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On Complete Digraphs Which Are Associated to Spheres

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1. Introduction

It is known that one can also construct a homotopy theory for categories of spaces having a structure weaker than a topology. For example, one can take the category of *prespaces* or *Čech closure spaces*.

To every digraph D one can associate, in a natural way, two finite prespaces $P(D)$ and $P^*(D)$; and vice versa, to every finite prespace one can associate two digraphs G and G^*, dually oriented. Hence one can identify the category of the digraphs with that of the finite prespaces. Therefore a homotopy theory can be defined for digraphs, by setting the regular homotopy group $Q_n(D)$ of D to be the homotopy group $\pi_n(P(D))$ of the associated prespace (see [4]).

In [2] Burzio and Demaria have proved that the groups $Q_n(D)$ are isomorphic to the classical homotopy groups $\pi_n(|K_D|)$, where $|K_D|$ is the poly-

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hedron of a suitable simplicial complex K_D associated with the digraph D.

To see the construction of the simplicial complex K_D, the reader is referred to [2].

In [4] we have characterized the complete digraphs C which are simply disconnected, that is $Q_1(C) \cong \pi_1(S^1)$. In the present paper, we study the complete digraphs C such that the associated polyhedron $|K_C|$ is homeomorphic to a sphere.

We recall (see [3]) that in the case of the complete digraphs C, the associated simplicial complex K_C has as 0–simplexes the vertices of C and the other simplexes are obtained by the transitive subtournaments contained in C.

In section 3, we define the notion of j–antisymmetric complete digraphs and we prove the following theorem, which motivates the definition of the *spheric complete digraph* of order n.

Theorem 3.6: Let C_n (with $n \geq 3$) be a complete digraph. The polyhedron $|K_{C_n}|$ is (homeomorphic to) the unit sphere $\mathbb{S}^{n-2}$ if and only if

(i) C_n is n–antisymmetric;

(ii) the oriented subdigraph $\mathcal{O}(C_n)$, formed by the simple arcs of C_n and their vertices, is an n–cycle.

In section 4, we introduce the concept of *expansion* of polyhedra and we prove the following result for unit spheres:

Theorem 4.4 and 4.5: Let $h \geq 2$. If we have h unit spheres $\mathbb{S}^{n_1}, \mathbb{S}^{n_2}, \ldots, \mathbb{S}^{n_h}$ of dimension $n_1, n_2, \ldots, n_h$, respectively, then their expansion is given by

$$\mathbb{S}^{n_1} * \mathbb{S}^{n_2} * \cdots * \mathbb{S}^{n_h} = \mathbb{S}^{n_1+\cdots+n_h+h-1}$$

In section 5, using the results from the previous sections we prove the following conclusive theorem:

Theorem 5.7: Let C_n (with $n \geq 3$) be a complete digraph. If the following conditions hold:

(i) C_n is k–antisymmetric, with $k \geq n$;

(ii) $\mathcal{O}(C_n)$ contains h pairwise disjoint maximal cycles $\Gamma^{(1)}, \Gamma^{(2)}, \ldots, \Gamma^{(h)}$, of order $n_1, n_2, \ldots, n_h$, respectively, and such that $n_1 + n_2 + \cdots + n_h = n$;

(iii) every complete subdigraph, determined by the maximal cycles $\Gamma^{(i)}$ for $i = 1, 2, \ldots, h$, is spheric.

Then $|K_{C_n}|$ is homeomorphic to the unit sphere $\mathbb{S}^{n-h-1}$.

2. Some Definitions and Notations

Definition 2.1. Let V be a finite non-empty set and E a set of ordered pairs $(u, v) \in V \times V$, such that $u \neq v$. We call the pair $D = (V, E)$ a *directed graph* or *digraph*. The elements of V are called the *vertices* of D, the cardinality of V the *order* of D and the elements of E the *arcs* of D. Moreover, we write $u \to v$ instead of (u, v), and we call u a *predecessor* of v and v a *successor* of u.

Remark. Given two distinct vertices u and v, we have a priori four possibilities, and then four types of arcs:

(1) There is no oriented arc between u and v, and then we shall denote the *null arc* by $u|v$;

(2) there is the oriented arc (u, v), but not the arc (v, u), and then we shall denote the *simple arc* by $u \to v$;

(3) there is the oriented arc (v, u), but not the arc (u, v), and then we shall denote the *simple arc* by $u \leftarrow v$;

(4) there are both oriented arcs (u, v) and (v, u), and then we shall denote the *double arc* by $u \leftrightarrow v$. A double arc is also called a *symmetric pair*.

Definition 2.2. A digraph is called *oriented* if between two distinct vertices there is at most one ordered arc, that is, the possible arcs are either simple arcs or null arcs. A digraph is called a *non oriented graph* if between two distinct vertices there is either a double arc or a null arc. A digraph is called *complete* if between two distinct vertices there is at least one ordered arc, the possible arcs in this case are either simple or double arcs.

Definition 2.3. A digraph T is called a *tournament* if between every pair of distinct vertices there is one and only one arc. A digraph D is called *hamiltonian* if it contains a spanning cycle, i.e., a cycle passing through all the vertices of D.

Definition 2.4.: A tournament T is said to be *transitive* if it contains $u \to w$, whenever it contains $u \to v$ and $v \to w$. We denote by Tr_m the transitive tournament of m vertices $v_1, v_2, \ldots, v_m$ such that $v_i \to v_j$ whenever $i < j$.

Remark. Let D be a digraph. We shall denote by $A \to B$, if every vertex in A is a predecessor of every vertex in B and no vertex of B is a predecessor of a vertex of A. We shall denote by $A \leftrightarrow B$, if we have double arcs $u \leftrightarrow v$, whenever $u \in A$ and $v \in B$. We shall denote by $A|B$, if we have $u|v$ whenever $u \in A$ and $v \in B$. If A is a subset of vertices or a cycle in D, we shall denote by $< A >$ the induced subdigraph in D.

Definition 2.5: The vertices of a subdigraph A of D_n are called *equivalent* if, for any $v \in D_n - A$, either $v \to A$, or $A \to v$, or $v \leftrightarrow A$, or $v|A$. If the vertices of D_n are partitioned into disjoint subdigraphs $S^{(1)}, S^{(2)}, \ldots, S^{(m)}$ of equivalent vertices and Q_m denotes the digraph on m vertices $w_1, w_2, \ldots, w_m$, in which $w_i \to w_j$, or $w_i \leftrightarrow w_j$, or $w_i|w_j$, if and only if $S^{(i)} \to S^{(j)}$, or $S^{(i)} \leftrightarrow S^{(j)}$, or $S^{(i)}|S^{(j)}$, then D_n is called the *composition* $D_n = Q_m(S^{(1)}, S^{(2)}, \ldots, S^{m)})$ of the m subdigraphs $S^{(i)}$, for $i = 1, 2, \ldots, m$ with the digraph Q_m. Q_m is called the *quotient digraph* of D_n and the $S^{(i)}$ are called the *components* of D_n.

Definition 2.6: We say a digraph is *simple* if the composition $D_n = Q_m(S^{(1)}, S^{(2)}, \ldots, S^{(m)})$ implies that $m = 1$ or $m = n$; that is, if the quotient Q_m or all the components $S^{(i)}$ coincide with the trivial digraph of order 1.

Remark. We recall that every digraph has a unique simple quotient.

3. The Spheric Complete Digraphs

Let C_n be a complete digraph of order n. Then C_n has $n(n-1)/2$ arcs. If all these arcs are double arcs, then we just have a (*non-oriented*) *graph*.

If all these arcs are simple arcs, then we have a *tournament*. In order to distinguish the intermediary cases, we need to introduce the following definitions.

Definition 3.1: Let C_n be a complete digraph of order n. C_n is said to be *k-symmetric* if it contains exactly k symmetric pairs (or equivalently, k double arcs). On the other hand, C_n is said to be *j-antisymmetric* if it contains exactly j simple arcs. We shall denote by $\mathcal{O}(C_n)$ the oriented subdigraph formed by these simple arcs and their vertices.

Lemma 3.2: Let C_n (with $n \geq 2$) be a complete digraph and v any vertex in C_n. $|K_{C_n}|$ is a cone with vertex v if and only if v is a predecessor (or a successor) of all the other vertices (that is, either $v \notin \mathcal{O}(C_n)$ or, if $v \in \mathcal{O}(C_n)$, then $v \to u$ or $u \to v$ for all u in $\mathcal{O}(C_n)$).

Proof: It is obvious. □

Lemma 3.3: Let C_n (with $n \geq 2$) be a complete digraph. $|K_{C_n}|$ is the $(n-1)$–cell E_{n-1} if and only if the subdigraph $\mathcal{O}(C_n)$ does not contain any cycle.

Proof: In fact, $|K_{C_n}|$ is the $(n-1)$–cell if and only if C_n contains a transitive subtournament of order n. □

Theorem 3.4: Let C_n (with $n \geq 2$) be a j–antisymmetric complete digraph. If we have $0 \leq j \leq n-1$, then $|K_{C_n}|$ is a cone, and hence all the homotopy groups $Q_s(C_n)$ are all trivial.

Proof: If $n = 2$, the result is obvious. Let's suppose $n > 2$. At each vertex of C_n we have $n-1$ arcs. If a vertex v does not cone all the others, then we must have at least two simple arcs in v, such that $u \to v \to w$. Now since we have n vertices, and at each vertex we have at least two simple arcs, then we must have at least n simple arcs in C_n. But that is impossible, since by hypothesis C_n is j–antisymmetric with $j \leq n-1$.□

Theorem 3.5: Let C_n (with $n \geq 3$) be an n–antisymmetric complete digraph. If we have a vertex $v \in \mathcal{O}(C_n)$ such that:

(i) at v we have at least three simple arcs;

(ii) for every vertex of C_n we have exactly two simple arcs, and v is the predecessor or successor of two vertices in $\mathcal{O}(C_n)$;

then $|K_{C_n}|$ is a cone.

Proof: In fact, in the first case there must exist another vertex u which is the extreme of at most one simple arc, and so $|K_{C_n}|$ is a cone of vertex u.

In the second case, $|K_{C_n}|$ is obviously a cone having as vertex v itself. □

It remains to consider the case of an n–antisymmetric complete digraph C_n, such that each vertex $v \in C_n$ is the predecessor in a unique simple arc and, at the same time, the successor in another unique simple arc. In this case the subdigraph $\mathcal{O}(C_n)$ admits a decomposition in h disjoint cycles.

If $\mathcal{O}(C_n)$ is a unique n–cycle, we have the following theorem.

Theorem 3.6: Let C_n (with $n \geq 3$) be a complete digraph. The polyhedron $|K_{C_n}|$ is (homeomorphic to) $\mathbb{S}^{n-2}$ if and only if

(i) C_n is n–antisymmetric;

(ii) the oriented subdigraph $\mathcal{O}(C_n)$ is an n–cycle.

Proof: Let's suppose by absurd that C_n does not satisfy the conditions. Then we have three cases to be considered:

(1) If $\mathcal{O}(C_n)$ does not contain any cycle, then by lemma 3.3 we have that $|K_{C_n}|$ is the $(n-1)$–cell E_{n-1}.

(2) If $\mathcal{O}(C_n)$ contains an h–cycle Γ_1, with $h < n$, then there is a subdigraph of order $n-1$ which contains Γ_1. Thus we have that the corresponding $(n-2)$–simplex does not belong to $|K_{C_n}|$.

(3) If $\mathcal{O}(C_n)$ contains an n–cycle, and if it has an extra oriented arc, then this extra arc determines in $\mathcal{O}(C_n)$ a k–cycle Γ_2, with $k < n$. Hence we get the case (2).

Therefore in any case we don't have the unit sphere $\mathbb{S}^{n-2}$.

Conversely, let's suppose (i) and (ii) hold, then by Lemma 3.3 we have that $|K_{C_n}|$ can not be the $(n-1)$–cell E_{n-1}, since C_n contains an n–cycle. Consider any vertex $v \in C_n$, and let D_{n-1} be the subdigraph $C_n - \{v\}$. Then

D_{n-1} is $(n-2)$ – antisymmetric and it does not contain any cycle. Hence by lemma 3.3 $|K_{D_n-1}|$ is the $(n-1)$–cell. Therefore $|K_{C_n}|$ is homeomorphic to $\mathbb{S}^{n-2}$, since it is the union of the maximal faces of the $(n-1)$–cell. □

Remark: The triangulation of the unit sphere $\mathbb{S}^{n-2}$ which was used here it is its *minimal triangulation* by n cells of dimension $n-2$.

Definition 3.7: A complete digraph of order $n \geq 3$ satisfying the conditions of the previous theorem it is called the *spheric complete digraph* of order n, and it will be denote by Σ_n.

4. Expansion of Polyhedra

Before we can study the case in which $\mathcal{O}(C_n)$ is the disjoint union of two or more cycles, we need to introduce some definitions.

Definition 4.1: Let H and K be two disjoint simplicial complexes, and let H^+ and K^+ be their augmented simplicial complexes (see [7]). We call the *expansion* of H and K the simplicial complex $H * K$, which has as vertices the union of the vertices of H and K, and as simplexes the ones which are obtained in the following way:

If $\sigma = (x_0 \dots x_r)$ is an r–simplex in H^+ and $\tau = (y_0 \dots y_s)$ is an s–simplex in K^+, then $\sigma\tau = (x_0 \dots x_r y_0 \dots y_s)$ is an $(r+s+1)$–simplex in $(H * K)^+$.

We observe that $H * K$ is obviously a simplicial complex, and that $H * K$ is equal to $K * H$, for we just have to change the order of the vertices.

Remark: If $\dim H = p$ and $\dim K = q$, then $\dim H * K = p + q + 1$. In the case H is a point v, then $v * K$ is just the cone of vertex v and base K.

Definition 4.2: Let P and Q be two disjoint polyhedra. Let H be a triangulation of P and K be a triangulation of Q. The *expansion* of P and $Q, P * Q$ is the polyhedron $|H * K|$, obtained by the expansion of H and K.

Remark: We observe that $P * Q$ is well–defined, that is, it is independent of the chosen triangulations, for $P * Q$ can be considered as the polyhedron obtained by joining each point in P to each point in Q by a line segment.

Definition 4.3: Let $K_1, \ldots, K_s$ be s pairwise disjoint simplicial complexes. We can define the *expansion* of them, $K_1 * K_2 * \cdots * K_s$, by considering as it vertices the union of all the vertices, and the other simplexes are obtained as $\sigma_1\sigma_2\ldots\sigma_s$, where $\sigma_j \in K_j^+, j = 1, \ldots s$. Now given s polyhedra $P_1, P_2, \ldots, P_s$, as before we define the *expansion* $P_1 * P_2 * \cdots * P_s$ by taking the expansion $K_1 * K_2 * \cdots * K_s$, where K_j is a triangulation of the polyhedron P_j.

Remark: We observe that the expansion is a commutative and associative operation.

Theorem 4.4: Let $\mathbb{S}^m$ and $\mathbb{S}^n$ be the unit spheres of dimension m and n, respectively. Then $\mathbb{S}^m * \mathbb{S}^n = \mathbb{S}^{m+n+1}$.

Proof: If $m = 0$ or $n = 0$, then we just have the suspension of a sphere, and hence the result holds.

Let's suppose that $m, n \geq 1$. If we consider the unit sphere $\mathbb{S}^m$ as the boundary of the $(m+1)$–cell, $|\overline{(x_0x_1\ldots x_{m+1})}|$, and $\mathbb{S}^n$ as the boundary of the $(n+1)$–cell, $|\overline{(y_0y_1\ldots y_{n+1})}|$, then we can take their minimal triangulations

$$\mathbb{S}^m = |\overline{(x_1x_2\ldots x_{m+1})}| \cup |\overline{(x_0x_2\ldots x_{m+1})}| \cup \cdots \cup |\overline{(x_0x_1\ldots x_m)}|,$$

with $m+2$ closed m–simplexes; and

$$\mathbb{S}^n = |\overline{(y_1y_2\cdots y_{n+1})}| \cup |\overline{(y_0y_2\cdots y_{n+1})}| \cup \cdots \cup |\overline{(y_0y_1\ldots y_n)}|,$$

with $n+2$ closed n–simplexes.

For the expansion $\mathbb{S}^m * \mathbb{S}^n$ we consider the triangulation given by the union of the $(m+2)(n+2)$ closed $(m+n+1)$–simplexes $|\overline{(x_0x_1\ldots\widehat{x}_i\ldots x_{m+1}y_0y_1\ldots\widehat{y}_j\ldots y_{n+1})}|$, where we delete x_i and y_j for all choices of i and j, with $0 \leq i \leq m+1$ and $0 \leq j \leq n+1$.

Now the unit sphere $\mathbb{S}^{m+n+1}$ can be considered as the dissection in two

$(m+n+1)$–cells: one given by the upper hemisphere, and the other given by the lower hemisphere. If the upper hemisphere is identified with the face $|\overline{(x_1x_2\ldots x_{m+1}y_1y_2\cdots y_{n+1})}|^+$, the result is true if we can show that the face $|\overline{(x_1x_2\ldots x_{m+1}y_1y_2\cdots y_{n+1})}|^-$, which is identified with the lower hemisphere, admits as a triangulation the union of all the faces that are different of $|\overline{(x_1x_2\ldots x_{m+1}y_1y_2\cdots y_{n+1})}|$.

To see this we consider the two points x_0 and y_0 in a convenient position in the interior of $|\overline{(x_1x_2\ldots x_{m+1}y_1y_2\cdots y_{n+1})}|^-$. For example, the point x_0 can be chosen to be the barycenter of $|\overline{(x_1x_2\ldots x_{m+1}y_1y_2\cdots y_{n+1})}|^-$ and the point y_0 is chosen to be the barycenter of the $(m+1)$–cell $|\overline{(x_0x_1\ldots x_{m+1})}|$.

By using the first point x_0 we have an initial triangulation given by the union of the following $(m+n+2)$ cells

$$|\overline{(x_1x_2\ldots x_{m+1}y_1y_2\cdots y_{n+1})}| = \bigcup_{i=1}^{m+1} |\overline{(x_0x_1\ldots \widehat{x}_i\ldots x_{m+1}y_1y_2\cdots y_{n+1})}|$$

$$\cup\Big(\bigcup_{j=1}^{n+1} |\overline{(x_0x_1\ldots x_{m+1}y_1y_2\cdots \widehat{y}_j\cdots y_{n+1})}|\Big).$$

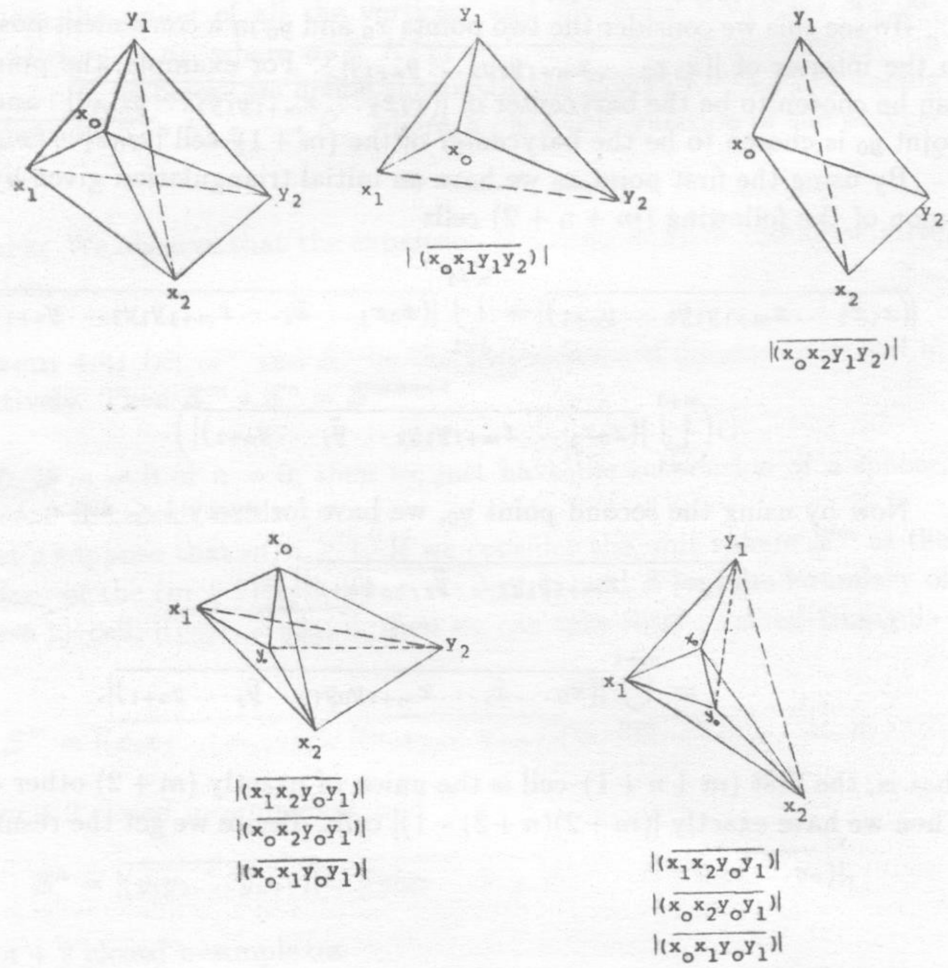
Now by using the second point y_0, we have for every $1\le j\le n+1$

$$|\overline{(x_0x_1\ldots x_{m+1}y_1y_2\cdots \widehat{y}_j\cdots y_{n+1})}| =$$

$$= \bigcup_{i=0}^{m+1} |\overline{(x_0\ldots \widehat{x}_i\ldots x_{m+1}y_0y_1\cdots \widehat{y}_j\cdots y_{n+1})}|,$$

that is, the first $(m+n+1)$–cell is the union of exactly $(m+2)$ other cells. Then we have exactly $[(m+2)(n+2)-1)]$ cells. Hence we get the result. □

In the case $m = n = 1$.



In the case we have more than two spheres, the following theorem holds:

Theorem 4.5: If we have h unit spheres $\mathbb{S}^{n_1}, \mathbb{S}^{n_2}, \ldots, \mathbb{S}^{n_h}$ of dimension $n_1, n_2, \cdots, n_h$, respectively, then

$$\mathbb{S}^{n_1} * \mathbb{S}^{n_2} * \cdots * \mathbb{S}^{n_h} = \mathbb{S}^{n_1+n_2+\cdots n_h+h-1}.$$

Proof: By induction on h. If $h = 2$, the result holds by the previous theorem. Let's suppose the result is true for $h > 2$. Now if in the expansion $\mathbb{S}^{n_1} * \cdots * \mathbb{S}^{n_h} * \mathbb{S}^{n_{h+1}}$, we associate the first h unit spheres, that is, if we take $(\mathbb{S}^{n_1} * \cdots * \mathbb{S}^{n_h}) * \mathbb{S}^{n_{h+1}}$ and we take as triangulation for $\mathbb{S}_1^{n} * \cdots * \mathbb{S}^{n_h}$ the minimal triangulation of the unit sphere $\mathbb{S}^{n_1+\cdots+n_h+h-1}$, then by the previous theorem we get the unit sphere of dimension $(n_1 + \cdots + n_h + h - 1) + n_{h+1} + 1$ which is equal to $n_1 + \cdots + n_{h+1} + (h + 1) - 1$. And hence the theorem is true. □

5. The Case $\mathcal{O}(C_n)$ Splits in Cycles

Now we shall study the case in which the oriented subdigraph $\mathcal{O}(C_n)$ admits a decomposition in h pairwise disjoint cycles, with $h \geq 2$.

First of all, let's consider the case of a complete digraph C_n having a decomposition in two disjoint subdigraphs A and B. We have the following theorem.

Theorem 5.1: If $C_n = A \leftrightarrow B$, then

$$K_{C_n} = K_A * K_B.$$

Proof: If σ^i is an i–simplex of K_A^+ and τ^j is a j–simplex of K_B^+, then $\sigma^i\tau^j$ belongs to $K_{C_n}^+$. Infact, the subdigraph $< \sigma^i >$, which is determined by the vertices of σ^i, must contain a transitive subtournament of order $i + 1$, and similarly $< \sigma^j >$ must contain a transitive subtournament of order $j + 1$. Since we have double arcs from A to B, then $< \sigma^i\tau^j >$ contains a transitive subtournament of order $i + j + 2$. Hence $K_A * K_B \subset K_{C_n}$.

Conversely, given an m–simplex μ of $K_{C_n}^+$, we let μ' be the set of vertices of μ which are contained in A, and let μ'' be the set of vertices of μ which are contained in B. Let us suppose that μ' and μ'' have h and k elements, respectively, with $h + k + 1 = m$. Since $< \mu >$ contains a transitive subtournament of order $m + 1$ in C_n, then we have that $< \mu' > \subset A$ contains a transitive subtournament of order $h + 1$; and similarly $< \mu'' > \subset B$ contains a transitive subtournament of order $k + 1$. That is, μ' is a simplex in K_A^+ and μ'' is a simplex in K_B^+. Therefore $K_{C_n}^+ \subset K_A * K_B$, and thence the result holds. □

Corollary 5.2: If C_n is a complete digraph such that $C_n = A \rightarrow B$, with A and B being two disjoint subdigraphs, then $K_{C_n} = K_A * K_B$. □

Remark: The two previous results actually hold for digraphs in general, not only for complete ones.

Corollary 5.3: If T_n is a tournament such that $T_n = A \rightarrow B$, with A and B being two disjoint subtournaments, then $K_{T_n} = K_A * K_B$. □

Theorem 5.4: Let C_n be an n–antisymmetric complete digraph such that $\mathcal{O}(C_n)$ is the disjoint union of two cycles $\Gamma_r^{(1)}$ and $\Gamma_s^{(2)}$ (with $r + s = n$), then the polyhedron $|K_{C_n}|$ is the unit sphere $\mathbb{S}^{n-3}$

Proof: By theorem 3.6, we have that $|K_{\Gamma_r^{(1)}}| = \mathbb{S}^{r-2}$ and $|K_{\Gamma_s^{(2)}}| = \mathbb{S}^{s-2}$. By Theorem 4.4, we have that $\mathbb{S}^{r-2} * \mathbb{S}^{s-2} = S^{r+s-3}$. Hence the result is true. □

Theorem 5.5: Let C_n be an n–antisymmetric complete digraph such that $\mathcal{O}(C_n)$ is the union of h pairwise disjoint cycles $\Gamma^{(1)}, \Gamma^{(2)}, \ldots, \Gamma^{(h)}$ of order $n_1, n_2, \ldots, n_h$, respectively. Then we have that $|K_{C_n}|$ is the unit sphere $\mathbb{S}^{n-h-1}$.

Proof: It is obvious by theorem 4.5. □

Remark: The complete digraph C_n we have just considered is unique (up to isomorphim) and it depends only of the order $n_1, n_2, \ldots, n_h$ of the h cycles, and not of the particular ordering they are considered.

So far we have determined some "maximal" complete digraphs such that their associated polyhedra are spheres (see Theorems 3.6, 4.4 and 4.5). The "maximality" here is to be understood in the sense that if we add to the digraph an extra double arc then the result is not true anylonger (see Theorem 3.4).

We would like now to consider some "minimal" complete digraphs such that their associated polyhedra are spheres. The "minimality" here is to be understood in the sense that if we omit a double arc, then the property does not hold anylonger.

Theorem 5.6: Let C_n be the composition $C_n = T_{\tau_h}(\Sigma^{(1)}, \Sigma^{(2)}, \ldots, \Sigma^{(h)})$ of h spheric complete subdigraphs $\Sigma^{(i)}$ of order n_i, with the transitive tournament of order h. Then $|K_{C_n}|$ is homeomorphic to the unit sphere $\mathbb{S}^{n-h-1}$.

Proof: By Corollary 5.2, we have that $K_{C_n} = K_{\Sigma^{(1)}} * K_{\Sigma^{(2)}} * \cdots * K_{\Sigma^{(h)}}$. Since for every i we have that $|K_{\Sigma^{(i)}}|$ is $\mathbb{S}^{n_i-2}$, then by Theorem 4.5 we get the result. □

Remark. (1) If we remove from C_n a double arc, then this double arc must have been removed from one of the $\Sigma^{(i)}$. Hence by Theorem 3.6 the corresponding $|K_{\Sigma_{(i)}}|$ can not be the sphere of dimension $n_i - 2$. Therefore $|K_{C_n}|$ can no longer be a sphere.

(2) The complete digraph C_n is not unique since it depends also on the ordering we consider the cycles and not only on the order $n_1, n_2, \ldots n_h$ of them. Actually the number of these "minimal" complete digraphs is exactly $h!/(h_1! \ldots h_p!)$, where h_j is the number of the distinct cycles having the same order, and $h_1 + h_2 + \cdots + h_p = h$.

Finally, from the previous results we can state the conclusive theorem.

Theorem 5.7: Let C_n (with $n \geq 3$) be a complete digraph. If the following conditions hold:

(i) C_n is k-antysimmetric, with $k \geq n$;

(ii) $\mathcal{O}(C_n)$ contains h pairwise disjoint maximal cycles $\Gamma^{(1)}, \Gamma^{(2)}, \ldots, \Gamma^{(h)}$ of order $n_1, n_2, \ldots, n_h$, respectively, and such that $n_1 + n_2 + \cdots + n_h = n$;

(iii) every complete subdigraph $<\Gamma^{(i)}>$, for $i = 1, 2, \ldots, h$, is spheric. Then $|K_{C_n}|$ is homeomorphic to the unit sphere $\mathbb{S}^{n-h-1}$.

Proof: If $k = n$ then the result follows from the theorems 3.6 and 5.5.

Let $k > n$, and consider the $r = k - n$ simple arcs $\alpha_1, \alpha_2, \ldots, \alpha_r$ which are not in the $\Gamma^{(i)}$.

If we replace the simple arcs α_i by double arcs we get the "maximal" complete digraph M_n which contains C_n as a subdigraph. Clearly the simplicial complex K_{C_n} is contained in K_{M_n}.

On the other hand there is a "minimal" complete digraph N_n which is contained in C_n, so that K_{N_n} is contained in K_{C_n}. Since $K_{N_n} = K_{M_n}$, the theorem is true if we exibit this digraph N_n.

First of all, we observe that every arc α_i must go from a vertex in a $<\Gamma^{(s)}>$ to a vertex in a $<\Gamma^{(t)}>$, with $s \neq t$. We also observe that all the other eventual arcs α_j between $<\Gamma^{(s)}>$ and $<\Gamma^{(t)}>$ must all go in the same direction as α_i does. Thus we can replace all the eventual double arcs between $<\Gamma^{(s)}>$ and $<\Gamma^{(t)}>$ by simple arcs in such manner to obtain $<\Gamma^{(s)}>\rightarrow<\Gamma^{(t)}>$.

If we follow this process for all α_i, with $1 \leq i \leq n$, at the end we get the composition $P_n = Q_h(<\Gamma^{(1)}>, <\Gamma^{(2)}>, \ldots, <\Gamma^{(h)}>)$.

Then we have two possibilities: either Q_h is a tournament or it is not a tournament.

In the case Q_h is a tournament, then Q_h is transitive. For otherwise it would contain a 3–cycle, and hence we would not have the maximality of the components $\Gamma^{(i)}$. Therefore, P_n is the digraph N_n we are looking for.

In the case Q_h is not a tournament, then we have $<\Gamma^{(i)}>\leftrightarrow<\Gamma^{(j)}>$ for some $i \neq j$. Then we replace the double arcs in Q_n by simple arcs in such a way that we avoid 3–cycles among the components $<\Gamma^{(i)}>$. And then we are back to the previous case, and we obtain N_n. □

Corollary 5.8: Let T_n be a tournament of order n. If $n = 3h$ (with $h \geq 1$) and T_n is the composition of h 3–cycles $\Gamma^{(1)}, \Gamma^{(2)}, \ldots, \Gamma^{(h)}$, by a transitive tournament (that is, $T_n = T_{r_h}(\Gamma^{(1)}, \Gamma^{(2)}, \ldots, \Gamma^{(h)})$). Then $|K_{T_n}|$ is homeomorphic to the unit sphere $\mathbb{S}^{2h-1}$. □

Remark: The tournament $T_n = T_{r_h}(\Gamma^{(1)}, \Gamma^{(2)}, \ldots, \Gamma^{(h)})$ given above is unique (up to isomorphism).

Up to this point we have found some sufficient conditions for a complete digraph to have the associated polyhedron being a unit sphere. We conjecture that these conditions are also necessary, and we hope to prove this in a future paper.

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