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Invariant subspaces of the polydisk

Subespaços invariantes do polidisco

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Dissertação apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Mestre em Matemática.

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"Understanding a question is half an answer"

(Socrates)

Resumo

O Problema do subespaço invariante é um dos problemas em aberto mais famosos em teoria dos operadores, ele pergunta se dado qualquer operador linear limitado em um espaço de Hilbert complexo, separável e de dimensão infinita, tal operador admite um subspaço invariante fechado não-trivial.

No presente trabalho, iremos introduzir a teoria básica do espaço de Hardy-Hilbert do polidisco, $H^2(\mathbb{D}^n)$, e seus operadores de Toeplitz e com isso exploraremos as conexões entre universalidade e operadores de Toeplitz T_ϕ sobre os espaços de Hardy-Hilbert do disco e do polidisco e relações com o problema do subespaço invariante, isto é, uma abordagem ao problema pelo estudo subespaços invariantes com relação a um operador de Toeplitz universal.

Palavras-chave: Operadores universais. Operadores de Toeplitz. Espaços de Hardy. Espaços de Hilbert. Subespaços invariantes.

Abstract

The invariant subspace problem (ISP) is one of the most famous open problems in operator theory, it asks if given any bounded linear operator on a infinite dimensional, complex and separable Hilbert space, does such operator admit a non-trivial closed invariant subspace?

In this present work, we will introduce the basic theory of the Hardy-Hilbert space over the polydisk, $H^2(\mathbb{D}^n)$, and its Toeplitz operators and we will explore the connections between universality and Toeplitz operators T_ϕ over the Hardy-Hilbert spaces of the disk and the polydisk and its relations with the invariant subspace problem, in other words, a approach to the problem by studying the invariant subspaces of a universal Toeplitz operators.

Keywords: Universal operators. Toeplitz operators. Hardy spaces. Hilbert spaces. Invariant subspaces.

List of symbols

$\mathbb{N}$	The set of natural numbers
$\mathbb{N}_0$	The set of positive integers
$\mathbb{C}$	The set of complex numbers
$\mathbb{C}[z_1, \ldots, z_n]$	Ring of polynomials in n variables with coefficients in $\mathbb{C}$
$\mathrm{Re}(z)$	Real part of the complex number z
$\mathrm{Im}(z)$	Imaginary part of the complex number z
$\mathbb{D}$	The unit disk $\{z \in \mathbb{C} : \lvert z\rvert < 1\}$
$\mathbb{D}^n$	Cartesian product of n copies of $\mathbb{D}$
$\mathbb{T}$	The unit circle $\{z \in \mathbb{C} : \lvert z\rvert = 1\}$
$\mathbb{T}^n$	Cartesian product of n copies of $\mathbb{T}$
$\mathcal{H}$	Infinite dimensional complex separable Hilbert space
$B(X)$	Bounded linear operators on a vector space X
$\lVert \cdot \rVert_p$	p-norm with $1 \leqslant p < \infty$
T^*	Adjoint of an operator T
I	Identity operator
ℓ^2	Hilbert space of square-summable sequences of complex numbers indexed by $\mathbb{N}$
$V^\perp$	The orthogonal complement of a subspace V
$H^2(\mathbb{D})$	Hardy-Hilbert space over the disk
$H^2(\mathbb{D}^n)$	Hardy-Hilbert space over the polydisk
$H^\infty(\mathbb{D})$	Hardy space of bounded holomorphic functions over $\mathbb{D}$
$H^\infty(\mathbb{D}^n)$	Hardy space of bounded holomorphic functions over $\mathbb{D}^n$
$L^2(\mathbb{T})$	Space of complex-valued square-integrable functions over $\mathbb{T}$
$L^2(\mathbb{T}^n)$	Space of complex-valued square-integrable functions over $\mathbb{T}^n$

$\tilde{H}^2$	Subspace of $L^2(\mathbb{T})$ whose coefficients are non null if are indexed by positive integers
$\tilde{H}^2(\mathbb{D}^n)$	Subspace of $L^2(\mathbb{T}^n)$ whose coefficients are non null if it multi-indexed by positive integers
$H^2(\mathcal{H})$	$\mathcal{H}$-valued Hardy-Hilbert space
$L^\infty(\mathbb{T}^n)$	Space of essentially bounded measurable functions over $\mathbb{T}^n$
M_z	Operator multiplication by z
T_ϕ	Toeplitz operator over $H^2(\mathbb{D}^n)$ with symbol ϕ
$Id_{\mathcal{H}}$	Identity operator on $\mathcal{H}$
$\ker(T)$	Kernel of a linear operator T
$\mathrm{Ran}(T)$	Range of a linear operator T
$\sigma(T)$	Spectrum of a linear operator T
$[\phi]$	Smallest invariant subspace containing ϕ
k_{z_0}	Reproducing kernel in $H^2(\mathbb{D})$ at $z_0 \in \mathbb{D}$
K_α	Reproducing kernel in $H^2(\mathbb{D}^n)$ at $\alpha \in \mathbb{D}^n$
$P_r(\theta)$	Poisson kernel on $\mathbb{D}$
$P(z, \zeta)$	Poisson kernel on $\mathbb{D}^n$
$P[d\rho](z)$	Poisson integral with respect to a measure ρ

Contents

Introduction

The Invariant Subspace Problem (ISP) is currently one of the most famous open problem from operator theory. It was stated in the mid 1900's after the works of (1) and a unpublished work by John von Neumann. So far, several different approaches to this problem have been studied. The recent monograph (2) introduces many modern approaches to the ISP. The problem may be formulated as follows: given a complex Banach space E with dimension greater than 1 and any bounded linear operator $T : E \to E$, does T admit a non-trivial closed invariant subspace?

Solutions for some cases have already been obtained. If E is finite dimensional, the problem may be solved by using Jordan Canonical form. If E is an infinite dimensional non-reflexive Banach Space, a counter-example was given by (3, 4, 5, 6). If E is an infinite dimensional non-separable Banach space, the closure of the orbit of any non-zero element in E works as a non-trivial closed subspace.

Several other results have been proven concerning Banach spaces, for instance it was proved by (7), that every compact operator in a Banach space with dimension greater or equal to 2 has a non-trivial closed invariant subspace, (8) proved that for polynomially compact operators, we also have a positive answer to the problem. Later, (9) proved that if an operator T commutes with a non-zero compact operator, then T has a non-trivial closed subspace.

The currently most important open forms of the problem involve complex reflexive Banach spaces and complex separable Hilbert spaces. Our focus in this work is on the separable Hilbert space case this gives us a good advantage in the form of Riesz-Fischer Theorem, which states that every separable Hilbert space is isometrically isomorphic to ℓ^2.

Of course, many techniques used are inspired by the one-dimensional case, where we have a complete characterization of the invariant subspaces of the shift on ℓ^2 due to (1). He proved that every invariant subspace of the unilateral shift of $H^2(\mathbb{D})$ are given by $\phi H^2(\mathbb{D})$, where ϕ is an bounded holomorphic function on the unit disk with $|\phi(e^{i\theta})| = 1$ almost everywhere on $\mathbb{T}$ and $H^2(\mathbb{D})$ is the Hardy-Hilbert space over the disk.

This work focus on the Hardy-Hilbert space over the polydisk, $H^2(\mathbb{D}^n)$. One of the advantages of using this space, is the use of Rota's universal operators. An operator U is said to be universal if given a non-zero operator $T \in B(\mathcal{H})$, there exists a non-trivial closed subspace M of $\mathcal{H}$ such that the restriction of U to M is similar to T in M. Verifying that a operator is universal is not a trivial task but the Caradus Criteria give us sufficient conditions to prove that a operator is universal and so we can present some examples, one of those are the adjoints of the shifts on $H^2(\mathbb{D}^n)$ which is convenient since we are studying shift invariant subspaces.

This work will be organized as follows:

In the first chapter, we will present a short introduction for function theory on $H^2(\mathbb{D})$ since we are going to make some similar arguments and comparisons with $H^2(\mathbb{D}^n)$ and also a short introduction to several complex variables, later we will present some general results about function theory in $H^2(\mathbb{D}^n)$, a short introduction to Toeplitz operators over $H^2(\mathbb{D}^n)$ and define universality on Hilbert spaces and its connections to the ISP.

In the second chapter, we study shifts in $H^2(\mathbb{D})$ and Beurling's theorem. Also, via vector-valued Hardy spaces, we present a connection to the polydisk case and some examples of invariant subspaces.

In the third chapter, we will study some recent advances in ISP via Toeplitz operators, in the disk and polydisk case and we will see why we prefer the polydisk case and the current struggles with this approach.

In the appendix, we will state some general results from functional analysis that will be use through the text.

This work aims to be a concise introduction to the study of ISP via universality of Toeplitz operators on $H^2(\mathbb{D}^n)$ and we assume that the reader has knowledge in functional analysis, complex analysis in one variable, rudiments of Fourier analysis and measure theory.

1 Preliminaries

In this chapter, we will introduce the basic concepts for the text, first a short introduction to the Hardy-Hilbert space over the disk and several complex variables, then we will discuss the Hardy-Hilbert space over the polydisk, results about Toeplitz operators over the polydisk and universality. This chapter is based on (10, 11, 12, 13, 14, 15).

1.1 The Hardy space over the disk

In this section, we will briefly introduce the Hardy-Hilbert space of the disk as a starting point for this work. This space was used to give a complete characterization for the shift invariant subspaces on ℓ^2 space by (1) (this characterization will be stated and proved in chapter 2) and will provide some intuition when we start to work with Hardy-Hilbert space of the polydisk since many properties are naturally generalized. First, we will introduce the most natural example of a Hilbert space, the space of square-summable complex sequences ℓ^2.

Definition 1.1.1. *We define ℓ^2 as follows*

$$\ell^2 = \left\{ (a_n)_{n\in\mathbb{N}_0} \in \mathbb{C} : \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}.$$

Of course, in this space, we have a natural definition of inner product as

$$\langle (a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0} \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}$$

and, naturally, a definition of norm

$$\|(a_n)_{n\in\mathbb{N}_0}\| = \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2}.$$

Now we present the Hardy-Hilbert space, which is isometrically isomorphic to ℓ^2, but it has a richer theory behind than ℓ^2. We define the disk as the set $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ as the boundary of $\mathbb{D}$.

Definition 1.1.2. *We define the Hardy-Hilbert space of the disk as holomorphic functions on the disk such that the coefficients in the power series representation are square-summable, i.e.,*

$$H^2(\mathbb{D}) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}$$

where we define the inner product for $f = f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g = g(z) = \sum_{n=0}^{\infty} b_n z^n$ as

$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}$$

It is clear now that the mapping

$$T : \ell^2 \to H^2(\mathbb{D}), \quad (a_n)_{n\in\mathbb{N}_0} \mapsto \sum_{n=0}^{\infty} a_n z^n$$

is a isometric isomorphism between ℓ^2 and $H^2(\mathbb{D})$ and therefore the Hardy-Hilbert space is separable. One might ask if every holomorphic function on the disk belongs in $H^2(\mathbb{D})$, but a clear example is the function

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

which is trivially a holomorphic function on $\mathbb{D}$, but the coefficients are not square-summable. When we said that this space has a richer theory, we mean in the sense that we may use the usual function theory of complex analysis and, of course, a class of vectors that are very useful in the development of theory that are not present in the ℓ^2 space, the reproducing kernels.

Definition 1.1.3. *Let $z_0 \in \mathbb{D}$, we define the function k_{z_0} as*

$$k_{z_0}(z) = \sum_{n=0}^{\infty} \overline{z_0}^n z^n = \frac{1}{1 - \overline{z_0} z}$$

We say that k_{z_0} is the reproducing kernel at z_0. By the power series representation is clear that $k_{z_0} \in H^2(\mathbb{D})$, moreover we have the following

Theorem 1.1.1. *(10, Theorem 1.1.8) For $z_0 \in \mathbb{D}$ and $f \in H^2(\mathbb{D})$, we have $\langle f, k_{z_0} \rangle = f(z_0)$ and $\|k_{z_0}\| = \dfrac{1}{\sqrt{1 - |z_0|^2}}$.*

Now, we want to give another possible characterization for the Hardy-Hilbert space of the disk, that is, by the use of some simple Fourier analysis.

Definition 1.1.4. *We define the Lebesgue space $L^2(\mathbb{T})$ as the space of square-integrable complex functions over $\mathbb{T}$ with respect to the normalized Lebesgue measure μ, i.e,*

$$L^2(\mathbb{T}) = \left\{ f : \int_{\mathbb{T}} |f|^2 d\mu < \infty \right\}.$$

In particular we know that the inner product defined in $L^2(\mathbb{T})$ is given for any $f, g \in L^2(\mathbb{T})$ as

$$\langle f, g \rangle = \int_{\mathbb{T}} f \overline{g} d\mu$$

and naturally, we define the norm as

$$\|f\| = \left(\int_{\mathbb{T}} |f|^2 d\mu\right)^{1/2}.$$

Let $n \in \mathbb{Z}$, define the function e_n in $\mathbb{T}$ such that $e_n(e^{i\theta}) = e^{in\theta}$ we can see that $\{e_n : n \in \mathbb{Z}\}$ forms an orthonormal basis for $L^2(\mathbb{T})$ since by the Fourier transform we have that for a given $f \in L^2(\mathbb{T})$ we write the Fourier coefficients of f as

$$\hat{f}(n) = \frac{1}{2\pi} \int_{\mathbb{T}} f(e^{i\theta}) e^{-in\theta} d\theta$$

and, of course we may write the function f as

$$f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$$

where $a_n = \hat{f}(n)$. Now define the space $\tilde{H}^2$ as

$$\tilde{H}^2 = \left\{f^* \in L^2(\mathbb{T}) : \langle f^*, e_n \rangle = 0 \text{ for } n < 0\right\}.$$

In other words, we say that this space is written as

$$\tilde{H}^2 = \left\{ f^* \in L^2(\mathbb{T}) : f^*(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{in\theta} \quad \text{and} \quad \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}.$$

From this previous definition, the reader must already think that we wish to say that $H^2(\mathbb{D})$ is isometrically isomorphic to $\tilde{H}^2$. In fact that is true, but since this is a short introduction, for a complete proof we refer (10, Section 1.1), however in the polydisk case we will see a similar characterization which we will be proved. A way to relate these two spaces is via radial functions defined as

$$f_r(\omega) = f(r\omega) \quad \text{for every } \omega \in \mathbb{T}$$

the idea is that for every function $f \in H^2(\mathbb{D})$ we can associate a function $f^* \in L^2(\mathbb{T})$ with the following relation

$$f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$$

for almost every θ, so this presents a connection with the boundary values of a function f that, from our original definition, we do not have enough information, in fact we may present examples of functions that belong in $H^2(\mathbb{D})$ that are not holomorphic in a determined point of $\mathbb{T}$, for instance, if we define for a fixed θ_0

$$h(z) = \sum_{n=0}^{\infty} \frac{e^{-in\theta_0}}{n} z^n.$$

We know that $h \in H^2(\mathbb{D})$ but when we approach the point $e^{i\theta_0}$ from inside the disk we see that $|h(z)|$ tends to infinity and therefore it can not be holomorphic at that point. Moreover we can produce an example of function that belongs in $H^2(\mathbb{D})$ but it is not holomorphic at any point of the unit circle and to this purpose we need to give more background. From our previous characterization for $H^2(\mathbb{D})$ being seen as a subspace of $L^2(\mathbb{T})$ we get the following:

Theorem 1.1.2. *(10, Theorem 1.1.12) Let $f \in Hol(\mathbb{D})$ then $f \in H^2(\mathbb{D})$ if and only if*

$$\sup_{0<r<1} \int_{\mathbb{T}} |f(re^{i\theta})|^2 d\mu < \infty$$

Also, we need to define another Hardy space as follows:

Definition 1.1.5. *We define the Hardy space of all bounded holomorphic functions on $\mathbb{D}$ as $H^\infty(\mathbb{D})$ and we give the norm of uniform convergence to this space, i.e.,*

$$\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|.$$

From Theorem 1.1.2, we get that $H^\infty(\mathbb{D})$ is a subspace of $H^2(\mathbb{D})$ by the fact that for any function on $H^\infty(\mathbb{D})$, the supremum relation is satisfied. In particular, this has some connections to the Lebesgue space of essentially bounded measurable functions.

Definition 1.1.6. *We say that a function f over $\mathbb{T}$ is essentially bounded function if there exists some $K > 0$ such that*

$$\mu(\{e^{i\theta} : |f(e^{i\theta})| > K\}) = 0$$

and we define $L^\infty(\mathbb{T})$ the Lebesgue space of all essentially bounded measurable functions with the essential norm

$$\|f\|_\infty = \inf\{K : \mu(\{e^{i\theta} : |f(e^{i\theta})| > K\}) = 0\}.$$

From basic measure theory and functional analysis we know that $L^\infty(\mathbb{T})$ is a Banach space. It is not hard to see that $H^\infty(\mathbb{D})$ with uniform convergence norm is also a Banach space and as in the Hardy-Hilbert space of the disk we may also give a connection between $H^\infty(\mathbb{D})$ and $L^\infty(\mathbb{T})$.

Theorem 1.1.3. *(10, Corollary 1.1.29) If $f \in H^\infty(\mathbb{D})$, then $f^* \in L^\infty(\mathbb{T})$.*

Now, we are ready to define a class of functions called Blaschke products.

Definition 1.1.7. *Let $(z_n)_{n \in \mathbb{N}_0}$ be a sequence of non zero complex numbers in $\mathbb{D}$ and assume that $\sum_{n=0}^{\infty}(1 - |z_n|) < \infty$. Let $s \in \mathbb{N}_0$, then the Blaschke product with zeros $(z_n)_{n \in \mathbb{N}_0}$ and multiplicity s zero at $z = 0$ is defined by*

$$B(z) = z^s \prod_{n=0}^{\infty} \frac{\overline{z_n}}{|z_n|} \frac{z_n - z}{1 - \overline{z_n} z}$$

It is easy to see that Blaschke products belong in $H^\infty(\mathbb{D})$. Remember that functions of the type

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}$$

with $\alpha \in \mathbb{D}$ are involutive automorphisms of the disk, that is

- $(\psi_\alpha \circ \psi_\alpha)(z) = z$ for all $z \in \mathbb{D}$.
- ψ_α is a conformal map from the disk to itself.

From the latter property, we get that $|\psi_\alpha(z)|$ have 1 as an upper bound in $\mathbb{D}$, thus bounded. Elements of the form $\dfrac{\overline{w}}{w}$ with $w \in \mathbb{C}$ is always bounded by 1 and of course, the function z^s with $s \in \mathbb{N}_0$ is always bounded by 1 in $\mathbb{D}$ by the Maximum module principle, so the Blaschke products are always the product of bounded elements it is a bounded holomorphic function. Since Blaschke products belong in $H^\infty(\mathbb{D})$, they belong in $H^2(\mathbb{D})$ and they can be used to construct the example of a function that is not holomorphic at any point of the unit circle but belongs in $H^2(\mathbb{D})$.

Remark 1.1.1. *Take $(a_n)_{n \in \mathbb{N}_0}$ a dense sequence in $\mathbb{T}$ and for each $n \in \mathbb{N}_0$ we set*

$$z_n = \left(1 - \frac{1}{n^2}\right) a_n.$$

Is easy to see that $|z_n| < 1$ and that $1 - |z_n| \leqslant \frac{1}{n^2}$ for each $n \in \mathbb{N}_0$, then $(z_n)_{n \in \mathbb{N}_0}$ are the zeros of a Blaschke product B. By construction of B, every $\omega \in \mathbb{T}$ is a limit point of $(z_n)_{n \in \mathbb{N}_0}$, and then if B is holomorphic in some point in $\mathbb{T}$ we obtain that for a neighborhood of this point, B is necessarily null which is a contradiction.

So, in the Hardy-Hilbert space, the boundary behaviour is important, but it is not necessary that we extend the domain of holomorphy to the boundary; in fact, we just need a "good" behaviour for almost every point. Now, we can introduce some special classes of functions that play a big role when we discuss the Hardy-Hilbert space of the disk, the inner and outer functions.

Definition 1.1.8. *A function $\phi \in H^\infty(\mathbb{D})$ that satisfies $|\phi^*(e^{i\theta})| = 1$ almost everywhere in $\mathbb{T}$ is said to be a inner function.*

A simple example of inner functions are the Blaschke products, and we will show that the inner functions play a role when we start discussing invariant subspaces, in particular Beurling's theorem proves that every shift invariant subspace of $H^2(\mathbb{D})$ is of the form $\phi H^2(\mathbb{D})$ where ϕ is an inner function. Now, we define outer functions.

Definition 1.1.9. *A function $F \in H^2(\mathbb{D})$ is an outer function if F is a cyclic vector for the operator multiplication by z on $H^2(\mathbb{D})$, i.e, the closed linear span of the elements $(z^n F)_{n\in\mathbb{N}_0}$ is equal to $H^2(\mathbb{D})$.*

It is easy to see that the constant function 1 is an example of an outer function. From this definition, we have some properties for outer functions, for instance:

Theorem 1.1.4. *(10, Theorem 2.3.2) If F is an outer function, then F has no zeros in $\mathbb{D}$.*

Also, we may give a full characterization for this class of function as:

Theorem 1.1.5. *(10, Theorem 2.7.10) The function $F \in H^2(\mathbb{D})$ is outer if and only if*

$$\log|F(0)| = \int_{\mathbb{T}} \log|F^*|d\mu$$

So we can give a canonical factorization in $H^2(\mathbb{D})$, the inner-outer factorization.

Theorem 1.1.6. *(10, Theorem 2.3.4) Let $f \in H^2(\mathbb{D})$ be a non-null function, then $f = \phi F$, where ϕ is an inner function and F is an outer function. This factorization is unique up to constant factors.*

1.2 Several complex variables

This section will be dedicated to a short introduction to several complex variables where we will show some primary results and some differences from the one-variable function theory. First of all, we will present some background and notations.

Similar to the one variable theory, we define the elements of $\mathbb{C}^n$ as the sum of the real part and imaginary part, but in this case, we have a n-tuple:

$$(z_1, \dots, z_n) = (x_1, \dots, x_n) + i(y_1, \dots, y_n)$$

where $z_j = x_j + iy_j$ for all $j = 1, \dots, n$.

We also need to define some particular sets; here in several variables, we see the idea of polydisks and balls:

Definition 1.2.1. *First take $\mathbb{D}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$ and $\overline{\mathbb{D}_r(z_0)} = \{z \in \mathbb{C} : |z - z_0| \leqslant r\}$, so we define $\mathbb{D}_{\boldsymbol{R}}(\omega) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_i \in \mathbb{D}_{r_i}(\omega_i) \quad \text{where} \quad \boldsymbol{R} = (r_1, \dots, r_n)\}$ for $\omega = (\omega_1, \dots, \omega_n)$.*

Usually, if $r_1 = \dots = r_n = 1$ and $\omega = (0, \dots, 0)$, we denote the polydisk as $\mathbb{D}^n$ and the ball in $\mathbb{C}^n$ has the same definition as in usual normed vector space, i.e.,

Definition 1.2.2. *We define* $B_r(\omega) = \{z \in \mathbb{C}^n : \|\omega - z\| < r\}$, *where* $\|.\|$ *is the usual euclidean norm in* $\mathbb{C}^n$, *the ball with radius* r *centered at* ω.

So, in the one variable case, the disk and the ball coincide, but note that the polydisk and ball are not equivalent. Now we are ready to define the main object of study in this section, the holomorphic functions, which have three equivalent definitions, for those take $\Omega \subseteq \mathbb{C}^n$:

Definition 1.2.3. *We say that a function* $f : \Omega \to \mathbb{C}$ *is said to be holomorphic if for every* $j = 1, \ldots, n$ *and each fixed* $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n$ *the function*

$$\omega \mapsto f(z_1, \ldots, z_{j-1}, \omega, z_{j+1}, \ldots, z_n)$$

is holomorphic in the sense of one variable holomorphic function.

Here, one might think about the Cauchy-Riemann equations, which have a generalized version for several complex variables. We define the differential operators

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \quad , \quad \frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

for $j = 1, \ldots, n$ and note that the Cauchy-Riemann equations over the variable z_j is equivalent to say that

$$\frac{\partial f}{\partial \overline{z_j}} = 0 \quad \text{on the set } D$$

where $D = \{\omega \in \mathbb{C} : (z_1, \ldots, z_{j-1}, \omega, z_{j+1}, \ldots, z_n) \in \Omega\}$. So this definition says that if the Cauchy-Riemann equations are valid for each variable separately, then the function is holomorphic.

Definition 1.2.4. *A function* $f : \Omega \to \mathbb{C}$ *is holomorphic for each* $\omega \in \Omega$ *if there exist an* $\boldsymbol{R} = (r_1, \ldots, r_n)$ *with each* $r_j > 0$ *such that* $\overline{\mathbb{D}_{\boldsymbol{R}}(\omega)} \subseteq \Omega$ *and* f *can be written as absolutely and uniformly convergent power series*

$$f(z) = \sum_{i_1, \ldots, i_n = 0}^{\infty} a_{i_1, \ldots, i_n}(z_1 - \omega_1)^{i_1} \ldots (z_n - \omega_n)^{i_n}$$

for all $z \in \mathbb{D}_{\boldsymbol{r}}(\omega)$.

Definition 1.2.5. *Let* $f : \Omega \to \mathbb{C}$ *be a continuous function in each variable separately and locally bounded. The function* f *is said to be holomorphic if for each* $\omega \in \Omega$ *there exist an* $\boldsymbol{R} = (r_1, \ldots, r_n)$ *with* $r_j > 0$ *for* $j = 1, \ldots, n$ *such that* $\overline{\mathbb{D}_{\boldsymbol{R}}}(\omega) \subseteq \Omega$ *and*

$$f(z) = \left(\frac{1}{2\pi i}\right)^n \int_{|\zeta_n - \omega_n| = r_n} \cdots \int_{|\zeta_1 - \omega_1| = r_1} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1) \ldots (\zeta_n - z_n)} d\zeta_1 \ldots d\zeta_n$$

for all $z \in \mathbb{D}_{\boldsymbol{R}}(\omega)$.

Now we aim to prove the equivalences between these holomorphicity definitions, so we may choose a more appropriate definition whenever needed. Here, we take Definition 1.2.3 as a primary definition of a holomorphic function, and in order to progress, we need to state one of the most famous theorems in complex analysis, the Cauchy's integral formula, in this case for polydisks.

Theorem 1.2.1. *Let $\omega \in \mathbb{C}^n$ and $r_1, \ldots, r_n > 0$. Suppose that f is a continuous function on $\overline{\mathbb{D}_{r_1}}(\omega_1) \times \cdots \times \overline{\mathbb{D}_{r_n}}(\omega_n)$ and f is a holomorphic in $\mathbb{D}_{r_1}(\omega_1) \times \cdots \times \mathbb{D}_{r_n}(\omega_n)$, then*

$$f(z) = \left(\frac{1}{2\pi i}\right)^n \int_{|\zeta_n - \omega_n| = r_n} \cdots \int_{|\zeta_1 - \omega_1| = r_1} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - \omega_1) \ldots (\zeta_n - \omega_n)} d\zeta_1 \ldots d\zeta_n$$

for all $z \in \mathbb{D}_{\boldsymbol{R}}(\omega)$.

Proof. In these conditions, since f is holomorphic, then f is holomorphic in each separate variable, so we may apply Cauchy's integral formula for one variable in the variable z_n, so we may write f as

$$f(z_1, \ldots, z_n) = f(z) = \frac{1}{2\pi i} \int_{|\zeta_n - \omega_n| = r_n} \frac{f(z_1, \ldots, z_{n-1}, \zeta_n)}{(\zeta_n - \omega_n)} d\zeta_n$$

Moreover, we may apply the Cauchy integral formula in the variable z_{n-1} and, therefore

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta_n - \omega_n| = r_n} \int_{|\zeta_{n-1} - \omega_{n-1}| = r_{n-1}} \frac{f(z_1, \ldots, \zeta_{n-1}, \zeta_n)}{(\zeta_n - \omega_n)} d\zeta_{n-1} d\zeta_n.$$

Applying for every variable, we get the desired result. □

So Theorem 1.2.1 implies Definition 1.2.5 of a holomorphic function, now we will use the theorem to prove the equivalence between Definition 1.2.4 and Definition 1.2.5.

Corollary 1.2.1.1. *If f is holomorphic in $\Omega \subseteq \mathbb{C}^n$, then f is of class C^∞ on Ω.*

Proof. Take a closed polydisk contained in Ω. Remember that in the one variable case, we had the following consequence for Cauchy's integral formula.

$$g^{(n)}(z) = \frac{n!}{2\pi i} \int_{|z - \omega| = r} \frac{g(\omega)}{(z - \omega)^{n+1}} d\omega$$

for a function g holomorphic in $\mathbb{D}_r(\omega)$ and continuous on the closure of this disk, meaning that we may differentiate under the integral sign for one variable. Now, to take any partial derivative, we take the Cauchy integral formula representation for a function f on Ω, and with this observation and the use of Fubini's theorem, we get that every partial derivative exists and it is continuous, and in particular every derivative is again differentiable on Ω. □

Corollary 1.2.1.2. *If f is a holomorphic function on $\Omega \subseteq \mathbb{C}^n$, then f has a convergent power series representation about each element $\omega \in \Omega$.*

Proof. Take a closed polysdisk $\overline{\mathbb{D}_{\mathbf{R}}}(\omega)$, $\mathbf{R} = (r_1, \ldots, r_n)$, that is contained in Ω and apply the Cauchy's integral formula for f in this polydisk. Fix a $z = (z_1, \ldots, z_n) \in \mathbb{D}_{\mathbf{R}}(\omega)$ so by the Cauchy integral formula we get

$$f(z) = \left(\frac{1}{2\pi i}\right)^n \int_{|\zeta_n - \omega_n| = r_n} \cdots \int_{|\zeta_1 - \omega_1| = r_1} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1)\ldots(\zeta_n - z_n)} d\zeta_1 \ldots d\zeta_n \tag{1.1}$$

Our idea now is, as in the one variable case, to rewrite the terms to get to the power series representation, so we write.

$$\frac{1}{(\zeta_1 - z_1)\ldots(\zeta_n - z_n)} = \frac{1}{(\zeta_1 - \omega_1)\ldots(\zeta_n - \omega_n)} \frac{1}{(1 - \frac{z_1 - \omega_1}{\zeta_1 - \omega_1})\ldots(1 - \frac{z_n - \omega_n}{\zeta_n - \omega_n})}.$$

Note that for each ζ_j and z_j for $j = 1, \ldots, n$ there exist $0 < \ell < 1$ such that

$$\left|\frac{z_j - \omega_j}{\zeta_j - \omega_j}\right| < \ell$$

and in particular, we also know that

$$\frac{1}{1 - \frac{z_j - \omega_j}{\zeta_j - \omega_j}} = \sum_{i_j = 0}^{\infty} \left(\frac{z_j - \omega_j}{\zeta_j - \omega_j}\right)^{i_j}$$

is an absolute convergent power series, hence in general, we may write that

$$\frac{1}{(\zeta_1 - z_1)\ldots(\zeta_n - z_n)} = \sum_{i_1, \ldots, i_n = 0}^{\infty} \frac{(z_1 - \omega_1)^{i_1} \ldots (z_n - \omega_n)^{i_n}}{(\zeta_1 - \omega_1)^{i+1} \ldots (\zeta_n - \omega_n)^{i_n}} \tag{1.2}$$

simply because we can interchange the summation signs of the product of these power series since they are all absolutely convergent by hypothesis and therefore, the resulting power series still is an absolute convergent series, by substituting (1.2) in (1.1) and the fact we may interchange integration and summation signs since we have an absolute convergent series we get that f can be written as

$$f(z) = \sum_{i_1, \ldots, i_n = 0}^{\infty} b_{i_1, \ldots, i_n} (z_1 - \omega_1)^{i_1} \ldots (z_n - \omega_n)^{i_n}$$

where

$$b_{i_1, \ldots, i_n} = \left(\frac{1}{(2\pi i)^n} \int_{|\zeta_n - \omega_n| = r_n} \cdots \int_{|\zeta_1 - \omega_1| = r_1} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - \omega_1)^{i_1 + 1} \ldots (\zeta_n - \omega_n)^{i_n + 1}} d\zeta_1 \ldots d\zeta_n\right)$$

□

Sometimes Corollary 1.2.1.2 is called Osgood's Lemma in the literature, and now we proved that Definition 1.2.5 implies Definition 1.2.4, but note that Definition 1.2.4 already implies Definition 1.2.3 so we proved that those three definitions are equivalent as they are in the one variable case. We will see next some results in several complex variables that have an analog in one variable.

Theorem 1.2.2. *If f and g are holomorphic functions in a connected open subset $\Omega \subseteq \mathbb{C}^n$ and if $f(z) = g(z)$ for all points in an open neighborhood $U \subseteq \Omega$, then $f(z) = g(z)$ for all $z \in \Omega$.*

Proof. Let $E \subseteq \Omega$ such that E is the interior of the set $\{z \in \Omega : f(z) = g(z)\}$, thus E is an open set, and it is not empty since by hypothesis $U \subseteq E$. Since Ω is connected by hypothesis, it suffices to show that E is closed in the induced topology of Ω as a topological subspace of $\mathbb{C}^n$ because then E will be an open and closed non-empty subset of Ω and therefore E must be equal to Ω. If $\omega \in \Omega \cap \overline{E}$, where $\overline{E}$ denotes the closure of E in the topology of Ω, take $r > 0$ sufficiently small such that $\mathbb{D}_{\mathbf{R}}(\omega) \subseteq \Omega$ with $\mathbf{R} = (r, \ldots, r)$ and now fix $\omega_0 \in \mathbb{D}_{\mathbf{r}/2}(\omega) \cap E$ and note that such ω_0 exists since $\omega \in \overline{E}$, therefore $\omega \in \mathbb{D}_{\mathbf{r}/2}(\omega_0)$. Now, the function $f - g$ has a power series representation centered at ω_0 converging for every point in $\mathbb{D}_{\mathbf{R}/2}(\omega_0)$ but remember that $f - g$ is null in this polydisk. Therefore the coefficients of the power series representation must be all null and hence $f - g$ is null in $\mathbb{D}_{\mathbf{R}/2}(\omega_0)$ and thus $\omega \in \mathbb{D}_{\mathbf{R}/2}(\omega_0) \subset E$, so E contains its accumulation points therefore $E = \overline{E}$ and this concludes the proof. □

The following theorem is the analog of the Maximum module principle for several variables.

Theorem 1.2.3. *If f is holomorphic in a connected open subset $\Omega \subseteq \mathbb{C}^n$ and if there exists $\omega \in \Omega$ such that $|f(z)| \leqslant |f(\omega)|$ for all z in a open neighbourhood of ω, then $f(z) = f(\omega)$ for every $z \in \Omega$.*

Proof. The proof of this theorem will not be done in this work since it uses some concepts of differential forms that were not introduced and will not be used. For a complete proof, see (14, Theorem 4, Page 6). □

Definition 1.2.6. *An open set $U \subseteq \mathbb{C}^n$ is called a domain of holomorphy if does not exist non-empty open subsets U_1, U_2, with U_2 connected, $U_2 \nsubseteq U$ and $U_1 \subseteq U \cap U_2$ such that for every holomorphic function h on U there exists a holomorphic function h_2 in U_2 such that $h = h_2$ on U_1.*

Note that every in one variable, every open set is a domain of holomorphy or simply a domain. Now it is time to introduce some of the differences that we talked about regarding adding new variables for holomorphic functions. In one variable we always knew that zeros were always isolated, and that is not the case over several variables all due to what is known in literature as Hartog's phenomenon, which states the following

Theorem 1.2.4. *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain and let K be a compact subset of Ω with the property that $\Omega \backslash K$ is connected. If f is holomorphic on $\Omega \backslash K$, then there exists a holomorphic function F such that $F|_{\Omega \backslash K} = f$.*

Proof. Again, this result relies heavily on differential forms theory, so we will refer (15, Theorem 1.2.6) as a reference for the proof. □

Now, by Hartog's phenomenon, we are able to show that every zero on several variables is not isolated, let $\Omega \subseteq \mathbb{C}^n$ be a connected domain and $f : \Omega \to \mathbb{C}$ be a holomorphic function. We suppose that exists only one zero in Ω, i.e., there exists unique $z_0 \in \Omega$ such that $f(z_0) = 0$. Since Ω is connected, then $\Omega \backslash \{z_0\}$ is connected and also the function $1/f(z)$ is holomorphic on $\Omega \backslash \{z_0\}$, so by Hartog's theorem there exists a holomorphic extension of $1/f(z)$ to all of Ω. However, suppose $1/f(z)$ is defined for all Ω. In that case, we obtain $f(z_0) \cdot 1/f(z_0) = 1$ by continuity of the product of continuous functions, which implies $f(z_0) \neq 0$, a contradiction, therefore f does not have only one zero.

However, we can construct in a more explicit manner by the use of Hurwitz theorem of one variable complex analysis, which we will state here for completeness

Theorem 1.2.5. *(16, Chapter VII, Theorem 2.5) Let $(g_n)_{n \in \mathbb{N}}$ be a sequence of holomorphic functions on an open connected set $D \subseteq \mathbb{C}$ that converges uniformly on compact subsets of D to a function g which is not constantly zero on D. If g has a zero of order m at z_0, then for a sufficiently small $\delta > 0$ and for a sufficiently large $k \in \mathbb{N}$, which depends on δ, g_n has precisely m zeros in the disk defined by $|z - z_0| < \delta$, including multiplicity.*

With the previous result, take $f : \Omega \to \mathbb{C}$ a holomorphic function, where $\Omega \subseteq \mathbb{C}^2$ is a open and connected set then take $f(z_1, z_2) = 0$ and define $f_n(z) = (z, z_2 + 1/n)$ and z_2 is fixed. Since f is holomorphic we get that $(f_n)_{n \in \mathbb{N}}$ converges uniformly on compact subsets to the function $g(z) = f(z, z_2)$ and by Hurwitz's theorem there exists a sufficiently large $N \in \mathbb{N}$ such that there exists a z_N with the property $|z_1 - z_N| < \varepsilon/2$ and $f_N(z_N) = 0$. Assuming without loss of generality that $N > 2/\varepsilon$, we get that $f(z_N, z_2 + 1/N) = 0$ and that $\|(z_1, z_2) - (z_N, z_2 + 1/N)\| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, therefore for any zero of a holomorphic function in several variables we have other arbitrarily close zeros thus no zero is isolated.

So this is one of the significant differences in the function theory of several complex variables. On a side note we also get that singularities are also never isolated for several complex variables, and these two facts can turn the behavior of some functions harder to study and so we finish our short introduction to several complex variables, if the reader wants to study more about this topic we refer (14, 15).

1.3 The Hardy space over the polydisk.

Definition 1.3.1. *Let $\mathbb{T}$ be the boundary of $\mathbb{D}$ we take $\mathbb{T}^n$, the cartesian product of n copies of $\mathbb{T}$, to be the distinguished boundary of $\mathbb{D}^n$, which is the cartesian product of the boundaries of each $\mathbb{D}$, and σ to be the normalized Lebesgue measure on $\mathbb{T}^n$. We define the Hardy space $H^2(\mathbb{D}^n)$ to be the Hilbert space of all holomorphic functions f over $\mathbb{D}^n$ such that:*

$$\|f\|_2^2 := \sup_{0<r<1} \int_{\mathbb{T}^n} |f(r\zeta)|^2 d\sigma(\zeta) < \infty$$

We also define $H^\infty(\mathbb{D}^n)$ to be the set of all the bounded holomorphic functions over $\mathbb{D}^n$. Note that through the text, we always suppose that $n \geqslant 2$, except mentioned otherwise.

We may also have two other possible characterizations of the Hardy space similar to the ones given for the disk

- Power Series: In this case, we exploit the holomorphic structure of the functions in $H^2(\mathbb{D}^n)$ to write the space as

$$H^2(\mathbb{D}^n) = \{f = \sum_{i_1,\dots,i_n=0}^{\infty} a_{i_1,\dots,i_n} z_1^{i_1} \dots z_n^{i_n} : f \in \text{Hol}\,(\mathbb{D}^n), \quad \sum_{i_1,\dots,i_n=0}^{\infty} |a_{i_1,\dots,i_n}|^2 < \infty\}$$

  This approach is very interesting to use in case one wants to use algebraic structures since, in this case, one can see $H^2(\mathbb{D}^n)$ as a "Hilbert" sub-module of $\mathbb{C}[z_1, \dots, z_n]$, in the following manner, given a set $\{T_1, \dots, T_n\}$ of commuting bounded linear operators, the n-tuple $(T_1, \dots, T_n)$ provides the module structure by:

$$\mathbb{C}[z_1, ..., z_n] \times H^2(\mathbb{D}^n) \to H^2(\mathbb{D}^n) \quad (p, f) \mapsto p(T_1, \dots, T_n)f$$

  where $p \in \mathbb{C}[z_1, ..., z_n]$ and $f \in H^2(\mathbb{D}^n)$. Usually when using this approach, the operators $M_{z_1}, \dots, M_{z_n}$, i.e, the bounded linear operators of multiplication by z_i, $i = 1, \dots, n$ are used in the place of $T_1, \dots, T_n$. We refer to the survey article (17) for a more in-depth introduction to this.

- Fourier series: Analogously to the one-variable case where we may define $H^2(\mathbb{D})$ as isometrically isomorphic to

$$\tilde{H}^2 = \{\tilde{f} \in L^2(\mathbb{T}) : \hat{f}(n) = 0 \quad \text{for} \quad n < 0\}$$

  where $\hat{f}(n)$ denotes the n^{th}-Fourier coefficient. In particular, we may see taking $\hat{f}(n) = a_n$ the functions of $\tilde{H}^2$ as

$$\tilde{f}(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{in\theta} \quad \text{with} \quad \sum_{n=0}^{\infty} |a_n|^2 < \infty$$

  So the idea for this is the equivalence of $f \leftrightarrow \tilde{f}$ for $f \in H^2(\mathbb{D})$ and $\tilde{f} \in L^2(\mathbb{T})$. So the analogous approach to $H^2(\mathbb{D}^n)$ can be given by

$$\tilde{H}^2(\mathbb{D}^n) = \{\tilde{f} \in L^2(\mathbb{T}^n) : \hat{f}(k) = 0 \quad \text{for} \quad |k| < 0\}$$

where $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ is a multi-index, $|k| = k_1 + \cdots + k_n$ and $\hat{f}(k)$ is k^{th}-Fourier coefficient, but note here that every $k_i \geqslant 0$ for all $i = 1, \ldots, n$. The proof of this characterization will be given by Theorem 1.3.2. It gives us a similar analytic characterization of $H^2(\mathbb{D}^n)$, which will be useful later when discussing compact Toeplitz operators.

Definition 1.3.2. *Let $z = (z_1, ..., z_n) \in \mathbb{D}^n$ and $\zeta = (\zeta_1, ..., \zeta_n) \in \mathbb{T}^n$ such that $z_j = r_j e^{i\theta_j}$ and $\zeta_j = e^{i\varphi_j}$ for $j = 1, \ldots, n$, we define the Poisson kernel $P(z, \zeta)$ as the product:*

$$P(z, \zeta) = \prod_{k=1}^{n} P_{r_k}(\theta_k - \varphi_k)$$

where $P_r(\theta)$ denotes the usual Poisson Kernel defined in $\mathbb{D}$.

Note that $P(z, \zeta) > 0$, since the Poisson kernel on the polydisk, is the product of Poisson kernels on the disk, which are always positive, and in particular, we have the following result

Lemma 1.3.1. *If $P(z, \zeta)$ is the Poisson kernel on the polydisk, then $\int_{\mathbb{T}^n} P(z, \zeta) d\sigma(\zeta) = 1$.*

Proof. We may rewrite this integral as:

$$\int_{\mathbb{T}^n} P(z, \zeta) d\sigma(\zeta) = \int_{\mathbb{T}} P_{r_1}(\theta_1 - \varphi_1) d\mu \ \ldots \int_{\mathbb{T}} P_{r_n}(\theta_n - \varphi_n) d\mu$$

where μ denotes the usual normalized Lebesgue measure on the circle.

We know that in the one-dimensional case, we have $\int_{\mathbb{T}} P_r(\theta - t) d\mu = 1$, where t is any real number; thus, the result follows. □

In particular, since we know that the Poisson Kernel on the polydisk is the product of Poisson kernels on the disk, we may write the following:

$$\begin{aligned} P(z, \zeta) = \prod_{j=1}^{n} P_{r_j}(\theta_j - \varphi_j) &= \prod_{j=1}^{n} \sum_{k_j \in \mathbb{Z}} r_j^{|k_j|} e^{ik_j(\theta_j - \varphi_j)} \\ &= \sum_{k_1 \in \mathbb{Z}} r_1^{|k_1|} e^{ik_1(\theta_1 - \varphi_1)} \cdots \sum_{k_n \in \mathbb{Z}} r_n^{|k_n|} e^{ik_n(\theta_n - \varphi_n)} \end{aligned}$$

Since we know that all the series above converges absolutely, we obtain that

$$P(z, \zeta) = \sum r_1^{|k_1|} ... r_n^{|k_n|} e^{ik\cdot(\theta - \varphi)} \tag{1.3}$$

where the summation extends over all lattice points $k = (k_1, ..., k_n) \in \mathbb{Z}^n$ and $k \cdot \theta = k_1\theta_1 + ... + k_n\theta_n$ and in particular we know that the family $\{e^{ik}\}_{k \in \mathbb{Z}^n}$ is a basis for $L^2(\mathbb{T}^n)$.

Definition 1.3.3. *If ρ is a complex Borel measure in $\mathbb{T}^n$, then we define its Poisson integral as the following function:*

$$P[d\rho](z) = \int_{\mathbb{T}^n} P(z, \zeta) d\rho(\zeta)$$

Henceforth, if $f \in L^1(\mathbb{T}^n)$, we will denote by $P[f]$ instead of $P[fd\sigma]$ to simplify the notation.

Theorem 1.3.1. *The following assertions hold:*

1. *If $f \in L^\infty(\mathbb{T}^n)$ and $z \in \mathbb{D}^n$, then $|P[f](z)| \leqslant \|f\|_\infty$.*

2. *If $f \in C(\mathbb{T}^n)$, then $P[f]$ extends to a continuous function on $\overline{\mathbb{D}^n}$.*

3. *If $1 \leqslant p < \infty$, $f \in L^p(\mathbb{T}^n)$, and $u = P[f]$, then $\|u_r\|_p \leqslant \|f\|_p$ and $\|u_r - f\|_p \to 0$ as $r \to 1$.*

where u_r is the function $u_r(w) = u(rw)$ and u is a function in $\mathbb{D}^n$ with $0 \leqslant r < 1$.

Proof. Let us prove the assertions.

1. This first item is just a straightforward application of Hölder's inequality

$$|P[f](z)| = \left|\int_{\mathbb{T}^n} P(z,\zeta)f(\zeta)d\sigma(\zeta)\right| \leqslant \int_{\mathbb{T}^n} P(z,\zeta)d\sigma(\zeta)\cdot\|f\|_\infty = \|f\|_\infty$$

2. By equation (1.3), item 2 holds trivially if f is a trigonometric polynomial. Now by Stone-Weierstrass theorem, every $f \in C(\mathbb{T}^n)$ is the uniform limit of a trigonometric polynomials, hence by applying the previous assertion, we have the result.

3. Now, since $u = P[f]$, we have the following:

$$|u_r(\zeta)|^p = \left|\int_{\mathbb{T}^n} P(r\zeta,\zeta')f(\zeta')d\sigma(\zeta')\right|^p = \left|\int_{\mathbb{T}^n} P(r\zeta,\zeta')^{1-\frac{1}{q}}P(r\zeta,\zeta')^{\frac{1}{q}}f(\zeta')d\sigma(\zeta')\right|^p$$

Now by Hölder's inequality follows that:

$$|u_r(\zeta)|^p \leqslant \left(\int_{\mathbb{T}^n} P(r\zeta,\zeta')^{\frac{1}{q}q}d\sigma(\zeta')\right)^{\frac{p}{q}}\left(\int_{\mathbb{T}^n} P(r\zeta,\zeta')^{(1-\frac{1}{q})p}|f(\zeta')|^p d\sigma(\zeta')\right)^{\frac{1}{p}p}$$

Since, $(1-\frac{1}{q})p = 1$, follows that $|u|^p \leqslant P[|f|^p]$, in particular we obtain $|u_r(\zeta)| \leqslant \int_{\mathbb{T}^n} P(r\zeta,\zeta')|f(\zeta')|^p d\sigma(\zeta')$. Now, by integrating over ζ we get that:

$$\int_{\mathbb{T}^n} |u(r\zeta)|^p d\sigma(\zeta) \leqslant \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} P(r\zeta,\zeta')|f(\zeta')|^p d\sigma(\zeta')d\sigma(\zeta)$$

By, Fubini's Theorem, we have:

$$\int_{\mathbb{T}^n} |u(r\zeta)|^p d\sigma(\zeta) \leqslant \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} P(r\zeta,\zeta')|f(\zeta')|^p d\sigma(\zeta)d\sigma(\zeta')$$

Since $\int_{\mathbb{T}^n} P(r\zeta,\zeta')d\sigma(\zeta) = 1$, it follows that $\|u_r\|_p \leqslant \|f\|_p$. Now, $\|u_r - f\|_p \to 0$ as $r \to 1$ by item 2, since $C(\mathbb{T}^n)$ is dense in $L^p(\mathbb{T}^n)$ for $1 \leqslant p < \infty$.

□

Now, we want to show that, in fact, the limit of the radial functions exists almost everywhere, and therefore, we have the same characterization as in the disk case. In order to do so, we need some definitions.

Definition 1.3.4. *For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we define the α-shaped box as the cartesian product $I_1 \times \cdots \times I_n$ of half-open arcs $[s_j, t_j)$ in $\mathbb{T}$ for all $j = 1, \ldots, n$ such that the arc lengths have the same ratio to each other as are the numbers $2^{\alpha_1}, \ldots, 2^{\alpha_n}$ and of course $t_j - s_j \leqslant 2\pi$ for every $j = 1, \ldots, n$.*

The ratio here means that if you take the length of two intervals we have the relation.

$$\frac{t_i - s_i}{t_j - s_j} = \frac{2^{\alpha_i}}{2^{\alpha_j}}$$

for all $1 \leqslant i, j \leqslant n$.

Let λ be a fixed positive measure on $\mathbb{T}^n$ and we put

$$g_\alpha(w) = \frac{\sup \lambda(B)}{\sigma(B)}$$

for $w \in \mathbb{T}^n$, B is α-shaped box and the supremum is taken over all α-shaped boxes with center at w and we set

$$G(w) = \sum_{\alpha \in \mathbb{N}^n} 2^{-|\alpha|} g_\alpha(w)$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. So, we are ready to state the first of the three necessary lemmas to prove the desired characterization for the Hardy space of the polydisk.

Lemma 1.3.2. *Define the set $\{G > t\} = \{w \in \mathbb{T}^n : G(w) > t\ ,\ t > 0\}$. Then $\sigma(\{G > t\}) \leqslant 35^n \|\lambda\|\, t^{-1}$, where $\|\lambda\|$ is the total variation of the measure λ.*

Proof. Fix $\alpha \in \mathbb{N}^n$ and let $E_\alpha(\xi) = \{g_\alpha > \xi\}$, so by the definition of g_α we get that every $w \in E_\alpha(\xi)$ is the center of some α-shaped box B_w and again by the definition of g_α we also get that for $w \in E_\alpha(w)$

$$\xi < g_\alpha(w) < \frac{\lambda(B_w)}{\sigma(B_w)}$$

which implies that $\lambda(B_w) > \xi\sigma(B_w)$. Given $\varepsilon > 0$, there exists a subset $A_\varepsilon \subseteq E_\alpha(\xi)$ such that $\sigma(A_\varepsilon) \geqslant (1-\varepsilon)\sigma(E_\alpha(\xi))$ which is covered by finitely many boxes, set $H = \{B_1, \ldots, B_r\}$ be the collection of theses boxes. From this finite collection we may extract a disjoint subcollection $\tilde{B}_1, \ldots, \tilde{B}_j$ whose union has measure greater or equal to $3^{-n}\sigma(\bigcup_{i=1}^r B_i)$. The way to get this subcollection is to pick the largest set, in terms of measure, take out all sets that intersect it, pick the second largest and repeat the process until we get the desired subcollection and the relation given in the measure holds since we are taking the intersections off the subcollection so the factor n is to take account for the dimension and

3 is to ensure that the measure will be less than the desired; also this choice will provide a simple factorization later on. Then

$$3^{-n}(1-\varepsilon)\sigma(E_\alpha(\xi)) \leqslant \sum_{i=1}^{j}\sigma(\tilde{B}_i) \leqslant \xi\sum_{i=1}^{j}\lambda(\tilde{B}_i)$$

which implies that

$$\sigma(E_\alpha(\xi)) \leqslant 3^n\|\lambda\|\xi^{-1}$$

noting that $\xi > 0$. Now, put $\eta = \sum_{\ell=0}^{\infty} 2^{-\ell/2} = 2+\sqrt{2}$ and then $\sum_{\alpha\in\mathbb{N}^n} 2^{-|\alpha|/2} = \eta^n$. So if $G(w) > t$ we obtain

$$t < \sum_{\alpha\in\mathbb{N}^n} 2^{-|\alpha|}g_\alpha(w) \leqslant \eta^n \sup_{\alpha\in\mathbb{N}^n}\{2^{-|\alpha|/2}g_\alpha(w)\}$$

hence follows that the set $\{G > t\}$ belongs in the union of the sets $E_\alpha(2^{|\alpha|/2}\eta^{-n}t)$ and therefore we get that the measure has an upper bound given by

$$\sigma(\{G > t\}) \leqslant 3^n\|\lambda\| \sum_{\alpha\in\mathbb{N}^n} 2^{-|\alpha|/2}\eta^n t^{-1} = (3\eta^2)^n\|\lambda\|t^{-1}.$$

□

Our goal now is to provide an upper bound for $P[d\lambda](rw)$; we saw in Theorem 1.3.1 that a candidate for the radial function is exactly the Poisson integral of f. Fix $0 < r < 1$, then there exist a $c = c(r)$ such that $1 \leqslant c \leqslant 2$ and $\frac{\pi}{(1-r)c}$ is an integer of the form 2^t with $t \in \mathbb{N}$. Put $x_0 = 0$, $y_0 = (1-r)c$, $x_i = 2^{i-1}(1-r)c$ and $y_i = 2x_i$ for $1 \leqslant i \leqslant t$. Let B_α be α-shaped box with center at ω whose sides have length $2y_{\alpha_1}, \ldots, 2y_{\alpha_n}$, and let Q_α be the set of all $(e^{is_1}, \ldots, e^{is_n}) \in B_\alpha$ such that $x_{\alpha_i} \leqslant |s_i| \leqslant y_{\alpha_i}$ for $i = 1, \ldots, n$. Note that Q_α is a union of 2^n boxes and that $\mathbb{T}^n = \bigcup_{\alpha\in\mathbb{N}^n} Q_\alpha$. Moreover, this is a finite union since the $\alpha_i \leqslant t$ for all α. Now,

$$\sigma(B_\alpha) = \frac{1}{(2\pi)^n}\prod_{i=1}^{n} 2y_{\alpha_i} = \frac{1}{\pi^n}\prod_{i=1}^{n} 2^{\alpha_i-1}(1-r)c = \frac{2^{|\alpha|}(1-r)^n c^n}{\pi^n}$$

and of course, by hypothesis we know that $Q_\alpha \subset B_\alpha$, then

$$\lambda(Q_\alpha) \leqslant \frac{2^{|\alpha|}(1-r)^n c^n g_\alpha(\omega)}{\pi^n}.$$

By taking the inequality $P_r(\theta) \leqslant (\pi/\theta)^2(1-r)$ for $|\theta| \leqslant \pi$, we get

$$P_r(x_i) \leqslant \frac{4\pi^2}{4^i\,(1-r)\,c^2}, \quad \text{for} \quad 0 \leqslant i \leqslant t.$$

Now, by simple Poisson kernel properties for the disk, we know that $P_r(s_i) \leqslant P_r(x_{\alpha_i})$,

$$\int_{Q_\alpha} P_r(s_1)\dots P_r(s_n)d\lambda \leqslant \lambda(Q_\alpha)\prod_{i=1}^{n} P_r(x_{\alpha_i})$$

so by the preceding inequalities, we obtain

$$\int_{Q_\alpha} P_r(s_1)\dots P_r(s_n)d\lambda \leqslant (4\pi)^n 2^{-|\alpha|} g_\alpha(\omega).$$

If we take the union of the collection of Q_α we may add the integrals over α and by changing ω to any $w \in \mathbb{T}^n$ we calculate the following estimate

$$P[d\lambda](rw) \leqslant (4\pi)^n \sum_{\alpha\in\mathbb{N}^n} 2^{-|\alpha|} g_\alpha(w) = (4\pi)^n G(w)$$

but note that only the left-hand side depends on r, which we can conclude the following two lemmas

Lemma 1.3.3. $\sigma(\{\sup_{0<r<1} P[d\lambda] > \xi\}) \leqslant (140\pi)^n \|\lambda\| \xi^{-1}$.

Proof. If we assume that $\sup_{0<r<1} P[d\lambda](rw) > \xi$, we get that $G(w) > \dfrac{\xi}{(4\pi)^n}$, then by Lemma 1.3.2 we obtain that this only happens if the measure of the set is less or equal to $35^n(4\pi)^n\|\lambda\|\xi^{-1} = (140\pi)^n\|\lambda\|\xi^{-1}$. □

Lemma 1.3.4. *If λ vanishes in some open set $V \subset \mathbb{T}^n$, then*

$$\lim_{r\to 1} P[d\lambda](rw) = 0 \quad \textit{almost everywhere on } V.$$

Proof. Assume that $w \in \mathbb{T}^n$ is such that $G(w) < \infty$, which is valid for almost every w by Lemma 1.3.2, then choose $\delta > 0$ such that a cube C centered at w and edge 2δ is contained in V. So the integrals over Q_α on which $y_{\alpha_i} < \delta$ are null in the preceding estimates and hence the upper bound for $P[d\lambda](rw)$ is obtained under the summation of α such that $2^{|\alpha|} \geqslant \dfrac{\delta}{1-r}$. When we take the limit of $r \to 1$, we start to take fewer terms of the convergent series, and therefore, in the limit, the upper bound tends to zero, and since λ is a positive measure, the result follows. □

So we are ready to state and prove our identification between the functions on $L^2(\mathbb{T}^n)$ with the radial functions.

Theorem 1.3.2. *If $f \in L^1(\mathbb{T}^n)$, λ is a measure on $\mathbb{T}^n$ which is singular with respect to σ and $u = P[f + d\lambda]$, then $u^*(w) = \lim_{r\to 1} u(rw) = f(w)$ almost everywhere on $\mathbb{T}^n$.*

Proof. Take $f \in L^1(\mathbb{T}^n)$ and define $B_f(w) = \limsup_{r\to 1} P[f](rw) - \liminf_{r\to 1} P[f](rw)$. If $f = g + h$ with $g \in C(\mathbb{T}^n)$ we obtain from Theorem 1.3.1 that $P[g]$ is continuous on $\overline{\mathbb{D}}^n$ therefore $B_g = 0$, so we get

$$B_f(w) = B_h(w)$$

and in particular, we know that

$$B_h(w) = \limsup_{r\to 1} P[h](rw) - \liminf_{r\to 1} P[h](rw) \leqslant |\limsup_{r\to 1} P[h](rw)| + |\liminf_{r\to 1} P[h](rw)|$$
$$\leqslant 2 \sup_{0<r<1} P[|h|](rw)$$

So by Lemma 1.3.3 we get that the set $\{B_f > 2\xi\}$ has a upper bound for its measure given by $(140\pi)^n\|h\|_1\xi^{-1}$. We can take $\|h\|_1$ to be arbitrarily small since the set $C(\mathbb{T}^n)$ is dense on $L^1(\mathbb{T}^n)$, so given $\varepsilon > 0$, there exists $g \in C(\mathbb{T}^n)$ such that

$$\|f - g\|_1 = \|h\|_1 < \varepsilon$$

hence we know that $B_f \leqslant 2\xi$ almost everywhere for every $\xi > 0$ and thus $B_f = 0$ almost everywhere which implies that $\lim_{r\to 1} P[f](rw)$ exists almost everywhere, then by Theorem 1.3.1 we obtain that this limit is equal to $f(w)$ for almost every $w \in \mathbb{T}^n$.

Now remember that a measure τ is said to be concentrated on A if for some set $A \in \Sigma$, Σ a sigma-algebra, $\tau(B) = \tau(A \cap B)$ for every $B \in \Sigma$. Choose $\varepsilon > 0$ and take $\lambda = \tau + \nu$ by the Lebesgue's decomposition theorem for measures, where τ is concentrated on a compact set K with $\sigma(K) = 0$, and $\|\nu\| < \varepsilon$. So by Lemma 1.3.4 we get that $\limsup_{r\to 1} P[d\lambda](rw) = \limsup_{r\to 1} P[d\nu](rw)$ almost everywhere and therefore by Lemma 1.3.3 we obtain

$$\sigma(\{\limsup_{r\to 1} P[d\lambda] > \xi\}) = \sigma(\{\limsup_{r\to 1} P[d\nu] > \xi\}) \leqslant (140\pi)^n\varepsilon\xi^{-1}$$

hence $\limsup_{r\to 1} P[d\lambda] \leqslant \xi$ almost everywhere for all $\xi > 0$, so we get $P[d\lambda] \to 0$ almost everywhere. □

Hence, we may identify a function $f \in H^2(\mathbb{D}^n)$ with a function $f^* \in L^2(\mathbb{T}^n)$ as follow:

$$f^*(\zeta) := \lim_{r\to 1^-} f(r\zeta)$$

for almost every $\zeta \in \mathbb{T}^n$, thus we can see $H^2(\mathbb{D}^n)$ as a linear subspace of $L^2(\mathbb{T}^n)$ and, in particular, $H^\infty(\mathbb{D}^n)$ of $L^\infty(\mathbb{T}^n)$ and therefore we can use the inner product structure of $L^2(\mathbb{T}^n)$ on $H^2(\mathbb{D}^n)$ as follows: Given $f, g \in H^2(\mathbb{D}^n)$:

$$\langle f, g\rangle = \int_{\mathbb{T}^n} f^*(\zeta)\overline{g^*(\zeta)}d\sigma(\zeta)$$

and of course, the norm is given by:

$$||f||_2^2 = \int_{\mathbb{T}^n} |f^*(\zeta)|^2 d\sigma(\zeta).$$

Since we have this identification, from now on, we will denote f^* by f to simplify the notation. We will state a few primary results from general $H^2(\mathbb{D}^n)$ function theory.

Theorem 1.3.3. *$H^\infty(\mathbb{D}^n)$ is a linear subspace of $H^2(\mathbb{D}^n)$.*

Proof. Let $f \in H^\infty(\mathbb{D}^n)$, then exists a $M > 0$ such that $||f||_\infty \leqslant M$, therefore:

$$||f||_2^2 = \sup_{0<r<1} \int_{\mathbb{T}^n} |f(r\zeta)|^2 d\sigma(\zeta) \leqslant M^2 \sup_{0<r<1} \int_{\mathbb{T}^n} d\sigma(\zeta) = M^2 < \infty$$

Thus, $f \in H^2(\mathbb{D}^n)$. □

Definition 1.3.5. *Let $\alpha \in \mathbb{D}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $\alpha_i \in \mathbb{D}$ for $i = 1, ..., n$. Then we define the reproducing kernel of $H^2(\mathbb{D}^n)$ to be:*

$$K_\alpha(z_1, ..., z_n) = \prod_{i=1}^{n} \frac{1}{1 - \overline{\alpha_i} z_i}.$$

So by this definition, we get that the reproducing kernel of $H^2(\mathbb{D}^n)$ is the product of n reproducing kernels of $H^2(\mathbb{D})$. Therefore, we see that its norm is given by the product of the norm of these n reproducing kernels. The reproducing kernel has the property that $\langle f, K_\alpha \rangle = f(\alpha)$ for all $f \in H^2(\mathbb{D}^n)$ and it defines a continuous linear functional. For completeness, take $\alpha \in \mathbb{D}^n$ and $f \in H^2(\mathbb{D}^n)$, then we have that

$$|f(\alpha)| = |\langle f, K_\alpha \rangle| \leqslant ||f|| \, ||K_\alpha||$$

Since both $||f||$ and $||K_\alpha||$ are bounded, in particular, the operator that maps f to $f(\alpha)$ is continuous.

Theorem 1.3.4. *Let $\{f_n\}_{n \in \mathbb{N}} \in H^2(\mathbb{D}^n)$ be a sequence of functions, if $f_n \to f$ in $H^2(\mathbb{D}^n)$, then $f_n \to f$ uniformly on compact subsets of $\mathbb{D}^n$.*

Proof. For a fixed $\alpha \in \mathbb{D}^n$, we have:

$$|f_n(\alpha) - f(\alpha)| = |\langle f_n - f, K_\alpha \rangle| \leqslant ||f_n - f|| \, ||K_\alpha||$$

If K is a compact subset of $\mathbb{D}^n$, then exists an M such that $||K_\alpha|| \leqslant M$ for all $\omega \in K$ Hence,

$$|f_n(\alpha) - f(\alpha)| \leqslant M \, ||f_n - f|| \text{ for all } \alpha \in K$$

Which implies the theorem. □

We may use the reproducing kernel, the Cauchy Kernel in the literature, to prove Cauchy's Integral Formula for the polydisk.

Theorem 1.3.5. *If f is a holomorphic function in an open set $\Omega \subset \mathbb{C}^n$ such that $\overline{\mathbb{D}^n} \subset \Omega$ and $\omega_0 = (\omega_1, ..., \omega_n) \in \mathbb{D}^n$, then*

$$f(\omega_0) = \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \frac{f(\zeta)}{(\zeta_1 - \omega_1) \ldots (\zeta_n - \omega_n)} d\sigma(\zeta)$$

Proof. A proof for this was provided in the first section but can also be proved by calculating $\langle f, K_{\omega_0}\rangle$ and using the $L^2(\mathbb{T}^n)$ inner product structure. □

Definition 1.3.6. *Let $\phi \in H^\infty(\mathbb{D}^n)$ such that $|\phi(\zeta)| = 1$ for almost every $\zeta \in \mathbb{T}^n$, we say that ϕ is a inner function.*

Theorem 1.3.6. *If ϕ is not a constant inner function, then $|\phi(\omega)| < 1$ for all $\omega \in \mathbb{D}^n$.*

Proof. If ϕ is not constant then we get that $|\phi(z)| < \|\phi\|_\infty$ by Theorem 1.2.3. Now we consider the Poisson integral of ϕ

$$\phi(r\zeta) = \int_{\mathbb{T}^n} \phi^*(\eta)P(r\zeta, \eta)d\sigma(\eta)$$

here, we use the radial function notation to avoid any misconception, so by taking the modulus on both sides, we get that

$$\begin{aligned} |\phi(r\zeta)| &= \left| \int_{\mathbb{T}^n} \phi^*(\eta)P(r\zeta, \eta)d\sigma(\eta) \right| \\ &\leqslant \int_{\mathbb{T}^n} |\phi^*(\eta)|P(r\zeta, \eta)d\sigma(\eta) \\ &< \int_{\mathbb{T}^n} P(r\zeta, \eta)d\sigma(\eta) \\ &= 1 \end{aligned}$$

for all $0 < r < 1$.

□

Definition 1.3.7. *A function $f \in H^\infty(\mathbb{D}^n)$ is called a generalized inner function if $1/f \in L^\infty(\mathbb{T}^n)$.*

Remark 1.3.1. *Note that every inner function is a generalized inner function. Since if f is inner, we have that $|f| = 1$ for almost every $\zeta \in \mathbb{T}^n$, hence $1/f \in L^\infty(\mathbb{T}^n)$.*

Example 1.3.1. *Let $p \in \mathbb{C}[z_1, \dots, z_n]$ be a polynomial such that is zero free in $\mathbb{D}^n$, then $p \in H^\infty(\mathbb{D}^n)$ and also, we get that $1/p \in L^\infty(\mathbb{T}^n)$ hence p is a generalized inner function.*

1.4 Basic properties of Toeplitz operators

In this section, we will define Toeplitz operators over the Hardy space of the polydisk and present some useful results. In particular, we will show that our interest shifts as Toeplitz operators. Every result stated here is also valid for Toeplitz operators over $H^2(\mathbb{D})$; for some, the proofs are the same.

Definition 1.4.1. *Let P be the orthogonal projection of $L^2(\mathbb{T}^n)$ onto $H^2(\mathbb{D}^n)$. We define the Toeplitz operator T_ϕ with symbol $\phi \in L^\infty(\mathbb{T}^n)$ by:*

$$T_\phi f := P(\phi f)$$

for all $f \in H^2(\mathbb{D}^n)$.

Note that in general, we will interchange the use of $H^2(\mathbb{D}^n)$ and $\tilde{H}^2(\mathbb{D}^n)$ to be possible to perform the product of a function in $L^\infty(\mathbb{T}^n)$ with a function on $H^2(\mathbb{D}^n)$.

Lemma 1.4.1. *Let $\phi \in L^\infty(\mathbb{T}^n)$, then the operator M_ϕ defined by $M_\phi f = \phi f$ for every $f \in L^2(\mathbb{T}^n)$ is a bounded linear operator.*

Proof. Let $f,g \in L^2(\mathbb{T}^n)$ and $\lambda \in \mathbb{C}$. Hence

$$M_\phi(f + \lambda g) = \phi(f + \lambda g) = \phi f + \lambda \phi g = M_\phi f + \lambda M_\phi g$$

Therefore, M_ϕ is linear.

Now, suppose $f \in L^2(\mathbb{T}^n)$ such that $\|f\|_2 = 1$ then

$$||M_\phi f||^2 = \int_{\mathbb{T}^n} |\phi f|^2 d\sigma \leqslant \int_{\mathbb{T}^n} |\phi|^2 d\sigma = ||\phi||_\infty^2$$

Thus, M_ϕ is a bounded linear operator. □

Proposition 1.4.1. *Let $\phi \in L^\infty(\mathbb{T}^n)$, then T_ϕ is a bounded linear operator over $H^2(\mathbb{D}^n)$.*

Proof. First, we prove that T_ϕ is a linear operator. Let $\phi \in L^\infty(\mathbb{T}^n)$ and let $f,g \in H^2(\mathbb{D}^n)$ and $\lambda \in \mathbb{C}$.

$$T_\phi(f + \lambda g) = P(\phi(f + \lambda g)) = P(\phi f + \lambda \phi g) = P(\phi f) + \lambda P(\phi g) = T_\phi f + \lambda T_\phi g$$

We must use the previous lemma to prove that T_ϕ is bounded.

$$\|T_\phi f\| = \|P(M_\phi f)\| \leqslant \|P\| \, \|M_\phi\| \, \|f\| = \|M_\phi\| \, \|f\|$$

Therefore, T_ϕ is a bounded linear operator.

□

Proposition 1.4.2. *Let $\phi \in L^\infty(\mathbb{T}^n)$ and T_ϕ a Toeplitz operator, then $\|T_\phi\| = \|\phi\|_\infty$.*

Proof. Based on the previous proposition, we already have an upper bound for the norm; let's establish a lower bound. One can show that, for reproducing kernels in $H^2(\mathbb{D}^n)$ we have:

$$\|K_\alpha\|_2 = \prod_{i=1}^{n} \frac{1}{\sqrt{1 - |\alpha_i|^2}}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{D}^n$. Take $\tilde{K_\alpha}$ to be the normalized reproducing kernel. Then, we obtain:

$$|\langle T_\phi \tilde{K_\alpha}, \tilde{K_\alpha} \rangle| \leqslant \|T_\phi \tilde{K_\alpha}\|_2 \|\tilde{K_\alpha}\|_2 \leqslant \|T_\phi\| \|\tilde{K_\alpha}\|_2^2 = \|T_\phi\|$$

On the other hand, we have the following:

$$\begin{aligned}\|T_\phi\| \geqslant |\langle T_\phi \tilde{K_\alpha}, \tilde{K_\alpha} \rangle| = |\langle P(\phi \tilde{K_\alpha}), \tilde{K_\alpha} \rangle| = |\langle \phi \tilde{K_\alpha}, \tilde{K_\alpha} \rangle| &= \left| \int_{\mathbb{T}^n} \prod_{i=1}^{n} \frac{1 - |\alpha_i|^2}{|\zeta_i - \alpha_i|} \phi(\zeta) d\sigma(\zeta) \right| \\ &= P[\phi](\alpha).\end{aligned}$$

Now, define $\alpha = r\zeta$ with $\zeta \in \mathbb{T}^n$ and $0 < r < 1$, so by the third assertion in Theorem 1.3.2, follows that $\|T_\phi\| \geqslant |\phi(\zeta)|$ for almost every $\zeta \in \mathbb{T}^n$ and hence $\|T_\phi\| \geqslant \|\phi\|_\infty$. ☐

Proposition 1.4.3. *Let ϕ, $\psi \in L^\infty(\mathbb{T}^n)$ and $\lambda \in \mathbb{C}$. Then,*

$$T_{\phi + \lambda\psi} = T_\phi + \lambda T_\psi.$$

In other words, a Toeplitz operator is linear in its symbol.

Proof. In fact, let $f \in H^2(\mathbb{D}^n)$ be arbitrary then

$$\begin{aligned}T_{\phi+\lambda\psi} f = P((\phi + \lambda\psi)f) = P(\phi f + \lambda\psi f) &= P(\phi f) + P(\lambda\psi f) \\ &= P(\phi f) + \lambda P(\psi f) = T_\phi f + \lambda T_\psi f\end{aligned}$$

☐

Proposition 1.4.4. *Let $\phi \in L^\infty(\mathbb{T}^n)$ and let T_ϕ be a Toeplitz operator with symbol ϕ, then $T_\phi^* = T_{\overline{\phi}}$.*

Proof. Let f, $g \in H^2(\mathbb{D}^n)$, then

$$\begin{aligned}\langle T_{\overline{\phi}} f, g \rangle = \langle P(\overline{\phi} f), g \rangle = \int_{\mathbb{T}^n} f \overline{\phi g} d\sigma = \langle f, \phi g \rangle &= \langle f, P(\phi g) \rangle \\ &= \langle f, T_\phi g \rangle \\ &= \langle T_\phi^* f, g \rangle\end{aligned}$$

Hence, $T_{\overline{\phi}} = T_\phi^*$

☐

Proposition 1.4.5. *Let $\phi \in L^\infty(\mathbb{T}^n)$. The Toeplitz operator T_ϕ is self-adjoint if, and only if, ϕ is a real-valued function.*

Proof. By the previous proposition, we have that $T_{\overline{\phi}} = T_\phi^*$. Now, if T_ϕ is self-adjoint, then $T_\phi = T_{\overline{\phi}}$ if, and only if, $\phi = \overline{\phi}$ ☐

Definition 1.4.2. *If $\phi \in H^\infty(\mathbb{D}^n)$, T_ϕ is said to be an analytic Toeplitz operator. Analogously, T_ϕ is said to be coanalytic if T_ϕ^* is analytic. If a Toeplitz operator T_ϕ is analytic, then we have:*

$$T_\phi f = \phi f$$

for all $f \in H^2(\mathbb{D}^n)$. Therefore, if T_ϕ is analytic, then this operator acts as a multiplication operator in $H^2(\mathbb{D}^n)$.

Proposition 1.4.6. *Let $\phi \in H^\infty(\mathbb{D}^n)\backslash\{0\}$, then* $\operatorname{Ran}(T_\phi^*)$ *is dense.*

Proof. Is easy to see that if T_ϕ is analytic, then it is injective, thus $\ker(T_\phi) = \{0\}$, and we by Theorem A.0.4 we get

$$\overline{\operatorname{Ran}(T_\phi^*)} = \ker((T_\phi^*)^*)^\perp = \ker(T_\phi)^\perp = \{0\}^\perp = H^2(\mathbb{D}^n)$$

□

Now, similarly as the characterization in (18), where we know that $T \in B(H^2(\mathbb{D}))$ is a Toeplitz operator if and only if $T_z^* T T_z = T$, here we have a characterization of all Toeplitz operators over $H^2(\mathbb{D}^n)$.

Theorem 1.4.1. *(19, Proposition 2.1) Let $T \in B(H^2(\mathbb{D}^n))$, then T is a Toeplitz operator if and only if $T_\phi^* T T_\phi = T$ for every inner function ϕ.*

Proof. Set

$$M = \{\overline{\phi} h : \phi \quad \text{are inner functions}; h \in H^2(\mathbb{D}^n)\}$$

Note that M is a dense subspace of $L^2(\mathbb{T}^n)$ since we may choose monomials of the form $z_1^{m_1} \dots z_n^{m_n}$ for all $m_1, \dots, m_n \in \mathbb{N}_0$ as the inner functions. Define the following map

$$\Phi : M \to \mathbb{C} \quad \Phi(\overline{\phi} h) = \langle h, T\phi \rangle$$

First we assume that $T_\phi^* T T_\phi = T$ for all inner functions ϕ, we want to show that Φ is well-defined and linear. In fact, if $\overline{\phi_1} h_1 = \overline{\phi_2} h_2$, then

$$\begin{aligned}
\Phi(\overline{\phi_1} h_1) = \langle h_1, T\phi_1 \rangle = \langle h_1, T_{\phi_2}^* T T_{\phi_2} \phi_1 \rangle &= \langle \phi_2 h_1, T\phi_2 \phi_1 \rangle \\
&= \langle \phi_1 h_2, T\phi_1 \phi_2 \rangle \\
&= \langle h_2, T_{\phi_1}^* T T_{\phi_1} \phi_2 \rangle \\
&= \langle h_2, T\phi_2 \rangle \\
&= \Phi(\overline{\phi_2} h_2)
\end{aligned}$$

We get that Φ is well-defined. Now, by an analogous argument, Φ is linear, so Φ is a linear functional, but note that from the definition of Φ we get,

$$|\Phi(\overline{\phi})h| \leqslant \|\phi\| \|Th\| \leqslant \|T\| \|h\| = \|T\| \|\overline{\phi} h\|$$

Thus, Φ is a bounded linear functional. Since M is dense in $L^2(\mathbb{T}^n)$, there exists a unique $\psi \in L^2(\mathbb{T}^n)$ such that

$$\Phi(\overline{\phi}h) = \langle \overline{\phi}h, \psi \rangle$$

and

$$|\langle \overline{\phi}h, \psi \rangle| = \left| \int_{\mathbb{T}^n} \overline{\phi \psi} h d\sigma \right| \leqslant \|T\| \|\overline{\phi}h\| = \|T\| \, \|h\|$$

hence, $\psi \in L^\infty(\mathbb{T}^n)$ since ψ is a measurable function and $\overline{\phi}h$ is a $L^1(\mathbb{T}^n)$ and by the definition of this operator we get that $\psi \in (L^1(\mathbb{T}^n))^* = L^\infty(\mathbb{T}^n)$ and we obtain

$$\langle h, T\phi \rangle = \Phi(\overline{\phi}h) = \langle \overline{\phi}h, \psi \rangle = \langle h, \phi\psi \rangle = \langle h, T_\psi \phi \rangle$$

and we know that the set of all finite linear combinations of inner functions is dense in $H^2(\mathbb{D}^n)$, therefore $T = T_\psi$ for some ψ. □

Generally, we may give a sharper statement as follows

Theorem 1.4.2. *(20, Theorem 3.1) Let $T \in B(H^2(\mathbb{D}^n))$. Then T is a Toeplitz operator if and only if $T_{z_j}^* T T_{z_j} = T$ for all $j = 1, \ldots, n$.*

The following result shows us a class of eigenvectors for all analytic Toeplitz operators, the reproducing kernels.

Proposition 1.4.7. *Let $\phi \in H^\infty(\mathbb{D}^n)$, T_ϕ be a Toeplitz operator and K_α be a reproducing kernel. Then $T_\phi^* K_\alpha = \overline{\phi(\alpha)} K_\alpha$.*

Proof. Let $f \in H^2(\mathbb{D}^n)$ be arbitrary, then

$$\langle f, T_\phi^* K_\alpha \rangle = \langle T_\phi f, K_\alpha \rangle = \phi(\alpha) f(\alpha) = \langle f, \overline{\phi(\alpha)} K_\alpha \rangle$$

Thus, $T_\phi^* K_\alpha = \overline{\phi(\alpha)} K_\alpha$. □

Theorem 1.4.3. *For $\phi \in L^\infty(\mathbb{D}^n)$, the following are equivalent:*

1. *$\phi = 0$ almost everywhere in $\mathbb{T}^n$.*
2. *T_ϕ is compact.*

Proof. Suppose 1. holds, then T_ϕ is the null operator which is trivially compact. Now, suppose that 2. holds then we know by Lemma A.0.1 that $Z^M = z_1^{m_1} \ldots z_n^{m_n}$, where $Z = (z_1, ..., z_n)$ and $M = (m_1, ..., m_n)$, converges weakly to 0 as $|M| = (m_1 + \cdots + m_n) \to \infty$ in $H^2(\mathbb{D}^n)$. Therefore, by Lemma A.0.3, we get that $\|T_\phi Z^N\| \to 0$. On the other hand, we have:

$$\|T_\phi Z^M\|_2^2 = \left\| P\left(\sum_{k \in \mathbb{Z}^n} \hat{\phi}(k) \zeta^{k+M} \right) \right\|_2^2 = \left\| \sum_{|k|=-|M|}^{\infty} \hat{\phi}(k) \zeta^{k+M} \right\|_2^2$$

Where $\zeta \in \mathbb{T}^n$ and $\hat{\phi}(k)$ is the k-th Fourier coefficient of ϕ. Now, by Theorem A.0.5, we obtain:

$$\left\| \sum_{|k|=-|M|}^{\infty} \hat{\phi}(k)\zeta^{k+M} \right\|_2^2 = \sum_{|k|=-|M|}^{\infty} |\hat{\phi}(k)|^2$$

Now, by taking the limit of $|M| \to \infty$, we get the norm of ϕ, which by the previous argument is equal to 0, then $\phi = 0$ almost everywhere in $\mathbb{T}^n$. □

Corollary 1.4.3.1. *For every $\phi \in L^\infty(\mathbb{T}^n)\backslash\{0\}$, $\mathrm{Ran}(T_\phi)$ is infinite dimensional.*

Note that since no Toeplitz operator is compact, we do not have a compact operators theory to characterize the invariant subspaces. Therefore, all the work toward characterization must use a different set of tools; in particular, we may not use Lomonosov's theorem, which we will state here for completeness.

Theorem 1.4.4. *(21, Theorem 10.20) Let $\mathcal{B}$ be a complex Banach space and let $T \in B(\mathcal{B})$, if there exists an operator $S \in B(\mathcal{B})$ that satisfies:*

- *S is a non-scalar operator.*
- *S commutes with T.*
- *S commutes with a non-zero compact operator.*

Then, T has a non-trivial invariant subspace.

1.5 Universality

We shall introduce the concept of universal operators, introduced by (22), which will be a fundamental tool for studying the ISP. The idea of using universal operators is exchanging the need to show that every bounded linear operator has an invariant subspace for characterization of the invariant subspaces of a universal operator U. More details about this will be given in this section. First, we must define our object of interest, the invariant subspaces.

Definition 1.5.1. *Let X be a normed space, $T \in B(X)$, and $M \subseteq X$ be a subspace. We say that M is a T-invariant subspace if M is closed and $T(M) \subseteq M$. We say that M is non-trivial if $M \neq X$ and $M \neq \{0\}$.*

Definition 1.5.2. *Let $\mathcal{B}$ be a Banach space, and U be a bounded linear operator on $\mathcal{B}$. Then U is said to be universal, in the sense of Rota, for $\mathcal{B}$ if for any bounded linear operator T on $\mathcal{B}$ there exists a constant $\alpha \neq 0$ and an invariant subspace $\mathcal{M}$ for U such that the restriction $U\big|_{\mathcal{M}}$ is similar to αT.*

In (23), a sufficient condition was given for an operator U to be universal for a complex separable infinite dimensional Hilbert space $\mathcal{H}$, which we will call the Caradus criteria. Throughout the text, a Hilbert space $\mathcal{H}$ will always denote a complex separable infinite dimensional Hilbert space.

The Caradus Criteria. Let $\mathcal{H}$ be a Hilbert space and U a bounded linear operator on $\mathcal{H}$. If U satisfies:

1. $\ker(U)$ is infinite dimensional.

2. U is surjective.

Then U is universal for $\mathcal{H}$.

Proof. Let $K = \ker(U)$, define $\tilde{U} := U|_{K^\perp} : K^\perp \to \mathcal{H}$. Note that $\tilde{U}$ is bijective because its domain is $K^\perp$, and by hypothesis, U is surjective. In fact, one can write $\mathcal{H} = K \oplus K^\perp$, define also $V = \tilde{U}^{-1} : \mathcal{H} \to K^\perp$ and we take $W : \mathcal{H} \to K$ a isometric isomorphism by the Riesz-Fischer Theorem, since $U \in B(\mathcal{H})$ and K is closed subspace of $\mathcal{H}$, therefore a Hilbert space and in particular, K is infinite dimensional by hypothesis. Hence, with these notations, we have that:

1. $U \circ V = Id_{\mathcal{H}}$.

2. $U \circ W = 0$.

3. $\ker(W) = \{0\}$.

4. $\operatorname{Ran}(V) = K^\perp$.

Now, let us check the definition of universality. Let $T \in B(\mathcal{H})$ and let $\alpha \in \mathbb{C}\backslash\{0\}$ such that $|\alpha|\|T\|\|V\| < 1$. Take $n = |\alpha|\|T\|\|V\|$. Assume $\sum_{k=0}^{\infty} \alpha^k V^k W T^k$ and note that this series is absolutely convergent because:

$$\sum_{k=0}^{\infty} \|\alpha^k V^k W T^k\| \leqslant \sum_{k=0}^{\infty} |\alpha|^k \|V\|^k \|W\| \|T\|^k \leqslant \|W\| \sum_{k=0}^{\infty} n^k < \infty$$

It is clear that $B(\mathcal{H})$ is a Banach space. Therefore, absolute convergence implies convergence; then it follows that exists a $J \in B(\mathcal{H})$ such that $J = \sum_{k=0}^{\infty} \alpha^k V^k W T^k$. One can see that:

$$W + \alpha V J T = J \tag{1.4}$$

$$UJ = U(W + \alpha V J T) = 0 + \alpha Id_{\mathcal{H}} J T = \alpha J T \tag{1.5}$$

In (1.4), is easy to see that:

$$W + \alpha VJT = W + \alpha V \left(\sum_{k=0}^{\infty} \alpha^k V^k W T^k \right) T = W + \sum_{k=0}^{\infty} \alpha^{k+1} V^{k+1} W T^{k+1} = J$$

We need to prove that $\text{Ran}(J)$ is closed and invariant by U and that $J : \mathcal{H} \to \text{Ran}(J)$ is a linear isomorphism, then equation (1.4) give us the universality of U.

- $\text{Ran}(J)$ is closed:

Let $\tilde{x} \in \overline{\text{Ran}(J)}$, choose $(x_n) \subset \mathcal{H}$ such that $J(x_n) \to \tilde{x}$. Take $P : \mathcal{H} \to K$ the orthogonal projection of $\mathcal{H}$ onto K. Applying P in the equation (1.4), we get:

$$PJ(x_n) = P(W(x_n) + \alpha VJT(x_n)) = W(x_n)$$

Because $W(x_n) \in K$ and $\text{Ran}(V) = K^{\perp}$, therefore it follows that $W(x_n) \to P(\tilde{x})$. Since W is isometric isomorphism, we get that $x_n \to x$ for some $x \in \mathcal{H}$ and this implies that $J(x_n) \to J(x)$ and hence $\tilde{x} = J(x)$.

- $\text{Ran}(J)$ is invariant by U.

By equation (1.5), we have that:

$$UJ(x) = \alpha JT(x) = J(\alpha T(x))$$

for all $x \in \mathcal{H}$. Then follows that $\text{Ran}(J)$ is invariant by U.

- $J : \mathcal{H} \to \text{Ran}(J)$ is a isomorphism.

It is clear that J is surjective and continuous. Now, if $J(x) = 0$ then $W(x) + \alpha VJT(x) = 0$ which implies that $W(x) = 0$. Since W is isometric isomorphism, follows that $x = 0$. By inverse application theorem, we get that J is an isomorphism.

□

In fact, for an operator U in a Hilbert space $\mathcal{H}$ to be universal, the hypothesis of $\dim \ker(U) = \infty$ is necessary.

Remark 1.5.1. *Let $\mathcal{H}$ be a Hilbert space and U be a universal operator for $\mathcal{H}$. Choose $T \in B(\mathcal{H}) \backslash \{0\}$ such that $\dim Ker(T) = \infty$, without loss of generality, suppose $T(e_i) = 0$ for every $i \in I$, where I is a infinite family of index contained in $\mathbb{N}$ and $(e_n)_{n \in \mathbb{N}}$ is a Hilbert basis for $\mathcal{H}$. By the hypothesis of U being universal, we have*

$$S^{-1} \alpha T = U|_M S^{-1}$$

Where $\alpha \in \mathbb{C} \backslash \{0\}$, M is a invariant subspace of U and $S : M \to \mathcal{H}$ is an isomorphism. Therefore,

$$S^{-1}\alpha T(e_i) = U|_M S^{-1}(e_i) = 0$$

for every $i \in I$. *But, since* S^{-1} *is a isomorphism, then* $\{S^{-1}(e_i) : i \in I\}$ *is linearly independent and infinite, hence* $\dim\ker(U) = \infty$.

One might wonder if every Hilbert space admits a universal operator; in fact, let $\mathcal{H}$ be a Hilbert space and define the space

$$\ell^2(\mathcal{H}) = \left\{ x = (x_n)_{n\in\mathbb{N}} : \ x_n \in \mathcal{H}\,, \forall n \in \mathbb{N} \text{ and } \sum_{n=1}^{\infty} \|x_n\|^2 < \infty \right\}.$$

Let us show that $\ell^2(\mathcal{H})$ is a separable Hilbert space. First of all, we define the inner product in $\mathcal{H}$ as follows:

$$\langle x, y\rangle = \left(\sum_{n=1}^{\infty} \langle x_n, y_n\rangle \right)^{\frac{1}{2}} \tag{1.6}$$

where $x = (x_n)_{n\in\mathbb{N}}$, $y = (y_n)_{n\in\mathbb{N}} \in \ell^2(\mathcal{H})$. Take $(e_i)_{i\in\mathbb{N}}$ to be an orthonormal basis of $\mathcal{H}$, now by Theorem A.0.2, we need to show that the basis for $\ell^2(\mathcal{H})$ is countable and orthonormal. Is easy to see that the elements $(e_{i,k})_{i,k\in\mathbb{N}}$

$$e_{i,k} = (0, \dots, 0, \underbrace{\quad e_i \quad}_{k\text{-th coordinate}}, 0, \dots)$$

are a basis of $\ell^2(\mathcal{H})$ from our definition; hence we must show that it is orthonormal and countable. Countable is clear from our indexes because we know that the cardinality of our basis is the same as $\mathbb{N}\times\mathbb{N}$, or even the same as $\mathbb{N}$ and thus countable and so to take under account orthogonality is easy to see that given indexes (i,k) and (j,ℓ)

$$\langle (e_{i,k})_{i,k\in\mathbb{N}}, (e_{j,\ell})_{j,\ell\in\mathbb{N}}\rangle = \delta_{k,\ell}\,\delta_{i,j}$$

where $\delta_{k,\ell}, \delta_{i,j}$ are Kronecker's deltas, simply because if $k \neq \ell$ every coordinate in the inner product described in equation (1.6) is multiplied by a zero, and if $k = \ell$ we have the original orthogonality relation of $\mathcal{H}$. It is orthonormal because

$$\|(e_{i,k})_{i,k\in\mathbb{N}}\|^2 = \langle (e_{i,k})_{i,k\in\mathbb{N}}, (e_{i,k})_{i,k\in\mathbb{N}}\rangle = \langle e_i, e_i\rangle = \|e_i\|^2 = 1$$

for all $i, k \in \mathbb{N}$. Define the operator $T : \ell^2(\mathcal{H}) \to \ell^2(\mathcal{H})$ by $T(x_1, x_2, x_3, ...) = (x_2, x_3, x_4, ...)$, note that

$$\|T(x)\| = \|(x_2, x_3, x_4, \dots)\| \leqslant \|(x_1, x_2, x_3, \dots)\| = \|x\| \tag{1.7}$$

So, T is a bounded linear operator. We want to show that T has an infinite dimensional kernel and is surjective.

- $\dim \ker T = \infty$

  Note that $\ker T = \{(z, 0, 0, 0, \dots) : z \in \mathcal{H}\}$, hence $\ker T \simeq \mathcal{H}$ and therefore it is infinite dimensional.

- T is surjective:

  Let $y = (y_1, y_2, y_3, ...) \in \mathcal{H}$, then $y = T(z)$, where $z = (0, y_1, y_2, y_3, ...) \in \mathcal{H}$, thus T is surjective.

Therefore, by the Caradus criteria, T is universal for $\mathcal{H}$. Now, by Riesz-Fischer theorem, exists a isometric isomorphism W between $\mathcal{H}$ and any other complex separable infinite dimensional Hilbert space $\tilde{\mathcal{H}}$, and hence $W \circ T \circ W^{-1}$ is universal for $\tilde{\mathcal{H}}$.

Theorem 1.5.1. *(24, Proposition 8.1.2) Let $\mathcal{B}$ be an infinite-dimensional complex Banach space and $U \in B(\mathcal{B})$ be a universal operator. Then the following are equivalent:*

1. *Every $T \in B(\mathcal{B})\backslash\{0\}$ has a non-trivial invariant subspace.*

2. *Every invariant subspace M by U that is isomorphic to $\mathcal{B}$ contains a non-trivial invariant subspace.*

Proof. To see that the first condition implies the second, we see that the restriction of U to M is similar to some operator $T \in B(\mathcal{B})$ with restriction to the subspace M by the universality hypothesis, but note that since M is isomorphic to $\mathcal{B}$ we get that the restriction of U to M is similar to T on $\mathcal{B}$. The first condition implies that there exists a non-trivial invariant subspace contained in M. Conversely, since U is universal, we get that every $T \in B(\mathcal{B})$ is similar to U on the restriction to some invariant subspace M. Thus, T has a non-trivial invariant subspace contained in M. □

Now, we will get the following corollary for Hilbert spaces: the connection between universality for Hilbert spaces and the ISP.

Corollary 1.5.1.1. *Let $\mathcal{H}$ be a Hilbert space and let $U \in B(\mathcal{H})$ be a universal operator, then the following statements are equivalent:*

1. *Every $T \in B(\mathcal{H})\backslash\{0\}$ has a non-trivial invariant subspace.*

2. *Every minimal invariant subspace of U is one-dimensional.*

Proof. Every infinite dimensional invariant subspace M for U is isomorphic to $\mathcal{H}$; thus, it's not minimal. If M is finite-dimensional, by Jordan canonical form, we get that there exists a one-dimensional invariant subspace contained in M, which is, in fact, minimal, so unless it is one-dimensional, it is not minimal □

This theorem gives us the connection between the invariant subspace problem and the universal operators. Note that by this previous result, we can provide a "simple" approach to the ISP since we may exchange proving that every operator on $B(\mathcal{H})$ has a non-trivial to prove that all minimal invariant subspaces of the universal operator are one dimensional. Hence, we only need to study a specific operator's invariant subspaces. In Chapter 3, we will present our main results for universality for Toeplitz operators, which will be our choice of operators to investigate universality.

2 Shift invariant subspaces

In this chapter, we will briefly introduce shift-invariant subspaces in the one-dimensional case as a motivation for what comes in the next chapter in the several variables case. This chapter is based on (10, 11, 25).

2.1 Beurling's theorem

Initially, it was an open problem characterizing all the non-trivial invariant subspaces of ℓ^2 under the "shift" operator.

$$S : \ell^2 \to \ell^2 , \quad S(a_0, a_1, a_2, \dots) = (0, a_0, a_1, a_2, \dots)$$

S does not have an eigenspace, is easy to see since if there exists a $v = (a_n)_{n\in\mathbb{N}_0}$ such that

$$Sv = \lambda v \tag{2.1}$$

for some $\lambda \in \mathbb{C}\backslash\{0\}$, then we get $\lambda a_0 = 0$, since we suppose that λ is non-zero, we obtain $a_0 = 0$. Now, we also know that $a_0 = \lambda a_1$ but $a_0 = 0$, applying the same argument again, we have $a_1 = 0$, thus recursively, we get that $v = 0$.

We say that $M \subset \ell^2$ is a shift-invariant subspace if $SM \subseteq M$, and by the previous observation, we know that characterizing the non-trivial invariant subspaces is no trivial task, so the idea was bringing the problem from ℓ^2 to another separable Hilbert space with more structure, that is a space with a well-developed function theory and well-defined classes of elements. By Theorem A.0.3, every separable Hilbert space is isometrically isomorphic to ℓ^2; in particular, we have that $\ell^2 \simeq H^2(\mathbb{D})$ and in the Hardy space, we have the following Shift:

$$M_z : H^2(\mathbb{D}) \to H^2(\mathbb{D}) , \quad (M_z f)(z) = zf(z)$$

In particular, we know:

Theorem 2.1.1. *The operator M_z on $H^2(\mathbb{D})$ is unitarily equivalent to the operator S over ℓ^2.*

Proof. If $T : \ell^2 \to H^2(\mathbb{D})$ is the unitary operator given by:

$$T(a_0, a_1, a_2, \dots) = \sum_{n=0}^{\infty} a_n z^n. \tag{2.2}$$

So it suffices to check that $TS = M_z T$. By simple computation we get that $(TS)(a_n)_{n\in\mathbb{N}_0} = \sum_{n=0}^{\infty} a_n z^{n+1}$, and in the other hand $(M_z T)(a_n)_{n\in\mathbb{N}_0} = \sum_{n=0}^{\infty} a_n z^{n+1}$ for any $(a_n)_{n\in\mathbb{N}_0} \in \ell^2$, thus the result follows. $\square$

So, we have an equivalence of characterizing the shift-invariant subspaces on ℓ^2 with the shift-invariant subspaces on $H^2(\mathbb{D})$. In (1), we have a complete characterization of the non-trivial shift-invariant subspaces as follows:

Lemma 2.1.1. *(26, Theorem 3.14) Let $\theta \in H^\infty(\mathbb{D})$ be a inner function and define $H^2(\mathbb{D}) \ominus \theta H^2(\mathbb{D}) = H^2(\mathbb{D}) \cap (\theta H^2(\mathbb{D}))^\perp$. Then $\dim(H^2(\mathbb{D}) \ominus \theta H^2(\mathbb{D})) = n$ if and only if θ is of the form*

$$\theta(z) = \prod_{j=1}^{n} \frac{\lambda_j - z}{1 - \overline{\lambda_j} z}$$

for $\lambda_1, \ldots, \lambda_n \in \mathbb{D}$. In other words, we have finite dimension if and only if θ is a finite Blaschke product.

Now we present a simple proof for Beurling's theorem made in (27):

Theorem 2.1.2. *(Beurling) Let M be a closed nontrivial subspace of $H^2(\mathbb{D})$. Then M is M_z-invariant if and only if $M = \phi H^2(\mathbb{D})$, where ϕ is an inner function in $H^2(\mathbb{D})$.*

Proof. Let $M \subset H^2(\mathbb{D})$ be a M_z-invariant subspace then we can define a reproducing kernel to M in the following manner:

$$k_\lambda^M(z) = P_M k_\lambda(z) \tag{2.3}$$

where $\lambda \in \mathbb{D}$ and P_M is the orthogonal projection from $H^2(\mathbb{D})$ onto M. We get that $(1 - \overline{\lambda} z) k_\lambda^M(z)$ is the reproducing kernel of $M \ominus zM$ (See (28, Section 3.2)) and by Lemma 2.1.1 we get that $\dim(M \ominus zM) = 1$, therefore we get that

$$(1 - \overline{\lambda} z) k_\lambda^M(z) = \overline{\phi(\lambda)} \phi(z) \quad (\lambda, z \in \mathbb{D}) \tag{2.4}$$

for a function $\phi \in M \ominus zM$ with $\|\phi\| = 1$ and hence

$$|\phi(\lambda)|^2 = (1 - |\lambda|^2) k_\lambda^M(\lambda) = (1 - |\lambda|^2) \|P_M k_\lambda\|^2 \leqslant 1$$

thus $\phi \in H^\infty(\mathbb{D})$ and note that by (2.4) we know that

$$P_M k_\lambda(z) = \frac{\overline{\phi(\lambda)} \phi(z)}{1 - \overline{\lambda} z} = T_\phi T_\phi^* k_\lambda(z)$$

where T_ϕ is the analytic Toeplitz operator with symbol ϕ over $H^2(\mathbb{D})$. Note that $\{k_\lambda : \lambda \in \mathbb{D}\}$ is a total set for $H^2(\mathbb{D})$, so $P_M = T_\phi T_\phi^*$ and then

$$T_\phi T_\phi^* T_\phi T_\phi^* = T_\phi T_{|\phi|^2} T_\phi^* = T_\phi T_\phi^*$$

because since T_ϕ is analytic we obtain $\ker(T_\phi) = 0$ which implies that $T_\phi^* T_\phi = I$ that is equivalent to $|\phi(e^{i\theta})| = 1$ almost everywhere on $\mathbb{T}$, that is, ϕ is a inner function. The result follows from the fact that $T_\phi T_\phi^*$ is the projection from $H^2(\mathbb{D})$ onto $\phi H^2(\mathbb{D})$. The converse is straightforward, we know that $M = \phi H^2(\mathbb{D})$ is always closed when ϕ is a inner function and is easy to see that

$$zM = z\phi H^2(\mathbb{D}) = \phi(zH^2(\mathbb{D})) \subset \phi H^2(\mathbb{D}) = M$$

□

So, we finally got an answer for all the invariant subspaces of the shift, but we do not have a general characterization for other operators over ℓ^2. But note that in the present work, we are interested in the Hardy space of the polydisk and one might ask if Beurling's theorem has a natural extension to the polydisk, that is, every invariant subspace of $H^2(\mathbb{D}^n)$ is of the form $\phi H^2(\mathbb{D}^n)$ for some inner function ϕ. The answer here is no, and we have the following example.

Let $[z_1 - z_2] = \overline{(z_1 - z_2)H^2(\mathbb{D}^2)}$. It is widely known in the literature, see (29) for instance, that $[z_1 - z_2]$ is a invariant subspace of $H^2(\mathbb{D}^2)$ not generated by a inner function. Let z_j, $j = 1, 2$ be the action of a shift, let us prove that $(z_1 - z_2)H^2(\mathbb{D}^2)$ is invariant.

$$z_j\left[(z_1 - z_2)H^2(\mathbb{D}^2)\right] \subset z_1(z_jH^2(\mathbb{D}^2)) - z_2(z_jH^2(\mathbb{D}^2))$$

It is clear that $H^2(\mathbb{D}^2)$ is trivially invariant by z_j, $j = 1, 2$ hence

$$z_j\left[(z_1 - z_2)H^2(\mathbb{D}^2)\right] \subset (z_1 - z_2)H^2(\mathbb{D}^2)$$

We have that $(z_1 - z_2)H^2(\mathbb{D}^2)$ is invariant, thus $[z_1 - z_2]$ is also invariant, but note also that $[z_1 - z_2]$ is not of the form $\phi H^2(\mathbb{D}^2)$ for any inner function ϕ. Note that we needed the set to be closed because, according to our definition, invariant subspaces are always closed.

But this does not mean we do not have a Beurling-type theorem for the polydisk. This comes from the fact that $H^2(\mathbb{D}^2)$, or even $H^2(\mathbb{D}^n)$ can be seen as vector-valued Hardy-Hilbert space as follows:

Let $\mathcal{H}$ be Hilbert space, we may define $H^2(\mathcal{H})$ as the space of all sequences $f = (h_0, h_1, \dots)$ of elements in $\mathcal{H}$ of which $\sum_{n=0}^{\infty} \|h_n\|^2 < \infty$ or equivalently, we can see as space of analytic functions given as:

$$H^2(\mathcal{H}) = \{f(z) = \sum_{n=0}^{\infty} h_n z^n : \text{f is analytic and } (h_n)_{n\in\mathbb{N}_0} \text{ is square-summable}\} \tag{2.5}$$

Now defining, this space as such, we obtain that $H^2(\mathcal{H})$ is a space of square-integrable functions on $\mathbb{T}$ valued at $\mathcal{H}$ and via Poisson integral, we may extend f to the disk, and thus we get the following property:

$$\sup_{0<r<1} \int_{\mathbb{T}} \|f(re^{i\theta})\|^2 d\mu \leqslant \infty$$

where μ is the normalized Lebesgue measure in $\mathbb{T}$, and hence we have a similar identification in $H^2(\mathcal{H})$ as the one in $H^2(\mathbb{D})$, i.e., we may identify a function defined in the disk to a function defined in the boundary circle of the disk. Moreover, we may identify the shift operator in $H^2(\mathcal{H})$ with the multiplication by z (M_z). We refer to (21, 30) for an introduction to vector-valued functions.

Naturally, we may define $H^2(\mathbb{D}^2)$ as $H^2(H^2(\mathbb{D}))$. In particular, the Hardy space over any polydisk can be written recursively as $H^2(H^2(\mathbb{D}^{n-1}))$. Now, the last few ingredients before Beurling's theorem for vector-valued Hardy space and the next results are found on (25).

Definition 2.1.1. *Let $\mathcal{H}$ be a Hilbert space and $T \in B(\mathcal{H})$. We say that $A \subset \mathcal{H}$ is a wandering subspace for T if $A \perp T^n A$ for all $n \in \mathbb{N}$.*

Note that for a fixed operator T and a wandering subspace A, we may associate an invariant subspace M in the following manner. Define

$$M = \bigcap_{n=0}^{\infty} T^n A.$$

Now, is easy to see that $TM = \bigcap_{n=1}^{\infty} T^n A$ and therefore $TM \perp A$, so A is the orthogonal complement of TM inside M, in other words $A = M \ominus TM$.

Lemma 2.1.2. *If T is an isometry on a Hilbert space $\mathcal{H}$ and $R = (T\mathcal{H})^{\perp}$, then $\left(\bigcap_{n=0}^{\infty} T^n R\right)^{\perp} = \bigcap_{n=0}^{\infty} T^n \mathcal{H}$.*

Proof. Suppose that $f \in (\cap_{n=0}^{\infty} T^n R)^{\perp}$, we claim that $f \in T^n\mathcal{H}$ for all $n \in \mathbb{N}_0$. In fact, for $n = 0$ we get that if $f \in \left(\bigcap_{n=0}^{\infty} T^n R\right)^{\perp}$, we get that $f \in \mathcal{H}$ because $\left(\bigcap_{n=0}^{\infty} T^n R\right)^{\perp} \subseteq \mathcal{H}$. Now if $f \in T^n\mathcal{H}$, then there exists $g \in \mathcal{H}$ such that $f = T^n g$. Since, $f \in (T^n R)^{\perp}$, we get that $T^n g \perp T^n R$ and since T is an isometry follows that $\langle T^n g, T^n h\rangle = \langle g, h\rangle$ for any $h \in R$,i.e., $g \perp R$ and thus $g \in T\mathcal{H}$ and finally $f = T^n g \in T^{n+1}\mathcal{H}$, so we get $\left(\bigcap_{n=0}^{\infty} T^n R\right)^{\perp} \subset \bigcap_{n=0}^{\infty} T^n \mathcal{H}$. Conversely, note that if $f \in M^{\perp}$ so $Tf \in TM^{\perp}$ and if $g \in M$, then $Tg \in TM$, so $Tf \perp Tg$ and in particular, $f \perp g$ and therefore $TM^{\perp} \subset (TM)^{\perp}$. With that in mind, we get:

$$T^{n+1}\mathcal{H} = T^n(T\mathcal{H}) = T^n R^{\perp} \subset (T^n R)^{\perp}$$

and the result follows. □

Note that the previous lemma implies that if T is an isometry on a Hilbert space $\mathcal{H}$ and $R = (T\mathcal{H})^{\perp}$, then $\bigcap_{n=0}^{\infty} T^n R$ is a reducing subspace for T and of course this implies that $\bigcap_{n=0}^{\infty} T^n R$ and its orthogonal complement are invariant.

Theorem 2.1.3. *Let M be a shift invariant subspace of $H^2(\mathcal{H})$, then there exists a wandering subspace A such that $M = \bigcap_{n=0}^{\infty} M_z^n A$, where M_z is the multiplication by z on $H^2(\mathcal{H})$.*

Proof. Write $A = M \cap (M_z M)^{\perp}$. We claim $M \cap \left(\bigcap_{n=0}^{\infty} M_z^n A\right)^{\perp}$ is reducing for M_z. By Lemma 2.1.2, $M \cap \left(\bigcap_{n=0}^{\infty} M_z^n A\right)^{\perp}$ is equal to $\bigcap_{n=0}^{\infty} M_z^n M$ and of course this subspace is invariant by M_z. In particular, $\left(\bigcap_{n=0}^{\infty} M_z^n M\right)^{\perp}$ must also be M_z-invariant and one can see that

$$M_z A = M_z M \cap M_z (M_z M)^{\perp} \subset M_z M \cap (M_z^2 M)^{\perp} \subset M \cap M^{\perp} = \{0\}$$

since M is invariant by M_z, any reducing subspace must be the zero subspace that implies the theorem. □

So, we produced a wandering space with our invariant subspace, which will be used later in this section, and here we present a technical lemma.

Lemma 2.1.3. *If $f, g \in H^2(\mathcal{H})$ and if $\langle M_z^n f, g\rangle = 0$ for all $n \in \mathbb{N}$, then $\langle f, g\rangle$ is constant almost everywhere.*

Proof. The hypothesis is equivalent to $\int_{\mathbb{T}} z^n \langle f(z), g(z)\rangle dz = 0$ for all $n \in \mathbb{N}$, which means that the inner product on the integrand must have its Fourier coefficients equal to zero with possible exception the constant term; therefore we get the result. □

Lemma 2.1.4. *If A is a wandering subspace for M_z, then $\dim A \leqslant \dim \mathcal{H}$.*

Proof. Let $\{f_i\}_{i\in J}$ be orthonormal basis for A, where J is some index family. Since $H^2(\mathcal{H})$ is separable, J is countable. Now, since A is a wandering subspace, $\langle M_z^n f_i, f_j\rangle \neq 0$ for all $n \in \mathbb{N}$. By Lemma 2.1.3, we get that $\langle f_i, f_j\rangle$ is constant almost everywhere and we get that $\langle f_i, f_j\rangle = \delta_{ij}$, where δ_{ij} is the Kronecker's delta, for almost every z because

$$\int_{\mathbb{T}} \langle f_i(z), f_j(z)\rangle dz = \langle f_i, f_j\rangle = \delta_{ij}$$

Finally, up to a measure zero set, there exists at least one z such that $\langle f_i(z), f_j(z)\rangle = \delta_{ij}$. This implies that $\mathcal{H}$ must contain an orthonormal set with at least the same cardinality of $\{f_i\}_{i\in J}$, and the proof is complete. □

Definition 2.1.2. *Let $\mathcal{H}$ be a separable Hilbert space and let $U \in B(\mathcal{H})$, we define the inflation operator $\hat{U}$ on $H^2(\mathcal{H})$ as follows:*

$$\hat{U} : H^2(\mathcal{H}) \to H^2(\mathcal{H}), \quad \hat{U}(\sum_{n=0}^{\infty} h_n z^n) = (\sum_{n=0}^{\infty} (U h_n) z^n)$$

The map $U \mapsto \hat{U}$ preserves linear operations, products, adjoints, and norms. In particular, this map is an embedding of $B(\mathcal{H})$ on $B(H^2(\mathcal{H}))$.

If we consider an operator-valued function $F : \mathbb{D} \to B(\mathcal{H})$, we may define generalized inflation $\hat{F}$ similarly to operators as above, which still preserves linear operations, products, adjoints, and norms. Still, in this case, it is an embedding of the bounded measurable $\mathcal{H}$-valued operator algebra into $B(H^2(\mathcal{H}))$.

Lemma 2.1.5. *Let F be an operator-valued function on $\mathcal{H}$ and let W be the subspace of constant functions on $H^2(\mathcal{H})$. If $F(z)$ is an isometry on $\mathcal{H}$ for almost every z, then $\hat{F}W$ is a wandering subspace for M_z.*

Proof. If we take $f(z) = F(z)u$ and $g(z) = F(z)v$ for some $u, v \in \mathcal{H}$ we are considering $f, g \in \hat{F}W$ since we may consider the inflations $\hat{F}u$ and $\hat{F}v$ as $F(z)u$ and $F(z)v$ respectively by Definition 2.1.2 and u, v are considered constants as on $H^2(\mathcal{H})$, then

$$\langle M_z^n f, g\rangle = \int_{\mathbb{T}} \langle z^n F(z)u, F(z)v\rangle dz = \int_{\mathbb{T}} z^n \langle F(z)u, F(z)v\rangle dz = \langle u, v\rangle \int_{\mathbb{T}} z^n dz = 0$$

for all $n \in \mathbb{N}$ so we proved that $\hat{F}W \perp M_z^n \hat{F}W$ and thus $\hat{F}W$ is a wandering subspace for M_z. □

We will use this definition of W throughout this section.

Definition 2.1.3. *Let $\mathcal{H}$ be a separable Hilbert space and $F \in B(\mathcal{H})$. We say that F is rigid if a subspace $V \subset \mathcal{H}$ exists, such that F is almost everywhere an isometry on V and null on $V^\perp$. In particular, we say that F is a rigid Taylor function if F is rigid and $\hat{F}W \subset H^2(\mathcal{H})$.*

From this definition and the previous lemma, we get that if F is a rigid Taylor function, then $\hat{F}W$ is a wandering subspace of M_z and $\hat{F}H^2(\mathcal{H})$ is an M_z-invariant subspace.

Lemma 2.1.6. *If A is a wandering subspace for M_z, then a rigid Taylor function exists such as $A = \hat{F}W$.*

Proof. By Lemma 2.1.4, we have $\dim A \leqslant \dim \mathcal{H}$, then there exists a subspace V such that $\dim V = \dim A$, consider $T : \mathcal{H} \to A$ such that T maps isometrically V to A and $V^\perp \subset \ker(T)$. Define $F(z)v = (Tv)(z)$ for every $v \in \mathcal{H}$, then F is a measurable operator-valued function. Now, by definition of T, we get that $Tv \in A$, so $\langle M_z Tv, Tv\rangle = 0$ for all $n \in \mathbb{N}$ since A is a wandering subspace by hypothesis. So Lemma 2.1.3 implies that $\|F(z)v\|$ is constant almost everywhere. Thus, F is bounded. Let $v \in V$, then

$$\|v\|^2 = \|Tv\|^2 = \int_{\mathbb{T}} \|(Tv)(z)\|^2 dz = \int_{\mathbb{T}} \|F(z)v\|^2 dz = \|F(z)v\|$$

so $\|F(z)v\| = \|v\|$ almost everywhere, hence F is an isometry on V. Analogously, if $v \in V^\perp$, $Tv = 0$ and therefore $F(z)$ annihilates $V^\perp$ which proves that F is a partial isometry on V so F is rigid and by construction of F we get that $\hat{F}W \subset H^2(\mathcal{H})$, since if $f \in W$, then $f(z) = w$ almost everywhere for some $w \in \mathcal{H}$ and we obtain

$$(\hat{F}f)(z) = F(z)v = (Tv)(z)$$

so $\hat{F}W \subset A$. Conversely, let $g \in A$, such that $g = Tv$ for some $v \in V$ and if $f(z) = v$ for all z, then $f \in W$ and $g(z) = (Tv)(z) = (\hat{F}f)(z)$ and then $A \subset \hat{F}W$ and proof is finished. □

Now, we show Beurling's theorem for vector-valued Hardy-Hilbert spaces.

Theorem 2.1.4. *If M is a shift-invariant subspace of $H^2(\mathcal{H})$, then there exists a rigid Taylor function F such that $M = \hat{F}H^2(\mathcal{H})$.*

Proof. By Theorem 2.1.3, there exists a wandering subspace A such that $M = \bigcap_{n=0}^{\infty} M_z^n A$, now by Lemma 2.1.6, $A = \hat{F}W$ for F a rigid Taylor function. Note that:

$$M = \bigcap_{n=0}^{\infty} M_z^n(\hat{F}W) = \hat{F} \bigcap_{n=0}^{\infty} M_z^n W = \hat{F}H^2(\mathcal{H})$$

and the proof is finished. □

We can summarize and restate Beurling's theorem as:

Theorem 2.1.5. *Let $\mathcal{H}$ be a Hilbert space, and let M be a closed non-trivial shift invariant subspace of $H^2(\mathcal{H})$. Then there exists a Hilbert space K and a function Ψ such that:*

1. *Ψ is an analytic function in the unit disk with values in $B(K, \mathcal{H})$.*

2. *If $z \in \mathbb{D}$ then $\|\Psi(z)\| \leqslant 1$ and $\|\Psi(e^{i\theta})\| = 1$ almost everywhere.*

3. *$M = \Psi H^2(K)$, in other words, M consists of the functions g such that*

$$g(z) = \Psi(z)f(z)$$

Where $f \in H^2(K)$.

So we have a possible characterization for all invariant subspaces of $H^2(\mathbb{D}^n)$; the problem now is to find a concrete characterization of the spaces K and operator valued function Ψ of which the theorem will be valid and the ISP will be solved. Note that this is not such a trivial task and is an open problem nowadays.

3 Some recent results in universality in Toeplitz operators

In this chapter, we will introduce our main approach to the ISP: investigating the properties of universal Toeplitz operators in order to fully characterize their invariant subspaces. However, a full characterization of all invariant subspaces has not been possible so far. In the first section, we will discuss some results regarding the disk case, including examples of symbols, such as the induced Toeplitz operator being universal. Many results can be seen in (12, 31, 32, 33). In the following section, we will present the primary goal of this work, which is universality among Toeplitz operators over the polydisk with the most recent results in the area that can be found on (13, 34, 35).

3.1 Universality in Toeplitz operators over the disk

The idea of using universality simplifies the argument towards a possible proof of the ISP once we only need to study the invariant subspaces of said operator. In this section, we will study some classes of symbols that induce a universal Toeplitz operator in the disk case.

Lemma 3.1.1. *If $\phi \in H^\infty(\mathbb{D})$ is inner, then $\ker T_\phi^* = (\phi H^2(\mathbb{D}))^\perp$.*

Proof. Note that $\operatorname{Ran}(T_\phi) = \phi H^2(\mathbb{D})$ which is a closed subspace of $H^2(\mathbb{D})$, then by Theorem A.0.4, we get that $(\ker T_\phi^*)^\perp = \operatorname{Ran}(T_\phi)$ since both sides are closed, we get that $\ker T_\phi^* = (\operatorname{Ran}(T_\phi))^\perp = (\phi H^2(\mathbb{D}))^\perp$.

□

Lemma 3.1.2. *If $\phi \in H^\infty(\mathbb{D})$ is inner, then $T_z^{*n}\phi \in (\phi H^2(\mathbb{D}))^\perp$ for all $n \in \mathbb{N}$.*

Proof. We will proceed by induction. For $n = 1$, we know that $f \in (\phi H^2(\mathbb{D}))^\perp$ if and only if $\langle \phi g, f\rangle = 0$ for every $g \in H^2(\mathbb{D})$. So we want to prove that $\langle T_z^*\phi, \phi g\rangle = 0$ for every $g \in H^2(\mathbb{D})$. Thus,

$$\begin{aligned}\langle T_z^*\phi, \phi g\rangle = \langle \phi, T_z\phi g\rangle = \langle \phi, z\phi g\rangle = \int_{\mathbb{T}} \overline{zg}\, d\mu &= \langle 1, zg\rangle \\ &= \langle 1, T_z g\rangle \\ &= \langle T_z^* 1, g\rangle \\ &= \langle T_z^* k_0, g\rangle \\ &= \langle 0, g\rangle \\ &= 0\end{aligned}$$

Now, suppose that $T_z^{*(n-1)}\phi \in (\phi H^2(\mathbb{D}))^\perp$ for $n-1$, we want to show that $T_z^{*n}\phi \in (\phi H^2(\mathbb{D}))^\perp$, in particular,

$$\begin{aligned}\langle T_z^{*n}\phi, \phi g\rangle = \langle T_z^* T_z^{*(n-1)}\phi, \phi g\rangle &= \langle T_z^{*(n-1)}\phi, T_z\phi g\rangle \\ &= \langle T_z^{*(n-1)}\phi, \phi z g\rangle \\ &= 0\end{aligned}$$

Since by hypothesis $T_z^{*(n-1)}\phi \in (\phi H^2(\mathbb{D}))^\perp$ and $\phi z g \in \phi H^2(\mathbb{D})$ and therefore the proof is finished. □

Theorem 3.1.1. *If $\phi \in H^\infty$ is inner and not a finite Blaschke product, then T_ϕ^* is universal.*

Proof. The idea is to use the Caradus criteria to prove that T_ϕ^* is universal. Note that for any $g \in H^2(\mathbb{D})$, then we know that $T_\phi^*\phi g = P(\overline{\phi}\phi g) = P(|\phi| g) = g$, since $|\phi| = 1$ almost everywhere on $\mathbb{T}$, and thus T_ϕ^* is surjective. We need to prove that $\ker T_\phi^*$ is infinite-dimensional. By the previous lemma, we know that $T_z^{*n} g \in (\phi H^2(\mathbb{D}))^\perp$ for all $n \in \mathbb{N}$, we claim that the set $\{T_z^{*n}\phi : n \in \mathbb{N}\}$ is linearly independent. In fact, for a fixed n suppose that exists constants $c_1, \ldots, c_n$ such that

$$\sum_{j=1}^{n} c_j T_z^{*j}\phi = 0$$

Or we may rewrite the previous relation as $T_p^*\phi = 0$, where $p(z) = \overline{c_1}z + \cdots + \overline{c_n}z^n$, and therefore $\phi \in \ker T_p^*$. By straightforward calculation, we get that $\ker T_p^*$ consists only of a class of rational functions of the form $\dfrac{1}{(1-\overline{\lambda_j}z)^k}$, where λ_j are the zeros of p and k varies over the multiplicty of said zero, so by hypothesis ϕ is not a rational function, follows that $c_j = 0$ for all $1 \leqslant j \leqslant n$ and hence $\{T_z^{*j}\phi : 1 \leqslant j \leqslant n\}$ is linearly independent for every $n \in \mathbb{N}$, thus $\ker T_z^*$ is infinite-dimensional. By the Caradus criteria, T_z^* is universal for $H^2(\mathbb{D})$. □

Recently, (31, 32, 33), observed some interesting results for the disk case as follows:

Lemma 3.1.3. *If $\phi \in H^\infty(\mathbb{D})$ and there exists $\ell > 0$ such that $|\phi(e^{i\theta})| \geqslant \ell$ almost everywhere on $\mathbb{T}$, then $1/\phi \in L^\infty(\mathbb{T})$ and the Toeplitz operator $T_{1/\phi}$ is the left-inverse of T_ϕ.*

Proof. Naturally, remember that if $\psi_1 \in L^\infty(\mathbb{T})$ and $\psi_2 \in H^\infty(\mathbb{D})$, we have that for any $f \in H^2(\mathbb{D})$

$$T_{\psi_1}T_{\psi_2}f = T_{\psi_1}\psi_2 f = P(\psi_1\psi_2 f) = T_{\psi_1\psi_2}f$$

Since ϕ by hypothesis is bounded away from zero, i.e., there exists a constant $\ell > 0$ such that $|\phi(e^{i\theta})| \geqslant \ell > 0$ then follows that $1/\phi \in L^\infty(\mathbb{T})$. Now, since $1/\phi \in L^\infty(\mathbb{T})$ and $\phi \in H^\infty(\mathbb{D})$, we get that

$$T_{1/\phi}T_\phi = T_{1/\phi\phi} = T_1 = I$$

Where I is the identity operator in $H^2(\mathbb{D})$. □

Corollary 3.1.1.1. *In the hypothesis of the previous lemma, the Toeplitz operator T_ϕ^* is surjective.*

Proof. We know that $T_{1/\phi}T_\phi = I$, but note that

$$I = I^* = (T_{1/\phi}T_\phi)^* = T_\phi^* T_{1/\phi}^*$$

thus, the result follows. □

Now, we want to define a "simple" symbol that gives us a universal symbol for the disk case. Let $\Omega = \{z \in \mathbb{C} : \operatorname{Im} z^2 > -1 \quad \text{and} \quad \operatorname{Re} z < 0\}$. Define the following map:

$$\sigma : \mathbb{D} \to \Omega \quad z \mapsto \frac{-1+i}{\sqrt{z+1}}$$

Note that σ is a conformal map between $\mathbb{D}$ and Ω (see (32) for a complete argument of this). Now, define

$$\phi(z) = e^{\sigma(z)} - e^{\sigma(0)} = e^{\sigma(z)} - e^{-1+i} \tag{3.1}$$

where we choose the branch of the square root to be the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ satisfying $\sqrt{1} = 1$. Also, in (32) was shown that there exists an $\ell > 0$ such that $|\phi(e^{i\theta})| \geqslant \ell$ almost everywhere on $\mathbb{T}$ so we are going to make use of Lemma 3.1.3 and Corollary 3.1.1.1 to prove that T_ϕ is universal for $H^2(\mathbb{D})$.

Theorem 3.1.2. *Let ϕ be as defined in* (3.1)*, then T_ϕ is universal for $H^2(\mathbb{D})$.*

Proof. Since $1/\phi \in L^\infty(\mathbb{T})$, by Corollary 3.1.1.1, we get that T_ϕ^* is surjective, we wish to prove that T_ϕ^* has an infinite dimensional kernel. Therefore, the result follows from the Caradus criteria. Now, let $\omega_n = -1 + i + 2\pi n i$ and note that $(\omega_n)_{n\in\mathbb{N}} \subset \Omega$. Since, σ is a conformal map, let $z_n = \sigma^{-1}(\omega_n)$ and consider K_{z_n} the reproducing kernel at z_n, then

$$T_\phi^* K_{z_n} = \overline{\phi(z_n)} K_{z_n} = \overline{(e^{-1+i+2\pi n i} - e^{-1+i})} K_{z_n} = 0$$

But note that $(K_{z_n})_{n\in\mathbb{N}}$ is linearly independent set contained in $\ker T_\phi^*$, hence $\ker T_\phi^*$ is infinite-dimensional, and the result follows from the Caradus criteria. □

So we have a concrete example of a universal operator over $H^2(\mathbb{D})$, but note that the universal operators over the disk are somewhat complicated; thus, our interest goes over to the polydisk, where we can find more easily universal symbols although the space structure can be a bit more complicated given that the function theory in several complex variables can be quite different from the one variable.

3.2 Universality in Toeplitz operators over the polydisk

We have reached our goal in this current presentation; in this section, we will present the main results for the polydisk. The main reason to use this space is that we have concrete examples of universality; in particular, we have that $T_{z_1}, \ldots, T_{z_n}$ are shifts (in the same sense of the disk case) over the polydisk and the prime example of universal operators are $T^*_{z_1}, \ldots, T^*_{z_n}$, in fact we will show that these operators are universal. Moreover, the Hardy space over the polydisk enables the most profound result in this section, the Ahern and Clark theorem, crucial to several fresh results in this area.

For simplicity, consider the case of the bidisk, $H^2(\mathbb{D}^2)$. We may write $H^2(\mathbb{D}^2) = H^2(H^2(\mathbb{D}))$, so we may write the elements of $H^2(\mathbb{D}^2)$ as follows:

$$H^2(\mathbb{D}^2) = \left\{ f(z,w) = \sum_{n=0}^{\infty} h_n(z) w^n : (h_n)_{n\in\mathbb{N}_0} \in H^2(\mathbb{D}) \quad \text{and} \quad \sum_{n=0}^{\infty} \|h_n(z)\|^2 < \infty \right\}$$

and we define $T_w f(z,w) = wf(z,w)$, an analytic Toeplitz operator, and we get that for some $g(z,w) \in H^2(\mathbb{D}^2)$

$$T^*_w wg(z,w) = P(\overline{w}wg(z,w)) = g(z,w)$$

where P is the orthogonal projection from $L^2(\mathbb{T}^2)$ onto $H^2(\mathbb{D}^2)$ and noting that $\overline{w}w = 1$ almost everywhere on $\mathbb{T}^2$, hence $\overline{w}wg(z,w) \in H^2(\mathbb{D}^n)$, thus T^*_w is surjective. We claim that T^*_w has infinite dimensional kernel. Note that $H^2(\mathbb{D}) \subset \ker T^*_w$ since if $f \in H^2(\mathbb{D})$, in the variable z:

$$T^*_w f = P(\overline{w}f) = 0.$$

We may also write the adjoint as:

$$T^*_w f(z,w) = \frac{f(z,w) - f(z,0)}{w - 0}.$$

hence, by the Caradus criteria, T^*_w is universal for $H^2(\mathbb{D}^2)$. And analogously, we get that T^*_z is also universal for $H^2(\mathbb{D}^2)$, and we may extend this argument naturally to the polydisk via vector-valued factorization of the Hardy space.

Remember that in the one variable case, we defined a subspace M as shift invariant if $zM \subset M$, which was our object of interest. Now, over the polydisk, we also want to study shift-invariant subspaces and our definition here generalizes the previous one as follows.

Definition 3.2.1. *Let $M \subset H^2(\mathbb{D}^n)$ be subspace, we say M is a shift invariant subspace if M is invariant under z_i, i.e., $z_i M \subset M$ for all $i = 1, \ldots, n$.*

Proposition 3.2.1. *The spectrum $\sigma(T^*_{z_j}) = \overline{\mathbb{D}}$ for all $j = 1, ..., n$.*

Proof. We know from basic functional analysis that $\|T^*_{z_j}\| = \|T_{z_j}\|$ and in particular that the spectral radii $r(T^*_{z_j}) \leqslant \|T^*_{z_j}\|$. Now, we know that $\|T_{z_j}\| = \|z_j\|_\infty = 1$, hence $\sigma(T^*_{z_j}) \subset \overline{\mathbb{D}}$.

It is easy to see that $\sigma_p(T^*_{z_j}) = \mathbb{D}$, but note that the spectrum must be a non-empty closed subset of $\mathbb{C}$ and in this case is also contained in $\overline{\mathbb{D}}$. One can see that the smallest closed set that contains $\mathbb{D}$ is $\overline{\mathbb{D}}$. Thus, the proposition follows.

□

Our interest invariant subspaces are the shift-invariant subspaces, and by Corollary 1.5.1.1, the ISP in $H^2(\mathbb{D}^n)$ resumes to show that every minimal invariant subspace of $H^2(\mathbb{D}^n)$ is one-dimensional or even, that all maximal invariant subspaces have codimension one, here minimality and maximality are in the sense of a lattice of invariant subspaces, so we only need to study the invariant subspaces of one universal operator to solve the ISP. Still, it has not been possible so far. Here, we will present the approach via Toeplitz operators of the polydisk using universality. At first glance, Toeplitz operators may look reasonably simple but still do not have a complete characterization of Toeplitz operator that is universal for $H^2(\mathbb{D}^n)$, in general, the adjoint of an analytic Toeplitz operator has dense range by Proposition 1.4.6 and since the Caradus criteria is the tool used to verify if an operator is universal, we get that our operator must be closed. However, we know examples of universal operators that do not have closed range; for instance, take $\mathcal{H}$ a Hilbert space and let $U \in B(\mathcal{H})$ be a universal operator and take $T \in B(\mathcal{H})$ to be an arbitrary operator without closed range, then $U \oplus T$ on $\mathcal{H} \oplus \mathcal{H}$ is universal but does not have closed range. Although the Caradus criterion gives us sufficient conditions for an operator to be universal, as seen before, only the infinite-dimensional kernel hypothesis is necessary; therefore, we do not have all the required conditions such that Toeplitz operators are universal in the sense of Rota, but we will show some results that imply universality for Toeplitz operators.

Now, our goal is to show the Ahern and Clark theorem, which is one of the most important results in this section, that gives a connection between finite codimension and invariant subspaces of $H^2(\mathbb{D}^n)$. Still, for this, we need the following results:

Lemma 3.2.1. *If M is a invariant subspace of $H^2(\mathbb{D}^n)$ with finite codimension, then M is contained in invariant subspace M_1 such that* $\dim(M_1/M) = 1$.

Proof. Note that $H^2(\mathbb{D}^n) = M \oplus N$, for some finite dimensional subspace N of $H^2(\mathbb{D}^n)$. Now note that every $p \in \mathbb{C}[z_1, ..., z_n]$ determines linear maps $H_p : N \to M$ and $h_p : N \to N$, which are just multiplication operators with respect to the polynomial p, such that $pg = H_p g + h_p g$, if $g \in N$. In particular, we may write:

$$\begin{aligned} pqg = p(H_q g + h_q g) &= H_p H_q g + h_p H_q g + h_p H_q g + h_p h_q g \\ &= (H_p H_q g + h_p H_q g) + (H_p h_q g) + h_p h_q g \\ &= (p H_q g + H_p h_q g) + h_q h_q g \end{aligned}$$

where $pH_q g + H_p h_q g \in M$ since by hypothesis M is invariant, also note that since $pq \in \mathbb{C}[z_1 \ldots, z_n]$ we get that

$$pqg = H_{pq}g + h_{pq}g \tag{3.2}$$

where $H_{pq}g \in M$ and $h_{pq}g \in N$ and by comparison with the previous characterization we obtain that $h_{pq} = h_p h_q$ since $H^2(\mathbb{D}^n) = M \oplus N$ by hypothesis. In particular, we know that $h_{pq} = h_{qp}$ and thus $\{h_p\}$ is a collection of commuting linear operators on N. Since $\{h_p\}$ is a commuting collection and N is finite-dimensional, there exists an $f \in N$ that is a common eigenvector for all h_p. Let λ_p be the eigenvalue of f, then $pf - \lambda_p f \in M$ for every $p \in \mathbb{C}[z_1, ..., z_n]$ because:

$$pf - \lambda_p f = H_p f + h_p f - \lambda_p f = H_p f$$

Let $M_1 = M \oplus [f]$, where $[f]$ is the subspace generated by f and thus M_1 has the desired properties, since $pf \in M_1$ for every $p \in \mathbb{C}[z_1, ..., z_n]$.

□

Theorem 3.2.1. *Suppose M is an invariant subspace of $H^2(\mathbb{D}^n)$, with codimension $k < \infty$ and let $R = \mathbb{C}[z_1, ..., z_n]$ be the ring of complex polynomials in n variables. Then $R \cap M$ is an ideal in R such that:*

1. *$R \cap M$ is dense in M.*
2. *$\dim(R/(R \cap M)) = k$.*
3. *The set of common zeros in $\mathbb{C}^n$ of the members of $R \cap M$ is finite and lies in $\mathbb{D}^n$*

Conversely, if I is an ideal in R whose common zeros form a finite subset V_I of $\mathbb{D}^n$, and if M is the closure of I in $H^2(\mathbb{D}^n)$, then M is a finite codimensional invariant subspace of $H^2(\mathbb{D}^n)$ and $I = R \cap M$.

Proof. It is easy to see that $M \cap R$ is an ideal since if $f \in M \cap R$, for every $p \in R$, we get that $f \cdot p \in R$ because $f \cdot p$ is again a polynomial and since M is invariant, $f \cdot p$ belongs to M, hence $f \cdot p \in M \cap R$.

Suppose that $k = 0$, follows that $M = H^2(\mathbb{D}^n)$ and therefore $R \cap M = R$. We will proceed by induction on k. Assume that $k > 0$ and that the first part of the theorem holds for all invariant subspaces with codimension $\leqslant k - 1$; for this we may choose M_1 as in Lemma 3.2.1 so that $M \subset M_1$ and $\dim(M_1/M) = 1$ and by our induction hypothesis follows that the conditions 1, 2 and 3 hold for M_1 since M_1 has codimension $k - 1$.

Now, since M is closed and condition 1 holds for M_1, there exists a $q \in R \cap M_1$ such that $q \notin M$ and, by Proposition A.0.1 there exists a linear functional φ such that $\varphi(q) = 1$ and $\varphi|_M = 0$. If $p \in R \cap M_1$, then $p - \varphi(p)q \in M$ since if $p \in M$ then $\varphi(p) = 0$ and if $p \notin M$ the preceding expression will be ensured to be in M since we are taking terms that have a linear combination with Q that does not belong in M and thus $R \cap M_1$ is generated by $R \cap M$ and q, so the second condition is proved for subspaces with codimension k.

In order to prove the first condition for M, by our induction hypothesis on M_1, if $f \in M$ (therefore also in M_1), we get that $f = \lim_{n\in\mathbb{N}} p_n$ for some sequence $(p_n)_{n\in\mathbb{N}} \in R \cap M_1$ and therefore $p_n - \varphi(p_n)q \to f$, since $\varphi(p_n) \to 0$ because by hypothesis $p_n \to f \in M$. We know that $p_n - \varphi(p_n)q \in M$ for all $n \in \mathbb{N}$; hence, the condition 1 is proved for M.

Let $\omega \in \mathbb{C}^n$ be a common zero of all polynomials in M. Define the evaluation linear functional γ_ω such that $\gamma_\omega(p) = p(\omega)$, so γ_ω annihilates $R \cap M$ since ω is a common zero on $R \cap M$. Now since $R \cap M$ has codimension 1 in $R \cap M_1$, so there exists a constant c such that $p(\omega) = c\varphi(p)$, so we get that γ_ω is continuous in $R \cap M_1$ in the usual topology of $H^2(\mathbb{D}^n)$. Suppose that ω does not lie in $\mathbb{D}^n$, then at least one of the coordinates of ω, say ω_i, has the property that $|\omega_i| \geqslant 1$, so by our induction hypothesis there exists a $q_0 \in R \cap M_1$ such that $q_0(\omega) = 1$ and if we define

$$p(z) = \frac{1}{2}\left(1 + \frac{\overline{\omega_i} z_i}{|\omega_i|}\right) \tag{3.3}$$

For $z = (z_1, ..., z_n) \in \mathbb{C}^n$. It is straightforward that $p^m Q_0 \in R \cap M_1$ for $m \in \mathbb{N}$, and in particular $\|p^m Q_0\|_2 \to 0$ as $m \to \infty$, but $p^m Q_0(\omega)$ does not converge to zero. Therefore, ω must lie in $\mathbb{D}^n$ by continuity.

Now, any set of evaluation maps at distinct points is linearly independent on R; hence, by condition 2, the cardinality of this set must be at most k. This set annihilates $R \cap M$, which concludes the proof of condition 3.

Conversely, assume that I is an ideal of R with V_I a finite subset of common zeros of I in $\mathbb{D}^n$ and M is the closure of I in $H^2(\mathbb{D}^n)$. Then $\dim(H^2(\mathbb{D}^n)/M) \leqslant \dim(R/I)$ since R is dense in $H^2(\mathbb{D}^n)$, we get that M is a finite codimension invariant subspace. Now, we can see that $I \subset R \cap M$ because $M = \overline{I}$ by hypothesis and of course $\overline{R \cap M} = \overline{I}$ in $H^2(\mathbb{D}^n)$, so by Lemma A.0.2, is enough to prove that every linear functional that annihilates I is continuous.

Now, since I is the finite intersection of primary ideals (we say that I is a primary ideal if $xy \in I \implies$ either $x \in I$ or $y^n \in I$ for some $n > 0$) for each point of V_I, there exists an integer such that the ideal I contains an ideal J of R and J is the ideal of all polynomials with zeros of order greater or equal to m at every point of V_I, so the linear functionals on R that annihilates J are a linear combination of finitely many partial derivatives, evaluated at points of V_I, and these are continuous, since $V_I \subset \mathbb{D}^n$, thus the proof is complete. □

This result was first proved in (36) using several arguments involving algebraic geometry, but this proof follows the version done in (13), which is more related to functional analysis. Usually, this is known as the Ahern and Clark theorem, but in this work, we refer to the following corollary as the Ahern and Clark Theorem:

Ahern and Clark Theorem: If $f_1, ..., f_k \in H^2(\mathbb{D}^n)$ with $k < n$, then the invariant subspace M generated by $f_1, ..., f_k$ is either all $H^2(\mathbb{D}^n)$ or $M^\perp$ is infinite dimensional.

This is one of the deepest results about invariant subspaces over the polydisk, and it allows us to state some strong results about universality and invariant subspaces.

Theorem 3.2.2. *Let $\phi \in H^\infty(\mathbb{D}^n)$, then $\phi H^2(\mathbb{D}^n)$ is a invariant subspace of $H^2(\mathbb{D}^n)$ if and only if ϕ is generalized inner function.*

Proof. Let T_ϕ be the Toeplitz operator with the symbol ϕ. Suppose that $\phi H^2(\mathbb{D}^n)$ is invariant. Since $\ker T_\phi = \{0\}$ and $\operatorname{Ran} T_\phi$ is closed, we get that T_ϕ is bounded below and therefore exists an $\delta > 0$ such that $|\phi| \geqslant \delta$ almost everywhere on $\mathbb{T}^n$. Suppose that $\sigma(\{\zeta \in \mathbb{T}^n : |\phi(\zeta)| < \delta\}) > 0$, then there exists a $\delta_0 \in (0, \delta)$ such that $\sigma(\{\zeta \in \mathbb{T}^n : |\phi(\zeta)| < \delta_0\}) > 0$. Fixing such δ_0 and let $E = \{\zeta \in \mathbb{T}^n : |\phi(\zeta)| < \delta_0\}$, we will construct a sequence of functions $(f_n)_{n\in\mathbb{N}} \subset C(\mathbb{T}^n)$ such that $0 < f_n \leqslant 1$ and $\lim_{n\to\infty} f_n = \chi_E$ almost everywhere on $\mathbb{T}^n$. By (13, Theorem 3.5.3), for each $n \in \mathbb{N}$, there exists $g_n \in H^\infty(\mathbb{D}^n)$ such that $|g_n| = f_n$ almost everywhere on $\mathbb{T}^n$. Thus, we get:

$$\delta^2 \int_{\mathbb{T}^n} |g_n|^2 d\sigma \leqslant \int_{\mathbb{T}^n} |\phi|^2 |g_n|^2 d\sigma$$

for all $n \in \mathbb{N}$. Now, applying Lebesgue-dominated convergence, we obtain:

$$\delta^2 \sigma(E) \leqslant \int_E |\phi|^2 d\sigma \leqslant \delta_0^2 \sigma(E) < \delta^2 \sigma(E)$$

A contradiction. Therefore $|\phi| \geqslant \delta$ almost everywhere on $\mathbb{T}^n$ and hence $1/\phi \in L^\infty(\mathbb{T}^n)$ Conversely, suppose ϕ is a generalized inner function. It is straightforward that $\phi H^2(\mathbb{D}^n)$ is invariant, so we must show that $\phi H^2(\mathbb{D}^n)$ is closed. Here suffices to show that T_ϕ is bounded below by Theorem A.0.7. In particular,

$$\|g\|_2 = \|1/\phi \cdot \phi g\|_2 \leqslant \|1/\phi\|_\infty \cdot \|\phi g\|_2 = \|1/\phi\|_\infty \cdot \|T_\phi g\|_2$$

Thus, T_ϕ is bounded below, and the proof is finished. □

This result by (37) in some sense found a similar characterization to Beurling's in $H^2(\mathbb{D}^n)$ but note that it is not a complete characterization of the invariant subspaces of $H^2(\mathbb{D}^n)$, since we already showed an invariant subspace not generated by a generalized inner function. For this following result, we say that $[\phi]$ is the smallest invariant subspace that contains ϕ in the sense that $[\phi] = \overline{\{T_{z_1}^{k_1} \cdots T_{z_n}^{k_n} \phi : (k_1, \ldots, k_n) \in \mathbb{N}_0\}}$, in other words, the closed orbits via our shifts.

Theorem 3.2.3. *(34, Teorema 4.1.19) Let $\phi \in H^\infty(\mathbb{D}^n)$ be a generalized inner function. Then T_ϕ is invertible or T_ϕ^* is universal for $H^2(\mathbb{D}^n)$.*

Proof. Since $1/\phi \in L^\infty(\mathbb{T}^n)$, by the previous theorem we know that $\mathrm{Ran}(T_\phi)$ is closed, then by Theorem A.0.4, we get that $\mathrm{Ran}(T_\phi^*)$ is also closed and by Proposition 1.4.6 we know that $\mathrm{Ran}(T_\phi^*)$ is dense, therefore T_ϕ^* is surjective. Now, note that $[\phi]$ is smallest subspace that contains ϕ and that $\phi H^2(\mathbb{D}^n) \subseteq [\phi]$, thus follows $\mathrm{Ran}(T_\phi) = [\phi]$. So, by the Ahern and Clark theorem, we get the following dichotomy:

- $[\phi] = H^2(\mathbb{D}^n)$
- $\mathrm{codim}([\phi]) = \infty$

in otherwords, either T_ϕ is surjective or $\mathrm{Ran}(T_\phi)$ has infinite codimension. The first will imply invertibility since $\phi \in H^\infty(\mathbb{D}^n)$ implies injectivity, and the second implies from Theorem A.0.4 that $\ker(T_\phi^*)$ is infinite dimensional and thus by the Caradus criteria, T_ϕ^* is universal. □

Corollary 3.2.3.1. *Let $\phi \in H^\infty(\mathbb{D}^n)$ be a non-constant inner function, then T_ϕ^* is universal.*

Proof. By Proposition 1.4.2, we have that $\|T_\phi\| = \|\phi\|_\phi = 1$, and hence T_ϕ is an isometry. Suppose that T_ϕ is invertible, in particular, for any $f, g \in H^2(\mathbb{D}^n)$:

$$\langle T_\phi f, T_\phi g\rangle = \int_{\mathbb{T}^n} \phi f \overline{\phi g} d\sigma = \int_{\mathbb{T}^n} f\overline{g} d\sigma = \langle f, g\rangle$$

So T_ϕ is in fact unitary, thus $T_\phi^* T_\phi = T_\phi T_\phi^* = I$, hence ϕ must be a constant, which is a contradiction. Therefore, by the previous theorem, T_ϕ^* is universal for $H^2(\mathbb{D}^n)$. □

Note that by the previous result, we get again that the adjoint of our shifts are universal.

Corollary 3.2.3.2. *The Toeplitz operators $T_{z_1}^*, ..., T_{z_n}^*$ are universal for $H^2(\mathbb{D}^n)$.*

Corollary 3.2.3.3. *(34, Corolário 4.1.21) Let $p \in \mathbb{C}[z_1, \ldots, z_n]$ such that p has zeros in $\mathbb{D}^n$ and is zero free in $\mathbb{T}^n$, then T_p^* is universal for $H^2(\mathbb{D}^n)$.*

Proof. By hypothesis, p is a generalized inner function. Since it has zeros belong in the polydisk, there is $\omega \in \mathbb{D}^n$ such that $p(\omega) = 0$. Now, by Proposition 1.4.7, we know that $T_p^* K_\omega = \overline{p(\omega)} K_\omega$ for some $\omega \in \mathbb{D}^n$, so we get that if ω is a zero of p, then $K_\omega \in \ker(T_p^*)$ and since the set of zeros of p is infinite we get that $\ker(T_p^*)$ is infinite dimensional and therefore by Caradus criteria, T_p^* is universal. □

Lemma 3.2.2. *Let T be a bounded linear operator in a Hilbert space $\mathcal{H}$. Then are equivalent:*

1. *T is left-invertible.*
2. *T^* is surjective.*
3. *T is injective and has a closed range.*

Proof. If T is left-invertible, there exists $S \in B(\mathcal{H})$ such that $ST = I_{\mathcal{H}}$. By adjoint properties, we get that $T^*S^* = I^*_{\mathcal{H}} = I_{\mathcal{H}}$ and thus T^* is surjective, hence condition 1 implies condition 2. Now, by Theorem A.0.6 condition 2 and condition 3 are equivalent. Now, if T is injective and has a closed range, then $T(\mathcal{H})$ is a Hilbert space and $T : \mathcal{H} \to T(\mathcal{H})$ is bijective, so by the open mapping theorem there exists $T^{-1} : T(\mathcal{H}) \to \mathcal{H}$, hence T is left-invertible and therefore condition 3 implies condition 1, so the proof is complete. □

Proposition 3.2.2. *Let T_ϕ be a analytic Toeplitz operator in $H^2(\mathbb{D}^n)$. Then T_ϕ is left-invertible if, and only if, ϕ is invertible in $L^\infty(\mathbb{T}^n)$.*

Proof. If ϕ is invertible in $L^\infty(\mathbb{T}^n)$, then $T_{1/\phi}$ is a Toeplitz operator, hence

$$T_{\frac{1}{\phi}}T_\phi f = T_{\frac{1}{\phi}}\phi f = P(\frac{1}{\phi}\phi f) = P(f) = f$$

for all $f \in H^2(\mathbb{D}^n)$, therefore T_ϕ is left-invertible. Conversely, by the Lemma 3.2.2, if T_ϕ is left-invertible, its range is closed. Then $E = \phi H^2(\mathbb{D}^n)$ is closed and satisfies $z_i E \subset E$ for all $i = 1, ..., n$ and this implies that E is a Shift invariant subspace, thus ϕ is invertible in $L^\infty(\mathbb{T}^n)$ by Theorem 3.2.3. □

Theorem 3.2.4. *Let $\phi \in H^\infty(\mathbb{D}^n)$. Then T^*_ϕ is surjective and* $\dim\ker(T^*_\phi) = \infty$ *if, and only if, ϕ is invertible in $L^\infty(\mathbb{T}^n)$ but not in $H^\infty(\mathbb{D}^n)$.*

Proof. Suppose that T^*_ϕ is surjective and has an infinite-dimensional kernel, then by Lemma 3.2.2, it follows that T_ϕ is left-invertible and by the Proposition 3.2.2 it follows that ϕ is invertible. By Theorem A.0.4, we know that $\phi H^2(\mathbb{D}^n)^\perp = \ker(T^*_\phi)$, thus it has infinite codimension, moreover we obtain $\phi H^2(\mathbb{D}^n) \neq H^2(\mathbb{D}^n)$, so $1/\phi \notin H^\infty(\mathbb{D}^n)$. Conversely, suppose that ϕ is invertible in $L^\infty(\mathbb{T}^n)$ but not in $H^\infty(\mathbb{D}^n)$ then again by the Proposition 3.2.2, we get that T_ϕ is left-invertible, and then by the previous lemma we have that T^*_ϕ is surjective. Now, since $1/\phi \notin H^\infty(\mathbb{D}^n)$, we get that $H^2(\mathbb{D}^n) \neq \phi H^2(\mathbb{D}^n)$ and hence it is a non-trivial subspace of $H^2(\mathbb{D}^n)$, so by the Ahern-Clark theorem we obtain that $\phi H^2(\mathbb{D}^n)^\perp$ is infinite dimensional, since T^*_ϕ is surjective, by Theorem A.0.4, follows that $\ker(T^*_\phi) = \phi H^2(\mathbb{D}^n)^\perp$ is infinite dimensional as desired. □

Corollary 3.2.4.1. *Let T_ϕ be a left-invertible analytic Toeplitz operator in $H^2(\mathbb{D}^n)$. Then either T_ϕ is invertible or T^*_ϕ is universal.*

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Appendix

APPENDIX A – Functional Analysis

Lemma A.0.1. *(Riemann-Lebesgue) Given a function* $f \in L^1(\mathbb{T}^n)$*, we have that* $|\hat{f}(m)| \to 0$ *as* $|m| \to \infty$*.*

Proof. See (38, Proposition 3.3.1) □

Proposition A.0.1. *Let* E *be a normed space,* M *be a closed subspace of* E*,* $x_0 \in E \backslash M$ *and* $d = d(x_0, M)$*. There exist a* $\varphi \in E^*$ *such that* $\|\varphi\| = 1$*,* $\varphi(x_0) = d$ *and* $\varphi(x) = 0$ *for all* $x \in M$*.*

Proof. See (39, Proposição 3.3.1) □

Lemma A.0.2. *Let* E *be a Banach space and let* $F \subset E$ *be a subspace such that* $\overline{F} \neq E$*. Then there exists* $\psi \in E^*$*,* $\psi \neq 0$*, such that*

$$\psi(x) = 0 \quad \forall x \in F$$

Proof. See (40, Corollary 1.8). □

Theorem A.0.1. *Let* X *be a compact Hausdorff space. If* $\mathcal{U}$ *is a closed self-adjoint subalgebra of* $C(X)$ *which separates points of* X *and contains the constant function 1, then* $\mathcal{U} = C(X)$*.*

Proof. See (41, Theorem 2.40). □

Lemma A.0.3. *Let* E*,* F *be normed spaces and* $T \in B(E, F)$*. If* $(x_n)_n \in E$ *converges weakly to* $x \in E$ *and* T *is compact, then* $T(x_n) \to T(x)$*.*

Proof. See (39, Proposição 7.2.8) □

Theorem A.0.2. *A Hilbert space* $\mathcal{H}$ *is separable if and only if it has a countable orthonormal basis.*

Proof. See (39, Teorema 5.4.3) □

Theorem A.0.3. *Every separable Hilbert space* $\mathcal{H}$ *is isometrically isomorphic to* ℓ^2*.*

Proof. See (39, Teorema 5.4.4) □

Theorem A.0.4. *Let* E, F *be Banach spaces and* $T : E \to F$ *be a bounded linear operator. The following are equivalent:*

- $\mathrm{Ran}(T)$ *is closed.*
- $\mathrm{Ran}(T^*)$ *is closed.*

- $\text{Ran}(T) = \ker(T^*)^{\perp}$.
- $\text{Ran}(T^*) = \ker(T)^{\perp}$.

Proof. See (40, Theorem 2.19) □

Theorem A.0.5. *Let $f \in L^2(\mathbb{T}^n)$, then*

$$\|f\|_{L^2}^2 = \sum_{m \in \mathbb{Z}^n} |\hat{f}(m)|^2 \tag{A.1}$$

Where $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$ a multi-index and $\hat{f}$ the Fourier transform of f.

Proof. See (38, Proposition 3.2.7). □

Theorem A.0.6. *Let X, Y be Banach spaces and $T \in B(X, Y)$. Then are equivalent:*

- *T is surjective*
- *T^* is injective and has closed range.*

Proof. See (42, Theorem 4.15) □

Theorem A.0.7. *Let X, Y be Banach spaces and $T \in B(X, Y)$. T is bounded below if and only if T is injective and has closed range.*

Proof. See (21, Theorem 2.5) □