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Invariant subspaces of an affine symbol composition operator

Subespaços invariantes de um operador de composição com símbolo afim

Campinas
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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

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Orientador : Sahibzada Waleed Noor
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Resumo

Nesse trabalho nós começaremos apresentando o problema do subespaço invariante e o conceito de operador universal, estabelecendo a conexão entre esses dois tópicos. Ainda, enunciaremos e provaremos alguns resultados clássicos da teoria dos espaços de Hardy, mais especificamente, do espaço de Hardy-Hilbert H^2. Apresentaremos um maquinário que começa de tópicos considerados clássicos como operadores de composição e fatorização canônica até tópicos mais avançados como a função de contagem de Nevannlina e o espectro de funções interiores.

Posteriormente, analisaremos os subespaços invariantes em H^2 de um operador de composição (denotado por C_{ϕ_a}) cujo símbolo que o define é afim. Os subespaços invariantes minimais, que sempre são gerados por uma função $f \in H^2$, estão relacionados a um dos problemas em aberto mais clássicos da teoria de operadores: o problema do subespaço invariante (PSI). Mostraremos ainda que sob determinadas condições envolvendo o comportamento de funções próximo ao ponto 1 temos respostas positivas ao PSI. Além disso, provaremos que o número de zeros da função f interfere na dimensão desses espaços e iremos propor uma abordagem para unificar dois casos conhecidos de universalidade.

Para finalizar, classificaremos totalmente quais espaços modelo são invariantes pelo operador de composição C_{ϕ_a}. Ainda, seguindo a mesma linha de raciocínio, mostraremos quais espaços de Beurling são invariantes por tal operador de composição; nesse caso obteremos um resultado dicotômico. Conexões desses resultados com o clássico operador de Cesàro e ideias para futuros trabalhos serão mencionadas no final do texto.

Palavras-Chave: operadores de composição; espaços de Hardy; operadores universais; subespaços invariantes.

Abstract

In this thesis we begin by talking about the invariant subspace problem, universal operators and the connection between this topics. Also, we state and prove some classic results from the theory of Hardy spaces, more specifically, of the Hardy-Hilbert space H^2. We present a machinery that contains topics considered classic such as composition operators and canonical factorization, but also has more advanced topics such as the Nevannlina counting function and the spectrum of inner functions.

Moreover, we analyze the invariant subspaces in H^2 of a composition operator (denoted by C_{ϕ_a}) whose defining symbol is affine. Minimal invariant subspaces, which are always generated by a function $f \in H^2$, are related to one of the most classic open problems in operator theory: *the Invariant Subspace Problem* (ISP). We show that under certain conditions involving the behavior of functions near to the point 1 we have positive answers to the ISP. Furthermore, we prove that the number of zeros of f and the dimension of these spaces are connected. We propose also an approach to unify two known cases of universal operators.

Finally, we classify which model spaces are invariant by the composition operator C_{ϕ_a} obtaining a complete characterization in this case. Furthermore, following the same line of reasoning, we tried to understand which Beurling spaces are invariant by such a composition operator; in this case we obtained a dichotomic result. These results are related with a classic operator called the Cesàro operator. Ideas for future works are mentioned in the end.

Keywords: composition operators; Hardy spaces; universal operators; invariant subspaces.

List of Symbols

$\mathbb{N}$	The set $\{1, 2, 3, \ldots\}$.
$\mathbb{N}_0$	The set $\{0, 1, 2, 3, \ldots\}$.
$\mathbb{C}$	The field of complex numbers.
$\mathbb{D}$	The set of complex numbers with modulus less than 1, i.e, $\{z \in \mathbb{C} \mid \lvert z\rvert < 1\}$.
$\mathbb{T}$	The set of complex numbers with modulus 1, i.e, $\{z \in \mathbb{C} \mid \lvert z\rvert = 1\}$.
$\subset$	Strictly contained.
$\subseteq$	Contained.
$\dot{\bigcup}$	The disjoint union.
$a.e$	Almost everywhere.
$B(\mathcal{B})$	The space of all bounded linear operators defined on the Banach space $\mathcal{B}$.
$Lat(T)$	The set of all invariant subspaces of an operator T.
$Im(T)$	The image of the operator T.
Id_H	The identity operator on H.
$\sigma(T)$	The spectrum of T.
$\sigma_p(T)$	The point spectrum of T.
H^2	The Hardy-Hilbert space of the disk.
κ_w	The reproducing kernel at the point w in the space H^2.
κ_w^Θ	The reproducing kernel at the point w in the space $(\Theta H^2)^\perp$.
H^∞	The space of bounded analytic functions defined on $\mathbb{D}$.
e_n	The canonical Hilbert basis of H^2.
u_n	The canonical Hilbert basis of $L^2(\mathbb{T})$.
$H^2(\mathbb{T})$	The space of functions f^* in $L^2(\mathbb{T})$ such that $\langle f^*, u_n\rangle = 0$ for all $n < 0$
$LFT(\mathbb{D})$	The set of all linear fractional transformations that leaves $\mathbb{D}$ invariant.
N_ϕ	The Nevannlina counting function associated to ϕ.
$\rho(f)$	The resolvent of a function f.
$\sigma(f)$	The spectra of a function f.
$\Re$	The real part of a function or a number.

$\delta_{n,m}$	The Kronecker's delta.
ΘH^2	The Beurling type space associated to the inner function Θ.
$(\Theta H^2)^{\perp}$	The Model space associated to the inner function Θ.
$Z(f)$	The set all of zeros of the analytic function f.
$\mathcal{S}$	The Schur class.
ϕ_a	The self map of the disk given by $\phi_a(z) = az + 1 - a$ where $a \in (0,1)$.
S_μ	A singular inner function associated to a measure μ.
$\mu \ll \nu$	The measure μ is absolutely continuous with respect to ν.
$supp(\mu)$	The support of a measure μ.
K_f	The orbit of f under C_{ϕ_a}, i.e, $K_f := span\{f, C_{\phi_a}f, C_{\phi_a}^2 f, \ldots\}$.
EB	Eventually bounded.
$Hol(\mathbb{D})$	The set of all analytic functions defined on $\mathbb{D}$.
$A(\mathbb{D})$	The disk algebra, i.e, the set $\{f : \overline{\mathbb{D}} \to \mathbb{C} \mid f$ is continuous and $f_{\vert\mathbb{D}} \in Hol(\mathbb{D})\}$.
D_n	The disk $B(1 - a^n, a^n)$ where $a \in (0,1)$ is fixed.
$\mathcal{C}$	The Cesàro operator.
$\mathcal{H}$	The Kriete-Trutt space.

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1 Introduction

In the field of function theory and operator theory, the *composition operators* are one of the main classes of operators. Imagine a collection of complex or real functions defined on a set X. If $\varphi : X \to X$ is any map, then it makes sense to define the operator $C_\varphi f := f \circ \varphi$. This kind of operator is always linear because

$$C_\varphi(\lambda f + g) = (\lambda f + g) \circ \varphi = \lambda f \circ \varphi + g \circ \varphi = \lambda C_\varphi(f) + C_\varphi(g)$$

where f, g are functions defined X and λ is a scalar. However, we can not go further in a general context; the other properties of C_ϕ depend on the space X and on the function φ.

There is an extensive literature that deal with these objects in many contexts. The books [13] and [39] present some of the classical spaces of analytic functions (for example, the *Hardy spaces* $H^p(\mathbb{D})$, the *Bergman spaces* $A^p(\mathbb{D})$ and the *Dirichlet space* $\mathcal{D}$) and develop a theory of composition operators on these spaces. In this text we will deal basically with the space $H^2(\mathbb{D}) = H^2$. As a field of research, this area has been much explored and a kind of result that is central in the area is the following:

$$C_\varphi \text{ has the property } (A) \text{ if, and only if, } \varphi \text{ has the property } (B).$$

Of course, one-side implications are very common also. Concrete results are for example [30, Theorem 5.1.15] which classifies all the composition operators that are normal and [5, Theorem 2.2] which is about the cyclicity and hypercyclicity of composition operators.

In the field of general functional analysis one of the major open questions is the Invariant Subspace Problem (ISP). It belongs to a class of problems that can be stated in relatively simple terms but the complete solution remains a mystery. The general version of the ISP is the following: let $\mathcal{B}$ be a Banach space, if $T \in B(\mathcal{B})$ (the space of bounded linear operators on $\mathcal{B}$), is it true that T has a non-trivial invariant subspace? By a non-trivial invariant subspace we mean a closed subspace $M \subseteq \mathcal{B}$ such that $M \neq \{0\}$, $M \neq \mathcal{B}$ and $T(M) \subseteq M$. Due to the effort of many mathematicians, actually we can deal with the following important particular case of the above question:

Let H be a complex, separable and infinite dimensional Hilbert space. If $T \in B(H)$, is it true that T has a non-trivial invariant subspace?

In the next chapter, we will explain how the above version of the ISP is obtained. Note that the simplification above imposes certain hypotheses over the underlying space H and nothing is assumed about the operator T. Thus, one way to reason in a slightly distinct direction is to consider properties of the operator T to obtain solutions for particular cases. Historically, this approach was very successful. For example, in 1973 Lomonosov ([29]) proved that if there exists a compact operator S such that $ST = TS$ then T has a non-trivial invariant subspace. This result today is known as Lomonosov's Theorem. In 1978 Brown ([8]) showed that every subnormal operator (i.e, an operator that admits a normal extension) has a non-trivial invariant subspace. Recently, in 2019 Tcaciuc ([42]) showed that every operator $T \in B(H)$ admits a rank-one perturbation $F \in B(H)$ such that $T + F$ has a non-trivial invariant subspace.

There are a lot of possible approaches to the ISP; we refer to the detailed monograph of Partington and Chalendar [10] for some of the modern approaches. One of these methods (which will be used in this text) is based on the concept of a *universal operator* introduced by Rota in [37]; universal operators can be thought of as operators with a very special family of invariant subspaces. We comment also that while we were writing this text, Per H. Enflo published a preprint in arXiv in which he claims the solution of the ISP; Enflo solution's appears to use only "basic" functional analysis and thus these ideas are not directly related with our approach here. But how exactly does our approach work? How composition operators and universal operators work together?

Due to the remarkable work of Nordgren, Rosenthal and Wintrobe (see [33]) we know that if $\phi : \mathbb{D} \to \mathbb{D}$ is a *hyperbolic automorphism* (i.e, the two fixed points belong to the unit circle $\mathbb{T}$) then $C_\phi - \lambda$ is universal for every λ in the interior of the spectra of ϕ. Recently, this result was extended by Carmo and Noor [9, Theorem 3.1] to non-automorphic hyperbolic self-maps of the disk. This recent result leads us to study a special type of composition operator denoted by C_{ϕ_a} which will be the central point of our discussion in the next chapters. All these concepts can be connected in the following statement:

The ISP has a positive solution if, and only if, every minimal invariant subspace of C_{ϕ_a} *has dimension 1.*

This claim will be proved in the next chapter. With this is mind, it is clear that the invariant subspaces of C_{ϕ_a} are directly related with a major open question. But these operators are interesting by themselves because it is not trivial to classify which spaces are invariant under a composition operator. Moreover, another famous operator, called the Cesàro operator, will play a role. Thus, in general, this text can be viewed as an effort to understand the invariant subspaces of C_{ϕ_a}. After this introduction, the reader will find three chapters and one appendix.

The first of them is called *Preliminaries*, as the name suggests, here we exhibit some background results involving the ISP and the Hardy space H^2. We start by explaining and proving results related with the ISP and the simplification that we comment above, we also provide a detailed proof of Caradus criterion and applications. Next, we present the Hardy-Hilbert space H^2 showing facts about this space that we thought to be necessary for the next chapters. Once H^2 is introduced, we focus on composition operators in this space and in particular, we will show a detailed proof of Littlewood's subordination theorem. After this, we will present also a classical tool used by J. Shapiro in his seminal work ([40]) that helps us to compute the norm of composition operators. We finalize the chapter talking about the canonical factorization as well Beurling and Model spaces. All the statements in this chapter are known in the literature.

The next chapter is called *Minimal Invariant subspaces of a universal operator*. This chapter is based in a recent work of Noor and Carmo. Using the results proved by them in [9] we will study the ISP using the operator C_{ϕ_a}. All the minimal invariant subspaces are generated by some function $f \in H^2$ and this leads us to study properties of these generators to obtain properties of the corresponding invariant subspaces. Our first original contributions appear in this chapter. We will prove the veracity of the equivalence

$$M \text{ is minimal } \iff \dim M = 1$$

for some subspaces M that have functions with some specific properties. The behaviour of these functions at the point 1 will be crucial and an important hypothesis (that we will call **EB**) related with the derivative of this function will be considered. Moreover, the zero set of the functions will play a role and we end this chapter suggesting a unified approach to simplify the proofs that hyperbolic composition operators have universal translates.

In the last chapter, called *About some types of invariant subspaces of* C_{ϕ_a} our main question is the following: which Model spaces and which Beurling type spaces can be invariant under C_{ϕ_a}? To obtain the answers we used some recent results. Some of them appear in a paper due to S. Bose, P. Muthukumar and J. Sarkar (see [3] and [32]) and the others are in the work of E. Gallardo-Gutiérrez, J. Partington and W. Ross (see [21] and [22]). For the case of model spaces, we obtain a complete characterization. In the case of Beurling type spaces we obtain a dichotomic result that again shows us the relevance of the point 1 in the same way as the last chapter. To finish this chapter we highlight a connection between these results and the invariant subspaces of the Cesàro operator. We comment also about ideas for future works.

The goal of the appendix is to provide a precise reference for some results that we used in the text. All of these results are in general proved in courses of Functional Analysis, Measure Theory or Complex Analysis.

We will finish this introduction with the following general comment: one of the goals of our text is to be pleasant to read even to readers that have no previous contact with the area. This approach makes sense if we think that the PhD thesis is some kind of legacy for future students and we believe sincerely in this approach. We will prefer at many times write the details rather than hide them to obtain a smaller text. In the case of theorems that we will not prove, we provide a *precise* reference. We wish all readers a good experience with this text.

2 Preliminaries

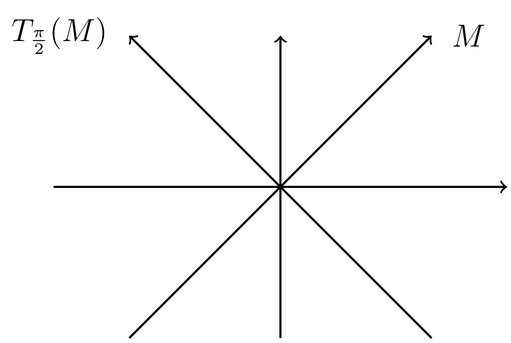
The goal of this chapter is to provide the reader some background results to understand the chapters 3 and 4. We start talking about the general version of the ISP and then we focus on the space H^2, which will be one of our main objects in the text.

2.1 The Invariant Subspace Problem and Universal Operators

Definition 2.1.1. *Let $\mathcal{B}$ be a Banach space and $T \in B(\mathcal{B})$ (the space of all bounded linear operators defined on $\mathcal{B}$). A subspace M of $\mathcal{B}$ is called T**-invariant** or simply **invariant** if M is closed and $T(M) \subseteq M$. The subspaces $\{0\}$ and $\mathcal{B}$ are called the **trivial** invariant subspaces. A T-invariant subspace, M, is called **non-trivial** if $M \neq \{0\}$ and $M \neq \mathcal{B}$.*

This concept is enough to know the following version of the ISP: let $\mathcal{B}$ be a Banach space and $T \in B(\mathcal{B})$ a non-null operator. Does T have a non-trivial invariant subspace?

If we allow the real scalar field then the ISP has a negative answer. A classical example is any rotation operator T_α in $\mathbb{R}^2$ (with angle α not equal to a entire multiple of 2π rad) . In fact, if M is a subspace of $\mathbb{R}^2$, $M \neq \{0\}$ and $M \neq \mathbb{R}^2$ then $M \cap T(M) = \{0\}$. So, the only invariant subspaces of T are the trivial ones.



The operator $T_{\frac{\pi}{2}}$

Of course if dim $\mathcal{B} = 1$ the only subspaces are $\{0\}$ and $\mathcal{B}$. So, in this chapter we will suppose always that dim $\mathcal{B} > 1$ and that the scalar field of $\mathcal{B}$ is $\mathbb{C}$. Only to clarify the terminology, we will say that **the ISP is true for $\mathcal{B}$** if every operator $T \in B(\mathcal{B})$ admits a non-trivial invariant subspace. Similarly, we will say that **the ISP is false for $\mathcal{B}$** if there exists an operator $T \in B(\mathcal{B})$ such that the only invariant subspaces of T are the trivial ones. Moreover, the notation $Lat(T)$ will appear in some moments, this is the collection of all invariant subspaces of some operator T, i.e,

$$Lat(T) = \{M \mid M \text{ is a } T\text{-invariant subspace } \}.$$

Let us narrow our range of spaces now. In 1980 Per H. Enflo [15] constructed a **non-reflexive** Banach space, $\mathcal{B}$, and an operator $T \in B(\mathcal{B})$ such that the only T-invariant subspaces are the trivial ones. In 1985 Charles J. Read [36] found a more concrete counter-example in l^1 (which is not reflexive). So the ISP has a general negative answer when we allow non-reflexive Banach spaces.

Due to these results, actually we can deal with one of the following versions of the ISP:

- Let $\mathcal{B}$ be a reflexive Banach space and $T \in B(\mathcal{B})$ a non-null operator. Does T have a non-trivial invariant subspace?

- Let H be a Hilbert space and $T \in B(H)$ a non-null operator. Does T have a non-trivial invariant subspace?

Note that the first version is more general because every Hilbert space is reflexive but our choice here is the second case. From now on we will deal with Hilbert spaces. As we will see, this case has some benefits. The first benefit is a consequence of the next proposition.

Proposition 2.1.2. *Let H be a Hilbert space (with dimension greater than 1). Then:*

a) If H is non-separable then the ISP is true for H.

b) If $T \in B(H)$ is such that $\sigma_p(T) \neq \emptyset$ then T has a non-trivial invariant subspace.

c) If dim $H < \infty$ then the ISP is true for H.

Proof. a) Let $T \in B(H)$, choose $x \in H - \{0\}$ and consider $M = span\{x, Tx, T^2x, \ldots\}$. It is clear that $T(M) \subseteq M$. Let us prove that $\overline{M}$ is T-invariant. If $m \in \overline{M}$ then there exists a sequence of elements $(m_k)_{k\in\mathbb{N}} \subseteq M$ such that $m_k \xrightarrow{k\to\infty} m$. The continuity of T ensure that $Tm_k \to Tm$. So $Tm \in \overline{M}$ because $(Tm_k)_{k\in\mathbb{N}} \subseteq M$. Moreover, $\overline{M} \neq \{0\}$ because $x \in M$ and $\overline{M} \neq H$ otherwise, by Lemma A.0.3 applied with $A = \{x, Tx, T^2x, \ldots\}$ we conclude that H is separable. Thus $\overline{M}$ is a non-trivial invariant subspace.

b) Let $\lambda \in \sigma_p(T)$. If f is an eigenvector associated to λ then consider $M := span\{f\}$. Note that M is non-trivial, it is closed (because is finite dimensional) and is invariant because $T(\alpha f) = \alpha\lambda f \in span\{f\}$ for any $\alpha \in \mathbb{C}$.

c) In the finite dimensional case we have $\sigma_p(T) = \sigma(T)$ for any operator $T \in B(H)$. We are dealing only with the complex scalar case, so $\sigma(T) \neq \emptyset$ due to Theorem A.0.2. Item c) follows now from b).

■

A consequence of the proof of item a) is that if $T(M) \subseteq M$ then $T(\overline{M}) \subseteq \overline{M}$ where M is any subset of H. A more interesting consequence of this proposition is the following: the only unsolved case of the ISP is when H is infinite dimensional and separable. In this case, the Riesz-Fischer Theorem (see Theorem A.0.4) ensure that H is isometrically isomorphic to

l^2. So we can focus our effort in l^2 or at our favorite separable infinite dimensional Hilbert space due to the following proposition.

Proposition 2.1.3. *Let H_1, H_2 be isomorphic Hilbert spaces. Then the ISP is true for H_1 if, and only if, is true for H_2.*

Proof. Suppose that the ISP is true for H_1 and let $T \in B(H_2)$ an arbitrary non-null operator. Consider $S : H_1 \to H_2$ an isomorphism, then the operator $S^{-1} \circ T \circ S : H_1 \to H_1$ has a non-trivial invariant subspace that we call M. Then $S(M)$ is a non-trivial invariant subspace of T ($S(M)$ is closed because S is an isomorphism and non-trivial because S is, in particular, a bijection). We are done. ∎

There are a lot of possible approaches to the ISP at this moment. We suggests the book [10] to learn about different methods. In this text, we will use one of these methods, based in the following concept developed by Gian Carlo Rota in [37].

Definition 2.1.4 (Rota's universal operator)**.** *Let H be a Hilbert space. An operator $U \in B(H)$ is said to be* ***universal*** *for H if for any non-null operator $T \in B(H)$ there exists M an invariant subspace of U, α a non-null complex scalar and $S : M \to H$ an isomorphism such that $\alpha T = SU_{|M}S^{-1}$.*

The existence of universal operators is not trivial, but as proved in [37], it is always possible to find universal operators in separable Hilbert spaces. It is hard to prove that an operator is universal using the definition. Until recently the main method for identifying a universal operator has been the **Caradus criterion** (see Theorem 2.1.6). Pozzi (see [35]) generalized this classical result and obtained the following theorem that we will call the **Caradus-Pozzi criterion**.

Theorem 2.1.5 (Caradus-Pozzi criterion)**.** *Suppose that $U \in B(H)$ satisfies:*

1. *$Ker\ U$ is infinite-dimensional.*
2. *$Im(U)$ has finite codimension.*

Then U is universal.

Recall that the codimension of a subspace $M \subseteq H$ is defined as the dimension of the quotient space H/M which is equal to the dimension of $M^{\perp}$ if M is a closed subspace. For the sake of completeness and historical relevance, we present here also the classical Caradus Criteria and a proof for it.

Theorem 2.1.6 (Caradus Criterion)**.** *Let H be an infinite dimensional separable Hilbert space and $U \in B(H)$. If $ker(U)$ is infinite-dimensional and U is surjective then U is universal for H.*

Proof. Let $K = Ker(U)$. We define $\mathcal{U} = U_{|K^\perp} : K^\perp \to H$, note that $\mathcal{U}$ is one-to-one since the domain is $U_{|K^\perp}$ and $\mathcal{U}$ is surjective: in fact, if $y \in H$ then $U(x) = y$ for some $x \in H$, as K is closed we can write $H = K \oplus K^\perp$ and then $x = x_1 + x_2$ where $x_1 \in K$ and $x_2 \in K^\perp$. Moreover, $y = U(x) = U(x_1) + U(x_2) = U(x_2) = \mathcal{U}(x_2)$. Define $\mathcal{V} = \mathcal{U}^{-1} : H \to K^\perp$. Furthermore, let $\mathcal{W} : H \to K$ be an isometric isomorphism (Riesz-Fischer Theorem). With this notation we have the following properties:

$$U\mathcal{V} = Id_H, \quad U\mathcal{W} = 0, \quad Ker\ \mathcal{W} = \{0\}, \quad and \quad Im(\mathcal{V}) = K^\perp.$$

Now we will check the definition of a Universal Operator. Let $T \in B(H)$. Choose $\lambda \in \mathbb{C}^*$ such that $|\lambda|\|T\|\|\mathcal{V}\| < 1$ and let $\eta = |\lambda|\|T\|\|\mathcal{V}\|$. Note that the series $\sum_{k=0}^{\infty} \lambda^k \mathcal{V}^k \mathcal{W} T^k$ is absolutely convergent because

$$\sum_{k=0}^{\infty} \|\lambda^k \mathcal{V}^k \mathcal{W} T^k\| \leq \sum_{k=0}^{\infty} |\lambda^k| \|\mathcal{V}\|^k \|\mathcal{W}\| \|T\|^k \leq \|\mathcal{W}\| \sum_{k=0}^{\infty} \eta^k < \infty.$$

As $B(H)$ is a Banach space, absolutely convergence implies convergence, so $\sum_{k=0}^{\infty} \lambda^k \mathcal{V}^k \mathcal{W} T^k =: J \in B(H)$. Note that $\mathcal{W} + \lambda \mathcal{V} J T = J$ and

$$UJ = U(\mathcal{W} + \lambda \mathcal{V} J T) = 0 + \lambda Id_H JT = \lambda JT.$$

If we can prove that $Im(J)$ is closed, $Im(J) \in Lat(U)$ and $J : H \to Im(J)$ is an isomorphism, then the above equation show us that U is universal.

- $Im(J)$ is closed: Let $y \in \overline{Im(J)}$, choose a sequence $(x_n)_{n\in\mathbb{N}} \subseteq H$ such that $J(x_n) \to y$. Let $P : H \to K$ be the orthogonal projection, applying P in the equation $J(x_n) = \mathcal{W}(x_n) + \lambda \mathcal{V} J T(x_n)$ we obtain

  $$PJ(x_n) = P(\mathcal{W}(x_n) + \lambda \mathcal{V} J T(x_n)) = \mathcal{W}(x_n)$$

  because $\mathcal{W}(x_n) \in K$ and $Im(\mathcal{V}) = K^\perp$. So, $\mathcal{W}(x_n) \to Py$. Since $\mathcal{W}$ is an isometry, so $x_n \to x$ for some $x \in H$ which implies that $J(x_n) \to J(x)$ and then $y = J(x)$.

- $Im(J) \in Lat(U)$: This is clear from $UJ(x) = \lambda JT(x) = J(\lambda T(x))$ for all $x \in H$.

- $J : H \to Im(J)$ is an isomorphism. Of course J is surjective and continuous. If $J(x) = 0$ then $\mathcal{W}(x) + \lambda \mathcal{V} J T(x) = 0$ and so $\mathcal{W}(x)$ and then $x = 0$. The open mapping theorem ensures that J is in fact a isomorphism (H and $Im(J)$ are Banach).

This finishes the proof of the Caradus Criterion. ■

Remark 2.1.7. *The first hypothesis of Caradus criterion is a necessary condition to an operator U be universal. In fact, suppose that U is universal and choose a non-null operator*

$T \in B(H)$ such that $ker(T)$ is infinite dimensional (for example choose $T(e_i) = 0$ for every odd number i where $\{e_i\}_{n\in\mathbb{N}}$ is a Hilbert basis for H). Then

$$S^{-1}\alpha T = U_{|M}S^{-1}$$

for some $\alpha \in \mathbb{C}$, $M \in Lat(U)$ and $S : M \to H$ an isomorphism. Then $U_{|M}S^{-1}(e_i) = 0$ for every i odd. Since S^{-1} is an isomorphism the set $\{S^{-1}(e_i) \mid i \text{ odd }\}$ is linearly independent and infinite. This show us that $Ker\ U$ is infinite dimensional.

Example 2.1.8. *Let H be a separable infinite dimensional Hilbert space. We define:*

$$l^2(H) := \{(h_j)_{j\in\mathbb{N}} \mid h_j \in H \text{ and } \sum_{j\in\mathbb{N}} \|h_j\|_H^2 < \infty\}.$$

It is not hard to show that $l^2(H)$ is a separable Hilbert space with the inner product given by:

$$\langle x, y\rangle_{l^2(H)} = \sum_{n=0}^{\infty}\langle h_n, g_n\rangle_H$$

and is clear that the backward shift operator $S(h_0, h_1, h_2, \ldots) = (h_1, h_2, ...)$ is a bounded linear operator in this space. Note that:

- *$Ker(S)$ is infinite dimensional: If $\{d_1, d_2, \ldots\}$ is a Hilbert basis for H (countable and infinite since H is separable and infinite-dimensional). Consider $x_i = (d_i, 0, 0, \ldots) \in l^2(H)$ for all $i \in \mathbb{N}$. Note that $S(x_i) = 0$ for all $i \in \mathbb{N}$ and $\{x_i\}_{i\in\mathbb{N}}$ is a linearly independent set because $\{d_i\}_{i\in\mathbb{N}}$ is linearly independent.*

- *S is surjective: If $(h_1, h_2, \ldots) \in l^2(H)$ then $S(0, h_1, h_2, \ldots) = (h_1, h_2, \ldots)$.*

By Caradus Criterion, S is a universal operator in $l^2(H)$.

Let us establish the connection between Rota's Universal Operator and the ISP. Let $T \in B(H)$ be any operator, in $Lat(T)$ consider the order given by inclusion of sets. We say that $\{0\} \neq M \in Lat(T)$ is **a minimal invariant subspace** if it is minimal with respect to this order, i.e, $M \in Lat(T)$ is minimal if for all $N \in Lat(T)$ such that $N \subseteq M$ we have $N = \{0\}$ or $N = M$. We begin with following simple but crucial remark about the minimal invariant subspaces.

Remark 2.1.9. *Let $T \in B(H)$. If $M \in Lat(T)$ is a minimal invariant subspace then $\dim\ M = \infty$ or $\dim\ M = 1$. In fact, if $\dim\ M > 1$ but it is finite, then consider the operator $S = T_{|M} : M \to M$. The operator S is an operator in a finite dimensional space with dimension greater than 1, so it has a non-trivial invariant subspace N by Proposition 2.1.2. In particular, $N \in Lat(T)$ and $\{0\} \subset N \subset M$; this contradicts the minimality of M.*

The connection between the ISP and the concept of Universal Operator is given by the following theorem:

Theorem 2.1.10. *Let H be a separable infinite dimensional Hilbert space and $U \in B(H)$ be a universal operator. Then, are equivalent:*

1. *The ISP is true for H, i.e, every non-null operator $T \in B(H)$ has a non-trivial invariant subspace.*

2. *The minimal invariant subspaces of U are one-dimensional.*

Proof. 1) $\implies$ 2). Let $M \in Lat(U)$ such that $dim\ M = \infty$, by Riesz-Fischer Theorem (see Theorem A.0.4) there exists an isometric isomorphism $S : M \to H$. Then the operator $R = S \circ U_{|M} \circ S^{-1} \in B(H)$ has a non-trivial invariant subspace N by the hypothesis. Note that $S^{-1}(N)$ is closed because S is an isomorphism. Moreover,

$$\{0\} \subset S^{-1}(N) \subset M$$

where the equalities never occur because otherwise $N = \{0\}$ or $N = S(M) = H$; that is not possible because N is not-trivial. So, if M is minimal the only possibility is that $dim\ M = 1$.

2) $\implies$ 1). Using the universality of U we can choose $\alpha \in \mathbb{C}$, $M \in Lat(U)$ and $S : M \to H$ an isomorphism such that $\alpha T = S \circ U_{|M} \circ S^{-1}$. So, M is not minimal because $dim\ M = \infty$ and we can choose $\{0\} \subset N \subset M$ such that $N \in Lat(U)$. Note that $\alpha T(S(N)) \subseteq S(N)$ because $U_{|M}(N) \subseteq N$, moreover $\{0\} \subset S(N) \subset N$ otherwise $N = \{0\}$ or $N = S^{-1}(H) = M$. As $S(N)$ is closed (S is an isomorphism) we conclude that $S(N)$ is a non-trivial invariant subspace for αT. It is clear that $Lat(\alpha T) = Lat(T)$ and we are done. ■

There are some points that we want to emphasize about the above proposition: this equivalence gives us the possibility to work with only **one** operator instead of analyze all operators. However, this operator must be universal and of course this is a non-trivial property. As in general a universal operator is not unique, our choice of a specific universal operator can smooth or not the situation, depending on "how good" is the operator. In chapter 3 we will work with a specific universal operator and comment about others. For now, we will finish this section making a connection with the adjoint operator.

Remark 2.1.11. *Let $T \in B(H)$. Then a closed subspace M of H is T-invariant if, and only if, $M^\perp$ is T^*-invariant. In fact, suppose that M is T-invariant. For every $y \in M^\perp$ and $x \in M$ we have*

$$\langle x, T^*y\rangle = \langle Tx, y\rangle = 0$$

because $Tx \in M$. So $T^*y \in M^\perp$ and $M^\perp$ is T^*-invariant. The same argument proves the other direction.

The application $M \to M^{\perp}$ defines a bijection between the set of minimal invariant subspaces of T and the set of maximal invariant subspaces of T^*, this can be checked easily using the definitions and the above proposition. So, Theorem 2.1.10 can be rephrased in terms of maximal invariant subspaces: the ISP is true for H if, and only if, all the maximal invariant subspaces of U^* have codimension equal to 1.

2.2 The Hardy-Hilbert Space

In the last section we saw that the ISP for Hilbert spaces is equivalent to the ISP for one separable infinite dimensional Hilbert space. Perhaps the most popular example is l^2, but our choice in this text will be the Hardy-Hilbert space of the disk, some kind of complex analysis brother of l^2. In chapter 3 we will connect the concept of universal operator with a special kind of operators called the composition operators.

This section is devoted to present properties of this space and some important concepts involving it. We will enunciate some classical structural results as the Poisson integral formula and the existence of the radial limit almost everywhere ($a.e$). Some properties will be proved but the most part will be referenced. The main reference for this section is [30] and we suggest this book for anyone who wants a solid and detailed introduction to the area.

Definition 2.2.1. *The Hardy-Hilbert space of the disk, denoted by $H^2(\mathbb{D})$ or simply H^2, is defined as:*

$$H^2(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} \mid f \text{ is analytic and } f(z) = \sum_{n=0}^{\infty} a_n z^n \ , \sum_{n=0}^{\infty} |a_n|^2 < \infty\}.$$

So the functions in the Hardy-Hilbert space are precisely the analytic functions defined on $\mathbb{D}$ whose expansion in Taylor series around 0 produces a sequence of square-summable coefficients. Remember that the coefficients of the expansion of an analytic function around any point are unique, so the definition make sense. Of course this is a complex vector space with the classical definitions and it becomes a Hilbert space if we define a inner product given by

$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. A short way to proof that is considering the map $T : H^2 \to l^2$ given by

$$T(f) = (a_0, a_1, \dots) \text{ where } f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

It is not difficult to prove that this map is an isometric isomorphism between these two spaces, so H^2 is a separable infinite dimensional Hilbert space.

A very special class of functions in H^2 are called the reproducing kernels defined as follows. For every $w \in \mathbb{D}$ we define the **reproducing kernel** at w as the function $\kappa_w : \mathbb{D} \to \mathbb{C}$ given by

$$\kappa_w(z) = \frac{1}{1 - \overline{w}z}.$$

The reproducing kernels will play a central role in the future. We will show the main properties of these functions and some consequences. We comment that the next lines show us that H^2 is a Reproducing Kernel Hilbert Space (RKHS) (see [34] for the definition and more details). For any $w \in \mathbb{D}$ the expression

$$\kappa_w(z) = \frac{1}{1 - \overline{w}z} = 1 + \overline{w}z + (\overline{w})^2 z^2 + \dots$$

show us $\kappa_w \in H^2$ because $|w| < 1$. Moreover for any $f \in H^2(\mathbb{D})$ a direct computation implies that

$$\langle f, \kappa_w \rangle = \sum_{n=0}^{\infty} a_n \overline{\overline{w^n}} = \sum_{n=0}^{\infty} a_n w^n = f(w).$$

As a consequence of this basic properties we can prove the following estimate:

Lemma 2.2.2. *For any $f \in H^2$ and any $w \in \mathbb{D}$ we have*

$$|f(w)| \leq \frac{\|f\|}{\sqrt{1 - |w|^2}}$$

Proof. Note that $\|\kappa_w\|^2 = \langle \kappa_w, \kappa_w \rangle = \frac{1}{1-|w|^2}$. By Cauchy-Schwarz inequality we have

$$|f(w)| = |\langle f, \kappa_w \rangle| \leq \|f\| \|\kappa_w\| \leq \frac{\|f\|}{\sqrt{1 - |w|^2}}.$$

■

Moreover, this growth estimate allow us to find a relation between the convergence of functions in H^2 and the classical locally uniform convergence of functions. The next result will be used many times in the text.

Theorem 2.2.3. *If $(f_n)_{n\in\mathbb{N}} \subseteq H^2$ is such that $f_n \to f$ in H^2 then $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$.*

Proof. Let $K \subseteq \mathbb{D}$ be a compact set. Fix $\epsilon > 0$, as K is a compact subset of $\mathbb{D}$ there exists

$0 < R < 1$ such that $|z| < R$ for any $z \in K$. Let $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ we have

$$\|f_n - f\| < \epsilon\sqrt{1 - R^2}.$$

So

$$|f_n(z) - f(z)| \leq \frac{\|f_n - f\|}{\sqrt{1 - |z|^2}} < \frac{\|f_n - f\|}{\sqrt{1 - R^2}} < \epsilon \quad \forall z \in K \text{ and } \forall n \geq n_0.$$

■

There is an alternative characterization of when an analytic function belongs to H^2. The proof consists in writing $|f(re^{i\theta})|^2$ as a series and uses the following well known identity:

$$\frac{1}{2\pi}\int_0^{2\pi} e^{ni\theta}\overline{e^{mi\theta}}d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{(n-m)i\theta}d\theta = \delta_{n,m}.$$

Theorem 2.2.4. *Let $f : \mathbb{D} \to \mathbb{C}$ be an analytic function. Then $f \in H^2$ if, and only if*

$$\sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty.$$

Moreover, if this is the case then

$$\|f\| = \left(\sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta\right)^{\frac{1}{2}}.$$

Proof. See [30, Theorem 1.1.12]. ■

A direct consequence of this alternative characterization, which is not trivial from the definition, is that $H^\infty \subseteq H^2$ where

$$H^\infty = \{f : \mathbb{D} \to \mathbb{C} \mid f \text{ is analytic and bounded }\}.$$

As usual the sup norm is denoted by $\|f\|_\infty$ and by definition $\|f\|_\infty := \sup_{z\in\mathbb{D}} |f(z)|$ where $f \in H^\infty$.

Now, let us show that is possible to see H^2 as a subspace of some space of square summable functions: consider the space $L^2(\mathbb{T}) = L^2(\mathbb{T}, \frac{m}{2\pi})$ where m is the Lebesgue measure. It is known that $L^2(\mathbb{T}, \frac{m}{2\pi})$ is a Hilbert space with the inner product given by:

$$\langle g, h\rangle = \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta})\overline{h(e^{i\theta})}d\theta$$

where $g, h \in L^2(\mathbb{T})$. For each $n \in \mathbb{Z}$ the function $u_n : \mathbb{T} \to \mathbb{C}$ given by $u_n(e^{i\theta}) = e^{in\theta}$ belongs to $L^2(\mathbb{T})$ and the collection $\{u_n \mid n \in \mathbb{Z}\}$ is a Hilbert basis for $L^2(\mathbb{T})$ (see for example [38]). This means that if $g \in L^2(\mathbb{T})$ then we can write

$$g = \sum_{n\in\mathbb{Z}} \langle g, u_n\rangle u_n.$$

We define $H^2(\mathbb{T})$ as the closed subspace of $L^2(\mathbb{T})$ such that the negative terms of the above sum vanishes, i.e,

$$H^2(\mathbb{T}) = \{f^* \in L^2(\mathbb{T}) \mid \langle f^*, u_n\rangle = 0 \quad \forall n < 0\}.$$

By Bessel inequality, $\sum\limits_{n=0}^{\infty} |\langle f^*, u_n\rangle|^2 < \infty$. So for each $f^* \in H^2(\mathbb{T})$ we know that $f^* = \sum\limits_{n\in\mathbb{N}} \langle f^*, u_n\rangle u_n$ and we can associate the function $f \in H^2$ given by $f(z) = \sum\limits_{n=0}^{\infty} \langle f^*, u_n\rangle z^n$.

It is possible to see that the map $T(f^*) = f$ defined as above is an isometric isomorphism between $H^2(\mathbb{T})$ and H^2. For any $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n \in H^2$ and $0 < r < 1$ we define $f_r : \mathbb{T} \to \mathbb{C}$ given by

$$f_r(e^{i\theta}) = f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}.$$

Clearly $f_r \in H^2(\mathbb{T})$. The precise relation between f and f^* is more deep (see Theorem 2.2.8 below); for now the following proposition that relates f_r and f^* is enough for our purposes.

Proposition 2.2.5. *Let* $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n \in H^2$. *Then*

$$\lim_{r\to 1^-} \|f^* - f_r\|_{H^2(\mathbb{T})} = 0$$

Proof. For a given $\epsilon > 0$ we can choose $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0}^{\infty} |a_n|^2 < \frac{\epsilon}{2}$$

because the series $\sum\limits_{n=0}^{\infty} |a_n|^2$ converges. Moreover, choose $\delta \in (0, 1)$ such that for all $r \in (\delta, 1)$

$$\sum_{n=0}^{n_0-1} |a_n|^2(1 - r^n)^2 < \frac{\epsilon}{2}.$$

The following estimate finishes the proof:

$$\begin{aligned}
\|f^* - f_r\|_{H^2(\mathbb{T})} &= \left\| \sum_{n=0}^{\infty} (a_n - a_n r^n) u_n \right\|^2 \\
&= \sum_{n=0}^{\infty} |a_n|^2 (1 - r^n)^2 \\
&= \sum_{n=0}^{n_0 - 1} |a_n|^2 (1 - r^n)^2 + \sum_{n=n_0}^{\infty} |a_n|^2 (1 - r^n)^2 \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

■

It is a known result of measure theory that L^2 convergence implies that a subsequence converges almost everywhere (see [17, Chapter II]). So the last proposition implies that there exists an increasing sequence of real numbers $(r_k)_{k\in\mathbb{N}}$ such that $f^*(e^{i\theta}) = \lim\limits_{k\to\infty} f_{r_k}(e^{i\theta})$ almost everywhere. In particular, if $f \in H^2$ is analytic at a neighborhood of $\overline{\mathbb{D}}$ then $f^*(e^{i\theta}) = \lim\limits_{k\to\infty} f_{r_k}(e^{i\theta}) = f(e^{i\theta})$.

For all $r \in [0, 1)$ we define the **Poisson kernel** as the function $P_r : \mathbb{R} \to \mathbb{R}_+$ given by

$$P_r(\psi) = \frac{1 - r^2}{1 - 2r\cos\psi + r^2}.$$

Of course $1 - r^2 > 0$; moreover as $\cos\psi \leq 1$ it follows that $0 < (1 - r)^2 \leq 1 - 2r\cos\psi + r^2$ and thus P_r is in fact a well defined positive function. The next theorem uses the Poisson kernel to establish a connection between f and f^*.

Theorem 2.2.6 (Poisson Integral Formula)**.** *If $f \in H^2$ and $re^{it} \in \mathbb{D}$ then*

$$f(re^{it}) = \frac{1}{2\pi} \int_0^{2\pi} f^*(e^{i\theta}) P_r(\theta - t) d\theta.$$

Proof. For all $z_0 \in \mathbb{D}$ the function κ_{z_0} is analytic at a neighborhood of $\overline{\mathbb{D}}$ and as we commented above

$$\kappa^*_{z_0}(z) = \frac{1}{1 - \overline{z_0} e^{i\theta}}$$

Thus

$$f(z_0) = \langle f, \kappa_{z_0} \rangle_{H^2} = \langle f^*, \kappa^*_{z_0} \rangle_{L^2} = \int_{\mathbb{T}} f^*(e^{i\theta}) \left(\frac{1}{1 - z_0 e^{-i\theta}} \right) d\theta.$$

In view of the geometric series

$$\frac{1}{1-z_0e^{-i\theta}} = 1 + z_0e^{-i\theta} + z_0^2e^{-2i\theta} + z_0^3e^{-3i\theta} + \dots$$

we note that

$$\int_{\mathbb{T}} f^*(e^{i\theta})\overline{\left(\frac{1}{1-z_0e^{-i\theta}} - 1\right)}d\theta = 0$$

because $f^* \in H^2(\mathbb{T})$. Consequently

$$f(z_0) = \int_{\mathbb{T}} f^*(e^{i\theta})\left(\frac{1}{1-z_0e^{-i\theta}} + \frac{1}{1-\overline{z_0}e^{i\theta}} - 1\right)d\theta$$

and the result follows from computation below, where $z_0 = re^{it}$;

$$\begin{aligned}\frac{1}{1-z_0e^{-i\theta}} + \frac{1}{1-\overline{z_0}e^{i\theta}} - 1 &= \frac{1-r^2}{1-re^{-it}e^{i\theta} - re^{it}e^{-i\theta} + re^{it}e^{-i\theta}re^{-it}e^{i\theta}} \\ &= \frac{1-r^2}{1-r(e^{-it}e^{i\theta} + e^{it}e^{-i\theta}) + r^2} \\ &= \frac{1-r^2}{1-2r\cos(\theta - t) + r^2} \\ &= P_r(\theta - t).\end{aligned}$$

■

An immediate consequence of the above formula is that $\frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t)d\theta = 1$, where $r \in [0,1)$ and $t \in \mathbb{R}$. This follows by considering f as the constant function 1.

Let us back to the duality between H^2 and $H^2(\mathbb{T})$; maybe the most important fact about this is some kind of "pointwise" relation between f and f^*. Given a function $f \in H^2$ the **radial limit** of f at a point $e^{i\theta} \in \mathbb{T}$ is defined as

$$\lim_{r\to 1^-} f(re^{i\theta}).$$

The above limit may not exist at some point $e^{i\theta} \in \mathbb{T}$ as the following example show us. First, note that the function $f(z) = (1-z)^{-\pi i}$ belongs to H^2 because $f \in H^\infty$, we also comment that if $n \in \mathbb{N}$ and $n \geq 2$ then $log(n) \in \mathbb{R} - \mathbb{Q}$ (to prove this you can argue by contradiction or use the seminal Lindemann-Weierstass Theorem).

Example 2.2.7. *The radial limit of $f(z) = (1-z)^{-\pi i}$ does not exist at the point 1. Consider, for example the sequence $(1 - \frac{1}{2^n})_{n\in\mathbb{N}}$. Of course, this sequence converges to 1 as $n \to \infty$ and*

$$\left(1-\left(1-\frac{1}{2^n}\right)\right)^{-\pi i}=\left(\frac{1}{2^n}\right)^{-\pi i}=e^{-\pi i(log(1)-log(2^n))}=e^{ni\pi log(2)}.$$

So the sequence $(f(1-\frac{1}{2^n}))_{n\in\mathbb{N}}$ is equal to the orbit of the irrational rotation $R_{\pi log(2)}:\mathbb{T}\to\mathbb{T}$ given by $R_{\pi log(2)}(e^{i\theta})=e^{i(\pi log(2))}e^{i\theta}$ at the point $1\in\mathbb{T}$. This orbit is dense in the circle $\mathbb{T}$ by the irrational rotation theorem (see [23, p.6-26]). In particular, the limit $\lim_{r\to 1^-} f(r)$ does not exist.

Nevertheless, the set of points in the circle such that this limit does not exist is small in the measure sense. This fact is a consequence of Poisson Integral Formula proved above and the remarkable Fatou's Theorem (see [30, p. 15]). With these tools, we obtain a precise relation between f and f^*:

Theorem 2.2.8. *If $f\in H^2$ then the radial limit $\lim_{r\to 1^-} f(re^{i\theta})$ exists for almost all θ and*

$$\lim_{r\to 1^-} f(re^{i\theta})\overset{a.e}{=}f^*(e^{i\theta})$$

Proof. See [30, Corollary 1.1.28]. ∎

2.3 Composition Operators

In this section we will see the basic facts about composition operators in the Hardy-Hilbert space. In particular, we will present a complete proof for the fact that composition operators are well defined on H^2. In the end of the section we focus in some terminology about the symbols that define the operators.

Let $\phi:\mathbb{D}\to\mathbb{D}$ be an analytic function. We define the **composition operator with symbol** ϕ, denoted by C_ϕ, as the operator defined on H^2 and given by $C_\phi f=f\circ\phi$ for all $f\in H^2(\mathbb{D})$. Of course $C_\phi f\in Hol(\mathbb{D})$ because the composition of analytic functions remains analytic; a direct computation show us that C_ϕ is a linear operator and that $C_\phi(f\cdot g)=(C_\phi f)\cdot(C_\phi g)$ if $f\cdot g, f, g\in H^2$.

Two non-trivials facts that we will need here about this operator are the following: $Im(C_\phi)\subseteq H^2$ and C_ϕ is a bounded linear operator. This is the content of the next theorem, sometimes called **Littlewood's Subordination Theorem** in the literature. To prove it we need the following lemma.

Lemma 2.3.1. *If $f\in H^2$ then for all $re^{it}\in\mathbb{D}$ we have*

$$|f(re^{it})|^2\le\frac{1}{2\pi}\int_0^{2\pi}|f^*(e^{i\theta})|^2P_r(\theta-t)d\theta.$$

Proof. By Poisson Integral Formula (see Theorem 2.2.6) we can write

$$f(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} f^*(e^{i\theta})P_r(\theta - t)d\theta.$$

The measure μ defined by $\mu(A) = \frac{1}{2\pi}\int_A P_r(\theta - t)d\theta$ is such that $\mu \ll m$ where m is the Lebesgue measure, moreover it is clear that the Radon-Nikodym derivative $\dfrac{d\mu}{dm}$ is given by $\frac{1}{2\pi}P_r(\theta - t)$. This implies

$$f(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} f^*(e^{i\theta})P_r(\theta - t)d\theta = \int_0^{2\pi} f^*(e^{i\theta})d\mu(\theta).$$

As r is fixed and $0 \le r < 1$ we have $1 - 2r\cos(\theta - t) + r^2 \ge (1-r)^2 > 0$ and then $P_r(\theta - t)$ is bounded above by some constant K_r. Thus

$$\int_0^{2\pi} |f^*(e^{i\theta})|^2 d\mu(\theta) = \frac{1}{2\pi}\int_0^{2\pi} |f^*(e^{i\theta})|^2 P_r(\theta - t)d\theta \le K_r\|f^*\|^2 < \infty.$$

where $\|f^*\|^2$ means the norm of f^* in the space $L^2(\mathbb{T}, \frac{m}{2\pi})$. This implies $f^* \in L^2(\mathbb{T}, \mu)$. Applying Holder's inequality to the functions f^* and 1 we obtain

$$\begin{aligned}
|f(re^{it})| &\le \int_0^{2\pi} |f^*(e^{i\theta})|d\mu(\theta) \\
&\le \left(\int_0^{2\pi} |f^*(e^{i\theta})|^2 d\mu(\theta)\right)^{\frac{1}{2}} \left(\int_0^{2\pi} 1 d\mu(\theta)\right)^{\frac{1}{2}} \\
&= \left(\frac{1}{2\pi}\int_0^{2\pi} |f^*(e^{i\theta})|^2 P_r(\theta - t)d\theta\right)^{\frac{1}{2}}
\end{aligned}$$

because $\frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t)d\theta = 1$. The result follows by squaring both sides of the above inequality. ■

In the next theorem some facts about harmonic functions will be used. We refer to [11] or [41] if the reader is not familiar with this topic and wants a precise reference.

Theorem 2.3.2. *If $\phi : \mathbb{D} \to \mathbb{D}$ is an analytic function then the operator $C_\phi : H^2 \to H^2$ is well defined and it is a bounded linear operator. Moreover,*

$$\|C_\phi\| \leq \sqrt{\frac{1+|\phi(0)|}{1-|\phi(0)|}}.$$

Proof. Consider the function $u : \mathbb{D} \to \mathbb{R}$ given by

$$u(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} |f^*(e^{i\theta})|^2 P_r(\theta - t)d\theta.$$

Using the previous lemma we conclude that

$$|f(re^{it})|^2 \leq u(re^{it}) \qquad \forall re^{it} \in \mathbb{D}.$$

As $\phi(\mathbb{D}) \subseteq \mathbb{D}$ we obtain $|f(\phi(re^{it}))|^2 \leq u(\phi(re^{it}))\ \ \forall re^{it} \in \mathbb{D}$ and then

$$\frac{1}{2\pi}\int_0^{2\pi} |f(\phi(re^{it}))|^2 dt \leq \frac{1}{2\pi}\int_0^{2\pi} u(\phi(re^{it}))dt \qquad \forall re^{it} \in \mathbb{D}.$$

Now, note that

$$\begin{aligned}
u(re^{it}) &= \frac{1}{2\pi}\int_0^{2\pi} |f^*(e^{i\theta})|^2 P_r(\theta - t)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} |f^*(e^{i\theta})|^2 P_r(t-\theta)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} |f^*(e^{i\theta})|^2 \Re\left(\frac{1+re^{i(t-\theta)}}{1-re^{i(t-\theta)}}\right)d\theta \\
&= \Re\left(\frac{1}{2\pi}\int_0^{2\pi} |f^*(e^{i\theta})|^2 \left(\frac{1+re^{i(t-\theta)}}{1-re^{i(t-\theta)}}\right)d\theta\right) \\
&= \Re\left(\frac{1}{2\pi}\int_0^{2\pi} |f^*(e^{i\theta})|^2 \left(\frac{e^{i\theta}+re^{it}}{e^{i\theta}-re^{it}}\right)d\theta\right)
\end{aligned}$$

where we used that P_r is an even function and made some computations. Writing $re^{it} = z$ we note that

$$g(z) := \frac{1}{2\pi}\int_0^{2\pi} |f^*(e^{i\theta})|^2 \left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)d\theta$$

is an analytic function due to [41, Theorem 5.4].

Recalling that the real part of an analytic function is harmonic the above discussion show us that u is harmonic. Moreover, from basic complex analysis it is known that the composition of an analytic function with a harmonic function is again a harmonic function, thus $u \circ \phi$ is harmonic. Hence the mean value property for harmonic functions (see [11, Chapter X]) implies that

$$u(\phi(0)) = \frac{1}{2\pi} \int_0^{2\pi} u(\phi(re^{it}))dt \qquad \forall re^{it} \in \mathbb{D}$$

and consequently

$$\|C_\phi f\|^2 = \|f \circ \phi\|_{H^2}^2 = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(\phi(re^{it}))|^2 dt \leq u(\phi(0)) \quad (\star).$$

Moreover, we note that for all $r \in (0, 1]$ we have

$$P_r(\theta - t) = \frac{1 - r^2}{1 - 2rcos(\theta - t) + r^2} \leq \frac{1 - r^2}{(1 - r)^2} = \frac{1 + r}{1 - r}$$

which implies that

$$u(re^{it}) = \frac{1}{2\pi} \int_0^{2\pi} |f^*(e^{i\theta})|^2 P_r(\theta - t)d\theta \leq \left(\frac{1 + r}{1 - r}\right) \|f^*\|_{L^2}^2 = \left(\frac{1 + r}{1 - r}\right) \|f\|_{H^2}^2 \qquad \forall re^{it} \in \mathbb{D}$$

or more briefly

$$u(z) \leq \left(\frac{1 + |z|}{1 - |z|}\right) \|f\|_{H^2}^2 \qquad \forall z \in \mathbb{D}.$$

Using $(\star)$ and the above equation we conclude that

$$\|C_\phi f\|^2 \leq u(\phi(0)) \leq \left(\frac{1 + |\phi(0)|}{1 - |\phi(0)|}\right) \|f\|_{H^2}^2$$

which implies that C_ϕ is a bounded linear operator. The promised estimate about the norm follows by considering the supremum over all $f \in H^2$ with $\|f\|_{H^2} = 1$.

■

The last theorem establishes that C_ϕ is always a bounded linear operator if ϕ is an analytic self-map of the disk. It is natural to expect that properties of the operator C_ϕ can be deduced from the symbol ϕ and vice-versa, in fact, we have a lot of examples of this approach in the literature. We refer to the books [13], [30] and [39] for concrete results about compacity, normality, fixed points and related properties.

Recall that for every analytic nonconstant map $\phi : \mathbb{D} \to \mathbb{D}$ the set $\phi(\mathbb{D})$ is a domain (an open connected set) by the open mapping theorem for analytic functions. Thus if $f \in Ker(C_\phi)$ then $f(\phi(\mathbb{D})) = \{0\}$ and this implies $f = 0$ by the principle of analytic continuation. In particular, no composition operator can be a universal in the sense of Rota by Remark 2.1.7. However, we will see in section 3 that some translations of them are universal.

Another important property of the composition operators, is the relation between their adjoints and the reproducing kernels: if $\phi : \mathbb{D} \to \mathbb{D}$ is analytic and $w \in \mathbb{D}$ then for all $f \in H^2$ we have

$$\langle f, C_\phi^* \kappa_w \rangle = \langle C_\phi f, \kappa_w \rangle = \langle f \circ \phi, \kappa_w \rangle = f(\phi(w)) = \langle f, \kappa_{\phi(w)} \rangle.$$

From this computation we conclude that:

Proposition 2.3.3. *If $\phi : \mathbb{D} \to \mathbb{D}$ is an analytic map and $w \in \mathbb{D}$ then $C_\phi^* \kappa_w = \kappa_{\phi(w)}$.*

To finish this section we will establish some terminology about the symbol ϕ that will be used in the next chapter. It is a well known result in complex analysis that every automorphism of the disk (i.e, an analytic bijection of the disk) has the form

$$\psi(z) = \lambda \frac{a - z}{1 - \bar{a} z}$$

where λ is a unimodular constant and $a \in \mathbb{D}$. If ψ is not the identity transformation and ψ has a fixed point in $\mathbb{D}$ then we say that ψ is **an elliptic automorphism**. We recall also the famous Denjoy-Wolff fixed point theorem.

Theorem 2.3.4 (Denjoy-Wolff)**.** *Let ψ be an analytic self map of $\mathbb{D}$ other than an elliptic automorphism.*

- *If ψ has a fixed point $q \in \mathbb{D}$ then $\psi^n \to q$ uniformly in the compact parts of $\mathbb{D}$ and $|\psi'(q)| < 1$.*
- *If ψ has no fixed point in $\mathbb{D}$ then there is a point $p \in \mathbb{T}$ (called the **Denjoy-Wolff** point of ψ) such that $\psi^n \to p$ uniformly in the compact parts of $\mathbb{D}$. Moreover the radial limit of ψ at the point p is p and the point p is called the **attractive fixed point**.*

Proof. See [39, Chapter V]. ■

Remember that a nonconstant map ϕ defined on $\mathbb{C}$ is called a **linear fractional transformation** if there exists $a, b, c, d \in \mathbb{C}$ such that

$$\phi(z) = \frac{az + b}{cz + d} \qquad \forall z \in \mathbb{C}.$$

Each element ϕ as above can be extended to a function defined on $C_\infty = \mathbb{C} \cup \{\infty\}$ (the one-point compactification of $\mathbb{C}$) in the following way: if $c = 0$ we define $f(\infty) = \infty$ and if $c \neq 0$ then we define $f(-\frac{d}{c}) = \infty$ and $f(\infty) = \frac{a}{c}$. If $c = 0$ then ∞ is clearly a fixed point of ϕ, moreover the other fixed point is given by the solution of $(a-d)z + b = 0$; if we suppose that ϕ is not the identity function (which means that $a = d$ and $b = 0$ can not occurs simultaneously) then the last equation has at most 1 solution. If $c \neq 0$ then the solutions of $\phi(z) = z$ are the solutions of a quadratic equation and thus $\phi(z) = z$ has at most two solutions in $\mathbb{C}$.

Thus every linear fractional transformation distinct from the identity has at least one and at most two fixed points in $\mathbb{C}_\infty$. In our work we will deal with linear fractional transformations that preserves the disk, i.e, we will consider all the linear fractional transformations ϕ such that $\phi(\mathbb{D}) \subseteq \mathbb{D}$. This set will be denoted by $LFT(\mathbb{D})$. All this discussion leads us to the following definition:

Definition 2.3.5. *Let $\phi \in LFT(\mathbb{D})$ without fixed points in $\mathbb{D}$. We say that*

- *ϕ is* ***parabolic*** *if the unique fixed point of ϕ is the attractive fixed point in $\mathbb{T}$.*
- *ϕ is* ***hyperbolic*** *if ϕ has an attractive fixed point in $\mathbb{T}$ and another fixed point that belongs to $C_\infty - \mathbb{D}$.*

Example 2.3.6. *The maps $\phi(z) = \frac{z+1}{2}$ and $\psi(z) = \frac{z-\frac{1}{2}}{1-\frac{1}{2}z}$ are both hyperbolic. The first has 1 and ∞ as fixed points while the second has 1 and -1 as fixed points. The map $\varphi(z) = \frac{1+z}{3-z}$ is an example of a parabolic map with the only fixed point being 1.*

2.4 Nevannlina Counting Function and Shapiro Theorems

Another crucial point in our work is to use the remarkable paper of Shapiro where he characterizes the essential norm (i.e, the norm of an element in the Calkin algebra) of a composition operator (see [40]). We will define in the next pages the **Nevannlina counting function**, a tool that appears in Shapiro's work. We need some preliminary arguments to ensure that this function is well defined.

Lemma 2.4.1. *Let $0 < R < 1$ be a constant, if $R < x < 1$ then $1 - x \approx log(\frac{1}{x})$. I.e, there exists positive constants C_1, C_2 (that does not depend on x) such that*

$$C_1(1-x) \leq log\left(\frac{1}{x}\right) \leq C_2(1-x) \ \ \forall x \in (R,1).$$

Proof. Remember the integral definition of the logarithm function for $x > 0$:

$$log(x) = \int_1^x \frac{1}{t} dt \quad \overset{x<1}{\Longrightarrow} \ -log(x) = \int_x^1 \frac{1}{t} dt.$$

For $R < x \leq t \leq 1$ we have

$$1 \leq \frac{1}{t} \leq \frac{1}{x}$$

and integrating in t over $[x, 1]$ we obtain

$$\int_x^1 1dt \leq \int_x^1 \frac{1}{t}dt \leq \int_x^1 \frac{1}{x}dt \implies 1 - x \leq -log(x) \leq \frac{1}{x}(1-x).$$

As $R < x$ and $-log(x) = log\left(\frac{1}{x}\right)$ we conclude that

$$1 - x \leq log\left(\frac{1}{x}\right) \leq \frac{1}{R}(1-x).$$

Thus it is enough to choose $C_1 = 1$ and $C_2 = \frac{1}{R}$ to finish the proof.

■

Definition 2.4.2. *Let $\phi : \mathbb{D} \to \mathbb{D}$ be a non constant analytic map. We define the **Nevannlina counting function**, denoted by N_ϕ, for each $w \in \mathbb{D} - \phi(0)$ as*

$$N_\phi(w) = \begin{cases} \sum\limits_{z \in \phi^{-1}\{w\}} log\left(\frac{1}{|z|}\right), & \text{if } w \in \phi(\mathbb{D}) - \{\phi(0)\}. \\ 0, & \text{if } w \notin \phi(\mathbb{D}). \end{cases}$$

with each point $z \in \phi^{-1}\{w\}$ occurring as many times as its multiplicity.

Let us explain why the Nevannlina function is well defined. Of course, for $w \notin \phi(\mathbb{D})$ this is clear. Now, fix $w \in \phi(\mathbb{D}) - \{\phi(0)\}$. The sum

$$\sum_{z \in \phi^{-1}\{w\}} log\left(\frac{1}{|z|}\right)$$

is a series because we are summing over all $z \in \mathbb{D}$ such that $\phi(z) - w = 0$; since the number of zeros of an analytic function is always countable (apply the compact exhaustion principle in $\mathbb{D}$ and the analytic continuation principle) this sum is a series and we need to check the convergence.

For that, we will use Lemma 2.4.1. We can write $\{z_n\}_{n \in \mathbb{N}} = \phi^{-1}\{w\}$. There exists $R > 0$ such that $|z_n| > R$ for all $n \in \mathbb{N}$, otherwise, since ϕ is analytic at 0 we can use the analytic continuation principle again to conclude that $\phi(z) = \phi(0)$ for all $z \in \mathbb{D}$ and so ϕ is constant. Using Lemma 2.4.1 we obtain a constant $C_2 > 0$ such that

$$\sum_{z \in \phi^{-1}\{w\}} log\left(\frac{1}{|z|}\right) = \sum_{n=1}^{\infty}\left(log\frac{1}{|z_n|}\right) < C_2\left(\sum_{n=1}^{\infty}(1 - |z_n|)\right).$$

Moreover, $\phi \in H^\infty$ and so $\phi - w \in H^\infty$. In particular $\phi - w \in H^2$. So, $\{z_n\}_{n\in\mathbb{N}}$ are the zeros of a function in the Hardy-Hilbert space. In particular, this sequence satisfies Theorem 2.5.7 and then

$$C_2\left(\sum_{n=1}^{\infty}(1-|z_n|)\right) < \infty.$$

We conclude the convergence of the desired series. To prove the results that motivate this section the Nevannlina counting function is one of the main tools. The other is the following identity due to Littlewood and Paley (recall that $dA(z) := \frac{1}{\pi}dxdy$).

Theorem 2.4.3. *(Littlewood-Paley Theorem) Let $f \in H^2$, then*

$$\|f\|^2 = |f(0)|^2 + 2\int_{\mathbb{D}} |f'(z)|^2 log\left(\frac{1}{|z|}\right) dA(z)$$

where A is the area measure in $\mathbb{D}$.

Proof. See [18, Theorem 4.35]. ■

We will use explicitly (and prove now) two results proved by Shapiro related with the Nevannlina counting function. But first, remember that every analytic function ϕ can be written as $\phi(x,y) = u(x,y) + iv(x,y)$ where u and v are real functions. A simple computation shows us that for any $z_0 \in \mathbb{D}$ we have $\phi'(z_0) = \frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0)$. Using the Cauchy-Riemann equations we deduce that

$$|\phi'(z_0)|^2 = \frac{\partial u}{\partial x}^2(z_0) + \frac{\partial v}{\partial x}^2(z_0) = \det\begin{bmatrix} \frac{\partial u}{\partial x}(z_0) & \frac{\partial u}{\partial y}(z_0) \\ \frac{\partial v}{\partial x}(z_0) & \frac{\partial v}{\partial y}(z_0) \end{bmatrix}.$$

Thus the change of variable theorem (see Theorem A.0.7) takes the following form:

$$\int_{\phi(\Omega)} f(x,y)dxdy = \int_{\Omega} (f\circ\phi)(x,y)|\phi'(x,y)|^2 dxdy$$

where Ω is an open set of $\mathbb{C}$ and $\phi : \Omega \to \mathbb{C}$ is analytic.

Theorem 2.4.4. *If g is a non-negative measurable function in $\mathbb{D}$ and ϕ is a nonconstant analytic self-map of $\mathbb{D}$ then*

$$\int_{\mathbb{D}} g(w)N_\phi(w)dA(w) = \int_{\mathbb{D}} g(\phi(z))|\phi'(z)|^2 \log\frac{1}{|z|}dA(z)$$

Proof. By the principle of analytic continuation the analytic function ϕ' vanishes on an at most countable set and all the zeros of ϕ' are isolated. Let $\mathcal{Z}$ be the set of zeros of ϕ'; the restriction of ϕ' to the open connected set $\mathbb{D} - \mathcal{Z}$ never vanishes and thus for every point of $\mathbb{D} - \mathcal{Z}$ there is an open set containing this point such that ϕ is a homeomorphism (by the inverse function theorem). Note that $\mathbb{D} - \mathcal{Z}$ can be written as the countable union of disjoint semi-closed polar rectangles, that we call $\mathcal{R}_n$, and on each $\mathcal{R}_n$ the function ϕ is one-to-one. Now we denote by ψ_n the inverse function of the restriction $\phi : \mathcal{R}_n \to \phi(\mathcal{R}_n)$. The change of variable formula and the discussion before the theorem show us that

$$\int\limits_{\mathcal{R}_n} g(\phi(z))|\phi'(z)|^2 \log\left(\frac{1}{|z|}\right) dA(z) = \int\limits_{\phi(\mathcal{R}_n)} g(w) \log\left(\frac{1}{|\psi_n(w)|}\right) dA(w)$$

$$= \int\limits_{\mathbb{D}} g(w)\chi_{\phi(\mathcal{R}_n)}(w) \log\left(\frac{1}{|\psi_n(w)|}\right) dA(w)$$

As $\mathbb{D} = \dot{\bigcup}_{n\in\mathbb{N}}\mathcal{R}_n$ if we consider the sum of the above integrals with respect to n we obtain:

$$\int\limits_{\mathbb{D}} g(\phi(z))|\phi'(z)|^2 \log\left(\frac{1}{|z|}\right) dA(z) = \int\limits_{\mathbb{D}} g(w) \left(\sum_{n=1}^{\infty} \chi_{\phi(\mathcal{R}_n)}(w) \log\left(\frac{1}{|\psi_n(w)|}\right)\right) dA(w).$$

To finish the proof, it is enough show that

$$\sum_{n=1}^{\infty} \chi_{\phi(\mathcal{R}_n)}(w) \log\left(\frac{1}{|\psi_n(w)|}\right) = N_\phi(w) \qquad a.e$$

To do this, note that if $w \in \mathbb{D} - \{\phi(0)\}$ is such that $w \notin \phi(\mathbb{D})$ then by definition $N_\phi(w) = 0$ and the equality $\phi(\mathbb{D}) = \bigcup_{n\in\mathbb{N}} \phi(\mathcal{R}_n)$ show us that the left side of the above equation is also 0. Now, suppose that $w \in \phi(\mathbb{D}) - (\{\phi(0)\} \cup \phi(\mathcal{Z}))$; if $z \in \phi^{-1}\{w\}$ there exists exactly one set $\mathcal{R}_n$ such that $z \in \mathcal{R}_n$. Thus $\phi(z) = w \in \phi(\mathcal{R}_n)$ and the term $log(\frac{1}{|\psi_n(w)|}) = log(\frac{1}{|z|})$ appears once in the sum in the left side of the above equation. Moreover, since $\phi(z) = w \notin \phi(\mathcal{Z})$ we see that the multiplicity of z is one and then the term $log(\frac{1}{|z|})$ appears exactly once in the sum that defines $N_\phi(w)$. Since $\phi(\mathcal{Z})$ is countable it has zero measure and thus we obtain the equality almost everywhere. ∎

Theorem 2.4.5. *If $\phi : \mathbb{D} \to \mathbb{D}$ is analytic then $\forall g \in H^2(\mathbb{D})$ we have*

$$\|C_\phi g\|^2 = 2\int\limits_{\mathbb{D}} |g'(w)|^2 N_\phi(w) dA(w) + |g(\phi(0))|^2.$$

Proof. From the Littlewood-Paley Theorem applied for the function $g \circ \phi$ it follows that

$$\begin{aligned}\|g \circ \phi\|^2 - |g(\phi(0))|^2 &= 2 \int_{\mathbb{D}} |(g \circ \phi)'(z)|^2 \; log\left(\frac{1}{|z|}\right) dA(z) \\ &= 2 \int_{\mathbb{D}} |g'(\phi(z))|^2 |\phi'(z)|^2 \; log\left(\frac{1}{|z|}\right) dA(z).\end{aligned}$$

But from Theorem 2.4.4 applied to $f = |g'|^2$ we deduce that

$$2 \int_{\mathbb{D}} |g'(\phi(z))|^2 |\phi'(z)|^2 \; log\left(\frac{1}{|z|}\right) dA(z) = 2 \int_{\mathbb{D}} |g'(w)|^2 N_{\phi}(w) dA(w)$$

and the result follows. ■

2.5 Canonical Factorization, Beurling and Model spaces

One of the major results from the theory of Hardy spaces is called the canonical factorization. In the present section, using this factorization, we will define some concepts (in particular, some subspaces of H^2) that will be important in chapter 4.

On the way to enunciate this theorem for H^2 functions we need to introduce some concepts. The **shift operator** in H^2 is denoted by $M_z : H^2 \to H^2$ and defined by $M_z(f) = zf(z)$ where $f \in H^2$. It is not difficult to see that M_z is well defined and is a bounded linear operator (in fact an isometry) in H^2.

Definition 2.5.1. *Let $f \in H^2$. We say that:*

1. *f is an **outer function** if $\overline{span\{f, M_z f, M_z^2 f, \dots\}} = H^2$.*
2. *f is an **inner function** if $|f^*(e^{i\theta})| \overset{a.e}{=} 1$.*
3. *f is a **singular-inner function** if f is nonconstant, inner and f is zero-free in $\mathbb{D}$.*

There are some alternative characterizations of these concepts.

Theorem 2.5.2. *A function $S \in H^2$ is a singular inner function if, and only if, S can be written in the form*

$$S(z) = \lambda \exp\left(-\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta)\right) \qquad \forall z \in \mathbb{D}$$

where λ is a unimodular constant, μ is a positive, finite and regular Borel measure in $[0, 2\pi]$. Moreover, $\mu \perp m$ where m is the Lebesgue measure in $[0, 2\pi]$.

Proof. See [30, Theorem 2.6.5]. ■

Using Theorem A.0.8 for the map $f : (0, 2\pi] \to \mathbb{T}$ given by $f(\theta) = e^{i\theta}$ we see that

$$\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta) = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\nu(\xi).$$

where ν is the pushforward measure of μ by f. Since f is a bijection and μ is singular to m the measure ν is singular to the Lebesgue measure in $\mathbb{T}$, which is defined as the pushforward measure of the Lebesgue measure m defined on $(0, 2\pi]$ by the map f. Thus the above theorem can be restated using measures in $\mathbb{T}$.

Example 2.5.3. *Let $\eta \in \mathbb{T}$ and $L > 0$. Consider the point mass measure defined on $\mathbb{T}$ by $\nu(A) = L$ if $\eta \in A$ and $\nu(A) = 0$ if $\eta \notin A$. In this case:*

$$-\frac{1}{2\pi} \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\nu(\xi) = -K \left(\frac{\eta + z}{\eta - z} \right)$$

where K is a positive constant. Thus the functions of the form

$$e^{-K\left(\frac{\eta+z}{\eta-z}\right)} \qquad (K > 0)$$

are all singular inner functions.

Theorem 2.5.4. *A function $G \in H^2$ is outer if and only if there exists a unimodular constant λ such that*

$$G(z) = \lambda \exp \left(\frac{1}{2\pi} \int_0^{2\pi} \left(\frac{e^{i\theta} + z}{e^{i\theta} - z} \right) \log |G^*(e^{i\theta})| d\theta \right)$$

for all $z \in \mathbb{D}$.

Proof. See [30, Corollary 2.7.8]. ∎

Definition 2.5.5. *Let $\{z_k\}_{k\in\mathbb{N}} \subseteq \mathbb{D}$. We say that $\{z_k\}_{k\in\mathbb{N}}$ satisfies the* ***Blaschke condition*** *if*

$$\sum_{k=1}^{\infty} (1 - |z_k|) < \infty.$$

Moreover, if $s \in \mathbb{N} \cup \{0\}$ we define the Blaschke product with zeros $\{z_k\}_{k\in\mathbb{N}} \subseteq \mathbb{D}$ and multiplicity s at 0 as the function

$$B(z) = z^s \prod_{k=1}^{\infty} \frac{\overline{z_k}}{|z_k|} \left(\frac{z_k - z}{1 - \overline{z_k} z} \right)$$

Theorem 2.5.6. *Let $\{z_k\}_{k\in\mathbb{N}} \subseteq \mathbb{D}$ a Blaschke sequence and $s \in \mathbb{N}$. Then the function B defined as above is an inner function whose zeros are exactly the sequence $\{z_k\}_{k\in\mathbb{N}}$, counting multiplicities, and 0 which is a zero of multiplicity s.*

Proof. See [30, Theorem 2.4.13]. ∎

If $z_1, \dots, z_n$ are elements of $\mathbb{D}$ then we define the Blaschke product B with zeros $z_1, \dots, z_n$ and multiplicity s at 0 in the same way. In this situation, as the product is finite, it is possible to see that B is in fact analytic in a neighborhood of $\overline{\mathbb{D}}$.

Every inner function Θ can be written as a unimodular constant multiple of $\Theta = BS_\nu$ where B is the Blaschke product formed by the zeros of Θ and S_ν is a singular inner function (ν is the measure that appears in the representation given by Theorem 2.5.2). See [30, Corollary 2.4.14].

The next theorem tells us that the zeros of functions in H^2 must go to the boundary with a certain speed.

Theorem 2.5.7. *Let $f \in H^2$ not null. If $\{z_k\}_{k\in\mathbb{N}}$ are the zeros of f then $\{z_k\}_{k\in\mathbb{N}}$ satisfies the Blaschke condition, i.e,*

$$\sum_{k=1}^{\infty}(1 - |z_k|) < \infty.$$

Proof. See [30, Corollary 2.4.10]. ∎

With these concepts, we present the canonical factorization theorem:

Theorem 2.5.8. *If $f \in H^2$ is not null then we have a unique factorization*

$$f = \lambda BSO$$

where λ is a unimodular constant, B is a Blaschke products formed exactly by the zeros of f, S is a singular inner function and O is an outer function. Moreover,

$$\|f\| = \|O\|.$$

Proof. See [18, Theorem 4.19]. ∎

The canonical factorization, as we said, is one of the major results about H^2 and can be naturally generalized to the Hardy spaces H^p. Another major result of this area is the celebrated Beurling theorem which gives us a complete characterization of the invariant subspace of the shift operator M_z.

Theorem 2.5.9 (Beurling's Theorem)**.** *Every invariant subspace of M_z other than $\{0\}$ has the form ΘH^2, where Θ is a inner function.*

Proof. See [30, Corollary 2.2.12]. ■

Motivated by the above theorem, it is common to consider two special subspaces of H^2.

Definition 2.5.10. *Let Θ be an inner function. Then:*

- *A space of the form ΘH^2 is called a* ***Beurling type space.***
- *A space of the form $(\Theta H^2)^\perp$ is called a* ***Model space****.*

The model spaces are, by definition, the orthogonal complements of Beurling type spaces. By Beurling Theorem and by Remark 2.1.11 we conclude that the model spaces are exactly the M_z^*-invariant subspaces.

To finish this chapter we will make some comments about the boundary behaviour of some H^∞ functions. Consider a function $f \in H^\infty$ such that $\|f\|_\infty \leq 1$. We say that a point $z \in \overline{\mathbb{D}}$ is a **regular point** of f if $z \in \mathbb{D}$ and $f(z) \neq 0$ or if $z \in \mathbb{T}$ and f has an analytic continuation in a neighborhood V of z with $|f(w)| = 1$ for every $w \in V \cap \mathbb{T}$.

Definition 2.5.11. *Let $f \in H^\infty$. We define the* ***resolvent*** *of f as:*

$$\rho(f) := \{z \in \overline{\mathbb{D}} \;|\; z \textit{ is a regular point of } f\}$$

and we define the ***spectrum*** *of f as:*

$$\sigma(f) = \overline{\mathbb{D}} - \rho(f).$$

Let $\Theta \in H^\infty$ be an inner function. Then $z \in \overline{\mathbb{D}}$ is a **regular point** of Θ if, and only if, $z \in \mathbb{D}$ and $\Theta(z) \neq 0$ or if $z \in \mathbb{T}$ and Θ has an analytic continuation in a neighborhood V. In fact, to check this we only need to verify that $|\Theta(w)| = 1$ for every $w \in V \cap \mathbb{T}$. This is clear because we know that $|\Theta(e^{i\theta})| \overset{a.e}{=} 1$, so given $w \in V \cap \mathbb{T}$ we can choose a sequence $(w_n)_{n\in\mathbb{N}} \subseteq \mathbb{T}$ such that $w_n \to w$ and $|\Theta(w_n)| = 1$. Thus $|\Theta(w)| = 1$ because Θ is analytic at w.

The following theorem will be important for us in the next sections.

Theorem 2.5.12. *Consider $f \in H^\infty$ such that $\|f\|_\infty \leq 1$ and let $\lambda \in \overline{\mathbb{D}}$. Then are equivalent:*

1. *$\lambda \in \rho(f)$*
2. *Either $\lambda \in \mathbb{D}$ and $f(\lambda) \neq 0$, or $\lambda \in \mathbb{T}$ and there exists $\epsilon > 0$ such that*

$$|f^*(\eta)| \overset{a.e}{=} 1, \quad \eta \in \mathbb{T} \cap B(\lambda, \epsilon)$$

and

$$\inf\{|f(z)| : z \in \mathbb{D} \cap B(\lambda, \epsilon)\} > 0.$$

Moreover, if $f = \Theta$ is a inner function with singular inner part S_ν then

$$\sigma(\Theta) = \left\{ z \in \overline{\mathbb{D}} \;\mid\; \liminf_{w \to z,\ w \in \mathbb{D}} |\Theta(w)| = 0 \right\} = \overline{Z(\Theta)} \cup supp(\nu)$$

Proof. See [18, Theorem 5.4] ■

In the above theorem $supp(\nu)$ is the support of the measure ν, defined as:

$$supp(\nu) := \{\xi \in \mathbb{T} \mid \nu(\{\xi e^{it} \mid -\epsilon < t < \epsilon\}) > 0 \;\; \forall \epsilon > 0\}.$$

Thus the support of μ consists on the points in the circle such that any open symmetric arc centered at this point has positive measure. The functions $f \in H^\infty$ such that $\|f\|_\infty \leq 1$ are called the **Schur** functions and the set of all such functions are denoted by $\mathcal{S}$. Thus the above theorem holds exactly for the functions in the class

$$\mathcal{S} = \{f \in H^\infty \mid \|f\|_\infty \leq 1\}.$$

3 Minimal Invariant subspaces of a universal operator

The main purpose of this chapter is to study the minimal invariant subspaces of a specific universal operator, denoted by C_{ϕ_a}, which was studied recently by Carmo and Noor in [9]. It is not hard to see (and we will prove soon) that if M is a minimal invariant subspace of C_{ϕ_a} then

$$M = K_f := \overline{span\{f, C_{\phi_a} f, C_{\phi_a}^2 f, \ldots\}}$$

for every $f \in M - \{0\}$.

With this remark in mind it makes sense to study the spaces K_f using known properties of the function f. This leads us to introduce the notion of **eventually bounded** functions and prove the main results of the chapter. We will exhibit a class of functions for which the claim

$$K_f \text{ is minimal } \Leftrightarrow dim\ \ K_f = 1.$$

is true. In a general way these hypotheses are related with the behaviour of the function f near to the point 1. Moreover we will show some relations between the number of zeros of f, the dimension of the space K_f and the ISP. In the end of this section we provide a universality criterion based on the work of Pozzi (see [35]).

3.1 Carmo-Noor Universal Operator

For each $0 < a < 1$ we define the function ϕ_a as $\phi_a(z) = az + 1 - a$ for every $z \in \mathbb{D}$. Note that $\phi_a \in LFT(\mathbb{D})$ because

$$|az + 1 - a| \leq |az| + |1 - a| < a + 1 - a = 1.$$

A simple computation show us that the fixed points of ϕ_a are 1 and ∞ which means that ϕ_a is of hyperbolic type. By Theorem 2.3.2 the composition operator C_{ϕ_a} given by

$$C_{\phi_a} f = f \circ \phi_a$$

is a bounded linear operator in H^2.

For any function $f \in H^2$ we define K_f as the closure of the span of the orbit of f under C_{ϕ_a}, more precisely:

$$K_f := \overline{span\{f, C_{\phi_a} f, C_{\phi_a}^2 f, \ldots\}} = \overline{span\{f, f \circ \phi_a, f \circ \phi_a^2, \ldots\}}.$$

This definition will play a central role from now on due to the next proposition.

Proposition 3.1.1. *If $f \in H^2$ then K_f is a C_{ϕ_a}-invariant subspace. Moreover if M is a minimal C_{ϕ_a}-invariant subspace, then $M = K_f$ for any $f \in M - \{0\}$.*

Proof. The fact that K_f is a C_{ϕ_a}-invariant subspace follows exactly by the same argument that we used in item a) of Proposition 2.1.2. For the second claim, note that if $f \in M - \{0\}$ then

$$\{0\} \subset span\{f, f \circ \phi_a, f \circ \phi_a^2, \dots\} \subseteq M$$

because f belongs to M, M is C_{ϕ_a}-invariant and M is a subspace. Since M is closed considering the closure in the inclusions above we obtain $\{0\} \subset K_f \subseteq M$ and then the minimality implies $K_f = M$ as desired. ∎

Note that for all $n \in \mathbb{N}$ and $z \in \mathbb{D}$ we have

$$\begin{aligned} \phi_a^n(z) &= \phi_a^{n-1}(az + 1 - a) \\ &= \phi_a^{n-2}(a(az + 1 - a) + 1 - a) \\ &= \phi_a^{n-2}(a^2 z + 1 - a^2) \\ &= \qquad \vdots \\ &= a^n z + 1 - a^n = \phi_{a^n}(z). \end{aligned}$$

This property is called the semigroup property and allows us to write

$$K_f = \overline{span\{f, f \circ \phi_a, f \circ \phi_{a^2}, \dots\}}.$$

In 2022 Carmo and Noor generalized a famous result of Nordgren, Rosenthal and Wintrobe (see [33, Theorem 6.2]) proving the following theorem.

Theorem 3.1.2. *[9, Theorem 3.1]. If $\phi \in LFT(\mathbb{D})$ then $C_\phi - \lambda$ is universal on H^2 for some $\lambda \in \mathbb{C}$ if, and only if, ϕ is hyperbolic.*

Moreover, it is not hard to see that for every $\lambda \in \mathbb{C}$ and $T \in B(H)$ we have $Lat(T) = Lat(T - \lambda)$ and thus the minimal invariant subspaces of an operator T and the translations of T are the same. Combining this fact with Theorem 2.1.10 we obtain the following Corollary which is crucial for the next section:

Corollary 3.1.3. *Let $a \in (0,1)$ and $\phi_a : \mathbb{D} \to \mathbb{D}$ as defined above. Then the ISP has a positive solution if, and only if, every minimal invariant subspace of C_{ϕ_a} has dimension 1.*

3.2 The minimal invariant subspaces K_g

The aim of this section is to study the minimal invariant subspaces M (which are of the form $M = K_g$ by Proposition 3.1.1) of C_{ϕ_a}, using properties of the function g. By Corollary 3.1.3 if we understand all these spaces we have the answer to the ISP.

Our first original contribution is the result below, it says that if g has a "good" behaviour around 1 then we have an answer in the positive direction. This theorem resembles another theorems obtained by Matache (see [31, Theorem 2]), Gallardo-Gutiérrez and Gorkin (see [20, Proposition 2.1]) while they work with the hyperbolic automorphisms.

Theorem 3.2.1. *If $g \in H^2(\mathbb{D})$ with $\lim_{z\to 1} g(z) = L \neq 0$ ($z \to 1$ within $\mathbb{D}$), then K_g contains the constants. In particular if K_g is minimal, then $g \equiv L$ and* $\dim K_g = 1$.

Proof. There exists a $\delta > 0$ such that if $z \in \mathbb{D}$ and $|z-1| < \delta$ then $|g(z) - L| < 1$. In particular, for every $z \in B(1,\delta) \cap \mathbb{D}$ we obtain

$$|g(z)| \leq |g(z) - L)| + |L| < 1 + |L| =: K$$

As $\phi_{a^n}(\mathbb{D}) = a^n\mathbb{D} + 1 - a^n$ are circles with center $1 - a^n$ (converging to 1) and radius a^n (converging to 0) there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\phi_{a^n}(\mathbb{D}) \subseteq B(1,\delta) \cap \mathbb{D}$. So, given $re^{i\theta} \in \mathbb{D}$ we conclude that $a^n re^{i\theta} + 1 - a^n \in B(1,\delta) \cap \mathbb{D}$ and so

$$|g \circ \phi_{a^n}(re^{i\theta})|^2 = |g(a^n re^{i\theta} + 1 - a^n)|^2 \leq K^2 \quad \forall n \geq n_0.$$

Consequently we get

$$\|C_{\phi_{a^n}} g\|^2 = \|g \circ \phi_{a^n}\|^2 = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |g \circ \phi_{a^n}(re^{i\theta})|^2 d\theta \leq K^2 \quad \forall n \geq n_0.$$

This shows that the sequence $(C_{\phi_{a^n}} g)_{n\in\mathbb{N}}$ is bounded. Moreover, for each $z \in \mathbb{D}$ we have $a^n z + 1 - a^n \to 1$ as $n \to \infty$ and then

$$g \circ \phi_{a^n}(z) = g(a^n z + 1 - a^n) \to L.$$

So $(C_{\phi_{a^n}} g)_{n\in\mathbb{N}}$ converges pointwise to L. By [13, Corollary 1.3] $(C_{\phi_{a^n}} g)_{n\in\mathbb{N}}$ converges weakly to the constant function L. So L belongs to the weak closure of the convex set

$$span\{g, C_{\phi_a} g, C_{\phi_a}^2 g, \ldots\}$$

which is equal to the norm closure by Mazur's Theorem (see Theorem A.0.5). So $L \in K_g$ and hence $K_g = K_L$ since $L \neq 0$. ∎

Remark 3.2.2. *The proof above works in the same way if $\lim_{z\to 1} g(z) = 0$ but the conclusion about minimality is not true, in fact the conclusion in that case is a trivial information: $0 \in K_g$.*

Every function $g \in A(\mathbb{D})$ (where $A(\mathbb{D})$ denotes the disk algebra) such that $g(1) \neq 0$ satisfies the above hypothesis; thus for such functions K_g is minimal if and only if g is constant. Note that we can not apply the theorem for simple H^2 functions like $g(z) = z^4 + z^3 - z^2 - z^1$. We will see soon some ways to deal with this situation.

For now we can at least solve this problem for polynomials due to the following basic but interesting property of C_{ϕ_a}: if p is a polynomial of degree N, say $p(z) = \sum_{j=0}^{N} a_j z^j$ then

$$C_{\phi_a} p(z) = p(az + 1 - a) = \sum_{j=0}^{N} a_j (az + 1 - a)^j$$

is again a polynomial of degree N. Thus the orbit of p by C_{ϕ_a}, i.e, the set $\{p, C_{\phi_a} p, C_{\phi_a}^2 p, \ldots\}$ contains only polynomials of degree N and consequently the set $K_p = \overline{span\{p, C_{\phi_a} p, C_{\phi_a}^2 p, \ldots\}}$ is contained in the finite dimensional space of polynomials of degree at most N. We summarize this discussion in the following proposition for future citations.

Proposition 3.2.3. *If $a \in (0, 1)$ and p is a polynomial of degree N then $C_{\phi_a} p$ is again a polynomial of degree N. In particular K_p is always a finite dimensional space and thus is minimal if and only if* $\dim K_p = 1$.

Proof. Follows from the above discussion and from Remark 2.1.9. ∎

Now we will focus in more general situations. In [9] the authors already had the idea that the behaviour of g around 1 is a key point and they used the notion of radial limit at 1 to divide the problem in three cases. Here, corroborating this intuition with Theorem 3.2.1 we propose a similar but new division according to the behaviour of the sequence of complex numbers $(g(1 - a^n))_{n \in \mathbb{N}}$. More precisely, we will consider three cases:

A) $g(1 - a^n)$ converges to a number $L \neq 0$.

B) $g(1 - a^n)$ converges to 0.

C) $g(1 - a^n)$ does not converges.

Of course this three cases covers all the possibilities. We define, for each $n \in \mathbb{N}$, D_n as the open disk with center $1 - a^n$ and radius a^n, i.e, $D_n := B(1 - a^n, a^n)$ (see the picture below for a geometric vision). We say that an analytic function f defined on $\mathbb{D}$ is **eventually bounded** (**EB**) if there exists a number $n_0 \in \mathbb{N}$ such that f is bounded in D_{n_0} (and consequently, it is bounded in every D_n for $n \geq n_0$).

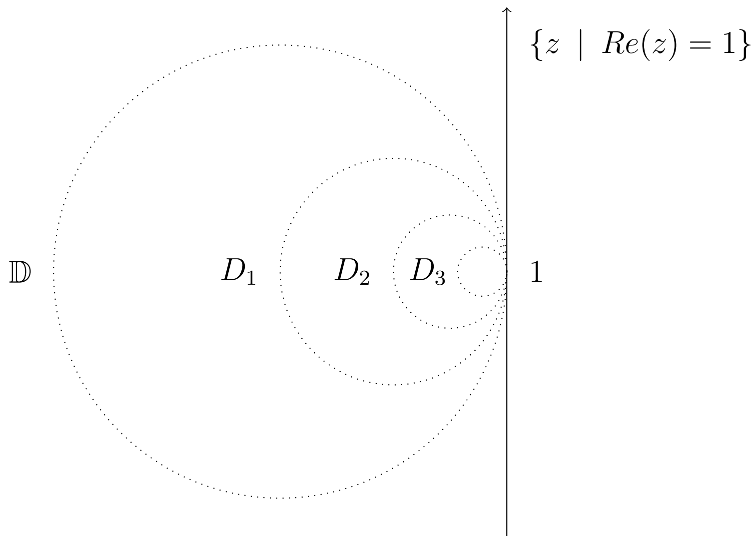


Figure 2: The disks $D_n = \phi_{a^n}(\mathbb{D})$ shrinking to the point 1.

To obtain our results, we begin with technical lemma.

Lemma 3.2.4. *If $g \in H^2$ is such that g' is eventually bounded then*

$$\int_{\mathbb{D}} |g'(w)|^2 \, N_{\phi_{a^n}}(w) dA(w) \to 0 \quad \textit{as } n \to \infty.$$

Proof. Let $n_0 \in \mathbb{N}$ such that g' is bounded in D_{n_0}. Applying Theorem 2.4.4 for $f = |g'|^2$ and $\phi = \phi_{a^n}$ we obtain:

$$\begin{aligned}\int_{\mathbb{D}} |g'(w)|^2 N_{\phi_{a^n}}(w) dA(w) &= \int_{\mathbb{D}} |g'(a^n z + 1 - a^n)|^2 |\phi'_{a^n}(z)|^2 \log \frac{1}{|z|} dA(z) \\ &\leq \int_{\mathbb{D}} C a^{2n} \log \frac{1}{|z|} dA(z) = a^{2n} K \qquad (n \geq n_0)\end{aligned}$$

where C is a constant such that $|g'(a^n z + 1 - a^n)|^2 \leq C$ for all $n \geq n_0$ and K is the constant given by C times the integral of $\log \frac{1}{|z|}$ over $\mathbb{D}$. So, if $n \to \infty$ then $a^{2n} \to 0$ and we conclude the proof. ∎

Another important tool is the following family of functions: for each $s \in \mathbb{C}$ Hurst showed in [25, Lemma 7] that the functions $f_s(z) := (1 - z)^s$ belong to H^2 if and only if $\Re(s) > -\frac{1}{2}$. For $\Re(s) > -\frac{1}{2}$ these functions are eigenvectors of C_{ϕ_a}:

$$\begin{aligned}C_{\phi_a} f_s(z) = f_s \circ \phi_a(z) &= f_s(az + 1 - a) \\ &= (1 - az - 1 + a)^s \\ &= (a(1 - z))^s = a^s f_s(z).\end{aligned}$$

With this in mind, we are able to proof a central result in the section. Recall that a sequence of complex numbers $(z_n)_{n\in\mathbb{N}}$ is called bounded away from zero if there exists a constant M such that $|z_n| \geq M > 0$ for all $n \in \mathbb{N}$.

Theorem 3.2.5. *Suppose that $f = f_s g$ for some $g \in H^2$ and $Re(s) \geq 0$. If g' is eventually bounded and there exists a subsequence of $(g(1-a^n))_{n\in\mathbb{N}}$ which is bounded away from zero, then $f_s \in K_f$. So, K_f is minimal, if and only if,* $\dim K_f = 1$.

Proof. By hypothesis $f \in H^2$ (because $f_s \in H^\infty$) and we can choose a subsequence $(g(1-a^{n_k}))_{k\in\mathbb{N}}$ which is bounded away from zero, i.e, there exists a constant M such that $|g(1-a^{n_k})| \geq M > 0$. As f_s is an eigenvector we obtain by a direct calculation the following relation:

$$C_{\phi_a}^{n_k} f = C_{\phi_a}^{n_k} f_s g = a^{n_k s} f_s C_{\phi_a}^{n_k} g.$$

Using the expression obtained above, Theorem 2.4.5 and Lemma 3.2.4 we obtain:

$$\begin{aligned}
\left\| \frac{C_{\phi_a}^{n_k} f}{a^{n_k s} g(1-a^{n_k})} - f_s \right\|_2^2 &= \left\| \frac{f_s C_{\phi_a}^{n_k} g}{g(1-a^{n_k})} - f_s \right\|_2^2 \\
&\leq \|f_s\|_\infty^2 \left\| C_{\phi_{a^{n_k}}} \left(\frac{g}{g(1-a^{n_k})} - 1 \right) \right\|_2^2 \\
&= 2\|f_s\|_\infty^2 \int_{\mathbb{D}} \left| \frac{g'(w)}{g(1-a^{n_k})} \right|^2 N_{\phi_{a^{n_k}}}(w) dA(w) + \left| \frac{g\left(1-a^{n_k}\right)}{g\left(1-a^{n_k}\right)} - 1 \right|^2 \\
&\leq \frac{2\|f_s\|_\infty^2}{M^2} \int_{\mathbb{D}} |g'(w)|^2 N_{\phi_{a^{n_k}}}(w) dA(w) \xrightarrow{n\to\infty} 0.
\end{aligned}$$

This means that f_s is the limit (with respect to the H^2 norm) of a sequence of elements living in span$\{f, C_{\phi_a} f, \ldots\}$, i.e, $f_s \in K_f$ and by minimality, $K_f = K_{f_s}$. We are done because f_s is an eigenvector. ∎

Corollary 3.2.6. *Suppose that $g \in H^2$ is such that g' is eventually bounded and there exists a subsequence of $(g(1-a^n))_{n\in\mathbb{N}}$ which is bounded away from zero then $1 \in K_g$. So, K_g is minimal, if and only if, dim $K_g = 1$.*

Proof. Apply the above theorem to $s = 0$. ∎

The result above corrects an error in [9, Theorem 4.2] which rendered the conclusion incorrect. The hypothesis of g' being **EB** is in fact necessary, as the following example demonstrates.

Example 3.2.7. *Choose $g(z) = f_t(z) = (1-z)^{\frac{2\pi i}{\log a}}$ where $t = \frac{2\pi i}{\log a}$ in Corollary 3.2.6. Then*

$$g(1-a^n) = (a^n)^{\frac{2\pi i}{\log a}} = e^{\frac{2\pi i}{\log a}\log a^n} = e^{2\pi i n} = 1$$

for all $n \in \mathbb{N}$ *and in particular* $(g(1-a^n))_{n\in\mathbb{N}}$ *is bounded away from zero. But for* $r \in (0,1)$

$$|g'(r)| = \left|\frac{2\pi i}{\log a}\frac{1}{r-1}e^{\frac{2\pi i}{\log a}\log(1-r)}\right| = \left|\frac{2\pi}{\log a}\right|\left|\frac{1}{r-1}\right| \xrightarrow{r\to 1} \infty$$

which shows that g' *is not* **EB***. Then* $K_g = \mathbb{C}g$ *because* g *is a* C_{ϕ_a}*-eigenvector and* $1 \notin K_g$*.*

As a consequence of these results, we have a partial solution from case A) and C):

Theorem 3.2.8. *If* $g \in H^2$ *belongs to case* (A) *or* (C) *defined above and* g' *is* **EB***, then* K_g *is minimal if and only if dim* $K_g = 1$*.*

Proof. If g belongs to case (A) then $(g(1-a^n))_{n\in\mathbb{N}}$ converges to a non-zero number, say L. Then, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $|g(1-a^n) - L| < \frac{|L|}{2}$ and then $|g(1-a^n)| \geq \frac{|L|}{2} > 0$ for $n \geq n_0$. If g belongs to case (C) then the radial limit does not exist, and in particular it is not 0. So we can find an $\epsilon > 0$ and a subsequence $(g(1-a^{n_k}))_{k\in\mathbb{N}}$ such that $|g(1-a^{n_k})| \geq \epsilon > 0$. The result follows by Corollary 3.2.6. ■

As the reader may have noticed, the case B), when $g(1-a^n) \to 0$ is actually our problem. However (B) (and another cases) can be solved if there is some generator g which is analytic at 1. These was one of the main results (proved incorrectly) in [9].

Theorem 3.2.9. *Suppose that* $g \in H^2$ *is analytic at* 1*. So,* K_g *is minimal, if and only if, dim* $K_g = 1$*.*

Proof. Notice that as g is analytic at 1 then g' is analytic at 1, so g' is eventually bounded. If $g(1) \neq 0$ then we are in the case A) and we are done by the above theorem. Suppose that $g(1) = 0$, then $g(z) = (1-z)^K h(z)$ in some on some open ball V around 1, where K is multiplicity of the zero 1, h analytic at V and $h(1) \neq 0$. Choose $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $D_n \subset V$. Note that h' is eventually bounded because h' is also analytic at V. Moreover,

$$g \circ \phi_{a^n}(z) = a^{nK}(1-z)^K h \circ \phi_{a^n}(z).$$

So $h \circ \phi_{a^n} \in H^\infty$ because h is bounded D_n (in particular, $h \circ \phi_{a^n} \in H^2$). As the function $f = f_K h \circ \phi_{a^{n_0}} = \frac{g\circ\phi_{a^{n_0}}}{a^{n_0}K}$ satisfies the hypotheses of Theorem 3.2.5, we conclude that $f_K \in K_{f_K h\circ\phi_{a^{n_0}}} = K_{g\circ\phi_{a^{n_0}}} \subseteq K_g$. So $K_g = K_{f_K}$ by minimality and we are done. ■

We conclude this section by considering case (B) more carefully. Even with the **EB** hypothesis over g', case (B) appears to be delicate. However if $g(1-a^n)$ has a subsequence that converges to 0 at a *sufficiently* slow rate, then we can still obtain a positive result.

Theorem 3.2.10. *Suppose that* $g \in H^2$ *is such that* g' *is* **EB** *and there exists* $0 < \epsilon < 1$ *and a constant* $L > 0$ *such that* $|g(1-a^{n_k})| \geq La^{n_k(1-\epsilon)}$ *for some subsequence* $(n_k)_{k\in\mathbb{N}}$*. Then* $1 \in K_g$*, and* K_g *is minimal if and only if* dim $K_g = 1$*.*

Proof. The computations are very similar to the last result:

$$\begin{aligned}
\left\|\frac{C_{\phi_a}^{n_k} g}{g(1-a^{n_k})}-1\right\|_2^2 &= \left\|\frac{C_{\phi_a}^{n_k} g}{g(1-a^{n_k})}-1\right\|_2^2 \\
&= \left\|C_{\phi_{a^{n_k}}}\left(\frac{g}{g(1-a^{n_k})}-1\right)\right\|_2^2 \\
&= 2\int_{\mathbb{D}}\left|\frac{g'(w)}{g(1-a^{n_k})}\right|^2 N_{\phi_{a^{n_k}}}(w)dA(w)+\left|\frac{g\left(1-a^{n_k}\right)}{g\left(1-a^{n_k}\right)}-1\right|^2 \\
&= 2\int_{\mathbb{D}}\frac{|g'(w)|^2}{|g(1-a^{n_k})|^2}N_{\phi_{a^{n_k}}}(w)dA(w) \\
&= 2\int_{\mathbb{D}}\frac{|g'(a^{n_k}w+1-a^{n_k})|^2}{|g(1-a^{n_k})|^2}|\phi'_{a^{n_k}}(w)|^2\log\frac{1}{|w|}dA(w) \\
&\leq 2\int_{\mathbb{D}} C^2\frac{1}{L^2a^{2n_k}a^{-2n_k\epsilon}}a^{2n_k}\log\frac{1}{|w|}dA(w) \\
&= a^{2n_k\epsilon}M \xrightarrow{k\to\infty} 0
\end{aligned}$$

where C is a upper bound for the values of g' in some open ball $D_{n_{k_0}}$ and M is a constant. This proves the result. ■

To obtain a complete solution for case (B) under the **EB** hypothesis over g', it is natural to ask what happens if $g(1-a^n)\to 0$ faster than required by Theorem 3.2.10. For instance if

$$|g(1-a^n)|\leq a^{\frac{n}{2}}$$

then the next result shows that the series $\sum_{n=1}^{\infty} C_{\phi_{a^n}} g$ converges in H^2.

Proposition 3.2.11. *Let $g\in H^2$ with g' an* **EB** *function. Then $h:=\sum_{n=1}^{\infty} C_{\phi_{a^n}} g$ converges absolutely if, and only if, $\sum_{n=1}^{\infty}|g(1-a^n)|$ converges in $\mathbb{C}$. Moreover $h\in K_g$.*

Proof. Suppose $\sum_{n=1}^{\infty}|g(1-a^n)|$ converges in $\mathbb{C}$. Using Theorems 2.4.5 and 2.4.4 we conclude that for all $n\geq n_0$ (where D_{n_0} is a ball in which g' is bounded)

$$\begin{aligned}
\|C_{\phi_{a^n}} g\|^2 &= \int_{\mathbb{D}}|g'(w)|^2 N_{\phi_{a^n}}(w)dA(w)+|g(1-a^n)|^2 \\
&= \int_{\mathbb{D}}|g'(a^nz+1-a^n)|^2|\phi'_{a^n}(z)|^2\log\frac{1}{|z|}dA(z)+|g(1-a^n)|^2 \\
&\leq a^{2n}L+|g(1-a^n)|^2
\end{aligned}$$

where L is a positive constant. So

$$\|C_{\phi_{a^n}} g\| \leq \sqrt{a^{2n}L + |g(1-a^n)|^2} \leq a^n\sqrt{L} + |g(1-a^n)| \quad \text{for } n \geq n_0.$$

Since $a < 1$ and $\sum\limits_{n=1}^{\infty} |g(1-a^n)| < \infty$, the comparison test implies

$$\sum_{n=1}^{\infty} \|C_{\phi_{a^n}} g\| < \infty.$$

The reciprocal follows from

$$\|C_{\phi_{a^n}} g\|^2 = \int\limits_{\mathbb{D}} |g'(w)|^2 N_{\phi_{a^n}}(w) dA(w) + |g(1-a^n)|^2 \geq |g(1-a^n)|^2$$

and by the comparison test again. ■

Under the hypotheses of Proposition 3.2.11 if we define $h_k := \sum_{n=k}^{\infty} C_{\phi_{a^n}} g$, then $h \in H^2$ obviously implies all $h_k \in H^2$ for $k \geq 1$. In this case, if K_g is a minimal invariant subspace for C_{ϕ_a} then we must have $K_g = K_{h_k}$ for all $k \in \mathbb{N}$. Notice that if some h_l were independent of the rest (say $h_l \notin \overline{span\{h_{l+1}, h_{l+2}, \ldots\}}$), then K_{h_l} would properly contain K_{h_k} for $k > l$ and in particular K_g can not be minimal. This leads us to conjecture that

If $g \in H^2$ and $\sum\limits_{n=1}^{\infty} C_{\phi_{a^n}} g \in H^2$, then K_g is minimal if and only if g is a C_{ϕ_a}-eigenvector.

3.3 The role of $Z(f)$

In this section we present some results showing how the cardinality of the zero set of f affects the dimension of K_f. The basic idea is that the shrinking property of the orbits allows us to consider the behaviour of the functions in smaller disks. We will use the notation $Z(f)$ to denote the set of zeros of f in $\mathbb{D}$ and by $|Z(f)|$ the cardinality of this set.

Proposition 3.3.1. *If $0 < |Z(f)| < \infty$ then dim $K_f \geq 2$.*

Proof. By the hypothesis of finite zeros, we can choose a number $0 < K < 1$ such that f is zero-free in the annulus $\mathbb{D} - \overline{B(0,K)}$ (for example, $K := \max\{|z| \mid z \in Z(f)\}$). Moreover, we can choose $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $f \circ \phi_{a^n}(\mathbb{D}) \subseteq \mathbb{D} - \overline{B(0,K)}$.

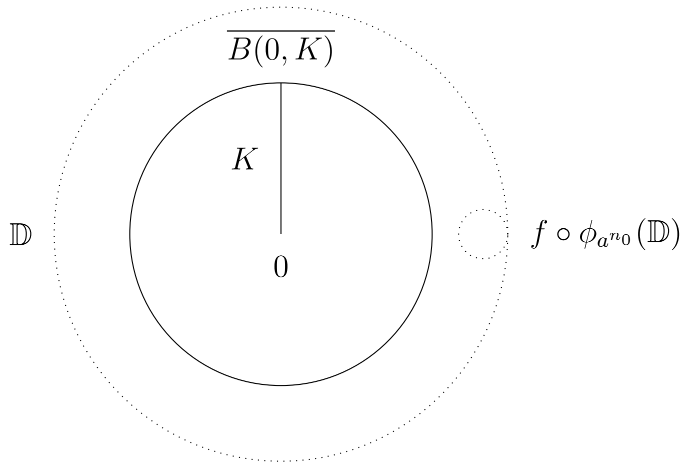


Figure 3: Choosing K and n_0.

If $n \geq n_0$ we claim that $\{f, f \circ \phi_{a^n}\}$ is a linearly independent set. In fact, consider any scalars $\alpha, \beta \in \mathbb{C}$ such that $\alpha f + \beta f \circ \phi_{a^n} = 0$. If $z_0 \in Z(f)$ we have

$$\beta f \circ \phi_{a^n}(z_0) = \alpha f(z_0) + \beta f \circ \phi_{a^n}(z_0) = 0$$

and since $f \circ \phi_{a^n}$ is zero-free, we conclude that $\beta = 0$ and, consequently, $\alpha = 0$. ■

This result has the following interesting consequence: If the ISP is true, then every K_f with f satisfying the above hypotheses is necessarily non-minimal. On the other hand if K_f is minimal for such an f, then the ISP is false. Next we study the cases when $|Z(f)| = 0$ or $|Z(f)| = \infty$, then K_f can be either minimal or not. The first case is more simple:

Example 3.3.2. *Consider $g_1(z) = z^2 + 1$ and $g_2(z) = 1 - z$. Both functions are zero-free in the disk; moreover K_{g_2} is minimal because g_2 is an eigenvector and K_{g_1} is not minimal due to Theorem 3.2.1 or Theorem 3.2.9.*

For the case when $|Z(f)| = \infty$ we will use the following result:

Proposition 3.3.3. *Let $w \in \mathbb{D}$. If $f(w) \neq 0$ but there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $f(a^n w + 1 - a^n) = 0$ then K_f is not minimal.*

Proof. In fact, if K_f is minimal then $K_f = K_{f\circ\phi_{a^{n_0}}}$ and consequently $(K_f)^\perp = (K_{f\circ\phi_{a^{n_0}}})^\perp$. Considering the reproducing kernel κ_w we observe that $\kappa_w \in (K_{f\circ\phi_{a^{n_0}}})^\perp$ since $f(a^n w + 1 - a^n) = 0$ $(n \geq n_0)$ but $\kappa_w \notin (K_f)^\perp$ because $f(w) \neq 0$. This is a contradiction. ■

Example 3.3.4. *The sequence $\{\phi_{a^n}(0)\}_{n\geq 2} = (1 - a^n)_{n\geq 2}$ is a Blaschke sequence because $\sum_{n=2}^{\infty} |1 - (1 - a^n)| = \sum_{n=2}^{\infty} a^n < \infty$. Consider B the Blaschke product associated to this sequence (in particular B has infinitely many zeros). We can write B as*

$$B(z) = \prod_{n=2}^{\infty} \left(\frac{1 - a^n - z}{1 - \overline{(1 - a^n)} z} \right).$$

Note that $B(0) \neq 0$ *but* $B(a^n 0 + 1 - a^n) = B(1 - a^n) = 0$ *for* $n \geq 2$, *so by the previous remark* K_B *is not minimal.*

The only remaining case is an example of some $f \in H^2$ such that K_f minimal and $|Z(f)| = \infty$.

Example 3.3.5. *Let* $a = \frac{1}{2}$. *Consider the function* $f = e_0 + f_s$ *where* $s = \frac{2\pi i}{\log a}$ *and* e_0 *is the constant function* 1. *Since*

$$C_{\phi_a}(e_0 + f_s) = e_0 + a^s f_s = e_0 + a^{\frac{2\pi i}{\log a}} f_s = e_0 + f_s.$$

it follows that $e_0 + f_s$ *is a* C_{ϕ_a} *eigenvector. So* K_f *is minimal. Moreover, note that*

$$(e_0 + f_s)(1 - \sqrt{2}) = 1 + f_s(1 - \sqrt{2}) = 1 + e^{\frac{2\pi i}{-2\log 2} \log 2} = 1 + e^{-\pi i} = 0.$$

Now, consider the sequence $\{\phi_{a^n}(1 - \sqrt{2})\}_{n \in \mathbb{N}} \subseteq \mathbb{D}$. *Then*

$$(e_0 + f_s)(\phi_{a^n}(1 - \sqrt{2})) = C^n_{\phi_a}(e_0 + f_s)(1 - \sqrt{2}) = (e_0 + f_s)(1 - \sqrt{2}) = 0$$

and thus we conclude that f *has infinitely many zeros in* $\mathbb{D}$.

The next result of this section strengthens Proposition 3.3.1 and shows we can always find a function in K_f that is orthogonal to another certain functions in K_f.

Proposition 3.3.6. *Suppose that* $0 < |Z(f)| < \infty$ *and let* $z_0 \in Z(f)$. *There exists a non-zero* $g \in K_f$ *such that* $\langle g, h \rangle = 0$ *for every* $h \in K_f$ *such that* $h(z_0) = 0$.

Proof. Let $z_0 \in \mathbb{D}$ a zero of f and consider the continuous map $E_{z_0} : K_f \to \mathbb{C}$ which is exactly the restriction of the evaluation map at z_0 defined on H^2. Note that E_{z_0} is surjective because there exists $n \geq n_0$ such that $C^n_{\phi_a} f(z_0) \neq 0$. So,

$$K_f \ominus Ker_{E_{z_0}} \simeq \frac{K_f}{Ker_{E_{z_0}}} \simeq \mathbb{C}.$$

which implies that $K_f \ominus Ker_{E_{z_0}}$ is a one-dimensional space. Let $g \in K_f \ominus Ker_{E_{z_0}}$ a non-null element. Thus $\langle g, h \rangle = 0$ whenever $h \in K_f$ is such that $h(z_0) = 0$.

∎

We end this chapter with a result that highlights a connection between cyclicity and universality. We note that the best known examples of universal operators are adjoints of analytic Toeplitz operators $T_\phi f = \phi f$ for $\phi \in H^\infty$, $f \in H^2$ or those that are similar to them (see [12] and [20]), and all such coanalytic Toeplitz operators are cyclic (see [43]).

Our universality result is based in the Caradus-Pozzi criteria and give a direction for a future work (see the discussion after the next theorem). The definition of a cyclic operator will appear in the next chapter (see Definition 4.1.1) but we will use it now.

Theorem 3.3.7. *If $T \in B(H)$ is a cyclic operator with closed range and with infinite dimensional kernel then T is universal.*

Proof. By the Caradus-Pozzi criterion (Theorem 2.1.5) it is enough to prove that $T(H)$ has finite codimension. In fact for any cyclic T the dimension of $T(H)^\perp$ is either 0 or 1. Although this is known to experts, we provide a proof for the sake of completeness. Let f be a cyclic vector for $T \in B(H)$ and define $N := \overline{\text{span}_{n\geq 1}\{T^n f\}}$. Let $P : H \to N^\perp$ be the orthogonal projection onto $N^\perp$. Now let g be any element in $T(H)^\perp$. Then we have $\langle g, T^n f\rangle = 0$ for $n \geq 1$ and consequently $g \in N^\perp$. Since f is a cyclic vector, we can find a sequence $g_n \in \overline{\text{span}_{n\geq 0}\{T^n f\}}$ such that $g_n \to g$. If we write $g_n = \alpha_n f + t_n$ where the α_n are scalars and $t_n \in N$, then we obtain $\alpha_n f + t_n \to g$. Applying P to this we conclude that $\alpha_n P f \to Pg = g$ and hence $g = \alpha P f$ for some $\alpha \in \mathbb{C}$. Since g is an arbitrary element of $T(H)^\perp$, the latter space is at most one-dimensional. Finally codim $T(H) = \dim T(H)^\perp \leq 1$ because T has closed range and the result follows. ■

Recall that if $\phi \in LFT(\mathbb{D})$ is hyperbolic then $C_\phi - \lambda$ is universal for all eigenvalues λ. The classical proof of the automorphic case (see [33]) and the recent proof of the non-automorphic case (see [9]) are both elaborate and involved. An elegant proof of the automorphic case was found recently by Cowen and Gallardo-Gutiérrez (see [12]). Theorem 3.3.7 suggests a possible approach to simplify both proofs: this approach is based on showing that the range of $C_\phi - \lambda$ is closed for some eigenvalue λ, rather than proving surjectivity. It is known that when $\phi \in LFT(\mathbb{D})$ is a hyperbolic map then the composition operator C_ϕ is hypercyclic, and in particular cyclic (see [1, Theorem 1.47]). Also for every $\lambda \in \mathbb{C}$ and bounded linear operator T we have

$$\text{span}\{f, Tf, T^2 f, \dots\} = \text{span}\{f, (T-\lambda)f, (T-\lambda)^2 f, \dots\}.$$

It follows that T is cyclic if and only if $T - \lambda$ is cyclic. Thus for all $\lambda \in \mathbb{C}$ the operator $C_\phi - \lambda$ is cyclic whenever ϕ is hyperbolic. Moreover for λ in the point spectrum of ϕ, the kernel of $C_\phi - \lambda$ is infinite dimensional ([13, Lemma 7.24 and Theorem 7.4]). Therefore the closure of the range of $C_\phi - \lambda$ gives universality by Theorem 3.3.7. For now, we do not have a simple proof of this fact.

4 About some types of invariant subspaces of C_{ϕ_a}

Recall from the last chapter that the minimal elements of $Lat(C_{\phi_a})$ are related with the Invariant Subspace Problem. The main goal of this chapter is to understand which Model and Beurling type spaces are invariant under C_{ϕ_a}, i.e, to understand which of these spaces are in $Lat(C_{\phi_a})$. Moreover, we will see some interesting connections with another topics including linear dynamics and the Cesàro operator that are not directly related with the ISP.

4.1 Some aspects about the dynamic of C_{ϕ_a}

In this section we begin by proving some dynamical properties of the operator C_{ϕ_a}. One of the most classical definitions in linear dynamics is the following:

Definition 4.1.1. *Let T be a bounded linear operator in a Banach space X. We say that:*

- *T is **hypercyclic** if there exists $x \in X$ such that $Orb(x,T) := \{x, Tx, T^2x, \dots\}$ is dense in X. In this case, x is called a **hypercyclic vector** for T.*
- *T is **cyclic** if there exists $x \in X$ such that $span\{x, Tx, T^2x, \dots\}$ is dense in X. In this case, x is called a **cyclic vector** for T.*

We commented in the end of the last section that the composition operator C_{ϕ_a} is a hypercyclic operator (see [1, Theorem 1.47]). It is interesting that the hypothesis that we used during the last section excludes the hypercyclic possibility, more precisely:

Proposition 4.1.2. *Let $f \in H^2$ be a hypercyclic vector for C_{ϕ_a} where $a \in (0,1)$.*

- *If $n \geq 1$ then $f^{(n)}$ (the n-th derivative of f) is not eventually bounded* (**EB**).
- *$\lim_{n\to\infty} f(1-a^n)$ does not exist. In particular, f is not analytic at 1.*

Proof.

- Let f be a hypercyclic vector for C_{ϕ_a}. Suppose on the contrary that $f^{(n)}$ is **EB** for some $n \geq 1$ and consider $e_n(z) = z^n$. Then there exists a subsequence $(C_{\phi_a}^{k_l} f)_{l\in\mathbb{N}}$ such that $C_{\phi_a}^{k_l} f \to e_n$ as $l \to \infty$. Note that for every $g \in H^2$ we have $\langle g, e_n\rangle = \frac{g^{(n)}(0)}{n!}$, so

$$\langle C_{\phi_a}^{k_l} f, e_n\rangle = \frac{\left(C_{\phi_{a^{k_l}}} f\right)^{(n)}(0)}{n!} = \frac{(a^{k_l})^n f^{(n)} \circ \phi_{a^{k_l}}(0)}{n!} \xrightarrow{k\to\infty} 0$$

because $f^{(n)}$ is **EB** and $(a^{k_l})^n = (a^n)^{k_l} \to 0$ as $l \to \infty$. On the other hand

$$\langle C_{\phi_a}^{k_l} f, e_n\rangle \to \langle e_n, e_n\rangle = \|e_n\|^2.$$

Thus $e_n = 0$ which is absurd. This contradiction establishes the result.

- We will argue by contradiction again. Suppose that $\lim_{n\to\infty} f(1-a^n) = L$ and pick a function $g \in H^2$ such that $g(0) \neq L$. As f is hypercyclic there exists a subsequence $(C_{\phi_a}^{n_k} f)_{k\in\mathbb{N}}$ such that $C_{\phi_a}^{n_k} f \to g$. Since H^2 convergence implies pointwise convergence we obtain

$$f(1-a^{n_k}) = (C_{\phi_a}^{n_k} f)(0) \to g(0).$$

This is a contradiction.

■

Remark 4.1.3. *Note that the proof of the second item above shows us that any hypercyclic vector f has the property that $(f(1-a^n))_{n\in\mathbb{N}}$ is a sequence that contains subsequences converging to any complex number. This means that $\overline{\{f(1-a), f(1-a^2), \ldots\}} = \mathbb{C}$. Until now, we are not able to give a concrete example of a hypercyclic vector; however, this seems to be a general difficult problem in linear dynamics as commented in [1, p. 264].*

In the above theorem we saw some hypotheses that are not compatible with hypercyclicity, we can ask now if the same is true when we change hypercyclicity by cyclicity. The answer is no and this can be viewed as a consequence of the characterization of when a reproducing kernel is a cyclic vector. This will be our next theorem.

There is a classical way to prove that some vector is cyclic: by Proposition A.0.1: $f \in H^2$ is cyclic (i.e, $K_f = H^2$) if and only if $\{f, f\circ\phi_a, f\circ\phi_{a^2}, \ldots\}^\perp = \{0\}$. We will use this idea many times without explicit remarks.

Theorem 4.1.4. *Let $\kappa_\alpha \in H^2$ be a reproducing kernel ($\alpha \in \mathbb{D}$). Then κ_α is a cyclic vector for C_{ϕ_a} if and only if $\alpha \neq 0$.*

Proof. If κ_α is cyclic then $\alpha \neq 0$, otherwise $\kappa_\alpha = \kappa_0 = 1$ and K_{κ_α} is the space of constants functions. For the other direction, let κ_α with $\alpha \neq 0$. Note that $1-\overline{\alpha}+\overline{\alpha}a \neq 0$, otherwise $1 = \overline{\alpha}(1-a)$ and this is not possible because $\overline{\alpha}, (1-a) \in \mathbb{D}$. Thus

$$\kappa_\alpha \circ \phi_a(z) = \frac{1}{1-\overline{\alpha}(az+1-a)} = \frac{1}{1-\overline{\alpha}+\overline{\alpha}a-\overline{\alpha}az} = \frac{1}{1-\overline{\alpha}+\overline{\alpha}a}\left(\frac{1}{1-\frac{\overline{\alpha}az}{1-\overline{\alpha}+\overline{\alpha}a}}\right).$$

We observe that a function of the form $h(z) = \frac{1}{1-\overline{y}z}$ where $y \in \mathbb{C}$ belongs to H^2 if, and only if, $y \in \mathbb{D}$; so $\frac{\overline{\alpha}a}{1-\overline{\alpha}+\overline{\alpha}a} \in \mathbb{D}$. Consequently, for every $a \in (0,1)$ we have

$$\kappa_\alpha \circ \phi_a = \frac{1}{1-\overline{\alpha}+\overline{\alpha}a}\kappa_{\frac{\alpha a}{1-\alpha+\alpha a}}.$$

Now let $f \in H^2$ such that $\langle f, \kappa_\alpha \circ \phi_{a^n}\rangle = 0$. Then $f(\frac{\alpha a^n}{1-\alpha+\alpha a^n}) = 0$ for every $n \in \mathbb{N}$. As the sequence $\{\frac{\alpha a^n}{1-\alpha+\alpha a^n}\}_{n\in\mathbb{N}}$ is a sequence of distinct points (because $\alpha \neq 0$), $\frac{\alpha a^n}{1-\alpha+\alpha a^n} \to 0$ and f is analytic at 0 we conclude that $f = 0$. Thus $K_{\kappa_\alpha} = H^2$. ■

Every reproducing kernel κ_w is analytic at a neighborhood of $\overline{\mathbb{D}}$. In particular, the same is true for all derivatives of κ_w. As a consequence, $\kappa_w^{(l)}$ is eventually bounded for all $l \in \mathbb{N}$ and $\kappa_w(1-a^n) \to \kappa_w(1) = \frac{1}{1-\overline{w}}$ as $n \to \infty$. So, as promised, this show us that Proposition 4.1.2 is not more true when we change hypercyclic vectors by cyclic vectors. Another class of functions that are cyclic vectors of C_{ϕ_a} and are analytic in a neighborhood of $\overline{\mathbb{D}}$ are the following Blaschke products.

Theorem 4.1.5. *Let $B(z) = \frac{z_1 - z}{1-\overline{z_1}z}$ where $z_1 \in \mathbb{D} - \{0\}$. Then B is a cyclic vector of C_{ϕ_a}.*

Proof. We will argue as in the above proof. Let $f \in H^2$ such that $\langle f, B\circ\phi_{a^n}\rangle = 0$ for all $n \in \mathbb{N}$. Our objective is to prove that $f = 0$. First, note that

$$\begin{aligned}
B\circ\phi_{a^n}(z) &= \frac{z_1 - a^n z - 1 + a^n}{1-\overline{z_1}(a^n z + 1 - a^n)} \\
&= \frac{z_1 - 1 + a^n}{1-\overline{z_1}a^n z - \overline{z_1} + \overline{z_1}a^n} - \frac{a^n z}{1-\overline{z_1}a^n z - \overline{z_1} + \overline{z_1}a^n} \\
&= \frac{z_1 - 1 + a^n}{1-\overline{z_1} + a^n\overline{z_1}}\left(\frac{1}{1-\frac{\overline{z_1}a^n z}{1-\overline{z_1}+a^n\overline{z_1}}}\right) - \frac{a^n}{1-\overline{z_1}+a^n\overline{z_1}}\left(\frac{z}{1-\frac{\overline{z_1}a^n z}{1-\overline{z_1}+a^n\overline{z_1}}}\right)
\end{aligned}$$

Let us define for each $n \in \mathbb{N}$ the number $\eta_{a,n} := \frac{z_1 a^n}{1-z_1+a^n z_1}$ for notation reasons. By the proof of the above theorem, this is an element of $\mathbb{D}$ and then

$$B\circ\phi_{a^n} = \frac{z_1 - 1 + a^n}{1-\overline{z_1}+a^n\overline{z_1}}\kappa_{\eta_{a,n}} - \frac{a^n}{1-\overline{z_1}+a^n\overline{z_1}}M_z\kappa_{\eta_{a,n}}$$

where M_z denotes the shift operator (multiplication by z) in H^2. Thus

$$\begin{aligned}
0 = \langle f, B\circ\phi_{a^n}\rangle &= \left\langle f, \frac{z_1 - 1 + a^n}{1-\overline{z_1}+a^n\overline{z_1}}\kappa_{\eta_{a,n}}\right\rangle - \left\langle f, \frac{a^n}{1-\overline{z_1}+a^n\overline{z_1}}M_z\kappa_{\eta_{a,n}}\right\rangle \\
&= \frac{\overline{z_1} - 1 + a^n}{1-z_1+a^n z_1}\langle f, \kappa_{\eta_{a,n}}\rangle - \frac{a^n}{1-z_1+a^n z_1}\left\langle \frac{f-f(0)}{z}, \kappa_{\eta_{a,n}}\right\rangle \\
&= \frac{\overline{z_1} - 1 + a^n}{1-z_1+a^n z_1} f(\eta_{a,n}) - \frac{a^n}{1-z_1+a^n z_1}\left(\frac{f-f(0)}{z}\right)(\eta_{a,n}). \\
&= \frac{\overline{z_1} - 1 + a^n}{1-z_1+a^n z_1} f(\eta_{a,n}) - \frac{a^n}{1-z_1+a^n z_1}\left(\frac{f(\eta_{a,n})-f(0)}{\frac{z_1 a^n}{1-z_1+a^n z_1}}\right). \\
&= \frac{\overline{z_1} - 1 + a^n}{1-z_1+a^n z_1} f(\eta_{a,n}) - \frac{1}{z_1}\left(f(\eta_{a,n}) - f(0)\right) \qquad (\star)
\end{aligned}$$

where we used that $M_z^*(g) = \frac{g-g(0)}{z}$ for every $g \in H^2$. Now, note that $\lim_{n\to\infty}\eta_{a,n} = 0$. So, if we let $n \to \infty$ in the above equality we obtain:

$$0 = \lim_{n\to\infty}\left(\frac{\overline{z_1} - 1 + a^n}{1 - z_1 + a^n z_1} f(\eta_{a,n}) - \frac{1}{z_1}\left(f(\eta_{a,n}) - f(0)\right)\right) = \left(\frac{\overline{z_1} - 1}{1 - z_1}\right) f(0)$$

because f is in particular continuous at 0. Then $f(0) = 0$ and by $(\star)$

$$0 = \left(\frac{\overline{z_1} - 1 + a^n}{1 - z_1 + a^n z_1} - \frac{1}{z_1}\right) f(\eta_{a,n}) \Rightarrow \left(\frac{\overline{z_1} - 1 + a^n}{1 - z_1 + a^n z_1}\right) f(\eta_{a,n}) = \frac{1}{z_1} f(\eta_{a,n})$$

for each $n \in \mathbb{N}$ which implies $f(\eta_{a,n}) = 0$ for at least infinitely many numbers $n \in \mathbb{N}$ because $\left(\frac{\overline{z_1}-1+a^n}{1-z_1+a^n z_1}\right)_{n\in\mathbb{N}}$ converges to a number of modulus 1 and the modulus of $\frac{1}{z_1}$ is greater than 1. Then f is null in a sequence of distincts points that goes to 0. The analicity of f at 0 is enough to conclude that $f = 0$. We are done. ■

Remark 4.1.6. *The above theorem shows a difference between the non-automorphic and the automorphic hyperbolic case. It was shown in [19, Proposition 3.4] that no finite Blaschke product can be a cyclic vector of a composition operator induced by a hyperbolic automorphism. Moreover, note that hypotheses of the above two theorems excludes exactly the functions* 1 *and* $-z$*, respectively. These vectors are never cyclic due to Proposition 3.2.3.*

The result above is not more true for general Blaschke products as the following example show us.

Example 4.1.7. *Consider the sequence* $(1 - a^n)_{n\in\mathbb{N}}$*; note that this is a Blaschke sequence because* $\sum_{n=0}^{\infty} 1 - |1 - a^n| = \sum_{n=0}^{\infty} a^n < \infty$*. So the Blaschke product whose zeros are* 0 *(with multiplicity one) and* $(1 - a^n)_{n\in\mathbb{N}}$ *is well defined and we call it* B*.*

Consider the constant function $1 = \kappa_0$*. Note that* $\langle B, \kappa_0\rangle = B(0) = 0$ *and for every natural number* n *we have*

$$\langle B \circ \phi_{a^n}, \kappa_0\rangle = B \circ \phi_{a^n}(0) = B(1 - a^n) = 0$$

Thus B *is not a cyclic vector because* $\kappa_0 \in \{B, B \circ \phi_a, \ldots\}^{\perp}$*.*

4.2 Invariant Model spaces

In this section we will characterize which Model spaces are invariant under C_{ϕ_a}. A first step in this direction should be an example of a Model space that is invariant:

Example 4.2.1. *For all* $n \in \mathbb{N}_0$ *let* $e_n(z) = z^n$ *and consider the model space* $(e_n H^2)^{\perp}$*. If* $n = 0$*,* $(e_n H^2)^{\perp} = \{0\}$ *which is of course* C_{ϕ_a}*-invariant. If* $n \geq 1$ *we claim that* $(e_n H^2)^{\perp}$ *is the space of polynomials of degree at most* $n - 1$*: in fact, let* $p(z) = a_0 + \ldots + a_{n-1} z^{n-1}$ *and consider* $e_n g \in e_n H^2$*. Thus* $e_n(z) g(z) = b_0 z^n + b_1 z^{n+1} + \ldots$ *for certain scalars and we see that* $\langle p, e_n g\rangle = 0$ *and therefore* $p \in (e_n H^2)^{\perp}$*. This show us one inclusion, the other inclusion* *is true because if* $g \in (e_n H^2)^\perp$ *then* $\langle g, e_m \rangle = 0$ *for all* $m \geq n$ *by considering the products* $e_n e_l$ *for* $l \geq 0$*. By Proposition 3.2.3 whenever* p *is a polynomial* $C_{\phi_a}(p)$ *is again a polynomial with the same degree of* p*; thus for every* $n \in \mathbb{N}_0$ $(e_n H^2)^\perp$ *is* C_{ϕ_a}*-invariant.*

Despite the simplicity of this example we will show in the next pages that there are no model spaces of the form $(\Theta H^2)^\perp$ that are invariant subspaces of C_{ϕ_a} unless $\Theta = e_n$ for some $n \in \mathbb{N}_0$. To obtain this conclusion we will need two recent results due to S. Bose, P. Muthukumar and J. Sarkar.

Theorem 4.2.2. *[32, Theorem 2.1] Let* Θ *be an inner function and* φ *be a holomorphic self-map of* $\mathbb{D}$*. The following are equivalent:*

1. $\Theta H^2 \in Lat(C_\varphi)$.
2. $\dfrac{\Theta \circ \varphi}{\Theta} \in \mathcal{S}$ *(The Schur class).*
3. $\dfrac{\Theta \circ \varphi}{\Theta} \in H^\infty$.
4. $\dfrac{\Theta \circ \varphi}{\Theta} \in H^2$.

Theorem 4.2.3. *[32, Theorem 4.3] Let* Θ *be an inner function and let* $\varphi(z) = az + b$, $a \neq 0$*. Suppose* $\sigma(z) = \frac{\bar{a}z}{1-\bar{b}z}$, $z \in \mathbb{D}$*. The following are equivalent:*

1. $(\Theta H^2)^\perp \in Lat(C_\varphi)$.
2. $\dfrac{\Theta \circ \sigma}{\Theta} \in H^\infty$.
3. $\Theta H^2 \in Lat(C_\sigma)$.

The idea of the next pages is to apply these general theorems and to use the concrete symbol that we have to obtain the results. We proceed with the following lemma.

Lemma 4.2.4. *Let* $a \in (0,1)$*. Then for all* $\theta \in (0, 2\pi]$

$$\left| \frac{ae^{i\theta}}{1 - e^{i\theta} + ae^{i\theta}} \right| \leq 1$$

and if

$$\left| \frac{ae^{i\theta}}{1 - e^{i\theta} + ae^{i\theta}} \right| = 1$$

for some $\theta \in (0, 2\pi]$ *then* $e^{i\theta} = 1$.

Proof. We can write $e^{i\theta} = x + iy$ with $x^2 + y^2 = 1$. A direct computation show us that the above inequality is true if, and only if,

$$|a|^2 \leq |1 - e^{i\theta} + ae^{i\theta}|^2 = |1 - x - iy + ax + aiy|^2 = (1 - x + ax)^2 + (y(a-1))^2.$$

Computing the right site we obtain:

$$\begin{aligned}(1 - x + ax)^2 + (y(a-1))^2 &= 1 - 2x + x^2 + 2ax - 2ax^2 + a^2x^2 + y^2a^2 - 2ay^2 + y^2 \\ &\overset{x^2+y^2=1}{=} 2 - 2x + 2ax - 2a + a^2.\end{aligned}$$

Thus the inequality is true if, and only if,

$$a^2 \leq 2 - 2x + 2ax - 2a + a^2 \Leftrightarrow 0 \leq 1 - x + ax - a \Leftrightarrow (1-x)(1-a) \geq 0.$$

As $(1 - a) > 0$ and $(1 - x) \geq 0$ the first claim is true. For the second claim, note that this is equivalent to $a^2 = 2 - 2x + 2ax - 2a + a^2$, i.e, equivalent to

$$(1-x)(1-a) = 0$$

and this implies $x = 1$ and consequently $y = 0$. We are done. ∎

The next theorem show us that if $(\Theta H^2)^\perp$ is invariant then the only possible zero of Θ is 0.

Theorem 4.2.5. *Let Θ be an inner function. If $(\Theta H^2)^\perp$ is C_{ϕ_a}-invariant then $\Theta(z_1) \neq 0$ for all $z_1 \in \mathbb{D} - \{0\}$.*

Proof. For the sake of contradiction, suppose that $\Theta(z_1) = 0$ for some $z_1 \in \mathbb{D} - \{0\}$. Thus

$$\langle \Theta f, \kappa_{z_1} \rangle = \Theta(z_1) f(z_1) = 0 \ \ \forall f \in H^2$$

which implies $\kappa_{z_1} \in (\Theta H^2)^\perp$. By hypothesis, this is a C_{ϕ_a}-invariant subspace, so $K_{\kappa_{z_1}} \subseteq (\Theta H^2)^\perp$. By Lemma 4.1.4 κ_{z_1} is a cyclic vector (because $z_1 \neq 0$) and thus $H^2 = (\Theta H^2)^\perp$ which implies $\{0\} = \Theta H^2$. So Θ is the zero function which give us a contradiction. We conclude that Θ does not have zeros in $\mathbb{D} - \{0\}$ as desired. The second part follows from the Canonical Factorization Theorem (Theorem 2.5.8). ∎

By the above Theorem the only possible inner functions Θ such that $(\Theta H^2)^\perp$ is invariant have the form $\Theta(z) = \lambda z^n S(z)$. The next theorem analyzes the inner singular case.

Theorem 4.2.6. *Let $M = (\Theta H^2)^\perp$ be a model space where Θ is a singular inner function. Then M is not C_{ϕ_a}-invariant.*

Proof. Suppose that $M = (\Theta H^2)^{\perp}$ is C_{ϕ_a}-invariant, we will arrive at a contradiction. Consider the function $\sigma(z) = \frac{az}{1-(1-a)z}$ which is an analytic self map of $\mathbb{D}$. Note that

$$(\Theta \circ \sigma)^*(e^{i\theta}) = \lim_{r \to 1^-} \Theta \circ \sigma(re^{i\theta}) = \lim_{r \to 1^-} \Theta(\sigma(re^{i\theta})) = \lim_{r \to 1^-} \Theta \left(\frac{are^{i\theta}}{1-(1-a)re^{i\theta}} \right).$$

If $\theta \neq 2\pi$ then by Lemma 4.2.4 we have $\left(\frac{ae^{i\theta}}{1-(1-a)e^{i\theta}}\right) \in \mathbb{D}$. Using that Θ is continuous in $\mathbb{D}$ and $|\Theta(w)| < 1$ for all $w \in \mathbb{D}$ because Θ is nonconstant (see [30, Theorem 2.2.10]) we obtain:

$$|(\Theta \circ \sigma)^*(e^{i\theta})| \overset{a.e}{=} \left| \Theta \left(\frac{ae^{i\theta}}{1-(1-a)e^{i\theta}} \right) \right| \overset{a.e}{<} 1.$$

By Theorem 4.2.3 M is invariant under C_{ϕ_a} if, and only if, ΘH^2 is invariant under C_σ. Thus there exists $g \in H^2$ such that

$$\Theta \circ \sigma = C_\sigma(\Theta) = \Theta g \implies \Theta(\sigma(z)) = \Theta(z) g(z) \quad \forall z \in \mathbb{D}. \tag{1}$$

Passing to the radial limits and considering the modulus we conclude that

$$1 \overset{a.e}{>} |(\Theta \circ \sigma)^*(e^{i\theta})| \overset{a.e}{=} |g^*(e^{i\theta})|.$$

As $g \in H^2$ this implies that $|g(z)| \leq 1$ for every $z \in \mathbb{D}$ see for example ([30, Corollary 1.1.24]). But, if we consider $z = 0$ in (1) we conclude that

$$\Theta(0) = \Theta(\sigma(0)) = \Theta(0) g(0)$$

and thus g(0) = 1 because Θ is inner singular, in particular, zero-free. So by the maximum module principle, g is constant and $g \equiv g(0) = 1$. Looking at (1) again we conclude that $C_\sigma(\Theta) = \Theta$ and then Θ is a fixed point. But considering the equality

$$\Theta \circ \sigma = C_\sigma(\Theta) = \Theta,$$

passing to radial limits and using the estimates proved above we obtain

$$1 \overset{a.e}{>} |(\Theta \circ \sigma)^*(e^{i\theta})| \overset{a.e}{=} |\Theta^*(e^{i\theta})| \overset{a.e}{=} 1$$

which is a contradiction. So Θ is constant and we arrived at a contradiction. ∎

Corollary 4.2.7. *Let Θ be an inner function. If $\Theta(z) = \lambda z^n S(z)$ for some S inner singular and $n \geq 1$ then $(\Theta H^2)^{\perp}$ is not C_{ϕ_a}-invariant.*

Proof. By Theorem 4.2.3 $(\Theta H^2)^\perp$ is invariant under C_{ϕ_a} if and only if $\frac{\Theta\circ\sigma}{\Theta} \in H^\infty$ where $\sigma(z) = \frac{az}{1-(1-a)z}$. Note that

$$\frac{\Theta \circ \sigma(z)}{\Theta(z)} = \frac{a^n z^n S \circ \sigma(z)}{(1-(1-a)z)^n z^n S(z)} = \frac{a^n S \circ \sigma(z)}{(1-(1-a)z)^n S(z)}.$$

If this function belongs to H^∞ then multiplying by $\frac{(1-(1-a)z)^n}{a^n}$ which belongs to H^∞ we conclude that $\frac{S\circ\sigma}{S} \in H^\infty$. Thus $(SH^2)^\perp$ is C_{ϕ_a}-invariant using again Theorem 4.2.3. This contradicts Theorem 4.2.6. ■

We conclude that the singular inner part cannot appear in the decomposition of the inner function Θ above. Moreover, for any scalar $\lambda \neq 0$ the spaces $\lambda e_n H^2$ and $e_n H^2$ are the same; thus Example 4.2.1 and the work above implies the following consequence which is the main result of this section:

Corollary 4.2.8. *The only model spaces that are invariant under C_{ϕ_a} are of the form $(e_n H^2)^\perp$ for some $n \in \mathbb{N}_0$.*

In what follows we will show some corollaries of our results. Since a model space is a closed subspace of a reproducing kernel Hilbert space (RKHS), $(\Theta H^2)^\perp$ is itself a RKHS and the reproducing kernels are given by the functions

$$\kappa_\lambda^\Theta(z) = \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{1 - \overline{\lambda} z}$$

where λ is some fixed element in $\mathbb{D}$ and $z \in \mathbb{D}$ (see [18, Corollary 14.12]).

Proposition 4.2.9. *If Θ is inner and non constant and $n \geq 1$ the spaces $e_n(\Theta H^2)^\perp$ are not C_{ϕ_a}-invariant.*

Proof. Suppose that this space is invariant and let $e_n g \in e_n(\Theta H^2)^\perp$ where $g \in (\Theta H^2)^\perp$; then $C_{\phi_a}(e_n g) = e_n G$ for some $G \in (\Theta H^2)^\perp$. This means that

$$(az + 1 - a)^n g \circ \phi_a(z) = z^n G(z) \qquad \forall z \in \mathbb{D}.$$

Evaluating at 0 we conclude that $g(1-a) = 0$. But there are functions in $(\Theta H^2)^\perp$ that does not satisfies this condition: the reproducing kernel κ_0^Θ given by $\kappa_0^\Theta(z) = 1 - \overline{\Theta(0)}\Theta(z)$ is such that $\kappa_0^\Theta(1-a) = 1 - \overline{\Theta(0)}\Theta(1-a) \neq 0$ (because Θ is inner and non constant, which implies that $\overline{\Theta(0)}$ and $\Theta(1-a)$ are in $\mathbb{D}$). ■

Corollary 4.2.10. *If S is inner singular and $n \in \mathbb{N}$ then $(e_n S H^2)^\perp$ is the direct sum of an invariant and a non-invariant C_{ϕ_a}-subspace.*

Proof. It is known that

$$(e_n S H^2)^\perp = (e_n H^2)^\perp \oplus e_n (SH^2)^\perp$$

(see for example [18, Lemma 14.6]) and the result follows from the above proposition and Example 4.2.1. ■

To finish this section, we will present a consequence involving universality. Since the operator C_{ϕ_a} has some universal translates, the minimal elements of *Lat* C_{ϕ_a} are specially interesting because they are related with the ISP as we commented in the introduction. The next corollary show us that if some minimal invariant subspace is a model space, we have a positive answer related with the ISP.

Corollary 4.2.11. *Suppose that K_f is minimal. If K_f is a model space, then dim $K_f = 1$.*

Proof. It is clear that $K_f = (e_n H^2)^\perp$ for some $n \geq 1$ due to Corollary 4.2.8. The space $K_f = (e_n H^2)^\perp$ is finite dimensional and thus Remark 2.1.9 implies the result. ■

A more general question inspired by the above corollary is what happens if some minimal invariant subspace K_f contains a function $g \neq 0$ that belongs to some model space. In this case, of course $K_g = K_f$. We mention here a result that answer part of this question.

Proposition 4.2.12. *Suppose that Θ is a nonconstant inner function such that $1 \notin \sigma(\Theta)$ and let $f \in (\Theta H^2)^\perp$. If K_f is minimal, then dim $K_f = 1$.*

Proof. It is known that each function $f \in (\Theta H^2)^\perp$ has an analytic continuation across the point 1 because $1 \in \mathbb{T} - \sigma(\Theta)$ (see [18], Lemma 14.27). The result follows from Theorem 3.2.9. ■

4.3 Invariant Beurling type spaces

If Θ is a zero free inner function then $\Theta = \lambda S_\mu$ where λ is a unimodular constant and S_μ is a singular inner function. In this case, by Theorem 2.5.12 we conclude that $\sigma(\Theta) = supp(\mu) \subseteq \mathbb{T}$. If $supp(\mu)$ has only one point $\xi_0 \in \mathbb{T}$ then

$$S_\mu(z) = e^{-K\left(\frac{\xi_0 + z}{\xi_0 - z}\right)}$$

for some $K > 0$ (see Example 2.5.3). The function above is called the **atomic singular inner function** with atom ξ_0. Our aim in the next pages is to clarify when a Beurling type space is invariant under C_{ϕ_a} and the atomic inner function will play a role in this context. When the inner function is a Blaschke product, a recent characterization using multiplicity of zeros was proved in [3].

Theorem 4.3.1 ([3], Corollary 2.4)**.** *Let B be a Blaschke product whose set of zeros is denoted by $Z(B)$ and let ϕ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent:*

- *BH^2 is C_ϕ-invariant.*
- *$mult_B(w) \le mult_{B\circ\phi}(w)$ for every $w \in Z(B)$.*

Consider for example the Blaschke product formed by the (simple) zeros $(1-a^{2n})_{n\in\mathbb{N}}$. Every $w \in Z(B)$ has multiplicity equal to 1 and $B \circ \phi_{a^2}(1-a^{2n}) = B(1-a^{4n}) = 0$. This means that the zeros of B are all zeros of $B \circ \phi_{a^2}$ which implies by the above Theorem that BH^2 is $C_{\phi_{a^2}}$-invariant. Moreover, note that $B \circ \phi_a(1-a^2) = B(1-a^3) \neq 0$ and thus using again the above theorem we conclude that BH^2 is not C_{ϕ_a}-invariant. We summarize this in the following example.

Example 4.3.2. *The Blaschke product B formed by the zeros $(1-a^{2n})_{n\in\mathbb{N}} = (1-a^2, 1-a^4, 1-a^6, \ldots)$ all with multiplicity 1 is such that BH^2 a $C_{\phi_{a^2}}$-invariant subspace but is not C_{ϕ_a}-invariant.*

The above example can be adapted to orbits starting in any point: let $z_0 \in \mathbb{D}$ and let B be the Blaschke product formed by the simple zeros $(z_0, az_0+1-a, a^2z_0+1-a^2, \ldots)$. The space BH^2 is C_{ϕ_a}-invariant because $B \circ \phi_a(a^n z_0 + 1 - a^n) = B(a^{n+1}z_0 + 1 - a^{n+1}) = 0$ for all $n \in \mathbb{N}_0$. This basic fact helps us to obtain the following result:

Corollary 4.3.3. *Let Θ be an inner function such that ΘH^2 is C_{ϕ_a}-invariant. Then there exists an inner function Υ such that $\{0\} \subset \Upsilon H^2 \subset \Theta H^2$ and ΥH^2 is also C_{ϕ_a}-invariant. In particular, no minimal invariant subspace of C_{ϕ_a} can be a Beurling type space.*

Proof. Let $z_0 \in \mathbb{D}$ be any point and consider the Blaschke product B formed by the zeros $(z_0, az_0+1-a, a^2z_0+1-a^2, \ldots)$. By the above discussion BH^2 is C_{ϕ_a}-invariant. Define $\Upsilon = B\Theta$, of course $\{0\} \subset \Upsilon H^2 \subset \Theta H^2$ and Υ is inner.

To prove that ΥH^2 is C_{ϕ_a}-invariant we argue as follows: by Theorem 4.2.2 we have $\frac{B\circ\phi_a}{B} \in \mathcal{S} \subseteq H^\infty$. Moreover, if $\Upsilon g = B\Theta g \in \Upsilon H^2$ then

$$\frac{(B\circ\phi_a)}{B} B(\Theta\circ\phi_a)(g\circ\phi_a) = \frac{(B\circ\phi_a)}{B} B\Theta g_2 = B\Theta\frac{(B\circ\phi_a)}{B} g_2 = \Upsilon h.$$

where $h \in H^2$. Thus $C_{\phi_a}(\Upsilon g) = \Upsilon h \in \Upsilon H^2$ and we are done. The last claim follows directly by the definition of minimality.

∎

Going back to the general question of when a Beurling type space can be C_{ϕ_a}-invariant we present our main result of this section. To prove item 2) below we follow an idea that appears in [22, Chapter VII] for which we give the credit.

Theorem 4.3.4. *Let Θ be a non constant inner function and consider the Beurling type space ΘH^2. If ΘH^2 is C_{ϕ_a}- invariant, then:*

1. $1 \in \sigma(\Theta)$.

2. *If 1 is not a cluster point for the zeros of Θ, then $\sigma(\Theta) \cap S^1 = \{1\}$ and $1 \in supp(\mu)$, where μ is the singular measure associated with Θ.*

Proof. 1. Suppose by contradiction that $1 \notin \sigma(\Theta)$. By Theorem 4.2.2 ΘH^2 is C_{ϕ_a}-invariant if, and only if,

$$\frac{\Theta \circ \phi_a}{\Theta} \in \mathcal{S} \quad (\star)$$

where $\mathcal{S} = \{f \in H^\infty \mid \|f\|_\infty \leq 1\}$. Since $1 \notin \sigma(\Theta)$ we know, by the definition of $\rho(\Theta)$, that Θ has an analytic continuation across some open arc I in $\mathbb{T}$ with $1 \in I$ and $|\Theta(w)| = 1$ for all $w \in I$. Now, consider the sequence $(1 - a^n)_{n \in \mathbb{N}} = \{\phi_{a^n}(0)\}_{n \in \mathbb{N}}$. Note that there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, we have $|\Theta(1 - a^n)| \geq \frac{1}{2}$, otherwise we conclude that for some subsequence $(1 - a^{n_k})$ we have $\frac{1}{2} \geq |\Theta(1 - a^{n_k})| \to |\Theta(1)| = 1$ by continuity and this is a contradiction. So, for $n \geq n_0$ the condition $(\star)$ implies that

$$\ldots \leq |\Theta(1 - a^{n_0+2})| \leq |\Theta(1 - a^{n_0+1})| \leq |\Theta(1 - a^{n_0})| \leq 1$$

when we evaluate $\frac{\Theta \circ \phi_a}{\Theta}$ at the points $\{(1 - a^n)\}_{n \geq n_0}$. Thus

$$|\Theta(1 - a^{n_0+j})| \leq |\Theta(1 - a^{n_0})| \leq 1. \quad (\diamond)$$

As $(1 - a^{n_0+j}) \to 1$ then letting $j \to \infty$ in $(\diamond)$ we conclude that $|\Theta(1 - a^{n_0})| = 1$ which implies Θ constant because every nonconstant inner function has image contained in $\mathbb{D}$ (see [30, Theorem 2.2.10]). This is a contradiction.

2. We can write $\sigma(\Theta) = \overline{\{z_1, z_2, \ldots\}} \cup supp(\mu)$ where μ is the measure associated to the singular inner part of Θ and $(z_n)_{n \in \mathbb{N}}$ are the zeros of Θ. Let $\xi \in \mathbb{T} - \{1\}$ such that $\xi \in \sigma(\Theta)$. By Theorem 2.5.12 we know that

$$\liminf_{w \to \xi,\ w \in \mathbb{D}} |\Theta(w)| = 0$$

As 1 is not a cluster point of the zeros of Θ, we can choose $m_0 \in \mathbb{N}$ such that Θ is zero free in $\phi_{a^n}(\mathbb{D})$ for all $n \geq m_0$. By hypothesis ΘH^2 is C_{ϕ_a}-invariant and then ΘH^2 is also $C_{\phi_{a^{m_0+1}}}$-invariant. By Theorem 4.2.2 we can write $\Theta \circ \phi_{a^{m_0+1}} = \Theta g$ for some $g \in \mathcal{S}$. Note that $a^{m_0+1}\xi + 1 - a^{m_0+1} \in \phi_{a^{m_0}}(\mathbb{D})$ (because $\xi \neq 1$) and

$$\begin{aligned}\liminf_{w \to \xi,\ w \in \mathbb{D}} |\Theta \circ \phi_a^{m_0+1}(w)| &= \liminf_{w \to \xi,\ w \in \mathbb{D}} |\Theta(a^{m_0+1} w + 1 - a^{m_0+1})| \\ &= |\Theta(a^{m_0+1}\xi + 1 - a^{m_0+1})| \neq 0\end{aligned}$$

because Θ is zero-free in $\phi_{a^{m_0}}(\mathbb{D})$. So

$$\liminf_{w\to\xi,\ w\in\mathbb{D}} |g(w)| = \infty$$

which implies g unbounded which is a contradiction. So the only possible point in $\sigma(\Theta)\cap\mathbb{T}$ is 1. For the last claim, since $1 \notin \overline{\{z_1, z_2, \dots\}}$ the only possibility is $1 \in supp(\mu)$. ■

It is known that the space $e^{-K(\frac{1+z}{1-z})}H^2$ (where $K > 0$ is a constant) is C_{ϕ_a}-invariant, this follows from a more general result due to Cowen and Wahl (see [14, Theorem 6]). As a consequence of this result and of the above theorem we have:

Corollary 4.3.5. *Let $a \in (0,1)$. The only zero free inner functions Θ such that ΘH^2 is C_{ϕ_a}-invariant are $\Theta(z) = \lambda e^{-K(\frac{1+z}{1-z})}$ for some $K > 0$ and $\lambda \in \mathbb{T}$.*

Proof. By the above theorem, the unique possibility is $\sigma(\Theta) = supp(\mu) = \{1\}$ where μ is the measure associated to the singular inner part of Θ. Thus Θ is as desired. ■

Corollary 4.3.6. *Let $a \in (0,1)$ and $\Theta \in H^2$ inner. If ΘH^2 is C_{ϕ_a}-invariant then exactly one of the two following situations happens.*

- $\Theta(z) = \lambda e^{-K(\frac{1+z}{1-z})}$ *for some $K > 0$ and $|\lambda| = 1$.*
- Θ *has infinitely many zeros accumulating at* 1.

Proof. If Θ is zero free by the previous corollary we have the first case. If Θ has a zero z_0, as ΘH^2 is C_{ϕ_a}-invariant we have

$$\frac{\Theta \circ \phi_a}{\Theta} \in H^\infty$$

by Theorem 4.2.2. So $\Theta(az_0 + 1 - a) = 0$ otherwise this quotient is not an analytic function; But ΘH^2 is also $C_{\phi_{a^2}}$ invariant because $C_{\phi_{a^2}}(\Theta H^2) = C_{\phi_a}(C_{\phi_a}(\Theta H^2)) \subseteq C_{\phi_a}(\Theta H^2) \subseteq \Theta H^2$. Using again Theorem 4.2.2 we conclude that

$$\frac{\Theta \circ \phi_{a^2}}{\Theta} \in H^\infty$$

Thus $\Theta(a^2 z_0 + 1 - a^2) = 0$. Repeating this argument for each $n \in \mathbb{N}$ we obtain $\Theta(a^n z_0 + 1 - a^n) = 0$ which implies the desired result. ■

Remark 4.3.7. *Due to Example 4.3.2 there exists Beurling type spaces that are invariant under $C_{\phi_{\tilde{a}}}$ for some $\tilde{a} \in (0,1)$ but are not invariant under all C_{ϕ_a} where $a \in (0,1)$ and thus this functions are in the second case of the above theorem because the atomic inner functions generates Beurling spaces that are common invariant subspaces, meaning that it is invariant under all C_{ϕ_a} for all $a \in (0,1)$.*

4.4 A connection with the Cesàro Operator

In this section we will highlight a connection of our work in the last sections with a classical operator called the **Cesàro Operator**. The connection is based in a recent result proved by Gallardo-Gutiérrez and Partington in [21] and it is similar to results obtained in [22].

The Cesàro operator is defined as the operator $\mathcal{C} : H^2 \to H^2$ given by

$$\mathcal{C}f(z) = \frac{1}{z}\int_0^z \frac{f(\xi)}{1-\xi}d\xi \qquad \text{if } z \in \mathbb{D} - \{0\}$$

and $\mathcal{C}f(0) = f(0)$ where $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2$. Using the power series expansion of H^2 functions and integrating term by term we see that the Cesàro operator can be defined also as

$$(\mathcal{C}f)(z) = \sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n} a_k\right) z^n.$$

From the last expression it follows from the famous Hardy's inequality ([24], Chapter IX) that the operator $\mathcal{C}$ is a well defined bounded operator. There is an extensive literature concerning the Cesàro Operator; classical properties like subnormality, spectra, norm and many others was studied by many authors for several years (see [7], [27] and [28]). We mention here that it is possible to define the Cesàro operator similarly in another spaces like H^p for $1 < p < \infty$ but four our purposes H^2 is enough.

In the recent paper [21], the authors obtained the following result:

Theorem 4.4.1. *[21, Theorem 2.1] Let $\Phi = \{\varphi_t\}_{t\geq 0}$ be the holomorphic flow given by*

$$\varphi_t(z) = e^{-t}z + 1 - e^{-t} \qquad (z \in \mathbb{D}).$$

A closed subspace M in H^2 is invariant under the Cesàro operator if and only if its orthogonal complement $M^\perp$ is invariant under the semigroup of composition operators induced by Φ, namely, $\{C_{\varphi_t}\}_{t\geq 0}$.

The bijection between $(0,\infty)$ and $(0,1)$ given by $t \to e^{-t}$ show us that the families $\{C_{\varphi_t}\}_{t>0}$ and $\{C_{\phi_a}\}_{a\in(0,1)}$ are the same. From this we can deduce the following statement:

Corollary 4.4.2. *A Beurling type space M is invariant under the Cesàro operator if and only if $M = e_n H^2$ for some $n \in \mathbb{N}$, where $e_n(z) = z^n$.*

Proof. If M is a Beurling type space that is invariant under the Cesàro operator then by Theorem 4.4.1 and the above comment the model space $M^\perp$ is invariant under the family

$\{C_{\phi_a}\}_{a\in(0,1)}$. Thus Corollary 4.2.8 implies the $M^{\perp} = (e_n H^2)^{\perp}$ which proves the first implication. For the reverse implication, Corollary 4.2.8 implies that the spaces $(e_n H^2)^{\perp}$ are all invariant under the family $\{C_{\phi_a}\}_{a\in(0,1)}$ and thus under the family $\{C_{\varphi_t}\}_{t>0}$. Moreover, the operator $C_{\varphi_0} = Id$ clearly leaves this spaces invariant. This finishes the proof. ∎

Using the results of the last section, we can naturally obtain a result like the above corollary for Model spaces. In fact, such type of result was obtained by Gallardo-Gutiérrez, Partington and Ross in [22]:

Theorem 4.4.3. *[22, Theorem 7.7] If u is a non constant inner function and $(uH^2)^{\perp}$ is an invariant subspace for the Cesàro operator, then $u = u_\alpha$ for some $\alpha > 0$ where $u_\alpha(z) = e^{\alpha(\frac{z+1}{z-1})}$*

Note that the above theorem can now be viewed as the first case of the Corollary 4.3.6, in fact is clear from the proof of this result that if a Beurling type space is invariant for all C_{ϕ_a} and the function Θ is not zero free then $\Theta(az_0 + 1 - a) = 0$ for all $a \in (0,1)$ where z_0 is a zero of Θ. This is impossible by the analytic continuation principle. Thus the only option is the first case of the Corollary 4.3.6.

4.5 Future works

In this section we will talk about some ideas for future works. Motivated by the remarks in the end of the chapter 3 one of our future ideas is to obtain a simple proof for the fact that the operators $C_\phi - \lambda$ have closed range for ϕ hyperbolic and for appropriate λ. It is a known fact of functional analysis that an operator T has closed range if and only if T^* has the same property, so the question above can be reformulated using the adjoints.

A classical result (see [13, Theorem 9.2]) shows us how to write the adjoint of a composition operator using a Toeplitz operator and another composition operator. In the next proposition, we apply this theorem to obtain an explicit expression for C_{ϕ_a}.

Proposition 4.5.1. *For $a \in (0,1)$, $n \in \mathbb{N}$ and $f \in H^2(\mathbb{D})$ we have:*

$$C^*_{\phi_{a^n}} f(z) = \frac{1}{(a^n-1)z+1} f\left(\frac{a^n z}{(a^n-1)z+1}\right) \quad \forall z \in \mathbb{D}$$

Proof. Note that the normal form of ϕ_a is

$$\phi_a(z) = \frac{\sqrt{a}z \ + \frac{(1-a)}{\sqrt{a}}}{\frac{1}{\sqrt{a}}}$$

With this expression, we can apply [13, Theorem 9.2] directly to obtain the above formula. ∎

The central question here is if this concrete formula for the adjoint can help us to prove that some translations like $C^*_{\phi_a} - Id$ are closed range operators; we do not have a concrete answer for now. Despite that, the above expression can be used for example, to deduce the following results about the dynamics of C_{ϕ_a}.

Theorem 4.5.2. *Let $f \in H^2$, consider $C^*_{\phi_a}$ where $a \in (0,1)$ and consider any subsequence of $(C^*_{\phi_{a^n}} f)_{n\in\mathbb{N}}$. Then this subsequence converges to 0 or diverges.*

Proof. Consider $f \in H^2$ and pass to a arbitrary subsequence that we denote by $(C^*_{\phi_{a^n}} f)_{n\in\mathbb{N}}$ for notation reasons . Using Proposition 4.5.1 we conclude that for every $z \in \mathbb{D}$

$$C^*_{\phi_{a^n}} f(z) = \frac{1}{(a^n-1)z+1} f\left(\frac{a^n z}{(a^n-1)z+1}\right) \xrightarrow{n\to\infty} \frac{1}{1-z} f(0).$$

If $C^*_{\phi_{a^n}} f$ diverges, we are done. Otherwise, $C^*_{\phi_{a^n}} f \to g$ where $g \in H^2$, consequently we have pointwise convergence (i.e., $C^*_{\phi_{a^n}} f(z) \to g(z)$ for all $z \in \mathbb{D}$). So $g(z) = \frac{1}{1-z} f(0)$. If $f(0) \neq 0$ then g is a constant multiple of $\frac{1}{1-z}$ which give us a absurd because this function does not belongs to H^2. So $f(0) = 0$, this implies $g = 0$ and $C^*_{\phi_{a^n}} f \to 0$. ■

Corollary 4.5.3. *For every $a \in (0,1)$ the operator $C^*_{\phi_a}$ is not hypercyclic.*

Proof. Consider $C^*_{\phi_a}$ and let $f \in H^2$. Then if $g \in \overline{Orb(f, C^*_{\phi_a})}$ there exists a sequence $(C^*_{\phi_{a^{n_k}}} f)_{n_k\in\mathbb{N}}$ such that $C^*_{\phi_{a^{n_k}}} f \to g$ in H^2 norm. This implies pointwise convergence and in particular $C^*_{\phi_{a^{n_k}}} f(0) \to g(0)$. As $C^*_{\phi_{a^{n_k}}} f(0) = f(0)$ (look at Proposition 4.5.1) we conclude that $g(0) = f(0)$. This means that only H^2 functions that agree with f at 0 can belong to $\overline{Orb(f, C^*_{\phi_a})}$, in particular, $\overline{Orb(f, C^*_{\phi_a})} \neq H^2$ for every $f \in H^2$. ■

These two results show us at least the distinct behaviours of C_{ϕ_a} and $C^*_{\phi_a}$, since the orbits of $C_{\phi_a} f$ can converge for some functions f (see the proof of Theorem 3.2.1) and C_{ϕ_a} is hypercyclic. Of course another question based in chapter 3 is to complete the picture of the eventually bounded case and perhaps to extend these arguments.

About the content of section 4, we recently learned in [22] about the existence of a Reproducing Kernel Hilbert Space called the **Kriete-Trutt space**: for any complex number $w \in \mathbb{D}$ we define the function $q_w : \mathbb{D} \to \mathbb{C}$ given by $q_w(z) = (1-z)^{\frac{w}{1-w}}$. A computation writing w as the sum of the real and imaginary part show us that $\Re(\frac{w}{1-w}) > -\frac{1}{2}$ and thus by [25, Lemma 7] we conclude that $q_w \in H^2$. Noting that $q_{\frac{n}{n+1}}(z) = (1-z)^n$ we conclude also that $\{q_{\frac{n}{n+1}}\}_{n\in\mathbb{N}}$ spans all the polynomials and as a consequence, $\overline{span\{q_w \mid w \in \mathbb{D}\}} = H^2$ because the polynomials are dense in H^2. **Note the functions q_w are exactly the eigenvectors that we used in chapter 3.**

We define for each function $f \in H^2$ the following function:

$$Kf(w) = \langle f, q_{\overline{w}} \rangle \qquad \forall w \in \mathbb{D}.$$

The Kriete-Trutt space is defined as $\mathcal{H} = \{Kf \mid f \in H^2\}$. Since $\overline{span\{q_w \mid w \in \mathbb{D}\}} = H^2$ the map $K : H^2 \to \mathcal{H}$ given by $K(f) = Kf$ is one-to-one and is, of course, surjective. We consider in $\mathcal{H}$ the range norm, i.e, $\|Kf\|_{\mathcal{H}} = \|f\|_{H^2}$ and thus K is an isometry.

This space was one of the main tools developed in [27] and [28] to study the subnormality of the Cesàro operator and there are a lot of open questions related with it (see the last chapter of [22]). Moreover, Kriete and Trutt (see [27]) also proved that there exists a positive finite Borel measure in $\overline{\mathbb{D}}$ that we call μ such that

$$\int_{\mathbb{D}} |p|^2 d\mu = \|p\|_{\mathcal{H}}^2.$$

for every analytic polynomial p. An interesting point in the context of this text is that the support of this measure is a sequence of circles shrinking to 1 (see [22, Figure 4]) **in the same form that happened in section 3** (see the Figure 3.2).

With new results and approaches involving the Cesàro operator and the operators C_{ϕ_a} we believe that this space can be useful and maybe could be a central object in a future study.

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A Appendix

The objective of this appendix is to provide the reader the precise statement of some background results that we used explicitly in the text as well as the references for their proofs.

Proposition A.0.1. *Let H be a Hilbert space and $M \subseteq H$ such that $M \neq \emptyset$. Then $M^{\perp} = \{0\}$ if, and only if, $\overline{span \ M} = H$.*

Proof. See [26, Lemma 3.3 − 7]. ■

Theorem A.0.2. *If $X \neq \{0\}$ is a Banach space and $T \in B(X)$ then $\sigma(T) \neq \emptyset$.*

Proof. See [26, Theorem 7.5 − 3]. ■

Lemma A.0.3. *Suppose that E is a normed space. Then E is separable if, and only if, there exists a countable set $A \subseteq E$ such that $span \ A$ is dense in E.*

Proof. See [4, Lemma 1.6.3]. ■

Theorem A.0.4 (Riesz-Fischer)**.** *Every separable and infinite dimensional-Hilbert space is isometrically isomorphic to l^2.*

Proof. See [4, Theorem 5.4.4.]. ■

Theorem A.0.5 (Mazur's Lemma)**.** *Let E be a Banach space.*

a) If $K \subseteq E$ is convex then the closure of K in the norm topology is equal to the closure of K in the weak topology, i.e, $\overline{K}^{\| \ \|} = \overline{K}^{\sigma(E,E')}$.

b) Assume that $(x_n)_{n\in\mathbb{N}} \subseteq E$ converges weakly to $x \in E$. There exists a sequence $(y_n)_{n\in\mathbb{N}}$ made up convex linear combinations of the vectors $(x_n)_{n\in\mathbb{N}}$ such that $y_n \to x$ in the norm topology of E.

Proof. See [4, Theorem 6.2.1] for item a) and [6, Corollary 3.8] for part b). ■

Theorem A.0.6. *(The Principle of Analytic Continuation) Suppose that f and g are analytic in a domain $\mathbb{U}$ and $(z_n)_{n\in\mathbb{N}}$ is a sequence of distinct points in $\mathbb{U}$ such that $z_n \to z_0 \in \mathbb{U}$. If $f(z_n) = g(z_n)$ for every $n \in \mathbb{N}$ then $f = g$ in $\mathbb{U}$.*

Proof. See [16, Corollary 6.30]. ■

Theorem A.0.7. *Suppose that Ω is an open set in $\mathbb{R}^n$ and $G : \Omega \to \mathbb{R}^n$ is a C^1 diffeomorphism. If f is a Lebesgue measurable function in $G(\Omega)$ then $f \circ G$ is a Lebesgue measurable function on Ω. If $f \geq 0$ or $f \in L^1(G(\Omega), m)$ then*

$$\int\limits_{G(\Omega)} f(x)dx = \int\limits_{\Omega} (f \circ G)(x)|\det D_xG|dx$$

where D_xG is the matrix of partial derivatives given by $\frac{\partial g_i}{\partial x_j}$ with $G = (g_1, g_2, \ldots, g_n)$.

Proof. See [17, Theorem 2.47] ■

Theorem A.0.8. *Let μ be a non negative measure and $f : X \to Y$ be a measurable map. A measurable function g defined on Y is integrable with respect to $\mu \circ f^{-1}$ precisely when the function $g \circ f$ is integrable with respect to μ. In addition, one has:*

$$\int\limits_Y g \; d(\mu \circ f^{-1}) = \int\limits_X (g \circ f) d\mu$$

Proof. See [2, Theorem 3.6.1]. ■