_1 \leqslant N_0} \left(\frac{N_2}{N_1} \right) N_2^{-1/2} A(N_2) \\
&\lesssim \eta \sum_{\frac{N}{10} \leqslant N_1 \leqslant N_0} N_1^{-1/2} A(N_1).
\end{aligned} \tag{4.90}$$

Then,

$$N^{1/2} \| P_N f_k(\mathbf{u}) \|_{L_t^\infty L_x^{4/3}} \lesssim \left(\frac{N}{N_0} \right)^{1/2} + \eta \sum_{\frac{N}{10} \leqslant N_1 \leqslant N_0} \left(\frac{N}{N_1} \right)^{1/2} A(N_1) + \eta \sum_{N_1 < \frac{N}{10}} \left(\frac{N_1}{N} \right)^{1/2} A(N_1)$$

finishing the proof. □

This Lemma leads us directly to our first result.

Proposition 4.14. *Let* $\mathbf{u}$ *be as in Theorem 4.9. Then*

$$\mathbf{u} \in \mathbf{L}_t^\infty \mathbf{L}_x^p, \quad \frac{14}{5} \leqslant p < 3. \tag{4.91}$$

In addition,

$$\nabla f_k(\mathbf{u}) \in \mathbf{L}_t^\infty \mathbf{L}_x^r, \quad \frac{7}{6} \leqslant r < \frac{6}{5}. \tag{4.92}$$

Proof. Combining Lemma 4.13 with Lemma 2.8, we deduce

$$\|P_N \mathbf{u}\|_{\mathbf{L}_t^\infty \mathbf{L}_x^4} \lesssim N^{1-}, \quad \text{for } N \leqslant 10N_0, \tag{4.93}$$

by setting $N = 10 \cdot 2^{-j} N_0$, $x_j = A(2^{-j}N_0)$ and take η sufficiently small. Now, by interpolation, Lemma 2.20 with $g(u) = u$, (4.93) and (4.77)

$$\begin{aligned}
\|P_N \mathbf{u}\|_{\mathbf{L}_t^\infty \mathbf{L}_x^p} &\lesssim \|P_N \mathbf{u}\|_{\mathbf{L}_t^\infty \mathbf{L}_x^4}^{\frac{2(p-2)}{p}} \|P_N \mathbf{u}\|_{\mathbf{L}_t^\infty \mathbf{L}_x^2}^{\frac{4}{p}-1} \\
&\lesssim N^{\frac{2(p-2)}{p}-} \left(N^{-1} \|\nabla \mathbf{u}\|_{\mathbf{L}_t^\infty \mathbf{L}_x^2}\right)^{\frac{4}{p}-1} \\
&\lesssim N^{\frac{2(p-2)}{p}-} N^{1-\frac{4}{p}} \\
&\lesssim N^{3-\frac{8}{p}-}.
\end{aligned}$$

Now, if $14/5 \leqslant p < 3$, then $3 - 8/p \leqslant 1/7$. Thereby,

$$\|P_N \mathbf{u}\|_{\mathbf{L}_t^\infty \mathbf{L}_x^p} \lesssim N^{\frac{1}{7}-}. \tag{4.94}$$

for all $N \leqslant 10N_0$. On the other hand, notice that Lemma 2.16 with $s = 1$, $q = 2$ gives us $\theta = 3 - \dfrac{6}{p}$, consequently,

$$\|P_N \mathbf{u}\|_{\mathbf{L}_t^\infty \mathbf{L}_x^p} \lesssim \|\nabla (P_N \mathbf{u})\|_{\mathbf{L}_t^\infty \mathbf{L}_x^2}^{3-\frac{6}{p}} \|P_N \mathbf{u}\|_{\mathbf{L}_t^\infty \mathbf{L}_x^2}^{\frac{6}{p}-2} = \|P_N (\nabla \mathbf{u})\|_{\mathbf{L}_t^\infty \mathbf{L}_x^2}^{3-\frac{6}{p}} \|P_N \mathbf{u}\|_{\mathbf{L}_t^\infty \mathbf{L}_x^2}^{\frac{6}{p}-2}, \tag{4.95}$$

where we used the commutativity of Littlewood-Paley operators with gradient in the last inequality. By Lemma 2.6 and (4.77)

$$\|P_N (\nabla \mathbf{u}_c)\|_{\mathbf{L}_t^\infty \mathbf{L}_x^2} \lesssim \|\nabla \mathbf{u}_c\|_{\mathbf{L}_t^\infty \mathbf{L}_x^2} \lesssim 1. \tag{4.96}$$

Using Lemma 2.20 with $g(u) = u$, and again (4.77), we deduce

$$\|P_N \mathbf{u}\|_{\mathbf{L}_t^\infty \mathbf{L}_x^2} \lesssim N^{-1} \|\nabla \mathbf{u}\|_{\mathbf{L}_t^\infty \mathbf{L}_x^2} \lesssim N^{-1}. \tag{4.97}$$

Inserting (4.96) and (4.97) in (4.95), we obtain

$$\|P_N \mathbf{u}\|_{\mathbf{L}_t^\infty \mathbf{L}_x^p} \lesssim N^{2-\frac{6}{p}}. \tag{4.98}$$

Finally, by (4.94) and (4.98),

$$\begin{aligned}\|\mathbf{u}\|_{\mathbf{L}_t^\infty \mathbf{L}_x^p} &\leqslant \|P_{\leqslant N_0}\mathbf{u}\|_{\mathbf{L}_t^\infty \mathbf{L}_x^p} + \|P_{>N_0}\mathbf{u}\|_{\mathbf{L}_t^\infty \mathbf{L}_x^p}\\ &\leqslant \sum_{N\leqslant N_0} \|P_N\mathbf{u}\|_{\mathbf{L}_t^\infty \mathbf{L}_x^p} + \sum_{N>N_0} \|P_N\mathbf{u}\|_{\mathbf{L}_t^\infty \mathbf{L}_x^p}\\ &\lesssim \sum_{N\leqslant N_0} N^{\frac{1}{7}-} + \sum_{N>N_0} N^{2-\frac{6}{p}}\\ &\lesssim 1,\end{aligned}$$

as stated.

In particular, by Lemma 2.11 (iii), with $q = 2$, and (4.77),

$$\|\nabla f_k(\mathbf{u})\|_{\mathbf{L}_t^\infty \mathbf{L}_x^r} \lesssim \|\mathbf{u}\|_{\mathbf{L}_t^\infty \mathbf{L}_x^p}\|\nabla \mathbf{u}\|_{\mathbf{L}_t^\infty \mathbf{L}_x^2} \lesssim \|\mathbf{u}\|_{\mathbf{L}_t^\infty \mathbf{L}_x^p}.$$

Since $\frac{1}{r} = \frac{1}{p} + \frac{1}{2}$ and $\mathbf{u} \in \mathbf{L}_t^\infty \mathbf{L}_x^p$ for $\frac{14}{5} \leqslant p < 3$, it follows that $\nabla f_k(\mathbf{u}) \in \mathbf{L}_t^\infty \mathbf{L}_x^r$ for $\frac{7}{6} \leqslant r < \frac{6}{5}$. Finishing the proof. □

The second step to reach our goal will be done as in (KILLIP; VISAN, 2010), where we will prove (4.91) by using Lemma 4.10 twice.

Proposition 4.15. *(Some negative regularity) Let* $\mathbf{u}$ *be as in Theorem 4.9. If* $|\nabla|^s f_k(\mathbf{u}) \in L_t^\infty L_x^r$ *for some* $\frac{7}{6} \leqslant r < \frac{6}{5}$, $s \in [0,1]$ *and* $k = 1, \ldots, l$, *then there is* $s_0 = s_0(r) > 0$ *such that* $\mathbf{u} \in \mathbf{L}_t^\infty \dot{\mathbf{H}}^{s-s_0+}(\mathbb{R})$.

Proof. We first notice that

$$\||\nabla|^{s-s_0+}u_k\|_{L_t^\infty L_x^2} \leqslant \||\nabla|^{s-s_0+}P_{N\leqslant 1}u_k\|_{L_t^\infty L_x^2} + \||\nabla|^{s-s_0+}P_{N>1}u_k\|_{L_t^\infty L_x^2} := A + B. \quad (4.99)$$

We will work the cases separately. We start with A:

$$\begin{aligned}A &= \||\nabla|^{s-s_0+}\sum_{N\leqslant 1} P_N u_k\|_{L_t^\infty L_x^2}\\ &\lesssim \sum_{N\leqslant 1}\||\nabla|^{s-s_0+}P_N u_k\|_{L_t^\infty L_x^2}\\ &= \sum_{N\leqslant 1}\||\nabla|^{-s_0+}(|\nabla|^s P_N u_k)\|_{L_t^\infty L_x^2}\\ &\lesssim \sum_{N\leqslant 1} N^{-s_0+}\||\nabla|^s P_N u_k\|_{L_t^\infty L_x^2},\end{aligned} \quad (4.100)$$

where we used Bernstein's inequality in the last line. We will show that

$$\||\nabla|^s P_N u_k\|_{L_t^\infty L_x^2} \lesssim N^{s_0}, \quad N > 0 \quad s_0 := \frac{6}{r} - 5 > 0. \quad (4.101)$$

By time-translation, it suffices to prove

$$\||\nabla|^s P_N u_k(0)\|_{L_x^2} \lesssim N^{s_0}, \quad N > 0 \quad s_0 := \frac{6}{r} - 5 > 0. \quad (4.102)$$

By Duhamel's formula (4.79), both in the future and the past, for $k = 1, ..., l$, we write

$$\begin{aligned}
&\||\nabla|^s P_N u_k(0)\|_{L_t^\infty L_x^2} \\
&\quad = \lim_{T\to\infty}\lim_{T'\to-\infty}\left\langle i\int_0^T U_k(-t)P_N|\nabla|^s f_k(\mathbf{u}(t))dt, -i\int_{T'}^0 U_k(-\tau)P_N|\nabla|^s f_k(\mathbf{u}(\tau))d\tau\right\rangle \\
&\quad \leqslant \int_0^\infty\int_{-\infty}^0 |\langle P_N|\nabla|^s f_k(\mathbf{u}(t)), U_k(t-\tau)P_N|\nabla|^s f_k(\mathbf{u}(\tau))\rangle|\, dtd\tau.
\end{aligned} \tag{4.103}$$

We treat the integral in two ways. First, using Hölder's inequality and Lemma 2.9,

$$\begin{aligned}
&|\langle P_N|\nabla|^s f_k(\mathbf{u}(t)), U_k(t-\tau)P_N|\nabla|^s f_k(\mathbf{u}(\tau))\rangle| \\
&\qquad \lesssim \|P_N|\nabla|^s f_k(\mathbf{u}(t))\|_{L_x^r}\|U_k(t-\tau)P_N|\nabla|^s f_k(\mathbf{u}(\tau))\|_{L_x^{r'}} \\
&\qquad \lesssim |t-\tau|^{3-\frac{6}{r}}\||\nabla|^s f_k(\mathbf{u})\|^2_{L_t^\infty L_x^r}.
\end{aligned} \tag{4.104}$$

On the other hand, by Bernstein's inequality,

$$\begin{aligned}
&|\langle P_N|\nabla|^s f_k(\mathbf{u}(t)), U_k(t-\tau)P_N|\nabla|^s f_k(\mathbf{u}(\tau))\rangle| \\
&\qquad \lesssim \|P_N|\nabla|^s f_k(\mathbf{u}(t))\|_{L_x^2}\|U_k(t-\tau)P_N|\nabla|^s f_k(\mathbf{u}(\tau))\|_{L_x^2} \\
&\qquad \lesssim N^{2(\frac{6}{r}-3)}\||\nabla|^s f_k(\mathbf{u})\|^2_{L_t^\infty L_x^r}.
\end{aligned} \tag{4.105}$$

Then, combining (4.104) with (4.105), and using in (4.103), we deduce

$$\||\nabla|^s P_N u_k(0)\|^2_{L_x^2} \lesssim \||\nabla|^s f_k(\mathbf{u})\|^2_{L_t^\infty L_x^r}\int_0^\infty\int_{-\infty}^0 \min\{|t-\tau|^{-1}, N^2\}^{\frac{6}{r}-3}dtd\tau. \tag{4.106}$$

Now, if $t < 0 < \tau$ hence, $|t-\tau| = \tau - t$. If $|t-\tau|^{-1} \leqslant N^2$ then $\tau \geqslant t + \dfrac{1}{N^2}$ so

$$\begin{aligned}
\int_0^\infty\int_{-\infty}^0 \min\{|t-\tau|^{-1}, N^2\}^{\frac{6}{r}-3}dtd\tau &= \iint_{\{\tau\geqslant t+\frac{1}{N^2}\}} \frac{1}{(\tau-t)^{\frac{6}{r}-3}}dtd\tau \\
&= \int_0^{\frac{1}{N^2}}\int_{-\infty}^{\tau-\frac{1}{N^2}} \frac{1}{(\tau-t)^{\frac{6}{r}-3}}dtd\tau + \int_{\frac{1}{N^2}}^\infty\int_{-\infty}^0 \frac{1}{(\tau-t)^{\frac{6}{r}-3}}dtd\tau.
\end{aligned} \tag{4.107}$$

To simplify notation, we will write $q := \dfrac{6}{r} - 3$. Thus,

$$\begin{aligned}
\int_0^{\frac{1}{N^2}}\int_{-\infty}^{\tau-\frac{1}{N^2}} \frac{1}{(\tau-t)^q}dtd\tau &= \frac{1}{q-1}\int_0^{\frac{1}{N^2}} \left.\frac{1}{(\tau-t)^{q-1}}\right|_{-\infty}^{\tau-\frac{1}{N^2}} d\tau \\
&= \frac{1}{q-1}\int_0^{\frac{1}{N^2}} \frac{1}{(\tau-(\tau-N^{-2}))^{q-1}}d\tau \\
&= \frac{1}{q-1}\int_0^{\frac{1}{N^2}} N^{2q-2}d\tau \\
&= \frac{1}{q-1}N^{2q-4}.
\end{aligned} \tag{4.108}$$

Likewise,

$$\begin{aligned}\int_{\frac{1}{N^2}}^{\infty}\int_{-\infty}^{0}\frac{1}{(\tau-t)^q}dtd\tau &= \frac{1}{q-1}\int_{\frac{1}{N^2}}^{\infty}\left.\frac{1}{(\tau-t)^{q-1}}\right|_{-\infty}^{0}d\tau\\ &= \frac{1}{q-1}\int_{\frac{1}{N^2}}^{\infty}\frac{1}{\tau^{q-1}}d\tau\\ &= \frac{1}{(q-1)(q-2)}\left.-\frac{1}{\tau^{q-2}}\right|_{\frac{1}{N^2}}^{\infty}\\ &= \frac{1}{(q-1)(q-2)}N^{2q-4}.\end{aligned} \tag{4.109}$$

On the other hand, if $N^2\leqslant|t-\tau|^{-1}$, then $\tau\leqslant t+\dfrac{1}{N^2}$. So

$$\begin{aligned}\int_{0}^{\infty}\int_{-\infty}^{0}\min\{|t-\tau|^{-1},N^2\}^q dtd\tau &= \iint_{\{\tau\leqslant t+\frac{1}{N^2}\}}N^{2q}dtd\tau\\ &= \int_0^{\frac{1}{N^2}}\int_{-\frac{1}{N^2}}^{0}N^{2q}dtd\tau\\ &= N^{2q}\int_0^{\frac{1}{N^2}}\frac{1}{N^2}d\tau\\ &= N^{2q-4}.\end{aligned} \tag{4.110}$$

By (4.107), (4.108), (4.109) and (4.110), and noticing that $2q-4=2\left(\dfrac{6}{r}-3\right)-4=\dfrac{12}{r}-10=2s_0$, besides that $\dfrac{6}{r}-3>2$ since $r<\dfrac{6}{5}$, it follows that

$$\int_0^{\infty}\int_{-\infty}^{0}\min\{|t-\tau|^{-1},N^2\}^{\frac{6}{r}-3}dtd\tau\lesssim N^{2s_0}. \tag{4.111}$$

Replacing (4.111) in (4.106), we deduce

$$\||\nabla|^s P_N u_k(0)\|_{L^2_x}^2\lesssim N^{2s_0}\||\nabla|^s f_k(\mathbf{u})\|_{L^\infty_t L^r_x}^2.$$

Then (4.102) holds, and consequently

$$A\lesssim\sum_{N\leqslant 1}N^{-s_0+}N^{s_0}=\sum_{N\leqslant 1}N^{0+}. \tag{4.112}$$

To estimate B, by Lemma 2.19, Lemma 2.20 and (4.77), for $k=1,...,l$,

$$\||\nabla|^{s-s_0+}P_N u_k\|_{L^\infty_t L^2_x}\lesssim N^{s-s_0+}\|P_N u_k\|_{L^\infty_t L^2_x}\lesssim N^{s-s_0+}(N^{-1}\|\nabla u_k\|_{L^\infty_t L^2_x})\lesssim N^{(s-s_0+)-1}.$$

Thus

$$B\lesssim\sum_{N>1}\||\nabla|^{s-s_0^+}P_N u_k\|_{L^\infty_t L^2_x}\lesssim\sum_{N>1}N^{(s-s_0+)-1}. \tag{4.113}$$

Replacing (4.112) and (4.113) in (4.99), for $k=1,...,l$, we have

$$\||\nabla|^{s-s_0+}u_k\|_{L^\infty_t L^2_x}\lesssim\sum_{N\leqslant 1}N^{0+}+\sum_{N>1}N^{(s-s_0+)-1}\lesssim 1.$$

which completes the proof. □

Proof of Theorem 4.9. By proposition 4.14, we may apply Proposition 4.15 with $s = 1$ to show that $\mathbf{u} \in \mathbf{L}_t^\infty \dot{\mathbf{H}}^{1-s_0+}$ for some $s_0 > 0$. Using (2.4) we deduce,

$$\||\nabla|^{1-s_0+} f_k(\mathbf{u})\|_{\mathbf{L}_t^\infty \mathbf{L}_x^r} \lesssim \|\mathbf{u}\|_{\mathbf{L}_t^\infty \mathbf{L}_x^p} \|\mathbf{u}\|_{\mathbf{L}_t^\infty \dot{\mathbf{H}}^{1-s_0+}}.$$

Then, (4.91) guarantees that $|\nabla|^{1-s_0+} f_k(\mathbf{u}) \in \mathbf{L}_t^\infty \mathbf{L}_x^r$, for $\frac{7}{6} \leqslant r < \frac{6}{5}$ and $k = 1, ..., l$. Another application of Proposition 4.15 helps us to get $\mathbf{u} \in \mathbf{L}_t^\infty \dot{\mathbf{H}}^{1-2s_0+}$. Iterating this procedure finitely many times gives us $\mathbf{u} \in \mathbf{L}_t^\infty \dot{\mathbf{H}}^{-\epsilon}$ for some $0 < \epsilon < s_0$. □

4.6 Soliton

In this section we exclude the soliton-like solution. For this, we need to show that the critical solution has zero momentum and, from that, get some compactness properties. We first define the momentum associated to the solution $\mathbf{u}$ by

$$\mathcal{P}(\mathbf{u}) := 4 \sum_{k=1}^{l} \alpha_k \mathrm{Im} \int \nabla u_k \bar{u}_k dx.$$

Notice that if $\mathbf{u}(t, x)$ is a solution to (1.2) then the function $\mathbf{u}^\xi(t, x)$, called Galilean transformation, given by

$$u_k^\xi(t, x) := e^{ix\cdot\xi \frac{\alpha_k}{\gamma_k}} e^{-it|\xi|^2 \frac{\alpha_k}{\gamma_k}} u_k(t, x - 2t\xi), \quad k = 1, ..., l, \tag{4.114}$$

is also a solution to (1.2). This is a direct consequence of Gauge condition (see Lemma 2.13). The next Lemma gives us some properties of mass and kinetic energy of Galilean transformation.

Lemma 4.16. *For $\xi \in \mathbb{R}^6$, let $\mathbf{u}^\xi$ be a Galilean transformation. Then*

(i)

$$Q(\mathbf{u}^\xi(x)) = Q(\mathbf{u}(x)).$$

(ii)

$$\left|\nabla u_k^\xi(x)\right|^2 = \frac{\alpha_k^2}{\gamma_k^2}|\xi|^2|u_k(x)|^2 + 2\frac{\alpha_k}{\gamma_k}\xi \cdot \mathrm{Im}[\nabla u_k \bar{u}_k](x) + |\nabla u_k(x)|^2.$$

In particular,

$$K(\mathbf{u}^\xi(x)) = |\xi|^2 Q(\mathbf{u}(x)) + \xi \cdot \mathcal{P}(\mathbf{u}(x)) + K(\mathbf{u}(x)).$$

Proof. The proof follows by direct calculations, so we omit the details. □

In other words, we can write $E(\mathbf{u}) = E(\mathbf{u}^\xi(x)) + (4M(\mathbf{u}))^{-1}\mathcal{P}(\mathbf{u})^2$, which express that total energy can be decomposed as the energy viewed in the center of mass frame plus the energy arising from the motion of the center of the mass. (see (LANDAU; LIFSHITZ, 1976, §8)).

Lemma 4.17. *Assume that hypothesis (H3) and (H5) hold. Then, the momentum $\mathcal{P}(\mathbf{u})$ associated with the solution $\mathbf{u}$ is a conserved quantity.*

Proof. Suppose that $\mathbf{u}$ is a sufficiently regular solution. Then, formally, we multiply (1.2) by $\partial_{x_j}\bar{u}_k$, and integrate on $\mathbb{R}^6$ and taking the imaginary part to obtain

$$\alpha_k \mathrm{Im}\left[\int \partial_t u_k \partial_{x_j}\bar{u}_k dx\right] = \gamma_k \mathrm{Im}\left[i\int \partial_t \nabla u_k \partial_{x_j}\bar{u}_k dx\right] + \mathrm{Im}\left[i\int f_k(\mathbf{u})\partial_{x_j}\bar{u}_k dx\right]. \quad (4.115)$$

First, notice that by integrating by parts $i\int \partial_t \nabla u_k \partial_{x_j}\bar{u}_k dx$ agrees with the complex conjugate and then, it is a real number. Thus, the first integral on the right-hand side of (4.115) vanishes. Hence, (4.115) becomes

$$\alpha_k \mathrm{Im}\left[\int \partial_t u_k \partial_{x_j}\bar{u}_k dx\right] = \mathrm{Im}\left[i\int f_k(\mathbf{u})\partial_{x_j}\bar{u}_k dx\right].$$

Summing over $k = 1, ..., l$ in the last equality and using Lemma 2.14 (ii), we deduce

$$\sum_{k=1}^{l} \alpha_k \mathrm{Im}\int \partial_t u_k \partial_{x_j}\bar{u}_k dx = \int \partial_{x_j}\mathrm{Re}F(\mathbf{u})dx.$$

Integrating by parts, as a consequence of Lemma 2.14 (iii), the integral on right-hand side vanishes. Then,

$$\sum_{k=1}^{l} \alpha_k \mathrm{Im}\int \partial_t u_k \partial_{x_j}\bar{u}_k dx = 0. \quad (4.116)$$

Now, we use the following identity $\partial_t[u_k\partial_{x_j}\bar{u}_k] = \partial_t u_k \partial_{x_j}\bar{u}_k + u_k\partial_t\partial_{x_j}\bar{u}_k$, to write (4.116) as

$$\sum_{k=1}^{l} \alpha_k \mathrm{Im}\int \partial_t[u_k\partial_{x_j}\bar{u}_k]dx - \sum_{k=1}^{l} \alpha_k \mathrm{Im}\int u_k\partial_t\partial_{x_j}\bar{u}_k dx = 0. \quad (4.117)$$

Using integration by parts and that u_k satisfies (1.2), we may write

$$\alpha_k \int \bar{u}_k \partial_t \partial_{x_j} u_k dx = -\gamma_k \int i\Delta u_k \partial_{x_j}\bar{u}_k dx - i\int f_k(\mathbf{u})\partial_{x_j}\bar{u}_k dx.$$

Therefore, the second term in(4.117) can be written as

$$\begin{aligned} -\sum_{k=1}^{l} \alpha_k \mathrm{Im}\int u_k\partial_t\partial_{x_j}\bar{u}_k dx &= \sum_{k=1}^{l} \alpha_k \mathrm{Im}\int \bar{u}_k\partial_t\partial_{x_j}u_k dx \\ &= -\sum_{k=1}^{l} \gamma_k \int i\Delta u_k \partial_{x_j}\bar{u}_k dx - \int \partial_{x_j}\mathrm{Re}F(\mathbf{u})dx, \end{aligned} \quad (4.118)$$

where in the last integral on the right-hand side of (4.118) we applied Lemma 2.14 (iii). As before, the two integrals on the right-hand side vanishes. The result follows from this and (4.117). □

Proposition 4.18. *(Zero momentum). Assume that $\mathbf{u}_c$ is a blow-up solution to (1.2) with minimum kinetic energy and obeys $\mathbf{u}_c \in \mathbf{L}_t^\infty \mathbf{H}_x^1$. Then $\mathcal{P}(\mathbf{u}_c) = 0$.*

Proof. Suppose that $\mathbf{u}_c : I_c \times \mathbb{R}^6 \to \mathbb{C}^l$ is as in Proposition 4.18. We know that the mass $Q(\mathbf{u}_c)$ and the momentum $\mathcal{P}(\mathbf{u}_c)$ are conserved quantities. Besides that, $Q(\mathbf{u}_c) \neq 0$, otherwise we would have $\mathbf{u}_c = 0$, which is excluded beacuse $\mathbf{u}_c$ is a blow-up solution. Then, the vector given by $\xi_0 := -\dfrac{\mathcal{P}(\mathbf{u}_c)}{2Q(\mathbf{u}_c)}$ is well defined and the function $\mathbf{u}_c^{\xi_0}$ is a solution to (1.2) by invariance of the Galilean transformation.

By Lemma 4.16, we deduce

$$|\xi_0|^2 Q(\mathbf{u}_c(x)) + \xi_0 \cdot \mathcal{P}(\mathbf{u}_c(x)) = K(\mathbf{u}_c^{\xi_0}(x)) - K(\mathbf{u}_c(x)). \tag{4.119}$$

Now, $S_{I_c}(\mathbf{u}_c^{\xi_0}) = S_{I_c}(\mathbf{u}_c) = \infty$, hence $\mathbf{u}_c^{\xi_0}$ is also a blow-up solution to (1.2). Moreover, by hypothesis, $\mathbf{u}_c$ has minimum kinetic energy, and then, (4.119) implies that

$$|\xi_0|^2 Q(\mathbf{u}_c(x)) + \xi_0 \cdot \mathcal{P}(\mathbf{u}_c(x)) \geqslant 0.$$

On the other hand, by the definition of ξ_0, we deduce

$$0 \leqslant |\xi_0|^2 Q(\mathbf{u}_c(x)) + \xi_0 \cdot \mathcal{P}(\mathbf{u}_c(x)) = -\frac{|\mathcal{P}(\mathbf{u}_c(x))|^2}{4Q(\mathbf{u}_c(x))} \leqslant 0. \tag{4.120}$$

Since $Q(\mathbf{u}_c) \neq 0$, it follows from (4.120) that $\mathcal{P}(\mathbf{u}_c(x)) = 0$, as we desired. □

Lemma 4.19. *(Compactness in L^2) Let $\mathbf{u}_c$ be a soliton in the sense of Proposition 1.18. Then, for all $\eta > 0$, there is a constant $C(\eta) > 0$ such that*

$$\sup_{t\in\mathbb{R}} \sum_{k=1}^{l} \int_{|x-x(t)|\geqslant C(\eta)} |u_{ck}|^2 dx \lesssim \eta.$$

Proof. The argument takes place in a fixed t, in particular, we may assume $x(t) = 0$.

Initially, we control the contribution of low frequency. Using Bernstein's inequality, (2.6) and Theorem 4.9, we obtain, for $k = 1, ..., l$,

$$\|P_{<N} u_{ck}(t)\|_{L^2_x(|x|>R)} \leqslant \|P_{<N} u_{ck}(t)\|_{L^2_x} \lesssim N^{\epsilon} \|P_{<N} |\nabla|^{-\epsilon} u_{ck}\|_{L^\infty_t L^2_x} \lesssim N^{\epsilon} \||\nabla|^{-\epsilon} u_{ck}\|_{L^\infty_t L^2_x} \lesssim N^{\epsilon}.$$

This can be smaller than η choosing $N = N(\eta)$ sufficiently small.

For the high frequencies case, an application of Schur's test gives us the following: For some $m \geqslant 0$ (see (KILLIP; VISAN, 2010), page 408),

$$\left\|\chi_{|x|\geqslant 2R} \Delta^{-1} \nabla P_{\geqslant N} \chi_{|x|\leqslant R}\right\|_{L^2\to L^2} \lesssim N^{-1} \langle RN \rangle^{-m}$$

uniformly in $R, N > 0$. On the other hand, by Bernstein's inequality,

$$\|\chi_{|x|\geqslant 2R} \Delta^{-1} \nabla P_{\geqslant N} \chi_{|x|\geqslant R}\|_{L^2\to L^2} \lesssim N^{-1}.$$

Together, the above inequalities give us

$$\int_{|x|\geqslant 2R} |P_{\geqslant N} u_{ck}|^2 dx \lesssim N^{-2} \langle RN \rangle^{-2} \|\nabla u_{ck}(t)\|^2_{L^2_x} + N^{-2} \int_{|x|\geqslant R} |\nabla u_{ck}|^2 dx.$$

Choosing R as large as necessary, we can make the first term on the right-hand side smaller than η. The same holds to the second term because $\mathbf{u}_c$ is almost periodic modulo symmetries

$$\sup_{t\in\mathbb{R}}\int_{|x-x(t)|\geqslant C(\eta)}|\nabla u_{ck}|^2dx\leqslant\eta.$$

The result follows combining the estimates of $P_{<N}u_{ck}$ and $P_{\geqslant N}u_{ck}$. □

Corollary 4.20. *(Control of $x(t)$). Let $\mathbf{u}_c$ be a soliton solution in the sense of Proposition 1.18. Then*

$$|x(t)|=o(t),\quad t\to\infty.$$

Proof. We argue by contradiction. Suppose that there exist $\delta>0$ and a sequence $t_n\to\infty$ such that

$$|x(t_n)|>\delta t_n,\quad\forall n\geqslant 1. \tag{4.121}$$

By spatial-translation symmetry, we may assume $x(0)=0$.

Let $\eta>0$ be a constant that will be chosen later. By Remark 1.15 and Lemma 4.19,

$$\sup_{t\in\mathbb{R}}\sum_{k=1}^{l}\int_{|x-x(t)|>C(\eta)}\left(|\nabla u_{ck}(t,x)|^2+|u_{ck}(t,x)|^2\right)dx\leqslant\eta. \tag{4.122}$$

Define

$$T_n:=\inf_{t\in[0,t_n]}\{|x(t)|=|x(t_n)|\}\leqslant t_n\quad\text{and}\quad R_n:=C(\eta)+\sup_{t\in[0,T_n]}|x(t)|. \tag{4.123}$$

Let ϕ be a smooth, radial function such that

$$\phi(r)=\begin{cases}1, & r\leqslant 1\\ 0, & r\geqslant 2,\end{cases}$$

and define the "truncated" position

$$X_R(t):=\int_{\mathbb{R}^6}x\phi\left(\frac{|x|}{R}\right)|u_{ck}(t,x)|^2dx.$$

By Theorem 4.9, $u_{ck}\in L_t^\infty L_x^2$. Thus, by (4.123), if $|x|\leqslant C(\eta)$ then $\frac{|x|}{R_n}\leqslant 1$, hence, $\phi\left(\frac{|x|}{R_n}\right)=1$ and

$$\begin{aligned}\left|\int_{|x|\leqslant C(\eta)}x\phi\left(\frac{|x|}{R_n}\right)|u_{ck}(0,x)|^2dx\right|&\leqslant\int_{|x|\leqslant C(\eta)}|x|\left|\phi\left(\frac{|x|}{R_n}\right)\right||u_{ck}(0,x)|^2dx\\&\leqslant C(\eta)\int_{\mathbb{R}^6}|u_{ck}(0,x)|^2dx\lesssim C(\eta)Q(\mathbf{u}).\end{aligned} \tag{4.124}$$

On the other hand, if $|x| \geqslant 2R_n$ then $\phi\left(\frac{|x|}{R_n}\right) = 0$. Thus, using (4.122), we deduce

$$\left|\int_{|x|\geqslant C(\eta)} x\phi\left(\frac{|x|}{R_n}\right)|u_{ck}(0,x)|^2dx\right| \lesssim 2R_n\int_{|x|\geqslant C(\eta)}|u_{ck}(0,x)|^2dx \lesssim 2R_n\eta. \tag{4.125}$$

Therefore, combining (4.124) and (4.125),

$$\begin{aligned}|X_{R_n}(0)| &\leqslant \left|\int_{|x|\leqslant C(\eta)} x\phi\left(\frac{|x|}{R_n}\right)|u_{ck}(0,x)|^2dx\right| + \left|\int_{|x|\geqslant C(\eta)} x\phi\left(\frac{|x|}{R_n}\right)|u_{ck}(0,x)|^2dx\right| \\ &\lesssim C(\eta)Q(\mathbf{u}) + 2\eta R_n.\end{aligned} \tag{4.126}$$

On the other hand,

$$\begin{aligned}X_{R_n}(T_n) &= \int_{\mathbf{R}^6} x\phi\left(\frac{|x|}{R_n}\right)|u_{ck}(T_n,x)|^2dx \\ &= \int_{\mathbf{R}^6} x\phi\left(\frac{|x|}{R_n}\right)|u_{ck}(T_n,x)|^2dx + x(T_n)Q(\mathbf{u}) - x(T_n)Q(\mathbf{u}) \\ &\quad + \int_{\mathbf{R}^6} x(T_n)\phi\left(\frac{|x|}{R_n}\right)|u_{ck}(T_n,x)|^2dx - \int_{\mathbf{R}^6} x(T_n)\phi\left(\frac{|x|}{R_n}\right)|u_{ck}(T_n,x)|^2dx \\ &= x(T_n)\left[Q(\mathbf{u}) - \int_{\mathbb{R}^6}\left[1-\phi\left(\frac{|x|}{R_n}\right)\right]|u_{ck}(T_n,x)|^2dx\right] \\ &\quad + \int_{\mathbf{R}^6}[x - x(T_n)]\phi\left(\frac{|x|}{R_n}\right)|u_{ck}(T_n,x)|^2dx \\ &= x(T_n)\left[Q(\mathbf{u}) - \int_{\mathbb{R}^6}\left[1-\phi\left(\frac{|x|}{R_n}\right)\right]|u_{ck}(T_n,x)|^2dx\right] \\ &\quad + \int_{|x-x(T_n)|<C(\eta)}[x - x(T_n)]\phi\left(\frac{|x|}{R_n}\right)|u_{ck}(T_n,x)|^2dx \\ &\quad + \int_{|x-x(T_n)|\geqslant C(\eta)}[x - x(T_n)]\phi\left(\frac{|x|}{R_n}\right)|u_{ck}(T_n,x)|^2dx.\end{aligned}$$

By triangle inequality combined with (4.122) and (4.123),

$$\begin{aligned}|X_{R_n}(T_n)| &\gtrsim |x(T_n)|\left[Q(\mathbf{u}) - \left|\int_{\mathbb{R}^6}\left[1-\phi\left(\frac{|x|}{R_n}\right)\right]|u_{ck}(T_n,x)|^2dx\right|\right] \\ &\quad - \left|\int_{|x-x(T_n)|\leqslant C(\eta)}[x - x(T_n)]\phi\left(\frac{|x|}{R_n}\right)|u_{ck}(T_n,x)|^2x\right| \\ &\quad - \left|\int_{|x-x(T_n)|\geqslant C(\eta)}[x - x(T_n)]\phi\left(\frac{|x|}{R_n}\right)|u_{ck}(T_n,x)|^2x\right| \\ &\gtrsim |x(T_n)|[Q(\mathbf{u}) - \eta] - C(\eta)Q(\mathbf{u}) - \eta[2R_n + |x(T_n)|] \\ &\gtrsim |x(T_n)|[Q(\mathbf{u}) - \eta] - C(\eta)Q(\mathbf{u}) - \eta[2C(\eta) + 2|x(T_n)| + |x(T_n)|] \\ &\gtrsim |x(T_n)|[Q(\mathbf{u}) - 4\eta] - 3C(\eta)Q(\mathbf{u}),\end{aligned} \tag{4.127}$$

where in the last inequality we used that (4.122) implies $\eta \gtrsim Q(\mathbf{u})$. Thus, from (4.126) and (4.127), taking $\eta > 0$ sufficiently small (depending on $Q(\mathbf{u})$),

$$|X_{R_n}(T_n) - X_{R_n}(0)| \gtrsim |x(T_n)| - C(\eta). \tag{4.128}$$

Note that

$$X_R'(t) = 2\text{Im}\int \phi\left(\frac{|x|}{R}\right)\nabla u_{ck}(t)\overline{u_{ck}(t)}dx + 2\text{Im}\int \frac{x}{|x|R}\phi'\left(\frac{|x|}{R}\right)x\cdot\nabla u_{ck}(t)\overline{u_{ck}(t)}dx.$$

By Lemma 4.18 $\mathcal{P}(\mathbf{u}_c) = 0$; this together with Cauchy-Schwarz's inequality and (4.122), give

$$\begin{aligned}
|X_{R_n}'(t)| &\leqslant \left|2\text{Im}\int\left[1-\phi\left(\frac{|x|}{R_n}\right)\right]\nabla u_{ck}(t)\overline{u_{ck}(t)}dx\right| \\
&\quad + \left|2\text{Im}\int \frac{x}{|x|R_n}\phi'\left(\frac{|x|}{R_n}\right)x\cdot\nabla u_{ck}(t)\overline{u_{ck}(t)}dx\right| \\
&= \left|2\text{Im}\int_{|x|>R_n}\left[1-\phi\left(\frac{|x|}{R_n}\right)\right]\nabla u_{ck}(t)\overline{u_{ck}(t)}dx\right| \\
&\quad + \left|2\text{Im}\int_{R_n\leqslant|x|\leqslant 2R_n} \frac{x}{|x|R_n}\phi'\left(\frac{|x|}{R_n}\right)x\cdot\nabla u_{ck}(t)\overline{u_{ck}(t)}dx\right| \\
&\lesssim 2\int_{|x-x(t)|\geqslant C(\eta)}|\nabla u_{ck}(t)\overline{u_{ck}(t)}|dx \\
&\quad + 2\int_{R_n\leqslant|x|\leqslant 2R_n}\frac{|x|^2}{2R_n^2}\cdot|\nabla u_{ck}(t)\overline{u_{ck}(t)}|dx \\
&\lesssim \int_{|x-x(t)|>C(\eta)}\left(|\nabla u_{ck}(t,x)|^2 + |u_{ck}(t,x)|^2\right)dx \\
&\lesssim \eta,
\end{aligned}$$

for all $t \in [0, T_n]$. Hence, using (4.128) and the fundamental theorem of calculus

$$|x(T_n)| - C(\eta) \lesssim |X_{R_n}(T_n) - X_{R_n}(0)| \lesssim \int_0^{T_n}|X_{R_n}'(t)|dt \lesssim \eta T_n.$$

Since $|x(T_n)| = |x(t_n)| > \delta t_n \geqslant \delta T_n$, we have

$$\delta < \eta + \frac{C(\eta)}{T_n}.$$

Taking $\eta < \delta/2$ and making $n \to \infty$ we get $\delta < \delta/2$, which is a contradiction. □

We now are in position to exclude the soliton-like solution. When $x(t) = 0$, as in the radial case, the necessary argument can be found in (KENIG; MERLE, 2006).

Theorem 4.21. *There is no solution to* (1.2) *which is soliton-like, in the sense of Proposition 1.18.*

Proof. Let $\mathbf{u}_c : \mathbb{R}\times\mathbb{R}^6 \longrightarrow \mathbb{C}^l$ be a soliton like solution. By definition of almost periodicity and the embedding $\dot{H}^1(\mathbb{R}^6) \hookrightarrow L^3(\mathbb{R}^6)$, for any $\eta > 0$, there exists $C(\eta) > 0$ such that

$$\sup_{t\in\mathbb{R}}\int_{|x-x(t)|\geqslant C(\eta)}\sum_{k=1}^{l}(|\nabla u_{ck}|^2 + |u_{ck}|^3)dx \leqslant \eta. \tag{4.129}$$

Corollary 4.20 guarantees that there exists $T_0 = T_0(\eta) \in \mathbb{R}$ such that

$$|x(t)| \leqslant \eta t, \quad \forall t \geqslant T_0. \tag{4.130}$$

Setting $\phi(x)$ to be radial, smooth and obeying

$$\phi(x) = \begin{cases} r, & r \leqslant 1 \\ 0, & |x| \geqslant 2, \end{cases}$$

let $\psi(x) = R^2 \phi\left(\frac{|x|^2}{R^2}\right)$, where $R > 0$ will be chose later. We define

$$V_R(t) = \int \left(\sum_{k=1}^{l} \frac{\alpha_k^2}{\gamma_k} |u_{ck}|^2\right) \psi(x) dx.$$

By Proposition 2.33, we deduce

$$V_R'(t) = 2 \sum_{k=1}^{l} \alpha_k \mathrm{Im} \int \phi'\left(\frac{|x|^2}{R^2}\right) \nabla u_{ck} \overline{u_{ck}} dx.$$

It follows from Theorem 4.9 that $\mathbf{u}_c \in \mathbf{L}_t^\infty \mathbf{L}_x^2$. By Hölder's inequality and (1.32),

$$|V_R'(t)| = 2 \left|\sum_{k=1}^{l} \alpha_k \mathrm{Im} \int \phi'\left(\frac{|x|^2}{R^2}\right) \bar{u}_{ck} \nabla u_{ck} dx\right| \lesssim RK(\mathbf{u}_c)Q(\mathbf{u}_c) \lesssim R, \tag{4.131}$$

for all $t \in \mathbb{R}$ and $R > 0$. Using (2.35), Lemma 2.14, and the fact that, for $|x| \leqslant R$, we have $\partial_j \partial_i \phi(x) = 2\delta_{ij}$, $\Delta\phi(x) = 12$, $\Delta^2\phi(x) = 0$, we obtain

$$\begin{aligned}
V_R''(t) &= 4 \sum_{1 \leqslant m,j \leqslant 6} \mathrm{Re} \int \frac{\partial^2 \varphi}{\partial x_m \partial x_j} \left[\sum_{k=1}^{l} \gamma_k \partial_{x_j} \bar{u}_k \partial_{x_m} u_k\right] dx - \int \Delta^2 \varphi \left(\sum_{k=1}^{l} \gamma_k |u_k|^2\right) dx \\
&\quad - 2\mathrm{Re} \int \Delta\varphi F(\mathbf{u}) dx + 8 \sum_{k=1}^{l} \int_{|x|>R} \gamma_k |\nabla u_k|^2 dx - 8 \sum_{k=1}^{l} \int_{|x|>R} \gamma_k |\nabla u_k|^2 dx \\
&\quad + 24\mathrm{Re} \int_{|x|>R} F(u) dx - 24\mathrm{Re} \int_{|x|>R} F(u) dx \\
&= 8[K(\mathbf{u}_c) - 3P(\mathbf{u}_c)] + O\left(\int_{|x| \geqslant R} \sum_{k=1}^{l} |\nabla u_{ck}|^2 + |u_{ck}|^3 dx\right) \\
&\quad + O\left(\int_{R \leqslant |x| \leqslant 2R} \sum_{k=1}^{l} |u_{ck}|^3 dx\right)^{\frac{2}{3}}.
\end{aligned}$$

If for any $T_0 < T_1$, we choose

$$R = C(\eta) + \sup_{T_0 \leqslant t \leqslant T_1} |x(t)|,$$

then $|x| \geqslant R$ implies $|x - x(t)| \geqslant C(\eta)$ and, consequently, we may control the last two terms using (4.129). Taking $\eta > 0$ sufficiently small, by the conservation of energy, Lemma 2.30 and Lemma 2.31

$$V_R''(t) \gtrsim K(\mathbf{u}_c) \gtrsim E(\mathbf{u}_{c0}). \tag{4.132}$$

Applying the fundamental theorem of calculus in $[T_0, T_1]$, by (4.131), (4.132) and (4.130), we deduce

$$\begin{aligned}(T_1 - T_0)E(\mathbf{u}_{c0}) &\lesssim V_R'(T_1) - V_R'(T_0) \lesssim |V_R'(T_1)| + |V_R'(T_0)| \\ &\lesssim R = C(\eta) + \sup_{T_0 \leqslant t \leqslant T_1} |x(t)| \\ &\lesssim C(\eta) + \eta T_1, \quad \forall T_1 > T_0.\end{aligned}$$

Setting first η small enough and then making $T_1 \to \infty$ we get $E(\mathbf{u}_{c0}) = 0$. By the conservation of energy and Lemma 2.31 $E(\mathbf{u}_c(t)) = 0$, for all $t \in \mathbb{R}$, that is, $\mathbf{u}_c \equiv 0$, which contradicts $S_{\mathbb{R}}(\mathbf{u}_c) = \infty$. □

4.7 Low-to-high frequency cascade

In this part, we use negative regularity and some compactness properties to preclude the low-to-high frequence cascade.

Theorem 4.22. *There is no solution to* (1.2) *that is low-to-high frequency cascade, in the sense of Proposition 1.18.*

Proof. Let $\mathbf{u}_c : \mathbb{R} \times \mathbb{R}^6 \longrightarrow \mathbb{C}^l$ be a low-to-high frequency cascade solution. By negative regularity, we know that $\mathbf{u}_c \in \mathbf{L}_t^\infty \mathbf{L}_x^2$. By the mass conservation, we have for $t \in \mathbb{R}$,

$$0 \leqslant Q(\mathbf{u}_{c0}) = Q(\mathbf{u}_c(t)) := \sum_{k=1}^{l} \frac{\alpha_k^2}{\gamma_k} \|u_{ck}\|_{L^2}^2 < \infty,$$

or, equivalently

$$\|\mathbf{u}_c(t)\|_{\mathbf{L}^2}^2 < \infty, \quad \forall t \in \mathbb{R}.$$

Fixing $t \in \mathbb{R}$ and choosing $\eta > 0$ sufficiently small, according to Remark 1.15, we have

$$\sum_{k=1}^{l} \int_{|\xi| \leqslant C(\eta)N(t)} \gamma_k |\xi|^2 |\hat{u}_{ck}|^2 d\xi < \eta. \tag{4.133}$$

On the other hand, since $\mathbf{u}_c \in \mathbf{L}_t^\infty \dot{\mathbf{H}}_x^{-\epsilon}$ for some $\epsilon > 0$, we know that

$$\sum_{k=1}^{l} \int_{|\xi| \leqslant C(\eta)N(t)} \gamma_k |\xi|^{-2\epsilon} |\hat{u}_{ck}|^2 d\xi \lesssim 1. \tag{4.134}$$

By Hölder's inequality,

$$\begin{aligned}\sum_{k=1}^{l} \int_{|\xi| \leqslant C(\eta)N(t)} |\hat{u}_{ck}|^2 d\xi &\lesssim \sum_{k=1}^{l} \int_{|\xi| \leqslant C(\eta)N(t)} (|\xi||\hat{u}_{ck}|)^{\frac{2\epsilon}{\epsilon+1}} \left(|\xi|^{-\epsilon}|\hat{u}_{ck}|\right)^{\frac{2}{\epsilon+1}} d\xi \\ &\lesssim \left(\sum_{k=1}^{l} \int_{|\xi| \leqslant C(\eta)N(t)} \left((|\xi||\hat{u}_{ck}|)^{\frac{2\epsilon}{\epsilon+1}}\right)^{\frac{\epsilon+1}{\epsilon}} d\xi\right)^{\frac{\epsilon}{\epsilon+1}} \left(\sum_{k=1}^{l} \int_{|\xi| \leqslant C(\eta)N(t)} \left((|\xi|^{-\epsilon}|\hat{u}_{ck}|)^{\frac{2}{\epsilon+1}}\right)^{\epsilon+1} d\xi\right)^{\frac{1}{\epsilon+1}} \\ &\lesssim \eta^{\frac{\epsilon}{\epsilon+1}}.\end{aligned} \tag{4.135}$$

Moreover, by the fact that $\mathbf{u}_c$ has minimum kinetic energy, we deduce

$$\begin{aligned}\sum_{k=1}^{l}\int_{|\xi|\geqslant C(\eta)N(t)}|\hat{u}_{ck}|^2d\xi &\lesssim [C(\eta)N(t)]^{-2}\sum_{k=1}^{l}\int|\xi|^2|\hat{u}_{ck}|^2d\xi\\ &\lesssim [C(\eta)N(t)]^{-2}K(\mathbf{u}_c(t))\\ &\lesssim [C(\eta)N(t)]^{-2}K(\boldsymbol{\psi}).\end{aligned} \tag{4.136}$$

Combining (4.135) with (4.136) and using Plancherel's identity, we may estimate

$$0\leqslant Q(\mathbf{u}_c)\lesssim \eta^{\frac{\epsilon}{1+\epsilon}}+[C(\eta)N(t)]^{-2},\quad \forall t\in\mathbb{R}.$$

From definition of low-to-high frequency cascade, we are able to find a sequence $\{t_n\}\subset\mathbb{R}$ such that $t_n\to\infty$ and $N(t_n)\to\infty$ when $n\to\infty$. Thus,

$$0\leqslant \lim_{n\to\infty}Q(\mathbf{u}_c(t_n))\lesssim \eta^{\frac{\epsilon}{1+\epsilon}}.$$

Making $\eta\to 0$, we obtain $Q(\mathbf{u}(t_n))\to 0$ as $n\to\infty$, which implies $\mathbf{u}_c\equiv 0$, contradicting $S_{\mathbb{R}}(\mathbf{u}_c)=\infty$. □

4.8 Scattering and blow-up

This section is devoted to prove Corollary 1.11 and Theorem 1.12.

Proof of Corollary 1.11. Suppose that $I=(T_*,T^*)$. If $T_*,T^*<\infty$, then by Theorem 4.5 we have that $S_I(\mathbf{u})=\infty$. But this contradicts the fact that by Theorem 1.10 $S_I(\mathbf{u})<\infty$. So $I=\mathbb{R}$.

Now, for the scattering, we will only prove the statement for $\mathbf{u}^+$, since the $\mathbf{u}^-$ is analogous. Let us start by constructing the scattering state $\mathbf{u}^+$. This will be done by showing that $\mathbf{v}(t)$, where $v_k(t)=U_k(-t)u_k(t)$ for $t>0$ and $k=1,...,l$, converges in $\dot{\mathbf{H}}^1_x$ as $t\to\infty$, and then set $\mathbf{u}^+$ to be the limit. We start applying Duhamel's formula (1.26), for $k=1,...,l$, to obatin

$$v_k(t)=u_k(0)-i\int_0^t U_k(-s)\frac{1}{\alpha_k}f_k(\mathbf{u})ds. \tag{4.137}$$

Therefore, for $0<\tau<t$, $k=1,...,l$

$$v_k(t)-v_k(\tau)=-i\int_\tau^t U_k(-s)\frac{1}{\alpha_k}f_k(\mathbf{u})ds.$$

Then, by Strichartz's inequality, Lemma 2.11 and Hölder's inequality, we have for $k=1,..,l$,

$$\begin{aligned}\|v_k(t)-v_k(\tau)\|_{\dot{H}^1_x} &\lesssim \|\nabla(v_k(t)-v_k(\tau))\|_{L^\infty_tL^2_x}\\ &\lesssim \|\nabla f_k(\mathbf{u})\|_{L^2_tL^{3/2}_x([\tau,t]\times\mathbb{R}^6)}\\ &\lesssim \|\mathbf{u}\|_{\mathbf{L}^4_{t,x}([\tau,t]\times\mathbb{R}^6)}\|\mathbf{u}\|_{\mathbf{L}^4_t\mathbf{L}^{12/5}_x([\tau,t]\times\mathbb{R}^6)}\\ &\lesssim \|\mathbf{u}\|_{\mathbf{L}^4_{t,x}([\tau,t]\times\mathbb{R}^6)}\|\mathbf{u}\|_{\mathcal{S}^1([\tau,t]\times\mathbb{R}^6)}.\end{aligned} \tag{4.138}$$

Hence,

$$\|\mathbf{v}(t)-\mathbf{v}(\tau)\|_{\dot{\mathbf{H}}^1_x} \lesssim \|\mathbf{u}\|_{\mathbf{L}^4_{t,x}([\tau,t]\times\mathbb{R}^6)}\|\mathbf{u}\|_{\mathcal{S}^1([\tau,t]\times\mathbb{R}^6)}.$$

However, (1.29) implies that there is a constant $L>0$ such that $S_{\mathbb{R}}(\mathbf{u}) \leqslant L$ and then, by the same argument as in Lemma 4.3, we have $\|\mathbf{u}\|_{\mathcal{S}^1(\mathbb{R}\times\mathbb{R}^6)} \lesssim C(E,L)$, where E denotes the kinetic energy of the initial data $\mathbf{u}_0$. Also, by (1.29), for any $\eta>0$, there exists $t_\eta \in \mathbb{R}^+$ such that

$$\|\mathbf{u}\|_{\mathbf{L}^4_{t,x}([t,\infty)\times\mathbb{R}^6)} \lesssim \eta,$$

whenever $t>t_\eta$. Therefore,

$$\|\mathbf{v}(t)-\mathbf{v}(\tau)\|_{\dot{\mathbf{H}}^1_x} \to 0 \quad \text{as} \quad t,\tau\to\infty.$$

In particular, this implies that $\mathbf{u}^+$ is well defined. Also, looking at (4.137), one can see that, for $k=1,...,l$

$$u_k^+ = u_k(0) - i\int_0^\infty U_k(-s)\frac{1}{\alpha_k}f_k(\mathbf{u})ds \tag{4.139}$$

and thus

$$U_k(t)u_k^+ = U_k(t)u_k(0) - i\int_0^\infty U_k(t-s)\frac{1}{\alpha_k}f_k(\mathbf{u})ds. \tag{4.140}$$

By the same arguments as above, (4.140) and Duhamel's formula (1.26) imply that

$$\|\mathbf{u}(t)-\mathbf{U}(t)\mathbf{u}^+\|_{\dot{\mathbf{H}}^1_x} \to 0 \quad \text{as} \quad t\to\infty,$$

which completes the proof of Corollary 1.11. □

Now we turn our attention to Theorem 1.12. As we said before, the radial case was already considered in Theorem 4.1. (ii) of (NOGUERA; PASTOR, 2022). Therefore, it is left to prove the case $x\mathbf{u}_0 \in \mathbf{L}^2$.

Proof of Theorem 1.12. Suppose $x\mathbf{u}_0 \in \mathbf{L}^2$. Define

$$\tau(\mathbf{u}) = K(\mathbf{u}(t)) - 3P(\mathbf{u}(t)).$$

By definition of the energy

$$\tau(\mathbf{u}(t)) = \frac{3}{2}E(\mathbf{u}(t)) - \frac{1}{2}K(\mathbf{u}(t)).$$

It was shown in (NOGUERA; PASTOR, 2022), Lemma 4.4, that there exists $\delta>0$ such that $\tau(\mathbf{u}(t)) \leqslant -\delta < 0$. Besides, notice that defining

$$V(t) = \sum_{k=1}^{l}\frac{\alpha_k^2}{\gamma_k}\|xu_k(t)\|_{L^2}^2 = \sum_{k=1}^{l}\frac{\alpha_k^2}{\gamma_k}\int |x|^2|u_k(t,x)|^2dx,$$

by Proposition 2.32, we have

$$\begin{aligned} V''(t) &= 12E(\mathbf{u}(t)) - 4K(\mathbf{u}(t)) \\ &= 8\tau(\mathbf{u}(t)) \\ &\leqslant -8\delta. \end{aligned}$$

Hence, the graph of V lies under a parabola that is concave downward and, therefore, the solution $\mathbf{u}$ blows-up in both directions. □

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APPENDIX A

APPENDIX

A.1 Almost Periodic Solutions

For completeness of the work, we present here some basic facts about the frequency scale function $N(t)$ that were needed in the proof of Proposition 1.18. We reproduce here the proofs established in (KILLIP; TAO; VISAN, 2009). We start with the following definition.

Definition A.1. *(Convergence of solutions). Let* $\mathbf{u}^{(n)} : I^{(n)} \times \mathbb{R}^d \to \mathbb{C}$ *be a sequence of solutions to* (1.2)*, let* $\mathbf{u} : I \times \mathbb{R}^d \to \mathbb{C}$ *be another solution, and let* K *be a compact time interval. We say that* $\mathbf{u}^{(n)}$ *converges uniformly to* $\mathbf{u}$ *on* K *if we have* $K \subset I$ *and* $K \subset I^{(n)}$ *for all sufficiently large* n*, and furthermore,* $\mathbf{u}^{(n)}$ *converges strongly to* $\mathbf{u}$ *in* $\mathbf{L}_t^\infty \mathbf{H}_x^1(K \times \mathbb{R}^6) \cap \mathbf{L}_t^4 \mathbf{H}_x^{1,\frac{12}{5}}(K \times \mathbb{R}^6)$ *as* $n \to \infty$*. We say that* $\mathbf{u}^{(n)}$ *converges locally uniformly to* $\mathbf{u}$ *if* $\mathbf{u}^{(n)}$ *converges uniformly to* $\mathbf{u}$ *on every compact interval* $K \subset I$*.*

The first result about the frequency scale function is the following.

Lemma A.2. *(Quasi-uniquess of N) Let* $\mathbf{u}$ *be a non-zero solution to* (1.2) *with lifespan* I *that is almost periodic modulo symmetries with frequency scale function* $N : I \longrightarrow \mathbb{R}^+$ *and compact modulus function* $C : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *and also almost periodic modulo symmetries with frequency scale function* $N' : I \longrightarrow \mathbb{R}^+$ *and compact modulus function* $C' : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$*. Then we have*

$$N(t) \sim N'(t),$$

for all $t \in I$*.*

Proof. By symmetry, it suffices to establish the bound $N'(t) \lesssim N(t)$. We write $x'(t)$ for the spatial center function associated to N' and C'. To begin, fix t and let $\eta > 0$ to be choosen later. By Definition 1.14, for $k = 1, ..., l$, we have

$$\int_{|x-x'(t)|\geqslant C'(\eta)/N'(t)} |\nabla u_k(t,x)|^2 dx \lesssim \eta$$

and

$$\int_{|\xi|\geqslant C(\eta)N(t)} |\xi|^2 |\hat{u}_k(t,\xi)|^2 d\xi \lesssim \eta.$$

We split $u_k(t,x) = u_{k1}(t,x) + u_{k2}(t,x)$, where $u_{k1}(t,x) = u_k(t,x)\chi_{|x-x'(t)|\geqslant C'(\eta)/N'(t)}$ and $u_{k2}(t,x) = u_k(t,x)\chi_{|x-x'(t)|<C'(\eta)/N'(t)}$. Then, by Plancherel's theorem we have

$$\int_{\mathbb{R}^6} |\xi|^2 |\widehat{u}_{k1}(t,\xi)|^2 \lesssim \eta, \tag{A.1}$$

while by the Cauchy-Schwarz inequality we have

$$\sup_{\xi\in\mathbb{R}^6} |\xi|^2 |\widehat{u}_{k2}(t,\xi)|^2 \lesssim E(\mathbf{u})N'(t)^{-6}.$$

Integrating the last inequality over the ball $|\xi| \leqslant C(\eta)N(t)$ and using (A.1), we conclude that

$$\int_{\mathbb{R}^6} |\xi|^2 |\hat{u}_k(t,\xi)|^2 d\xi \lesssim \eta + O(E(\mathbf{u})N(t)^6 N'(t)^{-6}).$$

Then, by the Plancherel theorem and energy conservation,

$$E(\mathbf{u}) \lesssim \eta + O(E(\mathbf{u})N(t)^6 N'(t)^{-6}).$$

Choosing η to be small multiple of $E(\mathbf{u})$, we get the result. □

Lemma A.3. *(Quasi-continuous dependence of N on $\mathbf{u}$). Let $\mathbf{u}^{(n)}$ be a sequence of solutions to (1.2) with lifespans $I^{(n)}$, which are almost periodic modulo symmetries with frequency scale function $N^{(n)} : I^{(n)} \longrightarrow \mathbb{R}^+$ and compactness modulus functions $C : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, independent of n. Suppose that $\mathbf{u}^{(n)}$ converge locally uniformly to a non-zero solution $\mathbf{u}$ to (1.2) with lifespan I. Then $\mathbf{u}$ is almost periodic modulo symmetries with frequency scale function $N : I \longrightarrow \mathbb{R}^+$ and compactness modulus function C. Furthermore, we have*

$$N(t) \sim \liminf_{n\to\infty} N^{(n)}(t) \sim \limsup_{n\to\infty} N^{(n)}(t), \tag{A.2}$$

for all $t \in I$.

Proof. We first show that

$$0 < \liminf_{n\to\infty} N^{(n)}(t) \leqslant \limsup_{n\to\infty} N^{(n)}(t) < \infty, \tag{A.3}$$

for all $t \in I$. Indeed, if one of these inequalities fail for some t, the (by passing to a subsequence if necessary) $N^{(n)}(t)$ would converge to zero or to infinity as $n \to \infty$. Thus, by Definition 1.14, $\mathbf{u}^{(n)}(t)$ would converge weakly to zero, and hence, by the local uniform convergence, would converge strongly to zero. But, this contradicts the hypothesis that $\mathbf{u}$ is not identically zero. This establishes (A.3).

From (A.3), we see that for each $t \in I$ the sequence $N^{(n)}(t)$ has at least one limit point $N(t)$. Thus, using the local uniform convergence we easily verify that $\mathbf{u}$ is almost periodic modulo scaling with frequency scale function N and compactness modulus function C.

It remains to establish (A.2), which we prove by contradiction. Suppose it fails. Then given any $A = A_{\mathbf{u}}$, there exists a $t \in I$ for which $N^{(n)}(t)$ has at least two limit points which are separated by a ratio of at least A, and so $\mathbf{u}$ has two frequency scale functions with compactness modulus function C which are separated by this ratio. But this contradicts Lemma A.2 for A large enough depending on $\mathbf{u}$. Hence (A.2) holds. □

Lemma A.4. *(Compactness of almost periodic solutions) Let $\mathbf{u}^{(n)}$ be a sequence of solutions to (1.2) with lifespans $I^{(n)} \ni 0$, which are almost periodic modulo symmetries with frequency scale function $N^{(n)} : I^{(n)} \longrightarrow \mathbb{R}^+$ and compactness modulus functions $C : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$. Assume that we also have a uniform energy bound*

$$0 < \inf_n E(\mathbf{u}^{(n)}) \leqslant \sup_n E(\mathbf{u}^{(n)}) < \infty. \tag{A.4}$$

Then, up to a subsequence, there exists a non-zero maximal solution $\mathbf{u}$ to (1.2) which is almost periodic modulo symmetries such that $\mathbf{u}^{(n)}$ converge locally uniformly to $\mathbf{u}$.

Proof. By hyphotesis and Definition 1.14 we see that for every $\epsilon > 0$ there exists $R > 0$ such that

$$\int_{|x|\geqslant R} |\nabla u_k^{(n)}(0,x)|^2 dx \lesssim \epsilon$$

and

$$\int_{|\xi|\geqslant R} |\xi|^2 |\widehat{u}_k^{(n)}(0,\xi) d\xi \lesssim \epsilon,$$

for all n. From this, (A.4), and the Ascoli-Arzela Theorem, we see that the sequence $\mathbf{u}^{(n)}(0)$ is precompact in the strong topology of $\dot{\mathbf{H}}^1_x(\mathbb{R}^6)$. Thus, by passing to a subsequence if necessary, we can find $\mathbf{u}_0 \in \dot{\mathbf{H}}^1_x(\mathbb{R}^6)$ such that $\mathbf{u}^{(n)}(0)$ converge strongly to $\mathbf{u}_0$ in $\dot{\mathbf{H}}^1_x(\mathbb{R}^6)$. Again, by (A.4) we see that $\mathbf{u}_0$ is not identically zero. Now let $\mathbf{u}$ be the maximal solution to (1.2) corresponding to $\mathbf{u}_0$, with lifespan I. By Theorem 4.3, $\mathbf{u}^{(n)}$ converge locally uniformly to $\mathbf{u}$. □

Let $\mathbf{u}$ be a solution to (1.2) with lifespan $I \ni 0$, which is almost periodic modulo symmetries, with frequency scale function N and position center function x. We say that $\mathbf{u}$ is normalized if

$$N(0) = 1, \quad x(0) = 0.$$

We can define the normalization of $\mathbf{u}$ at time $t_0 \in I$ by

$$\mathbf{u}^{[t_0]} := T_{g_{-x_{\mathbf{u}(t_0)}N_{\mathbf{u}(t_0)}},N_{\mathbf{u}(t_0)}}(\mathbf{u}(\cdot + t_0)) = N^{-2}_{\mathbf{u}(t_0)}\mathbf{u}(N^{-2}_{\mathbf{u}(t_0)}t + t_0, N^{-1}_{\mathbf{u}(t_0)}(x + x_{\mathbf{u}(t_0)}N_{\mathbf{u}(t_0)})) \quad (A.5)$$

Observe that $\mathbf{u}^{t_0}$ is a normalized solution which is almost periodic modulo symmetries with lifespan

$$I^{[t_0]} := \{s \in \mathbb{R};\ t_0 + sN(t)^{-2} \in I\},$$

frequency scale and center spatial functions given by, respectively,

$$N_{\mathbf{u}^{t_0}}(t) = \frac{N_{\mathbf{u}}(t_0 + tN_{\mathbf{u}}(t_0)^{-2})}{N_{\mathbf{u}}(t_0)} \quad \text{and} \quad x_{\mathbf{u}^{t_0}}(t) = N_{\mathbf{u}}(t_0)[x_{\mathbf{u}}(t_0+tN_{\mathbf{u}}(t_0)^{-2})-x_{\mathbf{u}}(t_0)]. \quad (A.6)$$

and the same compactness modulus function as $\mathbf{u}$. Moreover, if $\mathbf{u}$ is maximal solution, then $\mathbf{u}(t_0)$ also is maximal solution.

Lemma A.5. *Let $\mathbf{u}$ be a non-zero maximal solution to (1.2) with lifespan I that is almost periodic modulo symmetries with frequency scale function $N : I \longrightarrow \mathbb{R}^+$. Then there exists $\delta > 0$, depending on $\mathbf{u}$ such that for every $t_0 \in I$ we have*

$$[t_0 - \delta N(t_0)^{-2}, t_0 + \delta N(t_0)^{-2}] \subset I \quad (A.7)$$

and

$$N(t) \sim N(t_0), \quad (A.8)$$

whenever $|t - t_0| \leqslant \delta N(t_0)^{-2}$.

Proof. Let us first establish (A.7). Assume that it fails. So, there exists sequences $t_n \in I$ and $\delta_n \to 0$ such that $t_n + \delta_n N(T_n)^{-2} \notin I$ for all n. Define the normalization $\mathbf{u}^{[t_n]}$ of $\mathbf{u}$ by (A.5). Then $\mathbf{u}^{[t_n]}$ are maximal normalized solutions where $I^{[t_n]}$ contain 0 but not δ_n. They are also almost periodic modulo symmetries with frequency scale functions $N^{[t_n]}$ given by

$$N^{[t_n]}(s) := N(t_n + sN(t_n)^{-2})/N(t_n) \quad (A.9)$$

and the same compactness modulus function as $\mathbf{u}$. By Lemma A.4, passing to a subsequence if necessary, we conclude that by Theorem 1.9, J is open and so contains δ_n for all sufficiently large n. This contradicts the local uniform convergence since, by hypothesis, δ_n does not belong to $I^{[t_n]}$. Hence (A.7) holds.

Now, we proceed to show (A.8). Again, assume that it is false no matter how small δ is. Then, we may find sequences $t_n, t'_n \in I$ such that $s_n := (t'_n - t_n)N(t_n)^2 \to 0$ but $N(t'_n)/N(t_n)$ converge to either zero or infinity. If we define $\mathbf{u}^{[t_n]}$ and $N^{[t_n]}$ as before and apply Lemma A.4, once again $\mathbf{u}^{[t_n]}$ converge locally uniformly to a maximal solution $\mathbf{v}$ with lifespan $J \ni 0$. But, then $N^{[t_n]}(s_n)$ converge to either zero or infinity. Hence, Definition 1.14 gives us that $\mathbf{u}^{[t_n]}(s_n)$ are converging weakly to 0. On the other hand, since $s_n \to 0$ and $\mathbf{u}^{[t_n]}$ are locally uniformly convergent to $\mathbf{v}$, we may conclude that $\mathbf{u}^{[t_n]}(s_n)$ converge strongly to $\mathbf{v}(0)$ in $\dot{\mathbf{H}}^1_x$. Therefore, $\mathbf{v}(0) = 0$ and $E(\mathbf{u}^{[t_n]})$ converge to $E(\mathbf{v}) = 0$. By conservation of energy, $\mathbf{u}$ must vanishes, which is a contradiction. So, (A.8) holds. ☐

Corollary A.6. *(Blow-up criterion). Let* $\mathbf{u}$ *be a non-zero maximal solution to* (1.2) *that is almost periodic modulo symmetries with frequency scale function* $N : I \to \mathbb{R}^+$. *If* T *is a finite endpoint of* I, *then* $N(t) \gtrsim |T-t|^{-1/2}$*; in particular,* $\lim_{t\to T} N(t) = \infty$.

Proof. Suppose without loss of generality that $T = \sup I$. By (A.7) we have that for $t \in I$,

$$|T-t| \geqslant |t - t + \delta N(t)^{-2}| = \delta N(t)^{-2} \Leftrightarrow N(t) \gtrsim |T-t|^{-1/2}.$$

In particular, $\lim_{t\to T} N(t) = \infty$. □

A.2 Compactness of almost periodic modulo symmetries

This section is devoted to discuss some compactness properties of almost periodic modulo symmetries functions. We start with the following definition.

Definition A.7. *A subset* A *of a metric space* X *is called totally bounded (pre-compact) if admits a finite cover consisting of open sets of diameter at most* ϵ, *for any* $\epsilon > 0$.

The next theorem gives us sufficient and necessary conditions for a subset of p-integrable functions space to be totally bounded.

Theorem A.8. *(Kolmogorov-Riesz-Sukadov). Let* $1 \leqslant p < \infty$. *A subset* $\mathcal{F}$ *of* $L^p(\mathbb{R}^6)$ *is totally bounded if, and only if,*

(i) for every $\epsilon > 0$ *there is* $R > 0$ *such that, for every* $f \in \mathcal{F}$,

$$\int_{|x|>R} |f(x)|^p dx < \epsilon^p,$$

(ii) for every $\epsilon > 0$ *there is* $\rho > 0$ *such that, for every* $f \in \mathcal{F}$ *and* $y \in \mathbb{R}^d$ *with* $|y| < \rho$,

$$\int_{\mathbb{R}^6} |f(x+y) - f(x)|^p dx < \epsilon^p.$$

Proof. See Theorem 1 in (HANCHE-OLSEN; HOLDEN; MALINNIKOVA, 2019). □

From the above Theorem, we can derive a similar result about totally bounded subsets of $\dot{H}^1(\mathbb{R}^6)$.

Corollary A.9. *A subset* $\mathcal{F}$ *of* $\dot{H}^1(\mathbb{R}^6)$ *is totally bounded if, and only if,*

(C1) for every $\epsilon > 0$ *there is* $R > 0$ *such that, for every* $f \in \mathcal{F}$,

$$\int_{|x|>R} |\nabla f(x)|^2 dx < \epsilon^2,$$

(C2) for every $\epsilon > 0$ there is $\rho > 0$ such that, for every $f \in \mathcal{F}$ and $y \in \mathbb{R}^6$ with $|y| < \rho$,

$$\int_{\mathbb{R}^6} |\nabla f(x+y) - \nabla f(x)|^2 dx < \epsilon^2.$$

Proof. Note that $\mathcal{F}$ is totally bounded in $\dot{H}^1(\mathbb{R}^6)$ if, and only if, the set $\{\nabla f, f \in \mathcal{F}\}$ is totally bounded in $L^2(\mathbb{R}^6)$. Hence, the result follows from Theorem A.8. □

The next corollary states alternative conditions for totally boundness of subset $\mathcal{F} \subset \dot{H}^1(\mathbb{R}^6)$.

Corollary A.10. *Let $\mathcal{F}$ be a bounded subset of $\dot{H}^1(\mathbb{R}^6)$. Then, $\mathcal{F}$ is totally bounded if, and only if,*

$$\lim_{r\to\infty} \sup_{f\in\mathcal{F}} \int_{|x|>r} |\nabla f(x)|^2 dx = 0 \tag{A.10}$$

and

$$\lim_{\rho\to\infty} \sup_{f\in\mathcal{F}} \int_{|\xi|>\rho} |\xi|^2 |\hat{f}(\xi)|^2 d\xi = 0 \tag{A.11}$$

Proof. Suppose that (A.10) and (A.11) hold. By Corollary A.9, it is sufficient to prove that $(C1)$ and $(C2)$ holds. Observe that $(C1)$ follows directly from the limit in (A.10). For $(C2)$, fix $\rho > 0$. By Plancherel's theorem,

$$\begin{aligned}
\int_{\mathbb{R}^6} |\nabla f(x+y) - \nabla f(x)|^2 dx &= \int_{\mathbb{R}^6} |\widehat{\nabla f(\cdot + y)}(\xi) - \widehat{\nabla f}(\xi)|^2 d\xi \\
&= \int_{\mathbb{R}^6} |\xi|^2 |e^{iy\xi}\hat{f}(\xi) - \hat{f}(\xi)|^2 d\xi \\
&= \int_{\mathbb{R}^6} |\xi|^2 |e^{iy\xi} - 1|^2 |\hat{f}(\xi)|^2 d\xi \\
&\leqslant \int_{|\xi|<\rho} |\xi|^2 |e^{iy\xi} - 1|^2 |\hat{f}(\xi)|^2 d\xi + 4\int_{|\xi|\geqslant\rho} |\xi|^2 |\hat{f}(\xi)|^2 d\xi.
\end{aligned}$$

By (A.11), for every $\epsilon > 0$, there exist $\rho > 0$ large enough such that for all $f \in \mathcal{F}$,

$$\int_{|\xi|\geqslant\rho} |\xi|^2 |\hat{f}(\xi)|^2 d\xi < \frac{\epsilon}{8}.$$

Moreover, since $\mathcal{F}$ is bounded, if $M > 0$ is such that $\|f\|_{\dot{H}^1} < M$, for all $f \in \mathcal{F}$, then

$$\int_{\mathbb{R}^6} |\nabla f(x+y) - \nabla f(x)|^2 dx \leqslant M^2 \sup_{|\xi|<\rho} |e^{iy\xi} - 1|^2 + \frac{\epsilon^2}{2}.$$

Now, using the fact that $|e^{i\theta} - 1| \leqslant |\theta|$, $\forall \theta \in \mathbb{R}$,

$$\int_{\mathbb{R}^6} |\nabla f(x+y) - \nabla f(x)|^2 dx \leqslant M^2 \sup_{|\xi|<\rho} |y\xi|^2 + \frac{\epsilon^2}{2} \leqslant M^2 |y|^2 \rho^2 + \frac{\epsilon^2}{2} \leqslant \epsilon^2,$$

provided $|y| < \dfrac{\epsilon}{M\rho\sqrt{2}} := \delta$. Then, $\mathcal{F}$ is totally bounded.

Now, suppose that $\mathcal{F}$ is totally bounded. By Corollary A.9, $(C2)$ holds, and then (A.10) follows immediately. It remains to show that (A.11) holds. To this end, we will follow the ideas presented in (PEGO, 1985), Theorem 4. First, observe that $(C2)$ is equivalent to

$$\lim_{y\to 0}\sup_{f\in\mathcal{F}}\int_{\mathbb{R}^6}|\nabla f(x+y)-\nabla f(x)|^2dx=0. \tag{A.12}$$

Let $\psi(x)=(2\pi)^{-d/2}e^{-|x|^2/2}$ and set $\psi_\rho(x)=\rho^6\psi(\rho x)$, $\rho>0$. Then, ψ and $\widehat{\psi}(\xi)=e^{-|\xi|^2/2}$ lies on Schwartz space and $\widehat{\psi}(0)=\displaystyle\int_{\mathbb{R}^6}\psi_\rho(x)dx=1$. Also, observe that for $|\xi|\geqslant 2\rho$, we have $\dfrac{1}{2}\leqslant 1-\widehat{\psi}_\rho(\xi)$, and then, for each $f\in\mathcal{F}$, by Plancherel's theorem,

$$\begin{aligned}\frac{1}{2}\left(\int_{|\xi|\geqslant 2\rho}|\xi|^2|\widehat{f}(\xi)|^2d\xi\right)^{1/2}&\leqslant\left(\int_{|\xi|\geqslant 2\rho}|\xi|^2|(1-\widehat{\psi}_\rho(\xi))\widehat{f}(\xi)|^2d\xi\right)^{1/2}\\&\leqslant\left(\int_{\mathbb{R}^6}|\nabla f(x)-\psi_\rho*\nabla f(x)|^2dx\right)^{1/2}\\&=\left(\int_{\mathbb{R}^6}\left|\int_{\mathbb{R}^6}(\nabla f(x)-\nabla f(x-y))\psi_\rho(y)dy\right|^2dx\right)^{1/2}.\end{aligned}$$

Since $\widehat{\psi}_\rho(0)=1$, Jensen's inequality applied to $t\to t^2$ together with Fubini's theorem,

$$\begin{aligned}\frac{1}{2}\left(\int_{|\xi|\geqslant 2\rho}|\xi|^2|\widehat{f}(\xi)|^2d\xi\right)^{1/2}&\leqslant\left(\int_{\mathbb{R}^6}\left(\int_{\mathbb{R}^6}|\nabla f(x)-\nabla f(x-y)|^2\psi_\rho(y)dy\right)dx\right)^{1/2}\\&=\left(\int_{\mathbb{R}^6}\left(\int_{\mathbb{R}^6}\left|\nabla f(x)-\nabla f\left(x-\frac{y}{\rho}\right)\right|^2dx\right)\psi(y)dy\right)^{1/2}\\&\leqslant\left(\int_{\mathbb{R}^6}H\left(\frac{y}{\rho}\right)\psi(y)dy\right)^{1/2},\end{aligned}$$

where H is the continuity modulo function in L^2 for $\mathcal{F}$, that is,

$$H(z)=\sup_{f\in\mathcal{F}}\int_{\mathbb{R}^6}|\nabla f(x-z)-\nabla f(x)|^2dx.$$

By (A.12), we have $H(y/\rho)\to 0$ as $\rho\to\infty$. Furthermore, since H is bounded (beacuse $\mathcal{F}$ is bounded), the dominated convergence theorem implies that the right-hand side of the last inequality goes to zero as $\rho\to\infty$. Hence, (A.11) holds. □

The Corollary A.10 tells us that if $\mathcal{F}$ is a bounded subset of $\dot{H}^1(\mathbb{R}^6)$ and if given $\epsilon>0$, there exist $\delta>0$ such that

$$\int_{|x|\geqslant\delta}|\nabla f(x)|^2dx+\int_{|\xi|\geqslant\delta}|\xi|^2|\widehat{f}(\xi)|^2d\xi<\epsilon,\ \forall f\in\mathcal{F},$$

then $\mathcal{F}$ is totally bounded in $\dot{H}^1(\mathbb{R}^6)$. This is equivalent to say that for every $\eta>0$, there is a function $C:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$\int_{|x|\geqslant C(\eta)}|\nabla f(x)|^2dx+\int_{|\xi|\geqslant C(\eta)}|\xi|^2|\widehat{f}(\xi)|^2d\xi<\eta,\ \forall f\in\mathcal{F}.$$

With this in hand, we have the following proposition.

Proposition A.11. *A family of functions $\mathcal{F}$ is totally bounded (or pre-compact) in $\dot{H}^1(\mathbb{R}^6)$ if, and only if, it is bounded and there exists a function $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\int_{|x|\geqslant C(\eta)} |\nabla f(x)|^2 dx + \int_{|\xi|\geqslant C(\eta)} |\xi|^2 |\widehat{f}(\xi)|^2 d\xi < \eta, \ \forall \eta > 0, \ \forall f \in \mathcal{F}.$$

Now, using Definition 2.23 of symmetry group G, we recall the fact that for $g \in G$,

$$\|g\mathbf{u}\|_{\dot{\mathbf{H}}^1_x} = \|\mathbf{u}\|_{\dot{\mathbf{H}}^1_x}.$$

Also, setting the transformation $T_g\mathbf{u}(t,x) := \lambda^{-2}\mathbf{u}(\lambda^{-2}t, \lambda^{-1}(x-x_0))$, we have that the map $\mathbf{u} \mapsto T_g\mathbf{u}$ maps a solution to (1.2) into a solution with the same energy and scattering size as $\mathbf{u}$.

Definition A.12. *We say that a family of functions $\mathcal{F}$ in $\dot{H}^1_x(\mathbb{R}^6)$ is pre-compact modulo symmetries if the set $G\mathcal{F} = \{gf;\, g \in G,\ f \in \mathcal{F}\}$ is pre-compact in $\dot{H}^1_x(\mathbb{R}^6)$.*

According to Proposition A.11, the set $G\mathcal{F}$ is pre-compact if, and only if, it is bounded and there exist a function $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\int_{|x|\geqslant C(\eta)} |\nabla (gf)(x)|^2 dx + \int_{|\xi|\geqslant C(\eta)} |\xi|^2 |\widehat{gf}(\xi)|^2 d\xi < \eta, \tag{A.13}$$

for all $\eta > 0$, $f \in \mathcal{F}$ and $g = g_{x_0,\lambda} \in G$, Also, if $gf(x) = \lambda^{-2}f(\lambda^{-1}(x-x_0))$, the first term in (A.13) gives us

$$\begin{aligned}
\int_{|x|\geqslant C(\eta)} |\nabla[\lambda^{-2}f(\lambda^{-1}(x-x_0))]|^2 dx &= \int_{|x|\geqslant C(\eta)} |\lambda^{-3}\nabla f(\lambda^{-1}(x-x_0))|^2 dx \\
&= \lambda^{-6}\int_{|x|\geqslant C(\eta)} |\nabla f(\lambda^{-1}(x-x_0))|^2 dx \\
&= \lambda^{-6}\int_{|\lambda y + x_0|\geqslant C(\eta)} ||\nabla f(y)|^2 \lambda^6 dy \\
&= \int_{|x+\frac{x_0}{\lambda}|\geqslant \frac{C(\eta)}{\lambda}} |\nabla f(x)|^2 dx.
\end{aligned}$$

To the second term in (A.13), since $\widehat{gf}(\xi) = \lambda^{-2}\widehat{f(\lambda^{-1}(\cdot - x_0))}(\xi) = \lambda^{-2}e^{-ix_0\cdot\xi}\lambda^{-6}\widehat{f}(\lambda\xi)$, we have

$$\begin{aligned}
\int_{|\xi|\geqslant C(\eta)} |\xi|^2|\widehat{gf}(\xi)|^2 d\xi &= \int_{|\xi|\geqslant C(\eta)} |\xi|^2|\lambda^{-2}e^{-ix_0\cdot\xi}\lambda^{-6}\widehat{f}(\lambda\xi)|^2 d\xi \\
&= \lambda^{8}\int_{|\xi|\geqslant C(\eta)} |\xi|^2|\widehat{f}(\lambda\xi)|^2 d\xi \\
&= \lambda^{8}\int_{|\lambda^{-1}\zeta|\geqslant C(\eta)} \lambda^{-2}|\zeta|^2|\widehat{f}(\zeta)|^2\lambda^{-6} d\zeta \\
&= \int_{|\xi|\geqslant C(\eta\lambda)} |\xi|^2|\widehat{f}(\xi)|^2 d\xi.
\end{aligned}$$

Hence, the set $\mathcal{F}$ is pre-compact in $\dot{H}^1(\mathbb{R}^6)$ modulo symmetries if, and only if, there exist a function $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for any $\eta > 0$, $f \in \mathcal{F}$, $x_0 \in \mathbb{R}^6$ and $\lambda > 0$,

$$\int_{\left|x+\frac{x_0}{\lambda}\right| \geqslant \frac{C(\eta)}{\lambda}} |\nabla f(x)|^2 dx + \int_{|\xi| \geqslant C(\eta)\lambda} |\xi|^2 |\widehat{f}(\xi)|^2 d\xi < \eta.$$

Combining the above results, one can see that a solution $\mathbf{u} : I \times \mathbb{R}^6 \to \mathbb{C}$ to (1.2) is almost periodic modulo symmetries if, and only if, the orbit $\{\mathbf{u}(t);\, t \in I\} \subseteq \{\lambda^2 f(\lambda(x + x_0)) :\, \lambda \in (0, \infty),\, x_0 \in \mathbb{R}^6\, and\, f \in K\}$ for some compact subset K of $\dot{H}^1_x(\mathbb{R}^6)$.