UNIVERSIDADE ESTADUAL DE CAMPINAS

Instituto de Matemática, Estatística e Computação Científica

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Cocharacters of associative algebras and Specht property of varieties of graded Lie algebras

Cocaracteres de álgebras associativas e a propriedade de Specht de variedades de álgebras de Lie graduadas

Campinas

2023

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutora em Matemática.

Thesis presented to the Institute of Mathematics, Statistics and Scientific Computing of the University of Campinas in partial fulfillment of the requirements for the degree of Doctor in Mathematics.

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Este exemplar corresponde à versão final da Tese defendida pela aluna Daniela Martinez Correa e orientada pelo Prof. Dr. Plamen Emilov Kochloukov.

Campinas

2023

Ficha catalográfica
Universidade Estadual de Campinas
Biblioteca do Instituto de Matemática, Estatística e Computação Científica
Ana Regina Machado - CRB 8/5467

M366c

Martinez Correa, Daniela, 1992-
Cocharacters of associative algebras and Specht property of varieties of graded Lie algebras / Daniela Martinez Correa. – Campinas, SP : [s.n.], 2023.

Orientador: Plamen Emilov Kochloukov.
Coorientadores: Lucio Centrone e Vesselin Stoyanov Drensky.
Tese (doutorado) – Universidade Estadual de Campinas, Instituto de Matemática, Estatística e Computação Científica.

1. PI-álgebras. 2. Co-caracter. 3. Séries de Hilbert. 4. Propriedade de Specht. 5. Identidades polinomiais graduadas. I. Kochloukov, Plamen Emilov, 1958-. II. Centrone, Lucio, 1983-. III. Drensky, Vesselin Stoyanov. IV. Universidade Estadual de Campinas. Instituto de Matemática, Estatística e Computação Científica. V. Título.

Informações Complementares

Título em outro idioma: Cocaracteres de álgebras associativas e a propriedade de Specht de variedades de álgebras de Lie graduadas
Palavras-chave em inglês:
PI-algebras
Cocharacter
Hilbert series
Specht property
Graded polynomial identities
Área de concentração: Matemática
Titulação: Doutora em Matemática
Banca examinadora:
Plamen Emilov Kochloukov [Orientador]
Daniela La Mattina
Artem Lopatin
Onofrio Mario Di Vincenzo
Felipe Yukihide Yasumura
Data de defesa: 25-01-2023
Programa de Pós-Graduação: Matemática

Identificação e informações acadêmicas do(a) aluno(a)
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- Currículo Lattes do autor: http://lattes.cnpq.br/5731622490286689

Tese de Doutorado defendida em 25 de janeiro de 2023 e aprovada

pela banca examinadora composta pelos Profs. Drs.

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A Ata da Defesa, assinada pelos membros da Comissão Examinadora, consta no SIGA/Sistema de Fluxo de Dissertação/Tese e na Secretaria de Pós-Graduação do Instituto de Matemática, Estatística e Computação Científica.

To the loves of my life, Nejo and Chiquita.

Acknowledgements

First, thank you God for carrying me trough all the hard times and never leaving me. Without you nothing is possible.

Thanks a lot to my family for all support and love, especially my mother and my grandmother. You are the most important people in my life.

I would like to express my sincere gratitude to my supervisors: Professor Plamen Kochloukov, Professor Lucio Centrone and Professor Vesselin Drensky for their valuable advices and continued support. I am very grateful for the guidance, discussions and sharing their experience and expertise.

I thank my boyfriend for all support, Carlos Bassani, with whom I lived the best moments during this process. His love made me feel at home.

Finally, thanks to all professors, friends, colleagues who helped me during all these years.

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001.

"The task is not so much to see what no
one yet has seen, but to think what nobody
yet has thought about that which everybody sees."
Arthur Schopenhauer

Resumo

Sejam F um corpo de característica 0 e E a álgebra de Grassman de dimensão infinita sobre F. Na primeira parte desta tese, encontramos um algoritmo que calcula a função geratriz da sequência de cocaracteres de $UT_n(E)$, a álgebra das matrizes triangulares superiores com entradas em E, contida numa faixa de comprimento fixo. Logo, calculamos a série dupla de Hilbert de E e definimos a série de (k,l)-multiplicidades de uma PI-álgebra. Como aplicação do anterior encontramos um algoritmo para determinar a série de (k,l)-multiplicidades de $UT_n(E)$.
Para a segunda parte da tese, vamos considerar F um corpo infinito e $UT_n(F)$, a álgebra das matrizes triangulares superiores com entradas em F e denotemos por $UT_n(F)^{(-)}$ a álgebra de Lie sobre o espaço vetorial $UT_n(F)$ com o commutador usual de matrizes. Nesta parte do trabalho, damos uma resposta positiva ao problema de Specht para o ideal das identidades $\mathbb{Z}_n$-graduadas de $UT_n(F)^{(-)}$ com a graduação canônica quando a característica p de F é zero ou maior que $n-1$. Também mostramos que se F é um corpo infinito de característica $p = 2$ então as identidades $\mathbb{Z}_3$-graduadas de $UT_3^{(-)}(F)$ não satisfazem a propriedade de Specht.

Palavras-chave: PI-álgebras. Cocaracter. Séries de Hilbert. Propriedade de Specht. Identidades graduadas.

Abstract

Let F be a field of characteristic 0 and let E be the infinite dimensional Grassmann algebra over F. In the first part of this thesis we give an algorithm that calculates the generating function of the cocharacter sequence of $UT_n(E)$, the $n \times n$ upper triangular matrix algebra with entries in E, lying in a strip of a fixed size. Then, we compute the double Hilbert series $H(E; \mathrm{T}_k, \mathrm{Y}_l)$ of E and we define the (k,l)-multiplicity series of any PI-algebra. As an application, we derive from $H(E; \mathrm{T}_k, \mathrm{Y}_l)$ an algorithm determining the (k,l)-multiplicity series of $UT_n(E)$.

For the second part of this thesis, let $UT_n(F)$ be the algebra of the $n \times n$ upper triangular matrices and denote $UT_n(F)^{(-)}$ the Lie algebra on the vector space of $UT_n(F)$ with respect to the usual bracket (commutator), over an infinite field F. In this second part of this work, we give a positive answer to the Specht property for the ideal of the $\mathbb{Z}_n$-graded identities of $UT_n(F)^{(-)}$ with the canonical grading when the characteristic p of F is 0 or is larger than $n-1$. Moreover, we show that if F is an infinite field of characteristic $p = 2$ then the $\mathbb{Z}_3$-graded identities of $UT_3^{(-)}(F)$ do not satisfy the Specht property.

Keywords: PI-algebras. Cocharacter. Hilbert series. Specht property. Graded identities.

Contents

Introduction

In this thesis, we study two independent topics concerning PI-algebras. The first one is about cocharacters of associative algebras over a field of characteristic zero, while the second one deals with the Specht property of varieties of graded Lie algebras.

We will start by discussing the first topic. Consider F a field of characteristic 0 and A an associative algebra over F with unity, let $X = \{x_1, x_2, \ldots\}$ be a countable set of non-commuting indeterminates, then we denote by $F\langle X\rangle$ the free associative algebra freely generated by X over F. We say that $f(x_1, \ldots x_m) \in F\langle X\rangle$ is a polynomial identity of a given algebra A if $f(a_1, \ldots, a_m) = 0$ for all $a_1, \ldots, a_m \in A$. If the algebra A satisfies a non-trivial polynomial identity, then A is called a PI-algebra. It is well known that its set of polynomial identities, $T(A)$, of A is a T-ideal of $F\langle X\rangle$, that is, an ideal that is invariant under all endomorphisms of $F\langle X\rangle$.

A famous theorem of Kemer [45] says that if A is a PI-algebra over a field of characteristic 0, its T-ideal is finitely generated. We recall that the complete set of finite generators of T-ideals is known only for few algebras.

In the case F is of characteristic 0, all the polynomial identities follow from the multilinear ones. By a theorem of Regev [64], it turns out to be more efficient to study the set of multilinear polynomials which (in a certain sense) are not polynomial identities for a given algebra. More precisely, if P_n is the vector space of multilinear polynomials in the variables $\{x_1, \ldots, x_n\}$, we study the factor space

$$P_n(A) := P_n/(P_n \cap T(A))$$

for each n. We recall that P_n is also a left S_n-module under the canonical left action. Since T-ideals are invariant under permutations of the variables then $P_n \cap T(A)$ is a submodule, and hence $P_n(A)$ is an S_n-module too. It affords a character, $\chi_n(A)$, called *n-th cocharacter of A*. The sequence $(\chi_n(A))_{n\in\mathbb{N}}$ is called the *sequence of cocharacters of A*. We also observe that $P_n(A)$ is a finite dimensional vector space whose dimension $c_n(A)$ is called *n-th codimension* of A, and the sequence $(c_n(A))_{n\in\mathbb{N}}$ is called the *sequence of codimensions of A*. The above mentioned theorem of Regev states that of A is a PI algebra and A satisfies an identity of degree d then $c_n(A) \leqslant d^{2n}$ for every n. Since $\dim P_n = n!$ this justifies our phrase above: the exponential function grows much "slower" than the factorial.

In [35], [36] Giambruno and Zaicev proved that there always exists the limit

$$\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$$

and it is a non-negative integer called the *PI-exponent of A*. If we use the language of varieties, the variety generated by the algebra A is the class

$$\mathcal{V} = \mathcal{V}(A) = \{B \text{ associative algebra} \mid T(A) \subseteq T(B)\}.$$

A variety of algebras $\mathcal{V}$ is *minimal* with respect to its exponent whenever for any proper subvariety $\mathcal{U}$ of $\mathcal{V}$ we have that $\exp(\mathcal{U}) < \exp(\mathcal{V})$. We say that a PI-algebra is *minimal* if it generates a minimal variety.

If S is any commutative ring with 1, we denote by $UT_n(S)$ the ring of upper triangular matrices with entries in S. Let E be the infinite dimensional Grassmann algebra over F, then the T-ideals of the algebras $UT_n(F)$ and $UT_n(E)$ are examples of maximal T-ideals of a given exponent of the codimension sequences (and the corresponding varieties of algebras are minimal varieties of this exponent). Years before Kemer's works, Genov in [32], [33], and Latyshev in [53], proved that every algebra belonging to $\mathcal{V}(UT_n(F))$ has a finite basis of its polynomial identities. In [54] Latyshev and Popov in [60], generalized the previous result for PI-algebras satisfying the identity

$$[x_1, x_2, x_3] \cdots [x_{3n-2}, x_{3n-1}, x_{3n}]$$

which generates the T-ideal $T(UT_n(E)) = T(E)^n$ of the algebra $UT_n(E)$. For a long time, until Kemer developed his structure theory, the results of Genov, Latyshev and Popov covered most of the known examples of classes of PI-algebras with the finite basis property.

One has in characteristic 0 that for each $n \in \mathbb{N}$,

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda(A)\chi_\lambda,$$

where χ_λ is the irreducible S_n-character associated with the partition λ. Let us set $X_d := \{x_1, \ldots, x_d\}$ and let us consider $F_d(A) := F\langle X_d\rangle/(F\langle X_d\rangle \cap T(A))$. Moreover, if $\mathrm{T} = \{t_1, \ldots, t_d\}$ is a set of commutative variables, the Hilbert series $H(F_d(A), T_d)$ of $F_d(A)$ may be decomposed as

$$H(F_d(A), \mathrm{T}_d) = \sum_\lambda m_\lambda(A) S_\lambda(\mathrm{T}_d),$$

where λ is a partition in no more than k parts and $S_\lambda(\mathrm{T}_d)$ is the Schur function associated to λ in the variables from T_d. We shall refer to $H(F_d(A), \mathrm{T}_d)$ as the *Hilbert series of* A and we shall write $H(A, \mathrm{T}_d)$ instead of $H(F_d(A); \mathrm{T}_d)$. By a result of Berele [5] and Drensky [21], the $m_\lambda(A)$'s are the same as in the cocharacter sequence of A. Hence, in principle, the knowledge of the Hilbert series of A will give us the multiplicities $m_\lambda(A)$ of the cocharacter sequence of A, when λ is a partition in no more than d parts. So if A is finite dimensional, working with a sufficiently large set of variables will be enough to capture all the multiplicities. This is no longer true for infinite dimensional algebras.

The explicit form of the multiplicities in the cocharacter sequence of a PI-algebra is known in very few cases. Among them, the infinite dimensional Grassmann algebra E [59], the 2×2 matrix algebra $M_2(F)$, [22] and [29], the algebra $UT_2(F)$ of 2×2 upper triangular matrices [66], based on the approach of Berele and Regev [11], the tensor square $E \otimes E$ of the Grassmann algebra, [14] and [61], the algebra $UT_2(E)$ of 2×2 upper triangular matrices with entries from the Grassmann algebra E [15], the algebra $UT_n(F)$ of $n \times n$ upper triangular matrices [13], the algebra $R_{p,q}(F)$ of upper block triangular $(p + 2q) \times (p + 2q)$ when p and q are small values [27].

In [26] Drensky and Genov defined the *multiplicity series* of a PI-algebra A, that is the generating function of the cocharacter sequence of A which corresponds to the multiplicities $m_\lambda(A)$ when λ is a partition in no more than d parts. Then, coming back to upper triangular matrices and their central role in PI-theory, in [13] Boumova and Drensky found an easy algorithm with input the multiplicity series of a symmetric function, and output the multiplicity series of its Young-derived. Applying it, they found the explicit form of the multiplicity series of the Hilbert series of $UT_n(F)$. Following this line of research, in the first part of this thesis we work with $UT_n(E)$ and calculate its multiplicity series in d variables.

Due to the fact that E is infinite dimensional, we need more tools than the ones used by Boumova and Drensky in order to know all multiplicities of $UT_n(E)$. Using the idea of Berele [8], we work with *double Hilbert series* instead of Hilbert series of PI-algebras and, due to the analogue of the result of Berele and Drensky for double Hilbert series, it suffices to study the decomposition of the double Hilbert series of $UT_n(E)$ in order to achieve the explicit form of the cocharacter sequence of $UT_n(E)$. In the second part of this work, we generalize the definition of multiplicity series of a PI-algebra defining a (k, l)-*multiplicity series* which controls three sets of disjoint variables. Here (k, l) means that the partitions $\lambda = (\lambda_1, \dots, \lambda_m)$ satisfy the condition $\lambda_{k+1} \leqslant l$. In other words, their Young diagrams D_λ are in a hook of height k of the arm and wide l of the leg.

Afterwards we compute the double Hilbert series of E and, as a consequence, we build up an algorithm with output the (k, l)-multiplicity series of $UT_n(E)$. In the spirit of [15] we compute the $(2, 3)$-multiplicity series of $UT_2(E)$, which one contains all multiplicities of the cocharacter sequence of $UT_2(E)$, and finally we compute the $(1, 1)$-multiplicity series of $UT_3(E)$.

The second part of this thesis studies aspects of one of the most important problems in the theory of algebras with polynomial identities: determining the identities of specific algebras and studying the properties of the varieties that these algebras generate. The most significant part of the advances in this area has been obtained for associative algebras over fields of characteristic zero. Although the study of problems in positive characteristic and for other types of algebras has grown in the last decades, there are still very many questions to answer.

As we already mentioned, in 1984–1986, Kemer proved in [45] that for every associative algebra over a field of characteristic zero, its T-ideal is finitely generated as a T-ideal, thus providing a positive answer to the long standing Specht problem. To that end Kemer developed a sophisticated theory which described the structure of the ideals of identities in the free associative algebra. We observe that Kemer's theory has been shaping a good deal of the research in PI theory since then.

Here we recall that for a wide range of groups and algebras the analogue of the Specht problem was investigated and solved. We mention the paper by Oates and Powell [58], who proved that the variety of groups generated by a finite group admits a finite basis of its laws. It is also known that the variety generated by a finite ring satisfies the Specht property; this was obtained independently by Kruse [52] and by Lvov [55]. Bahturin and Ol'shanskiĭ proved in [4] that the identities of a finite Lie ring or Lie algebra also admit a finite basis of its identities.

On the other hand, if the base field is infinite but of positive characteristic, the Specht problem can have a negative answer. The first such examples for Lie algebras were obtained by Vaughan-Lee [72], in characteristic 2, and by Drensky [20], for every characteristic $p > 0$. The first examples in the case of associative algebras were obtained, much later, independently (and almost simultaneously) by Belov [44], Grishin [38], and Shchigolev [68].

In [2] and [70] it was proved that the Specht problem has positive answer for graded associative algebras over fields of characteristic zero when the grading group is finite.

Parts of the theory developed by Kemer do not work so well for non-associative algebras, even in characteristic 0. Thus, for example, one should impose certain restrictions on the classes of algebras when studying the Specht problem for Lie or Jordan algebras. In characteristic 0, Iltyakov [42] proved that if L is a finitely generated Lie algebra and A is an associative enveloping algebra for L such that A is PI then the weak polynomial identities for the pair (A, L) are finitely based. A consequence of this result is that if L is finitely generated and the adjoint Lie algebra $Ad(L)$ generates an associative PI algebra then the ideal of identities of the Lie algebra L satisfies the Specht property. Recall that this is the case when L is finite dimensional. Vaĭs and Zelmanov [73] proved that if J is a finitely generated Jordan PI-algebra, over a field of characteristic 0, then the ideal of identities of J satisfies the Specht property. Once again Iltyakov [41] established the Specht property for the ideals of identities of finitely generated alternative algebras in characteristic 0.

Therefore it is interesting to study the Specht problem for concrete varieties of Lie and Jordan algebras. The variety of Lie algebras generated by $sl_2(F)$, the simple 3-dimensional Lie algebra, in characteristic 0, satisfies the Specht property, this was proved by Razmyslov [62, 63]. Krasilnikov showed in [51] that the variety of Lie algebras defined by $UT_n^{(-)}(F)$, the Lie algebra of the $n \times n$ upper triangular matrices over a field F, satisfies the Specht property when F is infinite of characteristic 0 or $p \geqslant n$.

Now we recall some of the known results concerning group graded Lie and Jordan algebras and their identities. In [47] finite bases of the graded identities of $sl_2(F)$ were found, for an infinite field of characteristic different from 2, and an arbitrary grading. In [34] it was proved that, in characteristic 0, the variety of group graded Lie algebras generated by sl_2 has the Specht property.

In [48] and [17] the authors studied graded identities for UJ_2, the Jordan algebra of 2×2 upper triangular matrices. In [18] it was shown that the variety generated by UJ_2 has the Specht property when it is graded by any finite abelian group. In [49] the graded identities for any $\mathbb{Z}_2$-grading on the Jordan algebra of symmetric matrices of order two were obtained, and in [69] the Specht property for the finite dimensional Jordan algebra of a non-degenerate symmetric bilinear form, graded by $\mathbb{Z}_2$, in characteristic 0, was established. Here we recall that the more difficult situation where there is no grading at all, for this algebra, was settled by Iltyakov [40] in the finite dimensional case. The infinite dimensional Jordan algebra of a non-degenerate symmetric bilinear form also satisfies the Specht property in characteristic 0, this follows by combining results obtained by Vasilovsky [71] and by Koshlukov [46]. Recently, in [57] it was shown that for any grading, the variety of graded commutative algebras generated by $(UT_2, \circ)$ has the Specht property in characteristic 2.

We recall that if the grading group is infinite then, even in characteristic 0, the graded identities of an algebra need not be finitely based, see for example [31, 28].

In [50], a finite basis for the graded identities for $UT_n(F)^{(-)}$ was found when this algebra is endowed with the canonical grading of $\mathbb{Z}_n$ and the field F is infinite.

In that part of the thesis, we study the variety of graded Lie algebras generated by $UT_n(F)^{(-)}$, endowed with the canonical $\mathbb{Z}_n$-grading. We prove that, when the characteristic of F is 0 or is a prime $p \geqslant n$, it satisfies the Specht property. In order to achieve this we employ properties of partially well-ordered sets. Furthermore we prove that the restriction $p \geqslant n$ for the characteristic of the base field cannot be removed. Namely, we prove that the $\mathbb{Z}_3$-graded identities of $UT_3^{(-)}(F)$ do not satisfy the Specht property if F is an infinite field of characteristic 2. To the best of our knowledge this is the first example of a finite dimensional Lie algebra nontrivially graded by a finite group that does not satisfy the Specht property.

This thesis is organized as follows. In the first chapter, we define some notions, and state several important results concerning PI algebras, gradings and partially well ordered sets. These notions are necessary in what follows. Most of the proofs are omitted, and the respective references are given when appropriate.

The second chapter is about the multiplicity series in d variables of $UT_n(E)$, as consequence we find a partial algorithm to calculate the multiplicities $m_\lambda(UT_n(E))$ when λ is a partition with no more than d part. Finally, in order to show how the algorithm works, we compute the multiplicities $m_\lambda(UT_n(E))$ when λ is a partition with no more than 2 parts and $1 \leqslant n \leqslant 3$.

In the third chapter, we define (k, l)-multiplicity series of a PI-algebra. Then, we compute the double Hilbert series of E and, as a consequence, we build up an algorithm with output the (k, l)-multiplicity series of $UT_n(E)$. We want to highlight that this algorithm allows us to find all multiplicities $m_\lambda(UT_n(E))$ when $k = n$ and $l = 2n - 1$. In order to show how the algorithm works, we finish this chapter considering some particular cases.

Finally, in the fourth chapter we deal first with the graded identities of the Lie algebra $UT_2^{(-)}(F)$, and prove that the corresponding ideal of graded identities satisfies the Specht property. Afterwards we consider $UT_3^{(-)}(F)$. Initially we require F an infinite field of characteristic 0 or $p > 2$, and prove that the ideal of graded identities satisfies the Specht property. We also show that if F is of characteristic 2, then the Specht property fails in this case. In the last section we prove the Specht property for $UT_n^{(-)}(F)$ in case $\operatorname{char} F = 0$ or $\operatorname{char} F = p \geqslant n$. We chose to separate the general case from those when $n \leqslant 3$ for several reasons. One of them is that the case $n = 2$ is much simpler and transparent, and gives no clue how to treat the case of $n > 2$. Another is that when $n = 3$ we have two completely different situations: when $\operatorname{char} F = 2$ and when $\operatorname{char} F \neq 2$. And the last reason is that the arguments in the cases $n = 2$ and $n = 3$ are more transparent and that for $n = 3$ gives a better idea of the methods used in the general case.

The results contained in Chapters 2 and 3 were obtained in collaboration with L. Centrone and V. Drensky. The paper that contains these results, [16], was submitted for publication in January 2023. The contents of Chapter 4 was written in collaboration with P. Koshlukov, and the corresponding paper [19] was submitted for publication in August 2022.

1 Preliminaries

In this chapter we define some notions, and state several important results concerning PI algebras. We shall deal with algebras over a field. Unless otherwise stated, all algebras we refer to will be associative and usually with unity. We mention here that in the last chapter of this text we will be dealing with Lie algebras.

Most of the results in this chapter can be found in [23] and [37]; more specifically, we recommend chapters 1-5, 8 and 12 of [23]. Moreover we are not going to cite explicitly the statements from these books.

1.1 PI algebras

1.1.1 Free algebras

Let F be a field and consider X a non-empty set. The *free associative algebra* freely generated over F by the set X is the algebra $F\langle X\rangle$ of the polynomials in non-commuting variables $x \in X$. A linear basis of $F\langle X\rangle$ consists of all words in the alphabet X (including the empty word, denoted by 1). Such words are called *monomials* and the product of two monomials is given by juxtaposition. This product is extended by linearity to the polynomials of $F\langle X\rangle$. The elements of $F\langle X\rangle$ are called, as we said above, *polynomials* and, if $f \in F\langle X\rangle$, we write $f = f(x_1, \ldots, x_n)$ to indicate that $x_1, \ldots, x_n \in X$ are the only variables occurring in f. To indicate the elements of X we shall commonly use the symbols x, x_i.

We set $\deg u$ as the usual degree of a monomial u, that is the length of the word u. Moreover $\deg_{x_i} u$, the degree of u with respect to the variable x_i, counts how many times x_i occurs in u. Accordingly, the degree $\deg f$ of a polynomial $f = f(x_, \ldots, x_n)$ is the maximum degree of a monomial in f; $\deg_{x_i} f$, the degree of f in x_i, is the maximum of $\deg_{x_i} u$, for any monomial u of f.

The algebra $F\langle X\rangle$ is defined, up to isomorphism, by the following universal property: given an associative F-algebra A, every map from X to A can be uniquely extended to a homomorphism of algebras from $F\langle X\rangle$ to A. We shall call the *rank* of $F\langle X\rangle$ the cardinality of X. As a rule we are going to consider the free algebra $F\langle X\rangle$ of infinite countable rank on the set $X = \{x_1, x_2, \ldots\}$.

We shall also consider the *free nonunitary algebra* $F^+\langle X\rangle$ which consists of all polynomials from $F\langle X\rangle$ without constant terms. Notice that $F^+\langle X\rangle$ is a free algebra in the class of all algebras without unit.

Here we observe that the free associative algebra is canonically isomorphic to the *tensor algebra* of the F-vector space with a basis the set X.

Definition 1.1.1. *Let A be an F-algebra and $f = f(x_1, \ldots, x_n) \in F\langle X\rangle$. We say that $f \equiv 0$ (or simply f) is a polynomial identity of A if $f(a_1, \ldots, a_n) = 0$ for all a_1, $\ldots$, $a_n \in A$.*

Let Φ denote the set of all homomorphisms $\varphi\colon F\langle X\rangle \to A$. Then $f \equiv 0$ is a polynomial identity for A if and only if $f \in \bigcap_{\varphi\in\Phi} \ker\varphi$. We shall usually say that $f \equiv 0$ is an identity of A or that A satisfies $f \equiv 0$ (or simply f).

Definition 1.1.2. *We say that A is a PI-algebra if A satisfies a non-trivial polynomial identity $f \equiv 0$.*

This means that $f \neq 0$ in $F\langle X\rangle$ but $f \equiv 0$ on A.

Definition 1.1.3. *The (left-normed) Lie commutator of length n is a polynomial in $F\langle X\rangle$ defined inductively by*

$$[x_1, x_2] := x_1x_2 - x_2x_1,$$
$$[x_1, x_2, \ldots, x_n] := [[x_1, \ldots, x_{n-1}], x_n];\ n > 2$$

We shall give some examples of PI-algebras.

Example 1.1.1.

- If A is a commutative algebra then A is a PI-algebra because $[x_1, x_2] \equiv 0$ is an identity of A.
- The algebra A is nilpotent of class of nilpotency $\leqslant n$ if and only if it satisfies the polynomial identity $x_1 \cdots x_n \equiv 0$. Of course, A is non-unitary and we have to consider $x_1 \cdots x_n$ as an element of the free non-unitary algebra $F^+\langle X\rangle$.
- The algebra $M_2(F)$ of the 2×2 matrices over F satisfies the Hall identity $[[x, y]^2, x] \equiv 0$. To see this, recall that if $b \in M_2(F)$, its characteristic polynomial is

  $$x^2 - \operatorname{tr}(b)x + \det(b),$$

  where $\operatorname{tr}(b)$ and $\det(b)$ are the trace and determinant of b. In the case b is a commutator, $\operatorname{tr}(b) = 0$ and so, $b^2 + \det(b)I = 0$, where I is the identity 2×2 matrix. This says that $b^2 = -\det(b)I$, so the square of any commutator is a scalar matrix, hence central. Then $[[x, y]^2, z] \equiv 0$ is a polynomial identity of $M_2(F)$. By putting $z = x$ we get the required identity.

- Let $UT_n(F)$ be the algebra of the $n \times n$ upper triangular matrices over F. Then $UT_n(F)$ is a PI-algebra since it satisfies the identity $[x_1, x_2] \cdots [x_{2n-1}, x_{2n}] \equiv 0$. To see this, observe that the commutator of any two upper triangular matrices is a strictly upper triangular matrix (that is its diagonal entries are all equal to 0). But the set of the strictly upper triangular matrix is a nilpotent two-sided ideal I of $UT_n(F)$ such that $I^n = 0$, hence $[x_1, x_2] \cdots [x_{2n-1}, x_{2n}] \equiv 0$ is an identity of $UT_n(F)$. Observe that I is the Jacobson radical of $UT_n(F)$.

- If A is a finite dimensional algebra and $\dim A < n$, then A satisfies the *standard identity* of degree n

  $$s_n(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(n)} \equiv 0.$$

  The algebra A also satisfies the *Capelli identity in n alternating variables,*

  $$Cap_n(x_1, \ldots, x_n, y_1, \ldots, y_{n-1}) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) x_{\sigma(1)} y_1 x_{\sigma(2)} y_2 \cdots y_{n-1} x_{\sigma(n)} \equiv 0.$$

- Since the $n \times n$ matrix algebra $M_n(F)$ is of dimension n^2, it satisfies the standard identity of degree $n^2 + 1$ and also the Capelli identity $d_{n^2+1}(x_1, \ldots, x_{n^2+1}, y_1, \ldots, y_{n^2})$.

Example 1.1.2 (Grassmann algebra)**.** Let W be an infinite dimensional vector space with basis $\{e_1, e_2, \ldots\}$ over a field F of characteristic different from two. The *Grassmann algebra* $E = E(W)$ is the associative algebra over F generated by $\{e_1, e_2 \ldots\}$ with defining relations $e_i e_j + e_j e_i = 0$ for all $i, j \in \mathbb{N}$. Note that E is isomorphic to the factor algebra $F\langle X\rangle/J$ where $X = \{x_1, x_2, \ldots\}$ and the ideal J is generated by $x_i x_j + x_j x_i$ with $i, j \geqslant 1$. Observe that $E = E^{(0)} \oplus E^{(1)}$ where

$$E^{(0)} := \operatorname{span}_F\{1, e_{i_1} \cdots e_{i_{2k}} | 1 \leqslant i_1 < i_2 < \cdots < i_{2k} \; k > 0\}$$
$$E^{(1)} := \operatorname{span}_F\{e_{i_1} \cdots e_{i_{2k+1}} | 1 \leqslant i_1 < \cdots < i_{2k+1} \; k \geqslant 0\}.$$

It is easily checked that $E^{(0)}E^{(0)} + E^{(1)}E^{(1)} \subseteq E^{(0)}$ and $E^{(0)}E^{(1)} + E^{(1)}E^{(0)} \subseteq E^{(1)}$, hence the decomposition $E = E^{(0)} \oplus E^{(1)}$ is a $\mathbb{Z}_2$-grading of E. Notice that $E^{(0)}$ coincides with the center of E.

Notice that E satisfies the identity $[x_1, x_2, x_3] \equiv 0$. In fact, observe that $E^{(0)}$ is central, and every non zero commutator of two elements of E is a linear combination of monomials in the e_i's of even length. Thus $[E, E] \subseteq E^{(0)}$ and the conclusion follows.

1.1.2 T-ideals and varieties of algebras

In this section we will introduce the notion of T-ideal and variety of algebras. Given an algebra A, we define

$$T(A) = \{f \in F\langle X\rangle | f \equiv 0 \text{ in } A\}$$

the set of identities of A. Note that $T(A)$ is a two-sided ideal of $F\langle X\rangle$. Moreover, if $f(x_1,\dots,x_n)$ is any polynomial in $T(A)$, and $g_1,\dots,g_n$ are arbitrary polynomials in $F\langle X\rangle$, it is clear that $f(g_1,\dots,g_n)\in T(A)$. Since any endomorphism of $F\langle X\rangle$ is determined by mapping $x\mapsto g$, $x\in X$, $g\in F\langle X\rangle$, it follows that $T(A)$ is an ideal invariant under all endomorphisms of $F\langle X\rangle$. The ideals with this property are called *T-ideals*.

Definition 1.1.4. *An ideal I of $F\langle X\rangle$ is a T-ideal if $\varphi(I)\subseteq I$ for all endomorphisms φ of $F\langle X\rangle$.*

Hence $T(A)$ is a T-ideal of $F\langle X\rangle$. On the other hand, it is easy to check that all T-ideals of $F\langle X\rangle$ are of this type. Indeed, if I is a T-ideal, it can be easily proved that $T(F\langle X\rangle/I)=I$. Due to this fact, we have that any algebra A determines a T-ideal of $F\langle X\rangle$. A further remark is relevant. Many (different and even non-isomorphic) algebras may correspond to the same T-ideal. For this purpose, we need the notion of a variety of algebras.

Definition 1.1.5. *Let S be a non-empty set of $F\langle X\rangle$. The class of all algebras A such that $f\equiv 0$ on A for every $f\in S$ is called the variety $\mathcal{V}=\mathcal{V}(S)$ determined by S.*

A variety $\mathcal{V}$ is called *non-trivial* if $S\neq 0$ whereas $\mathcal{V}$ is said to be *proper* if it is non-trivial and contains a non-zero algebra. For example, the class of all commutative algebras forms a proper variety with $S=\{[x,y]\}$. Also, if $S=\{x^n\}$, then $\mathcal{V}(S)$ is the class of all algebras which are nil of exponent bounded by n. Observe that if $\mathcal{V}$ is the variety determined by the set S and $\langle S\rangle_T$ is the T-ideal of $F\langle X\rangle$ generated by S, then $\mathcal{V}(S)=\mathcal{V}(\langle S\rangle_T)$ and $\langle S\rangle_T=\bigcap_{A\in\mathcal{V}}T(A)$. Let us write $\langle S\rangle_T=T(\mathcal{V})$. Thus a T-ideal of $F\langle X\rangle$ corresponds to each variety; the converse is also true. In fact we have the following theorem, see for example [37], Theorem 1.2.5.

Theorem 1.1.1. *There is a one-to-one correspondence between T-ideals of $F\langle X\rangle$ and varieties of algebras. In this correspondence, a variety $\mathcal{V}$ corresponds to the T-ideal of identities $T(\mathcal{V})$ and a T-ideal I corresponds to the variety of algebras satisfying all the identities in I.*

1.1.3 Homogeneous and multilinear polynomials

When the base field F is infinite (or "large enough"), the study of the identities of a given algebra can be reduced to the study of homogeneous, multi-homogeneous, or multilinear polynomials as we will see below.

Let $F_n=F\langle x_1,\dots,x_n\rangle$ be the free algebra of rank n. Notice that this algebra can be naturally decomposed as

$$F_n=F_n^{(0)}\oplus F_n^{(1)}\oplus F_n^{(2)}\oplus\cdots$$

where for every $k \geqslant 0$, $F_n^{(k)}$ is the subspace spanned by all monomials of total degree k. Since $F_n^{(i)}F_n^{(j)} \subset F_n^{(i+j)}$ for all $i, j \geqslant 0$, we say that F_n is graded by the degree or it has the structure of a $\mathbb{Z}$-graded algebra. The $F_n^{(i)}$'s are called *homogeneous components* of F_n.

This decomposition can be further refined as follows: for every $k \geqslant 1$ write

$$F_n^{(k)} = \bigoplus_{i_1+\cdots+i_n=k} F_n^{(i_1,\ldots,i_n)}$$

where $F_n^{(i_1,\ldots,i_n)}$ is the subspace spanned by all monomials of degree i_t in x_t. It is clear that $F_n^{(i_1,\ldots,i_n)}F_n^{(j_1,\ldots,j_n)} \subseteq F_n^{(i_1+j_1,\ldots,i_n+j_n)}$. In this case we say that F_n is *multigraded*. Such decompositions extend in an obvious way to $F\langle X\rangle$ when X is countable.

Definition 1.1.6. *A polynomial f belonging to $F_n^{(k)}$ for some $k \geqslant 0$, is called homogeneous of degree k. If f belongs to some $F_n^{(i_1,\ldots i_n)}$, it will be called multihomogeneous of multidegree $(i_1,\ldots,i_n)$. We also say that a polynomial f is homogeneous in the variable x_i, if x_i appears with the same degree in every monomial of f.*

Observe that if $f(x_1,\ldots,x_n) \in F\langle X\rangle$, we can always write

$$f = \sum_{i_1\geqslant 0,\ldots,i_n\geqslant 0} f^{(i_1,\ldots,i_n)}$$

where $f^{(i_1,\ldots,i_n)} \in F_n^{(i_1,\ldots,i_n)}$ is the linear combination of all monomials in f where $x_1, \ldots, x_n$ appear with degree $i_1, \ldots, i_n$ respectively. The polynomials $f^{(i_1,\ldots,i_n)}$ which are non-zero are called the *multihomogeneous components* of f. The following result gives us a useful property of T-ideals in the case when F is an infinite field.

Theorem 1.1.2. *Let F be an infinite field. If $f \equiv 0$ is a polynomial identity for the algebra A, then every multihomogeneous component of f is a polynomial identity for A.*

We recall that one of the proofs of the latter theorem can be found in [37], Chapter 1, and it uses standard Vandermonde argument. The most important consequence of Theorem 1.1.2 is that over an infinite field, every T-ideal is generated by its multihomogeneous polynomials.

Definition 1.1.7. *A polynomial f is linear in the variable x_i if x_i occurs with degree 1 in every monomial of f. A polynomial which is multihomogeneous and linear in each of its variables is called multilinear.*

In other words, a polynomial $f(x_1,\ldots,x_n) \in F\langle X\rangle$ is multilinear if it is multihomogeneous of multidegree $(1,\ldots,1)$. Moreover, it is clear that this polynomial is always of the form

$$f(x_1,\ldots,x_n) = \sum_{\sigma\in S_n} \alpha_\sigma x_{\sigma(1)}\cdots x_{\sigma(n)}$$

where $\alpha_\sigma \in F$, and S_n is the symmetric group of order n. (That is S_n consists of all permutations of the set $\{1, 2, \ldots, n\}$; its order as a group is of course $n!$.) We state the following definition that we shall need later on.

Definition 1.1.8. *i) Let S be a set of polynomials in $F\langle X\rangle$ and $f \in F\langle X\rangle$. We say that f is a consequence of the polynomials in S (or f follows from the polynomials in S) if $f \in \langle S\rangle_T$, the T-ideal generated by the set S.*

ii) Two sets of polynomial identities are equivalent, if they generate the same T-ideal.

We will state a fundamental theorem that simplifies the study of the T-ideals to their multilinear parts:

Theorem 1.1.3. *If* char $F = 0$, *every non-zero polynomial $f \in F\langle X\rangle$ is equivalent to a finite set of multilinear polynomials.*

We can write the previous result in the language of T-ideals.

Corollary 1.1.1. *If* char $F = 0$, *every T-ideal is generated, as a T-ideal, by the multilinear polynomials it contains.*

1.2 Graded polynomial identities

In this section, we define some notions concerning graded algebras. We let G be an arbitrary group with multiplicative notation and unit element 1, and A an arbitrary (not necessarily associative) algebra. Most of this section can be found in [37].

Definition 1.2.1. *The algebra A is G-graded if there exist vector subspaces $\{A_g\}_{g\in G}$, where some of the A_g can be zero, such that*

$$A = \bigoplus_{g\in G} A_g$$

and $A_g A_h \subseteq A_{gh}$ for all g, $h \in G$. The subspaces A_g are called homogeneous and the non-zero elements $a \in A_g$ are homogeneous of degree g. We denote this as G-deg $a = g$, *or if not ambiguous, simply as* $\deg a = g$.

A vector subspace $B \subseteq A$ is called *graded* (or *homogeneous*) if $B = \bigoplus_{g\in G}(A_g \cap B)$. If $I \subseteq A$ is an ideal and graded subspace, we call it a *graded ideal*. In this case the quotient A/I inherits from A a natural structure of G-graded algebra.

Definition 1.2.2. *Let $A = \bigoplus_{g\in G} A_g$ and $B = \bigoplus_{g\in G} B_g$ be two G-graded algebras. The map $f\colon A \to B$ is a homomorphism of G-graded algebras if f is a homomorphism of algebras such that $f(A_g) \subseteq B_g$. Similarly one defines endomorphism, automorphism, isomorphism, of G-graded algebras.*

Note that A and B are isomorphic as G-graded algebras whenever there is an isomorphism of algebras from A to B which respects the gradings. We are going to recall the definition of the *free G-algebra*. Let $g \in G$ and consider $X_g := \{x_1^{(g)}, x_2^{(g)}, \ldots, \}$ a set variables. Put $X = \bigcup_{g \in G} X_g$ and form the free algebra (associative, or Lie, or whatever else) $F\{X_G\}$ in the variables of X. We can define naturally a G-grading on $F\{X_G\}$ by assigning degree g to all variables from X_g, $\deg x_i^{(g)} = g$, and then extending this to all monomials $m \in F\{X_G\}$ in the following way

$$\deg m = \begin{cases} g, & if \quad m = x_i^{(g)}, \\ (\deg m_1)(\deg m_2) & if \quad m = m_1 m_2. \end{cases}$$

In the case of associative or Lie algebras, we denote the corresponding free algebra by $F\langle X_G \rangle$ and by $\mathcal{L}\langle X_G \rangle$, respectively.

Definition 1.2.3. *Consider A a G-graded algebra and let $f(x_1^{(g_1)}, \ldots, x_n^{g^{(n)}}) \in F\{X_G\}$. We say that f is a G-graded polynomial identity for A if $f(a_1, \ldots, a_m) = 0$ for all a_1, ..., $a_m \in A$ such that $\deg a_i = g_i$ for every i.*

We denote by $T_G(A)$ the set of G-graded identities for A. Observe that $T_G(A)$ is closed under endomorphisms of $F\{X_G\}$ that respect the grading. Conversely every such ideal is the ideal of G-graded identities for some G-graded algebra.

Definition 1.2.4. *If $J \subseteq F\{X_G\}$ is a G-graded ideal that is closed under endomorphisms of $F\{X_G\}$, we say that J is a T_G-ideal.*

Given a non-empty set $S \subseteq F\{X_G\}$, the T_G-ideal generated by S, denoted by $\langle S \rangle_G$, is the intersection of all T_G-ideals of $F\{X_G\}$ such that S is contained in them.

Let J be a T_G-ideal and consider $S \subseteq T_G(A)$. If $J = \langle S \rangle_G$, we say that S is a *basis of J* as T_G-ideal. We draw the readers' attention that we do not require S to be minimal; thus the whole J is a basis of J. Clearly such a basis is of little value and does not contribute much to our knowledge; we are interested in "small" sets S. The following statements are direct analogues of their counterparts from the ordinary (non-graded) case, and their proofs remain the same.

Theorem 1.2.1. *Consider $f(x_1, \ldots, x_n) = \sum_{i=1}^{s} f_i$, where f_i is the homogeneous component of f such that $\deg_{x_1} f = i$. Then the followings statements hold*

i. If the base field F contains more that s elements (for example if F is infinite), then $f_i(x_1, \ldots, x_n) \in \langle f \rangle_{T_G}$.

ii. If $\operatorname{char} F = 0$, then $f_i(x_1, \ldots, x_n) \in \langle f \rangle_{T_G}$ has a basis of multilinear polynomials.

Corollary 1.2.1. *Let A be a G-graded algebra over an infinite field F. Then*

i. If f is a G-graded identity for A, then the multihomogeneous components of f are G-graded identities for A.

ii. If $\operatorname{char} F = 0$, then $T_G(A)$ is generated, as a T_G-ideal, by the multilinear G-identities of A.

Definition 1.2.5. *Let A be G-graded algebra. We say that $T_G(A)$ has the Specht property if any T_G-ideal J such that $T_G(A) \subseteq J$, has a finite basis, i.e., J is finitely generated as a T_G-ideal. A variety $\mathcal{V}$ has the Specht property if its T_G-ideal has the Specht property.*

1.3 Hilbert series

By Theorem 1.1.2, if the base field F is infinite, then a polynomial f is equivalent, as an identity, to its multihomogeneous components $f^{(i_1,\dots,i_n)}$. Hence every T-ideal I of $F\langle X\rangle$ is a homogeneous ideal, i.e., I is a direct sum of its multihomogeneous components. If we consider a (multi)graded vector space such that all homogeneous components are finite dimensional, it is convenient to use its Hilbert (or Poincaré) series to measure it.

Definition 1.3.1. *i) The vector space V is graded if it is a direct sum of subspaces V_m, $m \geqslant 0$, that is*

$$V = V_0 \oplus V_1 \oplus V_2 \oplus \cdots .$$

The subspaces V_m are called the homogeneous components of degree m of V. Similarly, V is multigraded if

$$V = \bigoplus_{m_i \geqslant 0} V_{(m_1,\dots,m_d)},$$

where $V_{(m_1,\dots,m_d)}$ is its homogeneous component of degree $(m_1,\dots m_d)$, and the direct sum runs over all d-tuples $(m_1,\dots,m_d)$ such that $m_i \geqslant 0$.

ii) The subspace W of the graded space $V = \bigoplus_{m \geqslant 0} V_m$ is graded (homogeneous) subspace if $W = \bigoplus_{m\geqslant 0}(W \cap V_m)$. In this case the factor space V/W can also be naturally graded and V/W inherits the grading of V.

We shall use the Hilbert series to compute the so-called cocharacters of some interesting PI algebras

Definition 1.3.2. *If $V = \bigoplus_{m\geqslant 0} V_m$ is a graded space and $\dim V_m < \infty$ for all $m \geqslant 0$, the formal power series*

$$H(V,t) = \sum_{m \geqslant 0} (\dim V_m) t^m$$

is called the Hilbert series of V. If the vector space $V = \bigoplus_{m_i \geqslant 0} V_{(m_1,\dots,m_d)}$ is multigraded, then the Hilbert series of V is

$$H(V, t_1, \dots, t_d) = \sum_{m_i \geqslant 0} (\dim V_{(m_1,\dots,m_d)}) t_1^{m_1} \cdots t_d^{m_d}.$$

Example 1.3.1. The polynomial algebra $F[x_1, \dots, x_d]$ is $\mathbb{Z}$-graded assuming that the homogeneous polynomials of degree m (in the usual sense) are the homogeneous elements of degree m. Similarly, $F[x_1, \dots, x_d]$ has a multi-grading, counting the entry of each variable in the monomials. Analogously, one can define a grading and a (multi)grading on the free associative algebra $F\langle x_1, \dots, x_d \rangle$ of finite rank d. Usually we shall assume that $F[x_1, \dots, x_d]$ and $F\langle x_1, \dots, x_d \rangle$ are equipped with these two gradings. Their Hilbert series are

- $H(F[x_1, \dots, x_d], t) = \dfrac{1}{1-t}$,
- $H(F[x_1, \dots, x_d], t_1, \dots, t_d) = \prod_{i=1}^{d} \dfrac{1}{1-t_i}$,
- $H(F\langle x_1, \dots, x_d \rangle, t) = \dfrac{1}{1-dt}$,
- $H(F\langle x_1, \dots, x_d \rangle, t_1, \dots t_d) = \dfrac{1}{1-(t_1 + \cdots + t_d)}$.

In order to deduce these formulas, one expands the geometric progressions on the right-hand side, and compares the obtained expressions with the dimensions of the corresponding vector spaces in the usual gradings on the algebras on the left-hand sides.

As we know from basic algebra, if M is a graded module over a graded commutative algebra A, then the Hilbert series of M determines it up to an isomorphism.

Hilbert series are related to the usual operations on graded vector spaces. In fact:

Proposition 1.3.1. *Let V, W be (multi)graded spaces and consider U a homogeneous subspace of V. Then*

i) $H(V \oplus W, t_1, \dots, t_d) = H(V, t_1, \dots, t_d) + H(W, t_1, \dots, t_d)$,

ii) $H(V \otimes W, t_1, \dots, t_d) = H(V, t_1, \dots, t_d) \cdot H(W, t_1, \dots, t_d)$,

iii) $H(V/U, t_1, \dots, t_d) = H(V, t_1, \dots, t_d) - H(U, t_1, \dots, t_d)$.

Let A be a PI-algebra over an infinite field F. It is well known that $T(A)$ is a multihomogeneous ideal of $F\langle X\rangle$. Then, if $\mathrm{T}_d = (t_1, \ldots, t_d)$, we denote by

$$H(A, \mathrm{T}_d) := H(F\langle x_1, \ldots, x_d\rangle/(T(A) \cap F\langle x_1, \ldots, x_d\rangle), t_1, \ldots, t_d)$$

the Hilbert series of the relatively free algebra in d variables.

Formanek in [30] gave a formula for the Hilbert series of the product of two T-ideals as a function of the Hilbert series of the factors.

Theorem 1.3.1. *Let U and V be multihomogeneous ideals of the free algebra $F\langle x_1, \ldots, x_d\rangle$. Then the Hilbert series of UV, U and V are related by the equation*

$$H(U, t_1, \ldots, t_d)H(V, t_1, \ldots, t_d) = H(UV, t_1, \ldots, t_d)H(F(x_1, \ldots, x_d), t_1, \ldots, t_d).$$

Corollary 1.3.1. *Let A, B and C be PI-algebras over an infinite field F such that $T(A) = T(B)T(C)$. Then the Hilbert series of the relatively free algebras of A, B and C satisfy the equation*

$$H(A, \mathrm{T}_d) = H(B, \mathrm{T}_d) + H(C, \mathrm{T}_d) + (t_1 + \cdots t_d - 1)H(B, \mathrm{T}_d)H(C, \mathrm{T}_d)$$

The following example is an application of Corollary 1.3.1.

Example 1.3.2. Consider the algebra $UT_2(F)$ of the 2×2 upper triangular matrices over an infinite field F. It is known that $T(UT_2(F)) = T(F)T(F)$ and $F\langle x_1, \ldots, x_d\rangle/(T(F) \cap F\langle x_1, \ldots, x_d\rangle) = F[x_1, \ldots, x_d]$. By Example 1.3.1 and Corollary 1.3.1 we get

$$H(UT_2(F), \mathrm{T}_d) = 2\prod_{i=1}^{d} \frac{1}{1 - t_i} + (t_1 + \cdots t_d - 1)\prod_{i=1}^{d} \frac{1}{(1 - t_i)^2}.$$

1.4 Representation theory of the symmetric group and the general linear group

In this section we shall describe some applications of the representation theory of the symmetric groups and the general linear group to the theory of PI-algebras.

1.4.1 Background of Representation theory of groups

In this subsection we recall the basic definitions and results of the representation theory of finite groups over an algebraically closed field of characteristic zero. For more information about representation theory, see [43], [37] and [67].

Let V be a vector space over a field F and let $GL(V)$ be the group of invertible endomorphisms of V. Recall the following

Definition 1.4.1. *A representation of a group G in V is a homomorphism of groups $\rho\colon G \to GL(V)$.*

Let us denote by $\mathrm{End}(V)$ the algebra of F-endomorphisms of V. If FG is the group algebra of G over F and ρ is a representation of G in V, it is clear that ρ induces a homomorphism of F-algebras $\rho'\colon FG \to \mathrm{End}(V)$ such that $\rho'(1_G) = 1$. Throughout the text we shall be dealing only with the case when $\dim V = n < \infty$, i.e., with finite dimensional representations. In this case n is called the dimension or the degree of the representation ρ. Notice that a representation of a group G uniquely determines a finite dimensional FG-module (or G-module) in the following way. If $\rho\colon G \to GL(V)$ is a representation of G, then V becomes a (left) G-module by defining

$$g \cdot v = \rho(g)(v)$$

for all $g \in G$ and $v \in V$. It is also clear that if M is a G-module which is finite dimensional as a vector space over F, then $\rho\colon G \to GL(M)$, such that

$$\rho(g)(m) = g \cdot m$$

for all $g \in G$ and $m \in M$, defines a representation of G in M.

Definition 1.4.2. *If $\rho\colon G \to GL(V)$ and $\rho'\colon G \to GL(W)$ are two representations of a group G, we say that ρ and ρ' are equivalent if V and W are isomorphic as G-modules. In this case we write $\rho \sim \rho'$.*

Definition 1.4.3. *Let $\rho\colon G \to GL(V)$ be a representation of V.*

i) ρ is irreducible, if V is an irreducible G-module.

ii) ρ is completely reducible, if V is the direct sum of irreducible G-modules.

One of the basic tools for studying the representations of a finite group in characteristic zero is Maschke's theorem. By this theorem every representation of G in $\mathrm{char}\, F = 0$ is completely reducible, equivalently the group algebra FG is semisimple. Hence by Wedderburn-Artin's Theorem

$$FG = M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k)$$

where $D_1, \ldots, D_k$ are finite dimensional division algebras over F. Moreover every irreducible G-module is isomorphic to a minimal left ideal of FG (and hence to a minimal left ideal of some $M_{n_i}(D_i)$), where G acts on FG by left multiplication.

Proposition 1.4.1. *If M is an irreducible representation of G, then $M = J_i$ a minimal left ideal of $M_{n_i}(D_i)$, for some $i \in \{1, \ldots, k\}$. Hence there exists a minimal idempotent $e \in FG$ such that $M = eFG$.*

Another key topic in representation theory is provided by the theory of characters. From now on assume that F has characteristic zero and let $\mathrm{tr}\colon \mathrm{End}(V) \to F$ be the trace map on $\mathrm{End}(V)$.

Definition 1.4.4. *Let $\rho\colon G \to GL(V)$ be a representation of G. Then the map $\chi_\rho\colon G \to F$ such that $\chi_\rho(g) = \mathrm{tr}(\rho(g))$ is called the character of the representation ρ and $\dim V = \deg \chi_\rho$ is called the degree of the character χ_ρ.*

We say that the character χ_ρ is *irreducible* if ρ is irreducible. Since $\chi_\rho(g) = \chi_\rho(hgh^{-1})$, χ_ρ is constant on the conjugacy classes of G, i.e. χ_ρ is a class function of G. Notice that $\chi_\rho(1) = \deg \chi_\rho$.

1.4.2 Representations of the symmetric group

In this subsection we describe the ordinary representation theory of the symmetric group S_n, $n > 1$, our principal references of this part are [37] and [67].

Since $\mathbb{Q}$, the field of rational numbers, is a splitting field for S_n, for any field F of characteristic zero, the group algebra FS_n has a decomposition into simple components which are algebras of matrices over the field F itself, that is

$$FS_n = M_{n_1}(F) \oplus \cdots \oplus M_{n_k}(F)$$

The non-isomorphic irreducible representations of the symmetric group (and hence the left irreducible S_n-modules) are in one-to-one correspondence with the conjugacy classes of S_n and are described in terms of partitions and Young diagrams.

Definition 1.4.5. *Let $n \geqslant 0$ be an integer. A partition λ of n is a finite sequence of integers $\lambda = (\lambda_1, \ldots, \lambda_r)$ such that $\lambda_1 \geqslant \cdots \geqslant \lambda_r \geqslant 0$ and $\sum_{i=1}^{r} \lambda_i = n$. In this case we write $\lambda \vdash n$.*

If $r = 1$, then $\lambda_1 = n$ and we write $\lambda = (n)$. For the partition λ with $\lambda_1 = \cdots = \lambda_n = k$, the notation $\lambda = (k^n)$ is commonly used. There is a natural correspondence between the partitions of m and the conjugacy classes of S_n: if $\sigma \in S_n$, we decompose σ into the product of disjoint cycles, including 1-cycles. This decomposition is unique if we require that

$$\sigma = \pi_1 \pi_2 \cdots \pi_r,$$

where $\pi_1, \ldots, \pi_r$ are disjoint cycles of length $\lambda_1 \geqslant \cdots \geqslant \lambda_r \geqslant 1$, respectively, up to the ordering of the cycles of the same length. Then the partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ uniquely determines the conjugacy class of σ.

Proposition 1.4.2. *Let F be any field of characteristic zero and $n > 1$. Then there is a one-to-one correspondence between irreducible S_n-characters and partitions of n. Let*

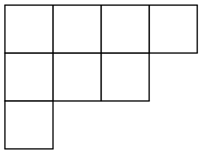
$\{\chi_\lambda \mid \lambda \vdash n\}$ be a complete set of irreducible characters of S_n and let $d_\lambda = \chi_\lambda(1)$ be the degree of χ_λ. Then

$$FS_n = \bigoplus_{\lambda \vdash n} I_\lambda \cong \bigoplus_{\lambda \vdash n} M_{d_\lambda}(F)$$

where $I_\lambda = e_\lambda FS_n \cong M_{d_\lambda}(F)$ and $e_\lambda = \sum_{\sigma \in S_n} \chi_\lambda(\sigma)\sigma$ is up to a scalar, the unit element of I_λ.

Definition 1.4.6. *The Young diagram D_λ of the partition $\lambda_1 = (\lambda_1, \ldots, \lambda_r)$ is the set of all knots (points) $(i, j) \in \mathbb{Z}^2$, such that $1 \leqslant j \leqslant \lambda_i$, $1 \leqslant i \leqslant r$.*

It is convenient to present the Young diagrams graphically as follows. We replace the knots with square boxes such that the first coordinate i (the index of the row) increases from top to bottom and the second coordinate j (the index of the column) increases from left to right. For example, consider the partition $\lambda = (4, 3, 1) \vdash 8$, its diagram is given in the figure below



Definition 1.4.7. *i) A Young tableau T_λ of the diagram D_λ with n boxes is a filling of the boxes of D_λ with the positive integers 1, 2, ..., n, without repetitions. If λ is a partition of n and $\sigma \in S_n$, we denote by $T_\lambda(\sigma)$ the tableau such that its first column contains the integers $\sigma(1), \ldots, \sigma(k_1)$ written in this order from top to bottom, the second column contains consequently written $\sigma(k_1 + 1), \ldots, \sigma(k_l + k_2)$, etc.*

ii) The tableau T_λ is called standard, if the integers written in each column and each row increase, respectively, from top to bottom and from left to right

Example 1.4.1. For the partition $\lambda = (4, 3, 1)$, consider the permutations

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 8 & 2 & 1 & 6 & 7 \end{pmatrix}, \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 6 & 2 & 4 & 5 & 7 & 8 \end{pmatrix}$$

Note that the tableau $T_\lambda(\sigma)$ is not standard while $T_\lambda(\tau)$ is standard:

$T_\lambda(\sigma) =$

3	8	1	7
4	2	6	
5			

$T_\lambda(\tau) =$

1	2	5	8
3	4	7	
6			

Definition 1.4.8. *Let $\lambda \vdash n$, $\sigma \in S_n$ and let $T = T_\lambda(\sigma)$ be the corresponding Young tableau. The row stabilizer of T is the subgroup $R(T)$ of all permutations ρ in S_n, such that i and $\rho(i)$ are in the same row of T, $i = 1, \ldots, n$. Similarly one defines the column stabilizer of T, $C(T)$.*

Example 1.4.2. From the above example,

$$T = T_\lambda(\sigma),\ \lambda = (4,3,1),\ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 8 & 2 & 1 & 6 & 7 \end{pmatrix}$$

the row stabilizer $R(T)$ is the subgroup $S_4 \times S_3 \times S_1$ of S_8, where S_4, S_3 and S_1 act respectively on the sets $\{3,8,1,7\}$, $\{4,2,6\}$ and $\{5\}$.

For each partition λ of n we denote by $M(\lambda)$ and χ_λ the corresponding irreducible S_n-module and its character, respectively.

Theorem 1.4.1. *i) Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition of n, $\tau \in S_n$ and let $T = T_\lambda(\tau)$ be the corresponding Young tableau. Up to a multiplicative constant the element of FS_n*

$$e_T = \sum_{\sigma \in R(T)} \sum_{\rho \in C(T)} \operatorname{sgn}(\rho)\sigma\rho$$

is a minimal idempotent which generates a submodule of FS_n isomorphic to $M(\lambda)$.

ii) The sum of all left S_n-modules $FS_n e_T$, where T runs over the set of standard λ-tableaux, is direct. It is equal to the minimal two-sided ideal I_λ of FS_n corresponding to λ.

iii) The dimension of $M(\lambda)$ is given by the hook formula

$$\dim M_\lambda = \frac{n!}{\prod(\lambda_i + \lambda'_j - i - j + 1)}$$

where λ'_j's are the lengths of the columns of D_λ and the product in the denominator is over all boxes of D_λ. The dimension $\dim M(\lambda)$ *is equal also to the number of standard λ-tableaux $T_\lambda(\tau)$, $\tau \in S_n$.*

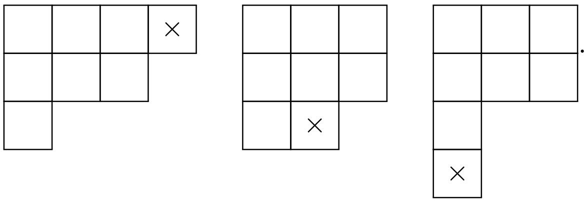
If H is a subgroup of the finite group G, and if W and V are respectively G- and H-modules, then we denote by $W \downarrow H$ the module W considered as an H-module and by $V \uparrow G$ the G-module *induced* by V. Recall that $V \uparrow G = FG \otimes_{FH} V$. If one observes that $V \subset V \uparrow G$ as H-modules via the embedding $V \to V \otimes_{FH} FG$, $v \to 1 \otimes v$, then $V \uparrow G$ has the following universal property: For every G-module W' and for every homomorphism of H-modules $\varphi\colon V \to W' \downarrow H$, there exists a unique homomorphism of G-modules $\psi\colon V \uparrow G \to W'$ which extends φ.

Identifying S_{n-1} with the subgroup of S_n fixing the symbol n, the branching theorem describes $M(\lambda) \downarrow S_{n-1}$, $\lambda \vdash n$, and $M(\mu) \uparrow S_n$, $\mu \vdash n-1$. Parts *i)* and *ii)* are equivalent by the *Frobenius reciprocity law.*

Theorem 1.4.2 (Branching Theorem)**.** *Let $\lambda \vdash n$, $\mu \vdash n-1$. Then* *i)* $M(\lambda) \downarrow S_{n-1} \cong \bigoplus M(\mu^{(i)})$ *where the direct sum runs over all partitions* $\mu^{(i)}$ *of* $n-1$ *such that their diagrams* $D_{\mu^{(i)}}$ *are obtained by deleting one box of the diagram* D_λ.

ii) $M(\mu) \uparrow S_n \cong \bigoplus M(\lambda^{(j)})$ *where the direct sum runs over all partitions* $\lambda^{(j)}$ *of* n *such that their diagrams* $D_{\lambda^{(j)}}$ *are obtained by adding one box of the diagram* D_μ.

Example 1.4.3. Consider $\lambda = (3^2, 1) \vdash 7$. Then the diagrams obtained from D_λ adding a box and applying Theorem 1.4.2 are



It follows that

$$M(\lambda) \uparrow S_8 \cong M((4,3,1)) \oplus M((3^2,2)) \oplus M((3^2,1^2)).$$

We embed the group $S_n \times S_m$ into S_{n+m} naturally. Recall that if M is an S_n-module and N is an S_m-module, then $M \otimes_F N$ has a natural structure of $S_n \times S_m$-module.

Definition 1.4.9. *If* M *is an* S_n*-module and* N *is an* S_m*-module, then the outer tensor product of* M *and* N *is defined as*

$$M \widehat{\otimes} N := (M \otimes N) \uparrow S_{n+m}.$$

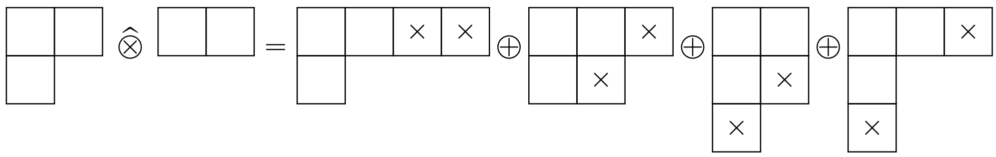
Theorem 1.4.3 (Young Rule)**.** *Let* $\lambda \vdash n$ *and* $m \geqslant 1$. *Then*

i.
$$M(\lambda) \widehat{\otimes} M((m)) = \sum_\mu M(\mu)$$
where the sum runs over all partitions μ *of* $n+m$ *such that* $\mu_1 \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \mu_{n+m} \geqslant \lambda_{m+n}$.

ii.
$$M(\lambda) \widehat{\otimes} M((1^m)) = \sum_\mu M(\mu)$$
where the sum runs over all partitions μ *of* $n+m$ *such that* $\mu_i = \lambda_i + \varepsilon_i$, $\varepsilon_i \in \{0,1\}$ *and* $1 \leqslant i \leqslant n+m$.

Note that in the first case the diagrams D_μ are obtained from the diagram D_λ by adding m boxes in such a way that no two new boxes are in the same column of D_μ. In the second case the Young diagrams D_μ are obtained from the diagram D_λ by adding m boxes in such a way that new boxes are not allowed to be in the same row.

Example 1.4.4. Consider the partition $\lambda = (2,1) \vdash 3$ and $m = 2$. By the Young rule, we have



Hence

$$M(\lambda)\widehat{\otimes}M((2)) \cong M((4,1)) \oplus M((3,2)) \oplus M((2^2,1)) \oplus M((3,1,1)).$$

In order to introduce the most general *Littlewood-Richardson rule*, which gives a decomposition $M(\lambda)\widehat{\otimes}M(\mu)$ for all λ, μ, we need some further definitions.

Definition 1.4.10. *An unordered partition of n is a finite sequence of positive integers $\alpha = (\alpha_1, \ldots, \alpha_t)$ such that $\sum_{i=1}^{t} \alpha_i = n$. In this case we write $\alpha \vDash n$.*

Definition 1.4.11. *Let $\lambda \vdash n$ and $\alpha \vDash n$. A (generalized) Young tableau of shape λ of content α is a filling of the diagram D_λ by positive integers in such a way that the integer i occurs exactly α_i times.*

For example, consider the partition $\lambda = (4,3,1)$ and set $\alpha = (2,3,1,2)$. Then

1	2	2	1
3	4	2	
2			

is a tableau of shape λ and contents α.

Definition 1.4.12. *A Young tableau is semistandard if the numbers are non-decreasing along the rows and strictly increasing down the columns.*

Example 1.4.5. Consider $\lambda = (4,3,1) \vdash 8$ and $\alpha = (2,3,1,2)$. Then a semistandard Young tableau of shape λ of contents α is

1	1	2	2
2	3	4	
4			

.

We now consider the natural partial order on the set of partitions. Let $\lambda = (\lambda_1, \ldots, \lambda_p) \vdash n$ and $\mu = (\mu_1, \ldots, \mu_q) \vdash m$, then $\lambda \geqslant \mu$ if and only if $p \geqslant q$ and $\lambda_i \geqslant \mu_i$ for all $1 \leqslant i \leqslant p$. In the language of Young diagrams $\lambda \geqslant \mu$ means that D_μ is a subdiagram of D_λ.

If $\lambda \geqslant \mu$, we define the skew-partition $\lambda/\mu = (\lambda_1 - \mu_1, \lambda_2 - \mu_2, \dots, \lambda_p - \mu_p)$; the corresponding diagram $D_{\lambda/\mu}$ is the set of boxes of D_λ which do not belong to D_μ. For example, consider $\lambda = (4, 3, 1) \vdash 8$ and $\mu = (2, 2, 1) \vdash 5$, notice that $\lambda/\mu = (2, 1, 0)$ and

$D_{\lambda/\mu} =$

*	*		
*	*		
*			

Definition 1.4.13. *A skew-tableau $T_{\lambda/\mu}$ is a filling of the boxes of the skew- diagram λ/μ with distinct positive integers. If repetitions occur, then we have the notion of (generalized) skew-tableau. We also have the natural notions of standard and semistandard skew-tableaux.*

Definition 1.4.14. *Let $\alpha = (\alpha_1, \dots, \alpha_t) \vDash n$. We say that α is a lattice permutation if for each j the number of i's which occur among $\alpha_1, \dots, \alpha_j$ is greater than or equal to the number of $(i+1)$'s for each i.*

Theorem 1.4.4 (Littlewood-Richardson Rule)**.** *Let $\lambda \vdash n$ and $\mu \vdash m$. Then*

$$M(\lambda)\widehat{\otimes}M(\mu) = \sum_{\nu \vdash n+m} k^{\mu}_{\nu/\lambda} M(\nu)$$

where $k^{\mu}_{\nu/\lambda}$ is the number of semistandard tableaux of shape ν/λ and content μ which yield lattice permutations when we read their entries from right to left and downwards.

Example 1.4.6. Consider the partitions $\lambda = (3, 2) \vdash 5$ and $\mu = (2, 1) \vdash 3$. We are going to find the decomposition of $M(\lambda)\widehat{\otimes}M(\mu)$ as a sum of irreducible S_8-modules using Theorem 1.4.4.

Below, we list the semistandard tableaux of shape ν/λ ($\nu \vdash 8$) and content μ which yield lattice permutations when we read their entries from right to left and downwards

i. $\nu = (5, 3) \vdash 8$

			1	1
		2		

ii. $\nu = (4, 3, 1) \vdash 8$

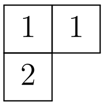
			1
		1	
2			

			1
		2	
1			

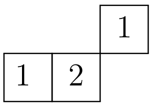
iii. $\nu = (5, 2, 1) \vdash 8$

			1	1
2				

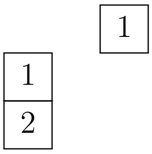
iv. $\nu = (3, 2^2, 1) \vdash 8$



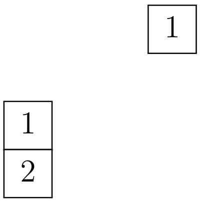
v. $\nu = (3^2, 2) \vdash 8$



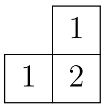
vi. $\nu = (3^2, 1^2) \vdash 8$



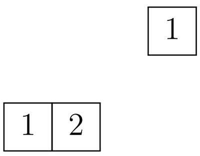
vii. $\nu = (4, 2, 1^2) \vdash 8$



viii. $\nu = (4^2) \vdash 8$



ix. $\nu = (4, 2^2) \vdash 8$



It follows that

$$M(\lambda)\widehat{\otimes}M(\mu) \cong M((5,3)) + M((5,2,1)) \oplus 2M((4,3,1)) \oplus M((4,2^2)) \oplus \\ M((4^2)) \oplus M((4,2,1^2)) \oplus M((3^2,2)) \oplus M((3^2,1^2)) \oplus M((3,2^2,1)).$$

1.4.3 S_n-actions on multilinear polynomials

In this section we introduce an action of the symmetric group S_n on the space of multilinear polynomials in n fixed variables.

Let A be a PI-algebra and $T(A)$ its T-ideal of identities. By Corollary 1.1.1, in characteristic zero, $T(A)$ is determined by its multilinear polynomials.

We denote by

$$P_n = \text{span}_F\{x_{\sigma(1)} \cdots x_{\sigma(n)} | \sigma \in S_n\},$$

the vector space of multilinear polynomials in $x_1, \ldots, x_n$ in the free algebra $F\langle X \rangle$. We define a map

$$\varphi \colon FS_n \to P_n$$

by setting

$$\varphi\left(\sum_{\sigma \in S_n} \alpha_\sigma \sigma\right) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}.$$

It is clear that φ is a vector space isomorphism. Observe that the symmetric group S_n acts from the left on the set P_n of multilinear polynomials of degree n as follows:

$$\sigma(x_{\tau(1)} \cdots x_{\sigma\tau(n)}) = x_{\sigma\tau(1)} \cdots x_{\sigma\tau(n)},$$

for all σ, $\tau \in S_n$. It follows that φ is a module isomorphism.

Since T-ideals are invariant under permutations of the variables, we obtain that $P_n \cap T(A)$ is a left S_n-submodule of P_n. Hence

$$P_n(A) := \frac{P_n}{P_n \cap T(A)}$$

has an induced structure of left S_n-module.

Definition 1.4.15. *For $n \geqslant 1$, the S_n-character of $P_n(A)$ is called the n-th cocharacter of A (or of the T-ideal $T(A)$) and is denoted $\chi_n(A)$.*

If we decompose the n-th cocharacter into irreducibles, we obtain

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda(A)\chi_\lambda,$$

where χ_λ is the irreducible S_n-character associated to the partition $\lambda \vdash n$ and $m_\lambda(A) \geqslant 0$ is the corresponding multiplicity.

Example 1.4.7. Let A be a (unitary) commutative algebra, then $\chi_n(A) = \chi_{(n)}$ for all $n \geqslant 1$. In fact, since the T-ideal of A coincides with the commutator ideal of $F\langle X \rangle$, the relatively free algebra $F(A)$ is isomorphic to the polynomial algebra $F[X]$ in infinitely many commuting variables. Hence $P_n(A)$ is spanned by the monomials $x_1 \cdots x_n$ and

$$\sigma(x_1 \cdots x_n) = x_1 \cdots x_n$$

for all $\sigma \in S_n$. Hence $P_n(A)$ is the trivial S_n-module.

Now we state without proof the following theorems about the partitions and the shapes of the Young diagrams corresponding to the irreducible characters in the cocharacter sequence of any PI-algebra.

Theorem 1.4.5 (Regev [65]). *The algebra A satisfies the Capelli identity in n skew-symmetric variables*

$$d_n(x_1, \ldots, x_n; y_1, \ldots y_{n-1}) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) x_{\sigma(1)} y_1 x_{\sigma(2)} y_2 \cdots y_{n-1} x_{\sigma(n)}$$

if and only if its cocharacter sequence is decomposed as

$$\chi_m(A) = \sum_{\lambda_n = 0} m_\lambda(A) \chi_\lambda,$$

i. e., the nonzero multiplicities correspond to partitions in less than n parts.

Note that by the previous theorem if A is an algebra such that $\dim A < n$, then its cocharacter sequences is completely determined by partitions λ with no more than n parts.

Theorem 1.4.6 (Amitsur-Regev [3]). *For every PI-algebra A there exist nonnegative integers k and l such that in the cocharacter sequence of A*

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda(A) \chi_\lambda,\ n = 1,\ 2,\ \ldots,$$

the partitions $\lambda = (\lambda_1, \ldots, \lambda_m)$ corresponding to non-zero multiplicities $m_\lambda(A)$, satisfy the condition $\lambda_{k+1} \leqslant l$. In other words, their diagrams D_λ are in a hook shape with height k of the arm and width l of the leg (see figure 1).

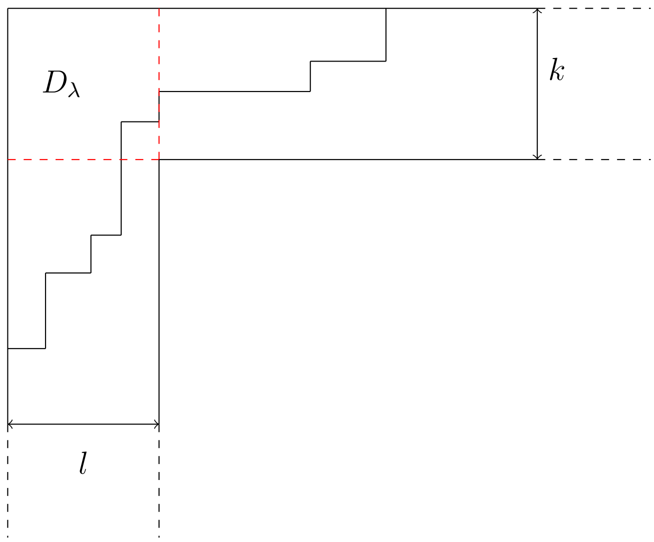


Figure 1 – The diagrams D_λ with $m_\lambda(A) \neq 0$.

1.4.4 The action of the general linear group

In this subsection we survey some results on representation theory of the general linear group in a form that is useful to our intent. For more details about this topic see [23] and [24].

We restrict most of our considerations to the case when $GL_d(F)$ acts on the free associative algebra of rank d. Let V be a vector space over F, we denote by $GL(V)$ the *general linear group of* V, i.e., the group of invertible linear transformations acting on V. When $\dim V = d < \infty$, we write

$$GL_d = GL_d(F) = GL(V)$$

and, for a fixed basis $\{e_1, \ldots, e_d\}$ of V, we identify GL_d with the group of invertible $d \times d$ matrices with entries from F.

Definition 1.4.16. *i. A representation of the general linear group GL_d*

$$\varphi\colon GL_d \to GL_s$$

is called polynomial (and V is a polynomial GL_d-module), if the entries $\varphi_{pq}(g)$ of the $s \times s$ matrices $\varphi(g) \in GL_s$ are polynomial functions of the entries a_{ij} for all $d \times d$ matrices $g = (a_{ij}) \in GL_d$.

ii. Let

$$D_d = \{g \in GL_d | g = g(b_1, \ldots, b_d) = b_1 e_{11} + b_2 e_{22} + \cdots + b_d e_{dd}\}$$

be the subgroup of diagonal matrices of GL_d. For every d-tuple $\alpha = (\alpha_1, \ldots, \alpha_d)$ of integers, we define the homogeneous component of weight α (or of degree α) of the GL_d-module V by

$$V^\alpha := \{v \in V \mid g(b_1, \ldots, b_d)v = b_1^{\alpha_1} \cdots b_d^{\alpha_d} v \ \text{ for all } \ g \in D_d\}$$

It is known that polynomial representations of the general linear groups behave much like representations of finite groups and have many common features with the representations of symmetric groups.

Theorem 1.4.7. *Let $\varphi\colon GL_d \to GL_s = GL(V)$ be a polynomial representation of GL_d. Then*

i. The GL_d-module V is completely reducible and is a direct sum of homogeneous polynomial modules.

ii. As a vector space V is a direct sum of its homogeneous components W^α, $\alpha \in \mathbb{Z}^d$.

iii. The Hilbert series of V

$$H(V, \mathrm{T_d}) = \sum_\alpha \dim V^\alpha t_1^{\alpha_1} \cdots t_d^{\alpha_d}$$

is a symmetric function of $t_1, \ldots, t_d$. If V is homogeneous of degree m, then $H(V)$ is also homogeneous of degree m.

iv. Two polynomial GL_d-modules V_1 and V_2 are isomorphic if and only if $H(V_1, \mathrm{T}_d) = H(V_2, \mathrm{T}_d)$.

Notice that by the previous result, the Hilbert series of the polynomial GL_d-module V plays the role of the character of V.

The description of the irreducible polynomial representations of GL_d is given by the following theorem.

Theorem 1.4.8. *i. The irreducible polynomial representations of GL_d are in a one-to-one correspondence with the partitions $\lambda = (\lambda_1, \ldots, \lambda_d)$ in not more than d parts. We denote by $V_d(\lambda)$ the irreducible GL_d-module corresponding to λ.*

ii. The dimension of $V_d(\lambda)$ is given by the formula

$$\dim V_d(\lambda) = \prod_{1 \leqslant i < \leqslant j \leqslant m} \frac{\lambda_i - \lambda_j + i - j}{j - i}.$$

iii. Let

$$V(\lambda) = \begin{vmatrix} t_1^{\lambda_1} & t_2^{\lambda_1} & \ldots & t_{d-1}^{\lambda_1} & t_d^{\lambda_1} \\ t_1^{\lambda_2} & t_2^{\lambda_2} & \ldots & t_{d-1}^{\lambda_2} & t_d^{\lambda_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_1^{\lambda_{d-1}} & t_2^{\lambda_{d-1}} & \ldots & t_{d-1}^{\lambda_{d-1}} & t_d^{\lambda_{d-1}} \\ t_1^{\lambda_d} & t_2^{\lambda_d} & \ldots & t_{d-1}^{\lambda_d} & t_d^{\lambda_d} \end{vmatrix}.$$

Then the symmetric polynomial

$$S_\lambda(\mathrm{T}_d) = H(V_d(\lambda), \mathrm{T}_d) = \sum_\alpha \dim V_d^\alpha(\lambda) t_1^{\alpha_1} \cdots t_d^{\alpha_d},$$

called the Schur function of $W_d(\lambda)$, is expressed as

$$S_\lambda(\mathrm{T}_d) = \frac{V(\lambda_1 + d - 1, \lambda_2 + d - 2, \ldots, \lambda_{d-1} + 1, \lambda_d)}{V(d-1, d-2, \ldots, 1, 0)}.$$

iv. *Up to a multiplicative constant, there exists a unique non-zero element of weight λ in $V_d(\lambda)$. It is called the highest weight vector of $V_d(\lambda)$ and is characterized by the property that it is invariant under the action of the subgroup $T_d(F)$ of GL_d of all upper triangular matrices with 1 on the diagonal.*

v. *Let V and V' be two submodules isomorphic to $V_d(\lambda)$ with highest weight vectors respectively v and v'. If $\varphi\colon V \to V'$ is a GL_d-module isomorphism, then $\varphi(v) = \alpha v'$ for some non-zero $\alpha \in F$. Conversely, for every $0 \neq \alpha \in F$, there exists a unique GL_d-module isomorphism $\varphi\colon V \to V'$ such that $\varphi(v) = \alpha v'$.*

Recall that the Schur functions multiply with the Littlewood-Richardson rule. So, translated in the language of the Schur functions, the Young rule can be stated as follows:

Case 1 Let $\mu = (\mu_1, \ldots, \mu_d)$ and (m) be partitions, then

$$S_{(m)}(\mathrm{T}_d) S_\mu(\mathrm{T}_d) = \sum_\lambda S_\lambda(\mathrm{T}_d)$$

where the summation is over all partitions λ such that

$$\lambda_1 + \cdots + \lambda_d = \mu_1 + \cdots + \mu_d + m,$$

$$\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \cdots \geqslant \lambda_d \geqslant \mu_d.$$

This means that the Young diagrams D_λ are obtained from the diagram D_μ by adding m boxes in such a way that no two new boxes are in the same column of D_λ.

Case 2 Let $\mu = (\mu_1, \ldots, \mu_d)$ and (1^m) be partitions with $m \leqslant d$, then

$$S_{(1^m)}(\mathrm{T}_d) S_\mu(\mathrm{T}_d) = \sum_\lambda S_\lambda(\mathrm{T}_d)$$

where the summation is over all partitions λ such that

$$\lambda_1 + \cdots + \lambda_d = \mu_1 + \cdots + \mu_d + m,$$

$$\mu_i = \lambda_i + \epsilon_i, \epsilon_i = 0, 1$$

In other words the Young diagrams D_λ are obtained from the diagram D_μ by adding m boxes in such a way that new boxes are not allowed to be in the same row.

Let V_d be a d-dimensional vector space with basis $\{x_1, \ldots, x_d\}$ and with the canonical action of GL_d, i.e., $GL_d = GL(V_d)$. The general linear group GL_d acts diagonally on the free associative algebra of rank d, $F_d = F\langle x_1, \ldots, x_d\rangle$, i.e., for every $g \in GL_d$

$$g(x_{i_1} \cdots x_{i_m}) = g(x_{i_1}) \cdots g(x_{i_m})$$

It is easy to see that the multi-homogeneous component of weight $(m_1, \dots, m_d)$ of the GL_d-module $F\langle x_1, \dots, x_d\rangle$, coincides with the vector subspace spanned by the elements $x_{i_1} \cdots x_{i_m}$ of degree m_j with respect to x_j.

Theorem 1.4.9. *i. Let U be a T-ideal of $F\langle X\rangle$. Then $U \cap F_d$ is GL_d-submodule of F_n and it is a direct sum of its homogeneous component $U \cap F_d^{(m)}$.*

ii. Let $\lambda = (\lambda_1, \dots, \lambda_d)$ be a partition of m. The irreducible polynomial GL_d-module $V_d(\lambda)$ is isomorphic to a submodule of $F_d^{(m)}$ and

$$F_d^{(m)} = \bigoplus_{\lambda \vdash m} d_\lambda V_d(\lambda),$$

where d_λ is the dimension of the corresponding S_m-module, $M(\lambda)$ and the the summation is on all partitions $\lambda = (\lambda_1, \dots, \lambda_d) \vdash m$.

The representations of the symmetric group and the polynomial representations of the general linear group are equivalent, and they have been used simultaneously in many branches of mathematics. This happened incidentally also in the theory of PI-algebras, the following theorem is a clear example (see [5] and [21]).

Theorem 1.4.10. *Let A be a PI-algebra and let*

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda(A)\chi_\lambda,\ n = 0,\ 1,\ \dots$$

be the cocharacter sequence of the T-ideal of A. Then, for any d, the relatively free algebra $F_d(A)$ is isomorphic, as a GL_d-module, to the direct sum

$$\sum_{n \geqslant 0} \sum_{\lambda \vdash n} m_\lambda(A) V_d(\lambda)$$

with the same multiplicities $m_\lambda(A)$ as in the cocharacter sequence (assuming that $V_d(\lambda) = 0$ if λ is a partition in more than d parts). Then the Hilbert series of $F_d(A)$ is

$$H(A, \mathrm{T}_d) = \sum_{n \geqslant 0} \sum_{\lambda \vdash n} m_\lambda(A) S_\lambda(\mathrm{T}_d).$$

1.5 Finite basis property for sets

Here we collect results and definitions concerning orders and the finite basis property for sets. For more details see [39].

Let P be a non-empty set. A relation $p_1 \leqslant p_2$ on P is a *quasi-order* if it is reflexive and transitive. It means that (i) $p_1 \leqslant p_1$ for every $p_1 \in P$, and (ii) $p_1 \leqslant p_2$ and $p_2 \leqslant p_3$ imply $p_1 \leqslant p_3$. If also $p_1 \leqslant p_2$ and $p_2 \leqslant p_1$ imply $p_1 = p_2$, the relation is an *order*; and if in addition for every p_1, p_2 either $p_1 \leqslant p_2$ or $p_2 \leqslant p_1$, it is a linear order.

If Q is a subset of the quasi-ordered set P, the closure of Q, written $\overline{Q}$, is the set of all elements $p \in P$ such that for some q in Q, $q \leqslant p$. A *closed* subset of P is one that coincides with its own closure. The quasi-ordered set P has the *finite basis property* (f.b.p.), or it is *well-quasi-ordered set*, if every closed subset of P is the closure of a finite set of elements. The following theorem was proved in [39] and gives useful equivalences to the f.b.p.

Theorem 1.5.1 (Higman [39]). *The following conditions on a quasi-ordered set P are equivalent.*

i. Every closed subset of P is the closure of a finite subset;

ii. If Q is any subset of P, there is a finite Q_0 such that $Q_0 \subseteq Q \subseteq \overline{Q}_0$;

iii. Every infinite sequence of elements $\{p_i\}_{i\geqslant 1}$ of P has an infinite ascending subsequence

$$p_{i_1} \leqslant p_{i_2} \leqslant \cdots \leqslant p_{i_k} \leqslant \cdots ;$$

iv. There exists neither an infinite strictly descending sequence in P nor an infinite one consisting of mutually incomparable elements of P.

Consider P a quasi-ordered set and let $D(P)$ be the set of finite sequences of elements of P. Then $D(P)$ is quasi-ordered by the rule: $x \leqslant y$ if x is majorized by a subsequence of y. In other words $x = (p_1, \ldots, p_n) \leqslant y = (q_1, \ldots, q_s)$ if there exists an order preserving injection $\varphi\colon \mathbb{N} \to \mathbb{N}$ such that $\varphi(n) \leqslant s$ and $p_i \leqslant q_{\varphi(i)}$ for every $i = 1, \ldots, n$.

As an example, take the set of positive integers $\mathbb{N}$, with respect to the usual order it is well ordered. It follows that $D(\mathbb{N})$ satisfies the f.b.p.

Theorem 1.5.2 (Higman, [39]). *Let P be a quasi-ordered set. If P has f.b.p., so has $D(P)$.*

The next result will be very useful in our paper.

Proposition 1.5.1 (Higman [39]). *Let $(P_1, \leqslant_1)$, $(P_2, \leqslant 2)$, $\ldots$, $(P_k, \leqslant_k)$ be quasi-ordered sets satisfying the f.b.p.*

i. The disjoint union of P_1, P_2, $\ldots$, P_k, endowed with the quasi-order where $p \leqslant q$ if and only if p, $q \in P_i$ and $p \leqslant_i q$ for some $i \in \{1, 2, \ldots, k\}$, satisfies the f.b.p.

ii. The Cartesian product $P_1 \times P_2 \times \cdots \times P_k$ endowed with the quasi-order given by $(p_1, p_2, \ldots, p_k) \leqslant (q_1, q_2, \ldots, q_k)$ whenever $p_i \leqslant_i q_i$ for every $i \in \{1, \ldots, k\}$, satisfies the f.b.p.

Example 1.5.1. Let k be a positive integer and consider $\mathbb{N}$ endowed with its natural order $\leqslant$. Then by Proposition 1.5.1, we have that $\mathbb{N}^k$ is partially well ordered with the following order

$$(n_1, n_2, \ldots, n_k) \leqslant'_k (m_1, m_2, \ldots, m_k) \text{ if } n_i \leqslant m_i \text{ for every } i \in \{1, \ldots, k\}.$$

Theorem 1.5.2 then implies that $(D(\mathbb{N}^k), \leqslant_k)$ has f.b.p. where the order $\leqslant_k$ is defined the following way: $(p_1, \ldots, p_n) \leqslant_k (q_1, \ldots, q_s)$ if there exists an order preserving injection $\varphi \colon \mathbb{N} \to \mathbb{N}$ such that $\varphi(n) \leqslant s$ and $p_i \leqslant'_k q_{\varphi(i)}$ for any $i = 1, \ldots, n$

2 Multiplicity series of $UT_n(E)$

In this chapter we construct an algorithm to compute the multiplicities of the cocharacter sequence of $UT_n(E)$ associated to partitions that have no more than d parts, The principal ideas come from the papers [13] and [27]. The most important new results of this chapter are Corollary 2.2.1, Theorem 2.2.4 and Theorem 2.3.1, and as stated in the Introduction, they are in preparation for submission.

2.1 Multiplicity series of a PI-algebra

We fix a positive integer d and consider the algebra

$$\mathbb{C}[[\mathrm{T}_d]] = \mathbb{C}[[t_1, \ldots, t_d]]$$

of formal power series in d commuting variables. Let $\mathbb{C}[[\mathrm{T}_d]]^{S_d} \subset \mathbb{C}[[\mathrm{T}_d]]$ be the subalgebra of symmetric functions. Every symmetric function $g(\mathrm{T}_d)$ can be represented in the form

$$g(\mathrm{T}_d) = \sum_{\lambda} m_\lambda S_\lambda(\mathrm{T}_d), m_\lambda \in \mathbb{C},\ \lambda = (\lambda_1, \ldots, \lambda_d),$$

where $S_\lambda(\mathrm{T}_d)$ is the Schur function related to the partition λ which has at most d parts, because the Schur functions in d variables form a basis for $\mathbb{C}[[\mathrm{T}_d]]^{S_d}$. For details on the theory of Schur functions see the monograph [56].

Let $g(\mathrm{T}_d) = \sum_{\lambda} m_\lambda S_\lambda(\mathrm{T}_d)$ be a symmetric function, then we define its *multiplicity series* as

$$M(g; \mathrm{T}_d) = \sum_{\lambda} m_\lambda {\mathrm{T}_d}^{\lambda} = \sum_{\lambda} m_\lambda {t_1}^{\lambda_1} \cdots {t_d}^{\lambda_d} \in \mathbb{C}[[\mathrm{T}_d]].$$

It is also convenient to consider the subalgebra $\mathbb{C}[[\mathrm{V}_d]] \subset \mathbb{C}[[\mathrm{T}_d]]$ of the formal power series in the new set of variables $\mathrm{V}_d = \{v_1, \ldots, v_d\}$, where

$$v_1 = t_1, \quad v_2 = t_1 t_2, \ldots, \quad v_d = t_1 \cdots t_d.$$

Then the multiplicity series $M(g; \mathrm{T}_d)$ can be written as

$$M'(g; \mathrm{V}_d) = \sum_{\lambda} m_\lambda {v_1}^{\lambda_1 - \lambda_2} \cdots {v_{d-1}}^{\lambda_{d-1} - \lambda_d} {v_d}^{\lambda_d} \in \mathbb{C}[[\mathrm{V}_d]].$$

We also call $M'(g; \mathrm{V}_d)$ the *multiplicity series of* g. The advantage of the mapping $M' \colon \mathbb{C}[[\mathrm{T}_d]]^{S_d} \longrightarrow \mathbb{C}[[\mathrm{V}_d]]$ defined by $M' \colon g(\mathrm{T}_d) \to M'(g; \mathrm{V}_d)$ is that it is a bijection.

Lemma 2.1.1 (Berele, [7]). *The functions $f(\mathrm{T}_d) \in \mathbb{C}[[\mathrm{T}_d]]^{S_d}$ and $M(f;\mathrm{T}_d)$ are related by the following equality. If*

$$f(\mathrm{T}_d)\prod_{i<j}(t_i - t_j) = \sum_{p_i \geqslant 0} b(p_1,\ldots,p_d)t_1^{p_1}\cdots t_d^{p_d}, \quad b(p_1,\ldots,p_d) \in \mathbb{C},$$

then

$$M(f;\mathrm{T}_d) = \frac{1}{t_1^{d-1}t_2^{d-2}\cdots t_{d-1}} \sum_{p_i > p_{i+1}} t_1^{p_1}\cdots t_d^{p_d},$$

where the summation is on all $p = (p_1,\ldots,p_d)$ such that $p_1 > p_2 \cdots > p_d$.

Remark 2.1.1. In general, it is difficult to find an explicit form of $M(f;\mathrm{T}_d)$ if we know $f(\mathrm{T}_d)$. But it is very easy to check whether the formal power series

$$h(\mathrm{T}_d) = \sum h(q_1,\ldots,q_d)t_1^{q_1}\cdots t_d^{q_d}, \quad q_1 \geqslant \cdots \geqslant q_d,$$

is equal to the multiplicity series $M(f;\mathrm{T}_d)$ of $f(\mathrm{T}_d)$ because $h(\mathrm{T}_d) = M(f;\mathrm{T}_d)$ if and only if

$$f(\mathrm{T}_d)\prod_{i<j}(t_i - t_j) = \sum_{\sigma\in S_d} t_{\sigma(1)}^{d-1}t_{\sigma(2)}^{d-2}\cdots t_{\sigma(d-1)}h(t_{\sigma(1)},\ldots,t_{\sigma(d)}).$$

This equation can be used to verify the computational results on multiplicities.

Definition 2.1.1. *Let A be a PI-algebra and consider its Hilbert Series*

$$H(A;\mathrm{T}_d) = \sum_{\lambda} m_\lambda(A)S_\lambda(\mathrm{T}_d),$$

where λ has at most d parts. We define the multiplicity series of A in d variables as

$$M(A;\mathrm{T}_d) = \sum_{\lambda} m_\lambda(A)\mathrm{T}_d^{\lambda} = \sum_{\lambda} m_\lambda(A){t_1}^{\lambda_1}\cdots {t_d}^{\lambda_d}.$$

Notice that if we know the multiplicity series of A in d variables, it is possible to find the multiplicities $m_\lambda(A)$. So if we have the Hilbert series $H(A;\mathrm{T}_d)$, the problem is to write the series as a linear combination of Schur functions, which in turn is equivalent to computing the multiplicity series of A.

The following two linear transformations play an important role in the development of an algorithm to compute the multiplicities in the cocharacter sequence of the algebra $UT_n(E)$.

Definition 2.1.2. *Let Y be the linear operator in $\mathbb{C}[[\mathrm{V}_d]]$ which sends the multiplicity series of a symmetric function to the multiplicity series of its Young-derived series. That is, let $g(T_d)$ be a symmetric function, then*

$$Y(M(g),\mathrm{T}_d) = M\left(\left(\prod_{i=1}^{d}\frac{1}{(1-t_i)}\right)g(\mathrm{T}_d);\mathrm{T}_d\right).$$

Definition 2.1.3. *Let $g(\mathrm{T}_d) \in \mathbb{C}[[\mathrm{T}_d]]^{S_d}$, then we define the linear operator $\widehat{Y}$ in $\mathbb{C}[[\mathrm{V}_d]] \subset \mathbb{C}[[\mathrm{T}_d]]$ as*

$$\widehat{Y}(M(g);\mathrm{T}_d) := M\left(g(\mathrm{T}_d)\left[\frac{1}{2}\prod_{i=1}^{d}(1-t_i)+\frac{1}{2}\prod_{i=1}^{d}(1+t_i)\right];\mathrm{T}_d\right).$$

The following proposition describes the multiple action of Y on 1.

Proposition 2.1.1. *For $d \geqslant k \geqslant 1$ the following decomposition holds*

$$\prod_{i=1}^{d}\frac{1}{(1-t_i)^k} = \sum_{\mu} S_\mu(\mathrm{T}_d),$$

where the summation is over all partitions $\mu = (\mu_1, \dots, \mu_k)$ and

$$\eta_\mu = S_\mu\underbrace{(1,\dots,1)}_{k\ \ times} = \dim V_k(\mu).$$

Equivalently,

$$Y^k(1) = \sum_{\mu} \dim V_k(\mu)\mathrm{T}_k^\mu, \quad \mu = (\mu_1,\dots,\mu_k), \quad k \geqslant 1.$$

Proof. Recall that

$$\prod_{i=1}^{d}\frac{1}{1-t_i} = \sum_{m\geqslant 0} S_m(\mathrm{T}_d),$$

then

$$\prod_{i=1}^{d}\frac{1}{(1-t_i)^k} = \sum S_{m_1}(\mathrm{T}_d)\cdots S_{m_k}(\mathrm{T}_d),$$

where the summation is over all k-tuples of non-negative integers $(m_1, \dots, m_k)$. By the Young rule

$$S_{(m_1)}(\mathrm{T}_d)S_{(m_2)}(\mathrm{T}_d) = \sum S_\pi(\mathrm{T}_d),$$

where the sum is over all partitions $\pi = (\pi_1, \pi_2) \vdash m_1 + m_2$ such that $\pi \geqslant m_1$ and the skew-diagram $D_{\pi/(m_i)}$ is a horizontal strip. We fill in the entries of $D_{(m_1)}$ and $D_{(m_2)}$ with $1'$s and $2'$s respectively. Then we fill in with $1'$s and $2'$s the boxes of D_π corresponding to the boxes of $D_{(m_1)}$ and $D_{(m_2)}$, respectively:

$$\begin{array}{|c|c|c|}\hline 1&\cdots&1\\\hline\end{array}\otimes\begin{array}{|c|c|c|}\hline 2&\cdots&2\\\hline\end{array}=\sum\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&\cdots&1&1&\dots&1&2&\cdots&2\\\hline 2&\cdots&2\\\cline{1-3}\end{array}$$

As a result, we obtain a bijection between the summands $S_\pi(\mathrm{T}_d)$ in the decomposition of the product $S_{(m_1)}(\mathrm{T}_d)S_{(m_2)}(\mathrm{T}_d)$ and the semistandard tableaux of contents (m_1, m_2). In the next step, the product of three Schur functions has the form

$$S_{(m_1)}(\mathrm{T}_d)S_{(m_2)}(\mathrm{T}_d)S_{(m_3)}(\mathrm{T}_d)) = \sum S_\rho(\mathrm{T}_d)$$

where the sum is over all partitions $\rho = (\rho_1, \rho_2, \rho_3) \vdash m_1 + m_2 + m_3$ which contain a partition $\pi = (\pi_1, \pi_2) \vdash m_1 + m_2$ such that the skew diagrams $D_{\pi/(m_1)}$ and $D_{\rho/pi}$ are horizontal strips. The Schur functions $S_\rho(\mathrm{T}_d)$ participate as many times as possible to choose the partition π. Hence $S_\rho(\mathrm{T}_d)$ appears in the sum with its multiplicity in the decomposition of $S_{(m_1)}(\mathrm{T}_d)S_{(m_2)}(\mathrm{T}_d)S_{(m_3)}(\mathrm{T}_d)) = \sum S_\rho(\mathrm{T}_d)$. Again, filling in the entries of $D_{(m_1)}$, $D_{(m_2)}$ and $D_{(m_3)}$ with $1's$, $2's$ and $3'$s, respectively, we obtain a bijection between the summands $S_\rho(\mathrm{T}_d)$ of $S_{(m_2)}(\mathrm{T}_d)S_{(m_3)}(\mathrm{T}_d))$ and the semistandard tableaux of contents (m_1, m_2, m_3).

$$\begin{array}{|c|c|c|}\hline 1 & \cdots & 1 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|}\hline 2 & \cdots & 2 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|}\hline 3 & \cdots & 3 \\ \hline \end{array} = \sum \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & \cdots & 1 & 1 & \ldots & 1 & 2 & \cdots & 2 & 3 & \cdots & 3 \\ \hline 2 & \cdots & 2 & 2 & \cdots & 2 & 3 & \cdots & 3 \\ \cline{1-9} 3 & \cdots & 3 \\ \cline{1-3} \end{array}$$

This bijection preserves the shape of the partitions and $S_\rho(\mathrm{T}_d)$ is mapped to a ρ-tableau. Carrying on this way we obtain a bijection between the summands $S_\mu(\mathrm{T}_d)$ in the decomposition of the product $S_{(m_1)}(\mathrm{T}_d) \cdots S_{(m_1)}(\mathrm{T}_d)$. This bijection counts the multiplicity of $S_\mu(\mathrm{T}_d)$ and the semistandard tableaux of content $(m_1, \ldots, m_k)$. Hence the multiplicity of $S_\mu(\mathrm{T}_d)$ in the decomposition of $\prod_{i=1}^{d} \frac{1}{(1-t_i)^k}$ is equal to the number of semistandard μ-tableaux, which in turn is equal to $S_\mu(1, \ldots, 1)$, and to the dimension of the GL_k-module $V_k(\mu)$. The equivalence of both statements follows from the definitions of the multiplicity series and the operator Y. □

In the general case there is an easy formula which translates the Young-derived operator to the language of multiplicity series.

Proposition 2.1.2 (Drensky and Genov [25])**.** *Let $g(\mathrm{T}_d) \in \mathbb{C}[[\mathrm{T}_d]]^{S_d}$. Then*

$$Y(M(g; \mathrm{T}_d)) = \prod_{i=1}^{d} \frac{1}{1-t_i} \sum (-t_2)^{\varepsilon_2} \cdots (-t_d)^{\varepsilon_d} M(g; t_1 t_2^{\varepsilon_2}, t_2^{1-\varepsilon_2} t_3^{\varepsilon_3}, \ldots, t_{d-1}^{1-\varepsilon_{d-1}} t_d^{\varepsilon_d}, t_d^{1-\varepsilon_d}),$$

where the summation runs over all $\varepsilon_2, \ldots, \varepsilon_d \in \{0, 1\}$.

Consider the operator $\widehat{Y}$ and observe that

$$\widehat{Y}^j(M(1); \mathrm{T}_d) = \left(\frac{1}{2} \prod_{i=1}^{d} (1-t_i) + \frac{1}{2} \prod_{i=1}^{d} (1+t_i) \right)^j .$$

Now we want to know which Schur functions participate in the decomposition of $\widehat{Y}^j(M(1); \mathrm{T}_d)$. This means we have to express $\widehat{Y}^j(M(1); \mathrm{T}_d)$ as a linear combination of Schur functions. First we are going to consider some particular cases of $\widehat{Y}^j(M(1); \mathrm{T}_d)$ with $j \in \{1, 2, 3\}$.

Example 2.1.1. Consider $j = 1$, since

$$\frac{1}{2} \prod_{i=1}^{d} (1-t_i) + \frac{1}{2} \prod_{i=1}^{d} (1+t_i) = \sum_{n \geqslant 0} e_{2n}(t_1, \ldots, t_d) = \sum_{n \geqslant 0} S_{(1^{2n})}(\mathrm{T}_d),$$

see [23], the decomposition of $\widehat{Y}(M(1);\mathrm{T}_d)$ as a sum of Schur functions is

$$\widehat{Y}(M(1);\mathrm{T}_d) = \sum_{n\geqslant 0} S_{(1^{2n})}(\mathrm{T}_d).$$

If $j=2$, we have that

$$\widehat{Y}^2(M(1);\mathrm{T}_d) = \left(\frac{1}{2}\prod_{i=1}^{d}(1-t_i) + \frac{1}{2}\prod_{i=1}^{d}(1+t_i)\right)^2 = \sum_{n_1,n_2\geqslant 0} S_{(1^{2n_1})}(\mathrm{T}_d)S_{(1^{2n_2})}(\mathrm{T}_d).$$

Note that by the Young rule, the Schur functions that participate in the decomposition of $S_{(1^{2n_1})}(\mathrm{T}_d)S_{(1^{2n_2})}(\mathrm{T}_d)$ are given by partitions π, whose diagrams D_π are obtained from the diagram $D_{(1^{2n_1})}$ by adding $2n_2$ boxes in such a way that no two of the new boxes are in the same row. Hence

$$S_{(1^{2n_1})}(\mathrm{T}_d)S_{(1^{2n_2})}(\mathrm{T}_d) = \sum_{\pi} S_\pi(\mathrm{T}_d)$$

where $\pi = (2^m, 1^l) \vdash 2(n_1+n_2)$, m, $l \geqslant 0$ and l is even. So if

$$\widehat{Y}^2(M(1);\mathrm{T}_d) = \sum_{n_i\geqslant 0} S_{(1^{2n_1})}(\mathrm{T}_d)S_{(1^{2n_2})}(\mathrm{T}_d) = \sum_{\rho} m_\rho S_\rho(\mathrm{T}_d)$$

and $m_\rho \neq 0$ then $\rho = (2^m, 1^l)$ with m, $l \geqslant 0$ and l is even.

If $j=3$, then we have

$$\widehat{Y}^3(M(1);\mathrm{T}_d) = \sum_{n_i\geqslant 0} S_{(1^{2n_1})}(\mathrm{T}_d)S_{(1^{2n_2})}(\mathrm{T}_d)S_{(1^{2n_3})}(\mathrm{T}_d) = \widehat{Y}^2(M(1);\mathrm{T}_d)\widehat{Y}(M(1);\mathrm{T}_d).$$

By the Young rule we have that the Schur functions participating in the decomposition of $S_{(2^m,1^l)}(\mathrm{T}_d)S_{(1^{2n_3})}(\mathrm{T}_d)$ are indexed by partitions ρ, whose diagrams D_ρ are obtained from the diagram $D_{(2^m,1^l)}$ by adding $2n_3$ boxes in such a way that no two of the new boxes can be in the same row. It follows

$$S_{(1^{2n_1})}(\mathrm{T}_d)S_{(1^{2n_2})}(\mathrm{T}_d)S_{(1^{2n_3})}(\mathrm{T}_d) = \sum_{\rho} S_\rho(\mathrm{T}_d)$$

where $\rho = (3^s, 2^m, 1^l) \vdash 2(n_1+n_2+n_3)$, s, m, $l \geqslant 0$ and $s \equiv l \pmod 2$. So

$$\widehat{Y}^3(M(1);\mathrm{T}_d) = \sum_{n_i\geqslant 0} S_{(1^{2n_1})}(\mathrm{T}_d)S_{(1^{2n_2})}(\mathrm{T}_d)S_{(1^{2n_3})}(\mathrm{T}_d) = \sum_{\rho} m_\rho S_\rho(\mathrm{T}_d),$$

and $m_\rho \neq 0$ then $\rho = (3^s, 2^m, 1^l)$ with s, m, $l \in \mathbb{Z}_{\geqslant 0}$.

Below, we have the general result.

Proposition 2.1.3. *Let $j \geqslant 1$ and let*

$$\left(\frac{1}{2}\prod_{i=1}^{d}(1-t_i) + \frac{1}{2}\prod_{i=1}^{d}(1+t_i)\right)^j = \sum_{\rho} m_\rho S_\rho(\mathrm{T}_d).$$

If $m_\rho \neq 0$, then $\rho = (j^{s_1}, (j-1)^{s_2}, \ldots, 1^{s_j}) \vdash 2(n_1+\cdots+n_j)$ and $(s_1,\ldots,s_j) \in \mathbb{Z}^j_{\geqslant 0}$. Equivalently, if $m_\rho \neq 0$ then ρ has at most j columns.

Proof. The proof will be done by induction on j.

The case $j = 1$ was dealt with in Example 2.1.1.

Assuming the result valid for j, let us prove it for $j + 1$. We have

$$\left(\frac{1}{2}\prod_{i=1}^{d}(1-t_i) + \frac{1}{2}\prod_{i=1}^{d}(1+t_i)\right)^{j+1} = \left(\frac{1}{2}\prod_{i=1}^{d}(1-t_i) + \frac{1}{2}\prod_{i=1}^{d}(1+t_i)\right)^{j}\left(\frac{1}{2}\prod_{i=1}^{d}(1-t_i) + \frac{1}{2}\prod_{i=1}^{d}(1+t_i)\right).$$

Suppose that

$$\left(\frac{1}{2}\prod_{i=1}^{d}(1-t_i) + \frac{1}{2}\prod_{i=1}^{d}(1+t_i)\right)^{j} = \sum_{\rho'} m_{\rho'} S_{\rho'}(\mathrm{T}_d).$$

By induction hypothesis, if $m_{\rho'} \neq 0$, then

$$\rho' = (j^{n_1}, \ldots, 1^{n_j})$$

with $(n_1, \ldots n_j) \in \mathbb{Z}^{j}_{\geqslant 0}$. Since

$$\frac{1}{2}\prod_{i=1}^{d}(1-t_i) + \frac{1}{2}\prod_{i=1}^{d}(1+t_i) = \sum_{n\geqslant 0} S_{(1^{2n})}(\mathrm{T}_d).$$

By the Young rule, we have

$$S_{\rho'}(\mathrm{T}_d)S_{(1^{2n})}(\mathrm{T}_d) = \sum_{\rho} S_{\rho}(\mathrm{T}_d)$$

where ρ has at most $j+1$ columns. This means $\rho = ((j+1)^{n_1}, \ldots, 1^{n_{j+1}})$ where $(n_1, \ldots, n_{j+1}) \in \mathbb{Z}^{j+1}_{\geqslant 0}$.

Hence, if

$$\left(\frac{1}{2}\prod_{i=1}^{d}(1-t_i) + \frac{1}{2}\prod_{i=1}^{d}(1+t_i)\right)^{j+1} = \sum_{\rho} m_{\rho} S_{\rho}(\mathrm{T}_d),$$

and $m_\rho \neq 0$, then $\rho = ((j+1)^{n_1}, \cdots, 1^{n_{j+1}})$ with $(n_1, \ldots, n_{j+1}) \in \mathbb{Z}^{j+1}_{\geqslant 0}$, as desired. □

Lemma 2.1.2. *Let $f(\mathrm{T}_d) \in \mathbb{C}[[\mathrm{T}_d]]^{S_d}$. Then*

$$\begin{aligned} M'(f(\mathrm{T}_d)S_{(1^2)}(\mathrm{T}_d); \mathrm{V}_d) =& v_2 M'(f(\mathrm{T}_d; \mathrm{V}_d) + v_1 \sum_{j=2}^{d-1} \frac{v_{j+1}}{v_j} g_j((M'(f(\mathrm{T}_d), v_1, \ldots, v_d)) \\ &+ \sum_{i=1}^{d-2} \frac{v_{i+2}}{v_i} g_i((M'(f(\mathrm{T}_d), v_1, \ldots, v_d)) \\ &+ \sum_{\substack{1\leqslant i,j\leqslant d-1 \\ i+1<j}} \frac{v_{i+1}}{v_i}\frac{v_{j+1}}{v_j} g_{ij}((M'(f(\mathrm{T}_d), v_1, \ldots, v_d)), \end{aligned}$$

where

$$g_j(M'(f(\mathrm{T}_d), v_1, \ldots, v_d)) = M'(f(\mathrm{T}_d), v_1, \ldots, v_d) - M'(f(\mathrm{T}_d); v_1, \ldots, v_{j-1}, 0, v_{j+1}, \ldots, v_d)$$

$$\begin{aligned} g_{ij}(M'(f(\mathrm{T}_d), v_1, \dots, v_d)) = & M'(f(\mathrm{T}_d), v_1, \dots, v_d) + \\ & M'(f(\mathrm{T}_d), v_1, \dots, v_{i-1}, 0, v_{i+1}, \dots, v_j, 0, v_{j+1}, \dots, v_d) - \\ & M'(f(\mathrm{T}_d); v_1, \dots, v_{i-1}, 0, v_{i+1}, \dots, v_d) - \\ & M'(f(\mathrm{T}_d); v_1, \dots, v_{j-1}, 0, v_{j+1}, \dots, v_d) \end{aligned}$$

Proof. Notice that it is sufficient to prove the lemma for $f(\mathrm{T}_d) = S_\mu(\mathrm{T}_d)$ where $\mu = (\mu_1, \dots, \mu_d)$ is a partition in no more than d parts. Rewriting $M'(S_\mu(\mathrm{T}_d, \mathrm{V}_d)$ as

$$M'(S_\mu(\mathrm{T}_d); \mathrm{V}_d) = v_1^{p_1} \cdots v_d^{p_d}$$

where $p_i = \mu_i - \mu_{i+1}$, $i = 1, \dots, d-1$ and $p_d = \mu_d$, by the Young Rule, we have $S_\mu(\mathrm{T}_d)S_{(1^2)}(\mathrm{T}_d)$ is a linear combination of $S_\lambda(\mathrm{T}_d)$, where

- $\lambda = (\mu_1 + 1, \mu_2 + 1, \mu_3, \dots, \mu_d)$.
- $\lambda = (\mu_1 + 1, \dots, \mu_j, \mu_{j+1} + 1, \dots, \mu_d)$, if $\mu_j > \mu_{j+1}$ and $j \geqslant 2$.
- $\lambda = (\mu_1, \dots, \mu_i, \mu_{i+1} + 1, \mu_{i+2} + 1, \dots, \mu_d)$, if $\mu_i > \mu_{i+1}$.
- $\lambda = (\mu_1 + 1, \dots, \mu_i, \mu_{i+1} + 1, \dots, \mu_j, \mu_{j+1} + 1, \dots, \mu_d)$, if $\mu_i > \mu_{i+1}$, $\mu_j > \mu_{j+1}$ and $i + 1 < j$.

In the language of multiplicity series, this means that $M'(S_\mu(\mathrm{T}_d)S_{(1,1)}(\mathrm{T}_d))$ is a linear combination of the following terms

- $v_2(v_1^{p_1} \cdots v_d^{p_d}) = v_2 M'(S_\mu(\mathrm{T}_d, \mathrm{V}_d)$,
- $v_1 \dfrac{v_{j+1}}{v_j} M'(S_\mu(\mathrm{T}_d, \mathrm{V}_d)$, if $\mu_j > \mu_{j+1}$ and $j \geqslant 2$,
- $\dfrac{v_{i+2}}{v_i} M'(S_\mu(\mathrm{T}_d, \mathrm{V}_d)$, if $\mu_i > \mu_{i+1}$,
- $\dfrac{v_{i+1}}{v_i} \dfrac{v_{j+1}}{v_j} M'(S_\mu(\mathrm{T}_d, \mathrm{V}_d)$, if $\mu_i > \mu_{i+1}$, $\mu_j > \mu_{j+1}$ and $i + 1 < j$.

Now, observe that

$$g_j((M'(f(\mathrm{T}_d); \mathrm{V}_d)) = \begin{cases} M'(f(\mathrm{T}_d); \mathrm{V}_d) & if \quad p_j > 0 \\ \\ 0 & if \quad p_j = 0 \end{cases}$$

$$g_{ij}((M'(f(\mathrm{T}_d); \mathrm{V}_d)) = \begin{cases} M'(f(\mathrm{T}_d); \mathrm{V}_d) & if \quad p_i > 0,\ p_j > 0 \\ \\ 0 & for \quad \text{all other cases} \end{cases}$$

Then the result follows easily. □

By Lemma 2.1.2, we have the following corollary

Corollary 2.1.1. *Let* $f(\mathrm{T}_2) \in \mathbb{C}[[\mathrm{T}_2]]^{S_2}$, *then*

$$\widehat{Y}(M(f;\mathrm{T}_2)) = (1+t_1t_2)M(f;T_2) = \left[\frac{1}{2}\prod_{i=1}^{2}(1-t_i) + \frac{1}{2}\prod_{i=1}^{2}(1+t_i)\right] M(f;T_2).$$

From Definitions 2.1.3 and 2.1.2 it follows that

$$\begin{aligned} Y(\widehat{Y}(M(g);T_d)) &= Y\left(M\left(g(T_d)\left[\frac{1}{2}\prod_{i=1}^{d}(1-t_i) + \frac{1}{2}\prod_{i=1}^{d}(1+t_i)\right]\right);\mathrm{T_d}\right) \\ &= M\left(\prod_{i=1}^{d}\frac{1}{1-t_i}g(T_d)\left[\frac{1}{2}\prod_{i=1}^{d}(1-t_i) + \frac{1}{2}\prod_{i=1}^{d}(1+t_i)\right];\mathrm{T_d}\right) \\ &= M\left(g(\mathrm{T}_d)\left[\frac{1}{2} + \frac{1}{2}\prod_{i=1}^{d}\frac{1+t_i}{1-t_i}\right];\mathrm{T}_d\right). \end{aligned}$$

Notice that the composition $Y(\widehat{Y}(M(g);T_d))$ is well-defined one because $g(\mathrm{T}_d)$ and also $\left(\frac{1}{2} + \frac{1}{2}\prod_{i=1}^{d}\frac{1+t_i}{1-t_i}\right)$ are symmetric functions. Hence $g(\mathrm{T}_d)\left[\frac{1}{2} + \frac{1}{2}\prod_{i=1}^{d}\frac{1+t_i}{1-t_i}\right]$ is also symmetric. Observe also that $Y \circ \widehat{Y} = \widehat{Y} \circ Y$.

We will define $Z := Y \circ \widehat{Y}$. This operator will appear in the computation of the multiplicity series of the algebra of $UT_n(E)$

2.2 Hilbert series and Multiplicity series of $UT_n(E)$

In this section, we will find the Hilbert series of $UT_n(E)$ and an algorithm to compute the multiplicities in the cocharacter sequence of this algebra using its multiplicity series.

First, we are going to state a result which gives a basis for the T-ideal of $UT_n(E)$. An important tool in the proof of this theorem is the well known Lewin's Theorem.

Theorem 2.2.1 (Abakarov [1])**.** *The T-ideal of $UT_n(E)$ is generated by the polynomial*

$$[x_1,x_2,x_3]\cdots[x_{3n-2},x_{3n-1},x_{3n}].$$

We also recall that in [12, Theorem 2.8] the authors give a more general version of the previous result.

The following results make reference to the Hilbert series and the multiplicities series of E, they are useful throughout this work.

Proposition 2.2.1. *Let E be the infinite dimensional Grassmann algebra over a field of characteristic zero. The Hilbert series of $F_d(E)$ in d variables is given by*

$$H(E;\mathrm{T}_d) = \frac{1}{2} + \frac{1}{2}\prod_{i=1}^{d}\frac{1+t_i}{1-t_i}.$$

A proof of the previous proposition can be found in [23].

Theorem 2.2.2 (Olsson and Regev [59])**.** *Let E be the infinite dimensional Grassmann algebra over a field of characteristic zero. Then the cocharacter sequence of E, for any $n \geqslant 1$ is given by*

$$\chi_n(E) = \sum_{p=1}^{n}\chi_{(p,1^{n-p})}.$$

Using Corollary 1.3.1 and Theorem 2.2.1, we have the following result

Theorem 2.2.3. *The Hilbert series $H(UT_n(E);\mathrm{T}_d)$ of the algebra $F_d(UT_n(E))$ is*

$$H(UT_n(E);\mathrm{T}_d) = \sum_{j=1}^{n}\binom{n}{j}\left(\frac{1}{2} + \frac{1}{2}\prod_{i=1}^{d}\frac{1+t_i}{1-t_i}\right)^{j}(t_1+\cdots t_d-1)^{j-1}.$$

Proof. We use an induction on n. For $n=1$, it is Proposition 2.2.1. Assuming the result true for n, let us prove it for $n+1$. By Theorem 2.2.1 we have

$$T(UT_{n+1}(E)) = T(UT_n(E))T(E).$$

Thus, applying Corollary 1.3.1 it follows that

$$H(UT_{n+1}(E);\mathrm{T}_d) = H(UT_n(E);\mathrm{T}_d)+H(E;\mathrm{T}_d)+(t_1+\cdots+t_d-1)H(UT_n(E);\mathrm{T}_d)H(E;\mathrm{T}_d).$$

By induction hypothesis, we have

$$H(UT_n(E);\mathrm{T}_d) = \sum_{j=1}^{n}\binom{n}{j}H(E;\mathrm{T}_d)^{j}(t_1+\cdots+t_d-1)^{j-1}.$$

Then

$$\begin{aligned} H(UT_{n+1}(E)); \mathrm{T}_d) &= \sum_{j=1}^{n} \binom{n}{j} H(E;\mathrm{T}_d)^j (t_1+\cdots+t_d-1)^{j-1} + H(E;\mathrm{T}_d) \\ &+(t_1+\cdots+t_d-1)\left(\sum_{j=1}^{n}\binom{k}{j} H(E;\mathrm{T}_d)^j (t_1+\cdots+t_d-1)^{j-1}\right) H(E;\mathrm{T}_d) \\ &= \sum_{j=1}^{n} \binom{n}{j} H(E;\mathrm{T}_d)^j (t_1+\cdots+t_d-1)^{j-1} + H(E;T_d) \\ &+\sum_{j=1}^{n} \binom{n}{j} H(E;\mathrm{T}_d)^{j+1} (t_1+\cdots+t_d-1)^{j} \\ &= (n+1)H(E;\mathrm{T}_d) + \sum_{j=2}^{n}\left(\binom{n}{j}+\binom{n}{j-1}\right) H(E;\mathrm{T}_d)^j (t_1+\cdots+t_d-1)^{j-1} \\ &+H(E;\mathrm{T}_d)^{n+1}(t_1+\cdots+t_d-1)^n \\ &= \binom{n+1}{1} H(E;\mathrm{T}_d) + \sum_{j=2}^{n}\binom{n+1}{j} H(E;\mathrm{T_d})^j (t_1+\cdots+t_d-1)^{j-1} \\ &+\binom{n+1}{n+1} H(E;\mathrm{T_d})^{n+1}(t_1+\cdots+t_d-1)^n \\ &= \sum_{j=1}^{n+1}\binom{n+1}{j} H(E;\mathrm{T_d})^j (t_1+\cdots+t_d-1)^{j-1} \end{aligned}$$

and we are done. □

Using the previous result and the operator Z, we get an expression for the multiplicity series of $UT_n(E)$.

Corollary 2.2.1. *The multiplicity series of $UT_n(E)$ is*

$$M(UT_n(E); \mathrm{T}_d)) = \sum_{j=1}^{n}\sum_{q=0}^{j-1}\sum_{\lambda\vdash q} (-1)^{j-1-q}\binom{n}{j}\binom{j-1}{q} d_\lambda Z^j(\mathrm{T}_d^\lambda),$$

where d_λ is the degree of the irreducible S_n-character χ_λ, $\mathrm{T}_d = t_1^{\lambda_1}\cdots t_d^{\lambda_d}$ and $Z = Y \circ \widehat{Y}$.

Proof. Note that

$$(t_1+\cdots+t_d-1)^{j-1} = \sum_{q=0}^{j-1}(-1)^{j-1-q}\binom{j-1}{q}(t_1+\cdots+t_d)^q$$

and expanding the expression of $H(U_k(E), \mathrm{T_d})$ from Proprosition 2.2.3, we get

$$H(UT_n(E); \mathrm{T}_d) = \sum_{j=1}^{n}\binom{n}{j}\left(\frac{1}{2}+\frac{1}{2}\prod_{i=1}^{d}\frac{1+t_i}{1-t_i}\right)^j \sum_{q=0}^{j-1}(-1)^{j-1-q}\binom{j-1}{q}(t_1+\cdots+t_d)^q.$$

Using the well-known equality

$$(t_1+\cdots+t_d)^q = S_{(1)}^q(\mathrm{T}_d) = \sum_{\lambda\vdash q} d_\lambda S_\lambda(\mathrm{T}_d),$$

where d_λ is the degree of the irreducible S_q-character χ_λ, we have

$$\begin{aligned} H(UT_n(E);\mathrm{T}_d) &= \sum_{j=1}^{n}\binom{n}{j}\left(\frac{1}{2}+\frac{1}{2}\prod_{i=1}^{d}\frac{1+t_i}{1-t_i}\right)^{j}\sum_{q=0}^{j-1}(-1)^{j-1-q}\binom{j-1}{q}\sum_{\lambda\vdash q}d_\lambda S_\lambda(\mathrm{T}_d)\\ &= \sum_{j=1}^{n}\sum_{q=0}^{j-1}\sum_{\lambda\vdash q}(-1)^{j-1-q}\binom{n}{j}\binom{j-1}{q}d_\lambda\left(\frac{1}{2}+\frac{1}{2}\prod_{i=1}^{d}\frac{1+t_i}{1-t_i}\right)^{j}S_\lambda(\mathrm{T}_d). \end{aligned}$$

Indeed, the multiplicity series of $S_\lambda(\mathrm{T}_d)$ is

$$M(S_\lambda(\mathrm{T}_d);\mathrm{T}_d) = {t_1}^{\lambda_1}\cdots {t_d}^{\lambda_d} = \mathrm{T}_d^{\lambda}.$$

Then

$$M\left(\left(\frac{1}{2}+\frac{1}{2}\prod_{i=1}^{d}\frac{1+t_i}{1-t_i}\right)^{j}S_\lambda(\mathrm{T}_d);\mathrm{T}_d\right) = Z^j(M(S_\lambda(\mathrm{T}_d);\mathrm{T}_d)) = Z^j(\mathrm{T}_d^{\lambda}).$$

Hence, the multiplicity series of $UT_n(E)$ equals

$$M(H(UT_n(E);\mathrm{T}_d)) = \sum_{j=1}^{n}\sum_{q=0}^{j-1}\sum_{\lambda\vdash q}(-1)^{j-1-q}\binom{n}{j}\binom{j-1}{q}d_\lambda Z^j(\mathrm{T}_d^{\lambda}). \qquad \square$$

We want to describe those partitions λ such that $m_\lambda(UT_k(E)) \neq 0$. In this way we obtain a better upper bound on the height of λ.

Theorem 2.2.4. *If $m_\lambda(UT_n(E)) \neq 0$ and $\lambda = (\lambda_1, \cdots, \lambda_d)$, then $\lambda_{n+1} \leqslant 2n-1$.*

Proof. By Theorem 2.2.3 and in the spirit of Corollary 2.2.1, the non-zero multiplicities $m_\lambda(UT_n(E))$ in the cocharacter sequence of $UT_n(E)$ come from the decomposition

$$\left(\frac{1}{2}+\frac{1}{2}\prod_{i=1}^{d}\frac{1+t_i}{1-t_i}\right)^{j}(t_1+\cdots+t_d)^q = \left(\prod_{i=1}^{d}\frac{1}{1-t_i}\right)^{j}\left(\frac{1}{2}\prod_{i=1}^{d}(1-t_i)+\frac{1}{2}\prod_{i=1}^{d}(1+t_i)\right)^{j}S_{(1)}(\mathrm{T}_d)^q,$$

$j \leqslant n$ and $q \leqslant n-1$, as a linear combination of Schur functions.
By Proposition 2.1.3, the Schur functions $S_\pi(\mathrm{T}_d)$ participating in the product

$$\left(\frac{1}{2}\prod_{i=1}^{d}(1-t_i)+\frac{1}{2}\prod_{i=1}^{d}(1+t_i)\right)^{j}$$

are indexed by partitions π having at most $j \leqslant n$ columns. By the Branching rule, the multiplication of $S_\pi(\mathrm{T}_d)$ by $S_{(1)}(T_d)$ is a linear combination of $S_\rho(\mathrm{T}_d)$ where the diagrams of ρ are obtained from the diagrams of π by adding a box. Multiplying q times by $S_{(1)}(\mathrm{T}_d)$ we add to the diagram of π no more than $q \leqslant n-1$ boxes in the first row.

The diagrams of the partitions ρ appearing in the decomposition of

$$\left(\frac{1}{2}\prod_{i=1}^{d}(1-t_i)+\frac{1}{2}\prod_{i=1}^{d}(1+t_i)\right)^{j}S_{(1)}(\mathrm{T}_d)^q,$$

have at most $j + q \leqslant 2n - 1$ boxes in the first row.

Due to the fact that

$$\left(\prod_{i=1}^{d} \frac{1}{1-t_i}\right)^j = \sum_{n_i \geqslant 0} S_{(m_1)}(\mathrm{T}_d) \cdots S_{(m_j)}(\mathrm{T}_d),$$

applying the Young rule iteratively, it follows that if the partition λ appears in the decomposition of

$$\left(\prod_{i=1}^{d} \frac{1}{1-t_i}\right)^j S_\rho(\mathrm{T}_d),$$

then λ is of the type $(m_1, \ldots, m_j, (j+q)^{s_1}, \ldots, 1^{s_{j+q}})$ with non-negative $m_1, \ldots, m_j$, $s_1, \ldots, s_{j+q}$. Hence $\lambda_{j+1} \leqslant j + q$.

Therefore, if $m_\lambda(UT_n(E)) \neq 0$, then $\lambda_{k+1} \leqslant 2n - 1$. □

It is worth mentioning that results about the cocharacter sequence of E and $UT_2(E)$ satisfy the bound given in the previous theorem. By Theorem 2.2.2, we know

$$\chi_n(E) = \sum_{k=0}^{n-1} \chi_{(n-k,1^k)}.$$

Note that the diagrams of the partitions $\lambda = (n-k, 1^k)$ have at most one box in the second column, that is, $\lambda_2 \leqslant 1$. Therefore, it agrees with Theorem 2.2.4.

In [15] Centrone proved that

$$H(UT_2(E); \mathrm{T}_d) = \sum m_\lambda(UT_2(E)) S_\lambda(\mathrm{T}_d),$$

where $\lambda = (m_1, m_2, 3, 2^m, 1^l)$ or $\lambda = (m_1, m_2, 2^m, 1^l)$. Hence the diagrams of the partitions λ have at most 3 boxes in the third row, that is, $\lambda_3 \leqslant 3$.

2.3 Application to the multiplicity series of $UT_n(E)$ in two variables

In this section, we shall compute the multiplicity series of $UT_n(E)$ in two variables for $n \in \{1, 2, 3\}$. As consequences, we obtain the multiplicities m_λ in the cocharacter sequences of $UT_n(E)$, where λ is a partition in no more than 2 parts.

Proposition 2.3.1. *Consider the algebras E and $UT_2(E)$, then*

i. The multiplicity series of E in two variables is

$$M'(E; \mathrm{V}_2) = \frac{1+v_2}{1-v_1}. \tag{2.1}$$

ii. The multiplicity series of $UT_2(E)$ in two variables is

$$M'(UT_2(E); \mathrm{V}_2) = \frac{2(1+v_2)}{1-v_1} + \frac{(1+v_2)^2(-1+v_1+2v_2-v_1v_2)}{(1-v_1)^2(1-v_2)}. \tag{2.2}$$

Proof. Let us start with the statement *i.*

i. By Corollary 2.2.1, we get

$$M(E,\mathrm{T}_2) = Z(1;\mathrm{T}_2).$$

Since $Z = Y \circ \widehat{Y}$, applying Proposition 2.1.2 and Corollary 2.1.1 we have

$$M(E,\mathrm{T}_2) = \frac{1+t_1t_2}{1-t_1}.$$

Recall that $v_1 = t_1$ and $v_2 = t_1t_2$. Hence

$$M'(E;\mathrm{V}_2) = \frac{1+v_2}{1-v_1}.$$

ii. By Corollary 2.2.1, we have

$$M(UT_2(E);\mathrm{T}_2) = 2Z(1) - Z^2(1) + Z^2(t_1).$$

Now, we will compute $2Z(1), Z^2(1)$ and $Z^2(t_1)$ using Propostion 2.1.2 and Corollary 2.1.1

- $2Z(1) = \dfrac{2(1+t_1t_2)}{1-t_1}$,
- $Z^2(1) = \dfrac{(1+t_1t_2)^2}{(1-t_1)^2(1-t_1t_2)}$,
- $Z^2(t_1) = \dfrac{(1+t_1t_2)^2(t_1+2t_1t_2-t_1^2t_2)}{(1-t_1)^2(1-t_1t_2)}$.

Hence

$$\begin{aligned} M(UT_2(E);\mathrm{T}_2) &= \frac{2(1+t_1t_2)}{1-t_1} - \frac{(1+t_1t_2)^2}{(1-t_1)^2(1-t_1t_2)} + \frac{(1+t_1t_2)^2(t_1+2t_1t_2-t_1^2t_2)}{(1-t_1)^2(1-t_1t_2)} \\ &= \frac{2(1+t_1t_2)}{1-t_1} + \frac{(1+t_1t_2)^2(-1+t_1+2t_1t_2-t_1^2t_2)}{(1-t_1)^2(1-t_1t_2)}. \end{aligned}$$

Finally, we have

$$M'(UT_2(E);\mathrm{V}_2) = \frac{2(1+v_2)}{1-v_1} + \frac{(1+v_2)^2(-1+v_1+2v_2-v_1v_2)}{(1-v_1)^2(1-v_2)},$$

and we are done. □

Now we are able to compute the multiplicity m_λ in the cocharacter sequences of E and $UT_2(E)$ when λ is a partition in no more than 2 parts. The next corollary shows how to compute the multiplicities by using Corollary 2.2.1 and Proposition 2.3.1

Corollary 2.3.1. *Let λ be a partition in no more than 2 parts.*

i. The multiplicity m_λ in the cocharacter sequence of E is given by

$$m_\lambda = \begin{cases} 1 & if\ \lambda = (n) \\ 1 & if\ \lambda = (\lambda_1, 1),\ \lambda_1 \geqslant 1 \\ 0 & for\ all\ other\ \lambda \end{cases}$$

ii. The multiplicity m_λ in the cocharacter sequences of $UT_2(E)$ is given by

$$m_\lambda = \begin{cases} 1 & if \quad \lambda = (n) \\ \lambda_1 & if \quad \lambda = (\lambda_1, 1),\ \lambda_1 \geqslant 1 \\ 3\lambda_1 - 4 & if \quad \lambda = (\lambda_1, 2), \lambda_1 \geqslant 2 \\ 4(\lambda_1 - \lambda_2 + 1) & if \quad \lambda = (\lambda_1, \lambda_2),\ \lambda_1 \geqslant \lambda_2 \geqslant 3 \end{cases}$$

Proof. i. By Proposition 2.3.1, we get

$$M'(E; \mathrm{V}_2) = \sum_{n\geqslant 0} v_1^n + \sum_{n\geqslant 0} v_1^n v_2.$$

From the first summand of the previous equality, we have that if $\lambda = (n)$ with $n \geqslant 0$ then $m_\lambda = 1$. Observe that $v_1^n v_2$ with $n \geqslant 0$ corresponds to the partition $\lambda = (n + 1, 1)$. It follows that if $\lambda = (\lambda_1, 1)$ where $\lambda_1 \geqslant 1$ then $m_\lambda = 1$.

ii. By Proposition 2.3.1, it follows that

$$\begin{aligned} M'(UT_2(E); \mathrm{V}_2) \quad = \quad & \sum_{n\geqslant 0} 2v_1^n + \sum_{n\geqslant 0} 2v_1^n v_2 - \sum_{m,n\geqslant 0} (n+1) v_1^n v_2^m - \\ & \sum_{n\geqslant 0, m\geqslant 1} 2(n+1) v_1^n v_2^m - \sum_{m\geqslant 2, n\geqslant 0} (n+1) v_1^n v_2^m + \\ & \sum_{m\geqslant 1, n\geqslant 0} 2(n+1) v_1^n v_2^m + \sum_{m\geqslant 2, n\geqslant 0} 4(n+1) v_1^n v_2^m + \\ & \sum_{m\geqslant 3, n\geqslant 0} 2(n+1) v_1^n v_2^m + \sum_{n\geqslant 1} n v_1^n + \sum_{n\geqslant 1} 2n v_1^n v_2 + \sum_{n\geqslant 1} n v_1^n v_2^2 \end{aligned}$$

Therefore

$$\begin{aligned} M'(UT_2(E); \mathrm{V}_2) = & \sum_{n\geqslant 0} v_1^n + \sum_{n\geqslant 0} (n+1) v_1^n v_2 + \sum_{n\geqslant 0} (3n+2) v_1^n v_2^2 + \\ & \sum_{n\geqslant 0, m\geqslant 3} 4(n+1) v_1^n v_2^m \end{aligned}$$

First, consider the first summand of the previous equality and observe that v_1^n corresponds to the partition $\lambda = (n)$. So, if $\lambda = (n)$, then $m_\lambda = 1$.

Now, notice that there is a bijection between $v_1^n v_2$ with $n \geqslant 0$ and the partition $\lambda = (n + 1, 1)$. It follows from the last equality that if $\lambda = (n, 1)$ where $n \geqslant 1$ then $m_\lambda = n$.

Finally, we have that $v_1^n v_2^m$ corresponds to the partition $\lambda = (n+m, m)$. Then since $m_\lambda = 4(n+1) = 4((n+m) - m + 1)$, it follows that $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 \geqslant \lambda_2 \geqslant 3$, and thus $m_\lambda = 4(\lambda_1 - \lambda_2 + 1)$.

The remaining case is treated similarly. □

We note that Corollary 2.3.1 agrees with the results presented in [59] and [15] when the partitions have no more than two parts.

Now, we will compute the multiplicity series of $UT_3(E)$ in two variables.

Theorem 2.3.1. *i. The multiplicity series of $UT_3(E)$ in two variables is*

$$\begin{aligned} M'(UT_3(E), \mathrm{V}_2) = & \frac{3(1+v_2)}{1-v_1} - \frac{3(1+v_2)^2}{(1-v_1)^2(1-v_2)} + \\ & 3\left(\frac{2(1+v_2)^2}{(1-v_1)^2(1-v_2)} + \frac{v_1(1+v_2)^2}{(1-v_1)^2}\right) + \\ & \frac{1}{(1-v_1)^3(1-v_2)^3}\big(1 - 2v_1 + v_1^2 - 2v_2 + 2v_1v_2 - 4v_2^2 + 8v_1v_2^2 \\ & - 3v_1^2v_2^2 + 7v_2^3 - 5v_1v_2^3 + 10v_2^4 - 13v_1v_2^4 + 3v_1^2v_2^4 - v_2^5 - v_1v_2^5 \\ & -3v_2^6 + 3v_1v_2^6 - v_1^2v_2^6\big) \end{aligned} \tag{2.3}$$

ii. Let λ be a partition in no more than 2 parts. The multiplicity m_λ in the cocharacter sequences of $UT_3(E)$ is given by

$$m_\lambda = \begin{cases} 1 & if \quad \lambda = (n) \\ \lambda_1 & if \quad \lambda = (\lambda_1, 1),\ \lambda_1 \geqslant 1 \\ \frac{1}{2}(\lambda_1+2)(\lambda_1-1) & if \quad \lambda = (\lambda_1, 2), \lambda_1 \geqslant 2 \\ \frac{1}{2}(16 - 17\lambda_1 + 5\lambda_1^2) & if \quad \lambda = (\lambda_1, 3), \lambda_1 \geqslant 3 \\ 14 - 16\lambda_2 + 4\lambda_2^2 + 2(\lambda_1-\lambda_2)(2 - 5(\lambda_1-\lambda_2)) & if \quad \lambda = (\lambda_1, \lambda_2),\ \lambda_1 \geqslant \lambda_2 \geqslant 4 \\ +4\lambda_2(\lambda_1-\lambda_2)(-3 + \lambda_2 + (\lambda_1-\lambda_2)) & \end{cases}$$

Proof. i. By Corollary 2.2.1, we have

$$M(UT_3(E), \mathrm{T}_2) = 3Z(1) - 3Z^2(1) + 3Z^2(t_1) + Z^3(1) - 2Z^3(t_1) + Z^3(t_1^2) + Z^3(t_1t_2).$$

Due to Corollary 2.1.1 and Proposition 2.1.2, we get

$$\begin{aligned} 3Z(1) - 3Z^2(1) + 3Z^2(t_1) = & \frac{3(1+v_2)}{1-v_1} - \frac{3(1+v_2)^2}{(1-v_1)^2(1-v_2)} + \\ & 3\left(\frac{2(1+v_2)^2}{(1-v_1)^2(1-v_2)} + \frac{v_1(1+v_2)^2}{(1-v_1)^2}\right) \end{aligned} \tag{2.4}$$

$$Z^3(1) - 2Z^3(t_1) + Z^3(t_1^2) + Z^3(t_1t_2) = \frac{1}{(1-v_1)^3(1-v_2)^3}\big(1 - 2v_1 + v_1^2 - 2v_2 + 2v_1v_2 - 4v_2^2 + 8v_1v_2^2 - 3v_1^2v_2^2 + 7v_2^3 - 5v_1v_2^3 + 10v_2^4 - 13v_1v_2^4 + 3v_1^2v_2^4 - v_2^5 - v_1v_2^5 - 3v_2^6 + 3v_1v_2^6 - v_1^2v_2^6\big) \tag{2.5}$$

From equalities (2.4) and (2.5), the result follows.

ii. We expand into a power series the expression for $M'(UT_3(E), \mathrm{V}_2)$ given in part *i.* using the following well known equalities.

$$\frac{v_1^{a_1}v_2^{a_2}}{1-v_1} = \sum_{n\geqslant a_1} v_1^n v_2^{a_2},$$

$$\frac{v_1^{a_1}v_2^{a_2}}{(1-v_1)^2} = \sum_{n\geqslant a_1} (n-a_1+1)v_1^n v_2^{a_2},$$

$$\frac{v_1^{a_1}v_2^{a_2}}{(1-v_1)^2(1-v_2)} = \sum_{n\geqslant a_1}\sum_{m\geqslant a_2} (n-a_1+1)v_1^n v_2^{m},$$

$$\frac{v_1^{a_1}v_2^{a_2}}{(1-v_1)^3(1-v_2)^3} = \sum_{n\geqslant a_1}\sum_{m\geqslant a_2} \binom{n-a_1+2}{2}\binom{m-a_2+2}{2} v_1^n v_2^{m}.$$

Easy manipulations give us the explicit expression for m_λ where λ is a partition in no more than two parts. In particular, if we want to compute the multiplicity of $\lambda = (\lambda_1, 1)$, we need to study the terms of type $v_1^n v_2$ in $M'(UT_3, \mathrm{V}_2)$. Hence we will study the following expression

$$\frac{3v^2}{1-v_1} - \left(\frac{3}{(1-v_1)^2(1-v_2)} + \frac{6v_2}{(1-v_1)^2(1-v_2)}\right) + \frac{6v_1v_2}{(1-v_1)^2} + \frac{6v_2}{(1-v_1)^2(1-v_2)} + \frac{1}{(1-v_1)^3(1-v_2)^3}\big(1 - 2v_1 + v_1^2 - 2v_2 + 2v_1v_2\big) \tag{2.6}$$

as a power series.

Notice that if $n \geqslant 2$ and $m = 1$, then $\lambda = (n+1, 1)$. We get

$$m_\lambda = 3 - (3(n+1) + 6(n+1)) + 6n + 6(n+1) + \frac{6(n+1)(n+2)}{4} - \frac{6n(n+1)}{2} + \frac{6(n-1)n}{4} - \frac{2(n+2)(n+1)}{2} + \frac{2n(n+1)}{2} = n + 1,$$

or equivalently, if $\lambda = (n, 1)$, with $n \geqslant 3$, then $m_\lambda = n$. Observe that if $m = 0$ and $n = 1$, then $\lambda = (1, 1)$. It follows from expression (2.6) that

$$m_\lambda = 3 - (3+6) + 6 + \frac{12}{4} - \frac{4}{2} = 3 - 9 + 6 + 3 - 2 = 1.$$

Finally, if $n = 1$ and $m = 1$ then $\lambda = (2, 1)$. Hence

$$m_\lambda = 3 - (6 + 12) + 6 + 12 + \frac{36}{4} - \frac{12}{2} - \frac{12}{2} + \frac{4}{2} = 3 - 18 + 18 + 9 - 6 - 6 + 2 = 2.$$

From the previous computations, we conclude that if $\lambda = (\lambda_1, 1)$ with $\lambda_1 \geqslant 1$ then $m_\lambda = \lambda_1$.

The remaining cases are treated similarly. □

3 (k,l)-multiplicity series of $UT_n(E)$

The multiplicity series of $UT_n(E)$ given in Corollary 2.2.1 gives us only the multiplicities m_λ when λ has no more than d parts. Our next goal is to find an algorithm that allows us to compute the multiplicities in the cocharacter sequence of $UT_n(E)$ having more "freedom" in the partitions λ. In other words we want to find m_λ without any restriction on the height of λ. Due to this, it is necessary to introduce some notions studied in [10] and [8]. The important new results of this chapter are Theorem 3.2.2, Theorem 3.4.1, Theorem 3.4.2 and Theorem 3.4.3.

3.1 Double Hilbert Series and Hook Schur Functions

Consider E, the infinite dimensional Grassmann algebra. Then

$$\mathcal{B} = \{1, e_{i_1}\cdots e_{i_m} \mid i_1 < \cdots < i_m,\ m = 1, 2, \ldots\}$$

is a basis of the vector space of E. We recall the action $*$ of S_n introduced by Berele and Regev in [10]. Given $1 \neq a = e_{i_1}\cdots e_{i_m} \in \mathcal{B}$, we write $l(a) = m$. Let $(a) = (a_1, \ldots, a_n)$, where $a_1, \ldots, a_n \in \mathcal{B}$, and define

$$I = \mathrm{Odd}(a) = \{i \mid l(a_i) \equiv 1 \pmod 2\}.$$

Remark 3.1.1. *Let $I \subseteq \{1, \ldots, n\}$ (possibly empty), $\sigma \in S_n$. Choose any $(a) = (a_1, \ldots, a_n)$, $a_i \in \mathcal{B}$, such that $a_1 \cdots a_n \neq 0$ and $\mathrm{Odd}(a) = I$. Then*

$$a_{\sigma(1)}\cdots a_{\sigma(n)} = \pm a_1 \cdots a_n.$$

Note that the sign $\pm$ depends on I and σ but does not depend on the concrete choice of a.

Definition 3.1.1. *Let $I \subseteq \{1, \ldots, n\}$ (possibly empty), $\sigma \in S_n$. Choose any $(a) = (a_1, \ldots, a_n)$, $a_i \in \mathcal{B}$, such that $a_1 \cdots a_n \neq 0$ and $\mathrm{Odd}(a) = I$. We define $f_I(\sigma) = \pm 1$ by the equality*

$$a_{\sigma(1)}\cdots a_{\sigma(n)} = f_I(\sigma) a_1 \cdots a_n.$$

Definition 3.1.2. *Fixing two non-commuting sets of variables $X = \{x_1, \ldots, x_k\}$ and $Z = \{z_1, \ldots, z_l\}$ and a vector space V with basis $X \cup Z = \{x_1, \ldots, x_k, z_1, \ldots, z_l\}$, the tensors $v_1 \otimes \cdots \otimes v_n$, $v_i \in X \cup Z$ form a basis of $V^{\otimes n}$. Given such $(v) = v_1 \otimes \cdots \otimes v_n$ we define the Z-indices of (v) by $IZ(v) = \{i \mid v_i \in Z\}$. Let $\sigma \in S_n$ and let us define the right action $*$ by*

$$(v_1 \otimes \cdots \otimes v_n) * \sigma = f_{IZ(v)}(\sigma) v_{\sigma(1)} \otimes \cdots v_{\sigma(n)}.$$

Finally, extend the $*$ action of σ to all $V^{\otimes n}$ by linearity.

Let us consider now double Hilbert series (or double Poincaré series) related to the polynomial identities of a PI-algebra A. In the above setup we identify the tensor algebra T_V of V with the free algebra

$$F\langle X, Z\rangle = F\langle x_1, \ldots, x_k, z_1, \ldots, z_l\rangle.$$

The latter algebra is a free superalgebra assuming, as usual, that $x_1, \ldots, x_k$ and $z_1, \ldots, z_l$ are, respectively, the even and the odd free generators. Let

$$\langle a; b\rangle = \langle a_1, \ldots, a_k; b_1, \ldots, b_l\rangle$$

where $a_1 + \cdots + a_k + b_1 + \cdots + b_l = n$, and let $V\langle a; b\rangle \subseteq V^{\otimes n}$ be the subspace of all polynomials which are homogeneous in each of $x_1, \ldots, x_k, z_1, \ldots, z_l$, of degree a_i in x_i and b_j in z_j.

Definition 3.1.3. *Let A be a PI-algebra and consider the following sets of commuting variables $T_k = \{t_1, \ldots, t_k\}$, $Y_l = \{y_1, \ldots, y_l\}$. The double Hilbert series of A is defined to be*

$$H(A; \mathrm{T}_k, \mathrm{Y}_l) = H(A; t_1, \ldots, t_k; y_1, \ldots, y_l) = \sum_{\langle a;b\rangle} \dim_F(V\langle a; b\rangle / V\langle a; b\rangle * Q_n) t_1^{a_1} \cdots t_k^{a_k} y_1^{b_1} \cdots y_l^{b_l},$$

where $Q_n = T(A) \cap P_n$.

Note that the variables t's and y's count, respectively, the degrees of the x's and z's.

There is another way to define double Hilbert series which is an exact analogue of the definition of Hilbert series of relatively free algebras. We recall that if A is a PI-algebra, then $A^M := A \otimes_F E$ inherits the superalgebra structure from the natural $\mathbb{Z}_2$-grading of E, i.e., $A_0 = A \otimes E_0$ and $A_1 = A \otimes E_1$.

If $T_2(A^M) \subseteq F\langle x_1, x_2, \ldots, z_1, z_2, \ldots\rangle$ is the T_2-ideal of the $\mathbb{Z}_2$-graded polynomial identities of A^M, then the relatively free $\mathbb{Z}_2$-graded algebra

$$F\langle x_1, \ldots, x_k, z_1, \ldots, z_l\rangle / (T_2(A^M) \cap F\langle x_1, \ldots, x_k, z_1, \ldots, z_l\rangle)$$

is the *magnum* of A. For more details on the magnum of a PI-algebra see [6]. The following result is well known (see [6]) and gives that the double Hilbert series related to the PI-algebra A coincides with the Hilbert series of the magnum of A.

Proposition 3.1.1. *Let A be a PI-algebra. If $\langle a; b\rangle = \langle a_1, \ldots, a_k; b_1, \ldots, b_l\rangle$ is such that $a_1 + \cdots + a_k + b_1 + \cdots + b_l = n$, then $V\langle a; b\rangle * Q_n = V\langle a; b\rangle \cap T_2(A^M)$.*

Now we discuss hook Schur functions and their relations with the double Hilbert series of a PI-algebra. We begin with a definition that generalizes the notion of semistandard tableau.

Fix integers k, $l \geqslant 0$, $k+l>0$ and $k+l$ variables $t_1, \ldots, t_k, y_1, \ldots, y_l$, so that $t_1 < \cdots < t_k < y_1 < \cdots y_l$. Let λ be a partition with Young diagram D_λ. Fill D_λ with elements from $\{t_1, \ldots, t_k, y_1, \ldots y_l\}$, allowing repetitions, to get a (k,l)-tableau T_λ. Such T_λ is said to be (k,l)-*semistandard* if

i. The "t part" (i.e., the cells filled with t_i's) of T_λ forms a tableau. (Thus the "y part" is a skew tableau.)

ii. The t_i's are increasing in rows (with possible repetitions), and strictly increasing in columns.

iii. The y_j's are increasing in columns (with possible repetitions), strictly increasing in rows.

Definition 3.1.4. *Let T_λ be a (k,l)-semistandard tableau, define $w^{T_\lambda} = t_1^{a_1} \cdots t_k^{a_k} y_1^{b_1} \cdots y_l^{b_l}$ where each a_i counts the number of entries of t_i in T_λ and each b_j counts the number of entries of y_j in T_λ. The hook Schur functions is defined by*

$$HS_\lambda(\mathrm{T}_k; \mathrm{Y}_l) = HS_\lambda(t_1, \ldots, t_k; y_1, \ldots, y_l) = \sum \{w^{T_\lambda} | T_\lambda \text{ is } (k,l)\text{-semistandard}\}.$$

Let $H(k,l;n) = \{\lambda = (\lambda_1, \lambda_2, \cdots) \vdash n | \lambda_{k+1} \leqslant l\}$ and

$$H(k,l) = \bigcup_{n \geqslant 0} H(k,l,n).$$

Note that if $\lambda \in H(k,l)$, then the Young diagram D_λ lies in the hook with width of the hand k and width of the leg l. It is not hard to see from the definition that $HS_\lambda(\mathrm{T}_k; \mathrm{Y}_l) \neq 0$ if and only if $\lambda \in H(k,l)$.

Example 3.1.1. Consider the hook $H(2,1)$ and let us calculate $HS_{(2,1,1)}(t_1, t_2; y_1)$. Notice that the $(2,1)$-semistandard tableux of shape $(2,1,1)$ are

$$\begin{array}{|c|c|}\hline t_1 & t_1 \\ \hline t_2 \\ \cline{1-1} y_1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline t_1 & y_1 \\ \hline y_1 \\ \cline{1-1} y_1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline t_1 & t_2 \\ \hline t_2 \\ \cline{1-1} y_1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline t_2 & t_2 \\ \hline y_1 \\ \cline{1-1} y_1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline t_1 & t_2 \\ \hline y_1 \\ \cline{1-1} y_1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline t_1 & t_1 \\ \hline y_1 \\ \cline{1-1} y_1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline t_2 & y_1 \\ \hline y_1 \\ \cline{1-1} y_1 \\ \cline{1-1}\end{array}$$

Hence

$$H_{(2,1,1)}(t_1, t_2, y_1) = t_1^2 t_2 y_1 + t_1 y_1^3 + t_1 t_2^2 y_1 + t_2^2 y_2^2 + t_1 t_2 y_1^2 + t_1^2 y_1^2 + t_2 y_1^3.$$

Theorem 1.4.6 shows that whenever k, l are large enough we can capture all partitions that have non-zero multiplicities in the cocharacter sequence of a PI-algebra A. i.e, if $m_\lambda(A) \neq 0$ then $\lambda \in H(k,l)$.

Let A be a PI-algebra, we write $\chi(A) \subseteq H(k,l)$ when the non-zero multiplicities $m_\lambda(A)$ in the cocharacter sequence $\chi_n(A)$, $n = 0, 1, 2 \ldots$, appear only for $\lambda \in H(k,l)$. By Theorems 6 and 11 of [9], we have the following generalization of Theorem 1.4.10.

Theorem 3.1.1 (Berele and Regev [9]). *Let A be a PI-algebra such that for any $n \geqslant 0$*

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda(A)\chi_\lambda.$$

Then

$$H(A; \mathrm{T}_k, \mathrm{Y}_l) = \sum_{n=0}^{\infty} \sum_{\lambda \in H(k,l;n)} m_\lambda(A) HS_\lambda(\mathrm{T}_k, \mathrm{Y}_l),$$

where k, l are non-negative integers.

Formanek expressed the Hilbert series of the product of two T-ideals in terms of the Hilbert series of the factors, (see Corollary 1.3.1). Here we have the following analogue for double Hilbert series.

Proposition 3.1.2. *Let A, A_1, A_2 be PI-algebras such that for any $n \in \mathbb{N}$ and $T(A) = T(A_1)T(A_2)$. Then*

$$H(A; \mathrm{T}_k, \mathrm{Y}_l) = H(A_1; \mathrm{T}_k, \mathrm{Y}_l) + H(A_2; \mathrm{T}_k, \mathrm{Y}_l) + (HS_{(1)}(\mathrm{T}_k, \mathrm{Y}_l) - 1)H(A_1; \mathrm{T}_k, \mathrm{Y}_l)H(A_2; \mathrm{T}_k, \mathrm{Y}_l).$$

Proof. Berele and Regev proved in [11] that if $T(A) = T(A_1)T(A_2)$ then

$$\chi_n(A) = \chi_n(A_1) + \chi_n(A_2) + \chi_{(1)} \widehat{\otimes} \sum_{j=0}^{n-1} \chi_j(A_1) \widehat{\otimes} \chi_{n-j-1}(A_2) - \sum_{j=0}^{n} \chi_j(A_1) \widehat{\otimes} \chi_{n-j}(A_2) \quad (3.1)$$

where $\widehat{\otimes}$ denotes the "outer" tensor product of characters. Recall that for irreducible characters $\widehat{\otimes}$ behaves according to the Littlewood-Richardson rule.

Due to Theorem 3.1.1, we have

$$H(A; \mathrm{T}_k, \mathrm{Y}_l) = \sum_{n=0}^{\infty} \sum_{\lambda \in H(k,l;n)} m_\lambda(A) HS_\lambda(T_k, Y_l)$$

$$H(A_1; \mathrm{T}_k; \mathrm{Y}_l) = \sum_{n=0}^{\infty} \sum_{\alpha \in H(k,l;n)} m_\alpha(A_1) HS_\alpha(T_k, Y_l)$$

$$H(A_2; \mathrm{T}_k, \mathrm{Y}_l) = \sum_{n=0}^{\infty} \sum_{\beta \in H(k,l;n)} m_\beta(A_2) HS_\beta(T_k, Y_l).$$

Since the hook Schur functions multiply according to the Littlewood-Richardson rule (see [10], section 6), we can conclude in the light of (3.1) that

$$H(A; \mathrm{T}_k, \mathrm{Y}_l) = H(A_1; \mathrm{T}_k, \mathrm{Y}_l) + H(A_2; \mathrm{T}_k, \mathrm{Y}_l) + (HS_{(1)}(\mathrm{T}_k, \mathrm{Y}_l) - 1)H(A_1; \mathrm{T}_k, \mathrm{Y}_l)H(A_2; \mathrm{T}_k, \mathrm{Y}_l),$$

as desired. □

Corollary 3.1.1. *Let A, C be PI-algebras such that $T(C) = T(A)^m$. Then*

$$H(C; \mathrm{T}_k, \mathrm{Y}_l) = \sum_{j=1}^{m} \binom{m}{j} H(A; \mathrm{T}_k, \mathrm{Y}_l)^j (S_{(1)}(\mathrm{T}_k; \mathrm{Y}_l) - 1)^{j-1}.$$

Proof. We will prove the statement using induction on m.

If $m=1$, then $\chi_n(A)=\chi_n(C)$. So by Theorem 3.1.1

$$H(C;\mathrm{T}_k;\mathrm{Y}_l)=H(A;\mathrm{T}_k;\mathrm{Y}_l).$$

Assuming the result true for $m-1$, let us prove it for m.

Let A_{m-1} be a PI-algebra such that $T(A_{m-1})=T(A)^{m-1}$, then $T(C)=T(A_{m-1})T(A)$. By Proposition 3.1.2, we get

$$\begin{aligned}H(C;\mathrm{T}_k;\mathrm{Y}_l)&=H(A_{m-1};\mathrm{T}_k;\mathrm{Y}_l)+H(A;\mathrm{T}_k;\mathrm{Y}_l)\\&\quad+(HS_{(1)}(\mathrm{T}_k,\mathrm{Y}_l)-1)H(A_{m-1};\mathrm{T}_k;\mathrm{Y}_l)H(A;\mathrm{T}_k;\mathrm{Y}_l).\end{aligned}$$

By induction hypothesis, we have

$$H(A_{m-1};\mathrm{T}_k;\mathrm{Y}_l)=\sum_{j=1}^{m-1}\binom{m-1}{j}H(A;\mathrm{T}_k;\mathrm{Y}_l)^j(S_{(1)}(\mathrm{T}_k;\mathrm{Y}_l)-1)^{j-1}.$$

Hence

$$\begin{aligned}H(C;\mathrm{T}_k;\mathrm{Y}_l)&=\sum_{j=1}^{m-1}\binom{m-1}{j}H(A;\mathrm{T}_k;\mathrm{Y}_l)^j(S_{(1)}(\mathrm{T}_k;\mathrm{Y}_l)-1)^{j-1}+H(A;\mathrm{T}_k;\mathrm{Y}_l)\\&\quad+(S_{(1)}(\mathrm{T}_k;\mathrm{Y}_l)-1)\left(\sum_{j=1}^{m-1}\binom{m-1}{j}H(A;\mathrm{T}_k;\mathrm{Y}_l)^j(S_{(1)}(\mathrm{T}_k;\mathrm{Y}_l)-1)^{j-1}\right)H(A;\mathrm{T}_k;\mathrm{Y}_l)\\&=mH(A;\mathrm{T}_k;\mathrm{Y}_l)+\sum_{j=2}^{m-1}\binom{m-1}{j}H(A;\mathrm{T}_k;\mathrm{Y}_l)^j(S_{(1)}(T_k,Y_l)-1)^{j-1}\\&\quad+\sum_{j=1}^{m-1}\binom{m-1}{j}H(A;\mathrm{T}_k;\mathrm{Y}_l)^{j+1}(S_{(1)}(\mathrm{T}_k;\mathrm{Y}_l)-1)^{j}\\&=mH(A;\mathrm{T}_k;\mathrm{Y}_l)+\sum_{j=2}^{m-1}\binom{m-1}{j}H(A;\mathrm{T}_k;\mathrm{Y}_l)^j(S_{(1)}(\mathrm{T}_k;\mathrm{Y}_l)-1)^{j-1}\\&\quad+\sum_{j=2}^{m}\binom{m-1}{j-1}H(A;\mathrm{T}_k;\mathrm{Y}_l)^j(S_{(1)}(\mathrm{T}_k;\mathrm{Y}_l)-1)^{j-1}\\&=mH(A;\mathrm{T}_k;\mathrm{Y}_l)+\sum_{j=2}^{m-1}\left(\binom{m-1}{j}+\binom{m-1}{j-1}\right)H(A;\mathrm{T}_k;\mathrm{Y}_l)^j(S_{(1)}(\mathrm{T}_k;\mathrm{Y}_l)-1)^{j-1}\\&\quad+H(A;\mathrm{T}_k;\mathrm{Y}_l)^m(S_{(1)}(\mathrm{T}_k;\mathrm{Y}_l)-1)^{m-1}\\&=mH(A;\mathrm{T}_k;\mathrm{Y}_l)+\sum_{j=2}^{m-1}\binom{m}{j}H(A;\mathrm{T}_k;\mathrm{Y}_l)^j(S_{(1)}(\mathrm{T}_k;\mathrm{Y}_l)-1)^{j-1}\\&\quad+H(A;\mathrm{T}_k;\mathrm{Y}_l)^m(S_{(1)}(\mathrm{T}_k;\mathrm{Y}_l)-1)^{m-1}\\&=\sum_{j=1}^{m}\binom{m}{j}H(A;\mathrm{T}_k;\mathrm{Y}_l)^j(S_{(1)}(\mathrm{T}_k;\mathrm{Y}_l)-1)^{j-1}\end{aligned}$$

and we are done. □

3.2 The Double Hilbert series of $UT_n(E)$

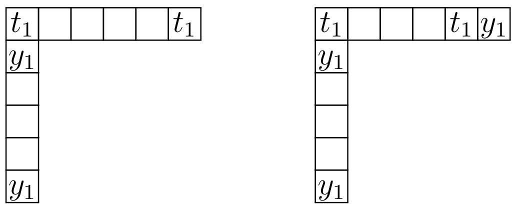
We know that the cocharacter sequence of E lies in the hook $H(1,1)$. The double Hilbert series $H(E;t_1,y_1)$ was computed in [9]. We present once again the computation of $H(E,t_1,y_1)$ as a direct application of the definition of hook Schur functions.

We also compute $H(E;\mathrm{T}_k,\mathrm{T}_l)$ for k, l non-negative integers. Moreover, we find an expression for the double Hilbert series of $UT_n(E)$. Finally, using $H(UT_n(E);\mathrm{T}_k,\mathrm{Y}_l)$ we give a description of the non-zero multiplicities m_λ in the cocharacters sequence of $UT_n(E)$.

Proposition 3.2.1. *Let E be the infinite dimensional Grassmann algebra. Then*

$$H(E;t_1,y_1) = \frac{1+t_1y_1}{(1-t_1)(1-y_1)}.$$

Proof. By Theorem 2.2.2 we know that for any $n \geqslant 1$, if $\lambda = (p, 1^n - p)$, we have $m_\lambda(E) = 1$. In light of Theorem 3.1.1, we have to compute $HS_\lambda(t_1,y_1)$ in order to determine $H(E;t_1,y_1)$. Note that the only $(1,1)$-semistandard tableaux of shape λ are



corresponding to the monomials $t_1^p y_1^{n-p}$ and $t_1^{p-1}y_1^{n-p+1}$, respectively. Hence

$$H(E;t_1,y_1) = 1 + \sum_{n=1}^{\infty}\sum_{p=1}^{n}(t_1^p y_1^{n-p} + t_1^{p-1}y_1^{n-p+1}).$$

Note that

$$\begin{aligned} 1 + \sum_{n=1}^{\infty}\sum_{p=1}^{n}(t_1^p y_1^{n-p} + t_1^{p-1}y_1^{n-p+1}) &= 1 + (t_1+y_1)\sum_{n=1}^{\infty}\sum_{p=1}^{n} t_1^{p-1}y_1^{n-p} \\ &= 1 + (t_1+y_1)\sum_{k=0}^{\infty}\sum_{n+p=k} t_1^p y_1^n \\ &= 1 + (t_1+y_1)\sum_{p=0}^{\infty} t_1^p \sum_{n=0}^{\infty} y_1^n \\ &= 1 + \frac{t_1+y_1}{(1-t_1)(1-y_1)} \\ &= \frac{1+t_1y_1}{(1-t_1)(1-y_1)}. \end{aligned}$$

It follows that

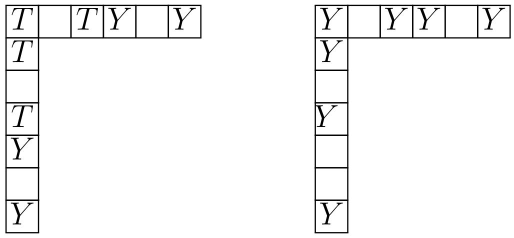
$$H(E;t_1,y_1) = \frac{1+t_1y_1}{(1-t_1)(1-y_1)}. \qquad \square$$

In a similar way, we compute $H(E;\mathrm{T}_k,\mathrm{Y}_l)$ for any k, $l \in \mathbb{N}$ using Definition 3.1.4 together with Theorems 3.1.1 and 2.2.2. The results is the following.

Proposition 3.2.2. *Let $k, l \in \mathbb{N}$. Then*

$$H(E;\mathrm{T}_k,\mathrm{Y}_l) = \frac{1}{2}\left(1 + \prod_{i=1}^{k}\prod_{j=1}^{l}\frac{(1+t_i)(1+y_j)}{(1-t_i)(1-y_j)}\right).$$

Proof. By the definition of (k,l)-semistandard tableau, we have only two types of tableaux for $\lambda = (p, 1^{n-p})$:



where with the symbol T, we mean "elements lying in T_k" and with the symbol Y "elements lying in Y_l". Recall that

- The elements in T_k are non-decreasing in rows and strictly increasing in columns.
- The elements in Y_l are non-decreasing in columns and strictly increasing in rows.

Hence the tableaux of the first type have $n \geqslant 1$ boxes and contain at least one symbol T. The tableaux of the second type do not contain symbols T and have $n \geqslant 0$ boxes.

Consider a tableau T_λ of type 1. The T-parts of T_λ form a semistandard tableau T_μ filled with elements from T_k where $\mu = (q, q^{m-q})$ for some $m \leqslant n$ and $q \leqslant p$. Hence the T-parts of such tableaux are in one-to-one correspondence with the semistandard μ-tableaux filled with elements from T_k where $\mu = (q, q^{m-q})$, $m \leqslant n$ and $q \leqslant p$. The sum on all μ of the products of the entries of T_μ is equal to the sum of the Schur functions $S_\mu(\mathrm{T}_k)$. If the Y-part of the arm of the tableau T_λ consists of $y_{j_1}, \ldots, y_{j_r}$, then $1 \leqslant j_1 \cdots < j_r \leqslant l$. Similarly, if the Y-part of the leg of the tableau of T_λ consists of $y_{m_1}, \ldots, y_{m_s}$ then $1 \leqslant m_1 \cdots \leqslant m_s \leqslant l$. Hence the sum of all monomials w^{T_λ}, when T_λ runs over all $(k-l)$-semistandard tableaux of type 1, is

$$\sum_{m\geqslant 1}\sum_{q=1}^{m} S_{(q,1^{m-q})}(\mathrm{T}_k)\sum_{c_i\geqslant 0} y_1^{c_1}\cdots y_l^{c_l}\sum_{j_1<\cdots<j_s} y_{j_1}\cdots y_{j_s}$$

$$= \sum_{m\geqslant 1}\sum_{q=1}^{m} S_{(q,1^{m-q})}(\mathrm{T}_k)\prod_{j=1}^{l}\frac{1}{1-y_j}\sum_{s=0}^{l} e_s(\mathrm{Y}_l) = \sum_{m\geqslant 1}\sum_{q=1}^{m} S_{(q,1^{m-q})}(\mathrm{T}_k)\prod_{j=1}^{l}\frac{1+y_j}{1-y_j}$$

where

$$e_s(\mathrm{Y}_l) = \sum_{j_1<\cdots<j_s} y_{j_1}\cdots y_{j_s}$$

is the s-th elementary symmetric function. By Theorem 2.2.2

$$\sum_{m\geqslant 1}\sum_{q=1}^{m} S_{(q,1^{m-q})} = H(E,\mathrm{T}_k)-1.$$

The explicit form of $H(E,\mathrm{T}_k)$ is given in Proposition 2.2.1. In particular, we have

$$H(E;\mathrm{T}_k) = \frac{1}{2}\left(1+\prod_{i=1}^{k}\frac{1+t_i}{1-t_i}\right).$$

In this way, the sum of all monomials w^{T_λ}, when T_λ runs over all (k,l)-semistandard tableaux of type 1, has the form

$$\frac{1}{2}\left(-1+\prod_{i=1}^{k}\frac{1+t_i}{1-t_i}\right)\prod_{j=1}^{l}\frac{1+y_j}{1-y_j}.$$

Now, we consider the (k,l)-semistandard tableaux $T_{(p,1^{n-p})}$ of type 2. Clearly, the transposed tableau $T'_{(p,1^{n-p})}$ is the tableau of shape $(n-p+1,1^{p-1})$. So the entries of $T'_{(p,1^{n-p})}$ do not decrease in the first row and strictly increase in the columns. Hence $T'_{(p,1^{n-p})}$ is a semistandard tableau in the ordinary sense. Applying the same argument as for the T-part of the sum for the tableaux of type (1), we have that the sum of the monomials w^{T_λ} over all $T^{(p,1^{n-p})}$ is equal to

$$1+\sum_{n\geqslant 1}\sum_{p=1}^{n} S_{(n-p+1,1^{p-1})}(\mathrm{Y}_l) = H(E;\mathrm{Y}_l) = \frac{1}{2}\left(1+\prod_{j=1}^{l}\frac{1+y_j}{1-y_j}\right).$$

Hence

$$\begin{aligned} H(E;\mathrm{T}_k,\mathrm{Y}_l) &= \frac{1}{2}\left(-1+\prod_{i=1}^{k}\frac{1+t_i}{1-t_i}\right)\prod_{j=1}^{l}\frac{1+y_j}{1-y_j}+\frac{1}{2}\left(1+\prod_{j=1}^{l}\frac{1+y_j}{1-y_j}\right)\\ &= \frac{1}{2}\left(-\prod_{j=1}^{l}\frac{1+y_j}{1-y_j}+\prod_{i=1}^{k}\prod_{j=1}^{l}\frac{(1+t_i)}{(1-t_i)}\frac{(1+y_j)}{(1-y_j)}+1+\prod_{j=1}^{l}\frac{1+y_j}{1-y_j}\right)\\ &= \frac{1}{2}\left(1+\prod_{i=1}^{k}\prod_{j=1}^{l}\frac{(1+t_i)}{(1-t_i)}\frac{(1+y_j)}{(1-y_j)}\right)\end{aligned}$$

and the proof follows. □

Corollary 3.2.1. *Let E be the infinite dimensional Grassmann algebra. Consider the algebra $UT_n(E)$ of $n\times n$ upper triangular matrices with entries in E. Then*

$$H(UT_n(E);\mathrm{T}_k,\mathrm{Y}_l) = \sum_{j=1}^{n}\binom{n}{j}\left(\frac{1}{2}\left[1+\prod_{i=1}^{k}\prod_{s=1}^{l}\frac{(1+t_i)}{(1-t_i)}\frac{(1+y_s)}{(1-y_s)}\right]\right)^{j}\left(\sum_{i=1}^{k}t_i+\sum_{s=1}^{l}y_s-1\right)^{j-1},$$

for some k, $l\in\mathbb{N}$.

Proof. By Proposition 3.2.2 and Corollary 3.1.1, we have

$$H(UT_n(E);\mathrm{T}_k,\mathrm{Y}_l)=\sum_{j=1}^{n}\binom{n}{j}\left(\frac{1}{2}\left[1+\prod_{i=1}^{k}\prod_{s=1}^{l}\frac{(1+t_i)}{(1-t_i)}\frac{(1+y_s)}{(1-y_s)}\right]\right)^{j}(S_{(1)}(\mathrm{T}_k;\mathrm{Y}_l)-1)^{j-1}.$$

Note that by the definition of hook Schur functions, it follows

$$S_{(1)}(\mathrm{T}_k;\mathrm{Y}_l)=\sum_{i=1}^{k}t_i+\sum_{s=1}^{l}y_s.$$

Hence $H(UT_n(E);\mathrm{T}_k,\mathrm{Y}_l)$ equals

$$\sum_{j=1}^{n}\binom{n}{j}\left(\frac{1}{2}\left[1+\prod_{i=1}^{k}\prod_{s=1}^{l}\frac{(1+t_i)}{(1-t_i)}\frac{(1+y_s)}{(1-y_s)}\right]\right)^{j}\left(\sum_{i=1}^{k}t_i+\sum_{s=1}^{l}y_s-1\right)^{j-1}.$$ □

By Theorem 1.4.6, there are positive integers k, l such that $\chi(UT_n(E))\subseteq H(k,l)$. By Theorem 2.2.4 it follows that $k=n$ and $l=2n-1$. So, we have the following result.

Proposition 3.2.3. *Let $n\geqslant 1$ and consider the algebra $UT_n(E)$. Then the partitions λ with non-zero multiplicities $m_\lambda(UT_n(E))$ in the cocharacter sequence of $UT_n(E)$ lie in the hook $H(n,2n-1)$.*

Using the double Hilbert series of $UT_n(E)$, we are able to give a better description of the partitions λ with $m_\lambda(UT_n(E))\neq 0$ than the one given in Theorem 2.2.4.

Theorem 3.2.1. *The hook Schur functions $HS_\pi(\mathrm{T}_k,\mathrm{Y}_l)$ participating in the product $H(E;\mathrm{T}_k,\mathrm{Y}_l)^j$ are indexed by partitions π lying in $H(j,j)$.*

Proof. We will prove the assertion by induction on j.

If $j=1$, the result follows from Theorems 3.1.1 and 2.2.2.

Assuming the result true for $j-1\geqslant 1$, let us prove it for j. Note that

$$H(E;\mathrm{T}_k,\mathrm{Y}_l)^j=H(E;\mathrm{T}_k,\mathrm{Y}_l)^{j-1}H(E;\mathrm{T}_k,\mathrm{Y}_l).$$

By induction hypotheses, we have

$$H(E;\mathrm{T}_k,\mathrm{Y}_l)^{j-1}=\sum\alpha_\lambda HS_\lambda(\mathrm{T}_k,\mathrm{Y}_l),$$

where $\lambda\in H(j-1,j-1)$ and $\alpha_\lambda\in\mathbb{C}$.

Let $\lambda\in H(j-1,j-1)$ and consider the following product

$$HS_\lambda(\mathrm{T}_k,\mathrm{Y}_l)HS_{(q,1^{m-q})}(\mathrm{T}_k,\mathrm{Y}_l)=\sum\theta_\alpha HS_\alpha(\mathrm{T}_k,\mathrm{Y}_l).$$

Suppose that there exists some partition α such that $HS_\alpha(\mathrm{T}_k, \mathrm{Y}_l)$ participates in the decomposition of the latter product, and $\alpha \notin H(j,j)$.

Note that $\alpha \notin H(j,j)$ implies $\alpha_{j+1} > j$, hence suppose that $\alpha_{j+1} = j+1$. Let $Q \subseteq \alpha/\lambda$ be the square formed by the boxes (j,j), $(j,j+1)$, $(j+1,j)$ and $(j+1,j+1)$ as in the picture below:

$$Q = \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}.$$

Of course, if we consider a semi-standard tableau of shape α/λ and content $(q, 1^{m-q})$, then Q must be filled as below

i.
$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline j & k \\ \hline \end{array},$$
where j, $k \neq 1$ and $j < k$.

ii.
$$\begin{array}{|c|c|}\hline 1 & i \\ \hline j & k \\ \hline \end{array},$$
where i, j, $k \neq 1$ e $i < k$ e $j < k$.

iii.
$$\begin{array}{|c|c|}\hline i & j \\ \hline k & l \\ \hline \end{array},$$
where i, j, k, $l \neq 1$, $i < j < l$ and $i < k < l$. Observe that those conditions are imposed because we are working with content $(q, 1^{m-q})$.

Note that i, j, k, l must be pairwise different since we consider a semi-standard tableau of shape α/λ and content $(q, 1^{m-q})$, then none of the cases above yields a lattice permutations when we read their entries from the right to the left and downwards. Hence by the Littlewood-Richardson rule, $HS_\alpha(\mathrm{T}_k, \mathrm{Y}_l)$ cannot participate in the decomposition of

$$HS_\lambda(\mathrm{T}_k, \mathrm{Y}_l) HS_{(q,1^{m-q})}(\mathrm{T}_k, \mathrm{Y}_l),$$

that is an absurd and we are done. □

In what follows we find the partitions with non-zero multiplicities participating in the decomposition of $H(UT_n(E); \mathrm{T}_k, \mathrm{Y}_l)$. Given $n \geqslant 1$, let $Q_2 = (n-1)^{n-1}$ be a square Young tableau of size $n-1$. We denote by $H(n,n) * Q_2$ the skew hook obtained when we identify the box $(1,1)$ of Q_2 with the $(n+1, n+1)$ (empty) box of $H(n,n)$, the box $(1,2)$ of Q_2 with the $(n+1, n+2)$ (empty) box of $H(n,n)$ and so on.

Proposition 3.2.4. *If $m_\lambda(UT_n(E)) \neq 0$, then $\lambda \in H(n,n) * Q_2$ and $|\lambda \cap Q_2| \leqslant n-1$.*

Proof. By Corollary 3.2.1, the non-zero multiplicities $m_\lambda(UT_n(E))$ in the cocharacter sequence of $UT_n(E)$ come from the decomposition, as an infinite sum of hook Schur functions, of

$$H(E;\mathrm{T}_k,\mathrm{Y}_l)^j\left(\sum_{i=1}^{k}t_i+\sum_{s=1}^{l}y_s\right)^q=H(E;\mathrm{T}_k,\mathrm{Y}_l)^jHS_{(1)}(T_k,Y_l)^q,$$

where $j\leqslant n$ and $q\leqslant n-1$, for some k, l. By Theorem 3.2.1, the hook Schur functions $HS_\pi(\mathrm{T}_k;\mathrm{Y}_l)$ participating in the product $H(E;\mathrm{T}_k;\mathrm{Y}_l)^j$ are indexed by partitions π lying in $H(n,n)$, then $\pi_{n+1}\leqslant n$. Note that by the Branching rule, the product $HS_\pi(\mathrm{T}_k;\mathrm{Y}_l)\cdot HS_{(1)}(T_k;Y_l)$ gives a sum of $HS_\rho(T;U)$ where the diagrams of ρ are obtained from the diagram of π by adding a box. It follows that the diagram of ρ has no more than one box in Q_2. Multiplying q times by $HS_{(1)}(\mathrm{T}_k;\mathrm{Y}_l)$ we add to the diagram of π no more than $q\leqslant n-1$ boxes in Q_2. This means if $m_\lambda(E)\neq 0$, then $|\lambda\cap Q_2|\leqslant n-1$.
If there is $\lambda\notin H(n,n)*Q_2$ such that $m_\lambda(E)\neq 0$, then $\lambda_{n+1}\geqslant 2n$ or $\lambda_{2n}\geqslant n+1$.

Suppose $\lambda_{n+1}\geqslant 2n$. We know that there is $\pi\in H(n,n)$ such that the diagram D_λ is obtained from the diagram D_π by adding $q\leqslant n-1$ boxes. So $\lambda_{n+1}\leqslant 2n-1$, which is an absurd.

If $\lambda_{2n}\leqslant n+1$, we know that there is $\pi\in H(n,n)$ such that the diagram D_λ is obtained from the diagram D_π by adding $q\leqslant n-1$ boxes. Note that the limit of cases is $\pi_i=n$ for $i\geqslant n+1$. Hence $\lambda_{2n}\leqslant n$, which is a contradiction and we are done. □

Remark 3.2.1. Note that the previous proposition gives a better description of the partitions $\lambda=(\lambda_1,\ldots,)$ such that $m_\lambda(UT_n(E))\neq 0$ than Theorem 2.2.4. In fact, by Proposition 3.2.4, we have that $\lambda\in H(n,2n-1)$, $\lambda_i\leqslant 2n-1$ for $n+1\leqslant i\leqslant 2n-1$ and $\lambda_i\leqslant n$ for $i\geqslant 2n$, whereas by Theorem 2.2.4, we only know $\lambda\in H(n,2n-1)$, that means $\lambda_i\leqslant 2n-1$ for $i\geqslant n+1$.

3.3 The (k,l)-multiplicity series

Note that by Theorem 1.4.5, If A is a finite dimensional PI-algebra, it suffices to work with a large enough set of variables T in order to capture all the multiplicities $m_\lambda(A)$ of its cocharacter sequence from its Hilbert series. But if A is an infinite dimensional algebra, knowing its multiplicity series is not enough to find all multiplicities $m_\lambda(A)$. Due to this fact, we want to generalize the idea of the multiplicity series defining the (k,l)-multiplicity series of a A. This series contains all the information about the multiplicities $m_\lambda(A)$ for λ in the hook $H(k,l)$.

As in [11], identifying a partition with its Young diagram, we can break each $\lambda\in H(k,l)$ into three parts $\lambda\to(\lambda_0,\mu,\nu)$ where λ_0 is the piece of the partition in the $k\times l$ rectangle (l^k), μ is a partition with at most k parts and it is the part λ to the right of λ_0 and ν is a partition with at most l parts and it is the conjugate of the part of λ below λ_0. (see figure 2)

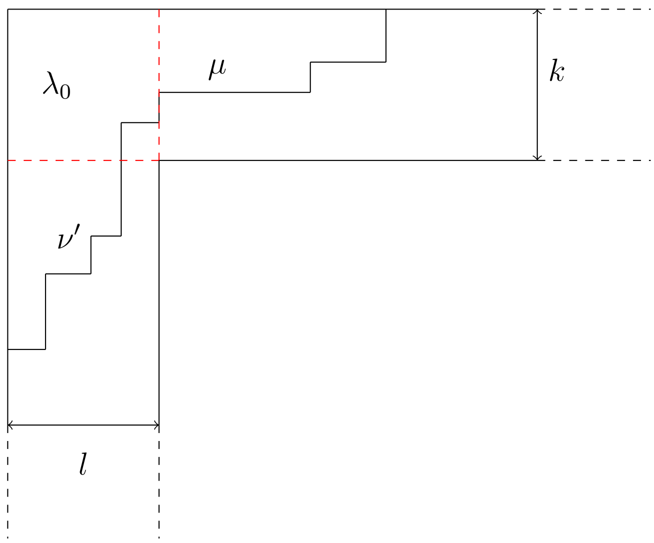


Figure 2 – Definition of λ_0, μ and ν

We fix two non-negative integers k, l such that $k + l \geqslant 1$. Let λ_0 be a partition such that $\lambda_0 \subseteq (l^k)$ and the set

$$H_{\lambda_0}(k,l) := \{\lambda \in H(k,l) | \lambda \cap \ (l^k) = \lambda_0\}.$$

Notice that $H(k,l) = \bigcup_{\lambda_0 \subseteq (l^k)} H_{\lambda_0}(k,l)$. Let $\mathrm{T}_k = \{t_1, \dots, t_k\}$, $Y = \{y_1, \dots, y_l\}$ and $V = \{v_1, \dots, v_k\}$ be three sets of commuting variables and consider the algebra

$$\mathbb{C}[[\mathrm{T}_k, \mathrm{Y}_l]] = \mathbb{C}[[t_1, \dots, t_k, y_1, \dots, y_l]]$$

of formal power series in $(k + l)$ commuting variables. Let

$$\Lambda^{(k,l,n)} := \{\sum_{\lambda} m_\lambda HS_\lambda(\mathrm{T}_k, \mathrm{Y}_l) | \lambda \in H(k,l;n) \, m_\lambda \in \mathbb{C}\},$$

the set $\{HS_\lambda(\mathrm{T}_k, \mathrm{Y}_l) | \lambda \in H(k,l,n)\}$ is a basis of $\Lambda^{(k,l,n)}$ as a vector space (see [10]). Now, define

$$\Lambda^{(k,l)} = \{\sum_{\lambda} m_\lambda HS_\lambda(\mathrm{T}_k, \mathrm{Y}_l) | \lambda \in H(k,l) \, m_\lambda \in \mathbb{C}\}.$$

Note that $\Lambda^{(k,l)}$ is a subalgebra of $\mathbb{C}[[\mathrm{T}_k, \mathrm{Y}_l]]$, since the hook Schur functions multiply with the Litlewood-Richarson rule. Given $g(\mathrm{T}_k, \mathrm{Y}_l) = \sum_{\lambda \in H(k,l)} m_\lambda HS_\lambda(\mathrm{T}_k, \mathrm{Y}_l) \in \Lambda^{(k,l)}$, we have

$$g(\mathrm{T}_k, \mathrm{Y}_l) = \sum_{\lambda_0 \subseteq (l^k)} \sum_{\lambda \in H_{\lambda_0}(k,l)} m_\lambda HS_\lambda(\mathrm{T}_k, \mathrm{Y}_l).$$

Definition 3.3.1. *Let* $g(\mathrm{T}_k,\mathrm{Y}_l)=\sum_{\lambda_0\subseteq(l^k)}\sum_{\lambda\in H_{\lambda_0}(k,l)} m_\lambda HS_\lambda(\mathrm{T}_k,\mathrm{Y}_l)\in\Lambda^{(k,l)}$*, we define the (k,l)-multiplicity series of g by*

$$\widehat{M}(g;\mathrm{V}_k,\mathrm{T}_k,\mathrm{Y}_l):=\sum_{\lambda_0\subseteq(l^k)}\sum_{\lambda\in H_{\lambda_0}(k,l)} m_\lambda \mathrm{V}_k^{\lambda_0}\mathrm{T}_k^{\mu}\mathrm{Y}_l^{\nu},$$

where $\mathrm{V}_k^{\lambda_0}=v_1^{\lambda_{01}}\cdots v_k^{\lambda_{0k}}$, $\mathrm{T}_k^{\mu}=t_1^{\mu_1}\cdots t_k^{\mu_k}$ *and* $\mathrm{Y}_l^{\nu}=y_1^{\nu_1}\cdots y_l^{\nu_k}$.

It is clear that $\widehat{M}(g;\mathrm{V}_k,\mathrm{T}_k,\mathrm{Y}_l)$ is an element of $\mathbb{C}[[\mathrm{V}_k,\mathrm{T}_k,\mathrm{Y}_l]]$ the algebra of formal power series in $(2k+l)$ variables. Observe that $\widehat{M}$ defines an injective linear map from $\Lambda^{(k,l)}$ to $\mathbb{C}[[\mathrm{V}_k,\mathrm{T}_k,\mathrm{Y}_l]]$ and set $\Lambda_{(2k,l)}:=\widehat{M}(\Lambda^{(k,l)})$.

Example 3.3.1. Consider the hook $H(2,1)$ and the partition $\lambda=(2,1^2)$. Then $\lambda_0=(1,1),\mu=(1)$ and $\nu=(1)$. It follows that

$$\widehat{M}(HS_\lambda(\mathrm{T}_2,\mathrm{Y}_1))=v_1v_2t_1y_1.$$

Now, if we consider the hook $H(3,1)$ and the same partition, we obtain $\lambda_0=(1^3)$, $\mu=(1)$ and $\nu=(0)$. Hence

$$\widehat{M}(HS_\lambda(\mathrm{T}_3,\mathrm{Y}_1))=v_1v_2v_3t_1.$$

Definition 3.3.2. *Let A be a PI-algebra. The formal series*

$$\widehat{M}(A;\mathrm{V}_k,\mathrm{T}_k,\mathrm{Y}_l)=\sum_{\lambda_0\subseteq(l^k)}\sum_{\lambda\in H_{\lambda_0}(k,l)} m_\lambda(A)\mathrm{V}_k^{\lambda_0}\mathrm{T}_k^{\mu}\mathrm{Y}_l^{\nu}$$

where $\lambda\in H(k,l)$ and $m_\lambda(A)$ is the multiplicity corresponding to χ_λ in the cocharacter sequences of A, is called the (k,l)-multiplicity series of A.

When the sets of variables are inferred from the context, we may also write $\widehat{M}(A)$ instead of $\widehat{M}(A;\mathrm{V}_k,\mathrm{T}_k,\mathrm{Y}_l)$.

Our next step is to find an expression for the (k,l)-multiplicity series of $UT_n(E)$. In the light of Proposition 3.2.2, we define the linear operator

$$G:\Lambda_{(2k,l)}\to\Lambda_{(2k,l)}$$

such that

$$G(\widehat{M}(g))=\widehat{M}\left(g\cdot\frac{1}{2}\left(1+\prod_{i=1}^{k}\prod_{s=1}^{l}\frac{(1+t_i)(1+y_s)}{(1-t_i)(1-y_s)}\right)\right),$$

where $g\in\Lambda^{(k,l)}$.

Remark 3.3.1. Notice that $HS_{(q,1^{m-q})}(\mathrm{T}_k,\mathrm{Y}_l)$ participates in the decomposition of $HS_{(q)}(\mathrm{T}_k,\mathrm{Y}_l)HS_{(1)}(\mathrm{T}_k,\mathrm{Y}_l)$ and $HS_{(q-1)}(\mathrm{T}_k,\mathrm{Y}_l)HS_{1^{(m-q+1)}}(\mathrm{T}_k,\mathrm{Y}_l)$ as sum of hook Schur functions. Hence $HS_{(q,1^{m-q})}$ appears with multiplicity 2 in the product

$$\sum_{n\geqslant 0}HS_{(n)}(\mathrm{T}_k,\mathrm{Y}_l)\cdot\sum_{m\geqslant 0}HS_{(1^m)}(\mathrm{T}_k,\mathrm{Y}_l).$$

It follows that

$$H(E;\mathrm{T}_k,\mathrm{Y}_l) = 1+\sum_{m\geqslant 1}\sum_{q=1}^{m} HS_{(q,1^{m-q})}(\mathrm{T}_k,\mathrm{Y}_l) = \frac{1}{2}\left(1+\sum_{n\geqslant 0} HS_{(n)}(\mathrm{T}_k,\mathrm{Y}_l)\sum_{m\geqslant 0} HS_{(1^m)}(\mathrm{T}_k,\mathrm{Y}_l)\right).$$

It is well known that

$$\sum_{n\geqslant 0} HS_{(n)}(\mathrm{T}_k,\mathrm{Y}_l) = \prod_{i=1}^{k}\prod_{s=1}^{l}\frac{(1+y_s)}{(1-t_i)};$$

$$\sum_{m\geqslant 0} HS_{(1^m)}(\mathrm{T}_k,\mathrm{Y}_l) = \prod_{i=1}^{k}\prod_{s=1}^{l}\frac{(1+t_i)}{(1-y_s)}.$$

Due to Remark 3.3.1, we define the following two operators $G_1 : \Lambda_{(2k,l)} \to \Lambda_{(2k,l)}$ and $G_2 : \Lambda_{(2k,l)} \to \Lambda_{(2k,l)}$ given by

$$G_1(\widehat{M}(g)) = \widehat{M}\left(g\cdot\prod_{i=1}^{k}\prod_{s=1}^{l}\frac{(1+y_s)}{(1-t_i)}\right)$$

and

$$G_2(\widehat{M}(g)) = \widehat{M}\left(g\cdot\prod_{i=1}^{k}\prod_{s=1}^{l}\frac{(1+t_i)}{(1-y_s)}\right),$$

where $g \in \Lambda^{(k,l)}$. Note that $G_1\circ G_2 = G_2\circ G_1$ and $G = \frac{1}{2}\left(\mathbb{1} + G_2\circ G_1\right)$ where $\mathbb{1}$ is the identity map.

3.3.1 The action of G_1 and G_2

Now we describe the action of G_1 and G_2 on $\Lambda_{2k,l}$. Let us start with the operator G_1. Using the notation of chapter 2, we define the following linear operator.

Definition 3.3.3. *Given a positive integer d. Let $f(\mathrm{T}_d) \in \mathbb{C}[[\mathrm{T}_d]]^{S_d}$, define the conjugate Young operator $\overline{Y}$ on $\mathbb{C}[[\mathrm{V}_d]]$ as*

$$\overline{Y}(M(f(\mathrm{T}_d)) := M\left(f(\mathrm{T}_d)\cdot\sum_{s=0}^{d} S_{(1^s)}(\mathrm{T}_d)\right).$$

Lemma 3.3.1. *Consider the hook $H(k,l)$ and let $\lambda \in H(k,l)$ be a partition such that $\lambda_0 = (l^k)$. Then*

$$\widehat{M}\left(HS_\lambda(\mathrm{T}_k,\mathrm{Y}_l)HS_{(n)}(\mathrm{T}_k,\mathrm{Y}_l)\right) = \mathrm{V}_k^{\lambda_0}\sum_{m=0}^{n} M(S_\mu(\mathrm{T}_k)S_{(m)}(\mathrm{T}_k))M(S_\nu(\mathrm{Y}_l)S_{(1^{n-m})}(\mathrm{Y}_l)) \quad (3.2)$$

Proof. Recall that $HS_\beta(\mathrm{T}_k,\mathrm{Y}_l) = 0$ if, and only if, $\beta \notin H(k,l)$. Since $\lambda_0 = (l^k)$, if $HS_\beta(\mathrm{T}_k,\mathrm{Y}_l)$ participates in the decomposition of

$$HS_\lambda(\mathrm{T}_k,\mathrm{Y}_l)HS_{(n)}(\mathrm{T}_k,\mathrm{Y}_l),$$

then $\beta_0 = \beta \cap (l^k) = (l^k)$. Hence, applying the Young rule to the partition λ is equivalent to applying the Young rule to μ and ν. So, we have

$$\widehat{M}\left(HS_\lambda(\mathrm{T}_k, \mathrm{Y}_l) HS_{(n)}(\mathrm{T}_k, \mathrm{Y}_l)\right) = v_1^l \cdots v_k^l \sum_{m=0}^{n} M(S_\mu(\mathrm{T}_k) S_{(m)}(\mathrm{T}_k)) M(S_\nu(\mathrm{Y}_l) S_{(1^{n-m})}(\mathrm{Y}_l)) \quad \square$$

Lemma 3.3.2. *Let λ be a partition in the hook $H(k,l)$ such that $\lambda_0 \neq (l^k)$, then*

$$\widehat{M}\left(HS_\lambda(\mathrm{T}_k, \mathrm{Y}_l) HS_{(n)}(\mathrm{T}_k, \mathrm{Y}_l)\right) = \sum_{\beta_0 \in \Pi} V_k^{\beta_0} \sum_{p=0}^{n-|D_{\beta_0} \backslash D_{\lambda_0}|} M\left(S_\mu S_{(p)}; \mathrm{T}_{\mathrm{r}_{\beta_0}}\right) M\left(S_\nu S_{(1^{n-|D_{\beta_0}\backslash D_{\lambda_0}|-p})}; \mathrm{Y}_c\right) \quad (3.3)$$

where $\Pi = \{\beta_0 \subseteq (l^k) | \beta$ participates in the decomposition of $HS_\lambda HS_{(n)}\}$, r_{β_0} is the number of the rows of β_0 of size l and c is the number of the columns of λ_0 of size k.

Proof. Note that the difference between this case and Lemma 3.3.1 is that if $\beta \in H(k,l)$ participates in the decomposition of

$$HS_\lambda(\mathrm{T}_k, \mathrm{Y}_l) HS_{(n)}(\mathrm{T}_k, \mathrm{Y}_l)$$

as sum of hook Schur functions, then β_0 is not necessarily λ_0. Consider the set Π and notice that if $\lambda_1 \geqslant l$ or $\lambda_1' \geqslant k$, then $\lambda_0 \in \Pi$. The possible β_0 are those whose diagrams are obtained from the diagram of λ_0 when we apply the Young rule to the partitions λ_0 and (m) for some $0 \leqslant m \leqslant n$ such that $D_{\beta_0} \subseteq D_{(l^k)}$.

Now, we identify λ with the triple (λ_0, μ, ν). Suppose that D_{β_0} is obtained from D_{λ_0} by adding m boxes. Note that $m = |D_{\beta_0} \backslash D_{\lambda_0}|$, that is, m is the number of boxes in the skew-diagram $D_{\beta_0} \backslash D_{\lambda_0}$. If we want to know what partitions β satisfy $\beta \cap (l^k) = \beta_0$, we have to add a total of $n - m$ boxes to the diagrams D_μ and D_ν using the Young rule. Consider the numbers r_{β_0} and c, observe that we can add boxes to the diagram D_μ up to line r_{β_0}. In the case of the diagram D_ν, it is only allowed to add boxes up to line c. Hence, the partitions $\beta \in H(k,l)$ participating in the decomposition of

$$\widehat{M}(HS_\lambda(\mathrm{T}_k, \mathrm{Y}_l) HS_{(n)}(\mathrm{T}_k, \mathrm{Y}_l))$$

such that $\beta \cap (l^k) = \beta_0$ are determined by the following expression

$$V_k^{\beta_0} \sum_{p=0}^{n-|D_{\beta_0} \backslash D_{\lambda_0}|} M\left(S_\mu S_{(p)}; \mathrm{T}_{\mathrm{r}_{\beta_{0,\mathrm{m}}}}\right) M\left(S_\nu S_{(1^{n-|D_{\beta_0}\backslash D_{\lambda_0}|-p})}; \mathrm{Y}_c\right).$$

It follows that $\widehat{M}\left(HS_\lambda(\mathrm{T}_k, \mathrm{Y}_l) HS_{(n)}(\mathrm{T}_k, \mathrm{Y}_l)\right)$ equals

$$\sum_{\beta_0 \in \Pi} V_k^{\beta_0} \sum_{p=0}^{n-|D_{\beta_0} \backslash D_{\lambda_0}|} M\left(S_\mu S_{(p)}; \mathrm{T}_{\mathrm{r}_{\beta_0}}\right) M\left(S_\nu S_{(1^{n-|D_{\beta_0}\backslash D_{\lambda_0}|-p})}; \mathrm{Y}_c\right). \quad \square$$

Using Lemmas 3.3.1, 3.3.1 and the linearity of $\widehat{M}$ and M, we obtain

Theorem 3.3.1. *Let λ be a partition in $H(k,l)$. The action of G_1 on $\widehat{M}(HS_\lambda(\mathrm{T}_k, \mathrm{Y}_l))$ can be described as follows*

i. If $\lambda_0 = (l^k)$, then

$$G_1(\widehat{M}(HS_\lambda(\mathrm{T}_k, \mathrm{Y}_l))) = \mathrm{V}_k^{\lambda_0} Y(M(S_\nu; T_k))\overline{Y}(M(S_\mu; \mathrm{Y}_l)).$$

ii. If $\lambda_0 \neq (l^k)$, then

$$G_1(\widehat{M}(HS_\lambda(\mathrm{T}_k; \mathrm{Y}_l))) = \sum_{\beta_0 \in \Omega} \mathrm{V}_k^{\beta_0} Y(M(S_\mu; \mathrm{T}_{r_{\beta_0}}))\overline{Y}(M(S_\nu); \mathrm{Y}_c)),$$

where $\Omega = \{\beta_0 \subseteq (l^k) | D_{\beta_0}$ is obtained from D_{λ_0} by the Young rule (Case 1)$\}$, r_{β_0} is the number of rows of the diagram of D_{β_0} of size l, and c is the number of colums of the diagram of D_{λ_0} of size k.

Note that if $r_{\beta_0} = 0$ or $c = 0$, then $\varnothing = Y_0 = T_0$. Hence $1 = Y(1, \mathrm{T}_0) = \overline{Y}(1, \mathrm{Y}_0)$.

Now, let us study G_2. Our goal is to describe G_2 in terms of the operators Y and $\overline{Y}$ defined above. Note that (n) and (1^n) are conjugate partitions, so from Theorem 3.3.1 we get the following result.

Corollary 3.3.1. *Let λ be a partition in $H(k,l)$. The action of G_2 on $\widehat{M}(HS_\lambda(\mathrm{T}_k, \mathrm{Y}_l))$ can be described as follows:*

i. If $\lambda_0 = (l^k)$ then

$$G_2(\widehat{M}(HS_\lambda(\mathrm{T}_k, \mathrm{Y}_l))) = \mathrm{V}_k^{\lambda_0} \overline{Y}(M(S_\nu; T_k))Y(M(S_\mu; \mathrm{Y}_l))$$

ii. If $\lambda_0 \neq (l^k)$ then

$$G_2(\widehat{M}(HS_\lambda(\mathrm{T}_k; \mathrm{Y}_l))) = \sum_{\beta_0 \in \Omega'} \mathrm{V}_k^{\beta_0} \overline{Y}(M(S_\mu; \mathrm{T}_r))Y(M(S_\nu); \mathrm{Y}_{c_{\beta_0}})),$$

where $\Omega' = \{\beta_0 \subseteq (l^k) | D_{\beta_0}$ is obtained from D_{λ_0} by Young rule (Case 2)$\}$, r is the number of rows of the diagram of D_{λ_0} of size l and c_{β_0} is the number of columns of the diagram of D_{β_0} of size k.

3.4 The (k, l)-multiplicity series of $UT_n(E)$

The following result is analogous to Theorem 2.2.3. We obtain an expression for the (k, l)-multiplicity series of $UT_n(E)$ using the linear operator G.

Theorem 3.4.1. *Let E be the infinite dimensional Grassmann algebra. Then*

$$\widehat{M}(UT_n(E);\mathrm{V}_k,\mathrm{T}_k,\mathrm{Y}_l)=\sum_{j=1}^{n}\sum_{q=0}^{j-1}\sum_{\lambda\vdash q}(-1)^{j-q-1}\binom{n}{j}\binom{j-1}{q}d_\lambda G^j(\mathrm{V}_k^{\lambda_0}\mathrm{T}_k^{\mu}\mathrm{Y}_l^{\nu}),$$

where d_λ is the degree of the S_λ-characters χ_λ and

$$T_k^{\mu}Y_l^{\nu}=t_1^{\mu_1}\cdots t_k^{\mu_k}y_1^{\nu_1}\cdots y_l^{\nu_l}$$

for $\mu=(\mu_1,\dots,\mu_k)$, $\nu=(\nu_1,\dots,\nu_l)$ partitions outside the rectangle (l^k).

Proof. Expanding the expression of $H(UT_n(E);\mathrm{T}_k,\mathrm{Y}_l)$ from Corollary 3.2.1 we obtain:

$$H(UT_n(E);\mathrm{T}k,\mathrm{Y}_l)\;=\;\sum_{j=1}^{n}\binom{n}{j}\left(\frac{1}{2}\left[1+\prod_{i=1}^{k}\prod_{j=1}^{l}\frac{(1+t_i)}{(1-t_i)}\frac{(1+y_j)}{(1-y_j)}\right]\right)^{j}\sum_{q=0}^{j-1}(-1)^{j-1-q}\binom{j-1}{q}\left(\sum_{i=1}^{l}t_i+\sum_{s=1}^{l}y_s\right)^{q}.$$

Since

$$\left(\sum_{i=1}^{l}t_i+\sum_{s=1}^{l}y_s\right)^{q}=HS_{(1)}(\mathrm{T}_k,\mathrm{Y}_l)^q=\sum_{\lambda\vdash q}d_\lambda HS_\lambda(\mathrm{T}_k,\mathrm{Y}_l),$$

where d_λ is the degree of the S_λ-characters χ_λ, it follows that

$$H(UT_n(E);\mathrm{T}k,\mathrm{Y}_l)=\sum_{j=1}^{n}\sum_{q=0}^{j-1}\sum_{\lambda\vdash q}(-1)^{j-q-1}\binom{n}{j}\binom{j-1}{q}d_\lambda\left(\frac{1}{2}\left[1+\prod_{i=1}^{k}\prod_{j=1}^{l}\frac{(1+t_i)}{(1-t_i)}\frac{(1+y_j)}{(1-y_j)}\right]\right)^{j}HS_\lambda(\mathrm{T}_k,\mathrm{Y}_l).$$

Recall that we can identify $\lambda\in H(k,l)$ with the partitions λ_0,μ,ν (see figure 2), so

$$G^j(\mathrm{V}_k^{\lambda_0}\mathrm{T}_k^{\mu}\mathrm{Y}_l^{\nu})=G^j(M(HS_\lambda(\mathrm{T}_k,\mathrm{Y}_l)))=M\left(\left(\frac{1}{2}\left[1+\prod_{i=1}^{k}\prod_{j=1}^{l}\frac{(1+t_i)}{(1-t_i)}\frac{(1+y_j)}{(1-y_j)}\right]\right)^{j}\cdot HS_\lambda(\mathrm{T}_k,\mathrm{Y}_l)\right).$$

Hence

$$\widehat{M}(UT_n(E);V_k,T_k,Y_l)=\sum_{j=1}^{n}\sum_{q=0}^{j-1}\sum_{\lambda\vdash q}(-1)^{j-q-1}\binom{n}{j}\binom{j-1}{q}d_\lambda G^j(V_k^{\lambda_0}T_k^{\mu}Y_l^{\nu}),$$

and the proof follows. □

Theorems 3.3.1, 3.3.1 and 3.4.1 give us an algorithm to compute the multiplicities in the cocharacter sequences of $UT_n(E)$. In order to show how the algorithm works, we consider some particular cases.

Proposition 3.4.1. *The $(1,1)$-multiplicity series of E is*

$$\widehat{M}(E;v,t,y)=1+\frac{v}{(1-t)(1-y)}.$$

Proof. Using Theorem 3.4.1, we have $\widehat{M}(E;v,t,y)=G(1)$. By theorem 3.3.1 we get

$$G_1(1)=G_1(\widehat{M}(HS_{\lambda_0});v,t,y)=Y(1;\mathrm{T}_0)\overline{Y}(1;\mathrm{Y}_0)+vY(1;\mathrm{T}_1)\overline{Y}(1;\mathrm{Y}_0)=1+\frac{v}{1-t}.$$

Now, we are going to compute $G_2(1)$ and $G_2\left(\frac{v}{1-t}\right)$. Using Corollary 3.3.1, we get

$$G_2(1) = G_2(\widehat{M}(HS_{(0)}); v, t, y) = \overline{Y}(1, \mathrm{T}_0)Y(1, \mathrm{Y}_0) + v\overline{Y}(1, \mathrm{T}_0)Y(1, \mathrm{Y}_1) = 1 + \frac{v}{1-y}$$

$$G_2\left(\frac{v}{1-t}\right) = v\overline{Y}\left(\frac{1}{1-t}; \mathrm{T}_1\right)Y(1; \mathrm{Y}_1) = \frac{v}{(1-t)(1-y)} + \frac{vt}{(1-t)(1-y)}.$$

Hence

$$G(1) = \frac{1}{2}\left(\mathbb{1} + G_2 \circ G_1(1)\right) = \frac{1}{2}\left(1 + 1 + \frac{2v}{(1-t)(1-y)}\right) = 1 + \frac{v}{(1-t)(1-y)},$$

as desired. □

Notice that at light of Theorem 3.2.1, the $(1,1)$-multiplicity series of E already contains the information about the multiplicities in the cocharacter sequence of E. In fact:

$$\frac{v}{(1-t)(1-y)} = \sum_{m,n\geqslant 0} vt^n y^m,$$

then, expanding its homogeneous component of degree m, we obtain

$$\sum_{q=1}^{m} vt^{q-1}y^{m-q}$$

which gives the exact multiplicities of the m-th cocharacter of E as in Theorem 2.2.2.

By Theorem 3.2.3, we have $\chi(UT_2(E)) \subseteq H(2,3)$. Hence we would like to compute the $(2,3)$ multiplicity series of $UT_2(E)$. We need the following technical lemma.

Lemma 3.4.1. *Consider the hook $H(2,3)$ and the set of variables $\{v_1, v_2, t_1, t_2, y_1, y_2, y_3\}$. Then:*

i. $G(1, \mathrm{V}_2, \mathrm{T}_2, \mathrm{Y}_3) = 1 + v_1 + v_1^2 + \dfrac{v_1v_2}{1-y_1} + \dfrac{v_1^2v_2}{1-y_1} + \dfrac{v_1^3}{1-t_1} + \dfrac{v_1^3v_2}{(1-t_1)(1-y_1)}.$

ii.

$$\begin{aligned}
G^2(1) \quad = \quad & 1 + 2v_1 + 3v_1^2 + v_1v_2\left(\frac{2}{1-y_1} + \frac{1}{(1-y_1)^2}\right) + v_1^2v_2\left(\frac{4}{1-y_1} + \right.\\
& \left.\frac{1+y_1}{(1-y_1)^2} + \frac{1}{(1-y_1)^2}\right) + v_1^3\left(\frac{3}{1-t_1} + \frac{1}{(1-t_1)^2}\right) + \\
& v_1^3v_2\left(\frac{5}{(1-t_1)(1-y_1)} + \frac{2(1+y_1)}{(1-t_1)(1-y_1)^2} + \frac{1+t_1}{(1-t_1)^2(1-y_1)} + \right.\\
& \left.\frac{1+t_1y_1}{(1-t_1)^2(1-y_1)^2}\right) + v_1^2v_2^2\left(\frac{2}{1-y_1} + \frac{2(1+y_1)}{(1-y_1)^2} + \frac{4y_1y_2}{(1-y_1)^2(1-y_1y_2)}\right) + \\
& v_1^3v_2^2\left(\frac{3}{(1-t_1)(1-y_1)} + \frac{3(1+y_1)}{(1-t_1)(1-y_1)^2} + \frac{6y_1y_2}{(1-t_1)(1-y_1)^2(1-y_1y_2)}\right.\\
& \left. + \frac{1+t_1}{(1-t_1)^2(1-y_1)} + \frac{(1+t_1)(1+y_1)}{(1-t_1)^2(1-y_1)^2} + \frac{2(1+t_1)y_1y_2}{(1-t_1)^2(1-y_1)^2(1-y_1y_2)}\right) +
\end{aligned}$$

$$v_1^3v_2^3\left(\frac{1}{(1-t_1)(1-y_1)}+\frac{1+y_1}{(1-y_1)^2(1+t_1)}+\right.$$
$$\frac{2y_1y_2}{(1-t_1)(1-y_1)^2(1-y_1y_2)}+\frac{(1+t_1+t_1t_2-t_1^2t_2)}{(1-t_1)^2(1-t_1t_2)(1-y_1)}$$
$$\left.+\frac{(1+t_1+t_1t_2-t_1^2t_2)(1+y_1)}{(1-t_1)^2(1-t_1t_2)(1-y_1)^2}+\frac{2(1+t_1+t_1t_2-t_1^2t_2)y_1y_2}{(1-t_1)^2(1-t_1t_2)(1-y_1)^2(1-y_1y_2)}\right).$$

iii.

$$G^2(v_1) = v_1++2v_1^2+v_1v_2\left(\frac{1}{1-y_1}+\frac{1}{(1-y_1)^2}\right)+v_1^2v_2\left(\frac{3}{1-y_1}+\frac{1+y_1}{(1-y_1)^2}+\right.$$
$$\left.\frac{2}{(1-y_1)^2}\right)+v_1^3v_2\left(\frac{4}{(1-t_1)(1-y_1)}+\frac{3(1+y_1)}{(1-t_1)(1-y_1)^2}+\right.$$
$$\left.\frac{2(1+t_1y_1)}{(1-t_1)^2(1-y_1)^2}+\frac{(1+t_1)}{(1-t_1)^2(1-y_1)}\right)+v_1^3\left(\frac{2}{1-t_1}+\frac{1}{(1-t_1^2)}\right)+$$
$$v_1^2v_2^2\left(\frac{2}{1-y_1}+\frac{3(1+y_1)}{(1-y_1)^2}+\frac{6y_1y_2}{(1-y_1)^2(1-y_1y_2)}+\frac{(1+y_1y_2)}{(1-y_1)^2(1-y_1y_2)}\right)$$
$$+v_1^3v_2^2\left(\frac{3}{(1-t_1)(1-y_1)}+\frac{5(1+y_1)}{(1-t_1)(1-y_1)^2}+\frac{10y_1y_2}{(1-t_1)(1-y_1)^2(1-y_1y_2)}\right.$$
$$+\frac{2(1+t_1)(1+y_1)}{(1-t_1)^2(1-y_1)^2}+\frac{4(1+t_1)y_1y_2}{(1-t_1)^2(1-y_1)^2(1-y_1y_2)}+$$
$$\left.\frac{1+y_1y_2+2t_1y_1y_2+t_1y_1-t_1y_1^2y_2}{(1-t_1)^2(1-y_1)^2(1-y_1y_2)}+\frac{1+3y_1y_2-y_1^2y_2+y_1}{(1-t_1)(1-y_1)^2(1-y_1y_2)}\right)$$
$$+v_1^3v_2^3\left(\frac{1}{(1-t_1)(1-y_1)}+\frac{2(1+y_1)}{(1-t_1)(1-y_1)^2}+\right.$$
$$\frac{4y_1y_2}{(1-t_1)(1-y_1)^2(1-y_1y_2)}+\frac{(1+t_1+t_1t_2-t_1^2t_2)}{(1-t_1)^2(1-t_1t_2)(1-y_1)}+$$
$$\frac{2(1+t_1+t_1t_2-t_1^2t_2)(1+y_1)}{(1-t_1)^2(1-t_1t_2)(1-y_1)^2}+\frac{4(1+t_1+t_1t_2-t_1^2t_2)y_1y_2}{(1-t_1)^2(1-t_1t_2)(1-y_1)^2(1-y_1y_2)}+$$
$$\frac{(1+t_1+t_1t_2-t_1^2t_2)(1-y_1^2y_2+3y_1y_2+y_1+2y_1y_2y_3)}{(1-t_1)^2(1-t_1t_2)(1-y_1)^2(1-y_1y_2)}$$
$$\left.\frac{(1-y_1^2y_2+3y_1y_2+y_1+2y_1y_2y_3)}{(1-t_1)(1-y_1)^2(1-y_1y_2)}\right).$$

Proof. i. The result follows directly from the definition of G.

ii. By the previous item and the linearity of G, we have

$$G^2(1) = G(1)+G(v_1)+G(v_1^2)+G\left(\frac{v_1v_2}{1-y_1}\right)+G\left(\frac{v_1^2v_2}{1-y_1}\right)+G\left(\frac{v_1^3}{1-t_1}\right)+$$
$$G\left(\frac{v_1^3v_2}{(1-t_1)(1-y_1)}\right). \tag{3.4}$$

We compute each part of the right hand side of the equality separately. Let us calculate $G(v_1)$. We start with computing $G_1(v_1)$. From Theorem 3.3.1, we get

$$G_1(v_1) = v_1+v_1^2+v_1v_2+v_1^2v_2+\frac{v_1^3v_2}{1-t_1}+\frac{v_1^3}{1-t_1}.$$

It follows that

$$G_2(G_1(v_1)) = G_2(v_1) + G_2(v_1^2) + G(v_1v_2) + G_2(v_1^2v_2) + G_2\left(\frac{v_1^3v_2}{1-t_1}\right) + G_2\left(\frac{v_1^3}{1-t_1}\right).$$

By Corollary 3.3.1, we have:

$$\begin{aligned} G_2(G_1(v_1)) =& v_1 + 2v_1^2 + \frac{2v_1v_2}{1-y_1} + \frac{2v_1^3}{1-t_1} + \frac{4v_1^2v_2}{1-y_1} + \frac{4v_1^3v_2}{(1-t_1)(1-y_1)} + \\ & \frac{2v_1^2v_2^2}{1-y_1} + \frac{2v_1^3v_2^2}{(1-t_1)(1-y_1)}. \end{aligned}$$

Hence

$$\begin{aligned} G(v_1) =& v_1 + v_1^2 + \frac{v_1v_2}{1-y_1} + \frac{2v_1^2v_2}{1-y_1} + \frac{v_1^3}{1-t_1} + \\ & \frac{2v_1^3v_2}{(1-t_1)(1-y_1)} + \frac{v_1^2v_2^2}{1-y_1} + \frac{v_1^3v_2^2}{(1-t_1)(1-y_1)}. \end{aligned} \tag{3.5}$$

To compute the remaining parts, we use Theorem 3.3.1 and Corollary 3.3.1. As the computations are too extensive, we only write the final results:

$$\begin{aligned} G(v_1^2) =& v_1^2 + \frac{v_1^2v_2}{1-y_1} + \frac{v_1^3}{1-t_1} + \frac{2v_1^3v_2}{(1-t_1)(1-y_1)} + \frac{v_1^2v_2^2}{1-y_1} + \frac{2v_1^3v_2^2}{(1-t_1)(1-y_1)} + \\ & \frac{v_1^3v_2^3}{(1-t_1)(1-y_1)}; \end{aligned} \tag{3.6}$$

$$\begin{aligned} G\left(\frac{v_1v_2}{1-y_1}\right) =& \frac{v_1v_2}{(1-y_1)^2} + v_1^2v_2\left(\frac{1+y_1}{(1-y_1)^2}\right) + \\ & v_1^2v_2^2\left(\frac{1+y_1}{(1-y_1)^2} + \frac{2y_1y_2}{(1-y_1)^2(1-y_1y_2)}\right) \\ & + v_1^3v_2\left(\frac{1+y_1}{(1-t_1)(1-y_1)^2}\right) + \\ & v_1^3v_2^2\left(\frac{1+y_1}{(1-t_1)(1-y_1)^2} + \frac{2y_1y_2}{(1-y_1)^2(1-y_1y_2)}\right); \end{aligned}$$

$$\begin{aligned} G\left(\frac{v_1^3}{1-t_1}\right) =& \frac{v_1^3}{(1-t_1)^2} + v_1^3v_2\left(\frac{1+t_1}{(1-t_1)^2(1-y_1)}\right) + v_1^3v_2^2\left(\frac{1+t_1}{(1-t_1)^2(1-y_1)}\right) + \\ & v_1^3v_2^3\left(\frac{1+t_1+t_1t_2-t_1^2t_2}{(1-t_1)^2(1-t_1t_2)(1-y_1)}\right); \end{aligned}$$

$$\begin{aligned} G\left(\frac{v_1^2v_2}{1-y_1}\right) =& \frac{v_1^2v_2}{(1-y_1)^2} + v_1^3v_2\left(\frac{1+y_1}{(1-t_1)(1-y_1)^2}\right) + v_1^2v_2^2\left(\frac{1+y_1}{(1-y_1)^2} + \right. \\ & \left.\frac{2y_1y_2}{(1-y_1)^2(1-y_1y_2)}\right) + v_1^3v_2^2\left(\frac{2(1+y_1)}{(1-t_1)(1-y_1)^2} + \right. \\ & \left.\frac{4y_1y_2}{(1-t_1)(1-y_1)^2(1-y_1y_2)}\right) + v_1^3v_2^3\left(\frac{(1+y_1)}{(1-t_1)(1-y_1)^2} + \right. \\ & \left.\frac{2y_1y_2}{(1-t_1)(1-y_1)^2(1-y_1y_2)}\right); \end{aligned}$$

$$
\begin{aligned}
G\left(\frac{v_1^3v_2}{(1-t_1)(1-y_1)}\right) =& v_1^3v_2\left(\frac{1+t_1y_1}{(1-t_1)^2(1-y_1)^2}\right)+v_1^3v_2^2\left(\frac{(1+t_1)(1+y_1)}{(1-t_1)^2(1-y_1)^2}+\right. \\
&\left.\frac{2(1+t_1)y_1y_2}{(1-t_1)^2(1-y_1)^2(1-y_1y_2)}\right)+ \\
&v_1^3v_2^3\left(\frac{(1+t_1+t_1t_2-t_1^2t_2)(1+y_1)}{(1-t_1)^2(1-t_1t_2)(1-y_1)^2}+\right. \\
&\left.\frac{2(1+t_1+t_1t_2-t_1^2t_2)y_1y_2}{(1-t_1)^2(1-t_1t_2)(1-y_1)^2(1-y_1y_2)}\right).
\end{aligned}
$$

By equality (3.4) and the previous computations, we obtain the desired result.

iii. From equality (3.5), we have

$$
\begin{aligned}
G^2(v_1) =& G(v_1)+G(v_1^2)+G\left(\frac{v_1v_2}{1-y_1}\right)+2G\left(\frac{v_1^2v_2}{1-y_1}\right)+G\left(\frac{v_1^3}{1-t_1}\right)+ \\
&2G\left(\frac{v_1^3v_2}{(1-t_1)(1-y_1)}\right)+G\left(\frac{v_1^2v_2^2}{1-y_1}\right)+G\left(\frac{v_1^3v_2^2}{(1-t_1)(1-y_1)}\right).
\end{aligned}
\tag{3.7}
$$

It only remains to calculate $G\left(\frac{v_1^2v_2^2}{1-y_1}\right)$ and $G\left(\frac{v_1^3v_2^2}{(1-t_1)(1-y_1)}\right)$, since the other summands have been found in item b).

The following equalities are obtained by Theorem 3.3.1 and Corollary 3.3.1.

$$
\begin{aligned}
G\left(\frac{v_1^2v_2^2}{1-y_1}\right) =& v_1^2v_2^2\left(\frac{1+y_1y_2}{(1-y_1)^2(1-y_1y_2)}\right)+v_1^3v_2^2\left(\frac{1+3y_1y_2-y_1^2y_2+y_1}{(1-y_1)^2(1-y_1y_2)(1-t_1)}\right)+ \\
&v_1^3v_2^3\left(\frac{1+3y_1y_2-y_1^2y_2+y_1+2y_1y_2y_3}{(1-y_1)^2(1-y_1y_2)(1-t_1)}\right).
\end{aligned}
$$

$$
\begin{aligned}
G\left(\frac{v_1^3v_2^2}{(1-t_1)(1-y_1)}\right) =& v_1^3v_2^2\left(\frac{1+y_1y_2+2t_1y_1y_2+t_1y_1-t_1y_1^2y_2}{(1-y_1)^2(1-y_1y_2)(1-t_1)^2}\right)+ \\
&v_1^3v_2^3\left(\frac{(1+t_1+t_1t_2-t_1^2t_2)(1+3y_1y_2-y_1^2y_2+y_1+2y_1y_2y_3)}{(1-t_1)^2(1-t_1t_2)(1-y_1)^2(1-y_1y_2)}\right).
\end{aligned}
$$

Now the result follows by a combination of equality (3.7) and the previous computations. □

Theorem 3.4.2. *The $(2,3)$-multiplicity series of $UT_2(E)$ is*

$$\begin{aligned}\widehat{M}(UT_2(E);\mathrm{V}_2,\mathrm{T}_2,\mathrm{Y}_3) =& 1 + v_1 + v_1^2 + \frac{v_1v_2}{1-y_1} + \frac{v_1^3}{1-t_1} + \frac{v_1^2v_2(2-y_1)}{(1-y_1)^2} + \\ & v_1^3v_2\left(\frac{1}{(1-t_1)(1-y_1)} + \frac{1+y_1}{(1-t_1)(1-y_1)^2} + \frac{1+t_1y_1}{(1-t_1)^2(1-y_1)^2}\right) \\ & + v_1^2v_2^2\left(\frac{2+y_1}{(1-y_1)^2} + \frac{4y_1y_2}{(1-y_1)^2(1-y_1y_2)}\right) + \\ & v_1^3v_2^2\left(\frac{2(1+y_1)}{(1-t_1)(1-y_1)^2} + \frac{4y_1y_2}{(1-t_1)(1-y_1)^2(1-y_1y_2)}\right. \\ & + \frac{(1+t_1)(1+y_1)}{(1-t_1)^2(1-y_1)^2} + \frac{2(1+t_1)y_1y_2}{(1-t_1)^2(1-y_1)^2(1-y_1y_2)} + \\ & \frac{1+y_1y_2+2t_1y_1y_2+t_1y_1-t_1y_1^2y_2}{(1-t_1)^2(1-y_1)^2(1-y_1y_2)} \\ & \left.+\frac{1+3y_1y_2-y_1^2y_2+y_1}{(1-t_1)(1-y_1)^2(1-y_1y_2)}\right) + v_1^3v_2^3\left(\frac{(1+y_1)}{(1-t_1)(1-y_1)^2}+\right. \\ & + \frac{2y_1y_2}{(1-t_1)(1-y_1)^2(1-y_1y_2)} + \frac{(1+t_1+t_1t_2-t_1^2t_2)(1+y_1)}{(1-t_1)^2(1-t_1t_2)(1-y_1)^2} + \\ & \frac{2(1+t_1+t_1t_2-t_1^2t_2)y_1y_2}{(1-t_1)^2(1-t_1t_2)(1-y_1)^2(1-y_1y_2)} + \\ & \frac{(1+t_1+t_1t_2-t_1^2t_2)(1-y_1^2y_2+3y_1y_2+y_1+2y_1y_2y_3)}{(1-t_1)^2(1-t_1t_2)(1-y_1)^2(1-y_1y_2)} \\ & \left.+\frac{(1-y_1^2y_2+3y_1y_2+y_1+2y_1y_2y_3)}{(1-t_1)(1-y_1)^2(1-y_1y_2)}\right).\end{aligned}$$

Proof. Due to Proposition 3.2.3 we have $\chi(UT_2(E)) \subseteq H(2,3)$. Hence we work with the set of variables $\{v_1, v_2, t_1, t_2, y_1, y_2, y_3\}$. By Theorem 3.4.1 we obtain

$$\widehat{M}(UT_2(E);\mathrm{V}_2,\mathrm{T}_2,\mathrm{Y}_3) = 2G(1) - G^2(1) + G^2(v_1). \tag{3.8}$$

From equality (3.8) and Lemma 3.4.1, the result follows. □

The next result was proved by Centrone in [15]. Now, we are going to prove it using the $(2,3)$-multiplicity series of $UT_2(E)$.

Corollary 3.4.1. *Let λ be a partition. The multiplicity m_λ in the cocharacter sequence of $UT_2(E)$ is given by the following expression:*

$$m_\lambda = \begin{cases} 1 & if \quad \lambda = (n) \\ 1 & if \quad \lambda = (1^m),\ m > 1 \\ m+1 & if \quad \lambda = (2,1^m),\ m \geqslant 1 \\ 3m+2 & if \quad \lambda = (2,2,1^m),\ m \geqslant 0 \\ 4(m+1) & if \quad \lambda = (2,2,2^s,1^m),\ m \geqslant 0,\ s > 0 \\ 2nm-3m-n+3 & if \quad \lambda = (n,1^m),\ n \geqslant 3,\ m \geqslant 1 \\ 6m(n-3)+9m+3(n-3)+5 & if \quad \lambda = (n,2,1^m),\ n \geqslant 3,\ m \geqslant 0 \\ (8(n-3)+12)(m+1) & if \quad \lambda = (n,2,2^s,1^m),\ n \geqslant 3,\ s \geqslant 1,\ m \geqslant 0 \\ 4(n_1-n_2+1)(2m+1) & if \quad \lambda = (n_1,n_2,1^m),\ n_1 \geqslant n_2 \geqslant 3,\ m \geqslant 0 \\ 12(n_1-n_2+1)(m+1) & if \quad \lambda = (n_1,n_2,2^s,1^m),\ n_1 \geqslant n_2 \geqslant 3, \\ & \quad\quad s \geqslant 1,\ m \geqslant 0 \\ 4(n_1-n_2+1)(m+1) & if \quad \lambda = (n_1,n_2,3,2^s,1^m),\ n_1 \geqslant n_2 \geqslant 3, \\ & \quad\quad s \geqslant 0,\ m \geqslant 0 \\ 0 & for \quad all\ other\ \lambda \end{cases}$$

Proof. Given a partition λ, by Theorem 3.2.3 we know that $m_\lambda = 0$ if $\lambda \notin H(2,3)$. Hence let $\lambda \in H(2,3)$. In order to compute the multiplicity m_λ, it is necessary to write the hook multiplicity series of $UT_2(E)$ as a power series.

Let $\lambda \in H(2,3)$ and consider the triple (λ_0, μ, ν). Notice that $D_{\lambda_0} \subseteq D_{(3,3)}$. It follows that

$$\lambda_0 \in \{(1),(2),(3),(1,1),(2,1),(3,1),(2,2),(3,2),(3,3)\}.$$

First, let λ be a partition such that $\lambda_0 \in \{(1),(2),(3)\}$. Using Theorem 3.4.2, we obtain $m_\lambda = 1$.

Consider now λ such that $\lambda_0 = (1,1)$. By Theorem 3.4.2, we have that λ_0 corresponds to the summand

$$\frac{v_1v_2}{1-y_1} = v_1v_2\left(\sum_{m\geqslant 0} y_1^m\right).$$

It follows that partitions of type $\lambda = (1,1,1^m)$ with $m \geqslant 0$ have multiplicity 1 or, equivalently, if $\lambda = (1^m)$, with $m > 1$ then $m_\lambda = 1$.

Now, let λ be such that $\lambda_0 = (2,1)$. Observe that

$$\frac{v_1^2v_2(2-y_1)}{(1-y_1)^2} = v_1^2v_2\left(2 + \sum_{n\geqslant 1}(m+2)y_1^m\right).$$

Hence, if $\lambda = (2,1)$, then $m_\lambda = 2$. Moreover, if $\lambda = (2,1,1^m)$, with $m \geqslant 1$, then $m_\lambda = m+2$ or, equivalently, if $\lambda = (2,1^m)$, with $m \geqslant 2$, then $m_\lambda = m+1$.

Let λ be a partition such that $\lambda_0 = (2,2)$. Then

$$v_1^2v_2^2\left(\frac{2+y_1}{(1-y_1)^2} + \frac{4y_1y_2}{(1-y_1)^2(1-y_1y_2)}\right) = v_1^2v^2\left(2 + \sum_{m\geqslant 1}(3m+2)y_1^m + \sum_{m\geqslant 1}\sum_{s\geqslant 1} 4my_1^{m+s-1}y_2^s\right).$$

Hence if $\lambda = (2,2)$, then $m_\lambda = 2$. If $\lambda = (2,2,1^m)$, then $m_\lambda = 3m+2$.

Observe that $y_1^{m+s-1}y_2^s$ is in one-to-one correspondence with the partition $\nu = (m+s-1, s)$, hence $\nu' = (2^s, 1^{m-1})$. So, if $\lambda = (2, 2, 2^s, 1^{m-1})$, with m, $s \geqslant 1$, then $m_\lambda = 4m$ or, equivalently, if $\lambda = (2, 2, 2^s, 1^m)$, with $s \geqslant 1$ and $m \geqslant 0$, then $m_\lambda = 4(m+1)$.

The remaining cases are treated similarly. □

Now we compute the multiplicities m_λ in the cocharacter sequence of $UT_3(E)$ when $\lambda \in H(1,1)$.

Theorem 3.4.3. *Let λ be a partition such that $\lambda \in H(1,1)$. The multiplicity m_λ in the cocharacter sequence of $UT_3(E)$ is given by the following expression:*

$$m_\lambda = \begin{cases} 1 & if \quad \lambda = (n), \ n \geqslant 0 \\ 1 & if \quad \lambda = (1^m), \ m > 1 \\ n & if \quad \lambda = (n,1), \ n \geqslant 2 \\ m+1 & if \quad \lambda = (2,1^m), \ m \geqslant 2 \\ \frac{1}{4}(76 - 90m + 26m^2 - 54n + 68mn - 20m^2n & if \quad \lambda = (n,1^m), \ n \geqslant 3, \\ +10n^2 - 12mn^2 + 4m^2n^2) & \qquad m \geqslant 2 \end{cases}$$

Proof. By Theorem 3.4.1, we have

$$\widehat{M}(UT_3(E); v, t, y) = 3G(1) - 3G^2(1) + 3G^2(v) + G^3(1) - 2G^3(v) + G^3(vt) + G^3(vy) \quad (3.9)$$

By Theorem 3.3.1 and Corollary 3.3.1, we obtain

- $G(1) = 1 + \dfrac{v}{(1-t)(1-y)}$,
- $G^2(1) = 1 + \dfrac{v}{(1-t)(1-y)} + \dfrac{v(1+ty)}{(1-t)^2(1-y)^2}$,
- $G^2(v) = \dfrac{v(1+2ty+t^2y^2)}{(1-t)^2(1-y)^2}$,
- $G^3(1) = 1 + \dfrac{v}{(1-t)(1-y)} + \dfrac{v(1+ty)}{(1-t)^2(1-y)^2} + \dfrac{v(1+2ty+t^2y^2)}{(1-t)^3(1-y)^3}$,
- $G^3(v) = \dfrac{v(1+3ty+3t^2y^2+t^3y^3)}{(1-t)^3(1-y)^3}$,
- $G^3(vt) = \dfrac{v(t+3t^2y+3t^3y^2+t^4y^3)}{(1-t)^3(1-y)^3}$,
- $G^3(vy) = \dfrac{v(y+3ty^2+3t^2y^2+t^3y^4)}{(1-t)^3(1-y)^3}$.

By Equation (3.9) the $(1,1)$-multiplicity series of $UT_3(E)$ in the variables v,t,y is

$$\widehat{M}(UT_3(E);v,t,y) = 1 + \frac{v}{(1-t)(1-y)} + \frac{v(1+4ty+3t^2y^2)}{(1-t)^2(1-y)^2} + \frac{v}{(1-t)^3(1-y)^3}\Big(-1+t + y - 4ty - 5t^2y^2 - 2t^3y^3 + 3t^2y + 3t^3y^2 + t^4y^3 + 3ty^2 + 3t^2y^3 + t^3y^4\Big) \tag{3.10}$$

Note that in order to compute the multiplicity m_λ where $\lambda \in H(1,1)$, it is necessary to write (3.10) as a power series. Recall that

$$\frac{t^{a_1}y^{a_2}}{(1-t)(1-y)} = \sum_{n\geqslant a_1}\sum_{m\geqslant a_2} t^n y^m,$$

$$\frac{t^{a_1}y^{a_2}}{(1-t)^2(1-y)^2} = \sum_{n\geqslant a_1}\sum_{m\geqslant a_2} (n-a_1+1)(m-a_2+1)t^n y^m,$$

$$\frac{t^{a_1}y^{a_2}}{(1-t)^3(1-y)^3} = \sum_{n\geqslant a_1}\sum_{m\geqslant a_2} \binom{n-a_1+2}{2}\binom{m-a_2+2}{2} t^n y^m.$$

Using the previous equations and making some algebraic manipulations, we obtain the following expression

$$\widehat{M}(UT_3(E);v,t,y) = 1 + v\left(\sum_{n\geqslant 0} t^n + \sum_{m\geqslant 1} y^m + \sum_{n\geqslant 1}(n+1)t^n y + \sum_{m\geqslant 2}(m+1)ty^m + \sum_{n\geqslant 2}\sum_{m\geqslant 2}\frac{(32-34(m+n)+10(n^2+m^2)-12(m^2n+n^2m)+44mn+4m^2n^2)}{4}t^n y^m\right) \tag{3.11}$$

By Equation (3.11), it follows that if $\lambda = (n)$ or $\lambda = (1^m)$ then $m_\lambda = 1$. Now, if $\lambda = (n+1,1)$ and $n \geqslant 1$ then $m_\lambda = n+1$, this means that if $\lambda = (n,1)$ with $n \geqslant 2$ then $m_\lambda = n$. Observe that if $\lambda = (2,1^m)$ with $m \geqslant 2$, its multiplicity is $m+1$.
Finally if $\lambda = (n+1,1^m)$ with n, $m \geqslant 2$, we have that

$$m_\lambda = \frac{32-34(m+n)+10(n^2+m^2)-12(mn^2+n^2m)+44mn+4m^2n^2}{4},$$

or equivalently if $\lambda = (n,1^m)$ with $n \geqslant 3$ and $m \geqslant 2$, we have that

$$m_\lambda = \frac{76-90m+26m^2-54n+68mn-20m^2n+10n^2-12mn^2+4m^2n^2}{4}.$$

□

Recall that if we want to know all multiplicities of the cocharacter sequence of $UT_3(E)$, we have to work with the hook $H(3,5)$ because by Theorem 3.2.3 we know that $\chi(UT_3(E)) \subseteq H(3,5)$. Hence the $(3,5)$-multiplicity series $\widehat{M}(UT_3(E), \mathrm{V}_3, \mathrm{T}_3, \mathrm{Y}_5)$ has 11 variables and the computations are very technical.

4 Specht property of varieties of graded Lie algebras

Let F be an infinite field and let $L = UT_n(F)^{(-)}$ be the Lie algebra of the $n \times n$ upper triangular matrices. If $a, b \in L$, denote the commutator $[a, b] = ab - ba$. The Lie brackets are asssumed left normed, that is, $[a, b, c] = [[a, b], c]$.

In this chapter we consider a particular but important grading on L. Suppose $G = \mathbb{Z}_n$ in additive notation and recall that e_{ij} stand for the matrix units: e_{ij} has an entry 1 at position (i, j) and 0 elsewhere, then the algebra L is G-graded by setting $L = \oplus_{k=0}^{n-1} L_k$ where L_k is the span of all e_{ij} such that $j - i = k$, this grading is called the *canonical* $\mathbb{Z}_n$*-grading*. The principal goal of this chapter is to prove that the $T_{\mathbb{Z}_n}$-ideal of graded identities of L has the Specht property when $\operatorname{char} F \geqslant n$ or $\operatorname{char} F = 0$. Moreover we prove that when $n = 3$ and F is of characteristic 2 then the corresponding ideal of graded identities does not satisfy the Specht property. Hence the restriction imposed on the characteristic cannot be removed. The most significant new results of this chapter are Theorem 4.1.1, Proposition 4.2.6, Theorem 4.2.1, Theorem 4.2.2 and Theorem 4.3.1.

The $T_{\mathbb{Z}_n}$-ideal of graded identities of L over an infinite field was described in

Theorem 4.0.1 (Koshlukov-Yukihide [50])**.** *The $\mathbb{Z}_n$-graded identities of the Lie algebra L of the upper triangular $n \times n$ matrices over F follow from*

$$\begin{cases} [x_1^{(i)}, x_2^{(j)}], & i + j \geqslant n \\ [x_1^{(0)}, x_2^{(0)}] & \end{cases} \tag{4.1}$$

4.1 The case $UT_2(F)^{(-)}$

First, we are going to study the Specht property when $n = 2$. Let Y and Z be two infinite countable sets, $Y = \{y_1, y_2, \ldots\}$ and $Z = \{z_1, z_2, \ldots\}$. Consider the free Lie algebra $\mathcal{L}(Y \cup Z)$ generated over F by $Y \cup Z$. Note that the algebra $\mathcal{L}(Y \cup Z)$ has a natural $\mathbb{Z}_2$-grading assuming the variables Y to be of degree 0 and those Z of degree 1. Consider the algebra $UT_2(F)$ endowed with the canonical $\mathbb{Z}_2$-grading, denote by I the ideal of $\mathbb{Z}_2$-graded identities for $UT_2(F)$. By Theorem 4.0.1 and the Jacobi identity, we get

$$[z, y_1, y_2] - [z, y_2, y_1] \in I, \tag{4.2}$$

hence the non-zero monomials in $L(Y \cup Z)/I$ are of the type

$$[z, y_{i_1}, y_{i_2}, \ldots, y_{i_k}]$$

where $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k$.

Notation 1. *We will denote* $[z, \underbrace{y_1, \ldots, y_1}_{a_1}, \ldots, \underbrace{y_n, \ldots, y_n}_{a_n}]$ *by* $[z, a_1y_1, \ldots, a_ny_n]$.

Consider the following set

$$\mathcal{B} = \{[z, a_1y_1, \ldots, a_ny_n] | n \geqslant 1,\ a_i > 0,\ \text{for all } i\}.$$

Given $f = [z, a_1y_1, \ldots, a_ny_n] \in \mathcal{B}$, we will define $V_f := (a_1 \ldots, a_n)$. Note that $\{V_f | f \in \mathcal{B}\} \subseteq D(\mathbb{N})$, hence $V_f = (a_1, \ldots, a_n) \leqslant V_g = (a'_1, \ldots, a'_m)$ if there exists an order preserving injection $\varphi\colon \mathbb{N} \to \mathbb{N}$ such that $\varphi(n) \leqslant m$ and $a_i \leqslant a'_{\varphi(i)}$ for any $i = 1, \ldots, n$.

Proposition 4.1.1. *Let* $f, g \in \mathcal{B}$. *If* $V_f \leqslant V_g$ *then* g *is a consequence of* f *modulo* I, *i.e.,* $g \in \langle f \cup I \rangle_{T_{\mathbb{Z}_2}}$.

Proof. Suppose that $V_f = (a_1, \ldots, a_n)$ and $V_g = (a'_1, \ldots, a'_m)$. By hypothesis, there is a subsequence $(a'_{i_1}, \ldots, a'_{i_n})$ of V_g such that $a_j \leqslant a'_{i_j}$ for all $j \in \{1, \ldots, n\}$. Let $f(z, y_{i_1}, \ldots, y_{i_n})$ the polynomial obtained replacing the variable y_j by y_{i_j} for each $1 \leqslant j \leqslant n$ in $f(z, y_1, \ldots, y_n)$. Then by equality (4.2), we get

$$[f(z, y_{i_1}, \ldots, y_{i_n}), (a'_{i_1} - a_1)y_{i_1}, \ldots, (a'_{i_n} - a_n)y_{i_n}] \equiv [z, a'_{i_1}y_{i_1}, \ldots, a'_{i_n}y_{i_n}] \pmod I$$

Now, let $\{l_1, \ldots, l_{m-n}\} = \{1 \ldots, m\} - \{i_1, \ldots, i_m\}$. Using again equality (4.2), we conclude that

$$[z, a'_{i_1}y_{i_1}, \ldots, a'_{i_n}y_{i_n}, a'_{l_1}y_{l_1}, \ldots, a'_{l_{m-n}}y_{l_{m-n}}] \equiv g \pmod I.$$

and the result follows. □

Definition 4.1.1. *Consider* $f, g \in \mathcal{B}$ *and define the following quasi-order em* $\mathcal{B}$: $f \leqslant_{\mathcal{B}} g$ *if* $V_f \leqslant V_g$.

Proposition 4.1.2. $(\mathcal{B}, \leqslant_{\mathcal{B}})$ *satisfies f.b.p.*

Proof. Suppose that there is an infinite sequence $\{f_i\}_{i\geqslant 0}$ of pairwise incomparable elements in $\mathcal{B}$ with respect to the order $\leqslant_{\mathcal{B}}$. The previous sequence defines the sequences $\{V_{f_i}\}_{i\geqslant 0}$ in $D(\mathbb{N})$, by definition 4.1.1, the elements of the sequence $\{V_{f_i}\}_{i\geqslant 0}$ are mutually incomparable, but this is a contradiction because by Theorem 1.5.1 $D(\mathbb{N})$ is partially-well ordered. □

Theorem 4.1.1. *Let* J *be a* $T_{\mathbb{Z}_2}$*-ideal such that* $I \subseteq J$, *then* J *is finitely generated as* $T_{\mathbb{Z}_2}$*-ideal.*

Proof. Since F is an infinite field, we know that J is generated as T_{Z_2}-ideal by multihomogeneous polynomials. Hence there exists a subset $\mathcal{B}' \subseteq \mathcal{B}$ such that

$$J = \langle \mathcal{B}' \rangle_{T_{\mathbb{Z}_2}} \pmod I.$$

By Proposition 4.1.2 and Theorem 1.5.1, there is $\mathcal{B}_0 \subseteq \mathcal{B}'$ such that $\mathcal{B}_0$ is a finite set and $\mathcal{B}_0 \subseteq \mathcal{B}' \subseteq \overline{\mathcal{B}}_0$. It follows that given $g \in \mathcal{B}'$, there exists $f \in \mathcal{B}_0$ such that $V_f \leqslant V_g$. By Proposition 4.1.1, $g \in \langle f \rangle_{T_{\mathbb{Z}_2}} \subseteq \langle \mathcal{B}_0 \rangle_{T_{\mathbb{Z}_2}}$ mod I, then

$$J = \langle \mathcal{B}_0 \rangle_{T_{\mathbb{Z}_2}} \pmod{I},$$

hence the $T_{\mathbb{Z}_2}$-ideal J is finitely generated. □

4.2 The case $UT_3(F)^{(-)}$

Let Y, Z and W be infinite countable sets, namely $Y = \{y_1, y_2, \ldots\}$, $Z = \{z_1, z_2, \ldots\}$ and $W = \{w_1, w_2, \ldots\}$. Consider the free Lie algebra $\mathcal{L}(Y \cup Z \cup W)$ generated over F by $Y \cup Z \cup W$. Note that the algebra $\mathcal{L}(Y \cup Z \cup W)$ has a natural $\mathbb{Z}_3$-grading assuming the variables Y, Z and W to be of degree $0, 1$ and 2 respectively. Consider the algebra $UT_3(F)$ endowed with the canonical $\mathbb{Z}_3$-grading, denote by I the ideal of $\mathbb{Z}_3$-graded identities for $UT_3(F)$.

By Theorem 4.0.1, the $\mathbb{Z}_3$-graded identities for $UT_3(F)$ follow from

$$\begin{cases} [y_1, y_2] \\ [z_1, z_2, z_3] \\ [w_1, w_2] \\ [z, w] \end{cases} \tag{4.3}$$

Using the previous identities and the Jacobi identity, we have the following equalities, (modulo I):

$$[z, y_1, y_2] = [z, y_2, y_1], \tag{4.4}$$

$$[w, y_1, y_2] = [w, y_2, y_1], \tag{4.5}$$

$$[z_1, y_1, y_2, z_2] = [z_1, y_2, y_1, z_2]. \tag{4.6}$$

It follows that the non-zero monomials in $\mathcal{L}(Y \cup Z \cup W)/I$ are the following types

I. $[z, y_{i_1}, \ldots, y_{i_k}]$;

II. $[w, y_{i_1}, \ldots, y_{i_k}]$;

III. $[z, y_{i_1}, \ldots, y_{i_k}, z, y_{j_1}, \ldots, y_{j_l}]$;

IV. $[z_1, y_{i_1}, \ldots, y_{i_k}, z_2, y_{j_1}, \ldots, y_{j_l}]$,

where $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k$ and $j_1 \leqslant i_2 \leqslant \cdots \leqslant j_l$. Observe that if f is a monomial of type III, then its second block of the variables Y can be empty. Moreover if f is a monomial of type IV, then its first or second block of the variables Y can be empty or both.

Let $f \in \mathcal{L}(Y \cup Z \cup W)/I$ be a multihomogeneous polynomial, then f is equivalent to a monomial of type I or II or f is a linear combinations of monomials of type III or type IV.

Notation 2.

$$[z, a_1y_1, \ldots, a_ny_n] = [z, \underbrace{y_1, \ldots, y_1}_{a_1}, \ldots, \underbrace{y_n, \ldots, y_n}_{a_n}]$$

$$[w, a_1y_1, \ldots, a_ny_n] = [w, \underbrace{y_1, \ldots, y_1}_{a_1}, \ldots, \underbrace{y_n, \ldots, y_n}_{a_n}]$$

$$[z_1, a_1y_1, \ldots, a_ny_n, z_2, b_1y_1, \ldots, b_ny_n] = [z_1, \underbrace{y_1, \ldots, y_1}_{a_1}, \ldots, \underbrace{y_n, \ldots, y_n}_{a_n}, z_2, \underbrace{y_1, \ldots, y_1}_{b_1}, \ldots, \underbrace{y_n, \ldots, y_n}_{b_n}]$$

Consider the following sets

$$\mathcal{B}_1 = \{[z, a_1y_1, \ldots, a_ny_n] \mid n \geqslant 1,\ a_i > 0,\ \text{for all } i\},$$
$$\mathcal{B}_2 = \{[w, a_1y_1, \ldots, a_ny_n] \mid n \geqslant 1,\ a_i > 0,\ \text{ for all } i\},$$
$$\mathcal{B}_{(1,1)} = \{[z_1, a_1y_1, \ldots, a_ny_n, z_2, b_1y_1, \ldots, b_ny_n] \mid n \geqslant 0,\ a_i, b_i \geqslant 0,\ \text{ for all } i\}.$$

Given $f \in \mathcal{B}_i$, we assign to it the finite sequence $V_f = (a_1, \ldots, a_n)$ where $1 \leqslant i \leqslant 2$. Note that $V_f \in D(\mathbb{N})$. So, we define the following order in $f \in \mathcal{B}_i$.

Definition 4.2.1. *Let f, $g \in \mathcal{B}_i$ where $i = 1$ or 2, we define $f \leqslant_{\mathcal{B}_i} g$ whenever $V_f \leqslant V_g$.*

Proposition 4.2.1. *Consider the set $\mathcal{B}_i$ and let f, $g \in \mathcal{B}_i$.*

i. If $f \leqslant_{\mathcal{B}_i} g$, then g is a consequence of f modulo I.

ii. $(\mathcal{B}_i, \leqslant_{\mathcal{B}_i})$ satisfies the f.b.p.

Proof. The argument is analogous to the proofs of Propositions 4.1.1 and 4.1.2. □

Corollary 4.2.1. *Let J be a $T_{\mathbb{Z}_3}$-ideal such that $I \subseteq J$ and let $\mathcal{A}_i = \{f \in J \mid f \in \mathcal{B}_i\}$, $i = 1$, 2. Then there exist finite subsets $\mathcal{A}_i' \subseteq \mathcal{A}_i$ such that*

$$\langle \mathcal{A}_i \rangle_{T_{\mathbb{Z}_3}} = \langle \mathcal{A}_i' \rangle_{T_{\mathbb{Z}_3}} \pmod{I}.$$

Proof. By item ii. of Proposition 4.2.1 and Theorem 1.5.1, there exist finite subsets $\mathcal{A}_i' \subseteq \mathcal{A}_i$ where $i = 1$, 2, such that

$$\mathcal{A}_i' \subseteq \mathcal{A}_i \subseteq \overline{\mathcal{A}_i'}.$$

It follows that given $g \in \mathcal{A}_i$, there exists $f \in \mathcal{A}_i'$ such that $f \leqslant_{\mathcal{B}_i} g$. Hence, by item i. of Proposition 4.2.1, g is a consequence of f modulo I. Then

$$\langle \mathcal{A}_i \rangle_{T_{\mathbb{Z}_3}} = \langle \mathcal{A}_i' \rangle_{T_{\mathbb{Z}_3}} \pmod{I}.$$ □

If $f \in \mathcal{B}_{(1,1)}$, we assign to it the finite sequence $V_f = ((a_1, b_1), \ldots, (a_n, b_n))$. Observe that $V_f \in D(\mathbb{N}_0{}^2)$. Recall that by Theorem 1.5.2 and Proposition 1.5.1, the set $D(\mathbb{N}_0{}^2)$ is partially well ordered.

Definition 4.2.2. *Let f, g be two polynomials in $\mathcal{B}_{(1,1)}$. Define the following quasi-order in $\mathcal{B}_{(1,1)}$: $f \leqslant_{\mathcal{B}_{(1,1)}} g$ whenever $V_f \leqslant_2 V_g$.*

Proposition 4.2.2. *If $f \leqslant_{\mathcal{B}_{(1,1)}} g$, then $g \in \langle f \cup I \rangle_{T_{\mathbb{Z}_3}}$.*

Proof. Suppose that $V_f = ((a_1, b_1), \ldots, (a_n, b_n))$ and $V_g = ((a'_1, b'_1), \ldots, (a'_m, b'_m))$. By hypothesis there exists a subsequence $((a'_{i_1}, b'_{i_1}), \ldots, (a'_{i_n}, b'_{i_n}))$ of V_g such that $a_k \leqslant a'_{i_k}$ and $b_k \leqslant b'_{i_k}$ for every $k \in \{1, \ldots, n\}$. Let $f(z_1, z_2, y_{i_1}, \ldots, y_{i_n})$ be the polynomial obtained replacing the variable y_k by y_{i_k} for each $1 \leqslant k \leqslant n$ in $f(z_1, z_2, y_1, \ldots, y_n)$. Then by equality (4.5), we get

$$\begin{aligned} f_1 &= [f(z_1, z_2, y_{i_1}, \ldots, y_{i_n}), (b'_{i_1} - b_1)y_{i_1}, \ldots, (b'_{i_n} - b_n)y_{i_n}] \\ &\equiv [z_1, a_{i_1}y_{i_1}, \ldots, a_{i_n}y_{i_n}, z_2, b'_{i_1}y_{i_1}, \ldots, b'_{i_n}y_{i_n}] \pmod{I}. \end{aligned}$$

Replacing the variable z_1 by $[z_1, (a'_{i_1} - a_1)y_{i_1}, \ldots, (a'_{i_n} - a_n)y_{i_n}]$ in f_1 and applying several times (4.4), we obtain

$$\begin{aligned} &f_1([z_1, (a'_{i_1} - a_1)y_{i_1}, \ldots, (a'_{i_n} - a_n)y_{i_n}], z_2, y_{i_1}, \ldots, y_{i_n}) \\ &\equiv [z_1, a'_{i_1}y_{i_1}, \ldots, a'_{i_n}y_{i_n}, z_2, b'_{i_1}y_{i_1}, \ldots, b'_{i_n}y_{i_n}]. \end{aligned}$$

Let $\{l_1, \ldots, l_{m-n}\} = \{1 \ldots, m\} \backslash \{i_1, \ldots, i_m\}$. It follows from equalities (4.4) and (4.5)

$$\begin{aligned} [[z_1, a'_{l_1}y_{l_1}, \ldots, a'_{l_{m-n}}y_{l_{m-n}}], a'_{i_1}y_{i_1}, \ldots, a'_{i_n}y_{i_n}, z_2, \\ b'_{i_1}y_{i_1}, \ldots, b'_{i_n}y_{i_n}, b'_{l_1}y_{l_1}, \ldots, b'_{l_{m-n}}y_{l_{m-n}}] \equiv g \pmod{I}. \end{aligned}$$

Hence g is a consequence of f modulo I. □

Proposition 4.2.3. *$(\mathcal{B}_{(1,1)}, \leqslant_{\mathcal{B}_{(1,1)}})$ is a quasi well ordered set.*

Proof. Suppose that there is an infinite sequence $\{f_i\}_{i \geqslant 0}$ of pairwise incomparable elements in $\mathcal{B}_{(1,1)}$ with respect to the order $\leqslant_{\mathcal{B}_{(1,1)}}$. The above sequence defines the infinite sequence $\{V_{f_i}\}_{i \geqslant 0}$ in $D(\mathbb{N}_0{}^2)$. Note that the elements of the sequence $\{V_{f_i}\}_{i \geqslant 0}$ are pairwise incomparable, but this is a contradiction since the set $D(\mathbb{N}_0{}^2)$ is partially well ordered. □

Lemma 4.2.1. *The commutators*

$$c = [z_1, y_1, \ldots, y_t, z_2, y_{t+1}, \ldots, y_{t+k}]$$

are linearly independent modulo the $\mathbb{Z}_3$-graded identities for $UT_3(F)^{(-)}$.

Proof. Here we use a technique based on generic matrices, adapted to our case. First, consider the substitution $z_1 = e_{12}$ and $z_2 = e_{23}$. Computing c with the above substitution of the z_i, and assuming $y_i = y_i^1 e_{11} + y_i^2 e_{22} + +y_i^3 e_{33}$ where the y_i^j are commuting independent variables (here j is an upper index, not an exponent), yields

$$c = \prod_{s=1}^{2} \prod_{i} (y_i^{s+1} - y_i^1) e_{13}.$$

Here in the second product, i ranges from 1 to t, if $s = 1$. If $s = 2$, i ranges from $t+1$ to $t+k$. Now make the following substitutions for the y_i.

First put all of the $y_i^j = 0$ **except for**:

- $i = 1, \ldots, t$: here define $y_i^2 = 1$.
- $i = t+1, \ldots, t+k$: here define $y_i^3 = 1$.

Such a substitution vanishes all commutators but c, and $c = e_{13}$. So if we suppose there is a linear combination among commutators of the type c (we assume them multihomogeneous; this is no loss of generality since the base field is infinite), we assume that c participates with a nonzero coefficient in it. Then make the substitution for the y_i as above. All commutators vanish except for c. This proves they are linearly independent. □

Lemma 4.2.2. *The commutators*

$$c = [z_1, a_1 y_1, \ldots, a_n y_n, z_2, b_1 y_1, \ldots, b_n y_n]$$

are linearly independent modulo the $\mathbb{Z}_3$-graded identities for $UT_3(F)^{(-)}$.

Proof. First, consider the substitution $z_1 = e_{12}$ and $z_2 = e_{23}$. Computing c with the above substitution of the z_i, and assuming $y_i = y_i^1 e_{11} + y_i^2 e_{22} + +y_i^3 e_{33}$, where the y_i^j are commuting independent variables as above, gives

$$c = \left(\prod_{i=1}^{n} (y_i^2 - y_i^1)^{a_i} \prod_{i=1}^{n} (y_i^3 - y_i^1)^{b_i} \right) e_{13}.$$

Now making $y_i^1 = 0$ for all i, we obtain

$$c = \left(\prod_{i=1}^{n} (y_i^2)^{a_i} \prod_{i=1}^{n} (y_i^3)^{b_i} \right) e_{13}. \tag{4.7}$$

Define the following monomial

$$m_c = \prod_{i=1}^{n} (y_i^2)^{a_i} \prod_{i=1}^{n} (y_i^3)^{b_i}.$$

Consider c, c' two different commutators and notice that $m_c \neq m_{c'}$. Hence, the elements of the set $\{m_c \mid c \in \mathcal{B}_{(1,1)}\}$ are linearly independent. So if we suppose there is a nontrivial linear combination among commutators of type c (we assume them multihomogeneous) such that

$$\sum_{i=1}^{t} \alpha_i c_i(z_1, z_2, y_1, \ldots, y_n) \in I.$$

By equality (4.7), we have

$$0 = \left(\sum_{j=1}^{t} \alpha_j m_{c_j} \right) e_{13}.$$

It follows that

$$0 = \sum_{j=1}^{t} \alpha_j m_{c_j},$$

hence $\alpha_j = 0$. This proves the commutators c are linearly independent. □

Definition 4.2.3. *Let f_i, $f_j \in \mathcal{B}_{(1,1)}$ be multihomogeneous polynomials of the same multidegree. Consider the finite sequences*

$$V_{f_i} = ((a_1, b_1), \ldots, (a_n, b_n)), \quad V_{f_j} = ((a'_1, b'_1), \ldots, (a'_n, b'_n)).$$

We define the order $f_j \leqslant' f_i$ as follows:

$$f_j \leqslant' f_i \text{ if } \sum_{i=1}^{n} a'_i > \sum_{i=1}^{n} a_i, \text{ or if } \sum_{i=1}^{n} a'_i = \sum_{i=1}^{n} a_i \text{ and } (a_1, \ldots, a_n) \leqslant_{lex} (a'_1, \ldots, a'_n).$$

The order $\leqslant'$ is linear on the polynomials of the same multidegree in $\mathcal{B}_{(1,1)}$.

Definition 4.2.4. *Let f be a multihomogeneous polynomial such that*

$$f = \sum_{i=1}^{n} \alpha_i f_i,$$

where $f_i \in \mathcal{B}_{(1,1)}$ and $\alpha_i \in F \backslash \{0\}$. We define the leading monomial of f by

$$ml(f) = \max_{\leqslant'} \{f_i \mid 1 \leqslant i \leqslant n\},$$

and the leading coefficient of f, denoted by $cl(f)$, as the coefficient of $ml(f)$.

By Lemmas 4.2.1 and 4.2.2, this way of writing f as a linear combination of elements of $\mathcal{B}_{(1,1)}$ is unique. For this reason, we can define the leading monomial of f with respect to the order $\leqslant'$.

Proposition 4.2.4. *Let $f(z_1, z_2, y_1, \ldots, y_n)$ be a multihomogeneous polynomial such that*

$$f(z_1, z_2, y_1, \ldots, y_n) = \sum_{i=1}^{n} \alpha_i f_i(z_1, z_2, y_1, \ldots, y_n),$$

where $f_i(z_1, z_2, y_1, \ldots, y_n) \in \mathcal{B}_{(1,1)}$ and $\alpha_i \in F \backslash \{0\}$. The following statements hold.

i. If $g(z_1, z_2, y_1, \ldots, y_n, y) = [f(z_1, z_2, y_1, \ldots, y_n), y]$, *then* $ml(g) = [ml(f), y]$.

ii. If $h(z_1, z_2, y_1, \ldots, y_n, y) = f([z_1, y], z_2, y_1, \ldots, y_n)$, *then*

$$ml(h) = ml(f)([z_1, y], z_2, y_1, \ldots, y_n).$$

Proof. It follows from Definitions 4.2.3 and 4.2.4. □

Proposition 4.2.5. *Let* f, g *be two multihomogeneous polynomials such that*

$$f = \sum_{i=1}^{t} \alpha_i f_i, \qquad g = \sum_{j=1}^{s} \beta_j g_j,$$

where α_i, $\beta_j \in F \backslash \{0\}$, f_i, $g_j \in \mathcal{B}_{(1,1)}$ *for all* $1 \leqslant i \leqslant t$ *and* $1 \leqslant j \leqslant s$. *Suppose that* $ml(f) \leqslant_{\mathcal{B}_{(1,1)}} ml(g)$, *then there exists* $h \in \langle f \rangle_{\mathbb{Z}_3}$ *modulo* I *such that* $ml(h) = ml(g)$ *and* $cl(h) = cl(f)$.

Proof. By Proposition 4.2.2, we know that $ml(g) \in \langle ml(f) \rangle_{\mathbb{Z}_3}$ mod I. Then, making the same computations done on $ml(f)$ to obtain $ml(g)$, we can obtain a consequence h from f. Moreover, by Proposition 4.2.4, $ml(h) = ml(g)$. It is clear that the polynomial h has the same leading coefficient as f. □

Definition 4.2.5. *Let* f *be a multihomogeneous polynomial that is a linear combination of polynomials in* $\mathcal{B}_{(1,1)}$. *Then* f *is called polynomial of type* $(1, 1)$.

Proposition 4.2.6. *There is no infinite sequence of polynomial* $\{f_i\}_{i \geqslant 1}$ *of type* $(1, 1)$ *such that*

$$f_i \notin \langle f_1, \ldots, f_{i-1} \rangle_{T_{\mathbb{Z}_3}} \pmod{I}$$

where $i \geqslant 2$.

Proof. Suppose, on the contrary, that there exists such an infinite sequence of polynomials $\{f_i\}_{i \geqslant 1}$. Moreover, suppose that the f_i's are of different multidegrees in the variables Y. Define the following sets

- $J_i = \langle f_1, \ldots, f_i \rangle_{T_{\mathbb{Z}_3}} \pmod{I}$;
- $R_i = \{f \in J_i \backslash J_{i-1} \mid f$ is of type $(1, 1)$ and of the same multidegree in the variables Y as $f_i\}$;
- $\widehat{R_i} = \{ml(f) \mid f \in R_i\}$.

Note that $f_i \in R_i$, so without loss of generality, we can suppose that

$$ml(f_i) = \min_{\leqslant'} \widehat{R_i}.$$

We denote $m_i := ml(f_i)$ where $i \geqslant 1$. Then we have the infinite sequence $\{m_i\}_{i\geqslant 1}$ in $\mathcal{B}_{(1,1)}$. By Proposition 4.2.3, we have that $(\mathcal{B}_{(1,1)}, \leqslant_{\mathcal{B}_{(1,1)}})$ is a quasi well ordered set. It follows from Theorem 1.5.1 that there exists an infinite subsequence $\{m_{i_k}\}_{k\geqslant 1}$ of the sequence $\{m_i\}_{i\geqslant 1}$ such that

$$m_{i_1} \leqslant_{\mathcal{B}_{(1,1)}} m_{i_2} \leqslant_{\mathcal{B}_{(1,1)}} \cdots \leqslant_{\mathcal{B}_{(1,1)}} m_{i_k} \leqslant_{\mathcal{B}_{(1,1)}} \cdots$$

where $i_1 < i_2 < \cdots i_k < \cdots$.

Let α_{i_k} be the leading coefficient of f_{i_k}, where $k \geqslant 1$. Take $s > 1$ such that

$$\sum_{l=1}^{s} \alpha_{i_l} \neq 0.$$

Recall that $m_{i_l} \leqslant_{\mathcal{B}_{(1,1)}} m_{i_{s+1}}$, where $l \in \{1, \dots s\}$. By Proposition 4.2.5, there exists $h_l \in \langle f_{i_l} \rangle_{T_{\mathbb{Z}_2}}$ such that

$$ml(h_l) = m_{i_{s+1}} \text{ and } cl(h_l) = \alpha_{i_l},$$

for every $1 \leqslant l \leqslant s$. Consider the polynomial

$$h = \sum_{l=1}^{s} h_l,$$

and notice that $ml(h) = m_{i_{s+1}}$ and $cl(h) = \sum_{l^1}^{s} \alpha_{i_l}$.

Observe that $h_l \in J_{i_l} \subseteq J_{i_{s+1}-1}$, where $1 \leqslant l \leqslant s$, and since $J_{i_{s+1}-1}$ is $T_{\mathbb{Z}_2}$-ideal, we have $h \in J_{i_{s+1}-1}$. Define

$$g = f_{i_{s+1}} - (\alpha_{i_{s+1}} cl(h)^{-1})h.$$

Then $ml(g) \neq m_{i_{s+1}}$ and $ml(g) \leqslant' m_{i_{s+1}}$. Note that $g \in J_{i_{s+1}}$ because $f_{i_{s+1}} \in J_{i_{s+1}}$ and $h \in J_{i_{s+1}-1} \subseteq J_{i_{s+1}}$. On the other hand, $g \notin J_{i_{s+1}-1}$ because $f_{i_{s+1}} \notin J_{i_{s+1}-1}$ and $h \in J_{i_{s+1}-1}$. It follows that $g \in R_{i_{s+1}}$, then $ml(g) \in \widehat{R}_{i_{s+1}}$, but $ml(g) <' m_{i_{s+1}}$ and $m_{i_{s+1}} = \min_{\leqslant'} \widehat{R}_{i_{s+1}}$, which is a contradiction. □

As a direct consequence of Proposition 4.2.6, we have the following corollary.

Corollary 4.2.2. *Let J be a $T_{\mathbb{Z}_3}$-ideal such that $I \subseteq J$. Consider the following set*

$$\mathcal{A}_{(1,1)} = \{f \in J \mid f \text{ is a polynomial of type } (1,1)\}.$$

Then there exists a finite subset $\mathcal{A}'_{(1,1)} \subseteq \mathcal{A}_{(1,1)}$ such that

$$\langle \mathcal{A}_{(1,1)} \rangle_{T_{\mathbb{Z}_3}} = \langle \mathcal{A}'_{(1,1)} \rangle_{T_{\mathbb{Z}_3}} \pmod{I}.$$

Now we deduce the Specht property for the ideal of $\mathbb{Z}_3$-graded identities for $UT_3^{(-)}(F)$ when F is an infinite field of characteristic different from 2.

Theorem 4.2.1. *Suppose that* $\operatorname{char} F \neq 2$. *If* J *is a* $T_{\mathbb{Z}_3}$*-ideal such that* $I \subseteq J$ *then* J *is finitely generated as* $T_{\mathbb{Z}_3}$*-ideal.*

Proof. Since F is infinite, J is generated by its multihomogeneous polynomials. As $\operatorname{char} F \neq 2$, using the multilinearization process, we can consider that every multihomogeneous polynomial with variables $z's$ is linear in these variables, because by the identities (4.3), they appear in the non-zero monomials of $\mathcal{L}(Y \cup Z \cup W)/I$ at most twice. Hence, J is generated as a $T_{\mathbb{Z}_3}$-ideal, modulo I, by the following sets

- $\mathcal{A}_i = \{f \in J \mid f \in \mathcal{B}_i\}$ where $1 \leqslant i \leqslant 2$;
- $\mathcal{A}_{(1,1)} = \{f \in J \mid f \text{ is a polynomial of type } (1,1)\}$.

Using Corollaries 4.2.1 and 4.2.2, we get that there exist finite subsets $\mathcal{A}'_i \subseteq \mathcal{A}_i$, where $i = 1, 2$, and $\mathcal{A}'_{(1,1)} \subseteq \mathcal{A}_{(1,1)}$ such that

$$\langle \mathcal{A}_i \rangle_{T_{\mathbb{Z}_3}} = \langle \mathcal{A}'_i \rangle_{T_{\mathbb{Z}_3}} \pmod{I};$$
$$\langle \mathcal{A}_{(1,1)} \rangle_{T_{\mathbb{Z}_3}} = \langle \mathcal{A}'_{(1,1)} \rangle_{T_{\mathbb{Z}_3}} \pmod{I}.$$

It follows

$$J = \langle \mathcal{A}'_1 \cup \mathcal{A}'_2 \cup \mathcal{A}'_{(1,1)} \rangle_{T_{\mathbb{Z}_3}} \pmod{I},$$

therefore $J = \langle \mathcal{A}'_1 \cup \mathcal{A}'_2 \cup \mathcal{A}'_{(1,1)} \cup I \rangle_{T_{\mathbb{Z}_3}}$, and J is finitely generated as a $T_{\mathbb{Z}_3}$-ideal. □

4.2.1 The case $UT_3^{(-)}(F)$ in characteristic 2

In this subsection we prove that the graded identities of $UT_3^{(-)}(F)$ do not satisfy the Specht property if F is an infinite field of characteristic 2.

Lemma 4.2.3. *Let* F *be an infinite field of characteristic* 2. *For* $k \geqslant 1$, *define the polynomial*

$$c_k = [z, y_1, \ldots, y_k, z].$$

Consider f *a consequence of* $c_k \pmod{I}$ *such that* f *is a multihomogeneous polynomial and* $\deg f > \deg c_k$. *Suppose that* f *has degree* 2 *in the variable* z *and it is multilinear in the variables* y*'s, then* f *is a linear combination of polynomials*

$$[z, y_{i_1}, \ldots, y_{i_t}, z, y_{i_t+1}, \ldots y_{i_l}],$$

where the rightmost block of variables Y *is not empty.*

Proof. Note that the multihomogeneous consequences of the polynomial $c_k \pmod{I}$, that have degree 2 in the variable z and degree 1 in all variables y, are obtained by applying a combination of the following rules:

- Making substitutions of type $z \mapsto z + [z, y_{i_1}, \ldots, y_{i_t}]$.
- Making $[c_k, y_{i_1}, \ldots, y_{i_t}]$.

Now replacing z by $z + [z, y_{k+1}, \ldots, y_{k+n}]$ in c_k and taking the homogeneous component of degree 1 in $y_{k+1}, \ldots, y_{k+n}$, we obtain the following polynomial

$$g = [z, y_1, \ldots, y_k, y_{k+1}, \ldots, y_{k+n}, z] + [z, y_1, \ldots, y_k, [z, y_{k+1}, \ldots, y_{k+n}]] \pmod I,$$

where we apply to the first summand several times the identity (4.4).

Let $C = \{k+1, \ldots, k+n\}$ and let $P(C) = \{S \mid S \subseteq C\}$ be the set of all subsets of C. If $S = \{i_1, \ldots, i_t\} \subseteq C$, and its complement in C is $S^c = C \backslash S = \{i_{t+1}, \ldots, i_n\}$, we suppose that $i_1 < \cdots < i_t$ and $i_{t+1} < \cdots < i_n$. Define

$$[z, y_1, \ldots, y_k, Y_S, z, Y_{S^c}] = [z, y_1, \ldots, y_k, y_{i_1}, \ldots, y_{i_t}, z, y_{i_{t+1}}, \ldots, y_{i_n}].$$

Recall that char $F = 2$, so using several times the fact that $ad\, y = [*, y]$ is a derivation in every Lie algebra, we have

$$[z, y_1, \ldots, y_k, [z, y_{k+1}, \ldots, y_{k+n}]] = \sum_{S \in P(C)} [z, y_1, \ldots, y_k, Y_S, z, Y_{S^c}],$$

therefore

$$\begin{aligned} g &= [z, y_1, \ldots, y_k, y_{k+1}, \ldots, y_{k+n}, z] + \sum_{S \in P(C)} [z, y_1, \ldots, y_k, Y_S, z, Y_{S^c}] \\ &= 2[z, y_1, \ldots, y_k, y_{k+1}, \ldots, y_{k+n}, z] + \sum_{S \in P(C), S \neq C} [z, y_1, \ldots, y_k, Y_S, z, Y_{S^c}] \\ &= \sum_{S \in P(C), S \neq C} [z, y_1, \ldots, y_k, Y_S, z, Y_{S^c}], \end{aligned}$$

and the result follows. □

Theorem 4.2.2. *If F is an infinite field,* char $F = 2$*, then I, the ideal of the $\mathbb{Z}_3$-graded identities of $UT_3^{(-)}$ does not have the Specht property.*

Proof. Given $k \geqslant 1$, as above, we consider the polynomials

$$c_k(z, y_1, \ldots, y_k) = [z, y_1, \ldots, y_k, z].$$

Note that by Theorem 4.0.1, $c_k \neq 0 \pmod I$. We perform the following substitutions in the polynomial c_k

- $z = e_{12} + e_{23}$,
- $y_i = \gamma_i^1 e_{11} + \gamma_i^2 e_{22} + \gamma_i^3 e_{33}$,

where γ_i^j are commuting independent variables. In this way we obtain the following expression

$$\left(\prod_{i=1}^{k}(\gamma_i^1+\gamma_i^2)+\prod_{i=1}^{k}(\gamma_i^2+\gamma_i^3)\right)e_{13} \tag{4.8}$$

Define

$$h=\left(\prod_{i=1}^{k}(\gamma_i^1+\gamma_i^2)+\prod_{i=1}^{k}(\gamma_i^2+\gamma_i^3)\right),$$

a polynomial in the commuting variables γ_i^l where $1 \leqslant i \leqslant k$ and $1 \leqslant l \leqslant 3$.

Let $I_k = \{1, \ldots, k\}$ and consider $S \subseteq I_k$ such that $S \neq \varnothing$ and $S \neq I_k$. We define the following polynomial

$$f_S = [z, y_{i_1}, \ldots, y_{i_t}, z, y_{i_{t+1}}, \ldots, y_{i_k}],$$

where $S = \{i_1, \ldots, i_t\}$, $S^c = \{i_{t+1}, \ldots, i_k\}$, $i_1 < \cdots < i_t$ and $i_{t+1} < \cdots < i_k$. By Theorem 4.0.1, $f_S \neq 0 \pmod I$. Making the above substitution in f_S, we get

$$\left(\prod_{i\in S}(\gamma_i^1+\gamma_i^2)+\prod_{i\in S}(\gamma_i^2+\gamma_i^3)\right)\prod_{j\in S^c}(\gamma_j^1+\gamma_j^3)e_{13}. \tag{4.9}$$

Consider

$$h_S=\left(\prod_{i\in S}(\gamma_i^1+\gamma_i^2)+\prod_{i\in S}(\gamma_i^2+\gamma_i^3)\right)\prod_{j\in S^c}(\gamma_j^1+\gamma_j^3).$$

Observe that h_S is a polynomial in the commuting variables γ_i^l where $1 \leqslant i \leqslant k$ and $1 \leqslant l \leqslant 3$. Notice that

$$h \notin \operatorname{span}_F\{h_S \mid S \subseteq I_k, S \neq I_k, S \neq \varnothing\},$$

because the monomials $\gamma_1^1 \prod_{i=2}^{k} \gamma_i^2$ and $\gamma_1^3 \prod_{i=2}^{k} \gamma_i^2$ appear in the polynomial h, but these monomials do not appear in any of the h_S.

Let $J_k = \langle c_1, \ldots c_k \rangle_{T_{\mathbb{Z}_3}} \pmod I$ and suppose $c_k \in J_{k-1}$. By Lemma 4.2.3, we get

$$c_k = \sum_{S\in\mathcal{P}} \alpha_S f_S$$

where $\alpha_S \in F$ and $S \in \mathcal{P} = \{S \subseteq I_k \mid S \neq I_k, S \neq \varnothing\}$. Then (4.8) and (4.9) imply

$$h = \sum_{S\in\mathcal{P}} \alpha_S h_S,$$

but $h \notin \operatorname{span}_F\{h_S \mid S \in \mathcal{P}\}$. Therefore $c_k \notin J_{k-1}$ and we have that the following ascending chain of $T_{\mathbb{Z}_3}$-ideals modulo I

$$J_1 \subset J_2 \subset \cdots \subset J_k \subset \cdots,$$

where $J_i = \langle c_1, \ldots c_i \rangle_{T_{\mathbb{Z}_3}}$, is not stationary (does not stabilize). Therefore the $T_{\mathbb{Z}_3}$-ideal I does not have the Specht property. □

4.3 The case $UT_n(F)^{(-)}$, $n > 3$

Let Z^{g_i} and Y be disjoint infinite countable sets, $Z^{g_i} = \{z_1^{g_i}, z_2^{g_i} \ldots ,\}$ where $g_i \in \mathbb{Z}_n \backslash \{0\}$ and $Y = \{y_1, y_2, \ldots\}$. Consider the free Lie algebra

$$\mathcal{L}_{\mathbb{Z}_n} = \mathcal{L}\Big(\bigcup_{g_i \in \mathbb{Z}_n \backslash \{0\}} Z^{g_i} \cup Y \Big)$$

freely generated over F by $\bigcup_{g_i \in \mathbb{Z}_n \backslash \{0\}} Z^{g_i} \cup Y$. Notice that the algebra $\mathcal{L}_{\mathbb{Z}_n}$ has a natural $\mathbb{Z}_n$-grading assuming the variables Y, Z^{g_i} to be of homogeneous degrees 0 and g_i, for $g_i \in \mathbb{Z}_n \backslash \{0\}$, respectively. Consider the algebra $UT_n(F)$ endowed with the canonical $\mathbb{Z}_n$-grading, and denote by I the ideal of $\mathbb{Z}_n$-graded identities for $UT_n(F)$.

By Theorem 4.0.1, we get that the non-zero monomials in the algebra $\mathcal{L}_{\mathbb{Z}_n}/I$ are of the following types

I. $[z^{g_i}, y_{i_1}, \ldots, y_{i_s}]$,

II. $[z^{g_{j_1}}, y_{i_1}^{(1)}, \ldots, y_{i_s}^{(1)}, z^{g_{j_2}}, y_{l_1}^{(2)}, \ldots, y_{l_m}^{(2)}, \ldots, z^{g_{j_k}}, y_{t_1}^{(k)}, \ldots, y_{t_n}^{(k)}]$,

where $\sum_{j=1}^{k} g_{i_j} \leqslant n - 1$ and the indices of the variables y's are ordered in non-decreasing way (that is their indices, in each group, increase with possible repetitions). Observe that some (or all) of the blocks of variables y's can be empty in monomials of type II.

Let $z_1, \ldots, z_k$ be variables of degree $g_1, \ldots, g_k$, respectively, such that $\sum_{i}^{k} g_i \leqslant n - 1$ and $g_i \neq 0$. Denote by

$$[z_i, a_1 y_1, \ldots, a_n y_n] = [z, \underbrace{y_1, \ldots, y_1}_{a_1}, \ldots, \underbrace{y_n, \ldots, y_n}_{a_n}]$$

$$[z_1, a_1^{(1)} y_1, \ldots, a_n^{(1)} y_n, \ldots, z_k, a_1^{(k)} y_1, \ldots, a_n^{(k)} y_n]$$
$$= [z_1, \underbrace{y_1, \ldots, y_1}_{a_1^{(1)}}, \ldots, \underbrace{y_n, \ldots, y_n}_{a_n^{(1)}}, \ldots, z_k, \underbrace{y_1, \ldots, y_1}_{a_1^{(k)}}, \ldots, \underbrace{y_n, \ldots, y_n}_{a_n{}^{(k)}}].$$

Define the following sets

$$\mathcal{B}_{g_i} = \{[z_i, a_1^{(1)} y_1, \ldots, a_n^{(1)} y_n] \mid a_i > 0,\ n \geqslant 1\};$$
$$\mathcal{B}_{(g_1,\ldots,g_k)} = \{[z_1, a_1^{(1)} y_1, \ldots, a_n^{(1)} y_n, z_{\sigma(2)}, a_1^{(2)} y_1, \ldots, a_n^{(2)} y_n, \ldots, z_{\sigma(k)}, a_1^{(k)} y_1, \ldots, a_n^{(k)} y_n]\},$$

where $\sigma \in S_k$, $\sigma(1) = 1$, $a_i^{(j)} \geqslant 0$, $1 \leqslant j \leqslant n$, and $n \geqslant 1$.

Definition 4.3.1. *Let $f = [z_i, a_1 y_1, \ldots, a_n y_n]$, $g = [z_i, a'_1 y_1, \ldots, a'_m y_m]$ be polynomials in $\mathcal{B}_{g_i}$. Consider the finite sequences in $D(\mathbb{N})$*

$$V_f = (a_1, \ldots, a_n), \quad V_g = (a'_1, \ldots, a'_m).$$

We define the following order on $\mathcal{B}_{g_i}$: $f \leqslant_{\mathcal{B}_{g_i}} g$ whenever $V_f \leqslant V_g$.

Proposition 4.3.1. *Consider the set $\mathcal{B}_{g_i}$ and let f, $g \in \mathcal{B}_{g_i}$ be two polynomials in $\mathcal{B}_i$.*

i. If $f \leqslant_{\mathcal{B}_{g_i}} g$, then g is a consequence of f modulo I.

ii. $(\mathcal{B}_{g_i}, \leqslant_{\mathcal{B}_{g_i}})$ satisfies the f.b.p.

iii. Let J be a $T_{\mathbb{Z}_n}$-ideal such that $I \subseteq J$. Consider the following sets

$$\mathcal{A}_{g_i} = \{f \in J \mid f \in \mathcal{B}_{g_i}\}.$$

Then there exist finite subsets $\mathcal{A}'_{g_i} \subseteq \mathcal{A}_{g_i}$ such that

$$\langle \mathcal{A}_{g_i} \rangle_{T_{\mathbb{Z}_n}} = \langle \mathcal{A}'_{g_i} \rangle_{T_{\mathbb{Z}_n}} \pmod{I}.$$

Proof. The proof of this proposition is completely analogous to the proofs of Proposition 4.2.1 and Corollary 4.2.1, and that is why we omit it. □

Definition 4.3.2. *Let f, $g \in \mathcal{B}_{(g_1,\dots,g_k)}$ be respectively the polynomials*

$$[z_1, a_1^{(1)}y_1, \dots, a_n^{(1)}y_n, z_{\sigma(2)}, a_1^{(2)}y_1, \dots, a_n^{(2)}y_n, \dots, z_{\sigma(k)}, , a_1^{(k)}y_1, \dots, a_n^{(k)}y_n],$$
$$[z_1, a_1'^{(1)}y_1, \dots, a_m'^{(1)}y_m, z_{\sigma(2)}, a_1'^{(2)}y_1, \dots, a_m'^{(2)}y_m, \dots, z_{\sigma(k)}, a_1'^{(k)}y_1, \dots, a_m'^{(k)}y_m]$$

where σ, $\tau \in S_k$ are such that $1 = \sigma(1) = \tau(1)$. Consider the finite sequences

$$V_f = ((a_1^{(1)}, \dots, a_1^{(k)}), \dots, (a_n^{(1)}, \dots, a_n^{(k)})),$$
$$V_g = ((a_1'^{(1)}, \dots, a_1'^{(k)}), \dots, (a_m'^{(1)}, \dots, a_m'^{(k)})).$$

If $V_f \leqslant_k V_g$ and $\sigma = \tau$, we write $f \leqslant_{\mathcal{B}_{(g_1,\dots,g_k)}} g$.

Recall that V_f, $V_g \in D({\mathbb{N}_0}^k)$. The order $\leqslant_k$ coincides with the order defined in Example 1.5.1.

Lemma 4.3.1. *Let $f \in \mathcal{B}_{g_1,\dots,g_k}$ be such that*

$$f(z_1, \dots, z_k, y_1, \dots, y_k) = [z_1, y_1, z_2, y_2, \dots, z_k, y_k].$$

Then the polynomial

$$g = [z_1, y_1, \dots, z_{i-1}, y_{i-1}, z_i, y, y_i, z_{i+1}, \dots, z_k, y_k] \in \langle f \rangle_{T_{\mathbb{Z}_n}} \pmod{I}.$$

Proof. If $i = 1$ then $g = f([z_1, y], z_2, \dots, z_k, y_1, \dots, y_k) \in \langle f \rangle_{T_{\mathbb{Z}_n}} \pmod{I}$. If $i = k$ then $g = [f, y]$ and g is a consequence of f.

Thus we suppose $1 < i < k$. Let f_s be the polynomial obtained replacing the variable z_s by $[z_s, y]$ in f, where $1 \leqslant s \leqslant i$. Notice that f_s equals

$$[z_1, y_1, \dots, z_s, y, y_s, z_{s+1}, y_{s+1} \dots, z_k, y_k] - [z_1, y_1, \dots, z_{s-1}, y_{s-1}, y, z_s, y_s, \dots, z_k, y_k],$$

where $s \in \{1, \ldots, i\}$. Recall that $[y, y'] = 0 \bmod I$, thus one has

$$g = \sum_{s=1}^{i} f_s \pmod{I}.$$

Therefore g is a consequence of f modulo I. □

The previous lemma has as a direct consequence the following result.

Proposition 4.3.2. *Let f, $g \in \mathcal{B}_{(g_1,\ldots,g_k)}$ be polynomials such that $V_f \leqslant_{\mathcal{B}_{(g_1,\ldots,g_k)}} V_g$. Then $g \in \langle f \rangle_{T_{\mathbb{Z}_n}}$ modulo I.*

Proposition 4.3.3. *The set $(\mathcal{B}_{(g_1,\ldots,g_k)}, \leqslant_{\mathcal{B}_{(g_1,\ldots,g_k)}})$ satisfies the f.b.p.*

Proof. Suppose that there is an infinite sequence $\{f_i\}_{i\geqslant 1}$ of pairwise incomparable elements in $\mathcal{B}_{(g_1,\ldots,g_k)}$ with respect to the order $\leqslant_{\mathcal{B}_{(g_1,\ldots,g_k)}}$. Since the symmetric group S_k is a finite set, we obtain an infinite subsequence $\{f_{i_j}\}_{j\geqslant i}$ from $\{f_i\}_{i\geqslant 1}$, such that the permutations σ_{i_j} are the same for all $j \geqslant 1$. Hence, the previous subsequence induces the sequence $\{V_{f_{i_j}}\}_{j\geqslant i}$ in $D(\mathbb{N}^k)$. By Definition 4.3.2, these elements are pairwise incomparable with respect to the order $\leqslant_k$. But this is absurd because $(D(\mathbb{N}_0{}^k), \leqslant_k)$ is partially well-ordered. □

Proposition 4.3.4. *Let $n \geqslant 3$ and take z_1, ..., z_k variables of degrees g_1, ..., g_k, respectively, such that $g_i \neq 0$ and $\sum_{i=1}^{k} g_i \leqslant n - 1$. Let y_i be variables of degree 0. Then the commutators*

$$c = [z_1, y_1, \ldots, y_{t_1}, z_2, y_{t_1+1}, \ldots, y_{t_1+t_2}, z_3, \ldots, z_k, y_{t_1+\cdots+t_{k-1}+1}, \ldots, t_{t_1+\cdots+t_k}]$$

are linearly independent modulo the graded identities for $UT_n(F)^{(-)}$.

Proof. Suppose that $m - 1 = g_1 + \cdots + g_k$ and consider first the substitutions:

$$z_1 = e_{1,g_1+1}, \quad z_2 = e_{g_1+1,g_1+g_2+1}, \ldots, \quad z_k = e_{g_1+\cdots+g_{k-1}+1,g_1+\cdots+g_k+1}.$$

Pay attention $g_1 + \cdots + g_k + 1 = m \leqslant n$.

A standard staircase argument shows that, fixing z_1 at the leftmost position in the commutator, the only permutation of the z_2, ..., z_k that yields a nonzero element will be as in the commutator c given in the statement of the proposition.

Computing c with the above substitution of the z_i, and assuming $y_i = y_i^1 e_{11} + y_i^2 e_{22} + \cdots + y_i^n e_{nn}$ where the y_i^j are commuting independent variables, yields

$$c = \prod_{s=1}^{k} \prod_{i} (y_i^{g_1+\cdots+g_s+1} - y_i^1) e_{1m}.$$

Here in the second product, i ranges:

- From 1 to t_1, if $s = 1$.
- From $t_1 + \cdots + t_{s-1} + 1$ to $t_1 + \cdots + t_s$, if $s \geqslant 2$.

Now make the following substitutions for the y_i.

First put all of the $y_i^j = 0$ **except for**:

- $i = 1, \ldots, t_1$: here define $y_i^{g_1+1} = 1$.
- $i = t_1 + 1, \ldots, t_1 + t_2$: here define $y_i^{g_1+g_2+1} = 1$.
- And so on, for $i = t_1 + \cdots + t_{s-1} + 1$ to $i = t_1 + \cdots + t_s$ define $y_i^{g_1+\cdots+g_s+1} = 1$.

Such a substitution vanishes all commutators but c, and c evaluates to $c = e_{1m}$.

Let us suppose there is a nontrivial linear combination among multihomogeneous commutators of the type c, such that c participates with a nonzero coefficient in it.

First we get, by means of the above substitution for the z_i, the correct order for the $z_2, \ldots, z_k$. Then we make the substitutions for the y_i as above. All commutators vanish except for c. This implies the coefficient of c must be 0, a contradiction. This proves the linear independence. □

Proposition 4.3.5. *Let $n \geqslant 3$ and let $z_1, \ldots, z_k$ be variables of degrees $g_1, \ldots, g_k$, respectively, such that $g_i \neq 0$ and $\sum_{i=1}^{k} g_i \leqslant n-1$. Take y_i variables of degree 0. Then the commutators*

$$c = [z_1, a_1^{(1)} y_1, \ldots, a_t^{(1)} y_t, z_2, a_1^{(2)} y_1, \ldots, a_t^{(2)} y_t, \ldots, z_k, a_1^{(k)} y_1, \ldots, a_t^{(k)} y_t]$$

are linearly independent modulo the graded identities of $UT_n(F)^{(-)}$.

Proof. Suppose that $m - 1 = g_1 + \cdots + g_k$ and consider first the substitution:

$$z_1 = e_{1,g_1+1}, \quad z_2 = e_{g_1+1,g_1+g_2+1}, \quad \ldots, \quad z_k = e_{g_1+\cdots+g_{k-1}+1,g_1+\cdots+g_k+1}.$$

(Pay attention that $g_1 + \cdots + g_k + 1 = m$.)

Once again, a staircase argument shows that, fixing z_1 at the leftmost position in the commutator, the only permutation of the $z_2, \ldots, z_k$ that yields a nonzero element will be as in the commutator c.

Computing c with the above substitution of the z_i, and assuming $y_i = y_i^1 e_{11} + y_i^2 e_{22} + \cdots + y_i^n e_{nn}$, where the y_i^j are commuting independent variables, gives

$$c = \prod_{s=1}^{k} \prod_{i=1}^{t} (y_i^{g_1+\cdots+g_s+1} - y_i^1)^{a_i^{(s)}} e_{1m}.$$

Now making the substitution $y_i^1 = 0$ for all i, we obtain:

$$c = \prod_{s=1}^{k}\prod_{i=1}^{t}(y_i^{g_1+\cdots+g_s+1})^{a_i^{(s)}} e_{1m}. \tag{4.10}$$

Define the following monomial

$$m_c = \prod_{s=1}^{k}\prod_{i=1}^{t}(y_i^{g_1+\cdots+g_s+1})^{a_i^{(s)}}.$$

Note that if c and c' are different commutators with the same permutation of the $z_2, \ldots, z_k$, we have that the monomials m_c and $m_{c'}$ are different. We suppose there is a nontrivial linear combination among commutators of type c

$$\sum_{i=1}^{r} \alpha_i c_i(z_1, \ldots, z_k, y_1, \ldots, y_t)$$

which is a graded identity for $UT_n(F)^{(-)}$. First we get, by means of the substitution for the z_i, the correct order for the $z_2, \ldots, z_k$ with respect to the commutator c_1. Then we make the substitution for the y_i as above. By Equality (4.10), we have

$$0 = \Big(\sum_{j=1}^{l} \alpha_{i_j} m_{c_{i_j}}\Big) e_{1m},$$

where the polynomials c_{i_j} have the same permutation of the $z_2, \ldots, z_k$ as the polynomial c_1. It follows that $0 = \sum_{j=1}^{l} \alpha_{i_j} m_{c_{i_j}}$.

But the monomials $m_{c_{i_j}}$ (in commuting variables) are linearly independent because they are different for each c_{i_j}, hence $\alpha_{i_j} = 0$. Using the same argument several times, we have that $\alpha_i = 0$ for every $1 \leqslant i \leqslant r$, and the claim follows. $\square$

Definition 4.3.3. *Let f_i, $f_j \in \mathcal{B}_{(g_1,\ldots,g_k)}$ be multihomogeneous polynomials of the same multidegree, as given below, respectively*

$$[z_1, a_1^{(1)} y_1, \ldots, a_n^{(1)} y_n, z_{\sigma(2)}, a_1^{(2)} y_1, \ldots, a_n^{(2)} y_n, \ldots, z_{\sigma(k)}, a_1^{(k)} y_1, \ldots, a_n^{(k)} y_n];$$
$$[z_1, {a'_1}^{(1)} y_1, \ldots, {a'_n}^{(1)} y_n, z_{\sigma(2)}, {a'_1}^{(2)} y_1, \ldots, {a'_n}^{(2)} y_n, \ldots, z_{\sigma(k)}, {a'_1}^{(k)} y_1, \ldots, {a'_n}^{(k)} y_n].$$

Consider the finite sequences

$$V_{f_i} = ((a_1^{(1)}, \ldots, a_1^{(k)}), \ldots, (a_n^{(1)}, \ldots, a_n^{(k)})),$$
$$V_{f_j} = (({a'_1}^{(1)}, \ldots, {a'_1}^{(k)}), \ldots, ({a'_n}^{(1)}, \ldots, {a'_n}^{(k)})).$$

We define the order $f_j \leqslant' f_i$ if some of the following conditions is met:

- $(\sigma(2), \ldots, \sigma(k)) <_{lex} (\tau(2), \ldots, \tau(k))$;

- $\sigma = \tau$ *and* $\sum_{s=1}^{k-1}\sum_{i=1}^{n} {a'_i}^{(s)} > \sum_{s=1}^{k-1}\sum_{i=1}^{n} {a_i}^{(s)}$;

- $\sigma = \tau$, $\sum_{s=1}^{k-1}\sum_{i=1}^{n} {a'_i}^{(s)} = \sum_{s=1}^{k-1}\sum_{i=1}^{n} {a_i}^{(s)}$ *and*
  $(\sum_{i=1}^{n} {a_i}^{(1)}, \ldots, \sum_{i=1}^{n} {a_i}^{(k-1)}) <_{lex} (\sum_{i=1}^{n} {a'_i}^{(1)}, \ldots, \sum_{i=1}^{n} {a'_i}^{(k-1)})$;

- $\sigma = \tau$, $\sum_{s=1}^{k-1}\sum_{i=1}^{n} {a'_i}^{(s)} = \sum_{s=1}^{k-1}\sum_{i=1}^{n} {a_i}^{(s)}$,
  $(\sum_{i=1}^{n} {a_i}^{(1)}, \ldots, \sum_{i=1}^{n} {a_i}^{(k-1)}) = (\sum_{i=1}^{n} {a'_i}^{(1)}, \ldots, \sum_{i=1}^{n} {a'_i}^{(k-1)})$, *and*
  $({a_1}^{(1)}, \ldots, {a_n}^{(1)}, \ldots, {a_1}^{(k-1)}, \ldots, {a_n}^{(k-1)}) \leqslant_{lex} ({a'_1}^{(1)}, \ldots, {a'_n}^{(1)}, \ldots, {a'_1}^{(k-1)}, \ldots, {a'_n}^{(k-1)})$.

Observe that this order is linear on the polynomials of the same multidegree in $\mathcal{B}_{(g_1,\ldots,g_k)}$.

Suppose $f = \sum_{i=1}^{t} f_i$ is a multihomogeneous polynomial, where $f_i \in \mathcal{B}_{(g_1,\ldots,g_k)}$ and $\alpha_i \in F\backslash\{0\}$. By Propositions 4.3.4 and 4.3.5, this way of expressing f as a linear combination of elements of $\mathcal{B}_{(g_1,\ldots,g_k)}$ is unique. For this reason, we can define the leading monomial of f with respect to the order $\leqslant'$.

Definition 4.3.4. *Let* $f = \sum_{i=1}^{t} \alpha_i f_i$ *be a multihomogeneous polynomial, where* $f_i \in \mathcal{B}_{(g_1,\ldots,g_k)}$ *and* $\alpha_i \in F\backslash\{0\}$. *We define the leading monomial of* f *by*

$$ml(f) = \max_{\leqslant'}\{f_i \mid 1 \leqslant i \leqslant n\},$$

and the leading coefficient of f, $cl(f)$, *as the coefficient of* $ml(f)$.

Proposition 4.3.6. *Consider two elements in* $\mathcal{B}_{(g_1,\ldots,g_k)}$

$$f_i = [z_1, a_1^{(1)}y_1, \ldots, a_n^{(1)}y_n, z_{\sigma(2)}, a_1^{(2)}y_1, \ldots, a_n^{(2)}y_n, \ldots, z_{\sigma(k)}, a_1^{(k)}y_1, \ldots, a_n^{(k)}y_n];$$
$$f_j = [z_1, {a'_1}^{(1)}y_1, \ldots, {a'_n}^{(1)}y_n, z_{\tau(2)}, {a'_1}^{(2)}y_1, \ldots, {a'_n}^{(2)}y_n, \ldots, z_{\tau(k)}, {a'_1}^{(k)}y_1, \ldots, {a'_n}^{(k)}y_n]$$

of the same multidegree and suppose that $f_j \leqslant' f_i$. *Let* $f_i^{(s)}$, $f_j^{(s)}$ *be the polynomials obtained by replacing the variable* z_s *by* $[z_s, y]$ *in* f_i *and* f_j, *respectively, where* $1 \leqslant s \leqslant k$. *Then* $ml(f_j^{(s)}) \leqslant' ml(f_i^{(s)})$.

Proof. Consider the finite sequences

$$V_{f_i} = ((a_1^{(1)}, \ldots, a_1^{(k)}), \ldots, (a_n^{(1)}, \ldots, a_n^{(k)})),$$
$$V_{f_j} = (({a'_1}^{(1)}, \ldots, {a'_1}^{(k)}), \ldots, ({a'_n}^{(1)}, \ldots, {a'_n}^{(k)})).$$

If $(\sigma(2), \ldots, \sigma(k)) <_{lex} (\tau(2), \ldots, \tau(k))$, the result follows.

Now consider $\sigma = \tau$ and without loss of generality suppose that σ is the identity. Hence

$$\begin{aligned} f_i^{(s)} = {} & [z_1, {a_1}^{(1)}y_1, \ldots, {a_n}^{(1)}y_n, \ldots, z_s, y, {a_1}^{(s)}y_1, \ldots, {a_n}^{(s)}y_n, z_{s+1}, \\ & {a_1}^{(s+1)}y_1, \ldots, {a_n}^{(s+1)}y_n \ldots, z_k, {a_1}^{(k)}y_1, \ldots, {a_n}^{(k)}y_n] \\ & - [z_1, {a_1}^{(1)}y_1, \ldots, {a_n}^{(1)}y_n \ldots, z_{s-1}, {a_1}^{(s-1)}y_1, \ldots, {a_n}^{(s-1)}y_n, y, z_s, \\ & {a_1}^{(s)}y_1, \ldots, {a_n}^{(s)}y_n, \ldots, z_k, {a_1}^{(k)}y_1, \ldots, {a_n}^{(k)}y_n]; \end{aligned}$$

$$\begin{aligned} f_j^{(s)} = {} & [z_1, {a_1'}^{(1)}y_1, \ldots, {a_n'}^{(1)}y_n, \ldots, z_s, y, {a_1'}^{(s)}y_1, \ldots, {a_n'}^{(s)}y_n, z_{s+1}, \\ & {a_1'}^{(s+1)}y_1, \ldots, {a_n'}^{(s+1)}y_n, \ldots, z_k, {a_1'}^{(k)}y_1, \ldots, {a_n'}^{(k)}y_n] \\ & - [z_1, {a_1'}^{(1)}y_1, \ldots, {a_n'}^{(1)}y_n \ldots, z_{s-1}, {a_1'}^{(s-1)}y_1, \ldots, {a_n'}^{(s-1)}y_n, y, z_s, \\ & {a_1'}^{(s)}y_1, \ldots, {a_n'}^{(s)}y_n, \ldots, z_k, {a_1'}^{(k)}y_1, \ldots, {a_n'}^{(k)}y_n]. \end{aligned}$$

By Definition 4.3.3, we have

$$\begin{aligned} ml(f_i^{(s)}) = {} & [z_1, {a_1}^{(1)}y_1, \ldots, {a_n}^{(1)}y_n, \ldots, z_s, y, {a_1}^{(s)}y_1, \ldots, {a_n}^{(s)}y_n, z_{s+1}, \\ & {a_1}^{(1)}y_1, \ldots, {a_n}^{(s+1)}y_n \ldots, z_k, {a_1}^{(k)}y_1, \ldots, {a_n}^{(k)}y_n]; \\ ml(f_j^{(s)}) = {} & [z_1, {a_1'}^{(1)}y_1, \ldots, {a_n'}^{(1)}y_n, \ldots, z_s, y, {a_1'}^{(s)}y_1, \ldots, {a_n'}^{(s)}y_n, z_{s+1}, \\ & {a_1'}^{(1)}y_1, \ldots, {a_n'}^{(s+1)}y_n \ldots, z_k, {a_1'}^{(k)}y_1, \ldots, {a_n'}^{(k)}y_n]. \end{aligned}$$

Observe that the s-th block of variables y's in both polynomials $cl(f_i^{(s)})$ and $cl(f_j^{(s)})$ is subjected to the same modification by the same variable y. Recall that

$$[z, y_1, y_2] = [z, y_2, y_1] \pmod I$$

for every variable z of degree different from 0. Hence the order is preserved, and this means $ml(f_j^{(s)}) \leqslant' ml(f_i^{(s)})$. □

Corollary 4.3.1. *Consider* $f = \sum_{i=1}^{t} \alpha_i f_i$ *a multihomogeneous polynomial, where* $f_i \in \mathcal{B}_{(g_1,\ldots,g_k)}$*, and suppose that* $ml(f) = f_1$*. Let* $f_i^{(s)}$ *be the polynomial obtained by replacing the variable* z_s *by* $[z_s, y]$ *in* f_i*, where* $1 \leqslant s \leqslant k$ *and* $1 \leqslant i \leqslant t$*. Then* $ml(f_i^{(s)}) \leqslant' ml(f_1^{(s)})$*.*

Proof. Recall that the commutators f_i have the same multidegree. So, applying the previous proposition, the result follows. □

Proposition 4.3.7. *Consider* f*,* g *two multihomogeneous polynomials such that*

$$f = \sum_{i=1}^{t} \alpha_i f_i, \qquad g = \sum_{j=1}^{s} \beta_j g_j,$$

where α_i, $\beta_j \in F\backslash\{0\}$, $f_i, g_j \in \mathcal{B}_{(g_1,\dots,g_k)}$ for every $1 \leqslant i \leqslant t$ and $1 \leqslant j \leqslant s$. Suppose that $ml(f) \leqslant_{\mathcal{B}_{(g_1,\dots,g_k)}} ml(g)$. Then there exists $h \in \langle f\rangle_{\mathbb{Z}_n}$ modulo I such that $ml(h) = ml(g)$ and $cl(h) = cl(f)$.

Proof. By Proposition 4.3.2, we have that $ml(g)$ is a consequence of $ml(f)$ modulo I. Then Corollary 4.3.1, making the same computations as in the case of $ml(f)$ in order to obtain $ml(g)$, we deduce a consequence h from f such that $ml(h) = ml(g)$. Moreover, the leading coefficient of h is the same as that of f. □

Definition 4.3.5. *Let f be a multihomogeneous polynomial that is a linear combination of polynomials in $\mathcal{B}_{(g_1,\dots,g_k)}$. Then f is called a polynomial of type $(g_1, \dots, g_k)$.*

Proposition 4.3.8. *There is no infinite sequence of polynomials $\{f_i\}_{i\geqslant 1}$ of type $(g_1, \dots, g_k)$ such that*

$$f_i \notin \langle f_1, \dots, f_{i-1}\rangle_{T_{\mathbb{Z}_n}} \pmod I$$

for every $i \geqslant 2$.

Proof. The proof is completely analogous to that of Proposition 4.2.5. □

As a consequence of the previous result, we have the following corollary

Corollary 4.3.2. *Let J be a $T_{\mathbb{Z}_n}$-ideal such that $I \subseteq J$. Consider the following set*

$$\mathcal{A}_{(g_1,\dots,g_k)} = \{f \in J \mid f \text{ is a polynomial of type } (g_1, \dots, g_k)\}.$$

Then there exists a finite subset $\mathcal{A}'_{(g_1,\dots,g_k)} \subseteq \mathcal{A}_{(g_1,\dots,g_k)}$ such that

$$\langle \mathcal{A}_{(g_1,\dots,g_k)}\rangle_{T_{\mathbb{Z}_n}} = \langle \mathcal{A}'_{(g_1,\dots,g_k)}\rangle_{T_{\mathbb{Z}_n}} \pmod{I}.$$

Theorem 4.3.1. *Suppose that $\operatorname{char} F = 0$ or $\operatorname{char} F \geqslant n$. If J is a $T_{\mathbb{Z}_n}$-ideal such that $I \subseteq J$, then J is finitely generated as a $T_{\mathbb{Z}_n}$-ideal.*

Proof. Since F is an infinite field, J is generated by its multihomogeneous polynomials. If $\operatorname{char} F \geqslant n$ or $\operatorname{char} F = 0$, using the multilinearization process, we can consider that any multihomogeneous polynomial is linear in the variables of degree different from 0, because by Theorem 4.0.1, each of them can appear in the non-zero monomials of $\mathcal{L}_{\mathbb{Z}_n}/I$ at most $n-1$ times. Hence, J is generated as $T_{\mathbb{Z}_n}$-ideal, modulo I, by the following sets

- $\mathcal{A}_{g_i} = \{f \in J \mid f \in \mathcal{B}_{g_i}\}$ where $g_i \in \mathbb{Z}_n\backslash\{0\}$;
- $\mathcal{A}_{(g_1,\dots,g_k)} = \{f \in J \mid f \text{ is a polynomial of type } (g_1, \dots, g_k)\}$, where $\sum_{i=1}^{k} g_i \leqslant n-1$ and $g_i \neq 0$.

Using Proposition 4.3.1 and Corollary 4.3.2, we get that there exist finite subsets $\mathcal{A}'_{g_i} \subseteq \mathcal{A}_{g_i}$, where $g_i \in \mathbb{Z}_n \backslash \{0\}$, and $\mathcal{A}'_{(g_1,\dots,g_k)} \subseteq \mathcal{A}_{(g_1,\dots,g_k)}$ such that

$$\langle \mathcal{A}_{g_i} \rangle_{T_{\mathbb{Z}_n}} = \langle \mathcal{A}'_{g_i} \rangle_{T_{\mathbb{Z}_n}} \pmod{I},$$
$$\langle \mathcal{A}_{(g_1,\dots,g_k)} \rangle_{T_{\mathbb{Z}_n}} = \langle \mathcal{A}'_{(g_1,\dots,g_k)} \rangle_{T_{\mathbb{Z}_n}} \pmod{I}.$$

It follows

$$J = \langle \mathcal{M} \rangle_{T_{\mathbb{Z}_n}} \pmod{I},$$

where $\mathcal{M}$ is a finite set. Then $J = \langle \mathcal{M} \cup I \rangle_{T_{\mathbb{Z}_n}}$ and since I has a finite basis, we can conclude that J is finitely generated as a $T_{\mathbb{Z}_n}$-ideal. □

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