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# MATHEMATICAL TEACHERS' SPECIALIZED KNOWLEDGE ON SUBJECTIVE PROBABILITY

CONHECIMENTO ESPECIALIZADO DE PROFESSORES DE MATEMÁTICA

SOBRE PROBABILIDADE SUBJETIVA


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# CONHECIMENTO ESPECIALIZADO DE PROFESSORES DE MATEMÁTICA SOBRE PROBABILIDADE SUBJETIVA

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## Abstract:

This thesis investigates the role of teachers' specialized knowledge in probability education, focusing on a professional development program aimed at enhancing teachers' proficiency in probabilistic concepts and their application in real-world contexts. Drawing on the subjectivist approach to probability, the study explores the decision-making processes of primary and lower secondary teachers engaged in a betting game designed to quantify the degree of confidence associated with each possible event. Using the framework of Mathematical Teachers' Specialized Knowledge (MTSK), the research examines the interplay between teachers' mathematical knowledge, pedagogical content knowledge, and their ability to navigate uncertainty within the context of probability education.

The thesis begins with an overview of classical, frequentist, and subjectivist approaches to probability, highlighting the increasing importance of probability education in national and international curricula. It then delves into the theoretical underpinnings of MTSK and its relevance to probability instruction. Through an analysis of prior research and teacher education programs, the study identifies gaps in teachers' specialized knowledge and the challenges they face in effectively teaching probability concepts.

The research methodology employs a Design Research approach, structured into two action phases: deepening teachers' Knowledge of Topic (KoT) on probability and designing a didactic path for students.

Through an analysis of teachers' behaviours and decision-making processes during the betting game, the study uncovers the complexities of probability assessment and the factors influencing teachers' confidence in assigning quotas. Insights from the analysis reveal the nuances of teachers' specialized knowledge, including their understanding of foundational concepts, procedural considerations, and the application of probabilistic reasoning in real-world scenarios.

The findings underscore the importance of intra-conceptual connections in guiding teachers' decisions and fostering a coherent understanding of probability. The subjectivist approach emerges as a significant theoretical perspective that enriches teachers' awareness of probability and informs their instructional practices. The study also highlights the potential of the betting game as educational tool for promoting probabilistic reasoning and facilitating the construction of probability measures. In conclusion, the thesis identifies key open questions in probability education and calls for further research to bridge the gap between theoretical understanding and practical application. By refining understanding of teachers' specialized knowledge and exploring effective educational strategies, we can empower students with the skills and understanding necessary to effectively explore uncertainty.

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# Introduction

Our daily life is full of events that we can't foresee for certain. Every human activity is dominated by uncertainty, resulting that a specific event can happen or not. It doesn't mean that the choice of an event should be left to the destiny, especially in specific situations (such as the launching of a new product on the market), but it is necessary to elaborate a rational evaluation, using the information that we have in that particular moment. This evaluation is largely influenced by the type of approach that the event requires. Nowadays, from the mathematical point of view, it is possible to recognize three main approaches which can be used and intertwined to model the uncertainty: classical approach, the frequentist one and the subjectivist one. Each of them is useful to mathematically face and handle, as far as possible, the unpredictability that rules our daily life.

Children are usually involved in situations where they need to act making decisions based on uncertainty, and the Probability Literacy is one of the competences that needs to be developed in order to be able to perform an active citizenship (e.g., OCSE, 2016). For this reason, for almost twenty years now, even if with the due differences and specificities according with the cultural context, most primary school curriculum includes an introduction to probability (e.g., OCSE, 2016; MIUR, 2012; MIUR, 2018; BRASIL, 2018).

In the last years the topic of Probability has become an increasingly central topic in the national and international curriculum around the world (e.g., Batanero *et al*., 2016). And, aligned with such an increase in curricular reforms, also in the field of mathematics education research is giving much more attention to this topic, focusing both on the pupils and the teachers (e.g., Ireland and Watson, 2009; Chernoff and Sriraman, 2014).

The Italian curriculum (MIUR, 2012; MIUR, 2018), for example, pose a real challenge to teachers: it is required to pass from a mathematics' teaching linked to memorization of rules and techniques to an idea of teaching mathematics where the techniques continue to have an important value, but as far as they allow to solve new problems (and not repetitive exercises). In short, it asks to move from a mathematical education focused on reproductive thought, to one that enhances and brings into play productive thought (Carotenuto, Di Bernardo & Di Martino, 2019). Saying that in a diverse way, it is necessary to develop students' procedural and conceptual knowledge (Hiebert & Lefevre, 1986), focusing both on "hows" and "whys", respectively, when dealing with mathematics. Indeed, such procedural knowledge is related to both object and mathematical symbols and is linked to symbolic operations skills. The conceptual knowledge, in turn, is sustained by establishing relationships between what one already knows and the new knowledge being developed.

A work focusing on developing students' procedural and conceptual mathematical knowledge requires a specific teacher's knowledge and involves, among others aspects, the type of tasks (Charalambous, 2010) and mathematical discussions (Ponte & Quaresma, 2016; Schoenfeld & Kilpatrick, 2008) teachers' implement on classes. This request involves the teacher mathematical knowledge, his experiences and especially his emotions towards mathematics and his teaching (Di Martino & Pezzia, 2018).

To overcome this challenge, focusing on teacher education, means not only to work with the aim to give opportunities to strengthen disciplinary's knowledge and teaching skills, but also to critically rethink about the personal experience with mathematics and, if necessary, to rebuild a relationship with mathematics: not to be afraid of mathematics and to teach it (Carotenuto, Di Bernardo, & Di Martino, 2019). In this vision, it is necessary to build examples of activities and for doing so one can consider a laboratory teaching and to have time to critically analyse it.

In recent years, probability has acquired an important role in mathematics curricula around the world with an increasing interest in research in the field of mathematical education: probability plays a crucial role in the Mathematical Literacy of individuals (e.g., OCSE, 2016; MIUR, 2012; MIUR, 2018) and probabilistic reasoning underlie many daily decision problems and interesting scientific issues.

Although the development of awareness of the assessment of a probability is something culturally and socially relevant, the research reveals difficulties on the part of the students but, given even more alarming, also on the part of the teachers (e.g., Batanero, Godino & Roa, 2004).

Some studies (e.g., Franklin & Mewborn, 2006; Chick & Pierce, 2008) point out that teachers' training in the field of probability is not enough to enable them to deal with challenges posed by the work with pupils. Several teachers present difficulties similar to their students' ones in managing basic concepts (Prodromou, 2012). Moreover, their pedagogical content knowledge in probability appears to be not appropriate (Batanero *et al*. 2004) and leads them to have scarce experience in conducting experimental activities with students (Stohl, 2005).

The different settings for the probability approaches (classical, frequentist and subjectivist) provide different systems of concepts and procedures that serve to analyze situations in the field of uncertainty. Teachers should be aware of the diversity of these approaches because they influence the reasoning of students in confronting problems inherent in probability topic. In this sense, it is essential to think of new ways/approaches to develop teacher's knowledge and awareness on probability, both during their educational path as prospective teachers as well as concerning continuous education.

In this view, Batanero *et al*. (2016) have highlighted the necessity of leading research on the components of teachers' knowledge and of designing materials for teachers' professional development concerning probability.

One of the approaches to the probability developed in mathematics education is the "classical" one, which looks at the probability as a ratio between the number of favourable cases and the number of possible cases (Laplace, 1995/1814). This approach offers many didactic benefits, however, if it is the only one studied, it can reduce understanding probability associated to a simple mathematical calculation, and it limits teacher's work to handling with prototypical problems, as those ones involving dices, coins and cards, not giving the opportunity to students activate their rational ability of making an evaluation or taking a decision.

On the contrary, it's important to explore contexts where we can apply different approaches to develop the probabilistic understanding.

Following Fischbein's (1982) approach, a central activity for learning probability entails predicting the outcomes of chance experiments by placing bets. To enhance the learning environment's engagement, this activity can be integrated into a gaming scenario, as illustrated in Aspinwall and Tarr's (2001) study. In our own research, we also recognize the significance of the betting game as a pivotal element of the learning process. Our study is grounded in the betting game.

The bet is one of these contexts, because it represents one of the main ways which lead to the discovery and to the understanding of the subjectivist approach of probability, as well as to the classical and frequentist ones.

In this thesis, exploring the way in which we face this crucial theme, I present a study in the context of a professional development course for primary and lower secondary teachers, designed in the frame of Mathematical Teachers' Specialised Knowledge – MTSK (Carrillo *et al*., 2018). In particular, taking inspiration from the subjectivist approach to probability (de Finetti, 1931), the teachers are involved in a context of betting games facing the problem of quantifying the *degree of confidence* of an event.

Teachers' knowledge plays a crucial role in students' learning process (e. g. Ball, Hill & Bass, 2005; Boyd *et al*., 2009; Nye, Konstantopoulos e Hedges, 2004). For this reason, the interest in teachers' knowledge has been an emerging focus of attention in the last years and, in this scenario, different approaches have been designed and conceptualized in order to help understanding the specificities of such knowledge. Teachers' knowledge can be perceived in a variety of manners and following a diversity of perspectives (e.g., the Mathematical Knowledge for Teaching – MKT (Ball, Thames, & Phelps, 2008); the Mathematics Teachers Specialized Knowledge – MTSK (Carrillo et al., 2018)). In the scope of the work, we develop we perceive such knowledge as specialized and we consider the MTSK conceptualization.

The MTSK conceptualization was proposed to better catch the complexity and the specialised nature of mathematics teachers' particular knowledge. It comprises two main domains: Mathematical Knowledge (MK) and the Pedagogical Content Knowledge (PCK), perceived in an intertwined manner, in which three subdomains compose each one of these domains.

In the particular case of probability, as in all the others, it is important to understand the content of specialised knowledge a teacher needs to have, and the content of the different dimensions of such specialized knowledge, in order to better understand the teachers practices and be able to devise ways to improve such practice and teacher education.

The construction of the probability theory from a subjectivist point of view has emerged from the intention to give the probability's meaning a psychological basis (Chernoff & Sriraman, 2014). In particular, the probability of an event is a quantitative measure of the *degree of confidence* based on the judgment that events occur. In this perspective, what really matters is not the concept *"what I foresee, will happen, because I foresaw it"* but, instead we should focus on the question *"why do I foresee that this event will happen?"* (de Finetti, 1931).

Teachers' preparation requires a task design that allows teachers to confront their inner ideas and to discuss and perform activities to increase their probabilistic knowledge and their didactic knowledge in this topic (Batanero, Biehler, Engel, Maxara, & Vogel, 2005; Batanero, *et al*., 2014). In this vision, a Design Research (Bakker, 2018) approach has been developed structured mainly into two action's phases: the first phase, in which the participant teachers have been able to deepen the Knowledge of Topic (Carrillo *et al*., 2018) on probability, and the second phase, in which they designed a didactic path for their students.

We consider then that, *if teachers are involved in a teacher education program focusing on developing their specialized knowledge on the elements of the subjective probability, then they* *will be in better position (knowledgeable) to develop specific processes of thinking improving their specialized knowledge on probability and thus, improving the tasks they propose and how they implement them (mathematical goals pursued).*

Regarding to prove as valid or false such hypothesis, the following research question is considered:

> *Which specialized mathematical knowledge in probability, demonstrated by teachers, impact their decision-making in a context of betting games and lead them to adopt a subjective probability view, to quantify the degree of confidence of an event?*

In order to deepening on specific aspects involved in such question, we proposed two foci to develop this investigation. Such a foci are formulated in terms of (sub)research questions:

*(i) Which elements of teacher's Knowledge of Topics (KoT) on probability are possible to trace on a context of a teacher's education program focused on establishing relationships amongst subjective, classic and frequentist perspectives?*

*(ii) To what extent do the theoretical elements of the subjectivist approach to probability contribute to the attribution of meaning to the degree of confidence that a teacher assigns to an event?*

In order to do so we follow a set of steps which guide the research being developed.

- Examine the Influence of Specialized Mathematical Knowledge: To investigate how teachers' specialized mathematical knowledge on probability affects their decision-making when engaging in betting games.
- Analyze Decision-Making Processes: To understand the decision-making processes employed by teachers, particularly focusing on the steps and factors involved in their choices during betting games.
- Use the Adoption of Subjective Probability: To assess the extent to which teachers adopt a subjective probability perspective when evaluating the confidence level in an event within the context of betting games.
- Identify Factors Influencing Confidence Assessment: To identify and explore the factors that influence teachers' confidence assessment in betting games, considering their specialized knowledge in probability.

In the first chapter, I provide an overview of three probabilistic approaches (classical, frequentist, and subjective probability), also exploring some key historical moments.

In the second chapter focus mainly on the MTSK conceptualization (Carrillo *et al*., 2018), which oriented this research is presented. Before introducing this conceptualization a discussion on Shulman (1986) categories and also the Mathematical Knowledge for Teaching – MKT (Ball et al., 2008) are discussed.

In the third chapter, we delve into educational considerations. This section aims to investigate the interplay between the mathematical perspective on probability and educational dimensions. The focus is on providing an overview of prior research in mathematical education, specifically addressing probability. The chapter revisit pivotal concepts and principles delineated in the literature, shedding light on the challenges and obstacles commonly faced by students in this domain. Furthermore, it will scrutinize the current landscape of teacher education programs with respect to probability instruction. Despite the increasing volume of research in mathematics education, there is a noticeable scarcity of studies dedicated explicitly to the teaching of probability.

In the fourth chapter, I discuss the educators and participant teachers who played a crucial role in the context of this study. The subsequent part will delve into the methodology, encompassing both Design Experiment (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) and Design Research (Bakker, 2018). This discussion sets the stage to present and justify, in the following section, the distinct phases that have shaped the course of this research. Lastly, the fourth section will detail the analysis methodology, elucidating the frameworks and approaches employed for a comprehensive understanding.

The fifth chapter is dedicated to the analysis' discussion, followed by subsequent conclusions.

In summary, this thesis delves into the realm of probability theory and its implications for teaching. We will explore various approaches to probability and discuss the impact of teachers' specialized knowledge on their decision-making during betting games. By examining factors that influence confidence assessment and the adoption of subjective probability, our aim is to provide insights to enhance mathematics education practices. Throughout the subsequent chapters, we will delve into the intricacies of probability education, offering practical recommendations to assist educators in their teaching efforts.

# 1. Probability: Historical and mathematical aspect

## 1.1 Historical background and different approaches to probability

For several centuries, in the course of history, several attempts have been developed in order to give a mathematical form to the uncertainty that rules our daily life. Nowadays, we accept the fact that there isn't a unique mathematical construct that can master all the issues regarding the uncertainty and prediction of events. The exploration of uncertainty and of logical reasoning exists as a *primordial* experience (Dehaene, 2019), but at certain time of history of science, the human kind started to look as something *measurable* linked to a numerical value. Today, this numerical value, thanks to Kolmogorov's axiomatization, is described by the following axioms:

1) The probability of event it's a number p included between 0 and 1;
2) The probability of an impossible event is 0 and the probability of a certain event is 1;
3) If two events A and B are incompatible, then the probability that one of them will happen it's the sum of their probabilities.

The different approaches to probability are linked to different ways of interpreting it and they are developed in different historical periods, by embodying and expressing different cultural and historical sensitiveness and needs (Radford, 1997).

The problems connecting the uncertainty, in particular to the gambling, are presented in different historical periods . Gambling and the counting problems connecting to its modelling were in fact the engine of the birth of this branch of mathematics and not by chance from these origins derives the word "*aleatory*" from the Latin word "*alea*", that means a game with dice, one of the most common adjectives used to define the type of events studied by the calculation of probabilities.

It's in the Latin poem of the XIII century, De Vetula, attributed Richard de Fournival, that we find the first and most ancient way to count and to put into series the number of possible arrangements of the faces of three dices (Bellhouse, 2000). In the three dices game, also known as *zara game* (the word "zara" may come from the Arabic word "*zahr*", which means dice, and from the same word may have derived the Italian saying "*gioco d'azzardo*"), the players must throw, taking turns, three equal six faces dices, and before that each dice reveals a number (included between 1 and 6), the player must pronounce loudly the number that in his opinion would have been the sum of the three numbers revealed by the dices (so a number included between 3 and 18).

This poem is the oldest known text establishing the link between observed frequencies and the enumeration of possible configurations (Bellhouse, 2000). Indeed, the author of the poem not only is able to correctly count the 56 possible not-order configurations of the faces of the three dices (see Figure 1) and the 216 order arrangements of the faces of the three dices (see Figure 2), but also to start to understand that each sum has its "weight", meaning a different "number of combinations related to its.":

> *"Sixteen compound numbers are produced. They are not, however, of equal value, since the larger and the smaller of them come rarely and the middle ones frequently"* (Bellhouse, 2000, p. 134).

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 6 | | | | | | | 3, 3 3 | 3, 3, 4 | 4, 4, 3 | 4, 4, 4 | | | | | | |
| 5 | | | | | | 2, 3, 3 | 1, 4, 4 | 2, 4, 4 | 5, 3, 3 | 6, 3, 3 | 5, 4, 4 | | | | | |
| 4 | | | | | 1, 3, 3 | 1, 3, 4 | 2, 3, 4 | 1, 4, 5 | 6, 3, 2 | 5, 4, 3 | 6, 4, 3 | 6, 4, 4 | | | | |
| 3 | | | | 2, 2, 2 | 2, 2, 3 | 2, 2, 4 | 1, 3, 5 | 2, 3, 5 | 5, 4, 2 | 6, 4, 2 | 5, 5, 3 | 5, 5, 4 | 5, 5, 5 | | | |
| 2 | | | 1, 2, 2 | 1, 2, 3 | 1, 2, 4 | 1, 2, 5 | 2, 2, 5 | 2, 2, 6 | 5, 5, 1 | 5, 5, 2 | 6, 5, 2 | 6, 5, 3 | 6, 5, 4 | 6, 5, 5 | | |
| 1 | 1, 1, 1 | 1, 1, 2 | 1, 1, 3 | 1, 1, 4 | 1, 1, 5 | 1, 1, 6 | 1, 2, 6 | 1, 3, 6 | 6, 4, 1 | 6, 5, 1 | 6, 6, 1 | 6, 6, 2 | 6, 6, 3 | 6, 6, 4 | 6, 6, 5 | 6, 6, 6 |
| | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |

*Figure 1: A possible modern visualization to arrange of all the 56 non-ordered terns by pulling three dice (in abscissas there are the 16 different sums and in ordinates there are the number of combinations related to its)*

Indeed, by counting, the author connects each of the 16 different sums (Figure 2) to its corresponding generative terns, achieving thus an implicit determination of their probabilities. He then advises the players to organize their bets according to their expected profit.

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 27 | | | | | | | | 1,3,6 | 1,4,6 | | | | | | | |
| 26 | | | | | | | | 1,4,5 | 1,5,5 | | | | | | | |
| 25 | | | | | | | 1,2,6 | 1,5,4 | 1,6,4 | 1,5,6 | | | | | | |
| 24 | | | | | | | 1,3,5 | 1,6,3 | 2,3,6 | 1,6,5 | | | | | | |
| 23 | | | | | | | 1,4,4 | 2,2,6 | 2,4,5 | 2,4,6 | | | | | | |
| 22 | | | | | | | 1,5,3 | 2,3,5 | 2,5,4 | 2,5,5 | | | | | | |
| 21 | | | | | | 1,1,6 | 1,6,2 | 2,4,4 | 2,6,3 | 2,6,4 | 1,6,6 | | | | | |
| 20 | | | | | | 1,2,5 | 2,1,6 | 2,5,3 | 3,2,6 | 3,3,6 | 2,5,6 | | | | | |
| 19 | | | | | | 1,3,4 | 2,2,5 | 2,6,2 | 3,3,5 | 3,4,5 | 2,6,5 | | | | | |
| 18 | | | | | | 1,4,3 | 2,3,4 | 3,1,6 | 3,4,4 | 3,5,4 | 3,4,6 | | | | | |
| 17 | | | | | | 1,5,2 | 2,4,3 | 3,2,5 | 3,5,3 | 3,6,3 | 3,5,5 | | | | | |
| 16 | | | | | | 1,6,1 | 2,5,2 | 3,3,4 | 3,6,2 | 4,2,6 | 3,6,4 | | | | | |
| 15 | | | | | | 2,1,5 | 2,6,1 | 3,4,3 | 4,1,6 | 4,3,5 | 4,3,6 | 2,6,6 | | | | |
| 14 | | | | | 1,1,5 | 2,2,4 | 3,1,5 | 3,5,2 | 4,2,5 | 4,4,4 | 4,4,5 | 3,5,6 | | | | |
| 13 | | | | | 1,2,4 | 2,3,3 | 3,2,4 | 3,6,1 | 4,3,4 | 4,5,3 | 4,5,4 | 3,6,5 | | | | |
| 12 | | | | | 1,3,3 | 2,4,2 | 3,3,3 | 4,1,5 | 4,4,3 | 4,6,2 | 4,6,3 | 4,4,6 | | | | |
| 11 | | | | 1,1,4 | 1,4,2 | 2,5,1 | 3,4,2 | 4,2,4 | 4,5,2 | 5,1,6 | 5,2,6 | 4,5,5 | | | | |
| 10 | | | | 1,2,3 | 1,5,1 | 3,1,4 | 3,5,1 | 4,3,3 | 4,6,1 | 5,2,5 | 5,3,5 | 4,6,4 | 3,6,6 | | | |
| 9 | | | | 1,3,2 | 2,1,4 | 3,2,3 | 4,1,4 | 4,4,2 | 5,1,5 | 5,3,4 | 5,4,4 | 5,3,6 | 4,5,6 | | | |
| 8 | | | | 1,4,1 | 2,2,3 | 3,3,2 | 4,2,3 | 4,5,1 | 5,2,4 | 5,4,3 | 5,5,3 | 5,4,5 | 4,6,5 | | | |
| 7 | | | | 2,1,3 | 2,3,2 | 3,4,1 | 4,3,2 | 5,1,4 | 5,3,3 | 5,5,2 | 5,6,2 | 5,5,4 | 5,4,6 | | | |
| 6 | | | 1,1,3 | 2,2,2 | 2,4,1 | 4,1,3 | 4,4,1 | 5,2,3 | 5,4,2 | 5,6,1 | 6,1,6 | 5,6,3 | 5,5,5 | 4,6,6 | | |
| 5 | | | 1,2,2 | 2,3,1 | 3,1,3 | 4,2,2 | 5,1,3 | 5,3,2 | 5,5,1 | 6,1,5 | 6,2,5 | 6,2,6 | 5,6,4 | 5,5,6 | | |
| 4 | | | 1,3,1 | 3,1,2 | 3,2,2 | 4,3,1 | 5,2,2 | 5,4,1 | 6,1,4 | 6,2,4 | 6,3,4 | 6,3,5 | 6,3,6 | 5,6,5 | | |
| 3 | | 1,1,2 | 2,1,2 | 3,2,1 | 3,3,1 | 5,1,2 | 5,3,1 | 6,1,3 | 6,2,3 | 6,3,3 | 6,4,3 | 6,4,4 | 6,4,5 | 6,4,6 | 5,6,6 | |
| 2 | | 1,2,1 | 2,2,1 | 4,1,1 | 4,1,2 | 5,2,1 | 6,1,2 | 6,2,2 | 6,3,2 | 6,4,2 | 6,5,2 | 6,5,3 | 6,5,4 | 6,5,5 | 6,5,6 | |
| 1 | 1,1,1 | 2,1,1 | 3,1,1 | 4,1,1 | 4,2,1 | 6,1,1 | 6,2,1 | 6,3,1 | 6,4,1 | 6,5,1 | 6,6,1 | 6,6,2 | 6,6,3 | 6,6,4 | 6,6,5 | 6,6,6 |
| | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |

*Figure 2: A possible modern visualization, made by author, to arrange of all the ordered terns by pulling three dices (in abscissas there are the 16 different sums and in ordinates there are the number of combinations related to its)*

But it was only in the XVII century that Galileo Galilei will shed light to this discovery. The XVII century was rich of advances and insights into the solution of problems related to gambling, found in the various epistolary exchanges between mathematicians of the time including Fermat and Pascal, which, although not yet systematized, represent the roots of the modern theory of probability, interpreted as a branch of mathematics which studies and analyzes situations ruled by uncertainty.

## 1.2 Classical approach

### 1.2.1 Towards the classical probability: configurations, permutations and combinations

Around 1620, the Grand duke of Tuscany presented to Galileo Galilei the first known problem of probability that was later solved and which was linked to the three dices game. In his work "About the discovery of the dices", the famous scientist analyses, from a mathematical point of view, the *zara game*. Galilei (1613-1623) deals with the problem in a very meticulous and precise way. In that time the problem was that considering the possible sums obtainable from the throw of three dices, that is the natural numbers between from 3 and 18 (extremes included), people wrongly believed that the sums 9, 10, 11 and 12 would display with the same frequency, considering the fact that they all are obtained with the same number of configurations (six different non-ordered terns). But by playing, people noticed that the numbers 10 and 11 manifested more frequently than the numbers 9 and 12, so the players would consider them as sums more likely to display compared to the other sums and preferred to bet on the sums 10 and 11.

Galilei faced the study of this problem starting to list all the possible cases. He affirms that the six *sorties* of a dice are "equiprobable", meaning that, when a dice is thrown, the dice can indifferently stop on one of his six faces: he translates, in this way, a symmetry of the dice.

It's possible to deduce that the *sorties* of each dice, and so the possible cases, are naturally independent and that for two or three dices there are respectively 6×6=36 and 6×36=216 possible combinations. The mistake that was made is the following: since the sums 9, 10, 11 and 12 can be obtained with three different triplets, people thought they had to manifest with the same frequency. But in this way the permutations weren't considered.

On 216 possible combinations, the sums 9 and 12 were obtained with 25 ordered configurations, while the sums 10 and 11 were obtained with 27 ordered configurations, that is with two additional ordered terns.

To prove this fact, Galilei (1613-1623) produced the following table (Figure 3) in which the bold numbers above each column represent the first eight possible sums of the three dice (sums ranging from 10 to 3). In parentheses, there are the sorties that can combine to achieve that sum, along with the number of possible permutations with those specific numbers (for example, under the sum 10, it reads (6; 3; 1) 6). Below each column, in bold, is the total number of the ordered terns to obtain that sum.

| **10** | **9** | **8** | **7** | **6** | **5** | **4** | **3** |
|---|---|---|---|---|---|---|---|
| (6,3,1) 6 | (6,2,1) 6 | (6,1,1) 3 | (5,1,1) 3 | (4,1,1) 3 | (3,1,1) 3 | (2,1,1) 3 | (1,1,1) 1 |
| (6,2,2) 3 | (5,3,1) 6 | (5,2,1) 6 | (4,2,1) 6 | (3,2,1) 6 | (2,2,1) 3 | **3** | **1** |
| (5,4,1) 6 | (5,2,2) 3 | (4,3,1) 6 | (3,3,1) 3 | (2,2,2) 1 | **6** | | |
| (5,3,2) 6 | (4,4,1) 3 | (4,2,2) 3 | (3,2,2) 3 | **10** | | | |
| (4,4,2) 3 | (4,3,2) 6 | (3,3,2) 3 | **15** | | | | |
| (4,3,3) 3 | (3,3,3) 1 | **21** | | | | | |
| **27** | **25** | | | | | | |

*Figure 3:Galileo's arrangement of the terns obtainable by the throw of three dices*

Galilei pointed out the existence of a certain symmetry for the sums following 10, in other words, observing that the sum 11 appears with the same numbers of configurations of the number 10, the number 12 appears with the same number of configurations of the number 9 and so on, he decided to visualize and calculate only the numbers of configuration for the sums minor of ten and obtaining the others using this symmetry recognition. In this way he can rationally explain the phenomenology of the game observed but that people previously couldn't explain: playing many times, it seems clear that the sums 10 and 11 result to be more frequent, and so more convenient, compared to the sums 9 and 12, exactly because 10 and 11 are associate to a bigger number of configurations, in other words they have more terns from which they can be originated.

### 1.2.2 The origins of the calculation of the classical probability

The mathematical formalization of the calculation of probability dates to 1654, when Antoine Gombaud Chevalier de Méré, writer and obstinate player of gambling, sent a letter to the French philosopher and mathematician Blaise Pascal with the following problem:

> *"Is it easier to obtain at least a 6 throwing a dice four times or to obtain at least two 6, that is 12, throwing two dices twenty-four times?"*

Gombaud tried to calculate the probability of both events, a calculation that made him obtain two estimates of probability that were identical.

Regarding the first event, Gombaud concluded that throwing one time a not altered dice, the probability of obtaining 6 was 1/6. Then he decided to sum up four times this probability, obtaining:

$$4 \times \frac{1}{6} = \frac{2}{3}$$

*Equation 1*

He makes the same steps also to calculate the probability of the second event: by throwing two not altered dice, he would obtain double 6, as well as sum 12, with probability 1/36. He decides to multiply 24 times this probability in order to obtain the same probability of the previous case with one dice:

$$24 \times \frac{1}{36} = \frac{2}{3}$$

*Equation 2*

Gombaud's conclusion was it's perfectly equal to bet on one of the two events: at least a 6 throwing a dice four times or at least two 6, that is 12, throwing two dices twenty-four times.

Gombaud focused his work only on the manifesting of the favourable events: 6 in the case of one dice and a double 6 in the case of two dices.

Pascal, together with the French mathematician Pierre de Fermat, managed to clarify Gombaud's doubts calculating how many the favourable cases were compared to the "*unfavourable*" cases, concluding that the double 6 out of 24 throws is a harder event to happen, so it's less likely to occur compared to a single 6 out of 4 throws.

A modern solution to this problem is linked to the concept of *complementary events*: two events called *E* and *E'* (with the probability P(*E*) and P(*E'*), respectively) are defined complementary if E or *E'* occur for sure, and it's excluded that both will happen simultaneously.

In this case we have:

$$P(E) + P(E') = 1$$

*Equation 3*

So, to calculate the probability of the event *E* knowing the one of *E'*:

$$P(E) = 1 - P(E')$$

*Equation 4*

Let's examine now the first part of the problem.

The probability of the event "*out of 4 throws of a dice will come out a 6 at least once*" is equivalent to 1 minus the probability of the complementary event: "*out of 4 throws will come out a 6 not even once"*. The event A consist in obtaining *at least* a 6 out of 4 throws of a dice alone, while the complementary event A' consist in obtaining 1 or 2 or 3 or 4 or 5 (that is not even one 6) out of 4 throws. In terms of the calculation of probability we have:

$$P(A') = \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \left(\frac{5}{6}\right)^4 = \frac{635}{1296} \cong 0{,}4822$$

*Equation 5*

So, based on the definition of complementary events:

$$P(A) = 1 - P(A') = 1 - \frac{625}{1296} \cong 0{,}5177 \cong \mathbf{52}\%$$

*Equation 6*

In the same way, the probability of the event: "*out of 24 throws of two dices comes out at least a double 6"* is equivalent to the opposite probability of the event: "*out of 24 throws of two dices comes out a double 6 not even once"*.

In this case, the event B consist in obtaining at least a 12 out of 24 throws of two dices, the complementary event B' consist in obtaining numbers between 2 and 11 (so obtaining not even a 12) out of 24 throws, that is:

$$P(B') = \frac{35}{36} \times \frac{35}{36} \times \dots \times \frac{35}{36} = \left(\frac{35}{36}\right)^{24} \cong 0{,}5085$$

*Equation 7*

For the relation that links the complementary events:

$$P(B) = 1 - P(B') = 1 - 0{,}5085 \cong 0{,}4914 \cong \mathbf{49}\%$$

*Equation 8*

It turns out, obviously, that's more probable to obtain at least a 6 out of 4 throws of a dice than two six out of 24 throws of two dices.

A change of perspective was necessary to solve the problem and in particular the emergence of the idea of the complementary event and the calculation of its probability that, in this case, it's simpler in comparison to the direct calculation of the probability of the event the original interest.

Another problem which challenged the mathematician's minds of that time, and which was linked with a matter of probability, was the one regarding the partition of the prize of a started game, but interrupted, knows as "*the problem of points*". The problem of the interrupted game, already introduced by the Italian mathematician and economist Luca Pacioli (1494) is the following:

> *"The players will play a game that will end when one of them will reach a prefixed number of victories (for example 6); the winner will earn a certain prize. If they decide to interrupt the game before arriving to 6, in our case with a score of 5 – 3, how would the prize be divided equally and honestly between the two players?"*

Pacioli suggested to divide the prize for the number of played games and to assign the parts based on how many games were been won by each player. A not totally equal solution. Niccolò Fontana noticed the mistake made by Pacioli; he was a Brescian mathematician also known as Tartaglia and in his work "*General Trattato",* published in 1556, he pointed out that, according to Pacioli's rule, if a player would have won a game and the other player would have won none, he should have acquired the whole prize, which is totally unfair. So, he tried to formulate a new solution, that also turned out to be unsatisfactory. His reasoning was the following: the difference of victories between the two players A and B was of two games, (5 – 3) which corresponded to 1/3 of the total prize, so the prize shall be divided in 3 parts, and just one would have been collected by B, while the other two would have been collected by A.

Once again, it was necessary a change of perception led by the mathematicians Pascal and Format, who, thanks to their illuminating contribution, managed to find a solution to the riddle "interrupted game". Starting from this exchange of ideas the two will keep in touch and will initiate a correspondence that will be famous in the field of the origins of the calculation of probability. They both understood that the key of the solution laid not in the number of games played and won by the two players, but in the number of games that were lacking to reach the victory, so the end of the match. In other words, Pascal and Format focused their attention not on what already happened, but on what haven't happened yet, that is the other cases that could manifest. Pascal formulated the right solution through the nowadays known *"Tartaglia's triangle"* (from this notion derive some basic mathematical concepts like the binomial coefficient), while Format solved the problem of the prize determining the maximum number of necessary games to complete the challenge, and also the calculation of the favourable cases for each player (creating the premises for the combinatorial calculation). They both came to the same conclusion, so that on July 29 of the year 1654 Pascal wrote to Fermat "*I see that the truth it's the same in Toulouse as much as in Paris"*.

Even if Pascal and Format never used the term "calculation of probability" their correspondence it's the beginning of the calculation of probability, a mathematical field that from the second half of the XVII century will attract the attention of many enthusiasts of the probabilistic problems, like Christiaan Huygens (1657). Huygens, inspired by Pascal, was the author of the first treatise of probability: *"De Ratiociniis in Ludo Alae"*, which was published in Latin in 1657. In this essay, Huygens introduces some aspects of the probability in an axiomatic way, like the concept of winning expectation with the term *"geometrica expectatio",* nowadays called "*mathematical hope"*. More in particular, Huygens claims that if $p$ is the probability of winning a certain amount $a$, and $q$ is the probability of winning an amount $b$, then we can expect to win the amount $pa+qb$.

Starting from Huygens' contrail, the eclectic Gottfried Wilhelm von Leibniz wrote:

> *«If a situation can lead to different advantageous results ruling out each other, the estimation of the expectation will be the sum of the possible advantages for the set of all these results, divided into the total number of results. »*

So, Leibniz suggests that, when a situation of uncertainty can lead to different convenient results, that exclude one another, the estimate of the winning expectation is the sum of the possible advantages for the totality of the all the results, divided for the total number of results.

With Leibniz, and even before, with Fermat, it's possible to witness the progressively development of what it's know as the first mathematical approach to the probability, which is the classical approach, whose first definition is contained in "*The Doctrine of Chances"*, the first book about the theory of probability, written by the French mathematician of the XVIII century Abraham de Moivre (1738/1967) and published for the first time in 1718. The definition is the following:

> *«Wherefore, if we constitute a Fraction whereof the Numerator is the number of Chances whereby an Event might happen, and the Denominator the number of all the chances whereby it may either happen or fail, that Fraction will be a proper definition of the Probability of happening. »*

In these lines emerges one of the first mathematical constructions of the concept of probability seen from the point of view of the classical approach.

These words, if transformed in mathematical symbols, create a fraction whose numerator represents the number of possibilities according to which an event can happen and the denominator expresses, instead, the number of all the possibilities through which that event can happen or not.

Of course, De Moivre's concept is one of the first definitions of the classical probability, but the one who created the classical method of the calculation of probability (in terms of ratio), universally known, is the French mathematician and noble Pierre Simon Laplace. Laplace (1812/1951), in his essay *"Essai philosophique sur les probabilités"*, published in 1814, points out the centrality of the subjective sight in the evaluation of the equiprobability, essential for the classical definition of probability, in concrete situations. In this regard, Laplace wrote:

> *«the theory of chance consists in reducing all events of the same kind to a certain number of equally possible cases, i.e. such that we can be equally undecided as to their existence»* (Laplace, 1812/1951)

So, we have the following definition, that is also the first principle of the classical conception of probability:

> *«the probability of an event is the ratio of the number of cases which are favourable to it to the number of all possible cases when there is no reason to believe that one of these cases should happen rather than the others»* (Laplace, 1812/1951)

The probability, in Laplace's opinion, is a fraction, associated to the meaning of a ratio, whose numerator represents the number of favourable cases, and the denominator is, instead, the number of possible cases. This is valid only when nothing makes us think that a case will happen more likely than the other one so the core idea of equiprobability. In other words, it's necessary that the events in question have the same probability of manifesting, so they all must be equally possible. Below the definition of classical probability:

$$Probability = \frac{number\ of\ favorable\ cases}{number\ of\ possible\ cases}$$

*Equation 9*

The classical approach represents a fundamental goal for the history of the calculation of probability. It can be applied in situations where it's possible to identify equiprobable configurations in which is possible to calculate in advance the number of combinations corresponding to a certain event. However, in many human or natural circumstances, where the characteristic of equiprobability decreases, this approach appears to be powerless. The one who shed light on this aspect was the Swiss mathematician and scientist Jacques Bernoulli (1713/1987), who, in his essay "*Ars conjectandi"*, wrote that what is not given in advance is at least possible in retrospect, so it can be possible to obtain it observing the result of many similar examples. This intuition will inspire Bernoulli himself to elaborate "The law of large numbers", paving the way to what will be a new way of approaching to the probability: the frequentist approach, an approach that will conquer mathematical legitimacy only centuries later, in 1900, thanks to the work of the Austrian mathematician and engineer Richard von Mises (1957).

## 1.3 Frequentist approach

Jacques Bernoulli launched some studies that will lead to the discovery of an alternative and totally new way of estimating a probability evaluation, based on an objective and frequentist estimation of the probability of a concrete event and on experience, which only appeared three centuries ago.

A reasonable valuation of the probability of an event can be obtained starting from the repeated observation, in the same conditions, of tests where an event can happen or not with a certain frequency. In other words, when repeating an experiment for a considerable number of times in the same conditions, the relative frequency of an event that actually happens represents a reasonable valuation of the probability of the event itself. By saying relative frequency, we mean the ratio between the absolute frequency (meaning the number of times that an event happens) and the number of accomplished proves.

The frequentist approach is needed when it isn't possible to know from the start the number of favourable cases without accomplishing some experiments for several times. It is based on a theorem which was elaborated by Bernoulli and which is known as *"Law of large numbers"***;**

It affirms that in a sequence of proves accomplished in the same conditions and whose events are independent, the frequency of an event moves closer to its probability and the approximation tends to improve with the increasing of the proves, tending to infinity. In modern words, this law can be formulated as follows: by the increasing of the number of experiments or proves accomplished, the value of the frequency tends to the theoretical value of probability, that is the value given to the classical probability.

If an experiment could accomplish for an infinite number of times, for example with the throwing of a coin, based on what said by the law of large numbers, the expectation would be that the value of the relative frequency of each event, heads or tails, tend to his value given by the classical probability. To be clearer, thanks to the classical probability we know that, throwing a not altered coin, the two faces of the coin, heads and tails, have the same probability of verifying, that is ½ = 0,5. Based on what said previously, throwing the coin for an always increasing number of times, we expect that the two relative frequencies of each event will stabilize, before or after, towards the value that we outlined earlier with the classical probability, indeed. So, by the increasing of the number of proves (which tends to infinity), the relative frequency will move always closer to the probability of each event. It's clear that such an experiment can't be actually repeated for an infinite number of times.

Some centuries later the Austrian mathematician R. von Mises (1957) will be the one to formalize the frequentist approach, employing a mathematical concept that conquered a place in the math's dictionary only in the XIX century: the limit.

Following, frequentist approach will be reported from the point of view of the frequentist conception:

$$probability = \lim_{n \to \infty} \frac{frequency}{number\ (n)\ of\ experiments}$$

*Equation 10*

The stabilization of the frequencies of an event by a random experiment, after a significant number of identical proves in the same conditions, has been for centuries a matter of interest.

But when is it convenient to resort to the classical probability, and when is it better to resort to the frequentist one?

First of all, the classical probability is established in advance, before consulting the facts.

In the frequentist conception the probability is obtained retrospectively; it's an estimate that is calculated after a series of independent proves whose results permit to study the frequency of a certain event, but also the relative frequency through which the event verified. In other words, we use the experience (relative frequency deduced through statistics) to form some opinions on the future (probability of future events).

So, it's the nature of the event itself to point out the best approach. The field of application of the frequentist approach is really ample, because it can be applied in all the situations where the events aren't equiprobable, but also in situations in which phenomena happen in similar conditions of which we have, or can produce, statistics data.

Also, the relative frequency, just like the probability, is a number contained between 0 and 1, with some differences of meaning. The frequency 0 doesn't imply that the event must be impossible, just like the frequency 1 doesn't imply necessarily that the event must be certainly (as intended in the field of classical probability). Indeed, when the relative frequency is 1, that means that for a certain number of proves N, the event has always verified. This fact doesn't implicate, in the following sequence of proves, that the same event will verify certainly with the same frequency, meaning with the frequency 1.

The same goes for the complementary event.

In conclusion, the frequentist approach defines the probability as relative frequency, it supposes that an aleatory experiment can be accomplished, in the same conditions, a large number of times that we call *N*. By the increasing of *N*, the ratio between "the number of times where the event has been successful" and "*N"* should converge to a number that we call probability.

Despite the differences between an approach and the other, both respond consistently to the axioms introduced by A. Kolmogorov (1933), which formalize mathematically the two approaches.

Indeed, in both approaches we can verify:

- The probability is a number contained between 0 and 1;
- The probability of the impossible event is 0, while the probability of the certain event is 1;
- In the classical vision:
  - 0 = impossible event;
  - 1 = certain event;

- In the frequentist conception:
    - 0 = possible event that hasn't verified yet;
    - 1 = possible event that has always verified on a number of proves N;
- Being A and B two incompatible events, then the probability that one of them will verify is given by the sum of their probabilities.

## 1.4 Subjectivist approach

### 1.4.1 Probability: the science of uncertainty

The probability, until now, has been treated like an intrinsic characteristic of events, that is revealed in the moment when a man tries to get to know it measuring it, in other words assigning a number to it, a parameter to those that are the cases through which it can actually manifest. In the two approaches that we discussed earlier, the calculation of probability is independent from man and leads to objective conclusions that aren't, in no way, conditioned from the opinions of who evaluates numerically the possibility that an event must verify. The history of probability that has been told until now has allowed to outline a type of logic subtended from the case, from uncertainty, from a not absolute truth, as well as the logic of uncertainty.

The concept of probability, before the XVII century, age when it became concretely object of investigation and scientific research, has always been used by men in its intuitive and vague form and been associated to events of everyday life marked by uncertain and unpredictable forms of knowledge. This uncertainty leads man, more or less consciously, to attribute a level of faith towards the happening of the event. The origins of this new mathematical science lay in the gambling.

Even in the title of the first real treaty of probability intended as mathematical calculation, that is the *Ars conjectandi* by J. Bernoulli, we can see the main characteristic of the probability, an element that makes it stick out compared to the other sciences: the uncertainty. This new science, indeed, is described as "the art of conjecturing", a definition that contrasts with the absolutism of the mathematical truth which ruled since the ancient times.

> *"Even the name alone of Calculation of Probabilities is a paradox: the probability, opposed to the certainty, is what we don't know, and how can we calculate something we don't know?"* (Poincaré, 1936).

And the only certainty this science is based on is the consciousness of the uncertainty.

Concerning the definitions that we introduced, the frequentist one and the classic one, De Finetti (1931) writes:

> *"They define nothing; even worse, they hide, with ramblings and arcane definitions, full of smoke and emptiness, the real sense with which the word is used by the last man of the street. The so-called definition based on partitions in equally probable cases requests that is already acquired, in a subjective sense, the concept of identical probability. And the one based on frequencies requires the same vicious cycle and also an intuition (necessarily approximated) of a connection between the observation of frequencies and the evaluation of probability (subjective), a connection of which only an adequate elaboration of the theory of probability can establish the meaning based on a real analysis of the circumstances at stake."*

These words said by Bruno De Finetti have started a deep and drastic change of perspective in the field of the calculation of probability. In the already described approaches, obviously different, but both marked by a scientific nature in the background, the probability is established numerically, entrusting exclusively a mathematical reasoning from which derives a calculation, more or less, correct.

As a matter of fact, both the definitions of probability are expressed in mathematical terms. We speak of "ratio" or "fraction" in the case of the classical probability and of "frequency" or "limit of frequency" in the case of the frequentist probability. Ramsey (1926) and Savage (1967), together with De Finetti, go against these conceptions outlining a new one, which includes the previous ones and that will mark an important turning point.

This change of perspective allows to evaluate the events not from the inside anymore, but from the outside: in this way, the point of view of the observer, who in that moment has some opinions and judgements regarding the events at stake, influences what will be the assignment of the probability of the event itself.

In other words, the probability corresponds to the

> *"grade of faith regarding the verifying of a certain event E, where the subjectivity is intended not as arbitrariness, but as "coherent" opinion of an expert deriving from the complex of information in possession of the subject who evaluates"* (De Finetti, 1931).

The above-mentioned definition constitutes the act of birth of a conception, not only new, but also revolutionary of the calculation of probability of events to which each man approaches in a totally unique and subjective way, with his own beliefs and judgments, a probability called, indeed, “subjective”.

### 1.4.2 Level of uncertainty and of dependability

In 1974, Bruno de Finetti, author of a true and real revolution in the field of probability, debuted with a pretty bizarre statement:

> *“The probability: who’s it? Before answering this question is surely opportune asking: does the probability really “exist”? and what should it be? I would answer no, it doesn’t exist”* (De Finetti, 1980, p. 1146).

In theoretical terms, we could support this statement. Theoretically speaking, the probability doesn’t exist. It doesn’t exist as absolute mathematical truth, as universal acknowledged and accepted definition. It subsists in the exact moment when man analyzes the reality which he knows to try to understand and to evaluate events that belong to it, through a quantitative measurement of the possibilities that an event will verify compared to other events. Basically, in the moment when a subject is called to decide between the verifying of an event or of another one, he puts into play a series of strategies to best ponder his decision.

To the answer given about the definition of probability, De Finetti (1931) adds:

> *“I could also say, vice versa and without contradiction, that the probability rules everywhere, that is, or at least should be, our “mentor in the thinking and acting”, and for this reason it interests me. Simply, it seems to me improper, and so it disappoints me, watching it being actualized in a substantive «probability», while I would believe more acceptable and more appropriate if we employed just the adjective “probable”, or even better, just the adverb «probably».”*

A paradoxical statement, but at the same time severely logic: the probability doesn’t exist or exists everywhere.

De Finetti considers and defines the probability as a “mentor in the thinking and acting”.

Probability intended as science of thought, a thought reasonably coherent. De Finetti has the urge, as he says, to establish psychologically the concept of probability.

Here’s what the probability is: not an intrinsic property of reality (as it was considered by objective positions), but a property that belongs inseparably to man, who will manage, thanks to the consciousness of the unpredictability of the events and thanks to his own critic and logical thought, to find the way in the labyrinth of uncertainty where he finds himself walking since his first steps.

In 1974, the professor Henry Kyburg wrote:

> *«Randomness is no longer a physical "objective" property but has a subjective character and probability does not measure a magnitude, such as length or weight, but only a degree of uncertainty, specific for each person. »* (Kyburg, 1974)

In this sense, the randomness, unlike the length and the weight, isn't a physics property objectively acknowledged and accepted. The randomness has a purely subjective character, of which the probability expresses not a magnitude, but a level of uncertainty, that is specific for each person.

In this sense, the probability doesn't appear as a "physics phenomenon" that we study and analyze, but as a typical characteristic of each man, which manifests every time that he finds himself faced with an unpredictable situation and it helps him to weight the consequences that could derive from a choice. For this reason, de Finetti ended saying *"the probability doesn't exist"*: it doesn't exist beyond the observer.

These ideas represent the manifesto of the subjective conception of probability.

For centuries, the scenario of the calculation of probability has been characterized by ideas and thoughts that led to the same goal, that is a number, a number that limits itself to quantify and measure the events of reality, but it doesn't suggest us "how" to read and interpret these data. This because the interpretation of data and of information is a task that concerns exclusively the subject, who has to make a choice, pondering potential risks and victories. Many of these information can be certainly obtained through the classical and frequentist approaches, but from this perspective, they can only provide operative modalities, tools of calculation through which obtain information that will be examined and evaluated directly by the subject. None approach could ever unveil the best choice we can make.

> *"So no science will make us say: this event will happen, things will go this way, because it's a consequence of a certain law, and this law is an absolute truth, but it won't either lead us to skeptically conclude: the absolute truth doesn't exist, and so this event can happen and cannot happen, it can go this way and it can also go in a totally different way, I know nothing about it."* (de Finetti, 1931)

The field in which the subjective conception of probability expresses in a satisfying way its own reason of being is the one of the *bet*, the same field in which the calculation of probability has its origins. It's in the bet, intended as equal and correct game, that man expresses his own level of trust (faith or fear) that he entrusts to the verifying of an event instead of others. This level of trust is translated into the field of betting.

### 1.4.3 Scheme of bets and following generalization

Based on the subjectivist approach, the definition of probability P of an event *E* can be given supposing that a bookmaker is obliged to accepts bets on a certain number of events, including the event *E.* The bookmaker (A) has the authority to decide the price p that a player (G) has to pay to bet on some events at stake and to collect the amount 1 in the case where the event *E* verifies. G has, so, the authority of deciding if he has to pay the price related to the chosen event to collect 1 or to pay S times that price ($p \times S$) to collect, in case of victory, S times that amount 1 ( $S \times 1$).

If G decides to bet on the verifying of the event *E,* A collects the price pxS from G who, in the case where the event *E* verifies, collects S. Let's indicate this situation as *pro E bet.*

In the Figure 4 are represented the transitions of money between G and A (the direction of the transfers is suggested by the arrows) both in the case where the event *E,* on which the bet took place, verifies (yellow full square) and also in the case where it doesn't verify (empty yellow square) and the connected variations of money at G's disposal.

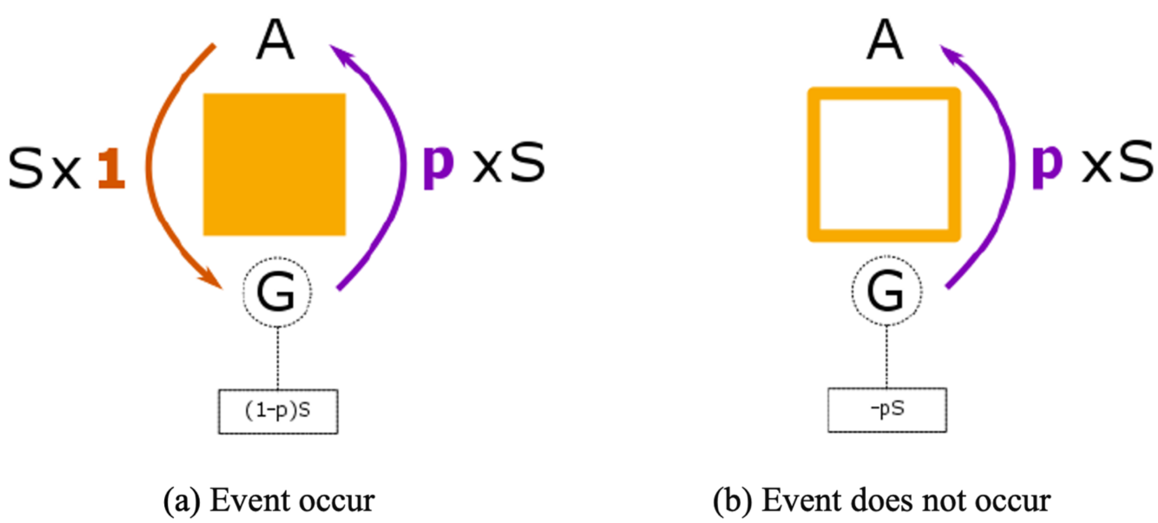



*Figure 4: Money transactions between G and A in a pro E bet*

In de Finetti's frame another possibility is provided: if G decides to bet against the verifying of *E,* B will be obliged to pay $p \times S$ to G if the event *E* doesn't verify (G-wins), or to collect $(1 - p)S$ if the event *E* verifies (G loses). In this way, G is forcing B to play money on an event using the prices set by B. Let's indicate this situation as *cons E bet*.

In de Finetti's scheme of bets (1931) the price p quantifies the *degree of confidence* of an event, attributing a measure that results to be the one of the probability.

This implies that is acquired the ability of comparing and order events according to the "*degree of confidence*" based on the *effects* caused by those same events.

In other terms, in this scheme the price p coincides with the probability of the event *E.*

In this way, the probability of the event is given by the relative price p in the case where A is *coherent,* meaning with this word the fact that A establishes prices $p_i$ for each event $E_i$ (where I = 1…n), so that G can't achieve a system of bets that will guarantee him a certain victory. This implies that those who have the possibility to establish the prices have to avoid to put themselves in the situation where they can certainly lose and have to assure to not be certainly exposed to negative credit transactions.

According to the scheme of bets, by virtue of the condition of coherence, de Finetti (1931) demonstrates that the probability p of an event *E* is bound to the condition:

$$0 \leq p \leq 1$$

*Equation 11*

Especially, he also demonstrates that p = 0 if and only if *E* is an impossible event and that p =1 if and only if *E* is a certain event.

In addition, he demonstrates (de Finetti, 1931) also that given a complete class of incompatible events $\{E_1, \ldots, E_n\}$, indicated with $p_i$ the probability of the event $E_i$ is obtained:

$$\sum_{i=1}^{n} p_i = 1$$

*Equation 12*

Two events are incompatible when the verifying of one of them implies that the other won't verify. A class of (incompatible) events is called complete if it is certain the verifying of one of them (that is if their logic sum is a certain event).

The validity of the relations reported above (which correspond to conditions that have to be satisfied by p in order to be a real probability) is demonstrable employing the fact that a player can play both *pro E* that *cons E* and that the bookmaker is coherent.

In some consecutive steps to the insertion of the scheme of bets, de Finetti (1931) generalizes the ideas that he introduced, to justify the insertion of a measure of probability in the form:

$$P(E) = \frac{m}{n}$$

where P(*E*) represents the probability of an event *E* for which there should be *m* favourable cases out of *n* possible cases.

It is, so, the way through which is usually defined the classical probability of *E,* where a definition of this type makes sense. In this regard, de Finetti (1937) writes:

> *There are two procedures that have been thought to provide an objective meaning for probability: the scheme of equally probable cases, and the consideration of frequencies. (...) In the case of games of chance, in which the calculus of probability originated, there is no difficulty in understanding or finding very natural the fact that people are generally agreed in assigning equal probabilities to the various possible cases, through more or less precise, but without doubt very spontaneous, considerations of symmetry. Thus the classical definition of probability, based on the relation of the number of favorable cases to the number of possible cases, can be justified immediately: indeed, if there is a complete class of n incompatible events, and if they are judged equally probable, then by virtue of the theorem of total probability each of them will necessarily have the probability p = 1/n and the sum of m of them the probability m/n. (...) However this criterion is only applicable on the hypothesis that the individual who evaluates the probabilities judges the cases considered equally probable; this is again due to a subjective judgment for which the habitual considerations of symmetry which we have recalled can furnish psychological reasons, but which cannot be transformed by them into anything objective.*

The possibility of playing both *pro E* and *cons E*, together with the conditions of *coherence,* allows de Finetti (1931) to impose and acknowledge bonds about the conferment of the prices $p_i$ and so, about the subjective choice of the level of reliability to attribute to the given events.

The bonds regarding the *degree of confidence* (that are listed as properties i, ii, iii, iv) correspond to characteristics considered generally as plausible in the employing of the *ordering relation.*

To explore these bonds, it's necessary to consider acquired the meaning of the relation "it's not less probable than".

The preposition "it's not less probable than *E'* ", represented with $E \geq E'$, is a relation that benefits of the following proprieties:

i. Given the events *E* and *E'*
   - is always valid $E \geq E'$ or $E' \geq E$,
   - if the two previous relations are valid together, the two events are said to be identically probable ($E \equiv E'$)
   - if is valid $E \geq E$ but isn't valid $E \equiv E$, it's possible to say that «*E* is more probable than *E'*» ($E > E'$);

ii. if A is a certain event and B is an impossible event, for each event *E* that is possible (not certain, either impossible) we have $A > E > B$;

iii. $E' \geq E$ and $E \geq E''$ implies $E' \geq E''$ (transitive property, that is valid also for the relations of equiprobability);

iv. If *E'* and *E''* are incompatible events with *E* and also $E' \geq E''$, we have that $E + E' \geq E + E''$ and inversely.

In the above listed properties, we refer to incompatible events, meaning with this word events for which the verifying of one of them implies the not verifying of the other (for example, in the single throw of a dice, the result "six" is incompatible with the result "two").

The writing of the type $E + E'$ refers to the logic sum of the events *E* and *E'* (for example, in the case of a single throw of a dice, the event "superior to four" corresponds to the logic sum of the event "five" and of the event "six").

The property iv, among those above listed, implicates a generalization of four events, of incompatible couples, that is not reported in this text.

The demonstration of the validity of the measure of probability in the form $P(E) = m/n$ is lead by de Finetti (1931) comparing events that belong to different spaces (for example comparing the event "result 2 throwing one dice" and the event "sum 12 throwing two dices").

Independently from the detail of the demonstration not reported in this text, the validity of the measure of probability in the form P($E$) = $m/n$ emerges

- from the introduction of the idea of probability as a measure to compare (and organize) levels of reliability of distinct events,
- from the imposition of the bonds i-iv to the way through which this comparison (and organization) is achieved
- from the reference to the comparison of probability of events belonging to different spaces
- from the employment of the relation of order in N, so from the ability to organize natural numbers.

While establishing this result, de Finetti (1931) generalizes the construct of subjective probability, doing without the scheme of bets and referring to the comparison between levels of reliability of events (without quantifying them as prices).

Moreover, he introduces a new definition of coherence, corresponding to the assumption of properties i-iv: an individual is defined coherent if he compares the level of probability of given events without contradicting the properties i-iv.

It's important to highlight that in this scheme we refer *primarily* to the organization between levels of probability, from which *descends* the conferring of a measure of probability to an event: this measure isn't defined minus of (and independently from) that ordering relation.

## 1.5 Kolmogorov's Axiomatic probability

In 1933 the mathematician A. Kolmogorov elaborates a system of axioms which provides the general characteristics of a probability that must always be valid, independently from how we chose to define the probability. In other words, all of the three approaches already presented, despite their discrepancies, respond consistently to the three axioms outlined by Kolmogorov.

In this paragraph the axiomatic probability is presented employing the language of the sets.

Before proceeding to the explanation of these concepts, we introduce some rudiments concerning the set theory, useful for the understanding of what will be presented.

### 1.5.1 Language of sets and language of events

**Definition 1**: We call **elementary events** all the possible results of an aleatory experiment, whose results aren't foreseeable with certainty, and **sampler space** the set of all the elementary events. The sampler space can be **discrete**, if its elements (meaning the elementary events) are an infinite number, **continuous** if it is more numerous.

**Definition 2**. Is Ω a discreet sampler space. We call event each subset of Ω. So the totality of the possible events is represented by the *set of parts of* Ω, **P(Ω),** that is the set that has as elements all the subsets of Ω.

The representation of the set theory of the events proves to be really efficient for the description of logic operations on the events.

Let's look at the following conformity between logic operations or relations on events and operations or logic relations of the set theory:

| ***Language of sets*** | ***Language of events*** |
|---|---|
| Ω, whole sampler space | "certain event" |
| ∅, empty set | "impossible event" |
| Set A | "A verifies" |
| Set $\bar{A}$ (A's complementary) | "A doesn't verify" |
| $A \cup B$ B (A union B) | "A or B verifies (or both)" |
| A ∩ B (A intersection B) | "A and B verify simultaneously" |
| A \ B (A minus B) | "A verifies B doesn't verify" |
| A ∩ B = ∅ (A and B are disjunct) | "the events A and B are incompatible" |
| B ⊆ A (B included in A) | "B implies A" |

### 1.5.2 The axiomatic definition and the properties of probability

Like the expression suggests, this definition is based on axioms, always valid, whichever approach is chosen for the calculation of probability:

**Positivity**: Ω is a discrete sampler space, we call probability on Ω any function that to each subset of Ω links a real number included between 0 and 1:

$$P: P(\Omega) \rightarrow [0,1]$$

**Certainty:** the probability of the certain event, so of the sampler space **Ω**, is always 1:

$$P(\Omega) = 1$$

**σ-additivity:** if $\{A_n\}_{n=1}^{\infty})$ is a sequence of events disjointed two by two (so $A_i \cap A_j = \emptyset$ if i≠ $j$), then:

$$P \bigcup_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} P(A_n)$$

From these axioms derive the fundamental theorems of the probability defined by A. Kolmogorov.

### 1.5.3 Properties of probability

$$P(\emptyset) = 0;$$

$$P(\bar{A}) = 1 - P(A)$$

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n),$$

as long as the events A are disjointed two by two;

$$P(A \cup B) = P(A) + P(B) - P(A \cap B),$$

for each couple of events A and B (even not disjointed).

# 2. Mathematics Teacher education and the specialized nature of mathematics teachers' knowledge

Starting from the seminal work of Shulman (1986) in the last three decades, the need for expanding research on Mathematics Teacher Education has been widely recognized, since preparing effective mathematics teachers (considering this preparation also as long life one) strongly affects the mathematics teaching itself and, therefore, the mathematical knowledge and skills of students. Starting from the $15^{th}$ ICMI Study (Even & Ball, 2009), a cross-cultural exchange of knowledge and information about the systems of teacher education have been developed at the international level, addressing fundamental themes that frame both the programs for mathematics teacher education and other professional development initiatives (e.g., Robutti *et al.*, 2016; Borko & Potari, 2020).

It is possible to recognize two main lines of research within the ongoing international debate on teacher education.

The first line is related to the different conceptualizations of Teacher knowledge and to the studies on the specialized nature of mathematics teachers' knowledge (e.g. Ball, *et al.*, 2008; Carrillo, *et al.*, 2018). The second line is the complex interplay between theory and practice within teacher education programs. With the aim of promoting teachers' shifts of attention towards constructs, theories, and practices that can inform and guide their teaching (Mason, 2008), in the last decades different research studies have focused on an active involvement of teachers in the analysis of practice through the lenses provided by theory. This has led to the setting up of teacher education processes that acquire the typical characteristics of a path of introduction to research, where teachers play the role of teacher-researchers within communities of inquiry (e.g. Jaworski, 2006; Mellone, 2011).

The study presented in this thesis is placed in the first line of studies, but always leaving the gaze open on the second line of research described above. In fact, as will be evident from the methodological system built, the intention was to work within the theoretical framework of the Mathematics Teacher Specialized Knowledge (Carrillo *et al.*, 2018), but at the same time to allow the emergence of a working group made by researchers and teachers as community of inquiry (Jaworski, 2006).

## 2.1 Specialized Content Knowledge: a large construct still in exploration

Before the seminal work of Shulman and his colleagues in 1986, where they focused on "Knowledge Growth in Teaching", the study and the categorization of the specific knowledge of teachers concerned essentially issues related to how to manage classrooms, the organization of activities, pupils' assessment and lessons planning (Shulman, 1986).

Shulman's (1986) critique about the literature built up until their study, was that several research papers on "*what to teach*" were conducted only in the field of cognitive psychology and the focus was from the point of view of the students. Therefore, a specifical baseline idea on different conceptualizations of teacher knowledge started with Shulman's research. According to this research approach, teacher knowledge branches into different fields (called dimensions) like knowledge of students and their characteristics, knowledge of educational contexts and educational purposes, knowledge of educational goals and values, general pedagogical knowledge, with particular reference to those principles and strategies of class management and organization that seem to transcend from the topic taught. However, according to Shulman (1986), they are not enough and "*mere content knowledge is likely to be as useless pedagogically as content-free skill*".

According with Shulman (1986) the analysis of these issues makes clear the problematic nature of the previous perspectives, indeed, the questions about teacher education that he considered "unanswered" were many:

> *"Where do teacher explanations come from? How do teachers decide what to teach, how to represent it, how to question students about it and how to deal with problems of misunderstanding?"* (Shulman, 1986, p. 8) *"What are the domains and categories of content knowledge in the minds of teachers? How, for example, are content knowledge and general pedagogical knowledge related? In which forms are the domains and categories of knowledge represented in the minds of teachers? What are promising ways of enhancing acquisition and development of such knowledge?"* (Shulman, 1986, p. 9).

Based on these questions, Shulman (1986) proposed to *"distinguish content knowledge* (p. 9)*"* into three different categories: Subject Matter Content Knowledge, Curricular Knowledge and Pedagogical Content Knowledge.

**Content Knowledge** is understood as the deep knowledge of the subject to be taught (both quantitatively and qualitatively). This implies that the teacher should not only know *"that* *something is so"*, but also *"why it is so"*. Therefore, in the teacher's mind there must be an organization of knowledge that allows the teacher to know also how to distinguish the central topics of a certain discipline from the peripheral ones.

**Curricular Knowledge** refers to the knowledge of the programs designed for the teaching of certain disciplines and to the knowledge of teaching materials to teach them. Shulman (1986) divides curricular knowledge into lateral and vertical curriculum knowledge: this means that the teacher must not only be master of this knowledge closely related to the discipline to teach and that specific curriculum, but he/she must also knows how to create connection with the topic that their pupils will learn over the next few years and within other disciplines. For this reasoning a mere knowledge of curriculum is not enough and the teacher "*must be familiar with the topics and issues that have been and will be taught in the same subject during the previous and subsequent years in the school, and with the materials that embody them*" (vertical curricular knowledge) and must also know the curriculum and teaching materials of other disciplines and classes (lateral curriculum knowledge).

**Pedagogical Content Knowledge** is the "new" (for those years) and crucial concept of pedagogical knowledge of the content linking Pedagogical Knowledge to Teaching Practice. Teachers need to learn the "psychologies" the topics they teach, making them accessible to everyone. Moreover, they need to know the suitable support to give to students in order to understand the topics specificity. This is a knowledge closely connected with subject matter content knowledge, because it includes the knowledge of different representations of the ideas that the teacher wants to teach and therefore of different illustrations, different examples, explanations, demonstrations and different interventions in response to the misconceptions that pupils can have. Indeed, teachers need to be aware of the concepts and preconceptions students have in relation to a particular topic. These elements are the result of the "wisdom of practice", but above all of the "research on teaching and on learning" that, in this circumstance, work together. This knowledge, according to Shulman (1986), should be at the heart of pedagogical understanding of subject matter.

These specific knowledge dimensions should be used for characterizing the research on teaching, especially because they can have a strong impact on state-level programs of teacher evaluation and teacher certification (Shulman, 1986).

Despite the great breakthrough in research by Shulman (1986) and his colleagues, at this first stage of its conceiving, pedagogical content knowledge appeared of limited utility and not adequately described to make it operative, especially because this research was theoretical and not empirical founded. As many researchers who contributed to the development of Shulman's idea observed: without empirical testing, ideas cannot improve teaching and learning. They in particular complain about the fact that the term pedagogical content knowledge appears not well defined and poorly developed, but with Shulman a new paradigm of research had opened up: teachers' knowledge needed to be more specialized. It was going to affirm the idea more and more clearly, clarified a few years later that: *"competence of a teacher in a particular area of ability (for example, throwing a ball) has been differentiated from the explicit knowledge of the ability necessary to teach it to students*" (Chen, 2002; Rovegno, Chen & Todorovich, 2003).

But unlike Shulman a few years earlier, to consider the knowledge that teaching entails, Ball and her colleagues (2008) have examined what teaching requires, working «from below upwards» starting with practice and focusing on the mathematical issue, instead of focusing on the curriculum or on standards for students learning. Their research is conducted through extensive qualitative analyses of teaching practice, with which hypothesis are made to design measures of mathematical knowledge.

> "*By Mathematical Knowledge For Teaching (MKT) it is emphasized the importance of focusing on the teaching process, and not on the teachers*: *MKT are the mathematical knowledge needed to carry out the work of teaching mathematics"* (Ball *et al.*, 2008).

With the term teaching, Ball and her colleagues (2008) mean everything that teachers must do to support the learning of their students: making interactive lessons, planning them and designing all the related tasks and so on. So, their research is based on what teachers need to know to make their teaching effective and on what are the recurrent tasks and problems in the mathematical teaching:

> *"What do teachers do when they teach math? What are the mathematical knowledge, skills and sensibilities needed to manage these tasks?"* (Ball *et al.*, 2008).

To answer these questions, they propose a diagram, refining Shulman's categories, in which "Pedagogical Content Knowledge" and "Subject Matter Knowledge" were the two principal domains of this model (Figure 5).

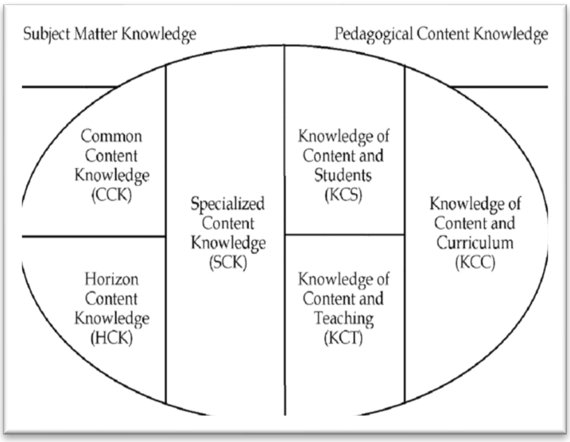



*Figure 5: Domains of Mathematical Knowledge for Teaching (Ball et al., 2008)*

Ball (2008) and her colleagues renamed the Curricular knowledge of Shulman (1986) into "Knowledge of Content and Curriculum (KCC)" to identify more clearly a type of mathematical knowledge, which has been incorporated in "Pedagogical Content Knowledge". KCC is understood as the knowledge of the programs designed for the teaching of certain disciplines, to allow the teacher to conduct linear mathematical paths through time, but that can also be interdisciplinary.

Teachers need to have awareness of how mathematical topics are related over the span of mathematics included in the curriculum, to allow them to make decisions on how to talk about a specific mathematical topic in order to simplify it and to be able to prepare the pupils to tackle future mathematical ideas. This type of skill is a mathematical knowledge identified by Ball (1993) as "Horizon Content Knowledge (HCK)", and which has been provisionally included like a third category within "Subject Matter Knowledge" (Ball *et al.,* 2008).

Pedagogical Content Knowledge also includes two other subdomains: "Knowledge of Content and Students (KCS)" and "Knowledge of Content and Teaching (KCT)".

KCS is a knowledge that allows teachers to create specific tasks to support students in overcoming their difficulties about what they could find confusing. When assigning these tasks, teachers who have this knowledge, will be able to anticipate and interpret students' answers, in order to provide them a more adequate feedback and support. With the experience, they can expand their familiarity with most common and potential students' mistakes, but also increase the set of more motivating and interesting ways to present more difficult topics. This knowledge, that for this reason is incorporated into the Pedagogical Content Knowledge, allows teachers to recognize the optimal moment during the growth of the students in which to propose new topics supported by their accurate learning skills.

KCT is a knowledge that combines the knowledge about teaching and the knowledge about mathematics, linked with related pedagogical problems: it concerns the knowledge that allows the teacher to design and evaluate a sequence of tasks (or interactive lessons) to deal with a topic, but also to take into consideration the comments of their students and use them to start a new mathematical discussion. Therefore, the teacher with this knowledge will know when to pause a task to make it clearer and to give time to students to ask questions about their doubts, but will also be able to be flexible on the scheduled lesson and to transform it if necessary.

The “Subject Matter Knowledge” domain, like Shulman’s Content Knowledge, satisfies mathematical demands of teaching and it’s composed by “Common Content Knowledge (CCK)” and “Specialized Content Knowledge (SCK)”, along with HCK.

CCK is a mathematical knowledge that allows anyone, and therefore also teachers, to give correct answers to mathematical problems, to be able to make correct calculations, to recognize the accuracy of an answer or a definition on a textbook. This kind of knowledge, however, is not unique for teaching, but of course it is absolutely necessary for a mathematics teacher.

The mathematical knowledge and skill, identified by Ball et al., (2008) as “Specialized Content Knowledge (SCK)”, require a kind of mathematical work that other do not. The decompression of the mathematical knowledge entails the use of a mathematical language according to the pupil’s level, as well as appropriate mathematical representations and effective mathematical ideas, thus being coherent with the mathematic CCK.

To explain the difference between CCK and SCK, the authors propose an example based on a simple subtraction computation in which there is a common error like in Figure 6.

| | | |
|---|---|---|
| 3 0 7 –<br>1 6 8 = | 2 9<br>~~3~~ ~~0~~ 7 –<br>1 6 8 =<br>1 3 9 | 3 0 7 –<br>1 6 8 =<br>2 6 1 |
| Subtraction | Correct computation. | Incorrect computation |

*Figure 6: Example based on a simple subtraction computation (Ball et al., 2008)*

Knowing how to perform this procedure (correct computation of subtraction) is a necessary but not sufficient condition to teach it (CCK). Knowing how to recognize the presence of an error in this procedure is something that anyone can see (CCK), but the diagnose of the error does not provide to the teacher a detailed mathematical understanding necessary for an effective treatment of the problems encountered by the students or for a development of a specific didactic action to prevent such error: the teaching requires more than the identification of a wrong answer (SCK).

The activity of interpreting is a daily task for teachers, who are faced every day with incorrect answers (both written and oral) but also with unpredicted solutions and non-standard strategies. According to the SCK, it is necessary to spend time in order to decompress them.

The management of errors in teaching mathematics is the central point of another conceptualization of teachers' knowledge, the Interpretative Knowledge – IK (e.g., Di Martino, Mellone and Ribeiro, 2020: Mellone et al., 2020), identified by a group of researchers who have worked in the MKT framework and which will be taken up later in the thesis.

## 2.2 Mathematics Teacher Specialized Knowledge (MTSK)

After an in-depth analysis of the MKT model, Carrillo and his colleagues (2018) questioned the positioning of SCK as a sub-domain of SMK, believing that SCK is also strictly connected to the PCK sub-domain.

These researchers pointed out that the specificities of teachers' knowledge, with respect to the model of the group of Michigan (Ball *et al.* 2008), must concern both mathematical content and teaching practice. Therefore, they proposed a new model, named Mathematics Teacher's Specialised Knowledge (MTSK), which considers all the knowledge that a teacher needs to have for teaching mathematics as specialised. A compact representation of the MTSK model is sketched in Figure 7.

According to them, the specificities required for mathematical teaching concerns *"meanings, the properties and definitions of particular topics, the means of building understanding of the subject, connections between content items and characteristics associated with learning mathematics, amongst others"* (Carrillo *et al*., 2018, p. 239).

Therefore, the model is mainly divided in two domains: the "Mathematical Knowledge" (MK) domain, that is a systemic combination of structured knowledge with its own rules and the "Pedagogical Content Knowledge" (PCK) domain that is strictly linked with classroom mathematic practice. These areas must be complementary to allow teachers to take decisions and to be reflective and critical about their actions in order to improve mathematics education.

It is important to emphasise that the PCK sub-domains are also closely related to the mathematical contents (as already suggested by the names of the three sub-domains), because the knowledge that belongs to them derives chiefly from mathematics: *"knowledge in which the mathematical content determines the teaching and learning which takes place"* (Carrillo *et al*., 2018).

Each domain is divided into three sub-domains. MK is divided into Knowledge of Topic (KoT), Knowledge of the Structure of Mathematics (KSM) and Knowledge of Practices in Mathematics (KPM). PCK, in turn, is divided into Knowledge of Mathematics Teaching (KMT), Knowledge of Features of Learning Mathematics (KFLM) and Knowledge of Mathematics Learning Standards (KMLS).

The centre of the model is composed by “Beliefs on mathematics” and “Beliefs on mathematics teaching and learning” which represent *“conceptions and beliefs* (Thompson, 1992) *about mathematics, how it is learned and how it should be taught”* (Carrillo, *et al*., 2018, p. 240): a philosophy of mathematics that influences (and is influenced by) the knowledge domains.

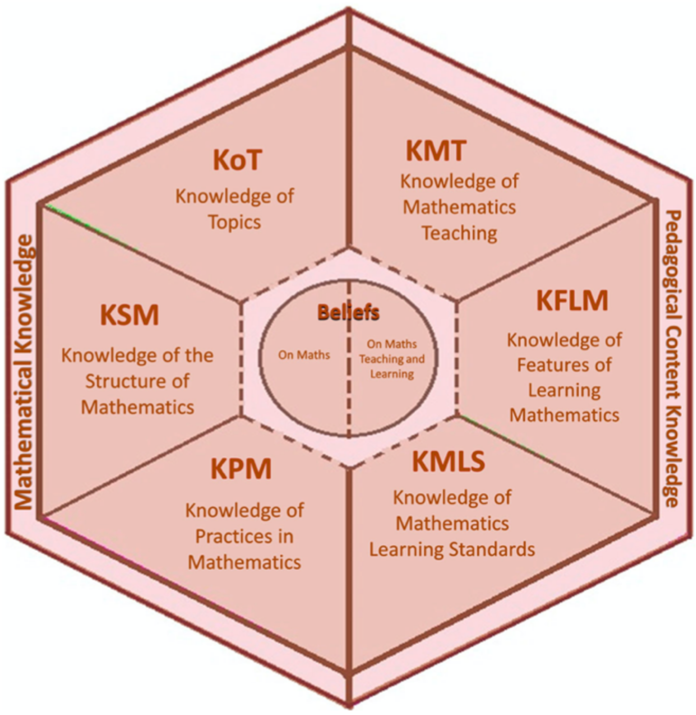



*Figure 7: The Mathematics Teacher’s Specialised Knowledge model (Carrillo, et al., 2018)*

The MTSK model arises from the need to identify the specific knowledge (organized in domains and sub-domains) required to the teacher to carry out his/her work of teaching:

> *“We consider the knowledge possessed by a mathematics teacher in terms of a scientific discipline within an educational context – the domain of Mathematical Knowledge (MK). We broaden the idea of Subject Matter Knowledge (Shulman, 1986), in that we consider characteristics of mathematics as a scientific discipline, and at the same time recognise a differentiation between Mathematics per se and School Mathematics. The other domain – Pedagogical Content Knowledge (PCK) – is comprised of the knowledge relating to mathematical content in terms of teaching- learning”* (Carrillo *et al*., 2018, p. 240).

To describe each sub-domain, Carrillo and colleagues (2018) refers strictly to what teachers need and use in their classroom practice, using categories. Therefore, they emphasize that teachers need knowledge about a particular sub-domain. However, this does not mean to have a predetermined list of contents, but that the teacher must have knowledge that can be located in those sub-domains.

For a better understanding of the meaning of each category, Carrillo and his colleagues (2018) proposed specific examples related to different mathematical topics. Referring to a single mathematical topic allows to distinguish more specifically the different categories. To this regard, Zakaryan and Ribeiro (2019) explored the KoT via a case study in the context of rational numbers, to characterize the teachers' specialized knowledge in a more refined manner and to investigate in depth the content of teachers' knowledge.

Hereafter, each sub-domain of MTSK will be presented in terms of its theoretical conceptualization. Given the research focus, examples of content related to teachers' knowledge on the topic of probability will be exclusively presented for the Knowledge of Topics sub-domain.

### 2.2.1 Knowledge of Topics (KoT)

KoT's sub-domain (Figure 8) concerns mathematics content itself and describes the "what" teachers know about the topic they teach, and so the definitions and notions about a specific mathematical topic, its own set of properties, the intra-conceptual connection present between single content items of the topic and the understanding of its foundation and history.

What makes the KoT a mathematical specialized knowledge is the way in which the teachers know the topic they teach, since KoT is defined as a deeper and broader perspective of content knowledge which is only required for math teachers. This sub-domain includes different representations system about it and different meanings connected with each operation related to that specific topic: these categories allow teachers to manage mathematical procedures in a more performing way exploring the characteristics of the resulting object linked to how, when and why it is obtained.

Mathematical epistemological aspects, and so all the forms of examples explorable in a real context or in connection with other disciplines, characterize the category of phenomenology and application, useful to contextualize a problem and a situation.

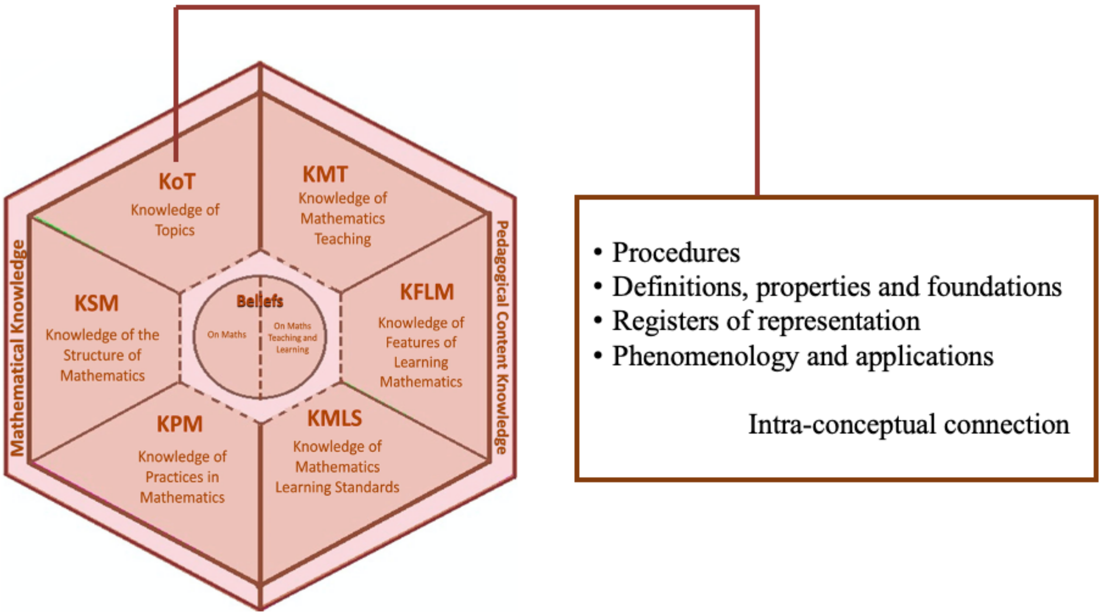



*Figure 8: KoT's categories*

In the field of probability, it is important to understand which kind of specialised knowledge is that a teacher needs to have, and which the sub-domains such a knowledge can be located.

It is desirable that a teacher has a broad and deep KoT, which involves, among other things, the fact that probability needs to be considered not only in the classic probability's vision, but also with the frequentist and subjectivist approach.

It is also relevant that teacher has a knowledge of the definition and the meaning of probability, which allows knowing the potential of each approach.

Let's take the example of the "dice game", that it is also the context of the discussion that we will analyse later, and consider the problem of the quantification of the *degree of confidence* of events "sum of faces of two six-sided dice" (KoT - phenomenology and application). This issue can be faced using all the three approaches of probability (KoT - definition, properties and foundation).

The classical approach to probability sees the probability's measure like a fraction between the number of favourable cases of a given event $E$ and the number of all possible cases of a given event space (KoT - definition). Thus, according to the classical approach, in Figure 9 are reported, with the black colour, the probability's measures associated with each sum (listed in green and unpacked in orange and blue) of the problem (KoT - procedures) presented.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | 6+1 | | | | | |
| | | | | 5+1 | 5+2 | 6+2 | | | | |
| | | | 4+1 | 4+2 | 4+3 | 5+3 | 6+3 | | | |
| | | 3+1 | 3+2 | 3+3 | 3+4 | 4+4 | 5+4 | 6+4 | | |
| | 2+1 | 2+2 | 2+3 | 2+4 | 2+5 | 3+5 | 4+5 | 5+5 | 6+5 | |
| 1+1 | 1+2 | 1+3 | 1+4 | 1+5 | 1+6 | 2+6 | 3+6 | 4+6 | 5+6 | 6+6 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ | $\frac{5}{36}$ | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ |

*Figure 9: probability's measures associated with sum from 2 to 12*

In the frequentist approach, if an experiment is repeated in the same conditions for a considerable number of times (KoT - phenomenology and application), the relative frequency of an event is considering a first reasonable valuation of the degree of reliability of the occurrence of the event itself (KoT - definition, properties and foundation). With "relative frequency" is meant the ratio between the number of occurrences of an event (meaning the number of times that an event happens, that can be "how many times the sum 6 came out of the dice") and the number of accomplished proves (that can be "the number of times we roll the dice"). By the increasing of the number of experiments or accomplished proves (potentially for an infinite number of times), the value of the frequency tends to the theoretical value of probability, that is the value given to the classical probability (law of large numbers - KoT - definition, properties and foundation).

The construction of the probability theory from a subjectivist point of view has emerged from the intention to give to the probability's meaning a psychological basis. In particular, considering teacher's knowledge of the probability of an event is a quantitative measure of the *degree of confidence* based on the judgment that events occur, is a knowledge associated to a foundation on probability theory (KoT - definition, properties and foundation). In this perspective, what really matters is not the concept *"what I foresee, will happen, because I foresaw it"* but, instead we should focus on the question *"why do I foresee that this event will happen?"* (de Finetti, 1931).

Indeed, the answers to the query "why different degrees of confidence can be attributed to different events" are various: reasoning based on sensations, on statistical analysis or on assessments that rely on the combinatorial calculation. Based on the subjectivist approach, the definition of probability P of an event *E* can be given supposing that a bookmaker is obliged to accept bets on a certain number of events, including the event $E$ (KoT - phenomenology and application).

It is still important to highlight that, the teacher’s awareness that the three approaches to probability are interconnected constitutes part of his/her specialized knowledge identified as intra-conceptual connection, within the subdomain KoT.

Indeed, mathematical connections are approached from three distinct perspectives. Firstly, they are considered intrinsic characteristics of the mathematical discipline, both in academic and school contexts. Secondly, they are seen as a mental construction or elaboration by mathematics students. Lastly, they are understood as a process of establishing associations through observation and seeking relationships among mathematical elements (Businskas, 2008).

The main difference between second and third perspectives lies in the nature of the activity involved in forming connections. In the second perspective, this activity is a mental function that occurs at a higher cognitive level, characterized by a process of "abstraction" (Skemp, 1989). On the other hand, in the third perspective, the activity is viewed more mechanically, where connections are identified through a deliberate process of search and observation of relationships.

In this way, while on one hand a mathematical connection can arise from intrinsic relationships among elements, that is, relationships that exist independently of being perceptible to an individual, on the other hand, it is possible to perceive that a mathematical connection is the result of the relationship that an individual establishes consciously and intentionally (or not) among these elements, based on their mathematical knowledge (Policastro, 2021).

In this context, to explore the teacher's knowledge regarding mathematical connections, it is essential to understand and consider them as "products", that is, "mental objetcts" (Businskas, 2008, p. 17), which can be identified, developed, recalled, and discussed.

Understanding, therefore, that mathematical connections are products of relationships established by individuals, whether consciously and deliberately or not, in the context of this research, we come to view connections as "*products of the relationships that are established between different constructs, concepts, properties, or foundations within the same topic and/or between different topics*" (Policastro, 2021, p. 79). This will be useful for assuming the intra-conceptual connections (Carrillo et al., 2018) in the context of probability as relationships established between different constructs, concepts, properties or foundations within the classical, frequentist or subjectivist approaches.

### 2.2.2 Knowledge of the Structure of Mathematics (KSM)

KSM’s sub-domain (Figure 10) includes the interlinking system which bind the contents of the topic with themselves in a temporal vision and the connections with other mathematical items. It follows that the knowledge of a mathematical topic must be broad in order to be able to approach a mathematical topic with different levels of depth, suited to the pupils’ level of comprehension.

A simple level of deepening must not, however, exclude any important element of that argument not to create crack when, successively over time, will renew that argument with a higher degree of complexity.  This type of knowledge allows the teacher to simplify or complicate the same topic and to connect with the pupils the different levels of complexity developed over time.

KSM refers to the teachers knowledge grounding making connections between different mathematical topics and involves four categories: connections based on simplification, connections based on complexification, auxiliary connections, transverse connections. Connections based on simplification and complexification are related to the teacher's knowledge that more advanced concepts build upon elementary ones in the case of simplification. Conversely, in the case of complexification, the teacher's knowledge is associated with elementary concepts, properties, and/or foundations related to a specific topic, viewed from an advanced standpoint (Klein, 1908), aiming to facilitate students' mathematical comprehension. Auxiliary connections, in turn, *“concerns the necessary participation of an item in larger processes”* (Carrillo *et al*., 2018, p.244), while transverse connections *“results when different content items have features in common such as is the case with the concepts of limit, derivative, local and global continuity”* (Carrillo *et al*., 2018, p. 244).

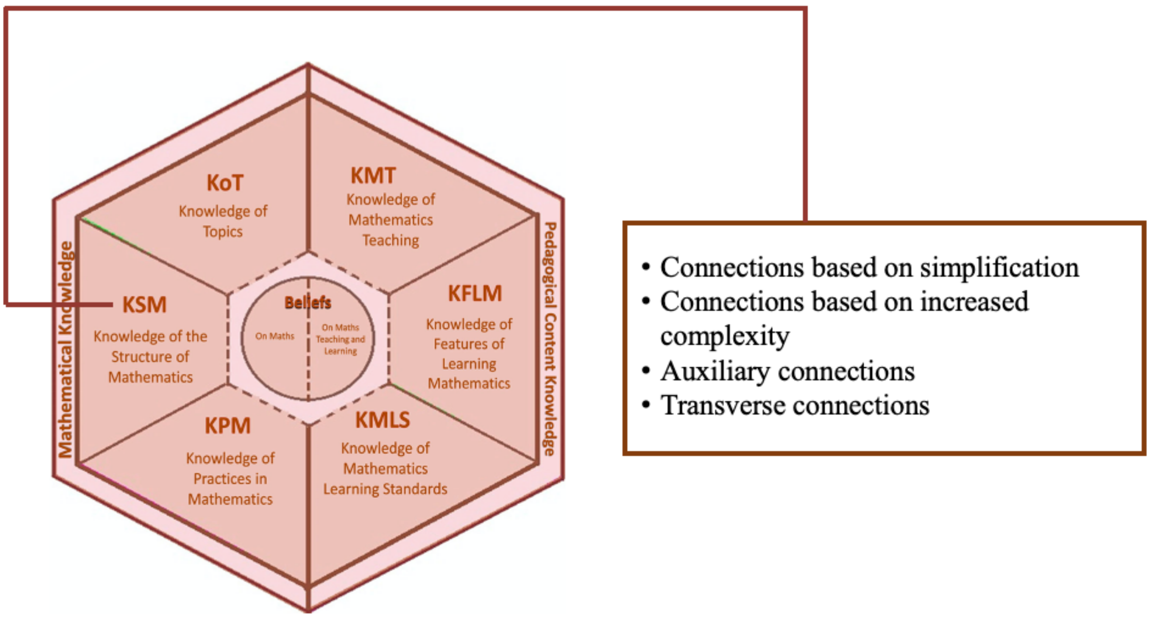



*Figure 10: KSM's categories*

### 2.2.3 Knowledge of Practices in Mathematics (KPM)

In the KPM sub-domain, "practices" refers to the functioning of mathematics and the syntactic knowledge (Schwab, 1978) of mathematics rather than to actions that are activated in the learning - teaching process. This sub-domain (Figure 11) includes *"knowing about demonstrating, justifying, defining, making deductions and inductions, giving examples and understanding the role of counterexamples. It also includes an understanding of the logic underpinning each of these practices* (Carrillo *et al*., 2018, p. 244).

> *In the MTSK model, Knowledge of Practices in Mathematics focuses specifically on means of production and mathematical functioning"* (Carrillo *et al.,* 2018, p. 245).

Knowledge of the Practices of Mathematics (KPM), falls under the umbrella of foundational mathematical activities. Its categories were recently delineated by Delgado-Rebolledo (2020).

These categories are: knowledge of the practice of demonstrating, knowledge of the practice of defining, knowledge of the practice of problem-solving, knowledge of the role of mathematical language.

These components form essential elements within the realm of mathematical creation, contributing to a deeper understanding of mathematical concepts and their practical applications.

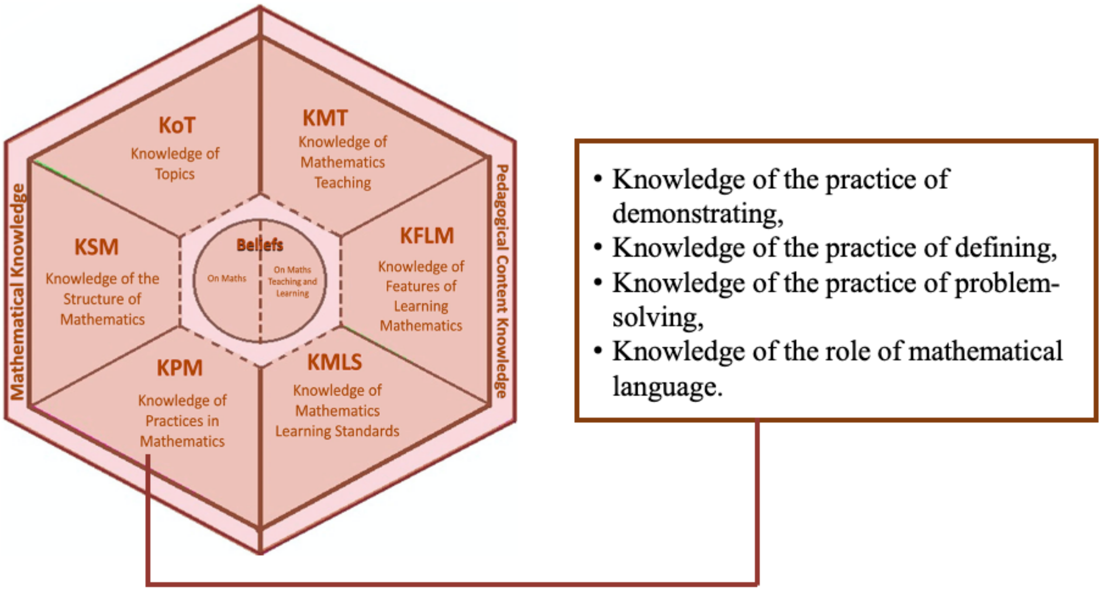



*Figure 11: KPM's descriptors*

### 2.2.4 Knowledge of Mathematics Teaching (KMT)

KMT's sub-domain (Figure 12), does not concern purely pedagogical knowledge, but *"knowledge intrinsically bound up with content"* (Carrillo *et al.*, 2018, p. 247). This knowledge is what enables teachers' to be aware of more effective activities, strategies and techniques to deal with a specific mathematical content. Teachers need to have critical awareness, and not only a mere knowledge and information, of resources and teaching materials like textbooks, manipulatives, technological resources or interactive whiteboards, in order to design efficient learning opportunities.

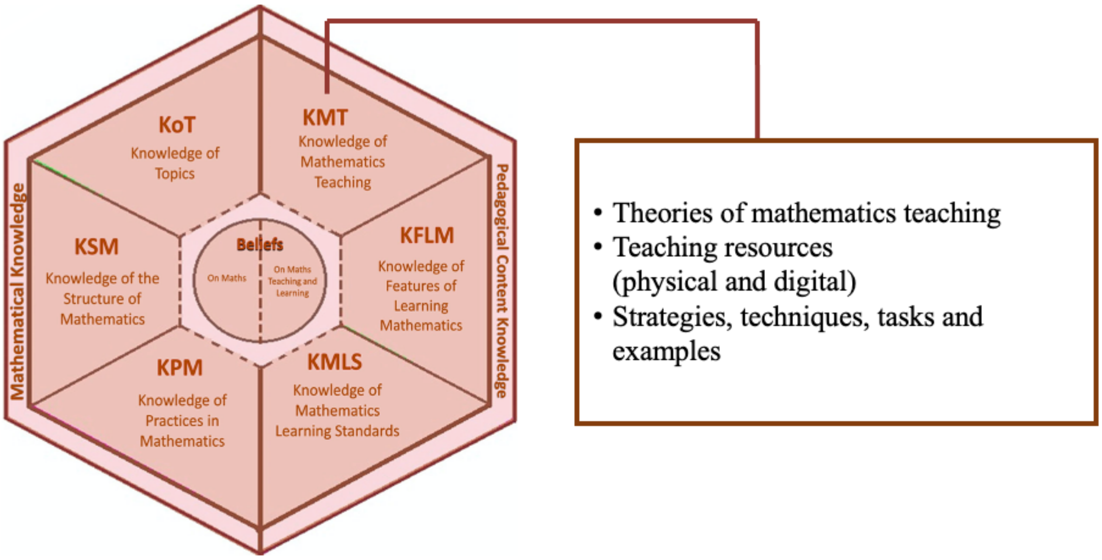



*Figure 12: KMT's categories*

## 2.2.5 Knowledge of Features of Learning Mathematics (KFLM)

KFLM (Figure 13) includes the knowledge of “how” the pupil learns (or can learn) mathematics. The mathematical topic, in this sub-domain, is intended as the object of pupils’ learning and, as such, the KFLM’s knowledge concerns the way in which students interact with that topic. To face that specific topic, the teacher will have to wonder how he/she can ease the student understanding, what will be the pupils’ reasoning and thoughts about the proposed task, what will be the possible answers, strategies, and errors adopted for the resolution of a problem, what are the strengths to start from.

This knowledge can be developed with the help of the mathematical education research and, especially, with a meticulous analysis of their teaching experience. Carrillo *et al.*, (2018, p. 247) consider that *“the final element of KFLM concerns the emotional aspects of learning mathematics* (Hannula, 2006). *At one extreme, this involves awareness of, for example, mathematics anxiety* (Maloney, Schaeffer, & Beilock, 2013)*, but it includes, too, such everyday things as what motivates the students, their interests and expectations of mathematics (both in general and in terms of specific areas)*”.

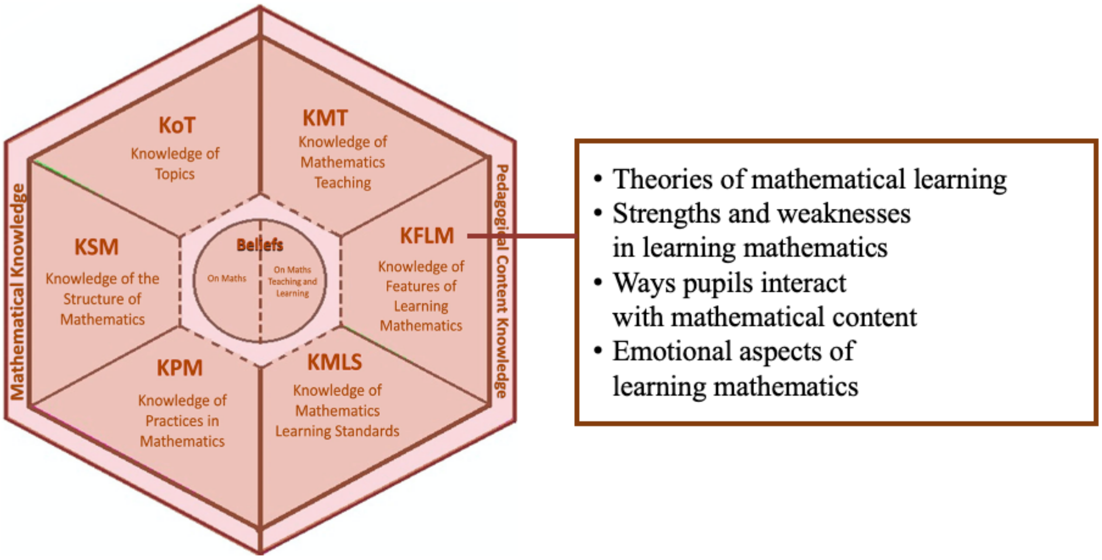



*Figure 13: KFLM's categories*

### 2.2.6 Knowledge of Mathematics Learning Standards (KMLS)

KMLS (Figure 14), in addition to what Ball and colleagues (2008) suggested, does not include only a mere knowledge of the curriculum, but a *"knowledge which enables the teacher to be critical and reflective in considering what the student should learn, and what focus should be taken, at any particular level, or period of development"* (Carrillo *et al.*, 2018, p. 246).

For this reasoning and to achieve this, the teacher needs to know how to search for information in unofficial documents (*e.g.*, curriculum specifications from other countries or research literature) and in what way teach mathematical content at any particular level.

This sub-domain also includes knowledge about the sequencing topics, considering the content to be taught "*both retrospectively, in terms of knowledge previously acquired, both prospectively, based on the knowledge that will have to be acquired to address subsequent topics*" (Carrillo *et al.* 2018, p. 247).

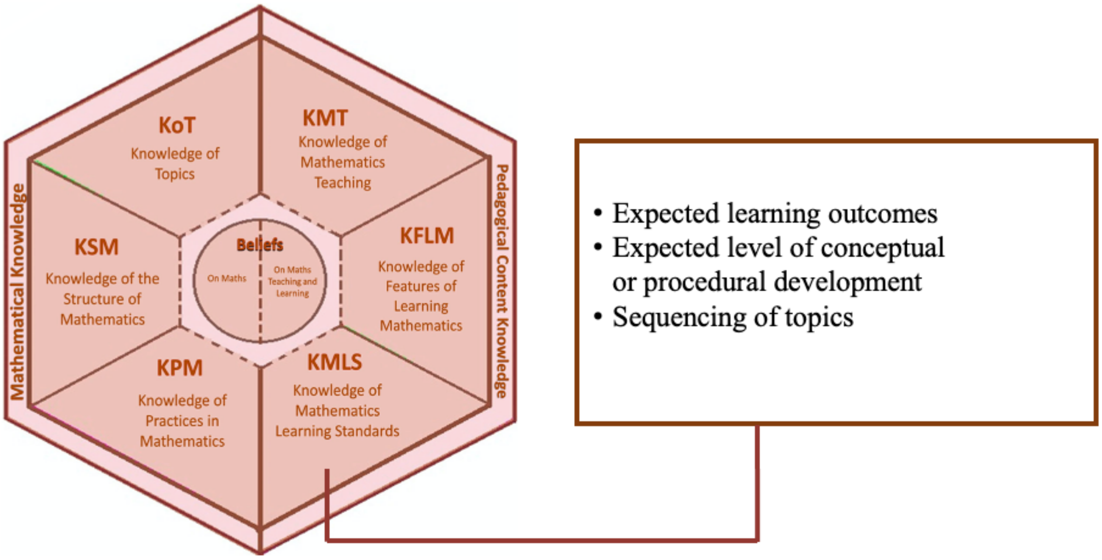



*Figure 14: KMLS's categories*

## 2.3 The dynamic way of perceiving teachers' knowledge as a process of knowing

In literature concerning mathematics teachers' knowledge, the researchers very often refer to a crucial action needed to move from the academic mathematics to the mathematics for teaching: the "unpacking" of a mathematics content, that involves a decompressions process in the form of examples, explanations and simplifications for making more extensive and richer the mathematics content and suitable to be worked in the teaching and learning process (Adler & Davis, 2006; Ball & Bass, 2000; Ma, 1999). This reference to the "unpacking process" emphasizes how the mathematical knowledge for teaching involves many processes, and perhaps it can be useful to look at itself as a process. Indeed, the authors of the MTSK model, alongside other authors, shifts the focus of attention to specialized "knowledge" of mathematic teachers introducing an alternative point of view that involves and highlights the process of "knowing" (Scheiner, Montes, Godino, Carrillo, & Pino-Fan, 2019).

> *"In our view, whether knowledge is specialized or not is a question of whether the knowledge is contextually adaptive (Hashweh, 2005), that is, functional on a moment- by-moment basis, and highly sensitive to the changing details of the situation as teachers interact with the environment and with the students around them"* (Scheiner *et al.*, 2019, p. 162).

Therefore, Scheiner and colleagues (2019) propose to focus on the process that teachers cross - or need to cross - to shape, and ensure specialized, the knowledge that they need: not only "what" teachers know, but "how" teachers' knowing comes into being.

> *"As such, a mathematics teacher's action is not a simple display of a static system of some certain knowledge types, but rather a highly contingent and continually adaptive and proactive response that shapes, and is shaped by, the environment in which the teacher interacts"* (Scheiner *et al.*, 2019, p. 163).

This change of perspective (Scheiner *et al.*, 2019), regarding mathematics teachers' styles of knowing, puts in constant interaction the various facets of the knowledge's organic whole that becomes specialized when teachers adapt it to the complex dynamics in which knowledges are useful.

Indeed, the authors emphasize viewing mathematics teacher knowledge more as a mindset rather than static traits or dispositions. The distinction proposed revolves around considering "knowledge about/of/for/in" a discipline versus "disciplinary knowing." While the former prompts inquiry into various types of knowledge, the latter focuses on a style of knowing shaped by specific activities, orientations, and recognizable disciplinary dynamics. There is a recognition of mathematics educational knowing as a specialization within mathematics teacher knowledge, illuminating a pathway for better integration of knowledge and action in teaching practice (Scheiner *et al.*, 2019).

The proposed dynamic point of view tries to overcome the static and additive vision of supplementary "item" knowledge needed to teach, but is rooted in the epistemology of knowledge and in the interactions that connect them (Scheiner *et al.*, 2019). In this perspective, the same teacher education cannot be static and detached from teaching practice, but it must necessarily take into account what happens in the classroom: the words of the pupils, the productions, the drawings, the answers to tasks, their gestures and their actions should be taken as learning opportunities for teachers.

In order to specialise the mathematical knowledge of teachers in a dynamic way, researchers involved in teacher education can use in a suitable way and according their goals the analysis of all the productions of the students. The process of understanding what is strange or wrong, the observation and investigation of the task answers different from what a teacher is accustomed to receive, are an integral part of the evolutionary and transformative process of knowledge.

In this perspective, for example, mathematical errors, but also non-standards reasoning and mathematical ambiguities should not be seen as "*something that make you lost*" (Borasi, 1996), but they can be used as opportunities in teacher educations contexts as in the real mathematics classes environments as firstly addressed in Ribeiro *et al*. (2013). Indeed, referring to the new approach to the errors in the field of mathematics education presented by Borasi (1996), a contribution to expand the construction of SCK in the sense of Ball's model (2008), was provided by Ribeiro *et al*. (2013) with the notion of "Interpretative Knowledge (IK)". It is focused on the vision of errors as learning opportunities and as a starting point for building students' mathematical knowledge and ways of mathematical thinking. In this trend of research, teacher education specific activities have been developed to broaden IK in preservice and in-service teachers:

> *"These activities are mainly based on open questions and structured into two phases. In the first phase, teachers have to solve a problem: in this way, they face with the difficulties of the problem, and their favourite strategy can emerge. In the*

*second phase, teachers are asked to evaluate several and different strategies developed by real students and to propose specific feedbacks for each strategy"* (Di Martino, Mellone and Ribeiro, 2020)

To activate the process of interpretation, teachers have the opportunity to study the proposed problem and solve it and later they receive pupils' strategies that have to be interpreted. As in a professional simulation, teachers become part of that dynamic experience in which the non-standard solution of pupils are evaluated; but in a checked situation, that is by having the time for interpreting and analyzing them, first individually or in small group and then collectively, and having the chance to come back to their initial interpretations and, in the case, to change these initial interpretations considering also that "*specialization is not a state of being but a process of becoming: mathematics teacher knowledge becomes specialized in its adaptive function in response to the dynamics and complexities in which it comes into being*" (Scheiner *et al.*, 2019, p. 155).

In order to size the source of an error, a mathematical knowledge tailored for teaching is required. Indeed, it features a decompression of mathematics unnecessary in other contexts and that goes beyond what is taught to students. Students have the task of exercising fluently with compressed mathematical knowledge, but the teacher must keep an unpackaged mathematical knowledge to teach it. For example, as we saw in the example shown in Figure 6, the teacher must know how to explore the mathematical error committed and must know how to recognize that the pupil has made the error in the part relating to ones and tens, inverting the minuend with subtrahend to perform the subtraction in a more comfortable way. In this way, the teacher will quickly recognize the motivations that led the pupil to make that mistake and will build a new path in the subtraction topic that satisfies that specific gap and, finally, propose new tasks to check whether that difficulty has been overcome.

> *"We hypothesise that (prospective and in- service) teachers' IK can be developed through training that focuses on what students actually know and how they know it, thus assisting teachers in understanding students' mathematical reasoning"* (Mellone *et al.*, 2020).

The interpretive process should, therefore, be at the core of teacher practice, so that what is wrong (or seems wrong) can become an opportunity to explore new strategies to solve a problem or mathematical exercise (Mellone *et al.*, 2020). Having a high level of this knowledge permits teachers to use the students reasoning to develop specific feedback in order to increase their learning.

> *"Interpretative Knowledge refers to a deep and wide mathematical knowledge that enables teachers to support students in building their mathematical knowledge starting from their own reasoning and productions, no matter how not standard or incorrect they might be"* (Di Martino, Mellone, & Ribeiro 2020).

This knowledge allows to give meaning to the students' productions, to understand their unexpected strategies or approaches and to make an unpacking of the typical or atypical error they have made, in the Borasi's sense.

With the studies carried out by Jakobsen et al. (2016), it was emphasized that having a strong CCK is not enough to develop a good level of IK, while having a solid SCK allows teachers to think ahead, and more quickly, alternative ways of solving a problem and proposing more constructive feedback when students make less common errors or productions.

# 3. Mathematic education in probability

This chapter aims to explore the relationship between the mathematical perspective on probability and educational aspects.

In particular, this chapter will provide a brief overview of previous research on mathematical education in term of probability. It will review the key concepts and principles that have been identified and discussed in the literature, as well as the main challenges and difficulties that students tend to encounter in this area. Additionally, it will examine the status of teacher education programs regarding the instruction of probability. While there is a growing body of research on mathematics education, studies specifically focused on the teaching of probability are relatively scarce (Alonso-Castaño, 2021). This is noteworthy given that the ability to understand random phenomena and make sound decisions under conditions of uncertainty is recognized as an important skill by many educational authorities (Batanero et al., 2016).

The amount of research on teacher education on probability is significantly lower than that focused on students. Moreover, much of the research on teacher education has been concentrated on preparing secondary school teachers (Alonso-Castaño, 2021). While probability is included in the curricula of many countries for primary and secondary education, (Batanero et al., 2016), there is still controversy around the interpretation of fundamental concepts such as the definition of probability (Batanero & Serrano, 1995). Besides, the complexity and lack of clarity of numerical and verbal expressions used to convey probability must be addressed (Blanco-Fernández et al., 2016; Rodríguez-Muñiz et al., 2021).

As probability has only recently been introduced into lower levels of education (e.g., BRASIL, 2018; MIUR, 2012), further research on teacher education is needed to improve training programs and ensure that teachers can offer comprehensive and effective instruction in probability from an early stage.

As usually, research in this field can be broadly categorized into two areas: one of them focusing on student learning processes and related activities in the context of probability; and the other, focusing on teaching processes and teacher knowledge of probability.

In the following paragraphs, it will briefly describe some of the research conducted in these two areas. In the final part of this chapter, specific research in the topic of subjective probability will be discussed: some researchers have the common aim of proposing some educational paths for the introduction of the concept of probability employing the subjectivist approach. During the analysis of the papers, it can be observed that each author will create different paths based on their "interpretation" of the subjectivist probability's approach. It is important to observe that an univocal didactical "ideal" path useful to keep together the contrasting opinions hasn't been proposed yet.

One of the main reasons that researchers have focused on this mathematical aspect is due to the centrality of concepts such as "uncertainty" and "risk" in daily life. In order to manage situations based on these concepts, it is essential for every citizen to develop strategies that enable them to make informed decisions in this field. While various didactical strategies with different approaches are employed in each paper, all the authors agree that probability is a useful tool for decision-making under conditions of uncertainty in everyday life. As such, the subjectivist approach could be an effective way to introduce probability in primary and secondary schools (Castilla et al., 2022; Homier & Martin, 2021; Rodríguez-Muñiz et al., 2022).

## 3.1 Research on Learning and Teaching probability

In the late 1980s and early 1990s, the Research Group on Statistical Education at the University of Granada began studying the teaching of probability. Godino et al. (1987) published their first work on the subject, proposing didactic principles and curricular proposals. In the 90s, the group investigated various aspects of probability, such as randomness (Batanero & Serrano, 1995), probabilistic reasoning among primary and secondary school students (Batanero et al., 1994; Cañizares & Batanero, 1997; Serrano et al., 1998), frequentist probability in Bachiller (Ortiz et al., 1996; Serrano et al., 1996), students' beliefs and conceptions of probability (Cañizares and Batanero, 1997; Serrano et al., 1999), and the idea of fair play in primary school (Cañizares et al., 1999). The group's research has covered a broad range of topics, including the teaching and learning of conditional probability (Contreras, 2011) and probabilistic content in textbooks (Gómez-Torres et al., 2014), and has made valuable contributions to the field of probability education.

Some curricula include probability since primary school, as it is the case of Australian Curriculum, Assessment and Reporting Authority; in contrast, other curricula delay to introduce the probability, only consider it on secondary education (Batanero *et al.*, 2016).

Primary school students typically approach probability by distinguishing "between the possible, the impossible, the certain, and what is possible but not certain" (Batanero, 2013) using everyday language that includes expressions related to probability.

However, some statistics education researchers recommend teaching statistical inference with an informal approach, which has led to a diminished emphasis on probability education. This change overlooks the importance of educating young children in probabilistic reasoning (Fischbein, 1975), and the multiple connections between probability and other areas of mathematics, as stated in the Guidelines for Assessment and Instruction in Statistics Education (GAISE) for pre-K-12 levels (Franklin et al. 2007). Therefore, Probability is a vital component of any mathematical education, as it enriches the subject as a whole through its interactions with other uses of mathematics (Batanero, 2013).

Classical and frequentist are the most common definitions of probability, and they appear most frequently in textbooks. These definitions are useful for laying the foundations for learning more complex topics in probability (Gómez-Torres et al., 2015).

Even if a low presence of subjective meanings even in secondary school textbooks are identified (Ortiz, 2002; Azcárate & Serradó, 2006), this is practically non-existent in primary education (Gómez-Torres et al., 2015).

Contrary, previous studies have specified that the different meanings of probability should appear since primary school, including the intuitive, frequency, subjective, and classical, and finally the axiomatic meaning in secondary school (Batanero, 2005; Batanero & Sánchez, 2005).

Many educational resources have been used to support probability education, and game-like scenarios that involve chance are a common method. Physical devices such as dice, coins, spinners, and marbles in a bag are often used to create these scenarios (Nilsson, 2014).

These resources are typically used to support a classical approach to probability, computing the probability of an event occurring a priori by examining the object and making assumptions about symmetry that often lead to equiprobable outcomes for a single trial (Batanero et al., 2016). These devices are often used together (e.g., two coins, a die, and a four-section spinner) to explore compound events and conditional probabilities (Martignon and Krauss, 2009). Organizational tools such as two-by-two tables and tree diagrams are also used to assist in enumerating sample spaces (Nunes et al., 2014) and computing probabilities.

Teachers have been increasingly using digital devices since the physical ones can be manipulated to induce chance events. For instance, experiments such as rolling, spinning, choosing, or dropping a marble can create relatively small samples and allow the recording of the frequencies of events.

These frequencies and relative frequencies are then used as estimates of probability in the frequentist perspective, and they are often compared to a priori computed probability based on the examination of the object (Batanero, 2016). Besides, when students are engaged in experiments with physical device, they tend to reveal specific and powerful mathematical thinking (Nilsson, 2009). The experiments that use physical devices to induce chance events may result in different probability estimates, depending on the perspectives of the teachers or the students involved. As a result, issues related to the sample size and the law of large numbers, as well as the distinction between frequency and probability, may or may not be discussed (Batanero et al., 2016). It is important for educators to be aware of these potential variations in perspectives and to address these issues as they arise in order to facilitate deeper understanding of probability concepts (Stohl, 2005).

Biehler (1991) suggested that technology could be utilized to enhance the learning and teaching of probability through various methods, such as the organization and analysis of data generated by a probabilistic model or through sampling, as well as the storage of data.

Indeed, understanding probability models and the data they produce is essential for a solid foundation in how probability is utilized in statistics, particularly when making inferences about populations and testing hypotheses. In certain cases, the models used in simulations are created by students or teachers, but they are open for inspection by students.

However, technology tools also offer the option to hide the model from the user, making the underlying probability distributions that control the simulation unknowable. While these "black-box" simulations may help students think about probability from a subjective or frequentist perspective, they are limited to only using the data generated from the simulation to make probability estimates for use in inference or decision-making scenarios (Batanero, 2013) For example, Lee et al. (2010) presented a scenario in which 11- and 12-year-olds students investigated whether a die company produces fair dice, to decide whether to buy dice from the company for a board game. This demonstrates how probability concepts and inference can be applied to real-life situations.

Research studies have shown that the prospective teachers' education on probability is inadequate and ineffective (Batanero et al., 2011; 2012) point out that teachers have poor training and misconceptions about probability, which can be attributed to the recent incorporation of probability in the compulsory school curricula. In order to improve the quality of probability education in schools, teacher training programs need to focus on developing teachers' probabilistic literacy and didactic skills.

Several studies have proposed training activities to help teachers overcome their difficulties in teaching probability. For example, some studies have proposed training programs that emphasize the development of skills in modelling probability and teaching it in an efficient and meaningful way. These programs have shown that pre-service teachers often have misconceptions about randomness and equiprobability, and they struggle to interpret the results of casual experiences (Arteaga et al., 2010; Batanero et al., 2012; Batanero & Arteaga, 2014; Batanero, Gómez-Torres, et al., 2014).

To improve the quality of probability education, it is crucial that teacher education programs provide prospective and practice teachers with adequate support in terms of stochastic and didactic education. This is especially important given that the teaching of probability in schools is often poor and ineffective, largely due to the lack of teacher education and knowledge of probability concepts (Alsina & Vásquez, 2016; Vásquez and Alsina, 2017a, 2017b). Thus, it is essential that teacher education programs focus on enhancing the professional development of teachers and promoting their probabilistic literacy.

The importance of teachers' knowledge in the scope of probability has been a focus of research from several perspectives. Contreras, Batanero et al. (2011), Ortiz et al. (2006), Fernándes et al. (2016), Mohamed and Ortiz (2012), and other researchers have shown that prospective teachers have a common content knowledge in probability but a poor specialized content knowledge in the subject. For example, Mohamed and Ortiz (2012) found that prospective teachers have difficulties in determining conditional probabilities and in perceiving the fairness of a game, while Ortiz et al. (2006) observed common prejudices in the interpretation of frequentist probability and in the use of heuristics.

The need to reform and improve the teachers education on probability is highlighted by these researchers and, in particular, they focus on the categories of the MKT conceptualization of "Common Content Knowledge", "Specialized Content Knowledge", and "Knowledge of Content and Students" in teaching probability, as well as the "Knowledge of Content and Students." However, the studies reveal poor knowledge of all these categories.

In a related study, Párraguez et al. (2017) observed that pre-service teachers struggle with the law of large numbers and the calculation of the sample space and its representation. They also found equiprobability biases among the pre-service teachers, as well as a poor understanding of the connection between classical and frequentist approaches, which is common in textbooks.

These studies highlight the need to improve teacher education in probability and the importance of addressing common misconceptions and biases in the subject. For example, teacher education programs could focus on providing specialized knowledge and training on topics such as conditional probability, the law of large numbers, and sample space calculation, as well as addressing equiprobability biases and common prejudices in the subject.

The importance of strengthening the (prospective) teachers education to improve the teaching of probability is emphasized in various studies (Afeltra et al., 2017; Cordani, 2014; Huerta, 2018; Perelli-D'Argenzio and Rigatti-Luchini, 2014). In one of these studies (Afeltra et al., 2017) proposes a task for teachers to interpret their students' reasoning when solving probability problems. This approach aims to develop the Interpretive Knowledge of teachers so they can help their students build their knowledge of probability from their reasoning. Another approach (Huerta, 2018) suggests that teachers should solve probability problems with a didactic approach. Positive effects on teachers' education related to content knowledge and didactical knowledge were found in experimental works conducted in a virtual class (Perelli-D'Argenzio and Rigatti-Luchini, 2014). Thus, working in labs and workgroups improves the teacher's vision of probability as an investigative approach to reality and can lead to remarkable changes in the students' learning.

Other studies focus on improving the knowledge of prospective teachers to teach probability, such as creating textbooks (Muñoz et al., 2014) or curricula (Naresh, 2014). Martin et al. (2018) investigate the way teachers deal with probability and their beliefs related to this topic. Gómez-Torres (2014) highlights the lack of dedicated hours to probability and its didactics in teacher education programs, making it essential to increase the number of hours of lessons to improve the skills of pre-service teachers in probability.

Contreras, Díaz et al. (2011) developed a research aligned with the perspective we assume in the research, having observed prejudices in the probabilistic reasoning of teachers, which improved after participating in training tasks in context intertwining the discussions on the mathematical topic as well as the pedagogical dimensions. This emphasizes the importance of proposing effective tasks for teacher education. Batanero, Contreras, and Díaz (2012) also emphasize the importance of working on experimental and didactic situations to develop the teaching skills of prospective teachers. For instance, Batanero, Gómez-Torres et al. (2012) and Díaz et al. (2012) have found prejudices in prospective secondary school teachers' understanding of conditional probability, suggesting that a high level of mathematical preparation alone is insufficient to overcome these prejudices. Therefore, they advocate for more attention to be given to the teaching of heuristics in problem-solving, which is particularly important.

Alonso-Castaño, Muñiz-Rodríguez, and Rodríguez-Muñiz (2019) propose training on heuristics for problem-solving for prospective teachers. As part of the proposal, prospective teachers were given the task of designing a probability problem adapted to a certain level of education, and the use of heuristics was found to be of support not only for solving the mathematical problems, but also for the design of appropriate problem statements. Prospective teachers who used heuristics in creating and solving their problems were able to propose more appropriate problems than those who did not use heuristics.

Overall, the studies discussed highlight the need for more research in the area of probability teaching and the implementation of teacher education programs to address shortcomings in this area. There is still a large span of space for development in the teaching and learning of probability, both at the national and international level.

## 3.2 Recognizing the importance of specialized teacher knowledge: moving from Intuitive to Subjective Probability

Much research takes a cross-cutting approach to definitions of probability, including classical, frequentist, and subjective perspectives. However, it is worth noting that there is a significant imbalance in the literature on probability education, with much fewer studies focusing on subjective probability compared to classical and frequentist approaches (Sharma, 2014). In elementary schools, the classical and frequentist (or experimental) approaches are the most commonly used definitions for teaching probability, while the subjective approach is often overlooked or barely mentioned in some contexts (Carranza & Kuzniak, 2009; Gómez-Torres et al., 2014). This highlights the need for more attention to be given to subjective probability in educational settings to provide a more balanced understanding of probability concepts.

However, it is important to note that subjective probability is highly relevant in daily life, as many events cannot be simplified into counting possible outcomes of a random experiment or repeating it under the same conditions. For example, subjective probability is commonly used when predicting sports results or assessing the risk of being infected by a virus, as these situations often involve a range of complex and uncertain factors that cannot be easily quantified using classical or frequentist probability approaches (Muñiz-Rodríguez et al., 2020).

The subjective approach can be a valuable tool as it enables individuals to make estimations based on **giving own opinion** (Castilla et al., 2022, Albert, 2006; Thibault & Martin, 2018).

This may involve formulating a hypothesis about the outcome that seems most likely in a random experiment. Additionally, the subjective approach can be used to **formulating a prediction**, according to Eichler and Vogel (2014) who suggested making a statement about the next outcome, or Homier and Martin (2021) who explored how it can be used to assess risk in a given situation.

For this reason, at early school ages, children's understanding of the subjective meaning of probability is closely linked to the use of verbal reasoning and chance language (Kazak & Leavy, 2018). This type of understanding is often referred to as the **intuitive meaning of probability** (Batanero, 2005), where the use of linguistic quantifiers and terms related to chance aids children in making qualitative probabilistic judgments (Rodríguez-Muñiz et al., 2022). The **intuitive approach** can provide a foundation for later learning of more formal approaches to probability, such as the classical and frequentist approaches. Moreover, research has shown that the development of probabilistic language and thinking skills during the early years can contribute to improved problem-solving abilities and decision-making skills later in life.

Research consistent with this assumption is presented by Castilla et al. (2022), who argue that the subjectivist approach views probability as a model for the partial information or uncertainty of the decision-maker. In terms of the example above *"this means that two different physicians could assign different probabilities (in fact, for possibly different diagnoses) and consider treatments and consequences in different ways according to their personal judgment"* (Castilla et al, 2022, p.)

Castilla and colleagues argue about two approaches in probability theory for modelling random contexts: the classical approach, which relies on equiprobability and counting methods, and the decision-theoretical approach, which takes into account personal information and coherence axioms. It is important to note that the acceptance of conditions that guarantee the limit frequency as a suitable assignment of probability for a given event is subjective and depends on the individual who is mathematically modelling the problem (mathematizing). In other words, the subjectivity of the assignment of probability implies that it can vary when updating our knowledge. Thus, both prior and posterior assignments must follow coherent consequential rules, which is equivalent to the relation between conditional probabilities and the Bayes rule (Castilla et al., 2022).

Probability is a complex branch of mathematics that presents several challenges to both students and teachers. Castilla et al. (2022) highlight that one of the primary reasons for this difficulty is the fact that probability deals with events for which there is no complete certainty. Unlike logical reasoning, which involves determining the truth or falsehood of a statement, probabilistic reasoning requires intricate mathematical calculations based on elaborate counting methods. This complexity can make it challenging for students to understand and apply probability concepts.

To address this challenge, Batanero et al. (2004) emphasize the importance of teacher education to focus on a deeper understanding of the inherent nature of probability and epistemological reflection on the concepts to be taught. Such an approach can help teachers convey the key concepts of probability more effectively to their students, enabling them to develop more robust mental models of probabilistic events. By understanding these concepts, students can improve their ability to reason about uncertain events and make better decisions. Furthermore, Castilla et al. (2022) argue that it is essential to examine how knowledge of subjectivist probability can influence people's decision-making and how personal modelling of events can impact the judgments of the subjects. This research emphasizes the need for a more nuanced understanding of how individuals use probabilistic reasoning in decision-making contexts, which can ultimately inform the development of more effective teaching strategies. Overall, improving the teaching of probability concepts can have a significant impact on students' ability to reason effectively and make informed decisions in their personal and professional lives.

In this sense, therefore, subjective probability estimation **begins** with giving an opinion from the information possessed. The idea underscores the importance of incorporating a subjective approach in the early stages of learning mathematics (Gouvernement du Québec, 2009). This approach can be applied through various concepts and processes, such as predicting results and discerning the difference between predicted outcomes and actual results derived from calculations or experiments *(*Homier & Martin, 2021).

In particular, Castilla and colleagues (2022) investigate the MTSK in relation to pre-service primary and secondary school teachers' understanding of probability. Their research aims to explore the teachers' concepts of randomness and about how – coherently or otherwise – they quantify (un)certainty, exploring to what extent preservice teachers consider probability as a measure of uncertainty or of partial information.

Castilla and colleagues (2022) investigate the pre-service teachers' understanding of the basic notion of probability, as well as their ability to model different contexts and quantify probability through various procedures. To assess the knowledge of the definition, properties, and foundation of probability (KoT), the authors examine the concept of randomness and explore how pre-service teachers associate it with uncertainty or partial information. They also question the contexts in which it is possible to calculate the exact probability, according to the analyzed subjects.

To analyze the knowledge connected to the ability of modelling the different presented contexts (KoT - phenomenology and application) the authors focus on the concept of randomness trying to understand if, according to the group of pre-service teachers examined, it is possible to link it to past and/or future events.

The pre-service teachers generally agree on what constitutes randomness and identify past events as "not random" because the results are already known, except for events related to draws.

Regarding the assignment of probabilities (KoT - method), the authors note that a significant number of teachers used combinatorial calculation and classic probability laws. While some pre-service teachers attempted to use Laplace's law for specific events, the corresponding values were chosen based on subjective considerations. Although the pre-service teachers received some notions of subjective probability, it did not seem to impact their understanding significantly. Their knowledge was comparable to that of secondary school students. To analyze the knowledge based on procedures of calculating probability (KoT - procedures) refer to the concept of coherence. This concept is introduced from a different point of view than that used by de Finetti (1931). According to the authors, a judging subject is defined as coherent if they assign the probability of a random event in the range of (0,1) or if they define the probability of a non-random event as equal to 0 or 1. A judging subject is defined as incoherent if they assign the probability of a non-random event in the range of (0,1). The authors define a non-random event as an event that is certain or impossible. Therefore, the judging subject is deemed incoherent if they estimate the probability of a certain or impossible event with a number between (0,1) and not equal to 0 if impossible, and equal to 1 if certain. According to this perspective, the authors analyze the emerged data from the experiment and observe that incoherence is common among the chosen teachers.

Castilla and her colleagues (2022) advocates for the inclusion of situations that directly challenge the conceptions of probability held by preservice teachers. They emphasize the importance of an approach that incorporate subjective probability, particularly for preservice primary and secondary school teachers who often rely on subjective reasoning to justify their probability assignments. By introducing such challenging situations, educators can encourage preservice teachers to critically examine and refine their understanding of probability, including its models and the assignment of probabilities.

Homier and Martin (2021) appear to be in agreement with Castilla and colleagues (2022) in emphasising that a subject assesses an uncertain situation based on available information or personal experience. Homier and Martin (2021) write, in fact, that from such an analysis emerges the **possible estimation** of probability in both quantitative and qualitative terms (an example is a **possible prediction** of the next outcome of the event considered).

A particularly significant aspect of the subjective approach, according to these authors, is the ability to **revise** probability estimation. These estimations can evolve as new relevant information about a probabilistic situation and its context is revealed (Eichler & Vogel, 2014; Kazak and Leavy, 2018). This type of estimatives, which can change in the light of new information, are referred to by them as **dynamic** (in contrast, estimates that are not subject to revision are referred to as **static**).

Rodríguez-Muñiz and colleagues (2022) shares de Finetti's view (1931) according to which probability is the *degree of confidence* of the judging subject in the occurrence of the event: probability represent, therefore, an estimate dependent on the decision-maker's judgement influenced by the owned information about the event under consideration. Coherently with Homier and Martin (2021), also Rodríguez-Muñiz *et al.*, (2022) wish to highlight the dynamic aspect intrinsic in the subject view of probability: the estimate, defined by the judging subject, change according to the owned information about the experiment and acquired as they went along.

Rodríguez-Muñiz and colleagues (2022) note and emphasize that although the subjective approach to probability depends on the decision-maker, **the subjective probability assignment** of a minimally informed person **should converge with** that obtained using **the classical** approach, in cases where it is applicable. Finally, Rodríguez-Muñiz and colleagues (2022) and Albert (2003) emphasize the importance of taking into account relevant information for the situation being evaluated when making a probabilistic estimate. In particular, when estimating probability, certain conditions of coherence based on mathematical laws (de Finetti, 1937) must be considered. Beyond mathematical coherence, some authors (Albert, 2003; Rodríguez-Muñiz et al., 2022) highlight the importance of justifying one's probabilistic estimate by explicitly naming the information and motivations that have been considered.

The study of probability is a vital subject in mathematics education that has significant implications for students' future success in various fields of study. The term "subjective probability" is a highly contested concept within mathematics education (Chernoff, 2009). There is a need to make a distinction between the general classifier and the specific theory of subjective probability to avoid confusion. Chernoff (2008) argued that subjective probability in mathematics education tends to be interpreted as personal or subjective, rather than logical. Therefore, he suggested that "personal probability" or other terms be used to describe the specific theory.

Despite the inclusion of probability topics in the school curriculum, there is little literature on the specialized knowledge domain of subjective probability. However, there are efforts to develop knowledge in this area, like Paparistodemou (2014) which used a specially designed computer game to help children develop and express probabilistic ideas. The importance of probability education in the elementary curriculum is also highlighted by Williams and Nisbet (2014). They investigated the use of probability games and activities to improve primary students' attitudes and understanding of probability concepts, while also helping teachers gain confidence and preparedness to teach probability using the activity approach. It is worth noting that the lack of emphasis on probability in the early grades is concerning, given that misconceptions about probability start forming in students as young as seven years of age, according to Fischbein (1975).

Mooney et al. (2014) developed probabilistic thinking models and frameworks, which they tested in small-group instructional settings. These frameworks provide a bridge between learning theory and instructional practices, highlighting the importance of teachers' knowledge and expertise in teaching probability effectively. The psychological and practical perspectives provide two lenses through which to observe and intervene in probabilistic thinking models and frameworks.

Gal (2005) suggests that teaching probability in real-life social contexts can enhance students' interest and motivation in the subject. Following Fischbein's (1975) approach, a central activity for learning probability involves predicting the outcomes of chance experiments through placing bets. To make the learning environment more engaging, this activity can be incorporated into a game scenario, as demonstrated in Aspinwall and Tarr's (2001) study.

In conclusion, the study of probability is a critical subject in mathematics education, with implications for students' success in various fields of study. The concept of subjective probability remains contested, and efforts to develop specialized knowledge in this area are ongoing. The lack of emphasis on probability in the early grades is concerning, and more research is needed to determine the feasibility of using probabilistic thinking models and frameworks in classroom settings. The importance of teachers' specialised knowledge in probability teaching cannot be underestimated.

# 4. Context and Method

The first part of the chapter presents the teacher educators and participants involved in the context of this research. In the second part, the methodology of Design Experiment (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) and Design Research (Bakker, 2018) is described. In the third part the phases in which the research has been structured are presented. In the fourth part, we will introduce the analysis methodology and outline the approaches through which the analysis is structured.

Let's remember our hypothesis and the research question that have led this thesis's work.

Hypothesis: if teachers are involved in an education program which focus on developing their knowledge on the elements of the subjective probability, then they are capable to develop specific processes of thinking improving their specialized knowledge on probability.

> *Which specialized mathematical knowledge in probability, demonstrated by teachers, impact their decision-making in a context of betting games and lead them to adopt a subjective probability view, to quantify the degree of confidence of an event?*
>
> (i) *Which elements of teacher's Knowledge of Topics (KoT) on probability are possible to trace on a context of a teacher's education program focused on establishing relationships amongst subjective, classic and frequentist perspectives?*
>
> (ii) *To what extent do the theoretical elements of the subjectivist approach to probability contribute to the attribution of meaning to the degree of confidence that a teacher assigns to an event?*

## 4.1 The participants of research

In this research, participated eight teachers (chosen on a voluntary basis): four of them teaching in primary school (Nando, Giorgio, Dayana and Luisa) and the other four, teaching in lower secondary school (Alba, Mirco, Paola and Maria) – names are pseudonymous.

The primary school teachers background in stochastics is the following: Nando and Giorgio teach based on their high school diplomas, as obtaining a university degree wasn't required for teaching in primary schools at the time of their graduation. However, Giorgio pursued further education, earning a master's degree in Law. Dayana and Luisa, on the other hand, possess master's degrees in Primary Teacher Education.

On the contrary, the academic backgrounds of secondary school teachers are as follows: Alba holds a PhD in Physics; Paola accomplished a master's degree in Mathematics, while Mirco obtained a master's degree in Geology.

Besides, the teacher educators, Rosa and Ciro, were the responsible of designing the tasks, and conducting the discussions with the teachers (I was one of the teacher educators). When referring to the context of data collection, the "we" form will be used to refer to Ciro and Rosa.

## 4.2 Methodology of the work management

The increasing of the cognitive skills of a human being and of his knowledge is a long and complex process, in which impact many factors, that may be individual or environmental; it isn't a linear process, but it is made of progresses and regressions, victories and defeats. It's often unpredictable in its specific configurations, but is often controllable in its general traits. This consciousness has highlighted how each evaluation or reasonable proposal can derive only from a hard work of observation focused on the evolutionary.

There is the need of specific methodological systems to perform observations and experimentations of this kind. For this research the "Design Experiment" (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) has been choose as methodology of research to design and improve the professional development course in which data has been collected.

As Bakker (2018) affirms, Design Experiment is part of a family of approaches including in educational design research (design based research, design experiments, formative experiments, design experimentation, design studies, development research, or developmental research).

There are 5 main characteristics of a design research (Bakker, 2018):

1. the purpose of developing theories regarding the learning and the tools which are designed to support that learning;
2. the interventionist nature. In a lot of researches, modifying and understanding a certain situation need;
3. design research is characterized by prospective and reflective components that have to be considered together, they can't be parted by a teaching experiment (Steffe &

Thompson, 2000). When implementing the prospective learning the researchers counterpose speculations with the actual understanding that they observe (reflective part). Reflection can be actualised after each lesson, also when the teaching experiment is no longer happening. This reflective analysis can lead to modifications of the primary plan for the following lesson;

4. the design research's cyclic nature: invention and revision create an interactive process. Many speculations on learning are often refused and alternative speculations can be created and tried. The cycle, which is generally made of preparation and design phase, teaching experiment, and retrospective analysis, can be repeated many times;
5. the developing theory needs to have an impact in the real work.

Bakker (2018) writes that *"a key characteristic of design research is that educational ideas for student or teacher learning are formulated in the design, but can be adjusted during the empirical testing of these ideas – for example if a design idea does not quite work as anticipated." (p. 5)*

For Cobb and colleagues (2003) *"one of the distinctive characteristics of the design experiment methodology is that the research team deepens its understanding of the phenomenon under investigation while the experiment is in progress."* (p. 12)

The context of learning, in this kind of research's methodology, is continuously adapted and readapted answering the demands of the ongoing researches and it is conceptualised like an interactive system in which the *canovaccio* (plot outline) of the design project is continually changing and it canonizes during its evolution.

The conducting a design experiment involves, on the planning and implantation, the combination of the theoretical and practical needs of the educational challenge: this is possible through the prefiguration of interventions that embody the assumptions of a theoretical disposition which derives from previous studies and the punctual check of their validity in the concrete context of the educational practice.

The Design Experiments are conducted to develop theories that don't have the presumption of working in every context, but that are useful to the aims of the learners' learning process.

This kind of methodological work allows the learners to create some schemes of reasoning, usable where educational necessities emerge in that specifical area.

In this perspective, the research team can improve the starting design and inspect the previous conjectures taken into account from the single meetings, from the observed difficulties or from the reached achievements.

To do so is necessary a continuous analysis both of the reasoning of the learners, than of the environment of learning.

There are four important operations that continually require the constant dedication both in the field of research and on the interpretative activity of what happened and on the planning of what will happen.

- In the first place, a clear vision of the planned paths of learning: the potential *canovaccio* has to be kept and communicated into the research team, also when the researchers react to the contingency.
- In the second place, the care of the most part of the designed experiments requires the nurturing of continuous relationships with the learners.
- In the third place, because of the reciprocal emphasis on the learning, the researchers of the design should try to develop a deep understanding of the ecology of learning, not only to simplify the logistics, but also because this understanding is a theoretical goal for the research. *"As part of the process of refining conjectures, subtle and often unanticipated cues need to be recognized and drawn into a larger perspective."* (Cobb *et al*., 2003, p. 12)
- In the fourth and last place, regular debriefing sessions are the fundamental moments in which past events can be interpreted and future events can be foreseen. *"These sessions are the sites where the intelligence of the study is generated and communicated."* (Cobb *et al*., 2003, p. 12)

According to Cobb and colleagues (2003), *"An educational accomplishment is characterized by contingency in which earlier events open up, enable, and also constrain the events that follow."* (p. 12). It is for this reason that the control of this process requires its theoretical or retrospective explanation, that can happen thanks to the comparison with the members of the team and thanks to the analysis of the recording of the meetings.

Besides,

> *"The purpose of design experimentation is to develop a class of theories about both the process of learning and the means that are designed to support that learning, be it the learning of individual students, of a classroom community, of a professional teaching community, or of a school or school district viewed as an*

*organization"* (Cobb *et al*., 2003, p. 10).

For this research in the field of teachers' knowledge on probability, a professional development course (PD) was designed and implemented and it involves primary school teachers and lower secondary school teachers together with their pupils. According to the presented framework, we have conceptualized the PD outlining the possible reachable goals in every meeting, but the structure of each one of them has been detailed under development, coherently with what emerged during the path itself. The meeting were audio recorded and all the teachers' productions have been collected – both for research as well as for the course design.

After every meeting with the teachers, we planned some debriefing sessions to analyze the recordings and the collected productions, with the aim of verifying the reaching of the planned goals and to outline precisely the changeable structure of the next meeting.

## 4.3 Exploring the research phases and procedures: a comprehensive overview of data gathering

The work associated to the data collection (PD) aimed at developing teacher's specialized knowledge, in the scope of the construct of probability, taking the steps of the subjectivist approach. Data gathering was made according to five phases were planned:

- Pilot phase (P): Reinterpretation of two experimental educational paths implemented with pupils of fourth and fifth grade (8, 9, 10 years old) in the field of probability (the first scenario implemented in the years of 2016 and 2017 and the second scenario in the years 2017 and 2018), from which raised the design of the PD object of this thesis;
- Phase A: make explicit the specialized mathematical Knowledge of Topic (KoT) in the field of probability together with the involved teachers;
- Phase B: improvement of PCK through the co-design and autonomous design of exemplary educational paths;
- Phase C: implementation of these educational paths with the pupils of the involved teachers;
- Phase D: Teachers' interpretation of the pupils' production.

In the following, each phase will be explained, either by the tasks used or by the objectives of the discussions carried out and all the phases above will be detailed in terms of the procedures data gathering.

Data collection refers to audio recordings and teachers' productions to tasks. To collect the data, all the teachers' registers for the tasks in each phase were collected at the end of each meeting and all the discussions were recorded in audio.

The audio recordings were subsequently transcribed. For privacy reasons we will use pseudonymous for the teachers' names.

The following Table 1 summarizes all the information about the meetings carried out in each phase, the data collected and the hours used for each meeting.

*Table 1: Overview of the research*

| Phase's name | Phases' aim | Meeting and codes | | Amount of hours (year of research) | Data gathering and Data analysed[1] |
|---|---|---|---|---|---|
| P Phase | Reinterpret the implemented pupils' research | 1st scenario | P1 | entire school year (2016-2017) | |
| | | 2nd scenario | P2 | entire school year (2017-2018) | |
| Phase A | Make explicit KoT in the field of probability with the teachers | 1st meeting | A1 | Three hours (June 2019) | audio recordings* and teacher productions to task 1* and task 2 |
| | | 2nd meeting | A2 | Three hours (June 2019) | audio recordings and teachers' productions |
| | | 3rd meeting | A3 | Three hours (June 2019) | audio recordings* and teachers' quotas* and notas* |
| | | 4th meeting | A4 | Three hours (July 2019) | audio recordings |
| | | 5th meeting | A5 | Three hours (July 2019) | audio recordings |
| Phase B | improve of PCK through the co-design and autonomous design of exemplary educational paths | 1st meeting | B1 | Three hours (October 2019) | audio recordings and teacher productions to tasks |
| | | 2nd meeting | B2 | Three hours (October 2019) | audio recordings |
| | | 3rd meeting | B3 | Three hours (December 2019) | audio recordings |
| Phase C | implementation of these educational paths | | C | - (2020) | interrupted due to the COVID-19 pandemic |
| Phase D | interpretation of the pupils' production | | D | - (2020) | interrupted due to the COVID-19 pandemic |

[1] Only the underlined data gathering (indicated with an asterisk) has been analyzed.

### 4.3.1 P phase

This was the "pilot phase". It's included because it helps to better understand the activities and tasks carried out with the teachers in the I and II phases. For this reason, the data for this phase will not be analysed.

We created and explored a didactic path for fourth grade pupils with the aim of building some telling points in the field of probability theory. This didactic path consists of two scenarios presented below, which correspond to the two activities carried out with pupils. The topic of probability has been obviously cared coherently with the level of knowledge of the involved pupils.

### P phase: First scenario (P1)

Since the beginning, pupils were put in a situation of game immersion. Before starting, we explained the game to the pupils: it consisted in betting on the sum of the two faces outcoming in the six-sided dice's roll (both white) and in establishing the quotas to be allocated to each possible sortie (sums from 2 to 12, including).

The quota is commonly understood as the multiplication factor that is applied to the player's bet to determine the amount that the same player will be entitled to collect if the bet event *E* occurs. The game's challenge with the pupils was to establish quotas so as to be coherent to de Finetti's vision (1931) in order to *"not enable competitors to win with certainty"* (de Finetti, 1931).

The setting of the classroom has been arranged in order to have three gaming stations using the game board depicted in Figure 15 that was designed and built for the pilot phase. Some children were given the role of bookmaker, those who set quotas on the game board (one for each gaming station), while the other pupils were given the role of bettor. Taking turns, each child played both roles in the game.

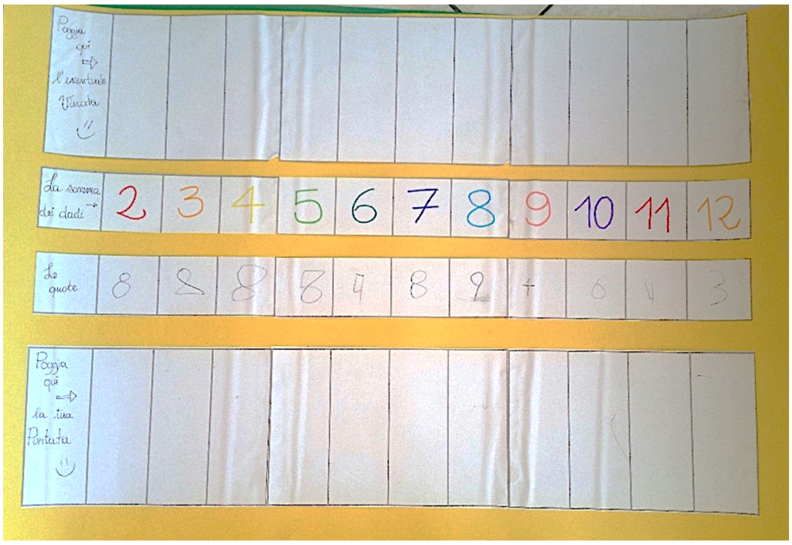



*Figure 15: P phase's game board*

To play the game the pupils (bookmakers and bettors) had fake coins. The challenge, both between the bookmakers' game stations and between the bettors, was to make choices that would allow them to earn as much money as possible. The bookmakers of each gaming station, before starting the game, transcribed the quotas on the game board. After that, the game started. The bookmakers asked the players on which sum they wanted to bet and how many coins they wanted to use. The bettors chose the amount of coins to pay and decided on which sum they wanted to bet, with the possibility of betting on more sums simultaneously. Each bettor could decide to play in the gaming station that he considered more appropriate and he could also decide to play in both gaming stations at the same time. The players put their coins in the specific square of the game board.

We remind that, to calculate the potential winning of the player, the paid amount is multiplied by the quota established by the bookmakers.

After the bets were collected, the bookmakers rolled the dice at the end of each round designated for betting. Rolled the dices, the winner(s) is (are) established and the bookmakers pay the eventual winnings and/or collect the losing bets on the game board.

During the betting sessions, the children had the opportunity to identify elements in the dynamics of the game to rethink the criteria with which they chose the quotas and to modify them in the second part of the game, if they wanted to.

After that all the players played as bookmakers, the game ended.

## P phase: Second scenario (P2)

This second research scenario has been thought with the goal of not being tied to the betting scheme, but to refer to a general measure of probability.

The pupils are divided in little groups and each group is given some instruments to fulfil the following task:

> Create a game where something happens by chance. Ensure that in the game things can happen that have a different probability. Ensure that the rules of the game are right.

Every group had to test the game and hand in a paper in which the rules are explained in detail.

In this thesis work we don't report all the created games, but we focus on a task in which we wanted to establish which one was the most probable ("easier" for children) among the following events:

- obtaining a "greater than four" from the rolling of a dice;
- obtaining "sum five" from the values of two cards drawn randomly, from a group of only one suit of cards with a value of between one and seven.

This game did not provide the replacement of the drawn card.

## P phase: Our choices and reflections on the games

In this first scenario, we recreated a situation similar to de Finetti's scheme of bet presented in the 1$^{st}$ chapter.

We used the betting game to reason on the opportunity of conferring a measure to a given event (the sums from 2 to 12), depending on its *degree of confidence*. This measure has been initially expressed in terms of quotas, useful, in a practical way, to pay for the verifying of an event.

This was a first variation, compared to de Finetti's scheme of bet, because we chose to use the quotas (as measure of the *degree of confidence*) associated to each event *E,* rather than the price to pay by the verifying of an event.

This choice doesn't correspond with the one of defining the measure of probability like the price p, in de Finetti's vision (1931).

The second variation was choosing of not to bet on *cons E.*

But the problem of "establishing quotas" has brought us, anyway, to compare different events according to their *degree of confidence*, to realise an order. The choice of using two dices, in the first scenario's game, is oriented by our need of exploring events for which it could be easy to locate configurations corresponding to them and for which it was possible to calculate a classical probability.

This fact has led us to introduce a measure of probability Q(E) as "function that to the event *E* associates the number of configurations corresponding to it".

To compensate for the lack of the betting *cons E,* we expected the presence of more bookmakers. Each one of them had the possibility to establish the quotas related to every event "sum" and had also the faculty of modifying these values. To make these changes, the bookmakers should follow the evaluations of the trend of the game: on one hand there were many bettors who had the chance of choosing a bookmaker to play with, moving freely among them during the duration of the game session and on the other one the bookmakers could know the values of the quotas established by every other bookmaker.

So, in a context of a game of this type, the quotas are subject to a mechanism of variation and selection.

We observe, however, that it isn't possible to establish in a deterministic way that such a **variational dynamic** should implicate the observance of de Finetti's proprieties (1931), nevertheless exploring its characteristics appears interesting.

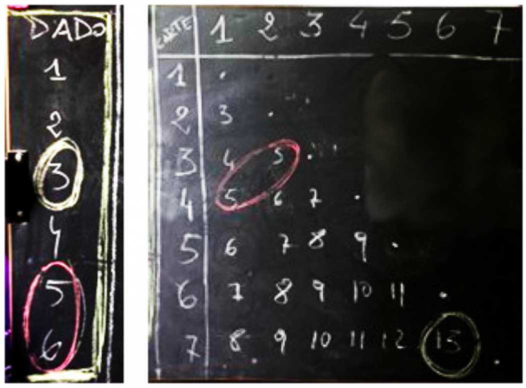
Together with the pupils, we have come to justify the opportunity of replacing the measure of probability, in the first scenario defined as "number of configurations corresponding to an event E", with a measure of probability defined as the fraction "favorable cases on possible cases".

We asked to compare the probability (always intended as *degree of confidence*) of events belonging to "different spaces".In both cases, indeed, the number of combinations corresponding to the considered event is equal to two (in the rolling of a dice there are 2 favorable configurations out of 6 possible configurations, while in the case of the cards there are 2 favorable configurations out of 21 possible configurations).

Using the measure of probability defined as "number of configurations corresponding to an event", the two considered events would result as equiprobable.

Along the development of the work, this assumption has been considered not plausible, basing this reasoning on the comparison between the favorable configurations and the possible ones.

The pupils built some techniques for the combinatory calculation, through the construction of representations like the ones reproduced in Figure 16 and a geometrical representation of fraction (that the pupils were used to using).




(a) Dado. (b) Somma dei valori di due carte.

*Figure 16: Representation used for combinatory calculation*

It was established that an appropriate measure of probability to compare those events was given by the ratio between favourable cases and possible cases.

What reproduced in Figure 17 is referring to the comparison between the sortie “three” of the rolling of a dice and “sum thirteen” of the values of two cards of the group described above.


*Figure 17: spatial representation of a fraction to compare events in distinct spaces*

Starting from our analysis of the P phase, we expanded our research, focusing not only on the children’s knowledge, but also on the specialized mathematical knowledge of the teachers.

### 4.3.2 Phase A

In this phase we met the eight teachers to highlight and develop the theoretical elements of the construct of probability. This phase consists of five meetings (A1, A2, A3, A4, A5), each one of them during around three hours.

The goal was to study the interactions created in the group of teachers to analyze their idea of probability and how they manage it. At the same time the topic of analysis of this phase was to understand how the specialized mathematical knowledge of the adults' group evolved.

### Phase A: first meeting (A1)

Data gathering: audio recordings and teacher productions to task 1 and 2

In the first meeting the teachers faced the background idea of the work in which they were involved.

Since the beginning, teachers were also put in a situation of game immersion and they were made to undergo similar interactions to the ones developed in the phase P.

Before starting, we explained the game to the players:

The game that we proposed was the first scenario's game of the P phase.

The setting of the classroom has been arranged in order to have two gaming stations using the game board depicted in Figure 18 that was designed and built for this professional development course.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| BET MONEY | | | | | | | | | | | |
| SUM OF THE VALUE OF TWO SIX-SIDES DICE'S FACE | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| QUOTES | | | | | | | | | | | |
| POTENTIAL WINNINGS | | | | | | | | | | | |

*Figure 18: teachers' game board*

Two teachers were given the role of bookmaker, those who set quotas (one for each gaming station), while the other six teachers were given the role of bettor. Taking turns, each teacher played both roles in the game.

Before getting into the thick of the game, the researchers presented task 1. In the first part of the task the researchers asked the teachers to decide the quotas that they are supposed to assign to each sum when they would play as bookmakers. The second part of the task would be used by the players during the game to report the eventual modifications of their established quotas.

TASK 1:

1.1 Write down the quotas that you are supposed to assign to each sum when you will play the role of bookmaker.

1.2 Write down the quotas that you chose to set as bookmaker, in case they are different from the ones you supposed previously.

To complete the first part of the task the teachers worked individually for 30 minutes.

After first question (Task 1.1) the role of each teacher was established.

The researchers explained the rules of the game and gave them the instruments to play. For each gaming station there was a game board. A certain amount of fake coins was given to the bettors and to the bookmakers to realize bets and to pay the potential winning, respectively.

The bookmakers of each gaming station, before starting the game, transcribed the quotas on the game board, coherently with the quotas established in Task 1. After that, the game started.

Also, the teachers had the possibility of rethinking the criteria with which they chose the quotas and to modify them.

The teachers could write down the modifications apported to their quotas in the second part of the task (Task 1.2).

The game ended when all the teachers played the role of bookmakers.

The last part of the meeting consisted of individual work in which the teachers had to answer, in writing, to Task 2:

TASK 2:

*1)* Which are the criterias through which you chose, and eventually changed, the quotas when you played the role of bookmaker?

2) Which are the criterias with which you took your decisions when you were playing as bettor?

3) Retracing your experience as a player, do you consider yourself a player who risks a lot or a little? Which behaviors or choices do you deduce this from?

4) In this game is it easy or difficult to win? Why do you think so?

This process of writing the first quotas and the following registration of the modification made to them, together with the answers given in task 2, have been analysed to study the adjustment process of quotas influenced by the competitive dynamics between the two bookmakers.

## Phase A: second meeting (A2)

Data gathering: audio recordings and teachers' autonomous productions

Considering our first approach on the data collected in the first meeting (A1), the second meeting was designed as presented following.

The teachers were divided into two groups that worked in two different rooms. To each group were given photocopies of all teachers' answers collected in task 1 and task 2. Each group had to identify elements useful to highlight similarities and differences among the answers of the players. The teachers were asked to not focus on the personal productions, but to realize an objective analysis of such a productions. To do this work the teachers had 20 minutes. The discussions in the two groups did not involve the researcher's participation and have been recorded. At the end of the teachers' reflection of productions, the two groups were reunited to make a report (15 minutes for each group) on what they discussed in the small groups. Then, starting from what was obtained in the report and from our previous analysis – related to the first meeting (A1) –, some challenging questions[2] were posed to each teacher. All teachers actively participated of the discussion.

## Phase A: third meeting (A3)

Data gathering: audio recordings and teachers' quotas

[2] Considering the focus of the research, these questions will not be presented here, since the teachers' answers will not be used as data of analyses.

In this meeting, teachers were involved in a simulated game session, that provided the possibility, for the first time, to do *pro E bet* but also *cons E bet*.

After remembering the *pro E bet* modes and explaining the dynamics of the *cons E bet*, each teacher took 10 minutes to set the quotas for each event "sum", taking into account the variation inserted in the game. Each teacher wrote down his/her quotas and spontaneous notes in a paper. Besides, teachers could share their choices with the group and write down their quotas on the game board.

The collected quotas could be commented with other teachers. The researchers' role was to orchestrate a discussion with the aim of not letting competitors win or lose with certainty, in order to be coherent, in de Finetti's vision (1931).

All the discussion was audio-recorded and the quotas written by the teachers on the game board were photographed to be gathered.

## Phase A: fourth meeting (A4)

Data gathering: audio recordings

In this meeting we described the theoretical aspects of the main theoretical framework, linking them with what collected in the previous meetings.

We narrated the experimentations made in the first and second scenario (P1 and P2 respectively), comparing the words and the actions of the children (P phase) with those collected in the meetings done with the teachers (PA1, A2 and A3).

The aim was to render explicit that we built the measure of the *degree of confidence* as an instrument that is necessary to "direct and guide the actions" and that, on a cognitive level, we don't vehiculate the idea that probability is an intrinsic property of the material systems.

His construction is based on considerations of "common sense".

## Phase A: fifth meeting (A5)

Data gathering: audio recordings

This fifth meeting is the conclusive one of the phase A, and is also the one that has introduced the teachers to the phase B of improvement of PCK through the co-design and autonomous design of exemplary educational paths.

The goal was to utilize the characteristics of the subjectivist probability to start building an educational path with the pupils of the involved teachers.

We reflected about the teaching approach (e.g., games, problem-solving) teachers were interested to work with; the mathematical goals that they considered they could reach and the essential traits the work environment ought to embody.

The specialized teacher's knowledge developed during the phase A is required for designing the educational path to develop in the classroom with the children. In the next section, we present the task proposed to teachers on the phase B, which was the motivation for the design of the educational path.

### 4.3.3 Phase B

This phase was based on three meetings and had the goal of outlining the key-steps for the designing of the work that would be developed in the teachers' classrooms (phase C).

The three mentioned meetings took place after two months since the ending of the phase A.

Each meeting lasted three hours and, also in this phase, all the discussions were audio recorded and all the teacher productions to tasks were gathered.

## Phase B: first meeting (B1)

Data gathering: audio recordings and teacher productions to tasks

Considering the lack of time between the last meeting on phase A to the first meeting on phase B, we decided to remember the teachers about, not only the main elements focused on each meeting of the phase A, but also the theoretical elements of subjective probability.

To begin collecting detailed ideas about the educational path, we gave the Table 2 to the teachers, which was composed of four columns.

In the first column we inserted the theoretical generalized elements of the subjective probability according to de Finetti's vision; in the second one we created some connections with de Finetti's bets' scheme; in the third column we inserted our choices about the setting used in the P Phase and/or in the Phase A; the fourth one was empty, to allow the teachers to write their ideas about the new game setting that we will implement in the Phase C.

The presence of more lines allowed to create some connections among the elements of the four columns.

On the same line of the table, so, it is possible to read the choice related to our setting linked to the theoretical motivation which brought us to make that choice and also the elements of the construction of the new game setting.

*Table 2: task to build the new game setting*

| Generalised | De Finetti bets' scheme | Our setting | New setting |
|---|---|---|---|
| Necessity of ordering events depending on the degree of confidence based on the effects caused by those events | Game of bets (Establishing prices) with transitions of credit | Game of bets on the sortie of the sum of two dices (establishing quotas) with transition of credit | |
| Introduction of a measure (of probability) to quantify the level of the degree of confidence that should be attributed to an event | Price | Number of combinations | |
| Necessity of imposing properties on a measure of probability | The bookmaker should/must avoid to establish prices that may put him in the condition of losing with certainty | Implicit (the bookmaker shouldn't enable competitors to win / lose with certainty | |
| | Pro E bet or cons E bet | Variational Dynamic<br>- competition among the bookmakers<br>- possibility of modifying the quotas | |
| Set of incompatible equiprobable events | General reference | Pair of dices' faces; combinations of cards | |
| Additional law on probability or on the measure of probability | Recognised as properties on prices | Implicit in the use of the probability combinatorial calculation | |
| Comparison among sets of different events | | Comparison among the "sorties of a dice" and the "combinations of cards" | |

After explaining the functioning of the table, the teachers had 30 minutes to think about the new setting and another 30 minutes to share individually their ideas with the group.

## Phase B: second and third meetings (B2 and B3)

Data gathering: audio recordings

In these two meetings we debated with the teachers about the details and the aspects of didactic mediation of the Phase C, starting from what emerged in the filled tables.

We asked the teachers to imagine a didactic path for their pupils. This didactic path had to be the same for all the involved classes, but using the framework of the design experiment, the *canovaccio* could change during the evolving of the work. Taking into account the emerging needs, the teachers worked together discussing with the resarchers.

### 4.3.4 Phase C and phase D

The C and D phases were planned respectively to implement the educational paths with the pupils of the involved teachers and the teachers' interpretation of the collected pupils' productions.

These phases of the PD were interrupted due to the COVID-19 pandemic. The description of theses phases will not be included in this thesis work.

## 4.4 Methodological analysis

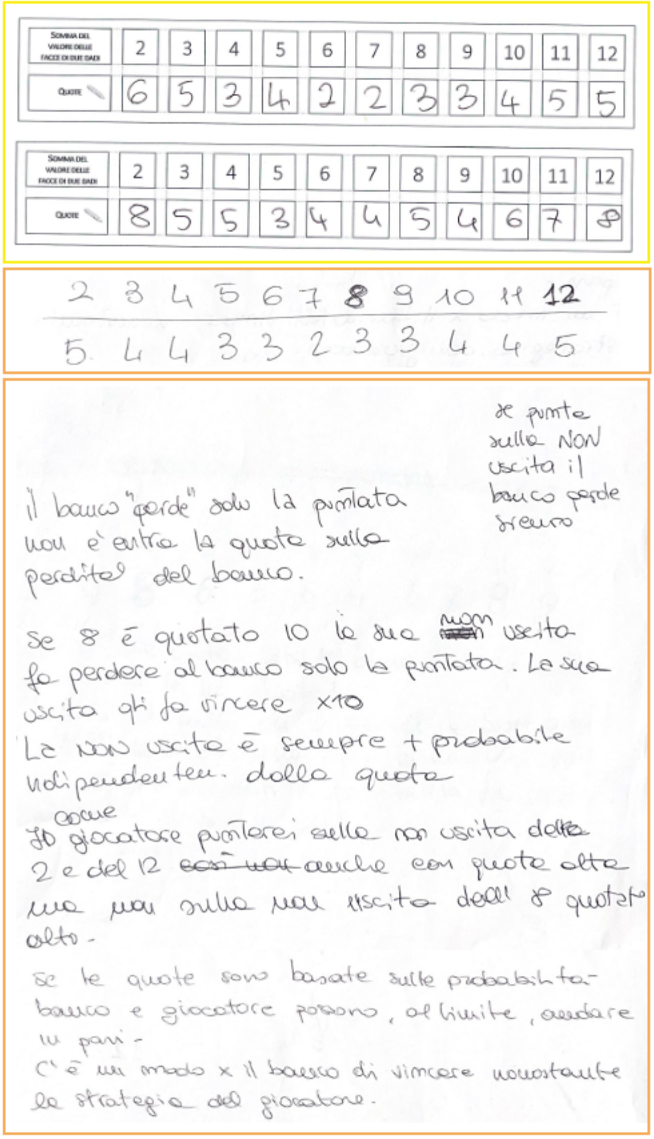
Here we outline the methodological approach employed to scrutinize the impact of teachers' specialized mathematical knowledge in probability on their decision-making within the dynamic context of betting games. The analysis unfolded through three distinct phases, each offering unique perspectives on the complex relationship between mathematical expertise and educators' decision-making processes. It will be presented consistently with this assumption in three following sections.

### 4.4.1 First step of the processes of analysis: examining quotas and teachers' notes

The first step of the process of the analysis involves a comprehensive examination of data gathered during the A1 and A3 meetings of the phase A. In the first meeting (A1), teachers engaged in a "*pro E* bet" session, generating quotas for potential sums (Task 1.1). Following a dynamic gaming session, teachers had the opportunity to refine their quotas (Task 1.2), vividly captured in yellow-framed boxes (Figure 19).

The subsequent introduction of a simulated game session in the third meeting of the phase A (A3), encompassing both "*pro E bet*" and "*cons E bet*," further enriched the analysis. Orange-framed box (Figure 19) showcase quotas assigned during such meeting (A3), along with spontaneous notes, providing insights into the decision-making processes of the teachers. Since the teachers' notes were written in the Italian language, the translation of the texts were included in a green box, on the right of the image.

Below is an example (Figure 19) illustrating how the evidences will be presented on the discussion section (Chapter 5).




*If you bet on "NOT-outcome" (cons E bet), the bookmaker loses with certain.*

*The bookmaker only "loses" the bet quota doesn't affect the bookmaker's loss.*

*If 8 is quoted at 10, its not-outcome only causes the bookmaker to lose the bet. Its outcome makes him win x10.*
*NOT-outcome is always more probable regardless of the quota.*
*As a player, I would bet on the not-outcome of 2 and 12, even with a high quota, but not would bet on the not-outcoming of 8 with a high quote.*
*If the quotas are based on probability, the bookmaker and the bettor can, at most, break even.*
*There's a way for the bookmaker to win despite the player's strategy.*

*Figure 19: An example ilustrating the boxes containing teachers' quotas and notes organized during the analysis process.*

During the process of analysis, all the quotas assigned for the teachers during A1 and A3 meetings have been reworked and plotted on linear graphs (Figure 20). The graphs were built considering the yellow and orange colours referred, respectively, to the A1 (Task 1.1 and 1.2) and A3 quotas boxes (Figure 20).

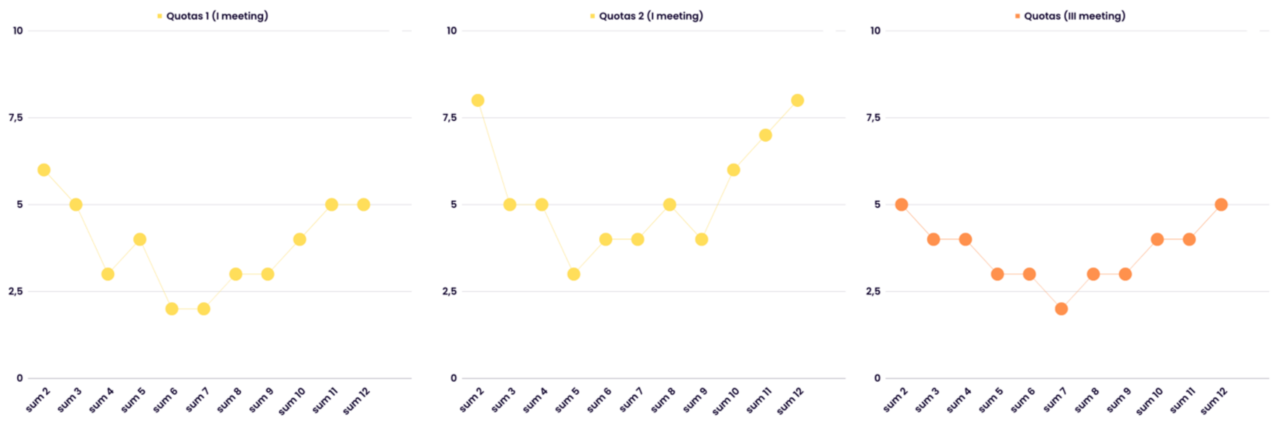



*Figure 20: An example of reworked quotas represented in linear graphics, as featured in the analysis.*

Graphical representations played a pivotal role in visually narrating the evolution and distribution of teachers' quotas. This aided in a clearer understanding of the trends within the data, facilitating a nuanced exploration of teachers' specialized knowledge in probability.

Consistently with De Finetti's perspective, which asserts, "not because what I predict will happen, but because I predict that it will happen" (1931), the creation of graphs served the purpose of not dwelling excessively on the numerical value assigned by teachers in this initial phase. Instead, the focus was on the qualitative trends of these values. Delving into why a teacher assigns the same quote to two events or why they assign different quotas to two events provided an opportunity not just to refine their measurements but also to cultivate the skill of contextual exploration. This exploration aimed to gather all necessary information for making informed decisions about the confidence level in a specific event.

It is important to highlight that, when a statement made by a teacher was considered as evidence of revealed specialized knowledge, it was presented in "*italic format*". Additionally, in the content of the analysis discussion, the identification of subdomains and their respective categories associated with MTSK and pertinent to the evidence of revealed specialized knowledge, will follow the format (Subdomain – Category – Particularity of the Knowledge) or [Intra-conceptual connection], as illustrated in the example below:

> Giorgio characterizes the low quotations on the "*sides*" as "*defensive*" suggesting a cautious stance towards events associated with extreme sums (KoT – Procedures – when to do something). This choice aligns with the idea of considering the non-occurrence of

2 and 12 more probable events than the non-occurrence of the sum 8 [Intra-conceptual connection].

### 4.4.2 Second step of the process of analysis: exploring the transcriptions of the discussion about the teachers' presentation of the quotas

Moving to the second phase, the analysis integrates transcriptions of discussions from the third meeting (A3). This exploration delves into how autonomous creation, modification, and deliberation around quotas initiated qualitative and quantitative processes within the teachers. These processes refined the measurement of confidence levels associated with potential events.

In transcribing, the process entails listening to the entirety of the audio files and then transcribing them in full, which means that all the discussion was transcribed *ipsis litteris*.

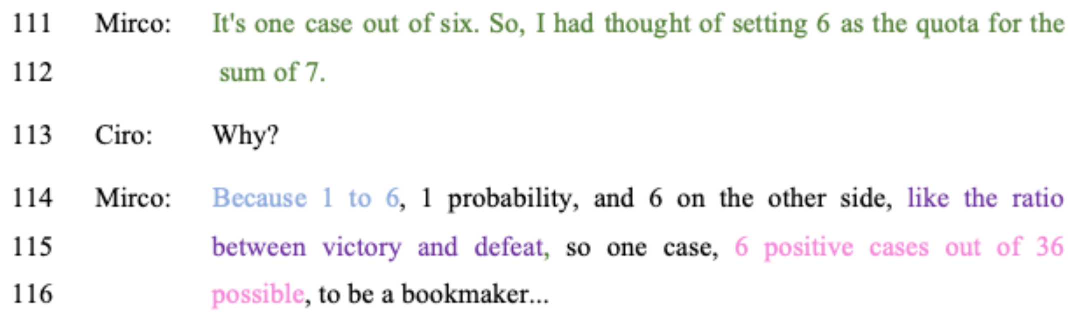
We chose to transcribe line by line (Schoenfeld, 2000) and numbered in a subsequent numeration, understanding that this method of organization contributed to identifying evidences of teachers' revealed specialized knowledge at different moments in their interactions during the discussion. Additionally, this method of organization was also crucial both in analysing the degree of confidence that teachers established and exploring a profound insight into teachers' decision-making beyond the quotas.

The transcriptions (Figure 21) were then organized and reported in the exploring and interpreting data chapter (Chapter 5), with each teacher referred to using pseudonyms to ensure anonymity and confidentiality.

| | | |
|---|---|---|
| 194 | Mirco: | So, I think this: given that this is the normal Gaussian, when you play against |
| 195 | | the event, the curve is opposite. So, in both modes of play, we set the |
| 196 | | same quota for everyone. |
| 197 | Paola: | No, I have to put higher quotas. No, it is not a good idea, because the |
| 198 | | bettor can always bet in both ways. |
| 199 | Mirco: | And so, this situation is impossible to be solved. |

*Figure 21: A screenshot depicting the transcription and line numbering as presented in the analysis.*

To enhance clarity and categorization, a color-coding system (Figure 22) was adopted in a separate document, associating different colours with distinct categories of knowledge identified during the interpretation of the transcriptions.



Mirco: It's one case out of six. So, I had thought of setting 6 as the quota for the sum of 7.

Ciro: Why?

Mirco: Because 1 to 6, 1 probability, and 6 on the other side, like the ratio between victory and defeat, so one case, 6 positive cases out of 36 possible, to be a bookmaker...

*Figure 22: A screenshot depicting the color-coding system to identify KoT's category in the transcription during the analysis'process*

The presentation of the exploration and interpretation of the content of the discussions conducted in meeting A3 follows a similar structure to that used for the data related to quotas and teachers' notes, as we mentioned in the previous section. Thus, when evidence of revealed specialized knowledge is identified in any excerpt of the transcription, it is presented as follows: "teacher's comment" or a discussion of comment (initial line - final line). Subsequently, as a result of the discussion conducted, the subdomain and category associated with the MTSK are identified, following the same approach as in the discussion of the data from the previous stage of analysis: (subdomain - category - particularity of knowledge), See the following examples:

> Alba adopts a qualitative reasoning (2 - 3) by examining the aspect of the *ordinal sorting* of quotas and the correlation between such *ordinal sorting* (KoT – Registers of representation).
>
> In navigating the contextual nuances, Mirco tells "*like the ratio between victory and defeat*" (114-115), he adeptly employs a logic (114-116) deeply rooted in a phenomenological method of extracting information (KoT - Phenomenology and Application).

### 4.4.3 Third step of analysis: the process of construction a coherent probability measurement

The final step of analysis extends the inquiry to the continuous discussion transcripts. Reading, re-reading, and re-interpreting, the analysis focuses on the part of the meeting where collective discussions revolved around measuring the *degree of confidence* for each event. In this phase, the exploration delves into intra-conceptual connections, guided by the notion that conscious mathematical knowledge may stem not only from phenomenological, procedural, or definitional knowledge but predominantly from these intra-conceptual connections.

This structured and detailed methodology, enriched by graphical representations and transcriptions, establishes a solid foundation for comprehending the intricate interplay between specialized mathematical knowledge and teachers' decision-making in the realm of probability.

# 5. Exploring and Interpreting the Data

This chapter focuses on the analysis and discussion of teachers' knowledge content mobilized and revealed during interactions with the game, among peers, and with the trainers.

In order to address the research question *“Which specialized mathematical knowledge in probability, demonstrated by teachers, impact their decision-making in a context of betting games and lead them to adopt a subjective probability view, to quantify the degree of confidence of an event?”* the analysis focused on identifying the content of teachers' knowledge associated to probability, specifically with respect to one of the Mathematical Knowledge subdomains of the Mathematics Teachers Specialized Knowledge – MTSK (Carrillo, et al., 2018), namely, Knowledge of Topics (KoT).

The purpose of phase A’s meetings was to study the dynamics, created in the group of teachers, to analyse the idea of probability that an adult has and how this idea of probability is managed and used.

Let’s remember that the simulation of betting games has been the background for this phase and that the game explored was the one related to betting on the sum of the two faces outcoming when rolling two six-sided dice.

We start by an in-depth analysis of the data gathered (quotas and notes) during the first and third meetings of the phase A (A1 and A3).

Notably, in the first meeting (A1), teachers engaged in a betting session featuring only the "*pro E bet*" mode. Each teacher initially had the opportunity to devise quotas for each possible sum. Subsequently, teachers were immersed in a gaming session, and after participating, they had the chance to reconsider and modify their quotas for refinement.

The initial quotas, along with the revised ones, are presented for each teacher within yellow-framed boxes.

During the third meeting (A3), teachers were involved in a simulated game session, that provided the possibility, for the first time, to engage in both *pro E bet* but also *cons E bet*.

After remembering the *pro E bet* modes and explaining the dynamics of the *cons E bet*, each teacher took some time to set the quotas for each event and to share them with the group.

Let's remember that the quotas from the first meeting (A1) are showcased within yellow-framed boxes, the quotas from the third meeting (A3) are showcased within orange-framed boxes. Alongside these quotas, we've included snapshots of spontaneous notes written by the teachers, providing insights into their rationales and considerations, showcased within orange-framed boxes. To facilitate a comprehensive understanding, English translations of these reflections are presented within green-framed boxes.

To maintain analytical consistency, we've exclusively reported and analyzed the quotas and contributions of the five teachers who participated in both of these meetings.

Quotas assigned by the teachers, along with the collected notes, should ideally mirror their perception of the probabilities associated with each event and will be analyzed to paint a picture of the starting point of their specialized knowledge.

We then follow to the analysis of teachers' written contributions (transcriptions of quotations and notes), excerpts from the transcriptions of the discussion that took place during the third meeting will also be analysed. It is essential to note that during this specific meeting, we delved into how the autonomous creation and modification of quotas, along with the ensuing deliberations about associated decisions and the stabilization of quotas, initiated qualitative and quantitative processes within the teachers. These processes have led to the refinement of the measurement of confidence levels associated with each potential event.

Building upon this analysis, we will explore the implications concerning the application of such settings to enhance teachers' knowledge in the field of probability.

Issues identifiable in the initial data presented may appear untreated or only partially discussed in the first paragraph of the analysis, but they have been addressed in subsequent paragraphs using transcript analysis, which has permitted a more comprehensive understanding of teachers' ideas and insights.

Analysing the trends of quotas, along with their written reflections and contributions during the discussions, allowed us to identify the specialized mathematical knowledge on this topic that needs to be connected to strengthen understanding of the probability topic.

## 5.1 Setting the Stage: initial measurements of degree of confidence

In the vast panorama of scientific research, the issue of forecasting and understanding future phenomena is a crucial knot. As individuals immersed in the flow of time, we constantly find ourselves trying to anticipate what will happen, whether it be weather forecasts, market trends, or the results of scientific experiments. However, this process is not as simple as it may seem at first glance. In its essence, we are called not only to predict future events but also to understand why we predict them.

As noted by Bruno de Finetti (1931), the fundamental difference lies in the attribution of "why". It is not simply a matter of predicting the fact itself, but of understanding why we are inclined to predict that it will occur. In this context, an intriguing perspective emerges: it is not so much the facts that require a cause to manifest themselves, but rather our thought that seeks to attribute causality to them in order to explain, coordinate, and make prediction possible.

This concept carries profound philosophical and scientific implications. Science itself is faced with the challenge of confronting the obvious objection that everything we mentally elaborate is intrinsically connected to our thought and our conception of the world: "We cannot reason about anything that is not our reasoning, we cannot contain anything that is external to us" (de Finetti, 1931).

In the context of the present analysis, we aim to explore how this perspective influences the measurement of the initial degree of confidence. In particular, we will focus on how the initial measurements of confidence can be influenced not only by the nature of the observed facts but also by our interpretation and perception of them. This will lead us to examine the crucial role that human thought plays in the process of forecasting and understanding.

### 1.1.2 Alba's first data

Below are reported the analysis of quotas and notes provided by Alba.


*Compared to the quotas established during the "first" game, I've raised these! I think that the fact that the player can bet on both the outcome of one or more numbers* (pro E bet) *and on NOT-OUTCOME* (cons E bet) *increases the player's chances of winning, and thus, the bookmaker must have higher quotas... ...I think...*

*Gianluca→ It makes me think that the bookmaker is a player.*

*Marco→A sense of fairness.*

*Figure 23: Alba's quotas and notes.*

Alba's assigned quotas exhibit a distinctive pattern that lacks complete symmetry (KoT – registers of representation – symmetry of quotas' distribution) in both the first and third meetings (Figure 23).

Notably, she frequently makes one-unit jumps when providing quotas in each assignment (A1 – quotas for tasks 1.1 and 1.2; A3 – quotas), resulting in increments or decrements of one. This phenomenon can be attributed to her consideration of the combinations for each event (KoT – procedures – how to do something). It is important to note that when rolling two dice and summing their outcomes, there's a clear pattern regarding the number of possible combinations. Initially, this number increases by one with each consecutive sum, until reaching a peak. After hitting the sum of 7, it then decreases by one for each subsequent sum.

As we delve deeper into Alba's approach, it becomes evident that her quota adjustments, in all three assignments, align with the changing combinations as we approach the "sum 7" event. As combinations' number increase, she tends to reduce the quotas, while she raises them as we approach the "sum 12" event (KoT – procedures – when to do something).

Also in the third meeting, Alba's assigned quotas to the different dice roll sums follow a sufficiently symmetrical trend (like in both A1 quota) with respect to the "sum 6" and "sum 7" events. Alba employs the quota "5" for events with different probability measures, such as the "sum 6" and "sum 7," but notably for the "sum 9" event as well. In line with the decision to utilize identical quotas for events with differing combinations, Alba allocates two distinct quotas to events that entail only one method of combination, occurring in the rolls of two dice.

Specifically, Alba assigns the quotas 10 and 9 to the "sum 2" and "sum 12" events, respectively. These quotas are nonetheless the highest among those assigned, suggesting her perception of considering them the least probable events of all (along with the "sum 3" event, to which she assigns the same quota of 9). This discrepancy in the quotas suggests an unorthodox view and an intriguing aspect of Alba's perception of probabilities.

The graphs (Figure 24) show as Alba's quotas remain more or less the same.

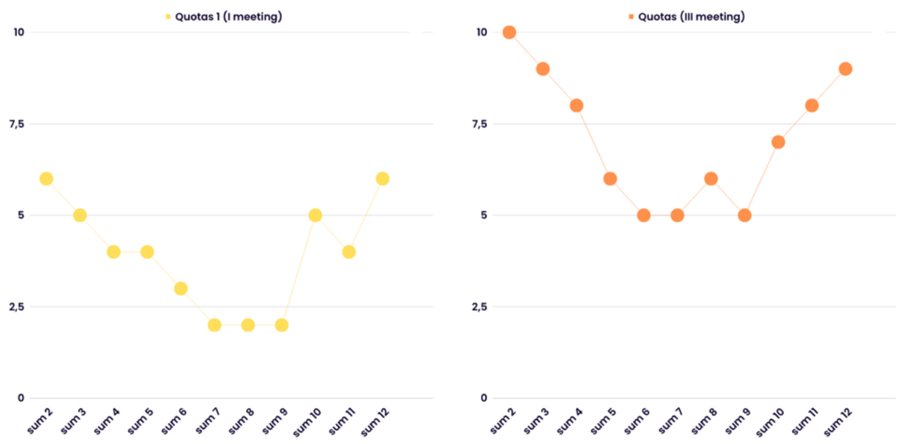



*Figure 24: Line chart depicting Alba's betting quotas.*

The shape of the graph is more or less the same merely shifted upwards because the quotas themselves are higher. Alba writes that she "thinks" that the possibility for players to bet on both the outcome of one or more numbers (*pro E bet*) and on the NOT-OUTCOME (*cons E bet*) increases the player's chances of winning. Consequently, she decides that the bookmaker must assign higher quotas.

The observation regarding Alba's behavior suggests a lack of coherence, albeit in a context where the term "coherent" is used in a less stringent manner than that introduced by de Finetti, who seeks to quantitatively formalize a way of integrating diverse opinions on the degree of confidence concerning each event. Alba's behavior appears inconsistent even in relation to how she sets the quotas. On one hand, she tends to keep the quotas low for easily achievable events, implying they offer a higher likelihood of winning. On the other hand, she argues that since *cons E bet* allows the player to win more easily, she must raise the quotas. This perhaps indicates that coherence – in the sense of de Finetti (1931) and, even before that, in a weaker sense – should be explicitly addressed as an element in educational contexts.

Alba's use of the term "think": this suggests that Alba's quotas, rather than being an effective measure of her *degree of confidence*, serve as a means to communicate qualitatively (KoT – Registers of representation – trend of the quotas) which events she has more or less confidence in. When people discuss an event's probability, they are talking about the level of surprise or trust they'd feel when it occurs. It's the degree of doubt or conviction we have when thinking about an uncertain event or outcome (de Finetti, 1931).

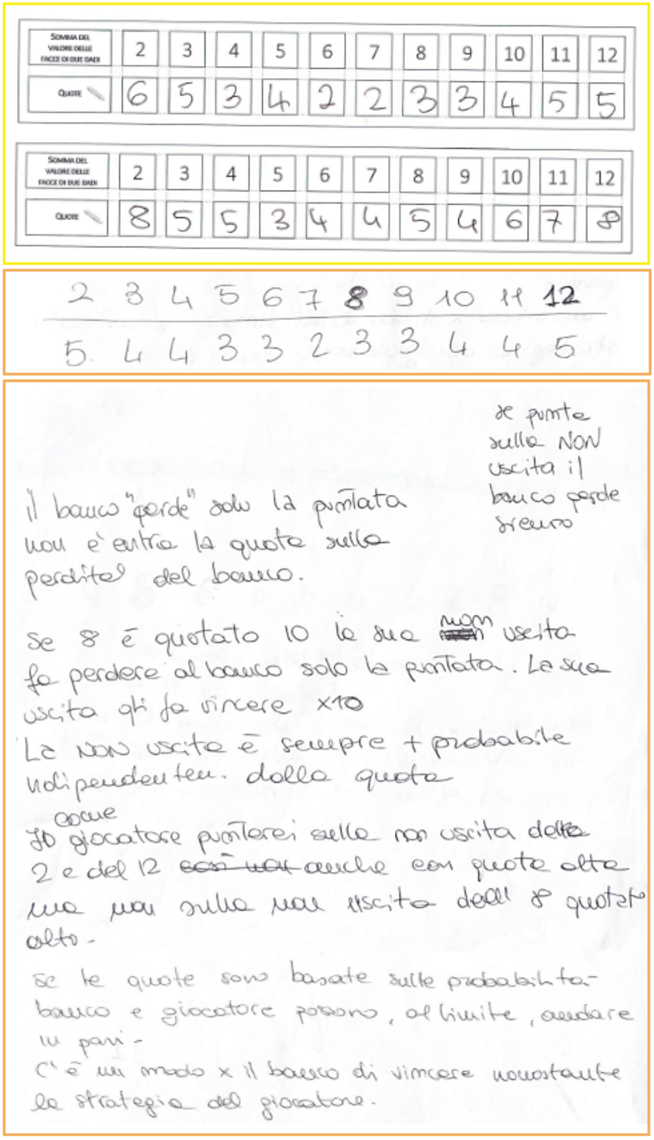
In that sense, Alba's approach appears to be more an *ordinal sorting* than a strategic quantification of the degree of confidence of these events, which is associated with Alba's knowledge of the category "how and when to do something" within the context of probability (KoT – Procedures – how and when to do something).

It's essential to highlight that Alba does not seem to use the relationships between events, nor does she differentiate how much she trusts certain events relative to others, such as her preference for "sum 9" over "sum 12." Her method appears to focus on the ordering of quotas, rather than establishing connections between different outcomes.

Alba's approach misses a systematic exploration of the interplay between different events, which could potentially limit the development of a more sophisticated strategy for quantifying her level of confidence in each event and that would entail (or at least concern) a coherent behaviour.

### 1.1.3 Paola's first data

Paola assigned non-uniform quotas (Figure 25) to the probable events of the sums obtained from the roll of two dice. This suggests that the teacher recognizes that some sums are more probable than others (KoT – Procedures – how to do something).




*If you bet on "NOT-outcome" (cons E bet), the bookmaker loses with certain.*

*The bookmaker only "loses" the bet quota doesn't affect the bookmaker's loss.*

*If 8 is quoted at 10, its not-outcome only causes the bookmaker to lose the bet. Its outcome makes him win x10.*
*NOT-outcome is always more probable regardless of the quota.*
*As a player, I would bet on the not-outcome of 2 and 12, even with a high quota, but not would bet on the not-outcoming of 8 with a high quote.*
*If the quotas are based on probability, the bookmaker and the bettor can, at most, break even.*
*There's a way for the bookmaker to win despite the player's strategy.*

*Figure 25: Paola's quotas and notes*

In the first meeting, Paola formulated quotas – in both instances – with a tendency toward a specific but not fully defined structure (quotas on the *tail* sums are consistently higher than the quotas on the *central* sums).

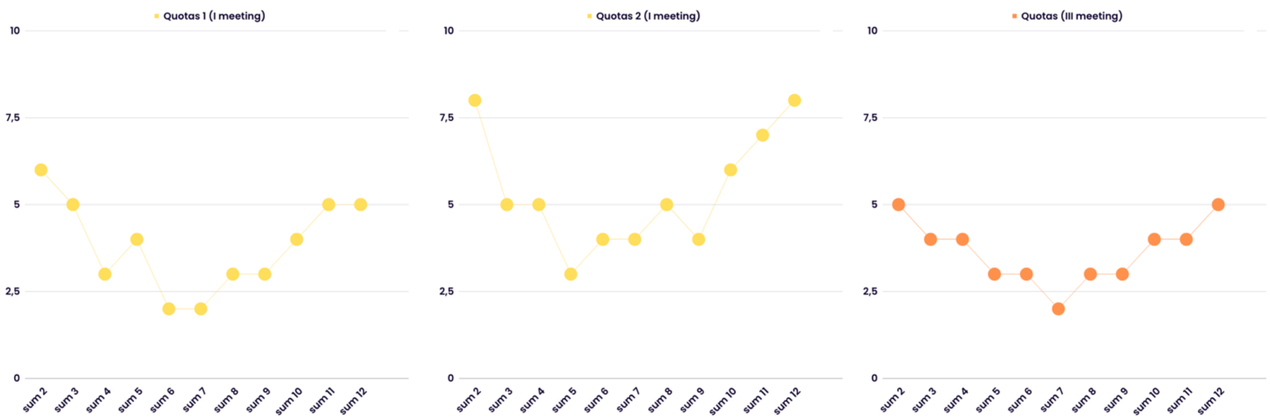



*Figure 26: Line chart depicting Paola's betting quotas.*

This precision in form (Figure 26) became more pronounced in the quotas established in the third meeting (KoT – registers of representation – symmetry of quotas' distribution). It appears that the experience of playing with the introduction of the "*cons E bet*" disrupted the initial quote structure. This adjustment in Paola's quote formation suggests a refinement and consolidation of her conceptualization, maybe influenced by the challenges posed by the *cons E bet* scenario. Notably, the quotas in the third meeting exhibit a lower magnitude compared to those in the first encounter.

The assignment of quotas, in the third meeting, reflects a perception of extremely low and high sums as less probable events (to which she assigns her highest quotas), while central sums are considered more probable (receiving lower quotas). The assigned quotas might also suggest that the teacher is influenced by the idea of a symmetric distribution, given the similarity of quotas to sums that are specular to 7 (e.g., sums 4 and 10 or sums 2 and 12). Assigning the lower quota to sum 7 could reflect the idea that it can be obtained from different ways (1+6, 2+5, 3+4, 4+3, 5+2, 6+1), making the probability of getting 7 relatively higher than other sums. These assigned quotas, thus show a certain degree of coherence with the probability distribution of the sums of two dice. For instance, lower quotas are assigned to central sums compared to less probable sums like 2 or 12. The teacher assigned higher quotas (quote 5 and quote 4, respectively) to extremely low sums (2 and 3) and assigned the same higher quotas (quote 4 and quote 5, respectively) to extremely low sums (11 and 12), indicating that she perceives these sums as relatively rare events and thus assigns them a lower probability. Paola assigned, for central sums, quotas ranging between 2 and 3, suggesting that these sums are perceived as more probable than lower sums. The lowest quote assigned to sum 7 might indicate that Paola sees this sum as one of the most probable (KoT – definition, proprieties and foundations – *degree of confidence*). The allocation of very similar quotas to sums with different probabilities might indicate that the teacher considers these sums equally probable.

Paola’s assigned quotas (A3) reveal an intriguing pattern: they increase and decrease by one, following the variations in the number of combinations associated with different dice sums (KoT – procedure – how to do something). Taking a closer look at sums 2 and 12, which boast the lowest probability and only one possible outcome (1+1 and 6+6), Paola designates quotas of 5 to both. For sums like 3 and 11, featuring slightly higher probabilities (calculated classically as 2/36), Paola assigns quotas of 4 each. This trend persists for almost all events, with quotas seemingly adapting consistently to changes in the number of combinations. However, it's essential to note that the increment of one in quotas corresponds to an increase of one in combinations, not necessarily reflecting the actual probabilities associated with them.

Moreover, Paola seems to be implementing a method for assessing measure to assess probabilities, trying to introduce quantitative elements into setting the quotas, which appear to be attributable to quantitative elements characterizing the combinations related to a sum (and thus the respective classical probabilities).but it's essential to note that this measure doesn't adhere to the relationships between events, nor does it apply constraints or the additive property.

Paola, in her notes, analyzes the role of the bookmaker in the proposed game, stating that if the bettor bets against the occurrence of a specific number, the loss is guaranteed. However, it is crucial to emphasize that this loss is confined to the initial bet and does not affect the bookmaker's quotas. This indicates that the bookmaker only loses what the player initially wagered.

In the context of a *cons E bet*, when the player bets *n* coins, Paola explains that the bookmaker is compelled to put *n* coins at stake and, consequently, might lose that amount if the event does not occur, and the bettor wins. This is in contrast to the *pro E bet*, in which the bookmaker only pays out the winnings when the event occurs.

Paola specifies that the "non-occurrence" of a particular event is always more probable than its occurrence (KoT – definition, properties and foundation – *degree of confidence*). Regardless of the assigned quotas, and in light of the aforementioned observation, Paola considers betting against a number a sensible choice for the bettor (KoT – phenomenology and application – process of betting).

The non-occurrence of an event is consistently more probable than its occurrence due to the distribution of combinations when rolling two dice. In most cases, there are more ways to obtain a combination that does not result in the desired outcome compared to the ways that lead to the desired outcome. For instance, considering the sum of 8, there are five possible ways to achieve this result (2+6, 3+5, 4+4, 5+3, 6+2). In contrast, there are thirty-one possible ways to not obtain the sum of 8. Even greater in number are the ways to not obtain the sums of 2 or 12, as they can only be achieved by combining 1+1 and 6+6.

This asymmetric distribution of combinations contributes to making the non-occurrence of certain sums more likely. Therefore, when betting against the occurrence of a specific sum in the roll of two dice, one is leveraging this asymmetry in the distribution of combinations and the higher probability of not obtaining the desired sum.

Paola seems to reinforce and expand on this concept when expressing a preference in her notes by comparing different events and their respective *cons E bet.*

She associates the event "sum 2" with the event "sum 12," strengthening her idea of considering them equally reliable (KoT – definition, proprieties and foundations – *degree of confidence*). She compares these two events with the "sum 8," stating that if the quotas were high on both events and she had to choose to bet against them, she would choose to bet on the *cons* 2 and the *cons* 12 and not on the *cons* 8 (KoT – phenomenology and application – process of betting). This choice aligns with the idea of considering the non-occurrence of 2 and 12 more probable events than the non-occurrence of the sum 8 [Intra-conceptual connection]. These observations highlight the subjective nature of probabilities in gambling, emphasizing the concept of " ordinal sorting".

Lastly, Paola raises a question regarding "quotas based on probability". She asserts that, under certain conditions, both the bookmaker and the player could break even. However, Paola emphasizes that there is a way for the bookmaker to win despite the player's strategy, thus highlighting the role of the bookmaker in gambling. These two statements appear to be in contrast.

### 5.1.3 Giorgio’s first data

In the exploration of quotas, Giorgio's evaluations (Figure 27) exhibit entirely different patterns from those examined thus far, unveiling a process of change that sheds light on his evolving perspectives.

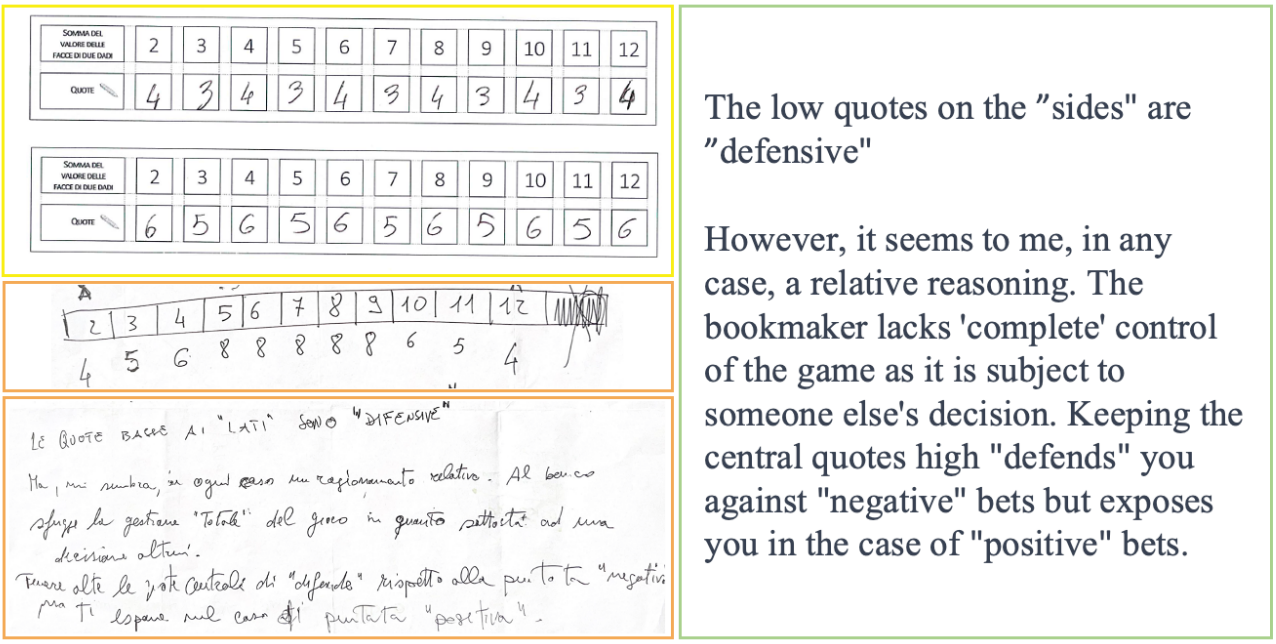



*Figure 27: Giorgio’s quotas and notes*

Giorgio's first quotas reveal a distinctive alternation between higher and lower values (Figure 28), reflecting a rhythmic pattern linked to the nature of dice sums, whether even or odd. This alternation might suggest a belief system where events “odd sums” are perceived as more probable than those “even sums”.

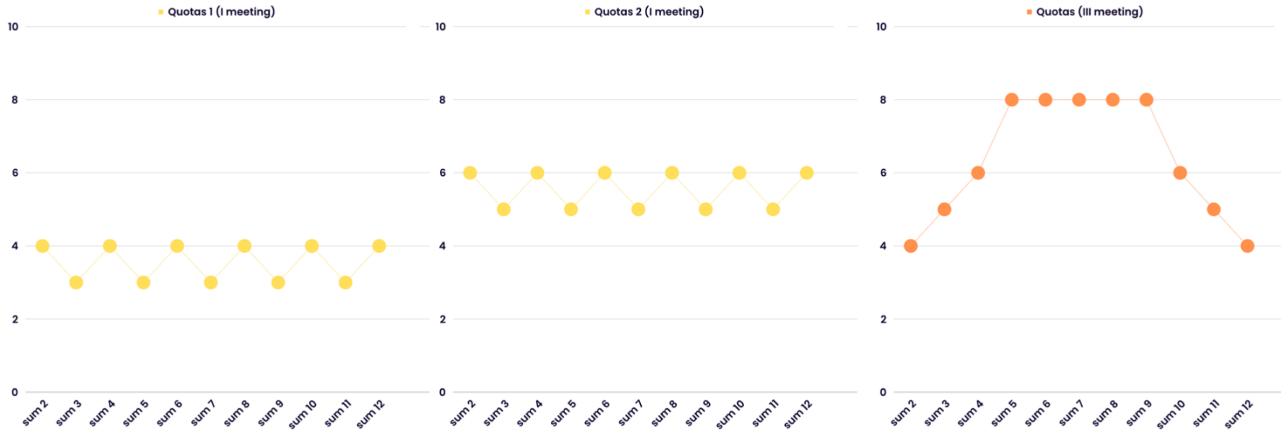



*Figure 28: Line chart depicting Giorgio's betting quotas*

The second set of quotations (Figure 28) maintains this alternating trend but undergoes an overall upward.

In the third meeting, a significant transformation unfolds as Giorgio's probability quotas adopt a bell-shaped distribution (KoT – Registers of representation - Trend of the quotas). This departure from the prior alternating model signals a revaluation of his perspectives on the likelihood of different dice sums. Giorgio characterizes the low quotations on the "*sides*" as "*defensive*" suggesting a cautious stance towards events associated with extreme sums (Kot – Procedures – when to do something).

Giorgio references to the events "sum 2" and "sum 12," assigning them the lowest quota and, expanding on this theme, he extends the defensive strategy to "sum 3 and sum 11" as well as "sum 4 and sum 10". Assigning the same quotas to these pairs of events, he's seeking a symmetry that didn't exist before (KoT - Registers of representation - symmetry of quotas' distribution).

Giorgio's strategy seems to involve an "ordinal sorting of the *degree of confidence* of events," linked to a knowledge "how to do something" in the probability topic (KoT – Procedures - how to do something).

It's as if Giorgio has grasped the notion that the initial approach (assigning high and low quotas to even and odd sums) might not be effective, but he still struggles to determine when to raise or lower the quotas. Giorgio expresses and encompasses in his action the fact that the *pro E bet* and the *cons E bet* have opposite effects on establishing the quotas.

### 5.1.4 Luisa’s first data

Luisa's pattern of assigning quotas stands out as markedly different from the other allocations.

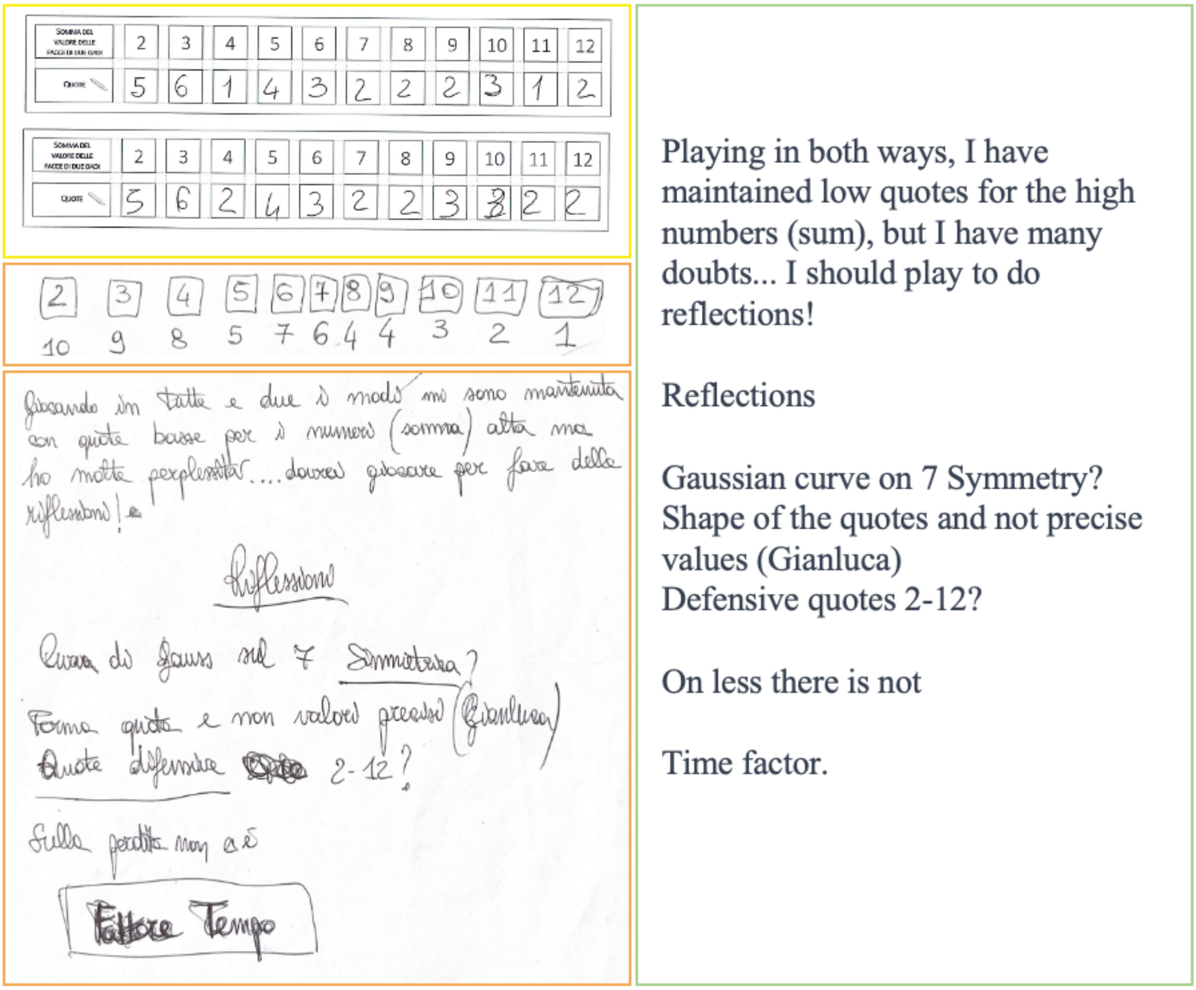



*Figure 29: Luisa’s quotas and notes*

In her notes (Figure 29), Luisa expresses numerous reservations and raises a series of questions, highlighting her difficulty in understanding the unfolding dynamics.

In the first meeting, Luisa maintains a consistent descending trend in quotas. However, there is an alteration: she substitutes the quotas “1”, previously assigned to the events “sums 4” and “sum 11”, with quota “2”. In the third encounter, she reintroduces quotas “1”, assigning it to the event “sum 12”.

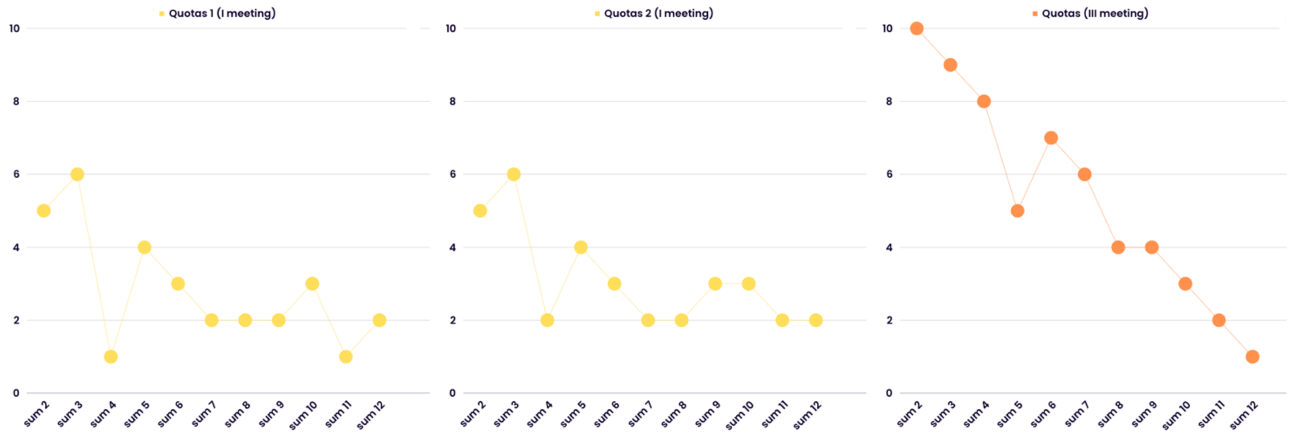



*Figure 30: Line chart depicting Luisa's betting quotas*

In the third meeting, the quotas exhibit a distinctly descending form (Figure 30). This descending trend could suggest Luisa's belief that achieving lower sums, such as 2 and 3, is more difficult and thus merits higher quotas. Conversely, higher sums like 10, 11, and 12 are deemed easier to attain, resulting in lower quotas.

Moreover, the gradual decrease in quotas (from the "sum 2" event to the "sum 7" event) might indicate a perception of probability based on the number of possible dice combinations for each sum. For instance, there are more ways to obtain a sum of 7 than a sum of 2, potentially influencing quotas allocation. However, Luisa's choice to attribute lower quotas to events with a decreasing number of combinations as the sum increases (from the "sum 8" event to the "sum 12" event) appears somewhat inconsistent with this assumption.

It appears that Luisa is indifferent to the number of combinations. In fact, she questions the reference to the bell curve distribution, as if she had not considered at all how the number of combinations is distributed.

Luisa's decision to assign quotas of 1 to the "sum 12" event (or any other probable and not certain event) wouldn't guarantee the bettor any winnings if the event occurred, as the bettor would only have the right to collect their initial bet.

## 5.1.5 Mirco's first data

In the first meeting, Mirco adopted a strategy of assigning low quotas (Figure 31 – yellow box).

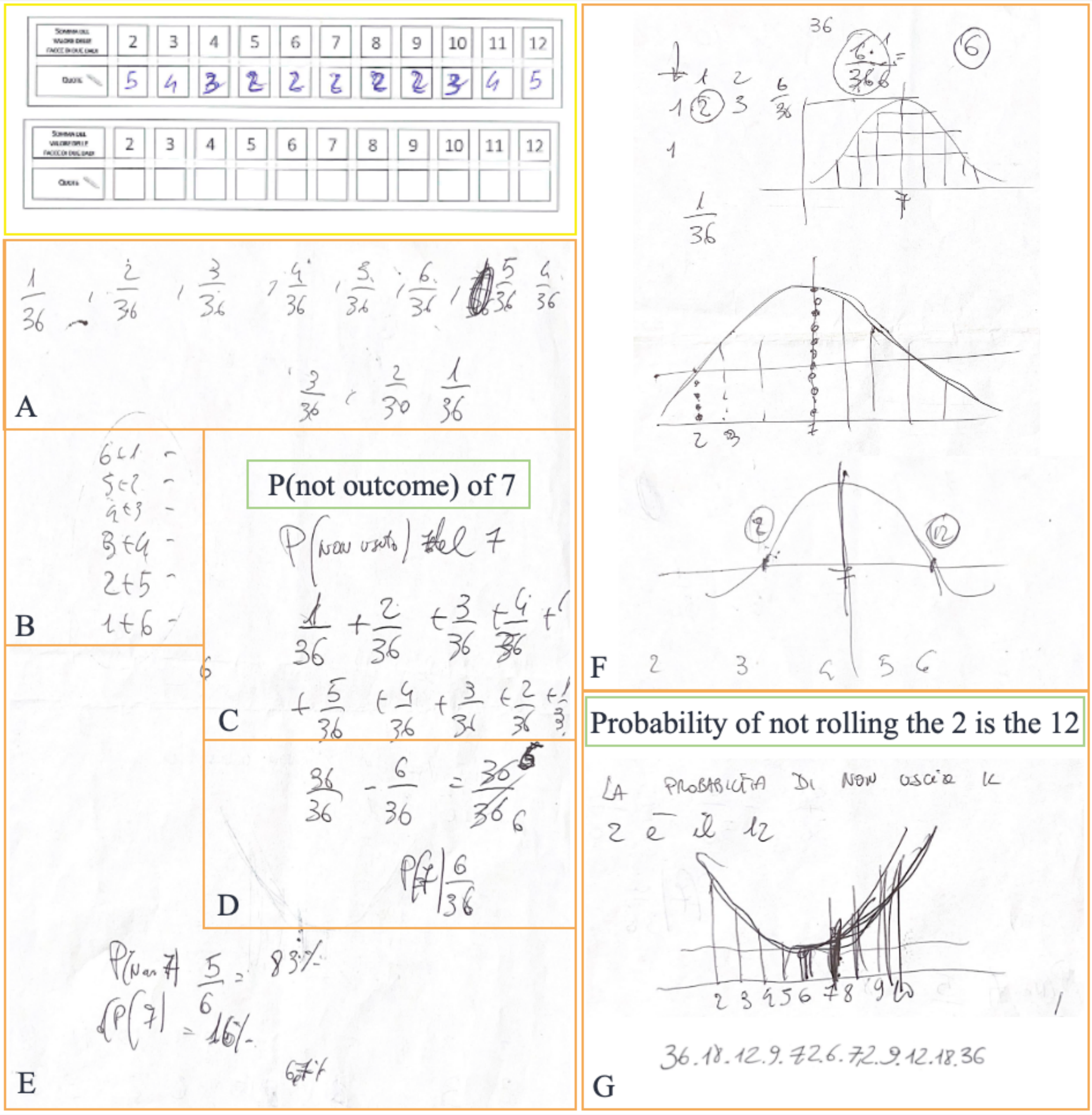



*Figure 31: Mirco's quotas and notes*

Notably, he chose to group together the central events, ranging from the sum of 5 to the sum of 9, all receiving an identical quote of 2. This grouping reflected his perception of these central sums as highly probable.

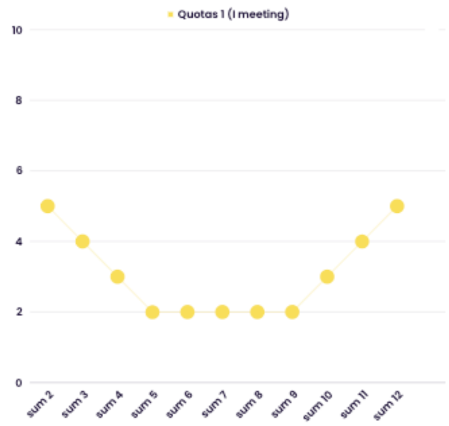



*Figure 32: Line chart depicting Mirco's betting first quotas*

In his quest for symmetry, Mirco assigned higher quotas, incremented by one for each successive event, as he moved towards the tails of the distribution (Figure 32). This deliberate progression in quotas along the tails suggested an effort to maintain an ordered representation of probabilities across different outcomes.

By keeping the central events uniform with a quote of 2, Mirco conveyed a sense of comparable likelihood for each of these events. However, his incremental adjustments towards the extremities underscored his nuanced awareness of the changing probabilities associated with different dice roll sums.

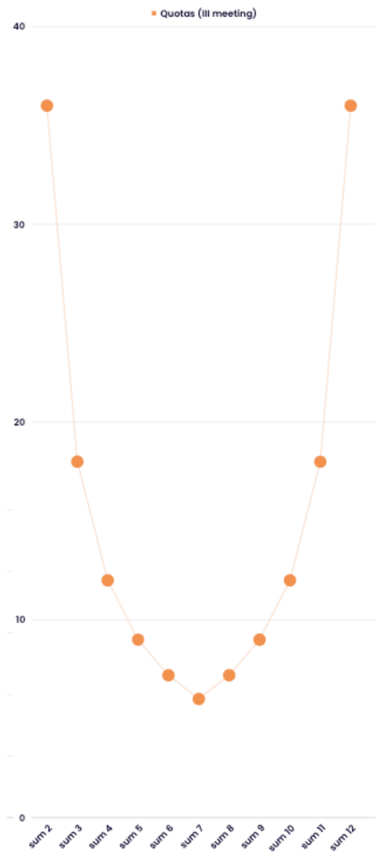



*Figure 33: Line chart depicting Mirco's betting second quotas*

Mirco's approach to quote assignment in the third meeting (Figure 33) indicates a nuanced perspective on the inverse probabilities (KoT – Procedures – how to do somethings) tied to dice sum outcomes. His method suggests a differentiated understanding of probabilities compared to his colleagues.

What is evident in the notes is Mirco's detailed grasp of the probabilities associated with various dice sum outcomes (A). It becomes apparent, although not explicitly mentioned, that Mirco is relying on the classical probability definition to compute the probabilities associated with each event (KoT – definition, proprieties and foundations - classical probability). The intentional assignment of non-uniform values to quotas (G) suggests a distinction in his varying degrees of confidence for each event. These seem to stem directly from the probability assignments just mentioned (KoT – Procedures – Why something is done in this way).

The quotas he assigns highlight a perception of extreme low and high sums as less probable, while central sums are seen as relatively more likely. This aligns with a probability distribution that peaks at the centre, indicative of a symmetrical probability perception (KoT – definition, properties and foundations - classical probability). Mirco assigns the highest quote, 36, to both sums 2 and 12, underscoring their rarity. This aligns with the low probabilities of these sums in the context of two dice rolls (1/36 each). Similarly, the relatively high quotas assigned to sums 3 and 11 (18) and sums 4 and 10 (12) suggest that Mirco perceives these sums as less probable compared to central sums.

In his notes, Mirco employs fractions (Figure 31 A – C – D) to represent probabilities (KoT – Registers of representation - fractions), calculates combinations (Figure 31 B) for the "sum 7" event (KoT – definition, proprieties and foundations - combinations), performs sum probabilities calculations for the "Non7" event and calculates the difference with the certain event (Figure 31 D) to verify the accuracy of his calculations (KoT – procedures - characteristics of the results), employs percentages (Figure 31 - KoT – Registers of representation - percentages) to verify the complementarity of the 'not-outcome 7' and 'sum 7' events (KoT – definition, proprieties and foundations – classical probability). The inclusion of Gaussian curves (Figure 31 F) and the depiction of the shape of assigned quotas (Figure 31 G) in the third meeting underscore a deep understanding of definition, proprieties and foundations in this context of probability topic.

Worth highlighting is Mirco's substantial modification of quotas between the first and third meetings.

## 5.2 The narrative of quotas: exploring a profound insight into teachers' decision-making beyond numbers

This second part of the discussion aims to examine the role and importance of the narrative surrounding teachers' quota-based decision-making. What do those quotas tell us? What do teachers convey when they discuss those numbers? This discussion transcends mere numerical quantification, offering an opportunity to delve deeper into the decision-making processes and the dynamics that influence them. What will be analyzed are segments of discussion from meeting A3, where each teacher provides a narrative of the quotas chosen and the motivations behind their choices.

As emphasized by Bruno de Finetti (1931), the fundamental question arises spontaneously: Does this instinct obey any laws? And why should it obey them? This question not only raises inquiries into the nature of instincts and human tendencies but also invites exploration into the reasons and motivations guiding teachers' decisions when assessing the level of reliability they attribute to an event.

The introduction of quotas and the narrative provided by teachers offer a unique opportunity to investigate the complexities of human decision-making and its interaction with cultural, social, and personal factors.

Through this discussion, our aim is to explore this insight, seeking to move beyond the numbers and grasp the richness of teachers' decision-making narratives within the realm of a betting game.

### 5.2.1 Ordinal Sorting: sequencing events based on degree of confidence

The issue of quotas cannot be reduced to a simple numerical question; rather, it represents an opportunity to explore the complex interactions between qualitative reasoning and quantitative measures. In this paragraph, we consider how the quality of teachers' decision-making narrative (particularly the narrative of Alba and Paola’s quotas), their specialized mathematical knowledge, and their assessment of future events dynamically intertwine with numerical considerations. According to Bruno de Finetti (1931), the laws governing and regulating the probabilities of events represent the relationships that must be respected to avoid internal contradiction among the different values attributed to these probabilities, regardless of their degree of demonstrability or acceptance. Furthermore, de Finetti (1931) distinguishes between instinct and caprice, suggesting that the determination of event probabilities should be guided by rational instinct rather than arbitrary preferences.

Alba: I have an idea about how to set them. The way you set the quotas changes because, on one hand, you might want to set them higher, as this discourages people from betting on the not-outcome. Why take a risk on this? In this sense, for example, it's enticing to play on the not-outcome of a number, but I would never bet on a number that the bank has quoted high because the risk for me is too high. So, I would bet on a lower quota.

Paola: But the not-outcome, so to speak, is very likely.

Alba: The not-outcome is very likely, I agree. But, do you understand how much the stake is? I mean, remember that the bank has to win and always wins.

Paola: So, you're betting on the not-outcome, so it's like you're betting on all numbers except 8.

Alba: And I agree. But to win something, betting on the not-outcome, which indeed encourages, you would never bet just one unit, since you only get back what you bet.

Paola: But it depends on the quota.

Alba: That's what I'm saying; you would never bet just one unit because you only get back what you bet.

Paola: Right.

In the discussed context, the probability of obtaining a particular sum is lower than the probability of not obtaining it. It is noteworthy that the approach to the probabilities of events and their not outcoming does not follow a symmetric treatment.

Alba adopts a qualitative reasoning (2 - 3) by examining the aspect of the *ordinal sorting* of quotas and the correlation between such *ordinal sorting* (KoT – Registers of representation - Trend of the quotas). She considers that an increase in the quota promotes the tendency to bet *pro E* but simultaneously inhibits the tendency to bet *cons E*. Alba addresses the idea that, in a context of quota arrangement, it is necessary to consider the possibility of bet *pro E* and bet *cons E* (KoT – phenomenology and application – process of betting).

A variation in quotas influences both directions of the betting tendency, both increase and decrease, as bets can be placed in both directions. Consequently, seek an *equilibrium* between these two directions becomes a crucial aspect [Intra conceptual connections]. Two ordering issues with contrasting directions are outlined. Alba's analysis focuses on qualitative assessments that mainly concern orderings.

On the contrary, Paola aims to introduce a quantitative metric to measure these concepts (15). She exposes the asymmetry linked to probabilities (7): the probability of an event occurring is inherently different from the probability of it not occurring. Consequently, the two probabilities cannot be treated symmetrically. Paola considers the need to develop quotas that take into account this asymmetry in the context of betting (KoT – Definition, proprieties and foundations – classical probability).

## 5.2.2 Introduction of a measure: Pressure, Compression, and Equilibrium

In exploring decision-making dynamics within the context of betting scenarios (particularly the narrative of Alba and Giorgio's quotas), teachers introduce an adjustment to the measure of the degree of confidence, which takes shape from the interaction of various "forces" influencing their actions. The concepts of *pressure*, *compression*, and *equilibrium* (to which the teachers will make explicit or implicit reference, explaining that the *pro E bet* and the *cons E bet* have opposite effects on establishing the quotas) will be used, to provide a key reference for analyzing the intricate dynamics observed during the deliberations of the individuals in question. Delving into n the detail of the transcripts, we will observe how these "forces" manifest and interact, shaping the decisions made and the strategies employed. This introductory measure will lay the groundwork for an examination of the factors driving decision-making processes within the gaming system in which the teachers are immersed.

Below is the discussion between Alba, Giorgio, and Ciro.

Alba: This time, since the bettor has a higher chance of winning, the bookmaker, I think, is forced to raise the quotas, partly to deter the bettor, and partly because, as the bettor's winning probability decrease, compared to the first way of betting, what bookmaker has to take must be more substantial.

Giorgio: Can I ask a question? Do you apply this reasoning to all numbers?

Alba: Yes, I raised the quotas.

Last time, my quotas were more symmetrical. There are some things that don't quite match up with the purely mathematical aspects, let's say those concerning the probability of a number coming up. If one thinks about the law of large numbers and has all the ratios between favourable cases and possible cases, it should be a Gaussian distribution peaked at 7. Because if one considers the combinations, the sum of 7 has more combinations, so it's symmetric around this point. I mean, if it truly wanted to be symmetrical, I would have given these to 5, these to 6, and these here to 7.

Ciro: Why didn't you do it?

Alba: I didn't do it because the law of large numbers doesn't apply in this case.

Alba's statement (19-22) provides insight into the rationale behind the higher quotas observed in the third meeting. She refers to "*this time*" (19), alluding to the introduction of the *cons E bet*. In Alba's initial statement (19-22), her focus lies in the applied realm, specifically discussing the bookmaker's strategic decision to raise quotas in response to an increased chance of the bettor winning. This expression reflects a preliminary phenomenological and applied approach, as Alba feels compelled to raise the quotas. This reveals an understanding of how practical considerations influence the bookmaker's actions within the dynamics of the betting scenario (KoT - Phenomenology and Application – process of betting). Her use of the term "*forced*" (20) suggests that the quotas are subject to an upward *pressure* that is external to her choices, dependent by the context of the betting game in which the teachers are immersed. This "*forced*" is closely connected to the *equilibrium* tackled in the previous paragraph.

Upon analyzing Alba's verbal expressions in conjunction with her corresponding quotas, it becomes evident that this *pressure* has an impact on all the quotas she assigns.

To clarify the concept of *pressure* referred to by Alba, the graphs in Figure 34 have been overlaid on a single chart. The blue arrows indicate the concept of *pressure* mentioned by Alba.

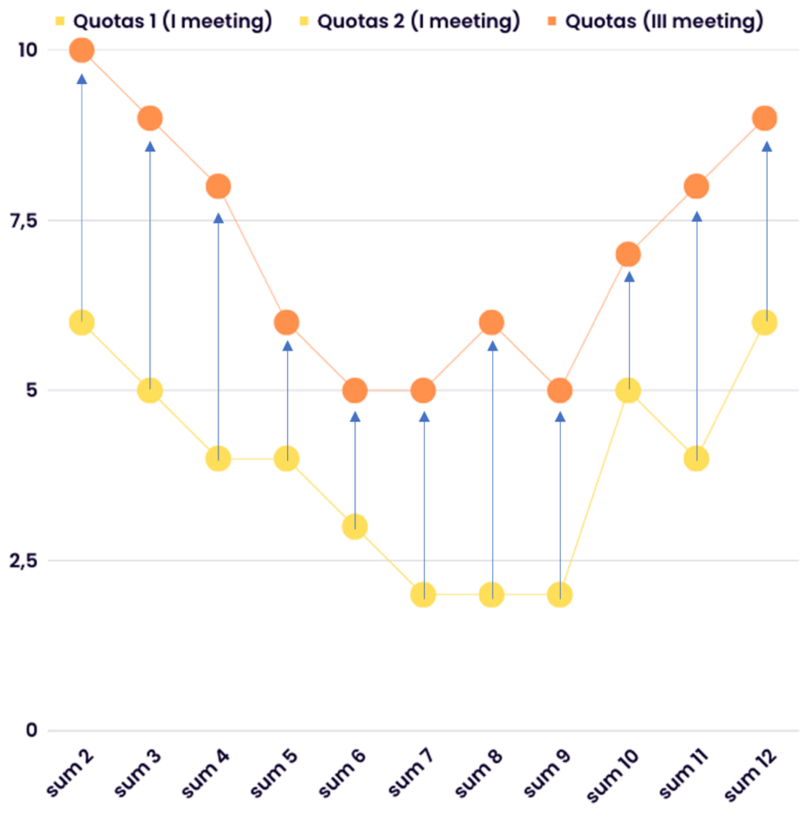



*Figure 34: Visual representation of pressure concept.*

It is important to emphasize, once again, that Alba prefaces the statement with "*I think*" (20), suggesting a degree of uncertainty (KoT- phenomenology and application – process of betting) in navigating this particular context she's attempting to explore.

When Giorgio poses a question (23), he ventures into the procedural domain by seeking to comprehend the reasoning behind Alba's approach. His inquiry delves into the principles or reasons guiding the application of a specific strategy to all numbers (KoT - Procedures - Why something is done this way).

Alba's subsequent response demonstrates a shift to the procedural aspect. She confirms the universal application of her reasoning and explicitly mentions raising the quotas. This showcases her practical knowledge in implementing a strategy, providing insights into the procedural aspects of decision-making within this context (KoT - Procedures - How to do something).

In Alba's expanded explanation (25-33), the focus deepens into the theoretical realm. She explores the law of large numbers, the classical probability, Gaussian distribution and the symmetry around the sum of 7. This intricate discussion reflects a understanding of the foundational principles of mathematics and their intricate connections to the dynamics of the betting scenario (KoT - Definition, Properties, and Foundations – law of large numbers).

Moving to Ciro's question (34), the discussion reiterates the foundational aspects. Alba's response (35) emphasizes that this law doesn't apply in this particular case, showcasing a nuanced discussion grounded in foundational principles (KoT - Definition, Properties, and Foundations - law of large numbers).

This analysis unveils a dynamic interplay of knowledge about phenomenology and application, procedural understanding, and a robust grasp of foundational mathematical principles throughout the conversation. It underscores the pivotal role of intra conceptual connection in specialized mathematical knowledge, shaping and motivating decisions related to a first refinement of the *degree of confidence* in events.

Giorgio: I approached this differently from Alba. I assigned lower side quotas, quoting 4, 5, 6, then all 8s up to the sum of 9, the central sums, and then descended with quotas 6, 5, 4. I didn't concern myself with raising or not raising, I reasoned more about which quotas I would keep low and which I would quote high. They could have been quotas 6, 7, 8, and then all 10s on the central sums; I didn't make a speech about the quantities of quotas. I asked myself: what kind of criterion would I use? and I would use this criterion.

Ciro: So, you thought about how the shape of the quota should be, not so much the specific values. But did you maintain symmetry in the attribution?

Giorgio: Absolutely, intentionally. So, I had this reasoning: first, unlike the previous rules, with this new rule, a significant portion of game control slips from the hands of the bookmaker; that is, the compression of the possibility to control the game is greatly reduced. Why is it greatly reduced? It's because the game's dynamics are conditioned. Even in the first version of the game, the game's dynamics are conditioned, just like with the players, but in this case, the game's dynamics are much more conditioned by how the player plays. So, the first consideration is the compression of the ability to manage. So, up to a certain point, the bookmaker can have a criterion, in my opinion. Because the criterion, so to speak, the rules that will govern the individual games, are managed to a percentage by the decisions of the house and to another percentage by the decisions of the player. That said, I put, so to speak, "defensive" quotas, why? Because since 2 and 12 are, so to speak... I took it as an axiom, which is yet to be proven.

Ciro: What is the axiom?

Giorgio: The axiom is that they come out less, that 2 and 12 have less probability of coming out. Don't ask me why; I stuck to the mathematicians who said this before. So, I play it on trust as well. So, this idea that 7is more likely to come out... but I still have some doubts as a player. So, having said that, if the curve is like that, and 2 and 12 have a lower chance of coming out, I have to defend myself. I have to make sure that the player doesn't play the non-outcome of those numbers (refers to 2, 3 4, 10, 11, 12 sums) because the non-outcome of those numbers is more probable. So, I keep the quotas low.

Paola: And why?

Alba: No!

Giorgio: I really want my reasoning to be understood.

Paola: If you have to follow your reasoning, you have to do as Alba did.

Giorgio: Yes, my upstream reasoning was that, then I made a mistake in writing the numbers of the quotas. As Paola says, I should have written the quotas exactly the opposite way. However, raising or lowering the quotas has a radically opposite effect depending on the type of bet, so everyone overlooks the control of the matter. What is called a pro bet E, when you raise the quota, has a positive effect on the bet where this number comes out, and has exactly the opposite effect on the cons bet E, in my opinion. Up to a certain point, you can give yourself a criterion. Raise or lower? Do you understand?

In the discussion, Giorgio's approach to setting quotas reflects a procedural decision-making process guided by evolving specialized mathematical knowledge in probability in a direction that seems to have roots in KoT - phenomenology and application – process of betting, and wants to grow towards KoT - Definition, proprieties and foundations – classical probability [Intra-conceptual connection].

Distinct from Alba's strategy (A3 quotas), he adopts an inverse approach, allocating lower quotas on sums in tails (36-40) and focusing on the overall distribution (40-43) rather than focusing on specific values (KoT - Procedures - How to do something). This procedural choice (40-43) indicates an evolving understanding of how to formulate strategies in this context of betting games. Giorgio's new decision to raise or lower quotas based on his evolving understanding of the probabilities associated with different sums indicates a procedural shift. Let’s remember that in the previous meeting (A1 - Figure 27), he placed high quotas on even sums and low quotas on odd sums, suggesting a procedural modification influenced by a changing perception of the *degree of confidence* of the different sums (KoT - Procedures - Characteristics of the result).

Giorgio's adjustment of quotas, raising some and lowering others, reflects a dynamic decision-making process influenced by the changed dynamics (46-55) of the game that he defines (48) (53) *compression* (KoT - Procedures - Why something is done this way). This concept, which we previously referred to as *equilibrium* with Alba and later renamed as *pressure*, is clear in his detailed explanation.

The acknowledgment of taking a particular assumption (59) (61-63) from mathematicians and stating that it is "yet to be proven" suggests an engagement with foundational principles in probability theory (KoT - Definition, proprieties and foundations – Classical probability). This uncertainty underscores a developing awareness of the need for a deeper understanding to develop his decision-making process.

The concept of "defence" emerges again in Giorgio's words - it had already emerged in his notes (Figure 27) and reappears in his discussion (64-66) and introduces a register of representation that combines qualitative and strategic elements in his decision-making (KoT - Registers of representation – Symmetry of quotas' distribution).

Giorgio delves into the considerations surrounding the adjustment of quotas in the context of betting games (74-82). His assertion that the effect of raising or lowering quotas is radically different depending on the type of bet underscores a deep understanding of the dynamics at play within the betting scenario. This reveals a specialized awareness of how stay in the probability scenario interfaces with decision-making processes (KoT - Phenomenology and Application – Process of betting).

The pivotal question of whether to raise or lower quotas serves as a linchpin in Giorgio's analysis. This decision-making point highlights his awareness of the complex interplay between mathematical principles and the practical application of these principles in betting scenario (KoT - Phenomenology and Application – Process of betting). The ability to discern the differential impact of raising or lowering quotas on "*pro E*" and "*cons E*" bets demonstrates an understanding of how mathematical knowledge guides decision-making in a dynamic environment.

The term "*compression*", previously introduced by Giorgio (48) (53) continues to be a focal point (KoT - Phenomenology and Application – Process of betting), especially when deliberating on whether to raise or lower quotas (81-82). This term represents the intricate *pressure* or constraints faced by the bookmaker in managing the game effectively (KoT - Phenomenology and Application – Process of betting). This *compression* is crucial in the context of the "*pro E*" and "*cons E*" dynamics, in which its application must be tailored to each outcome event. Giorgio does not recognize that this *pressure* must be differentiated for each event, and he does not acknowledge that this *pressure* is inherently tied to the *degree of confidence* in each sum. However, he implicitly suggests a critical observation in this context. Despite not openly stating the need for diversification of *pressure*, a hint of uncertainty emerges when he poses the crucial question: "Raise or lower? Do you understand?" This inquiry suggests that Giorgio is trying to grasp an aspect that might not be entirely clear in his analysis; his reflective question implies an awareness of a potential unresolved complexity. (KoT - Phenomenology and Application – Process of betting).

In essence, Giorgio's discussion embodies a phenomenological approach intertwined with practical considerations, showcasing how his specialized mathematical knowledge in probability informs the complexities of decision-making within the context of betting games.

Giorgio's discourse reflects a dynamic interplay of applied knowledge, evolving procedural understanding, and an emerging grasp of foundational mathematical principles.

### 5.2.3 Probability Experience: Does Exploration Enable Formalization of Thought?

In examining the dynamics of decision-making within the context of a betting scenario, we delve into the question of whether experiential exploration fosters the formalization of thought regarding probability. This inquiry is rooted in the foundational principles articulated by de Finetti (1931), who posits that probability serves as a fundamental tool of human cognition. His assertion underscores the notion that regardless of how individuals assess the probability of a given event, no experience can definitively prove or disprove their judgment. Instead, probability represents, for de Finetti (1931), the primary tool of our thought.

And perhaps here lies the crux of the matter: the concept of probability is inherently non-deterministic. The individual outcome of an event, whether positive or negative, cannot be used as a yardstick to assess the accuracy of its *degree of confidence*.

The subsequent analysis will explore how Luisa and Mirco navigate the realms of probability experience and formalized thinking. Specifically, we will examine their engagement with key concepts related to probability, such as symmetry in quotas and considerations of probability distributions. Through this exploration, we aim to uncover the extent to which experiential exploration informs the participants' understanding and application of probability concepts, shedding light on the intricate interplay and intra-conceptual connection between practical experience and formal knowledge in decision-making processes involving probability.

Luisa: I agree with the raised quota on 2, but I quoted 1 for the sum 12, so I went completely down the scale, I did the opposite operation.

The fact that Luisa (83-84) continues, in the A3 meeting, to refrain from adopting symmetry in her quotas (Figure 29) and consistently opts for lower quotas suggests a lack of awareness of basic principles and mathematical foundations that other teachers consider in their answers (KoT - Definition, Properties, and Foundations – *Degree of confidence*). The absence of considerations regarding probability and the question of whether a number can come out with greater or lesser probability highlights a lack of engagement with key concepts related to probability.

Luisa emerges as a participant contributing with a limited number of interventions in the discussion, and following this meeting, she will no longer attend the course regularly. Her statement, expressing agreement with the raised quota on 2 but followed by choosing to assign a quota of 1 for the sum 12, appears to lack thoughtful reflection on the dynamics of the game (KoT - Phenomenology and Application – Process of betting). Her input indicates our failure to construct phenomenological, theoretical, or procedural foundations to create connections and deepen her specialized knowledge [Intra - conceptual connection].

Mirco: What I did at school was give everyone the opportunity to roll two dice and observe the combinations, considering the probabilities of outcomes on each die. Analyzing the possibilities, I noticed that out of a total of 36 possibilities, we explored all possible combinations. We observed that out of 36 possibilities, there are 6 possible combinations, corresponding to outcomes with a sum of 7. Then, extending this approach to all numbers and sums (1, 2, 3, 4, 5), I highlighted that, generally, when rolling two dice, we get a sum of 2 when both dice show 1, a sum of 3 when one die shows 1 and the other shows 2, and so on. If we represent these combinations on a graph, we can transform them into percentages and decimals. For example, 6 cases out of 36 correspond to the centre, indicating the sum of 7, while the other combinations are symmetrically distributed around it. If we repeat this process by rolling two dice for an extended period with 20 participants, recording the results of each roll and counting how many times a sum of 2 occurs, we will obtain a distribution approximately similar to the previous one.

Alba: The higher the number of rolls, the closer it gets to this thing. If you don't adhere to the number of rolls, you might end up with a curve with a peak at a sum of 12.

In the discussion, Mirco outlines a procedural approach (KoT - Procedures - How to do something) he employed in a classroom setting to explore the outcomes of rolling two dice. By observing combinations and considering the probabilities on each die (87-93), he noted that, out of 36 possibilities, there are 6 combinations resulting in a sum of 7, consistently with what he had written in his notes (Figure 31). Extending this approach to all numbers and sums, Mirco highlights the general patterns when rolling two dice. He introduces (94) the idea of representing these combinations on a graph and transforming them into percentages and decimals (95). This procedural strategy provides a hands-on exploration of probability concepts (KoT - Procedures - Characteristics of the result).

Alba introduces (102) a consideration related to the number of rolls (KoT - Procedures - When to do something). She emphasizes that as the number of rolls increases, the distribution approaches a more accurate representation (102-104). Alba hints at the importance of a sufficient sample size in statistical analysis, addressing the potential issue of a skewed curve if this condition is not met. This insight demonstrates a procedural understanding of when to apply certain methods in a probabilistic context (KoT - Procedures - When to do something).

The discussion touches upon foundational aspects (KoT - Definition, Properties, and Foundations – *Degree of confidence*), as Mirco (96-101) and Alba (102-104) delve into the fundamental principles of probability. Mirco's approach involves a systematic exploration of all possible combinations (95-97), showcasing a foundational understanding of the concept. Alba's contribution (102-104) adds depth by emphasizing the importance of the number of trials in achieving a reliable distribution, aligning with foundational principles of probability.

The engagement with registers of representation (KoT - Registers of Representation - Graphical representation) is evident in Mirco's use of graphical representation (94-95) to depict probability outcomes. This visual representation aids in translating abstract concepts into tangible, visual forms, enhancing the understanding of probability.

The phenomenological application (KoT - Phenomenology and Application – Process of betting) is apparent as the teachers discuss the practical aspects of their approaches. Mirco's classroom activity involves physically rolling dice and observing outcomes, providing a tangible experience to comprehend abstract concepts. Alba contributes by linking the theoretical idea of probability distribution to practical considerations, emphasizing the significance of the number of rolls.

The participants haven't explicitly explored the intra-conceptual connections [Intra - conceptual Connection] between the different elements of probability theory and the context in which they are involved. There is a focus on individual aspects, such as combinations and sample size, but a deeper exploration of the interconnections between these elements is not explicitly addressed in this part of the discussion.

## 5.2.4 A Fair Bookmaker

"The narrative of quotas: exploring a profound insight into teachers' decision-making beyond numbers" chapter concludes with Mirco's discussion. He approaches the task of setting quotas as if it were a purely mathematical problem to be solved, distancing himself from the phenomenological context. He employs his formal mathematical knowledge of probability theory to tackle the issue. His narrative brings to mind de Finetti (1931), who posited that probability calculation is essentially the mathematical theory that teaches us to be coherent in our assessments. After calculating his quotas, he returns to the phenomenological context, acknowledging the unpredictable nature of individual events, as Poincaré (1950) suggests. Despite the formal mathematical nature of probability calculations, Mirco's emphasis on the unpredictability of events in betting games resonates with Poincaré's scepticism about calculating probabilities when faced with uncertainty and the unknown. This paragraph sets the stage for exploring Mirco's approach and the nuanced interconnection between formal mathematical reasoning and practical decision-making in the context of betting games.

Mirco: What I couldn't do in the first case was to find... for example, this one is... I had thought 6 out of 6 makes 36, one-sixth. And what is the value of the quota? How to determine it in relation to the probability of occurrence?

Ciro: You said, I have that number which is one-sixth, what does one-sixth tell us?

Mirco: It's one case out of six. So, I had thought of setting 6 as the quota for the sum of 7.

Ciro: Why?

Mirco: Because 1 to 6, 1 probability, and 6 on the other side, like the ratio between victory and defeat, so one case, 6 positive cases out of 36 possible, to be a bookmaker...

Ciro: To be a bookmaker how?

Mirco: It must be a bookmaker, like, scientific.

Giorgio: A decent bookmaker.

Mirco: One who respects a model of the game, a way of playing.

Ciro: But in what sense is this bookmaker decent, what does this bookmaker respect?

Mirco: It respects the rules of the probability of occurrence.

Alba: It's a fair bookmaker. He's not cheating.

Paola: I mean, he wants to justify all his outcomes.

Mirco: Clean. I use a logic that you can know too!

Alba: Clarity. It tells you: look, inform yourself.

Mirco: It's not illegal.

Alba: If the bookmaker doesn't do it like Mirco, one wonders why, for example, you give 4 to 8 and 9 to 5.

Ciro: Sorry, and if you go to a bookmaker and there are quotas like those of Alba or Giorgio, not symmetrical...

Mirco: Yes, what kind of game is it? I would think it's a strange game otherwise.

Ciro: Let's do this. Following your reasoning, do you want to set the quotas that a bookmaker, as Alba called it "fair," would set?

Mirco: I would set them like this: For the sum of 2, quotas 36; For the sum of 3, 18; For 4, 12; for 5, quotas 9; for 6, quotas 7.2; for 7, 6; for the sum of 8 again quotas 7.2, and it continues the same.

Mirco: After that, however, there are rules, ways of playing behind this. What happens makes it so that the bookmaker, in the end, still wins. That is, if we agree on how to bet our money and spread it properly on the board with the right criteria... the point is that when I roll a die, even if there have been 1000 rolls, I don't know where that other roll will go, what the outcome is, understand? I know it's balanced.

In this discussion among teachers, Mirco's approach (aligns with his notes Figure 31) reflects a deep understanding of probability theory (KoT - Definition, proprieties and foundations – *Degree of confidence*), aligning with De Finetti's notion of probability as a "*coherent opinion of an expert deriving from the complex of information in possession of the subject who* *evaluates*” (de Finetti, 1931). Mirco strives to be the "fair" bookmaker (111-112) (114-116), embodying fairness, legality, and transparency in establishing quotas (KoT - Procedures - Why something is done this way).

His attempt to set quotas (111-112, 114-116) based on the combinations of possible outcomes (Figure 31) indicates a coherent and systematic methodology (KoT - Procedures - How to do something). By considering all possible combinations and relating favourable outcomes to the total, Mirco exhibits a nuanced understanding of how to determine quotas coherently (KoT - Procedures - Characteristics of the result). The word "nuanced" is used because, although Mirko employs the inverse of classical probability (KoT - Procedures - How to do somethings) to calculate the odds, we cannot deem it "coherent" from de Finetti's standpoint. He should have explicitly stated that these odds are determined in such a way that the player does not have a guaranteed win, which Mirko neither mentions nor does. In the subsequent paragraph, we will indeed see that Mirko loses his footing when confronted with the actual context of the game.

Alba's endorsement of Mirco's approach as "fair" aligns (124) with the ethical considerations of a bookmaker respecting rules of probability, emphasizing clarity and justification for outcomes (KoT - Phenomenology and Application – Process of betting). This resonates with De Finetti's emphasis on probability as a property belonging to man and managed through consciousness and logical thought.

Giorgio's agreement further reinforces the shared understanding of fairness in Mirco's method (119), hinting at a collective appreciation for a systematic and scientific approach [Intra-conceptual connection].

Mirco exhibits a unique approach in tackling the practical challenge of setting quotas, essentially treating it as a mathematical puzzle that demands a solution (106-108) (118-120).

In navigating the contextual nuances, Mirco tells “*like the ratio between victory and defeat*” (114-115), he adeptly employs a logic (114-116) deeply rooted in a phenomenological method of extracting information (KoT - Phenomenology and Application – Process of betting).

As the conversation shifts to the actual determination of quotas (111-112) (139-144), Mirco seemingly deviates from the nuanced details of phenomenology and, instead, approaches the task by treating the acquired information as data for a mathematical problem (123). It seems that Mirco, in this instance, moves away from the intricate details of phenomenology. Instead, he treats the gathered information as if they were components of a mathematical problem, (115

- 116) applying his expertise and his knowledge (KoT - Definition, Properties, and Foundations – *Degree of confidence*).

Mirco reflects on the dynamics of betting games (139-144) and emphasizes the existence of underlying rules and strategies. He acknowledges the role of bookmakers, highlighting that despite players' efforts, the bookmaker ultimately prevails. Mirco suggests that effective betting involves agreement on how to bet money and strategic distribution across the board. However, he expresses uncertainty about predicting individual outcomes, using the example of rolling a die. Even after a significant number of rolls, Mirco contends that the outcome of each roll remains unpredictable, emphasizing the balanced nature of the die. He articulates a sense that individual outcomes are not easily forecasted based on past occurrences, underscoring the inherent unpredictability in such scenarios. No experience can definitively prove or disprove their judgment (de Finetti, 1931).

In considering Poincaré's perspective (1950), Mirco's discussion is aligned with the paradox inherent in the concept of calculating probabilities. Poincaré questions the feasibility of calculating something (probability) that is, by definition, associated with uncertainty and the unknown. Mirco's emphasis on the unpredictable nature of individual events in betting games resonates with Poincaré's scepticism about calculating probabilities when faced with the inherent uncertainty that defines them.

## 5.3 Intra-Conceptual Connections to Construct a Coherent Probability Measure

De Finetti's assertion (1931) regarding the pervasive role of probability as a guiding principle in human cognition emphasizes its fundamental importance. He contends that probability should serve as our mentor in thinking and acting. Instead, he highlights probability's role in structuring and formalizing our thoughts.

Furthermore, De Finetti (1931) argues for the coherence of probability evaluations, emphasizing the importance of measuring the degree of confidence in order not to enable competitors to win with certainty.

This underscores the necessity of establishing continual connections to assess the degree of confidence, maintaining immersion in phenomenological knowledge and the information derived from it.

While mathematical connections may originate from intrinsic relationships among elements or be consciously established based on individuals' mathematical knowledge, it's crucial to perceive them as “shared knowledge rather than personal ones” that can be identified, developed, and discussed together.

These connections represent relationships between various constructs, concepts, properties, or foundations within the same topic or across different topics.

Understanding these connections is vital for assuming intra-conceptual connections within the context of probability, especially within classical, frequentist, or subjectivist approaches.

Recognizing these connections enables the construction of a coherent probability measure aligning with De Finetti's vision of probability as a guiding principle in human cognition.

The following paragraph will elaborate on building shared intra-conceptual connections between emerging and future knowledge to raise awareness regarding the degree of confidence, ensuring that players are not put in a position to win with certainty.

Alba: In order to achieve the famous predictability and assert the law of large numbers, I have to play a certain number of times, let's say I have to play a million times. But let's consider 36rolls: I'm extrapolating the 36 rolls from the case of a million. So, out of 36 rolls: the "sum 2" comes out one time, the "sum 12" comes out one time, the "sum 7" comes out six times. Now, what should I do? In 36 rolls, I decide to bet all 36 bets only on one sum. Let's do the math!

Ciro: Does any of these allow you to do what you are saying?

Alba: Yes, Mirco's quotas.

Paola: Sorry, but if you always bet on the same number quoted 36, you lose 35 times, the last time you practically break even.

Alba: No, because think about everything I had to give you.

Paola: Okay, let's do this. I follow your reasoning, I bet one coin always on 12 quoted 36. One coin always on the same number, all 36 times. Twelve comes out once.

Alba: You lost 35 coins, and the time you won, you won 36 coins.

Paola: You placed a coin on 12, 36 times, and only once you won 36, so you placed 36 and recovered 36.

In the discussion, teachers connect into the intricacies of this betting games (considering the *pro E bets*) and the application of mathematical knowledge in probability.

Alba focuses on the procedural aspects of achieving predictability and mentions the law of large numbers (145-146) as a key principle (KoT - Procedures - Why something is done this way). She emphasizes (147-150) the need for a substantial number of plays (KoT – Procedure – When to do somethings) to assert the law of large numbers, using the example of playing a million times to demonstrate predictability. Her analysis (147-150) involves extrapolating data from a smaller subset of rolls (36 in this case) to draw conclusions about larger sets, showcasing a procedural method in probability analysis (KoT - Procedures - How to do something).

Alba introduces the concept of betting strategies within this framework. Her discussion involves making decisions based on the imagined frequencies of specific outcomes. In her example, she considers (147-150) the frequencies of different sums in 36 rolls and proposes a strategy of betting all 36 times on a single sum (KoT – Phenomenology and application – Process of betting), guided (152) by Mirco's quotas (KoT – Definition, proprieties and foundations – *Degree of confidence*).

Paola challenges in a critical evaluation (153-154) of Alba's strategy, questioning the effectiveness of consistently betting on a single number. This prompts a discussion about the potential losses and gains associated with such a strategy (KoT – Procedures - Characteristics of the result). The dialogue highlights the importance of understanding the underlying probabilities and the impact of betting decisions on outcomes.

Overall, the discussion showcases a blend of procedural considerations, strategic decision-making, and the application of foundational concepts in probability, laying the groundwork for further exploration of how specialized mathematical knowledge impacts decision-making in the context of betting games (KoT - Phenomenology and application – Process of betting).

The interplay between the phenomenological context and a solid understanding of foundational principles delineates the strength of Alba's analysis within this specific scenario. In fact, a correlation surfaces between the phenomenological context of the gambling situation, empowering Alba to steer this exploration, and her knowledge of definitions, properties, and foundations, that proficiency enables her to validate Mirco's quotas as reliable components of her analytical framework [Intra-conceptual connections].

Paola: Based on what we've said so far, in my opinion, the bookmaker can understand what the quotas should be to win consistently even when playing against E.

Ciro: What can it understand?

Paola: I mean, ensure that... if in the ideal scenario and with the possibility to bet pro and again an event, we said we break even, meaning all players must always bet on the same number, so that at the end of various rolls, we break even. So practically, the Bookmaker must now choose the quotas so that they are lower than the probability measure. In other words, if 2 has a probability of coming out 36 times, the quota should not be 36 but should be lower.

Alba: It would be 35 that makes the bookmaker wins.

Mirco: We never thought of putting the quotas that way.

Ciro: So, you're saying the Bookmaker must set quotas that are lower than the probabilities of coming out. So, even setting the quota at 35 for 12, quota 17 for sum 11, quota 11 for sum 10, quota 8 for 9, quota 6 for 8, quota 5 for 7 ensures you win every time.

Giorgio: Are we still talking about the old version of the game, the one where you can only play Pro?

Ciro: Why are you asking this?

Giorgio: Because if I get into a situation like this, the first thing I do is bet on the non-outcome, bet against any event. As a bettor, my first thought is "remember that you also play Cons E". The first thought that comes to me is "the bookmaker has forgotten that there's the new rule, so I have to play and bet against immediately."

Ciro: Why?

Giorgio: Because it's exactly the opposite. I flip the game and win. It is for this reason that I wrote it in the notes that this new rule of playing against an event, you no longer control the game.

Paola raises a procedural consideration (163-165) regarding the Bookmaker's ability to consistently win, both when playing in *pro E* bets mode and when playing in *cons E* bets mode. The discussion revolves around the determination of quotas that would enable the Bookmaker to secure an advantage, even in the scenario where all players consistently bet on the same number. Paola proposes that the Bookmaker should strategically set quotas lower than the respective probability measures, facilitating parity in the game dynamics.

Alba suggests (174) a specific quota (quota 35) that, according to Paola, ensures the bookmaker's victory. This brings attention to the procedural characteristics of determining quotas and their impact on the overall result (KoT - Procedures - Characteristics of the result).

The subsequent discussion involves specific examples of quota settings for various sums, emphasizing the importance of understanding and manipulating quotas. This represents a procedural analysis of how to determine a measure of the reliability of an event (KoT - Procedures - How to do something).

The discourse takes an intriguing turn when Giorgio introduces a unique perspective based on phenomenological and conceptual connections [Intra-conceptual connection]. Giorgio challenges Paola's idea of lowering the quotas and introduces the concept of consistently betting against an event, highlighting a fundamental shift in strategy. His approach involves exploiting the potential oversight of the Bookmaker regarding the new rule, playing against the non-outcome to gain an advantage. Giorgio has discerned that, contrary to what Alba and Paola stated, those lowered quotas are not favourable for the bookmaker but for the bettors. This introduces a nuanced understanding of the game dynamics, emphasizing the importance of situational awareness and adapting strategies based on evolving rules (KoT - Phenomenology and application – Process of betting).

Giorgio, with quotas lower than the probability measure, is suggesting betting against event *E,* meaning betting that a certain outcome will not occur.

Let's take an example using the quota for the sum 12, set at 35 provocatively by Ciro.

In the ideal scenario of 36 rolls, where sum 12 should occur only once, Giorgio could bet against the event 12 for 36 times with an amount of one coin each time: this means that the Bookmaker is obliged to bet one coin for each of the 36 rolls on the outcome of 12.

Let’s remember that the sum 12 should occur only once in 36 rolls.

If sum 12 does not occur in the first 35 rolls, the bookmaker would have bet and lost a total of 35 coins.

On the 36th roll, the bookmaker would be forced to bet another coin on sum 12 (for a total of 36 coins). When sum 12 finally occurs, the bookmaker would win 35 coins (because the set quota is 35). So, he would have paid 36 coins to earn 35. Consequently, Giorgio would have received 36 coins, paying only 35. Giorgio would earn one coin in total.

Therefore, Giorgio's strategy is to seize the opportunity to bet against events and consistently gain a profit.

Giorgio's contribution to the discussion underscores the depth of his specialized mathematical knowledge, particularly concerning the phenomenological aspects of the game (KoT – Phenomenology and applications – Process of betting). His ability to recognize and exploit the implications of the new rule, coupled with his reliance on "axioms" provided by mathematicians, reflects a heightened understanding in in the awareness of constructing a measure of a degree of reliability (KoT - Definition, properties, and foundations – Classical probability).

With this intervention, Giorgio is not yet introducing an alternative measure to the one proposed by Alba and Paola; instead, he questions and emphasizes the notion of complete control over the game.

Ciro: Using what Giorgio said: so, is the bookmaker at risk? How does the bookmaker respond to this move?

Mirco: So, I think this: given that this is the normal Gaussian, when you play against the event, the curve is opposite. So, in both modes of play, we set the same quota for everyone.

Paola: No, I have to put higher quotas. No, it is not a good idea, because the bettor can always bet in both ways.

Mirco: And so, this situation is impossible to be solved.

In the ongoing discussion, there is a noticeable tension between theoretical knowledge and its application in the phenomenological context. Paola suggests adjusting the quotas in response to the dynamics of the game. However, this proposal is met with scepticism, as Paola herself acknowledges potential flaws.

Mirco, who had previously demonstrated a solid understanding of definitions, procedures, foundations, representations, and characteristics of the result, seems to deviate from this wealth of theoretical knowledge. His assertion that "this situation is impossible to be solved" suggests a disconnection between his “profound” mathematical knowledge and the phenomenological nuances of the betting scenario he is immersed in [Intra-conceptual connection]. This allows us to emphasize that theoretical mathematical understanding may not necessarily translate into a keen awareness of the challenges posed by the dynamic and uncertain nature of probability contexts. In this case, Mirco appears to struggle with integrating his mathematical insights with the complexities of this scenario, underscoring the importance of a more robust intra-conceptual connection between theoretical knowledge and application in the phenomenological realm.

The discussion highlights the limitations of treating a phenomenological problem as a mathematical puzzle to be solved with rules derived from classical probability definitions, frequentist probability, or the law of large numbers. This tension between theory and application suggests that specialized mathematical knowledge in probability strongly needs to be rooted in the intra-conceptual connections of this topic.

Ciro: If you use lower quotas?

Alba: You lose.

Ciro: If you use higher quotas?

Mirco: You lose. So, you have to use equal quotas!

Ciro: Equal to what?

Paola: To 36 (referring to the "sum 2").

Ciro: Yes, quotas equal to the inverse of the probability value.

Giorgio: Yes, because if the quotas are low, I bet against and I win. If the quotas are high, I bet pro and I win. I have to stay on 36.

In this dialogue, all the participant teachers (except Luisa) engage in a discussion centred around the strategic determination of quotas.

Ciro's prompt about using lower quotas (200) triggers a consideration of potential outcomes in terms of wins and losses. This leads Alba to argue (201) that using lower quotas results in a loss. A similar response from Mirco (203) regarding the use of higher quotas concludes that this move also leads to a loss.

The subsequent recommendation to use equal quotas, as proposed by Paola (205), highlights a procedural strategy: choosing quotas that balance risks and benefits in the betting scenario (KoT - Procedures - Why something is done this way).

This strategic decision-making process indicates a nuanced understanding of the interaction between quotas and potential outcomes, reflecting the teachers' experience in navigating the complexities of probability in the explored context.

Ciro's further clarification (206), referencing Paola's intervention (equal to 36), introduces a connection to theoretical elements of probability (KoT - Definition, properties, and foundations – Classical probability). The idea that quotas are equal to the reciprocal of the probability value demonstrates an awareness of the mathematical relationship between quotas and probability, underscoring the teachers' proficiency in applying fundamental principles to decision-making in a betting context.

Giorgio's contribution (207 -208) introduces an element of adaptability and consideration of different betting strategies, demonstrating a sensitivity to the nuances of the gaming scenario (KoT - Phenomenology and application – Process of betting). His statement that staying on 36 for the event of sum 12 is strategically advantageous indicates an understanding of broader gaming dynamics.

In this discussion we could appreciate the collective knowledge development in constructing a criterion for establishing quotas (KoT - Phenomenology and application – Process of betting) in order to ensure that none of the competitors would win with certainty (de Finetti, 1931).

In summary, the discussion illuminates how teachers, through their involvement in quotas-related decisions, draw on their knowledge of definitions, procedural understanding, fundamental comprehension, and awareness of phenomenological knowledge in probability contexts. They develop the ability to navigate these aspects, impacting specialized mathematical knowledge in probability.

# 6. Conclusions

Historically, mathematics has accustomed us to a world of certainties, where every calculation and solution were inherently linked to direct experience. However, when we delve into the realm of probabilities, we face a different paradigm. And this is a problem that we still grapple with today. In classical mathematics, experimenting and solving meant obtaining a concrete answer, an absolute truth. But with probability, what we get is an anticipation, of what might happen. It's as if, in attempting to determine the probability of an event, we load that event with expectations and hopes, only to discover that reality may be different. We forget that we are navigating uncertainty, and this contradictory aspect makes probability a unique, challenging, and fascinating field. The experience associated with probability sharply contrasts with the repetition of other mathematical experiences. We cannot simply apply a formula and expect a consistent answer as in other branches of mathematics. Probability is intrinsically linked to our ability to anticipate uncertainty, making its study a constant balance between rigorous calculations and the awareness that what might happen is not always predictable.

This is a problem we still face today. Mathematics historically dictates that you undergo an experience and find a solution; you perform a calculation on that experience and arrive at the same solution. In probability, no! You make a calculation, expect something to happen, and the result that comes out is something different. Because we forget that we are calculating the probability of something happening, we load that something with expectations, and perhaps that something doesn't happen. The experience associated with probability is something different from the repetition of any other mathematical experience.This is an aspect still open, not thoroughly addressed in this research work.

This study explored the role of elements of Knowledge of Topics (KoT) in teachers' probability, focusing on a education program context aimed at establishing relationships between subjective, classical, and frequentist perspectives, trying to address two research questions:

(i) *Which elements of teacher's Knowledge of Topics (KoT) on probability are possible to trace on a context of a teacher's education program focused on establishing relationships amongst subjective, classic and frequentist perspectives?*

(ii) *To what extent do the theoretical elements of the subjectivist approach to probability contribute to the attribution of meaning to the degree of confidence that a teacher assigns to an event?*

Drawing inspiration from the subjectivist approach to probability proposed by de Finetti in 1931, we crafted a specific Professional Development (PD) program for primary and lower secondary teachers. This program engaged teachers in a betting game context where they were required to quantify the *degree of confidence* associated with each possible event in the game.

Through the analysis of quotas, notes, and transcriptions using the MTSK lens, several aspects emerge that can enrich future teacher education programs.

The analysis of teacher discussions on quotas in gambling highlights how the ability to strategically determine quotas needs a deeper awareness of knowledge related to the foundations of probability. Awareness of theoretical foundations emerges as an essential pillar built throughout the entire discussion on constructing quotas as *degree of confidence*.

In the first discussion's phase, we analyzed the behaviors and decision-making processes of five teachers engaged in a simulated betting game involving the rolling of two six-sided dice. Through an examination of their assigned quotas and accompanying notes across two meetings, we gained insights into their specialized knowledge of probability and its application in decision-making. However, the aspect that enriched us the most as researchers and granted us deeper access to their specialized knowledge was the narrative presentation of their notes and quotas.

What emerged from this part of the analysis allowed us to gain a clear picture of each teacher, summarized as follows:

- Alba's approach to setting quotas displayed a lack of coherence, characterized by inconsistent adjustments and a qualitative rather than quantitative consideration of confidence levels. Her perception of probabilities appeared to prioritize ordinal sorting over strategic quantification, neglecting to establish meaningful connections between different events.
- Paola demonstrated a nuanced understanding of probabilities, evidenced by her non-uniform quota assignments and thoughtful reflections on the asymmetrical distribution of dice sums. Her recognition of the relative likelihood of different outcomes and the impact of betting dynamics on player strategies showcased her depth of knowledge in probability theory.
- Giorgio's quota assignments evolved significantly over the course of the study, reflecting a transition from an initial alternating pattern to a bell-shaped distribution. While his approach initially lacked coherence, his later adjustments suggested a

growing appreciation for the complexities of probability assessment and the influence of betting dynamics.

- Luisa's quota assignments demonstrated a descending trend, reflecting her perception of the difficulty associated with achieving certain dice sums. Despite some inconsistencies, her considerations of the number of combinations and asymmetrical distribution of probabilities indicated a developing understanding of probability theory.
- Mirco exhibited a sophisticated understanding of probability concepts, as evidenced by his deliberate quota assignments, utilization of classical probability calculations, and detailed analyses of dice sum outcomes. His adept use of mathematical representations and rigorous approach to probability assessment highlighted his advanced knowledge in the field.

The elements of teacher's Knowledge of Topics (KoT) on probability that are possible to trace in this context of teacher's education program include:

- Definition, properties and foundation in Probability: both Paola and Mirco demonstrate an understanding of classical probability by assigning quotas based on perceived probabilities of different events, considering the likelihood of each outcome in a classical sense.
- Procedures: Alba, Paola, Giorgio, Luisa, and Mirco all exhibit various procedures for adjusting quotas, whether it's based on combinations of dice rolls, perceived probabilities, or other factors. This reflects their understanding of how to adjust betting quotas strategically.
- Phenomenology and Application: teachers like Alba, Paola, Giorgio, and Luisa engage in qualitative assessments of probabilities, considering factors such as symmetry of quotas' distribution and trends in their assigned quotas. They also demonstrate an understanding of the application of betting strategies such as pro E and cons E bets.
- Registers of Representation: Alba, Paola, Giorgio, and Luisa utilize different registers of representation in their quota assignments, including symmetry of quotas' distribution, descending trends, and ordinal sorting of confidence levels in events. Mirco also uses fractions, percentages, and graphical representations to depict probabilities and trends in quota assignments.

The analysis of teachers' behaviors and decision-making processes provided valuable insights into the complexities of probability assessment and its application in betting scenarios. While some participants demonstrated a nuanced understanding of probability theory and its practical implications, others exhibited inconsistencies and a need for further development in their specialized knowledge. These findings underscore the importance of targeted teacher education programs aimed at enhancing educators' proficiency in probability concepts and their application in real-world contexts. By addressing gaps in understanding and promoting coherent decision-making strategies, such initiatives can empower teachers to effectively support student learning in mathematics.

The analysis of the excerpt of the discussion showed how the refinement process of the *degree of confidence* associated to each possible event by the teachers involved both qualitative and quantitative aspects: qualitative, because teachers gave meaning to their knowledge; quantitative, because they used, especially at the end of the process, their knowledge of the probability definition as a tool to make conscious choices.

The intra-conceptual connection between fundamental concepts, their representation (KoT - Registers of representation), and the phenomenological aspects of the explored context (KoT - Phenomenology and applications) emerges as fundamental knowledge to create awareness of what one knows (as in the case of Mirco) or what one wants to expand (as in the case of Giorgio). Integrating this aspect into education programs can promote a more flexible approach to understanding and teaching probability.

Alba's emphasis on procedural aspects and betting strategies, coupled with Paola's critical evaluation, highlights the importance of understanding underlying probabilities and the consequences of betting decisions. Giorgio's unique perspective introduces a nuanced understanding of game dynamics, emphasizing the adaptability of strategies based on evolving rules and situational awareness. This demonstrates the depth of specialized mathematical knowledge, particularly in relation to phenomenological aspects of the game.

However, the discussion also exposes the limitations of solely relying on theoretical mathematical understanding without considering the complexities of the phenomenological context. Mirco's struggle to integrate his mathematical insights with the dynamic nature of the scenario underscores the necessity of fostering stronger intra-conceptual connections between theoretical knowledge and its application.

Overall, the collective knowledge development observed in the discussion reflects the teachers' proficiency in navigating various aspects of probability, from fundamental principles to strategic decision-making, ultimately contributing to the construction of a criterion for establishing quotas that ensures fairness and prevents certain players from gaining an unfair advantage. This highlights the importance of intra-conceptual connections in strengthening specialized mathematical knowledge and its practical application in real-world scenarios.

It emerges that intra-conceptual connections play a crucial role in guiding teachers' decisions in the context of gambling. In particular, the connection between phenomenology and applications acts as the linchpin that enables a conscious use of theoretical knowledge. This connection facilitates a practical and contextualized application of probability in real-life and problem-solving, emphasizing the importance of an integrated approach to probabilistic knowledge.

The analysis uncovers that de Finetti's subjective approach to probability, emphasizing the psychological rationale behind an individual's beliefs about the likelihood of events rather than solely numerical computations, fundamentally underpins the concept of probability. This perspective transcends mere numerical constraints and fosters a collective construction of knowledge within the learning group, thereby enriching teachers' specialized understanding.

In constructing a coherent (de Finetti, 1931) probability measure, intra-conceptual connections prove pivotal.

The discussion among the teachers reveals a multifaceted approach to understanding and applying specialized mathematical knowledge within the context of betting games. Through the exploration of procedural considerations, strategic decision-making, and the application of foundational concepts in probability, the participants demonstrate the intricacies involved in assessing the degree of confidence and mitigating the risk of competitors winning with certainty.

This study provides an in-depth overview of the dynamics of specialized knowledge in probability among teachers and underscores the importance of designing teacher education programs that integrate procedural, foundational, and phenomenological aspects. The subjective approach emerges as a significant theoretical perspective that contributes to developing a richer awareness of probability among teachers, preparing them to guide students in understanding this complex and dynamic concept.

With this study we are not proposing that the subjectivist definition of probability should be explicitly taught during PDs or students at school, but we are exploring its potentiality as educational innovative paths for teachers and students – to avoid proposing not trivial probability activities. It emerges that a betting game like the one we have used in the data collection context (which follows an idea by de Finetti's) actually generates a dynamic that leads to the coherent construction of a probability measure based on the choices of the players.

This provides a strategy for teachers to design educational paths where a (classical or frequentist) definition of probability is not given a priori, but it is established starting from the *degree of confidence* expressed by learners.

In conclusion, while significant strides have been made in the mathematical systematization of the subjectivist approach to probability, particularly in demonstrating its consistency with other probabilistic frameworks, there remains a critical gap in translating this theoretical understanding into effective pedagogical practices.

The dynamic and uncertain nature of probability contexts presents unique challenges that necessitate a nuanced approach to subjective probability.

Moving forward, future research in the field of probability education must address how to bridge this divide between theoretical understanding and practical application.

Key open questions include identifying effective educational strategies for fostering coherence in probabilistic reasoning among students, as well as exploring teachers' interpretations of student work to inform instructional practices: a refined and specific categorization of teachers' specialized knowledge in the field of probability education can further illuminate key open questions.

One avenue of inquiry involves identifying how teachers' specialized knowledge interact to shape instructional practices.

A more refined and specific categorization of teachers' specialized knowledge in this topic is crucial for addressing key open questions in probability education. One aspect to consider is how teachers' content knowledge in probability theory intersects with their pedagogical content knowledge (PCK), which encompasses their understanding of how to teach probabilistic concepts effectively.

Lastly, considering teachers' knowledge of instructional technologies and how they can be leveraged to enhance probabilistic reasoning instruction is also pertinent in exploring effective educational strategies. By delving into these nuanced dimensions of teachers' specialized knowledge, researchers can better understand how to support educators in fostering coherence in probabilistic reasoning among students and refining instructional practices to meet the diverse needs of learners.

Additionally, there is a need for further investigation into how to scaffold learning experiences that align with the complexities of real-world probabilistic scenarios.

Only by addressing these challenges can we fully realize the potential of probability education to empower students with the skills and understanding necessary to navigate uncertain environments effectively.

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